J. Differential Equations 254 (2013) 686–724

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonmonotone equations with large almost periodic forcing terms ☆

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ARTICLE INFO

Article history:
Received 14 June 2012
Revised 19 September 2012
Available online 3 October 2012

Keywords:
Almost periodic solutions
Bounded solutions
Haar measure
Compact groups

ABSTRACT

We consider the scalar differential equation $\dot{u} = f(u) + ch(t)$ where $f(u)$ is a jumping nonlinearity and $h(t)$ is an almost periodic function, while c is a real parameter deciding the size of the forcing term. The main result is that, if $h(t)$ does not vanish too much in some suitable sense, then the equation admits a (unique) almost periodic solution for large values of the parameter c. The class of the $h(t)$'s to which the result applies is studied in detail: it includes all the nontrivial trigonometric polynomials and is generic in the Baire sense.

1. The problem

This paper concerns the nonlinear scalar differential equation:

$$\dot{u} = f(u) + h(t) \tag{1.1}$$

where the forcing term $h(t)$ is almost periodic and the nonlinearity is a Lipschitz function which satisfies:

$$f(\pm\infty) = \pm\infty. \tag{1.2}$$

☆ The work is partially supported by project MTM2011-23652, Ministerio de Economía e Innovación, Spain and by the PRIN2009 grant "Critical Point Theory and Perturbative Methods for Nonlinear Differential Equations", Ministero dell'Istruzione, Università e Ricerca, Italy.

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http://dx.doi.org/10.1016/j.jde.2012.09.013

Indeed, stronger growth assumptions on the nonlinearity will be really needed, these will be discussed later on. On the contrary, the choice of the growth direction is purely conventional: it will be clear that nothing changes, when (1.2) is replaced by $f(\pm\infty) = \mp\infty$.

The problem investigated is the existence of almost periodic solutions to Eq. (1.1). Though explicit counter-examples are not available in the literature, the problem is not expected to be solvable for every almost periodic $h(t)$: many negative results are indeed known for similar problems, like for instance in [16,7,10] and [18]. This fact is in sharp contrast with the periodic and the bounded analogues, though the almost periodicity is in some sense intermediate between them: it is not difficult to check that bounded solutions always exist when $h(t)$ is bounded (see also Section 3) and it is well known that this implies the solvability in the periodic framework, when $h(t)$ is periodic too (see [13]).

The most classical existence result in the almost periodic framework concerns the case of nonlinearities which are monotone: see for instance Chapter 12 of Fink's book [6] or the more recent and more general paper by Bostan [3], where the full history of the problem is also presented. These results allow weak types of monotonicity, whose role in the proof is a bit cumbersome to describe. On the contrary, the consequences of a strong monotonicity assumption like:

$$\frac{f(u_2) - f(u_1)}{u_2 - u_1} \geqslant \alpha > 0 \quad \forall u_2 \neq u_1 \tag{1.3}$$

are very easily explained. In this case indeed, the bounded solution to (1.1) is unique and Favard's theory applies to show that it is also almost periodic. The original paper [5] by Favard only deals with linear equations, but the method extends to the nonlinear context: see Fink's book and also the final part of Section 5. Finally, it is worth noticing that at least another kind of monotonicity has been considered in the literature, which however does not fit into condition (1.2). Precisely, the case of a convex nonlinearity was treated in [1], proving that almost periodic solutions do exist as soon as the mean value of the forcing term:

$$\bar{h} = \lim_{T\to+\infty} \frac{1}{T} \int_0^T h(t)\,dt$$

is large enough, when compared with $h(t) - \bar{h}$.

When the nonlinearity is nonmonotone it is still possible to obtain some existence result, but the price to pay is to restrict the class of forcing terms. As far as we know, the common root of all the known results is the use of perturbative arguments. This is for instance true for the classical K.A.M. theory, where the problem is the persistence of invariant tori in a perturbed dynamical system. In this case, the forcing term is not simply almost periodic, but instead *quasi-periodic*, namely it writes as:

$$h(t) = H(\nu t)$$

where H is a continuous map on the torus $\mathbb{T}^N$. Here $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{R}^N$ is called the frequency vector and is assumed to be nonresonant: namely, its components are linearly independent over the rationals. With a little abuse of notations, the vector νt also stays for its equivalence class in the quotient space $\mathbb{T}^N$: when t varies, this vector winds around the torus on a dense orbit, due to the nonresonance condition. The existence of a quasi-periodic solution to (1.1) is granted by K.A.M. theory, under some restrictions on the frequencies and on the regularity and the size of H: see for instance [14].

A completely different use of perturbative arguments appears in [2], in order to obtain a generic type result. The forcing terms are now *limit periodic*: namely, $h(t)$ is obtained as the uniform limit, over the real line, of a sequence of purely periodic functions, typically with diverging minimal periods. Limit periodic orbits are rather important in the theory of dynamical systems: in [12] it is proved that they exist for the generic autonomous Hamiltonian system. In [2] the authors prove a similar result for the nonautonomous equation (1.1), obtaining a generic existence result for a special class of limit periodic $h(t)$'s. Roughly speaking, these limit periodic solutions originate around periodic solutions to (1.1), corresponding to periodic forcing terms: here is where the perturbative nature of the result comes into play.

The scope of the present paper is also within nonmonotone nonlinearities, with the aim of producing a new type of perturbative result: in some sense, perturbations from infinity will be considered and an existence theorem will be proved for a quite large class of almost periodic forcing terms. To introduce it, let us refer to the example:

$$f(\xi) = \begin{cases} \xi + 2 & \text{if } \xi \leqslant -1, \\ -\xi & \text{if } -1 \leqslant \xi \leqslant 1, \\ \xi - 2 & \text{if } \xi \geqslant 1. \end{cases} \tag{1.4}$$

The constant functions $u = -2, 0, +2$ are stationary solutions of the unforced equation $\dot{u} = f(u)$. They are nondegenerate, in the sense that the corresponding linearized equations have an exponential dichotomy. Thus, given an arbitrary almost periodic $h(t)$, classical roughness arguments apply to show that each stationary solution can be continued to an almost periodic solution of the equation:

$$\dot{u} = f(u) + ch(t) \tag{1.5}$$

for sufficiently small values of the parameter c. The natural question is whether or not these solutions may be continued in the large. As we already said, in general the answer is expected to be negative. The same conclusion is also suggested by the lack of a degree theory in the almost periodic framework: see [17]. Roughly speaking, the main result of the present paper is that, in spite of the previous considerations, almost periodic solutions to (1.5) still exist for *sufficiently large* values of the parameter c, as soon as the forcing term $h(t)$ *does not vanish too much.*

In order to give a precise statement, we need to introduce a couple of ingredients. The first one is a description of the general case of almost periodicity, which is similar to quasi-periodicity. It consists in thinking of the almost periodic forcing term $h(t)$ as to:

$$h(t) = H\big(\Psi(t)\big)$$

where H is a continuous function on some suitable metric, compact, connected and abelian topological group Ω, and $\Psi : \mathbb{R} \to \Omega$ is a continuous homomorphism with dense image. In this case, $h(t)$ is said to be *representable over* (Ω, Ψ). The quasi-periodic case corresponds to $\Omega = \mathbb{T}^N$ with $\Psi(t) = \nu t$. It is a standard fact that the representation is always possible, with Ω the so-called *hull* of $h(t)$. This is recalled in Section 5, based on some background material on topological groups, which is presented in Section 4.

The other ingredient we need is the Haar measure λ on Ω. This is classical tool in the literature and has a very simple expression in $\mathbb{T}^N$, where it coincides with the standard Lebesgue measure. A short introduction to the Haar measure is provided in Section 6 together with some nonstandard material, like a kind of Fubini-type decomposition along the minimal flow:

$$\omega \cdot t = \omega + \Psi(t) \tag{1.6}$$

generated on Ω by the homomorphism Ψ. This decomposition is used in the proof of the main theorem and in other parts of the paper. With these ingredients, and restricting to the toy nonlinearity above, our main result states as follows.

Theorem 1.1. *Let the nonlinearity f be defined by* (1.4)*. Under the assumption*:

$$\lambda\big(H^{-1}(0)\big) < 1/2 \tag{1.7}$$

Eq. (1.5) *admits, for c large enough, a unique almost periodic solution.*

This solution is representable over (Ω, Ψ) and is indeed the unique bounded solution. The result is proved in Section 8, for a general class of nonlinearities which satisfy:

$$\lim_{\xi\to\pm\infty} f'(\xi) = \alpha_\pm > 0. \tag{1.8}$$

There, the threshold value for the Haar measure of $H^{-1}(0)$ is also tuned on the specific nonlinearity, while the special case of (1.4) is fully treated in Appendix A. Notice that jumping nonlinearities are allowed by (1.8), overcoming the asymptotically linear character of (1.4) and then better exploiting the scalar character of the equation.

Moreover, and much more important, notice that (1.8) allows any kind of behavior in bounded regions: denoted indeed by α the smallest value between α_- and α_+, the strong monotonicity condition (1.3) anyway survives, but only for ξ large enough. This is a much stronger assumption than (1.2) and, as we will see, finally allows standard roughness methods for exponential dichotomies to apply in a rather nonstandard way. Very roughly speaking the idea is that, the larger is c and shorter is the time that the bounded solutions to (1.5) spend where $f(\xi)$ is nonmonotone: a careful implementation of this idea leads to show that all of them eventually collapse into a unique bounded solution, as in the monotone case. Of course, to trigger the device one needs a good control on the bounded solutions to (1.5): this is provided in Sections 2 and 3. The price to pay for that control is clearly stated in Theorem 3.6, but its ergodic nature becomes clear only when we interpret it in the light of flow (1.6). This happens in Section 7 and the final result is Proposition 7.4, which gives the basis to understand where condition (1.7) comes from.

Let us finally turn the attention to the main assumption (1.7). This is certainly verified when $H^{-1}(0) = \emptyset$ which however corresponds to a rather trivial case: see Remark 3.3 and Section 9. The most interesting case is when H changes sign, like for instance when it is nontrivial and satisfies:

$$\int_\Omega H\, d\lambda = 0.$$

Even with this restriction on, Theorem 1.1 has a rather wide range of application. For instance, in Section 9 we will prove that an assumption stronger than (1.7), namely:

$$\lambda\big(H^{-1}(0)\big) = 0 \tag{1.9}$$

is satisfied for a generic forcing term H. In some sense, this assumption makes the existence result suitable for every admissible nonlinearity, without bothering about analytical details.

It has to be stressed, however, that Theorem 1.1 is more than a generic-type existence result. Conditions (1.7) and (1.9) are indeed very explicit: given a concrete H, it is always possible to decide whether they are satisfied or not. The test becomes particularly simple in the quasi-periodic case, even when one starts from the knowledge of h instead of H. Just to make a concrete example, consider the most classical among quasi-periodic functions, namely:

$$h(t) = \sin(t) + \sin(\sqrt{2}t)$$

which is representable on $\mathbb{T}^2$ by the continuous function:

$$H(\theta_1, \theta_2) = \sin(2\pi\theta_1) + \sin(2\pi\theta_2).$$

The zero set of H is then the union of the two lines described in $\mathbb{T}^2$ by the equations:

$$\theta_2 = -\theta_1, \qquad \theta_2 = \theta_1 + \frac{1}{2}$$

which is easily seen to satisfy condition (1.9). The same quasi-periodic forcing term will be also considered in Appendix A, to show how to compute the threshold value for the parameter c in the case of the nonlinearity (1.4). In Section 9, moreover, it is shown that condition (1.9) is satisfied by some large classes of almost periodic forcing terms, including all the trigonometric polynomials and some specific limit periodic function.

Before concluding, we would like to thank R. Ortega for pointing out to us the paper [11] by G. Katriel. There the author considers the damped second order differential equation:

$$\ddot{u} + a\dot{u} + f(u) = ch(t)$$

where the nonlinearity is asymptotically linear in a weaker sense than (1.8), but the forcing term $h(t)$ is now purely periodic. The question is not the existence of a periodic solution, which is obvious, but instead its uniqueness for large values of the parameter c: in that, it seems very related to the present paper. The perturbative strategy is indeed exactly the same and also the assumptions on the forcing term $h(t)$ are clearly related to ours, when restricted to the periodic case. However, the proofs are quite different, inasmuch they are based on Riemann–Lebesgue type asymptotic results.

Notations

Given an additive topological group X and a function u on it, we set $u_\omega(\theta) = u(\theta + \omega)$. The symbols $C(X)$ and $B(X)$ stand for the classes of the continuous and the bounded functions on X, respectively, and we set moreover $BC(X) = C(X) \cap B(X)$. The space $B(X)$ is endowed with the standard sup-norm, namely $\|u\|_\infty = \sup_{x\in X} |u(x)|$. The same norm is used on the closed subspace $BC(X)$ and, when X is compact, also on $C(X)$.

Finally, $AP(\mathbb{R})$ stands for the class of the Bohr almost periodic functions, which is a closed subspace of $BC(\mathbb{R})$. When $u \in AP(\mathbb{R})$, its mean value is denoted by $\bar{u}$: the condition $\bar{u} = 0$ defines the closed subspace $AP_0(\mathbb{R})$.

2. An equation with a linear jumping

In this section we will study the equation:

$$\dot{y} = j(y) + h(t) \tag{2.1}$$

under the assumption that $h(t)$ is *bounded and continuous* and $j(\xi)$ is a *jumping linearity*, in the sense that:

$$j(\xi) = \begin{cases} \alpha_-\xi & \text{if } \xi \leqslant 0, \\ \alpha_+\xi & \text{if } \xi \geqslant 0. \end{cases}$$

The reason will be clear in the next section, where this equation will appear as the limit equation of (1.5), when the parameter c goes to infinity. We will assume that $j(\xi)$ is strictly increasing, namely that:

$$\alpha = \min\{\alpha_-, \alpha_+\} > 0 \tag{2.2}$$

so that the estimate:

$$\frac{j(\xi_2) - j(\xi_1)}{\xi_2 - \xi_1} \geqslant \alpha \tag{2.3}$$

holds for every $\xi_2 \neq \xi_1$.

Under these assumptions, Eq. (2.1) becomes a kind of a modified version of the linear equation:

$$\dot{y} = \alpha y + h(t)$$

whose homogeneous part exhibits an exponential dichotomy. For this equation, existence and uniqueness of the bounded solution are standard facts: this solution may be explicitly computed, and a priori estimates are easily obtained in terms of $\|h\|_\infty$. All that extends to Eq. (2.1), by replacing explicit formulas with some suitable differential inequalities. The proofs are very standard, and are sketched hereafter just for the sake of completeness.

Lemma 2.1. *For every $h \in BC(\mathbb{R})$ Eq. (2.1) admits bounded solutions and all of them satisfy*:

$$\|y\|_\infty \leqslant \frac{1}{\alpha}\|h\|_\infty.$$

Proof. All the solutions to (2.1) are globally defined. Take any constant $c > \|h\|_\infty/\alpha$ and set $\delta = \alpha c - \|h\|_\infty$. Let now $z(t)$ be any solution to (2.1) and assume that, for some value of τ, we know that $z(\tau) = c$: from the equation we deduce that $\dot{z}(\tau) \geqslant \delta$. Similarly, $\dot{z}(\tau) \leqslant -\delta$ as soon as $z(\tau) = -c$.

In particular $[-c, c]$ is negatively invariant, and this yields the existence of a bounded solution satisfying $\|y\|_\infty \leqslant c$. Start indeed from a sequence of solutions with initial data $z_n(\tau_n) = 0$, where $\tau_n \to +\infty$. These solutions satisfy the a priori bound $|z_n(t)| \leqslant c$ for every $t \leqslant \tau_n$. Their derivative is also uniformly bounded, from the equation. Given any compact set $K \subset \mathbb{R}$, the Ascoli–Arzelà theorem then guarantees that a subsequence is uniformly convergent on K. Repeat the same argument on an increasing sequence of compact sets, which exhausts all of $\mathbb{R}$, and use a diagonal argument to extract a subsequence of $(z_n)_{n\in\mathbb{N}}$ which converges uniformly on every compact subset of $\mathbb{R}$. Denote by $y(t)$ the limit: it is a solution, and the a priori bound $|y(t)| \leqslant c$ is true for all $t \in \mathbb{R}$.

In a similar way, the set $(-c, c)^C$ is positively invariant: if a solution satisfies $|z(\tau)| \geqslant c$ for some τ, then $|z(t)| \geqslant c$ for all $t \geqslant \tau$. Since $|j(z(t))| \geqslant \alpha c$ we have $|\dot{z}(t)| \geqslant \delta$ for the same t's, which prevents $z(t)$ to be bounded. In other words, every bounded solution to (2.1) must satisfy:

$$\left|y(t)\right| < c \quad \forall t \in \mathbb{R}.$$

The estimate in the statement follows by taking $c \to \|h\|_\infty/\alpha$. □

Imagine now that the forcing term is varying, in Eq. (2.1): the next lemma estimates the maximal variation of the corresponding bounded solutions.

Lemma 2.2. *Let $y_1(t)$ and $y_2(t)$ be any two bounded solutions to (2.1), which correspond to the forcing terms $h_1(t)$ and $h_2(t)$ respectively. Then*

$$\|y_1 - y_2\|_\infty \leqslant \frac{1}{\alpha}\|h_1 - h_2\|_\infty. \tag{2.4}$$

Proof. Set $z = y_1 - y_2$ and $k = h_1 - h_2$, consider any constant $c > \|k\|_\infty/\alpha$ and set $\delta = \alpha c - \|k\|_\infty$. It results:

$$\dot{z} = j(y_1) - j(y_2) + k(t)$$

and, arguing as in the proof of the previous lemma on the basis of (2.3), one deduces that $|\dot{z}(t)| \geqslant \delta$ for every $t \geqslant \tau$, as soon as $|z(\tau)| \geqslant c$. Since $z(t)$ is bounded by construction, the estimate $|z(t)| < c$ follows for every t. □

The lemma has two consequences. The first one follows by taking $h_1 = h_2 = h$, since one obtains that:

for every bounded and continuous $h(t)$, Eq. (2.1) admits exactly one bounded solution $y_h(t)$.

The function $y_h(t)$ plays a preeminent role in this paper, and some effort is devoted to investigate its zeroes and how they depend on the forcing term $h(t)$. The second consequence concerns the regularity of $y_h(t)$ as a function of $h(t)$: the estimates (2.4) guarantees the Lipschitz continuity of the functional $h \mapsto y_h$.

Remark 2.3. The solution y_h is also monotonically decreasing in h. Precisely, if for some $\rho \geqslant 0$

$$h_2(t) \geqslant h_1(t) + \rho \quad \forall t \in \mathbb{R}$$

then it results:

$$y_{h_2}(t) \leqslant y_{h_1}(t) - \frac{\rho}{\max\{\alpha_+, \alpha_-\}} \quad \forall t \in \mathbb{R}.$$

In particular, if $h(t)$ has a sign, either in the weak or in the strong sense, then the same happens to $y_h(t)$. The proof follows by arguments which are similar to those already used for Lemma 2.1 and Lemma 2.2: we omit it, since we will use this result just for comparative reasons.

We end the section by highlighting a trivial perturbation argument, which however will be crucial for the result. Consider the differential inequality:

$$\dot{w} \geqslant \{\alpha - \varphi(t)\} w \tag{2.5}$$

where $\varphi \in L^\infty(\mathbb{R})$. It has no positive solutions when $\varphi = 0$: the next lemma shows that this is again true, when φ is small in a suitable sense.

Lemma 2.4. *Assume that*:

$$\liminf_{T \to +\infty} \frac{1}{T} \int_0^T |\varphi(t)| \, dt < \alpha. \tag{2.6}$$

If $u(t)$ is a bounded solution to (2.5) then $u(t) \leqslant 0$ for every $t \in \mathbb{R}$.

Since the coefficients of (2.5) are not continuous, it is probably worth spending some words about the notion of solution. In the application of the lemma, we only need to work with $w(t)$ which are of class C^1. However, the proof below works for $w(t)$ which are absolutely continuous, in which case the inequality (2.5) is intended to be satisfied for almost every t.

Proof of Lemma 2.4. Since $A(t) = \int_0^t \{\alpha - \varphi(s)\} \, ds$ is a Lispchitz function, standard integration arguments apply to show that:

$$u(t) \geqslant u(\tau) e^{A(t) - A(\tau)}$$

for every $t \geqslant \tau$. But:

$$A(t) = t \left\{ \alpha - \frac{1}{t} \int_0^t \varphi(s) \, ds \right\} \geqslant t \left\{ \alpha - \frac{1}{t} \int_0^t |\varphi(s)| \, ds \right\}$$

which is unbounded as $t \to +\infty$. If $u(\tau) > 0$ for some τ, then $u(t)$ must be unbounded too as $t \to +\infty$. $\square$

Some remarks are worth mentioning about condition (2.6). This is clearly satisfied when $\|\varphi\|_\infty < \alpha$, which however is a quite uninteresting case: see Remark 3.4. The point of (2.6) is that it does not yield any restriction on $\|\varphi\|_\infty$, not even when φ exhibits some recurrence properties (which is the true case if interest): indeed, the condition is consistent with large values of $\varphi(t)$ on evenly spaced sets of t's of small measure.

The second remark is that condition (2.6) disregards the negative values of t. Of course, this is not really the truth when $\varphi(t)$ has a recurrent character. In the general case, this fact originates from the choice $\alpha > 0$: when the jumping linearity is decreasing, the negative values of t come into play.

Finally, in the next section we will construct the concrete perturbation term $\varphi(t)$ we are interested in. In particular, we will see that the estimate of $|\varphi(t)|$ heavily depends on the zeroes of the function $y_h(t)$, which in turn depends on the forcing term $h(t)$ under consideration: a considerable amount of efforts will be paid to deduce condition (2.6) directly from the knowledge of $h(t)$.

3. The uniqueness problem for the nonlinear equation

Consider the equation:

$$\dot{u} = j(u) + g(u) + ch(t) \tag{3.1}$$

where $h(t)$ is again *bounded and continuous* and $j(\xi)$ is the jumping linearity introduced in the previous section. Concerning the nonlinearity, it is assumed that $g(\xi)$ is *bounded, globally Lipschitz continuous* function which satisfies:

$$\lim_{|\xi| \to +\infty} g'(\xi) = 0. \tag{3.2}$$

The limit has to be intended in the set of ξ for which $g'(\xi)$ does exist: due to the Lipschitz condition, this set has full Lebesgue measure in $\mathbb{R}$. For the concrete application, we need to reformulate the vanishing of the derivative in a more convenient way, which involves the function:

$$K(r) = \sup\left\{\frac{|g(\xi_1) - g(\xi_2)|}{|\xi_1 - \xi_2|}\colon \xi_1 \neq \xi_2,\ |\xi_1| \geqslant r,\ |\xi_2| \geqslant r\right\}. \tag{3.3}$$

Notice that this function is nonincreasing with r and that:

$$K(0) = \|g'\|_\infty$$

is finite, since $g(\xi)$ is globally Lipschitz. The next lemma says what happens for large values of r.

Lemma 3.1. *Condition (3.2) is satisfied if and only if:*

$$\lim_{r \to +\infty} K(r) = 0. \tag{3.4}$$

For future reference, given an arbitrary $\varepsilon > 0$, we denote by r_ε any threshold such that:

$$r \geqslant r_\varepsilon \quad \text{implies} \quad K(r) \leqslant \varepsilon. \tag{3.5}$$

Proof of Lemma 3.1. Condition (3.4) clearly implies (3.2) where the derivative exists. The inverse implication will be proved by contradiction. Assume that (3.2) holds and that we can find two sequences $\xi_{1n} \neq \xi_{2n}$ with $|\xi_{1n}|, |\xi_{2n}| \to +\infty$ and $\varepsilon_0 > 0$ such that:

$$\big|g(\xi_{1n}) - g(\xi_{2n})\big| \geqslant \varepsilon_0 |\xi_{1n} - \xi_{2n}|$$

for every n. Since $g(\xi)$ is bounded, we know that:

$$|\xi_{1n} - \xi_{2n}| \leqslant \frac{2\|g\|_\infty}{\varepsilon_0}$$

is bounded too. Thus the interval:

$$I_n = \big\{s\xi_{1n} + (1-s)\xi_{2n}\colon 0 \leqslant s \leqslant 1\big\}$$

goes uniformly at infinity as $n \to +\infty$. Given an arbitrary $\varepsilon > 0$, we can now use (3.2) to show that:

$$\big|g'(\xi)\big| < \varepsilon$$

must eventually hold for almost all $\xi \in I_n$. Hence:

$$\big|g(\xi_{1n}) - g(\xi_{2n})\big| = \left|\int_{\xi_{1n}}^{\xi_{2n}} g'(\xi)\, d\xi\right| \leqslant \varepsilon |\xi_{1n} - \xi_{2n}|$$

is also eventually true, contradicting the assumption when $\varepsilon < \varepsilon_0$. □

Coming back to Eq. (3.1), notice that all the solutions are globally defined. We are mainly interested in its bounded solutions for large values of c. We study them under the additional assumption that:

$$c > 0. \tag{3.6}$$

The technical reason is to exploit the positive homogeneity of $j(\xi)$. Nevertheless, results for negative values of c may be recovered replacing $h(t)$ with $-h(t)$: as it can be easily checked along the paper, the methods are not affected by the change of sign in the forcing term.

It is not difficult to see that bounded solutions to (3.1) do exist for every value of c and satisfy the a priori bound:

$$\|u\|_\infty \leqslant \|g\|_\infty + c\|h\|_\infty.$$

This may be proved as in the first part of the proof of Lemma 2.1. We need to investigate how these bounded solutions behave for large values of c. To this aim, it is convenient to make the change of variable:

$$u = cx.$$

This change does not affect boundedness and transforms Eq. (3.1) into the new equation:

$$\dot{x} = j(x) + \frac{1}{c} g(cx) + h(t). \tag{3.7}$$

This is indeed the equation we will consider from now on, and all the results will refer to it. However, it will be clear how to translate them into results for the original equation (3.1).

Due to the boundedness of $g(\xi)$, the related term in (3.7) disappears when $c \to +\infty$: this way one obtains (but for the name of the variable) the *limit equation* (2.1). This equation has been already studied in the previous section and the same notations will be used also here. In particular, the function $y_h(t)$ will denote the only bounded solution to (2.1): the next lemma says that the bounded solutions to (3.7) approach it when c becomes large.

Lemma 3.2. *If $x(t)$ is a bounded solution to* (3.7) *then*:

$$\|x - y_h\|_\infty \leqslant \frac{1}{c}\|g\|_\infty.$$

Proof. Use Lemma 2.2 with $h_1(t) = c^{-1}g(cx(t)) + h(t)$ and $h_2(t) = h(t)$. □

Remark 3.3. Imagine that the forcing term has a sign, in the sense that:

$$\inf_t h(t) \geqslant \rho > 0.$$

Then Lemma 2.3 yields:

$$\sup_t y_h(t) \leqslant -\rho / \max\{\alpha_+, \alpha_-\}$$

and hence Lemma 3.2 implies that, for c sufficiently large:

$$\sup_t x(t) \leqslant -\rho / \max\{\alpha_+, \alpha_-\} + \|g\|_\infty / c < 0$$

holds for every bounded solution $x(t)$ to the differential equation (3.7). Possibly by taking a larger c, we may then assume that $cx(t)$ always lies in a region where the spatial term in the equation:

$$f_c(\xi) = j(\xi) + \frac{1}{c}g(c\xi)$$

is strictly monotone. In particular, the bounded solution is unique. Moreover, and more important, if $h(t)$ is known to be almost periodic, then standard arguments (see for instance [3]) apply to show that the same happens to the unique bounded solution of (3.7). In other words, the true target of the present paper are the forcing terms which change sign.

The main topic of this section is the question of the *uniqueness* of the bounded solution to (3.7), at least for large values of the parameter c: next we will show how to obtain it, under some suitable additional condition.

Start by assuming that $x_1(t)$ and $x_2(t)$ are both bounded solutions to Eq. (3.7), corresponding to the same forcing term $h(t)$ and the same value of c. They are ordered, so that we may assume that:

$$w(t) = x_1(t) - x_2(t) \geqslant 0 \tag{3.8}$$

for every $t \in \mathbb{R}$. Such $w(t)$ is a bounded function of class C^1 by construction and straightforward computations show that it satisfies the differential inequality:

$$\dot{w} \geqslant \alpha w + P_c(t) \tag{3.9}$$

where we settled:

$$P_c(t) = \frac{1}{c}\bigl\{g\bigl(cx_1(t)\bigr) - g\bigl(cx_2(t)\bigr)\bigr\}. \tag{3.10}$$

We made explicit the dependence on c, because we need soon to move it. The goal is to show that $w(t)$ is identically zero, and the way is to get some convenient Lipschitz-type estimates for $P_c(t)$. The simplest of these estimates is of course:

$$\left|P_c(t)\right| \leqslant \|g'\|_\infty \left|w(t)\right| \tag{3.11}$$

for every c and every t. The next remark explains why this is quite useless.

Remark 3.4. Inserting (3.11) into (3.9) yields the differential inequality:

$$\dot{w} \geqslant \left(\alpha - \|g'\|_\infty\right) w.$$

The only way to obtain that $w(t) \leqslant 0$ for every t, and hence that it is identically zero due to (3.8), is to require that $\|g'\|_\infty < \alpha$. However, this restriction makes things quite trivial: indeed, the nonlinearity $j(\xi) + g(\xi)$ is strictly increasing, in which case the expected existence result is well known to be true. Our assumptions require $g'(\xi)$ to be small only for large values of ξ, while it can be very large in finite regions.

A more convenient Lipschitz estimate of $P_c(t)$ follows from Lemma 3.2. Consider indeed the function:

$$\varphi_{\varepsilon\delta}(t) = \begin{cases} \|g'\|_\infty & \text{if } |y_h(t)| \leqslant \delta, \\ \varepsilon & \text{if } |y_h(t)| > \delta \end{cases} \tag{3.12}$$

where $\varepsilon \geqslant 0$ and $\delta \geqslant 0$ are arbitrary parameters. The next lemma shows that it provides the desired estimate, for large values of the parameter c.

Lemma 3.5. *For every $\delta > 0$ and $\varepsilon > 0$, it results*:

$$\left|P_c(t)\right| \leqslant \varphi_{\varepsilon\delta}(t) w(t) \tag{3.13}$$

for every $t \in \mathbb{R}$, as soon as:

$$c \geqslant \frac{\|g\|_\infty + r_\varepsilon}{\delta}. \tag{3.14}$$

Here the quantity r_ε is that defined by (3.5).

Proof. For the t's such that $|y_h(t)| \leqslant \delta$ there's nothing to prove. Consider now any t for which $|y_h(t)| > \delta$. Because of Lemma 3.2 we know that:

$$\left|cx(t)\right| \geqslant c\left|y_h(t)\right| - \|g\|_\infty > c\delta - \|g\|_\infty$$

where $x(t)$ is any bounded solution to (3.7), like for instance the $x_1(t)$ and $x_2(t)$ used to define $P_c(t)$. It is then sufficient to choose c according to (3.14) in order to be granted that $|cx(t)| \geqslant r_\varepsilon$ and hence:

$$\left|P_c(t)\right| = \frac{1}{c}\left|g\left(cx_1(t)\right) - g\left(cx_2(t)\right)\right| \leqslant \varepsilon\left|x_1(t) - x_2(t)\right| = \varepsilon w(t). \qquad \square$$

With the above estimate, we enter finally into the orbit of Lemma 2.4. This lemma is the perturbative core of the next uniqueness result, which is the main result of the present section.

Theorem 3.6. *Assume that, for some $\delta > 0$ and $\varepsilon > 0$, we have*:

$$\varepsilon + \|g'\|_\infty \liminf_{T\to+\infty}\left(\frac{1}{T} m\{t \in [0,T]:\ |y_h(t)| \leqslant \delta\}\right) < \alpha. \tag{3.15}$$

If the parameter c is chosen according to (3.14), then Eq. (3.7) admits a unique bounded solution.

The same conclusion clearly holds for the original equation (3.1).

Proof. It can be easily checked that:

$$\varphi_{\varepsilon\delta}(t) \leqslant \varphi_{0\delta}(t) + \varepsilon$$

and

$$\int_0^T \varphi_{0\delta}(t)\,dt = \|g'\|_\infty m\{t \in [0,T]:\ |y_h(t)| \leqslant \delta\}.$$

Then assumption (3.15) implies:

$$\liminf_{T\to+\infty} \frac{1}{T}\int_0^T \varphi_{\varepsilon\delta}(t)\,dt < \alpha. \tag{3.16}$$

Come now back to (3.8), defining the ordered difference $w \geqslant 0$ between any two bounded solutions to Eq. (3.7). According to (3.9) and to Lemma 3.5, this difference satisfies the differential inequality:

$$\dot{w} \geqslant \{\alpha - \varphi_{\varepsilon\delta}(t)\} w.$$

The estimate (3.16) and Lemma 2.4 then allow to conclude that $w \leqslant 0$. Hence one gets $w = 0$, proving that the two involved bounded solutions to (3.7) must indeed coincide. □

Condition (3.15) can be satisfied for some $\varepsilon > 0$ if and only if:

$$\|g'\|_\infty \liminf_{T\to+\infty}\left(\frac{1}{T} m\{t \in [0,T]:\ |y_h(t)| \leqslant \delta\}\right) < \alpha. \tag{3.17}$$

To estimate the left hand side is a main goal of the present paper, especially when the forcing term $h(t)$ is almost periodic: this is done in Section 7, on the basis of the arguments introduced in Sections 4, 5 and 6. We conclude this section by discussing how to verify (3.17) in a special bounded case, that is when the forcing term $h(t)$ has a limit at infinity.

Example 3.7. Assume that $h(t)$ is bounded and continuous and that moreover the limit $h(+\infty)$ exists. Then it is not difficult to check that also the limit $y_h(+\infty)$ does exist and satisfies:

$$j(y_h(+\infty)) + h(+\infty) = 0.$$

Since the jumping linearity j only vanishes at zero, it is clear that $h(+\infty) \neq 0$ if and only if $y_h(+\infty) \neq 0$. In this case one gets:

$$\lim_{T\to+\infty}\left(\frac{1}{T} m\{t \in [0,T]:\ |y_h(t)| \leqslant \delta\}\right) = 0$$

as soon as one takes $\delta < |y_h(+\infty)|$. Hence condition (3.17) is satisfied for the same value of δ.

On the contrary, when $h(+\infty)=0$ the above limit is 1 for every $\delta>0$. This fact provided a clear obstruction to the application of Theorem 3.6. Indeed condition (3.15) does no longer depend on δ and is satisfied if and only if $\|g'\|_\infty<\alpha$: we already explained in Remark 3.4 why this is an uninteresting case.

4. Topological groups and minimal flows

Almost periodic functions are well known to be tightly related to compact topological groups: in this section we will summarize some basic facts about the latter, which may be unfamiliar to people dealing with differential equations. The proofs of these facts may be found in classical textbooks like [19] and in the paper [18].

Let G denote a commutative topological group, which is metrizable and compact. The notations will be additive, namely the operation in G will be $+$ and the neutral element 0. The category of these groups will be denoted by $\mathcal{G}$, its morphisms being the continuous homomorphisms of groups. Besides the trivial group 0, the simplest element of $\mathcal{G}$ is the unit circle:

$$\mathbb{S}^1=\left\{z\in\mathbb{C}\colon\ |z|=1\right\}$$

though notations here are multiplicative. This is a connected and then perfect group, while the n-roots of unit provide an example of a discrete element of $\mathcal{G}$. In fact, it is well known that each element of $\mathcal{G}$ is either discrete or perfect.

A *character* is a morphism $G\to\mathbb{S}^1$. The set of all characters of G is itself a group, with respect to the pointwise product: it is called the dual group of G and usually denoted by G^*. The unit of G^* is the trivial character, which assigns the value 1 to every element of G. Nontrivial elements do exist when G is a nontrivial compact group: see [19, p. 241]. For instance, it is well known that:

$$\left(\mathbb{S}^1\right)^*=\left\{z\mapsto z^n\colon\ n\in\mathbb{Z}\right\}$$

and from this also the characters of the N-torus $\mathbb{T}^N$ may be easily obtained, where $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ is the additive group isomorphic to $\mathbb{S}^1$.

Characters give a way to construct new elements of $\mathcal{G}$ from a given one: if $G\in\mathcal{G}$ and $\varphi\in G^*$ then $\ker\varphi=\varphi^{-1}(0)$ is closed subgroup of G and then also an element of $\mathcal{G}$. The next result has been proved in [18].

Proposition 4.1. *Let $G\in\mathcal{G}$ and assume that $\varphi\in G^*$ is nontrivial. If $G\ncong\mathbb{S}^1$ then* $\ker\varphi$ *is perfect.*

In this paper we are mainly interested in elements of $\mathcal{G}$ that admits a one-parameter dense subgroup. More precisely, we shall consider pairs (Ω,Ψ) where $\Omega\in\mathcal{G}$ and $\Psi:\mathbb{R}\to\Omega$ is a continuous homomorphism whose image is dense in Ω: they are the objects of a new category which we denote by $\mathcal{P}$. Contrarily to the elements of $\mathcal{G}$, it is easily seen that those of $\mathcal{P}$ are connected sets. A morphism $(\Omega_1,\Psi_1)\to(\Omega_2,\Psi_2)$ between two elements of $\mathcal{P}$ is nothing else than a morphism $\chi:\Omega_1\to\Omega_2$ in the category $\mathcal{G}$, which preserves the dense subgroups, namely such that:

$$\chi\circ\Psi_1=\Psi_2. \tag{4.1}$$

Since $\chi(\Omega_1)$ must be closed:

$$\chi(\Omega_1)=\overline{\chi(\Omega_1)}\supset\overline{\chi\left(\Psi_1(\mathbb{R})\right)}=\overline{\Psi_1(\mathbb{R})}=\Omega_2 \tag{4.2}$$

and hence the morphisms of $\mathcal{P}$ are in fact epimorphisms.

The trivial group belongs to $\mathcal{P}$ by taking as Ψ the trivial homomorphism. Also $\mathbb{S}^1$ is an element of $\mathcal{P}$ together with the homomorphism:

$$\Psi(t) = e^{i\alpha t}$$

where $\alpha \neq 0$ is any real number. Such Ψ is a nontrivial periodic map: in [18] it is proved that this happens if and only if $\Omega \cong \mathbb{S}^1$.

Another important element of $\mathcal{P}$ is the torus $\mathbb{T}^N$ together with a homomorphism of the type:

$$\Psi(t) = (\nu_1 t, \ldots, \nu_N t). \tag{4.3}$$

Here the frequency vector $\nu = (\nu_1, \ldots, \nu_N)$ is a vector of $\mathbb{R}^N$ and, with a standard abuse of notation, we identify each real number $\nu_j t$ with its equivalence classes in $\mathbb{T}$. The classical Kronecker's theorem for diophantine approximations says that the density assumption is satisfied when ν is *nonresonant*, namely when the components of ν are independent over the $\mathbb{Z}$.

Coming back again to the general pair (Ω, Ψ), notice that the homomorphism Ψ induces a canonical flow on Ω, by means of:

$$\omega \cdot t = \omega + \Psi(t).$$

Since Ψ has dense image:

$$\overline{\omega \cdot \mathbb{R}} = \overline{\omega + \Psi(\mathbb{R})} = \omega + \overline{\Psi(\mathbb{R})} = \Omega$$

for every $\omega \in \Omega$. Namely, the flow is minimal. Equilibria or periodic orbits cannot exist unless $\Omega = 0$ or $\Omega \cong \mathbb{S}^1$, respectively.

Next we consider the problem of constructing global sections for this flow. Following [18], assume that $\varphi \in \Omega^*$ is nontrivial and define:

$$\Sigma = \big\{\omega \in \Omega \colon \varphi(\omega) = 1\big\}.$$

Notice that $\varphi \circ \Psi$ is also a nontrivial character of the additive group $\mathbb{R}$, endowed with the usual topology. Thus there exists a unique real number $\alpha \neq 0$ such that:

$$\varphi\big(\Psi(t)\big) = e^{i\alpha t} \quad \forall t.$$

In [18] it is proved that the minimal period of such function, namely:

$$S = \frac{2\pi}{|\alpha|} \tag{4.4}$$

acts as a *returning time* on Σ. Precisely, if we define $\tau(\omega)$ by means of:

$$0 \leqslant \tau(\omega) < S, \qquad \varphi(\omega) = e^{i\alpha\tau(\omega)}$$

then we have:

$$\omega \cdot t \in \Sigma \quad \Longleftrightarrow \quad t \in -\tau(\omega) + S\mathbb{Z}.$$

As a consequence, the restricted flow:

$$\Phi : \Sigma \times [0, S) \to \Omega, \qquad \Phi(\sigma, t) = \sigma \cdot t \tag{4.5}$$

is a continuous bijection, with inverse:

$$\Phi^{-1}(\omega) = \big(\omega \cdot \big(-\tau(\omega)\big), \tau(\omega)\big).$$

It is easily checked that this inverse fails to be continuous exactly at the ω's satisfying $\tau(\omega) = 0$. Thus Φ defines a homeomorphism $\Sigma \times (0, S) \cong \Omega \setminus \Sigma$.

We conclude the present section with a comment about Σ. In general, it is an element of $\mathcal{G}$ but not of $\mathcal{P}$, at least due to connectedness problems. However, it always belongs to a discrete version of the category $\mathcal{P}$, obtained by replacing $\mathbb{R}$ with $\mathbb{Z}$ in the homomorphism part. Define indeed $\psi : \mathbb{Z} \to \Sigma$ by means of:

$$\psi(n) = \Psi(nS). \tag{4.6}$$

Since Φ is a homomorphism, the same is true also for ψ. The point is that the image on ψ is dense in Σ: the proof is left to the reader, since we don't use this property (at least directly).

5. Almost periodic functions and Favard theory

Consider the general $(\Omega, \Psi) \in \mathcal{P}$ and a function $U \in C(\Omega)$. Using the compactness of Ω, it is not difficult to see that the function:

$$u(t) = U\big(\Psi(t)\big) \tag{5.1}$$

is almost periodic in the sense of Bohr. This u is said to be *representable over* (Ω, Ψ): by density arguments, it is clear that the representing function U is unique. It is manifest that the trivial group $\Omega = 0$ can only be used to represent constant functions. Another example is given $\Omega \cong \mathbb{S}^1$. Denoted by $T > 0$ the minimal period of the map Ψ, the formula (5.1) gives rise to a periodic function, with the same period of Ψ. It is not difficult to check that they are indeed the only functions which are representable over this pair. The case $\Omega = \mathbb{T}^N$ with Ψ defined as in (4.3) is much more interesting: in general, the composition rule (5.1) produces aperiodic functions, which are called quasi-periodic.

An important and well known point is, that any given almost periodic function u may be obtained as in (5.1) via the notion of hull. The hull $\mathcal{H}_u$ of the function u is defined by:

$$\mathcal{H}_u = \operatorname{cls}\{u_\tau \colon \tau \in \mathbb{R}\}$$

where $u_\tau(t) = u(t + \tau)$ and the closure is taken the topology of the uniform convergence over all the real line. This is indeed a metric topology which gives $\mathcal{H}_u$ the structure of a compact connected space. It becomes a topological group with the operation obtained as the extension by continuity of the rule $u_\tau + u_s = u_{\tau+s}$ (see the book [15] for a proof). The neutral element of $\mathcal{H}_u$ is u itself. If we define:

$$\Psi_u(\tau) = u_\tau$$

then the pair $(\mathcal{H}_u, \Psi_u)$ fits perfectly our framework, and then belongs to $\mathcal{P}$. The representation formula (5.1) holds with the function $U \in C(\mathcal{H}_u)$ defined by:

$$U(u_*) = u_*(0) \quad \forall u_* \in \mathcal{H}_u$$

which is sometimes called the 'extension by continuity' of the almost periodic function $u(t)$ to its hull $\mathcal{H}_u$. In [18] it is proved that this representation of u is minimal, in the sense given by the following lemma.

Lemma 5.1. *The almost periodic function $u(t)$ is representable over $(\Omega, \Psi) \in \mathcal{P}$ if and only if there exists a morphism $(\Omega, \Psi) \to (\mathcal{H}_u, \Psi_u)$ in the category $\mathcal{P}$.*

Assume now that $U \in C(\Omega)$ is given. The flow allows to define other almost periodic functions than (5.1), by means of:

$$u_\omega(t) = U(\omega \cdot t) = U\big(\omega + \Psi(t)\big) \tag{5.2}$$

where $\omega \in \Omega$. All these functions are representable over (Ω, Ψ) and then almost periodic. Moreover, it is not difficult to check that all of them have the same mean value, namely:

$$\bar{u}_\omega = \lim_{T \to +\infty} \frac{1}{T} \int_0^T U\big(\Psi(t)\big)\, dt \tag{5.3}$$

for every $\omega \in \Omega$. Notice finally that, as a whole, the functions (5.2) enjoy the invariance property:

$$u_{\omega \cdot \tau}(t) = u_\omega(t + \tau) \quad \forall \omega \in \Omega\ \forall t, \tau \in \mathbb{R} \tag{5.4}$$

and inherit from U the continuity property:

$$\text{the map } \omega \in \Omega \mapsto u_\omega(0) \in \mathbb{R} \text{ is continuous.} \tag{5.5}$$

It is a classical and important fact that the above procedure may be reversed. Assume indeed that a family of functions:

$$u_\omega : \mathbb{R} \to \mathbb{R}, \quad \omega \in \Omega \tag{5.6}$$

is given in such a way that the two conditions (5.4) and (5.5) are satisfied, and define:

$$U(\omega) = u_\omega(0).$$

Then $U \in C(\Omega)$ due to (5.5), while (5.4) implies:

$$u_\omega(t) = u_{\omega \cdot t}(0) = U(\omega \cdot t).$$

Summing up, all the u_ω's are almost periodic and representable over (Ω, Ψ). This simple fact is at the core of Favard theory, which is among the few general devices allowing to construct almost periodic solutions to almost periodic differential equations: for an introduction to the subject, see the original paper [5] or the more modern approach given in Fink's book [6]. The starting point is to consider, instead of a single equation, a family of them:

$$\dot{u} = F(\omega \cdot t, u) \tag{5.7}$$

where $\omega \in \Omega$. In order to guarantee global existence and uniqueness of the initial values problems, the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and globally Lipschitz in the second variable, uniformly with respect to the first one. The idea is to construct a representable almost periodic solution to (5.7) by using bounded solutions. The next result is a special (but relevant) case of Favard theory.

Theorem 5.2 *(Favard). Assume that, for each $\omega \in \Omega$, Eq.* (5.7) *admits a unique bounded solution u_ω. Then there exists $U \in C(\Omega)$ such that*:

$$u_\omega(t) = U(\omega \cdot t)$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}$.

The proof is classical and consists in showing that the family $\{u_\omega\}_{\omega\in\Omega}$ satisfies (5.4) and (5.5): the former property is quite obvious, while the second is more delicate but anyway follows from standard compactness arguments. Notice moreover that, due to the scalar character of Eq. (5.7), there are no other almost periodic solutions than the representable ones: the proof may be found in Chapter 12 of [6], under the name of module containment property (see for instance [18] to understand why this property is equivalent to the representability problem).

We end the section spending some words about the almost periodic sequences. They are less known than the corresponding functions, but share with them all the most relevant properties: the classical book [4] of Corduneanu is a good reference on the subject, see Section 1.6. The almost periodic sequences come naturally into play, when we consider a nontrivial character $\varphi\in\Omega^*$ and its kernel Σ, as we did in the final part of Section 4 and to which we refer for the terminology. Precisely, it is not difficult to check that:

$$v_n = V\big(\psi(n)\big)$$

is an almost periodic sequence as soon as $V\in C(\Sigma)$. Moreover, the limit:

$$\bar v = \lim_{n\to+\infty}\frac{v_k+v_{k+1}+\cdots+v_{k+n-1}}{n}$$

exists uniformly on $k\in\mathbb{Z}$ and is independent on k, so defining a good notion of mean value. This fact will be used in the next section.

6. Haar measure and Fubini-type decomposition along the flow

Each $G\in\mathcal{G}$, the category of compact commutative topological groups introduced in Section 4, has a unique invariant integral, usually called the *Haar integral* of G. That is, a normalized nonnegative linear functional $I_G: C(G)\to\mathbb{R}$ which moreover is invariant with respect to the addition on the group G, in the sense that:

$$I_G(F_w) = I_G(F) \tag{6.1}$$

for every $F\in C(G)$ and every $w\in G$. Here $F_w(z)=F(z+w)$ for every $w,z\in G$. This classical result is proved, for instance, in Section 29 of [19].

In turn, every invariant integral becomes from an invariant measure. More precisely, the Riesz representation theorem guarantees that there exists a unique regular Borel probability measure m_G such that:

$$I_G(F) = \int_G F\,dm_G$$

for every $F\in C(G)$. Notice that, since G is metric compact, each open set is σ-compact: thus every Borel probability measure is automatically regular. The proof of this fact and the explicit construction of the Riesz measure may be found in many textbooks, like for instance [9]. Looking at this construction, one immediately sees that the Riesz measure inherits the invariance property of the integral I_G, namely that:

$$m_G(B+w) = m_G(B)$$

for every Borel set B and every $w\in\mathcal{G}$. The invariant measure m_G is called the *Haar measure* of G and, as the integral, is unique. For instance, it can be easily seen that, when $G=\mathbb{S}^1$:

$$m_G(B) = \frac{1}{2\pi} m\big\{t\in[0,2\pi):\ e^{it}\in B\big\}$$

does perfectly the job, and then *is* the Haar measure on $\mathbb{S}^1$. Here, and in all the paper, the symbol m will stand for the Lebesgue measure on $\mathbb{R}$. The Haar measure on $\mathbb{T}$ and $\mathbb{T}^N$ may be easily deduced from that of $\mathbb{S}^1$.

Borel sets of Haar measure zero play a relevant role in the proof of our main result and they will be carefully investigated in Section 9. The next lemma says that detecting them is always a local problem: the proof follows from the classical Lindelöff theorem in topology.

Lemma 6.1. *Let B be a Borel set in $G \in \mathcal{G}$. Then $m_G(B) = 0$ if and only if, for every $w \in B$, there is an open $U_w \subset G$ such that $m_G(B \cap U_w) = 0$.*

Next we specialize to the situation we are more interested in, restring our attention from the category $\mathcal{G}$ to the category $\mathcal{P}$, also defined in Section 4. This passage does not affect the Haar measure of the involved groups, but the presence of a one dimensional dense subgroup allows to better describe the measure itself. Let us start by comparing the Haar measures corresponding to different elements of $\mathcal{P}$, when there is a morphism:

$$(\Omega_1, \Psi_1) \to (\Omega_2, \Psi_2). \tag{6.2}$$

Lemma 6.2. *Let $\chi : \Omega_1 \to \Omega_2$ be the morphism of $\mathcal{G}$ underlying the $\mathcal{P}$ morphism (6.2). Then*:

$$m_{\Omega_2}(B) = m_{\Omega_1}\big(\chi^{-1}(B)\big)$$

for every Borel set B in Ω_2.

The conclusion is manifestly false for morphisms in the category $\mathcal{G}$, as one may see by taking the trivial morphism.

Proof of Lemma 6.2. The rule:

$$m_2(B) = m_{\Omega_1}\big(\chi^{-1}(B)\big)$$

defines a (regular) probability Borel measure on Ω_2. To conclude by uniqueness it remains to show that m_2 is invariant with respect to the group operation in Ω_2. To this aim, let $\omega_2 \in \Omega_2$ and choose $\omega_1 \in \Omega_1$ such that $\chi(\omega_1) = \omega_2$. This is indeed possible since χ must be an epimorphism. Then:

$$\chi^{-1}(\omega_2 + B) = \omega_1 + \chi^{-1}(B)$$

and the invariance of m_{Ω_1} implies:

$$\begin{aligned} m_2(\omega_2 + B) &= m_{\Omega_1}\big(\chi^{-1}(\omega_2 + B)\big) = m_{\Omega_1}\big(\omega_1 + \chi^{-1}(B)\big) \\ &= m_{\Omega_1}\big(\chi^{-1}(B)\big) = m_2(B). \qquad \square \end{aligned}$$

We will focus now the attention on a single pair $(\Omega, \Psi) \in \mathcal{P}$ and denote by Λ the Haar integral of Ω and by λ the corresponding Haar measure.

A first consequence of the presence of Ψ, is that it allows an explicit representation of Λ. Indeed, it is well known that the equality:

$$\Lambda(U) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T U(\omega \cdot t)\, dt \tag{6.3}$$

holds for every $U \in C(\Omega)$ and every $\omega \in \Omega$. The proof is again by uniqueness of the Haar integral, the invariance following from (5.3).

Besides that, however, the presence of Ψ has another important consequence, which seems to be overlooked in the literature: it allows to decompose the Haar measure λ along the flow generated by Ψ, in a very convenient way for the computations. This decomposition is the main argument of this section. To start with, we need a transversal section to the flow. This is obtained as in Section 4, fixing a nontrivial $\varphi \in \Omega^*$ and taking:

$$\Sigma = \ker \varphi \neq \Omega.$$

As we said in Section 4, this is always possible when Ω is nontrivial, a condition which will be implicitly assumed from now on. Exactly as in Section 4, we denote by $S > 0$ the returning time on Σ and by:

$$\Phi : \Sigma \times [0, S) \cong \Omega \tag{6.4}$$

the continuous bijection given by the restricted flow $\Phi(\sigma, t) = \sigma \cdot t$. The map Φ defines a decomposition of Ω along the flow, which however is not continuous at Σ. Hereafter we will study the measurable properties of this decomposition.

The factor Σ in the decomposition (6.4) is an element of $\mathcal{G}$ and then is itself in the scope of Haar theory. Denote by M its Haar integral and by μ its Haar measure, so that we write:

$$M(V) = \int_{\Sigma} V \, d\mu$$

for every $V \in C(\Sigma)$. Similarly to Λ, the theory of almost periodic sequences (see the final part of Section 5) provides an explicit representation for M, namely:

$$M(V) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} V\bigl(\psi(k)\bigr) \tag{6.5}$$

where $\psi(k) = \Psi(kS)$ for every $k \in \mathbb{Z}$.

The second factor in (6.4) is the interval $[0, S)$, without any identification at the ends. It has a natural topology, inherited by $\mathbb{R}$, and also a natural probability measure:

$$m_*(I) = \frac{1}{S} m(I)$$

where m stands for the Lebesgue measure on $\mathbb{R}$. Exactly as λ and μ, also m_* is a regular Borel measure.

Having a measure on both factors, we may now endow $\Sigma \times [0, S)$ with a natural measure: the product measure $\mu \times m_*$. In principle, this measure is only defined on the σ-algebra generated by the open rectangles of $\Sigma \times [0, S)$, which are particular Borel sets. However, using that both the factors Σ and $[0, S)$ are separable metric spaces, it is not difficult to check that the two σ-algebras coincide. As a consequence, the product measure $\mu \times m_*$ is a (regular) Borel probability measure. Roughly speaking, the main goal of the present section is to show that it is the Haar measure on Ω.

Proposition 6.3. *The map Φ is an isomorphism of Borel spaces and measures, namely*:

$$\lambda(B) = (\mu \times m_*)\bigl(\Phi^{-1}(B)\bigr) \tag{6.6}$$

holds for every Borel set in Ω.

Proof. Since Φ is continuous, if B is a Borel subset of Ω then $\Phi^{-1}(B)$ is a Borel subset of $\Sigma \times [0, S)$. To prove that Φ maps Borel sets into Borel sets, it's enough to look at the image of an open rectangle $A \times I$. If $0 \notin I$ this image is open in $\Omega \setminus \Sigma$ and then in Ω, while the opposite case can be worked out by separating the contributions of $\{0\}$ and $I \setminus \{0\}$.

It remains to show that (6.6) holds true. To this aim, consider the measure in Ω defined by:

$$\lambda_\Phi(B) = (\mu \times m_*)\bigl(\Phi^{-1}(B)\bigr).$$

This is a Borel measure, due to the first part of the proof. Normalization and regularity are obvious. This measure induces an integral over $C(\Omega)$, namely:

$$\Lambda_\Phi(U) = \int\limits_{\Sigma\times[0,S)} U \circ \Phi \, d(\mu \times m_*).$$

Notice that, if we set:

$$V(\sigma) = \frac{1}{S}\int\limits_0^S U\bigl(\sigma + \Psi(t)\bigr)\,dt$$

then Fubini theorem applies to show that $\Lambda_\Phi(U) = M(V)$. To conclude the proof, it's enough to show that $\Lambda(U) = M(V)$ is also true. But this follows from (6.3) and (6.5), since:

$$\begin{aligned}
\Lambda(U) &= \lim_{n\to+\infty} \frac{1}{nS}\int\limits_0^{nS} U\bigl(\Psi(t)\bigr)\,dt = \lim_{n\to+\infty}\frac{1}{nS}\sum_{k=0}^{n-1}\int\limits_{kS}^{(k+1)S} U\bigl(\Psi(t)\bigr)\,dt \\
&= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1}\Biggl(\frac{1}{S}\int\limits_0^S U\bigl(\Psi(t) + \Psi(kS)\bigr)\,dt\Biggr) \\
&= \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} V\bigl(\Psi(kS)\bigr) = M(V). \qquad \square
\end{aligned}$$

Proposition 6.3 allows to use Fubini theory, when computing the Haar measure on Ω. The following characterization of the sets of Haar measure zero is a straightforward consequence: together with Lemma 6.1, it will represent a relevant step in the proof of our main result, in the next section.

Corollary 6.4. *Let B be a Borel set in Ω. Then $\lambda(B) = 0$ if and only if:*

$$m\bigl\{t \in [0, S)\colon \sigma \cdot t \in B\bigr\} = 0 \tag{6.7}$$

for μ-almost all $\sigma \in \Sigma$.

Two particular cases are worth to be mentioned here. The first one corresponds to the choice $B = \Sigma$. Since $\Phi^{-1}(\Sigma) = \Sigma \times \{0\}$, we clearly have:

$$\lambda(\Sigma) = 0. \tag{6.8}$$

The second is a kind of opposite case, and is the argument of the next lemma.

Lemma 6.5. *Assume that $\Omega \ncong \mathbb{S}^1$. Then $\lambda(\omega \cdot \mathbb{R}) = 0$ for every $\omega \in \Omega$.*

Proof. It is not restrictive to assume that $\omega \in \Sigma$. In this case $\omega \cdot t \in \mathbb{R}$ if and only if $t \in S\mathbb{Z}$, so that:

$$\omega \cdot \mathbb{R} = \{\omega \cdot (kS)\colon k \in \mathbb{Z}\} \times [0, S).$$

Since $\Omega \ncong \mathbb{S}^1$, Proposition 4.1 says that Σ is perfect and then contains infinitely many elements. This forces μ to be nonatomic. Hence $\mu\{\omega \cdot (kS)\colon k \in \mathbb{Z}\} = 0$ and Corollary 6.4 allows to conclude. □

7. Ergodic type results

By construction, the Haar measure λ is invariant under the flow on Ω. In fact, λ is the unique normalized Borel measure having this property: this is well known (see [8]) and it may be easily checked, using the density of $\Psi(\mathbb{R})$. As a standard consequence, the considered flow must be ergodic: in the literature, this is referred as a case of *unique ergodicity*.

Given a Borel set B, the Birkhoff Ergodic Theorem asserts that the amount of time that the flow spends in B (*time average*) generically coincides with the measure of B itself (*space average*). Precisely, it guarantees that the equality:

$$\lim_{T \to +\infty} \frac{1}{T} m\{t \in [0, T]\colon \omega \cdot t \in B\} = \lambda(B) \tag{7.1}$$

is true for almost every $\omega \in \Omega$, with respect to the measure λ. Here, the existence of the time averages for almost all the ω's is not an assumption, but instead part of the conclusion. In general, the equality may fail on a set of measure zero: next we provide an explicit example in our concrete framework.

Example 7.1. Take any nontrivial $\Omega \ncong \mathbb{S}^1$ and consider the set $\Psi(\mathbb{R})$: Lemma 6.5 says that $\lambda(B) = 0$. By setting $B = \Psi(\mathbb{R})^C$ we have an example where (7.1) fails *by defect* at some point: indeed, $\lambda(B) = 1$ while the left hand side vanishes for every $\omega \in \Psi(\mathbb{R})$.

A specular failure *by excess* is clearly obtained by setting $B = \Psi(\mathbb{R})$, but a more relevant example may be constructed by taking B an open set such that:

$$\Psi(\mathbb{R}) \subset B, \qquad \lambda(B) < 1.$$

This choice is granted by the regularity of λ. The time average is again 1 for all the starting points $\omega \in B$.

A relevant fact here is that (7.1) cannot fail by excess on a closed set. This is the sense of the next result.

Lemma 7.2. *Let $B \subset \Omega$ be a closed set. Then:*

$$\limsup_{T \to +\infty} \frac{1}{T} m\{t \in [0, T]\colon \omega \cdot t \in B\} \leqslant \lambda(B) \tag{7.2}$$

for every $\omega \in \Omega$.

Proof. Take $\varepsilon > 0$ and choose an open set $A \supset B$ such that:

$$\lambda(A) < \lambda(B) + \varepsilon.$$

This is possible due to the regularity of λ. Then choose an Urysohn function $V \in C(\Omega)$ satisfying:

$$V(\omega) = \begin{cases} 1 & \text{if } \omega \in B, \\ 0 & \text{if } \omega \notin A \end{cases}$$

so that:

$$\lambda(B) \leqslant \int_{\Omega} V \, d\lambda \leqslant \lambda(A) < \lambda(B) + \varepsilon.$$

Fix now an arbitrary $\omega \in \Omega$. Because of (6.3), we know that:

$$\begin{aligned} \limsup_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega \cdot t \in B\bigr\} &\leqslant \limsup_{T \to +\infty} \frac{1}{T} \int_0^T V(\omega \cdot t)\, dt \\ &= \lim_{T \to +\infty} \frac{1}{T} \int_0^T V(\omega \cdot t)\, dt = \int_{\Omega} V \, d\lambda < \lambda(B) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, the conclusion follows. $\square$

Though we don't really use this fact, it may be interesting to notice that Lemma 7.2 has a consequence on the validity of (7.1): the next result says that its failure is a kind of boundary effect.

Proposition 7.3. *Let B be any Borel set in Ω. If $\lambda(\partial B) = 0$ then the equality (7.1) holds for every $\omega \in \Omega$.*

In Example 7.1 the boundary of $\Psi(\mathbb{R})$ has full measure in Ω, so explaining the failure of (7.1). Moreover, notice that no failure is possible when B is a closed set and $\lambda(B) = 0$. This case is probably the most relevant in the applications, and it is considered in Section 9. There, however, a more convenient description will be presented, which makes use of the Fubini-type decomposition of λ, given in the previous section.

Proof of Proposition 7.3. Let ω be an arbitrary element of Ω. Using (7.2) on $\overline{B}$ we have:

$$\limsup_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega \cdot t \in B\bigr\} \leqslant \limsup_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega_0 \cdot t \in \overline{B}\bigr\} \leqslant \lambda(\overline{B}). \tag{7.3}$$

Do now the same, but starting from B^C. Since its closure is the set $\mathring{B}^C$ we get:

$$\begin{aligned} 1 - \lambda(\mathring{B}) = \lambda\bigl(\mathring{B}^C\bigr) &\geqslant \limsup_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega \cdot t \in B^C\bigr\} \\ &= 1 - \liminf_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega \cdot t \in B\bigr\} \end{aligned}$$

which may be written as:

$$\liminf_{T \to +\infty} \frac{1}{T} m\bigl\{t \in [0, T]\colon \omega \cdot t \in B\bigr\} \geqslant \lambda(\mathring{B}). \tag{7.4}$$

Now, the assumption of the lemma implies:

$$\lambda(\overline{B}) = \lambda(\mathring{B}) + \lambda(\partial B) = \lambda(\mathring{B}).$$

As a first consequence, we know that $\lambda(\mathring{B}) = \lambda(B) = \lambda(\overline{B})$. Moreover, taken together, the two estimates (7.3) and (7.4) show that the time average at ω exists, and that its value is exactly $\lambda(B)$. $\square$

Consider now $U \in C(\Omega)$ with the idea of estimating the time averages:

$$\frac{1}{T} m\{t \in [0, T]:\ |U(\omega \cdot t)| \leqslant \delta\}$$

for large values of T and small values of δ. As it can be easily guessed, our interest in this quantity is motivated by condition (3.17). In the light of Theorem 5.2, the focus here is on finding estimates which are uniform in $\omega \in \Omega$. The next result will provide them, by referring to the following couple of functions:

$$U^*(\delta, \omega) = \limsup_{T \to +\infty} \frac{1}{T} m\{t \in [0, T]:\ |U(\omega \cdot t)| \leqslant \delta\},$$

$$U_*(\delta, \omega) = \liminf_{T \to +\infty} \frac{1}{T} m\{t \in [0, T]:\ |U(\omega \cdot t)| \leqslant \delta\}.$$

These functions are manifestly monotone in δ, and then have a limit when δ tends to zero: the value of this limit and the way it is attained are specified in the following statement.

Proposition 7.4. *Assume that $U \in C(\Omega)$. Then, for every $\delta > 0$ and every $\omega \in \Omega$ one has:*

$$\lambda(U^{-1}(0)) \leqslant U_*(\delta, \omega) \leqslant U^*(\delta, \omega) \leqslant \lambda(U^{-1}([-\delta, \delta])) \tag{7.5}$$

where moreover:

$$\lambda(U^{-1}([-\delta, \delta])) \to \lambda(U^{-1}(0)) \tag{7.6}$$

as $\delta \to 0^+$.

Proof. The property (7.6) is a consequence of the Lebesgue Dominated Convergence Theorem. Moreover, the a priori bound on $U^*(\delta, \omega)$ follows from Lemma 7.2 with the choice:

$$B = U^{-1}([-\delta, \delta]).$$

To conclude the proof, it remains to show that:

$$U_*(\delta, \omega) \geqslant \lambda(U^{-1}(0))$$

for every $\delta > 0$ and every $\omega \in \Omega$. To this aim, given a $\delta > 0$ construct an Urysohn function $V \in C(\Omega)$ satisfying:

$$V(\omega) = \begin{cases} 1 & \text{if } U(\omega) = 0, \\ 0 & \text{if } |U(\omega)| \geqslant \delta. \end{cases}$$

Then observe that, for every $\omega \in \Omega$:

$$U_*(\delta,\omega) = \liminf_{T\to+\infty} \frac{1}{T} m\bigl\{t\in[0,T]:\ \bigl|U(\omega\cdot t)\bigr| \leqslant \delta\bigr\} \geqslant \liminf_{T\to+\infty} \frac{1}{T}\int_0^T V(\omega\cdot t)\,dt$$

$$= \lim_{T\to+\infty} \frac{1}{T}\int_0^T V(\omega\cdot t)\,dt = \int_\Omega V\,d\lambda \geqslant \lambda\bigl(U^{-1}(0)\bigr). \qquad \square$$

8. Statement and proof of the main result

In this section we come back to the nonlinear ordinary differential equation:

$$\dot{x} = j(x) + \frac{1}{c} g(cx) + h(t) \tag{8.1}$$

with the aim of stating and proving an existence result in the almost periodic framework. The notations and the assumptions on the jumping linearity j and the nonlinearity g are the same of Sections 2 and 3. To those assumptions we add now that the forcing term h is almost periodic and that a pair $(\Omega,\Psi)\in\mathcal{P}$ is given, such that a morphism:

$$(\Omega,\Psi) \to (\mathcal{H}_h,\Psi_h) \tag{8.2}$$

exists in the category $\mathcal{P}$. Thus Lemma 5.1 says that h can be represented on (Ω,Ψ), namely that a unique $H\in C(\Omega)$ does exist such that the equality:

$$h(t) = H\bigl(\Psi(t)\bigr) \tag{8.3}$$

holds for every t. Finally, as in the last two sections, the Haar measure on Ω will be denoted by λ.

Theorem 8.1. *Under the above assumptions, if moreover*:

$$\|g'\|_\infty \lambda\bigl(H^{-1}(0)\bigr) < \alpha \tag{8.4}$$

then Eq. (8.1) *admits, for c large enough, a unique almost periodic solution.*

Notice that, since by construction:

$$0 \leqslant \lambda\bigl(H^{-1}(0)\bigr) \leqslant 1$$

the assumption (8.4) is a kind of relaxed version of the condition:

$$\|g'\|_\infty < \alpha.$$

We recall from Section 2 that $\alpha > 0$ is the minimal growth rate of the jumping linearity j. Thus, as explained in Remark 3.4, the above condition corresponds to a well known existence result in the literature, which is even valid for every value of the parameter c. On the contrary (8.4) allows $\|g'\|_\infty$ to be much larger than α, as soon this fact is compensated by the smallness of $\lambda(H^{-1}(0))$: the price to pay here is that the existence result only holds for large values of c.

Not surprisingly, the almost periodic solution to (8.1) is representable over (Ω,Ψ). What will be really proved is that, for large values of c, all the differential equations:

$$\dot{x} = j(x) + \frac{1}{c} g(cx) + H(\omega\cdot t) \tag{8.5}$$

with $\omega \in \Omega$, admit a unique bounded solution. Then Theorem 5.2 will be invoked to show that these solutions write as $X(\omega \cdot t)$ for some suitable $X \in C(\Omega)$, so proving their almost periodicity. In all that, a key role will be played by the family of limit equations:

$$\dot{y} = j(y) + H(\omega \cdot t) \tag{8.6}$$

obtained by pushing c to infinity into (8.5). As already shown in Section 3, each of them admits a unique bounded solution which, again due to Theorem 5.2, writes as $Y_H(\omega \cdot t)$ where:

$$Y_H \in C(\Omega).$$

The next result says how the zeroes of Y_H are related to those of H.

Lemma 8.2. *The estimate*:

$$\lambda\big(Y_H^{-1}(0)\big) \leqslant \lambda\big(H^{-1}(0)\big)$$

holds for every $H \in C(\Omega)$.

Proof. Consider the following derivative along the flow:

$$D_\Psi Y_H(\omega) = \lim_{\tau \to 0} \frac{1}{\tau}\big\{Y_H(\omega \cdot \tau) - Y_H(\omega)\big\} = j\big(Y_H(\omega)\big) + H(\omega). \tag{8.7}$$

It is a continuous function on Ω and we can use it to write $Y_H^{-1}(0) = A \cup B$ where:

$$A = \big\{\omega \in Y_H^{-1}(0)\colon\ D_\Psi Y_H(\omega) = 0\big\},$$
$$B = \big\{\omega \in Y_H^{-1}(0)\colon\ D_\Psi Y_H(\omega) \neq 0\big\}.$$

Notice that $A \subset H^{-1}(0)$ due to (8.7) so that, to conclude the proof, it's enough to show that $\lambda(B) = 0$. This is obvious when $\Omega = 0$, since in this case H and Y_H must be constant: in particular $B = \emptyset$.

Assume now that Ω is nontrivial. Take a nontrivial $\varphi \in \Omega^*$ and use its kernel Σ to decompose λ along the flow: the terminology is that of Section 6. Because of (6.8) we have $\lambda(B) = \lambda(B \setminus \Sigma)$. Take now a point $\omega \in B \setminus \Sigma$ and use the continuity of $D_\Psi Y_H$ to select an open neighborhood U_ω such that:

$$D_\Psi Y_H(\theta) \neq 0 \tag{8.8}$$

for every $\theta \in U_\omega$. It is not restrictive to assume that $U_\omega \cap \Sigma = \emptyset$, so that we can write:

$$U_\omega = \Phi(A_\omega \times I_\omega)$$

where A_ω is open in Σ and I_ω is open in $(0, S)$. Because of (8.8) we know that, for every $\sigma \in A_\omega$, the function:

$$t \in I_\omega \mapsto Y_H(\sigma \cdot t)$$

is strictly monotone: thus it vanishes in at most one point. Thus Corollary 6.4 applies to show that:

$$\lambda\big((B \setminus \Sigma) \cap U_\omega\big) = 0$$

and the conclusion $\lambda(B \setminus \Sigma) = 0$ follows from Lemma 6.1. $\square$

Because of the previous lemma, the smallness condition (8.4) transfers to Y_H. This finally allows to trigger the uniqueness device provided by Theorem 3.6, on the basis of the computation rule provided by Proposition 7.4: that's the program followed hereafter.

Proof of Theorem 8.1. Because of Lemma 8.2, we know that:

$$\|g'\|_\infty \lambda\bigl(Y_H^{-1}(0)\bigr) < \alpha.$$

The second part of Proposition 7.4 says that $\delta > 0$ and $\varepsilon > 0$ exist such that:

$$\varepsilon + \|g'\|_\infty \lambda\bigl(Y_H^{-1}\bigl([-\delta, \delta]\bigr)\bigr) < \alpha \tag{8.9}$$

while, using the first part, we may conclude that:

$$\varepsilon + \|g'\|_\infty \sup_{\omega\in\Omega}\left(\liminf_{T\to+\infty}\frac{1}{T} m\bigl\{t \in [0,T]\colon \bigl|Y_H(\omega\cdot t)\bigr| \leqslant \delta\bigr\}\right) < \alpha$$

for every $\omega \in \Omega$. Finally, set:

$$c^* = \frac{\|g\|_\infty + r_\varepsilon}{\delta} \tag{8.10}$$

where the quantity $r_\varepsilon \geqslant 0$ only depends on the behavior of the nonlinearity g, as defined by (3.3) and (3.5). We stress that, since δ and ε are independent of $\omega \in \Omega$, then the same is true for the quantity c^*.

Assume now that $c \geqslant c^*$. Theorem 3.6 says that all Eqs. (8.5) have a unique bounded solution: then Theorem 5.2 allows to conclude. □

A comment is due to the real extent of Theorem 8.1. The main assumption (8.4) is manifestly dependent on the concrete choice of the pair (Ω, Ψ) which satisfies (8.2). The question is, that infinitely many choices are always available: thus, whereas (8.4) is an acceptable solvability condition for a given pair (Ω, Ψ), it seems nevertheless quite unsatisfactory when referred to the forcing term h. The next lemma shows that the question is indeed artificial. In the statement, $H_h \in C(\mathcal{H}_h)$ stands for the map which represents h over the pair $(\mathcal{H}_h, \Psi_h)$, namely the unique continuous map such that:

$$h(t) = H_h\bigl(\Psi(t)\bigr)$$

for all t. Moreover, λ_h denotes the Haar measure on $\mathcal{H}_h$.

Lemma 8.3. *If* (8.3) *is satisfied with $H \in C(\Omega)$ then necessarily*:

$$\lambda\bigl(H^{-1}(0)\bigr) = \lambda_h\bigl(H_h^{-1}(0)\bigr).$$

In other words, condition (8.4) just depends on the forcing term h, and not on the particular representation chosen for it.

Proof of Lemma 8.3. The function $h(t)$ is representable as in (8.3) if and only if (8.2) is satisfied: see Lemma 5.1. Denote by $\chi : \Omega \to \mathcal{H}_h$ the group epimorphism which shadows the morphism (8.2), the uniqueness of the representation yields:

$$H = H_h \circ \chi.$$

To conclude, apply Lemma 6.2 to the Borel set in $\mathcal{H}_h$ given by $B = H_h^{-1}(0)$, obtaining:

$$\lambda_h\big(H_h^{-1}(0)\big) = \lambda\big(\chi^{-1}\big(H_h^{-1}(0)\big)\big) = \lambda\big(H^{-1}(0)\big). \qquad \square$$

9. Sets of Haar measure zero

In this section a pair $(\Omega, \Psi) \in \mathcal{P}$ is assumed to be given and λ denotes, as usual, the Haar measure on Ω. The notations are the same of Section 6.

The aim here is to describe the Borel subsets B of Ω, which satisfy the condition:

$$\lambda(B) = 0. \tag{9.1}$$

The reason goes back to condition (8.4). Assume indeed that (9.1) is satisfied with $B = H^{-1}(0)$: then the conclusions of Theorem 8.1 become true *independently* on the concrete behavior of the jumping linearity j (represented by $\alpha > 0$) and of the size of the perturbation g (represented by $\|g'\|_\infty$). In other words, this is probably the most interesting case in the applications.

The following characterization is an enhancement of the ergodic theorem, with time averages on flow lines replaced by full Lebesgue measures: the Fubini-type decomposition of the Haar measure, given in Section 6, plays here a key role.

Proposition 9.1. *Condition* (9.1) *is satisfied if and only if*:

$$m\{t \in \mathbb{R}\colon\ \omega \cdot t \in B\} = 0 \tag{9.2}$$

for almost all $\omega \in \Omega$.

Notice that the information is no longer available for every $\omega \in \Omega$, also when B is assumed to be closed. For instance, this may be seen in the setting of Example 7.1, taking now $B = \Psi([-1, 1])$: hence B is compact and then closed.

Proof of Proposition 9.1. The result is true when Ω is trivial. In all the other cases, a nontrivial section Σ may be constructed and then the conclusions of Corollary 6.4 hold true. Notice also that condition (9.2) is invariant under the flow: if it is satisfied for a given $\omega \in \Omega$, then it is also satisfied for all the elements in $\omega \cdot \mathbb{R}$.

Assume now that there exists a Borel set $\Omega_0 \subset \Omega$ such that $\lambda(\Omega_0) = 1$ and condition (9.2) holds at every $\omega \in \Omega_0$. It is not restrictive to assume that Ω_0 is invariant under the flow, since otherwise we may replace it by $\Omega_0 \cdot \mathbb{R}$. Define:

$$\Sigma_0 = \Omega_0 \cap \Sigma.$$

Using the invariance of Ω_0, it can be easily checked that:

$$\Phi^{-1}(\Omega_0) = \Sigma_0 \times [0, S). \tag{9.3}$$

Proposition 6.3 then guarantees that $\mu(\Sigma_0) = \lambda(\Omega_0) = 1$, and Corollary 6.4 allows to conclude that $\lambda(B) = 0$.

To prove the reverse implication, start assuming that $\lambda(B) = 0$ and use again Corollary 6.4 to find a Borel set $\Sigma_0 \subset \Sigma$ such that $\mu(\Sigma_0) = 1$ and:

$$m\big\{t \in [0, S)\colon\ \sigma \cdot t \in B\big\} = 0$$

for every $\sigma \in \Sigma_0$. Then define, for every integer k:

$$\Sigma_k = \{\sigma \in \Sigma_0 \colon \sigma + \Psi(kS) \in \Sigma_0\}.$$

Notice that $\sigma + \Psi(kS) \in \Sigma_0$ if and only if $\sigma \in -\Psi(kS) + \Sigma_0$ so that we have indeed:

$$\Sigma_k = \Sigma_0 \cap \{-\Psi(kS) + \Sigma_0\}.$$

By the invariance of μ we know that $\mu(-\Psi(kS) + \Sigma_0) = \mu(\Sigma_0) = 1$ and hence also $\mu(\Sigma_k) = 1$. If we define finally:

$$E = \bigcap_{k \in \mathbb{Z}} \Sigma_k$$

we have that $\mu(E) = 1$. We claim that, for every $\sigma \in E$:

$$m\{t \in \mathbb{R} \colon \sigma \cdot t \in B\} = 0.$$

To show it, begin by noticing that:

$$\begin{aligned}
\{t \in \mathbb{R} \colon \sigma \cdot t \in B\} &= \bigcup_{k \in \mathbb{Z}} \{t \in [kS, (k+1)S) \colon \sigma \cdot t \in B\} \\
&= \bigcup_{k \in \mathbb{Z}} \bigl(kS + \{s \in [0, S) \colon \sigma + \Psi(kS) + \Psi(s) \in B\}\bigr).
\end{aligned}$$

Now, if $\sigma \in E$ then $\sigma + \Psi(kS) \in \Sigma_0$ for all $k \in \mathbb{Z}$. This implies that each set in the above countable union has zero Lebesgue measure. The same must then be true for the union itself, proving the claim.

To conclude the proof, observe that condition (9.2) is also satisfied at every point of $E + \Psi(\mathbb{R})$. But:

$$E + \Psi(\mathbb{R}) \supset E + \Psi([0, S)) = \Phi(E \times [0, S))$$

and hence Proposition 6.3 applies to show that:

$$\lambda(E + \Psi(\mathbb{R})) \geqslant \mu(E) = 1. \qquad \square$$

The second part of the section is devoted to functions. The aim is to describe the set properties of the $U \in C(\Omega)$ which satisfy:

$$\lambda(U^{-1}(0)) = 0 \tag{9.4}$$

and to exhibit some relevant examples.

The first question we consider is whether condition (9.4) may be expressed in terms of a single flow line. More precisely, fix $\omega_0 \in \Omega$ and set:

$$u_0(t) = U(\omega_0 \cdot t).$$

Since $\overline{\omega_0 \cdot \mathbb{R}} = \Omega$ and U is continuous in Ω, the action of U on all of Ω is completely determined by the knowledge of u_0. Thus it seems reasonable that (9.4) may be obtained from some suitable assumption on u_0. This is certainly true if $\Omega \cong \mathbb{S}^1$. In this case indeed, Ω is made a single periodic flow line and hence (9.4) simply means that the set:

$$\big\{t \in \mathbb{R}\colon\ U(\omega_0 \cdot t) = 0\big\} \tag{9.5}$$

has zero Lebesgue measure.

The situation changes drastically when $\Omega \not\cong \mathbb{S}^1$. Of course, it is again true that the strong sign condition:

$$U(\omega_0 \cdot t) \geqslant \delta > 0 \quad \forall t$$

implies $U \geqslant \delta$ on all of Ω, so that condition (9.4) is satisfied. However, no other weaker assumptions on the set (9.5) are suitable to it: this is the sense of the next example.

Example 9.2. Take any nontrivial $\Omega \not\cong \mathbb{S}^1$. Moreover choose $0 < \varepsilon < 1$ and an open set A in Ω such that:

$$\Psi(\mathbb{R}) \subset A, \qquad \lambda(A) < \varepsilon.$$

This is possible since the measure λ is regular and $\lambda(\Psi(\mathbb{R})) = 0$: see Lemma 6.5. Then define $U \in C(\Omega)$ by means of:

$$U(\omega) = d\big(\omega, A^C\big)$$

where d stands for a metric which generates the topology of Ω. Consider now the flow line passing trough $\omega_0 = 0$, namely $\Psi(\mathbb{R})$. Due to the choice of A we know that $U(\Psi(t)) > 0$ for every $t \in \mathbb{R}$: hence the set (9.5) is empty. However condition (9.4) fails, since $U^{-1}(0) = A^C$ and $\lambda(A^C) > 1 - \varepsilon > 0$ by construction. The point here is that the set:

$$\big\{t \in \mathbb{R}\colon\ U(\omega \cdot t) = 0\big\}$$

has a positive Lebesgue measure, for a set of ω's which has a positive λ-measure: this follows from Proposition 9.1, by taking $B = U^{-1}(0)$. In fact, the Birkhoff Ergodic Theorem (7.1) allows to say more: the time average of the above set is $\lambda(A^C)$ for almost every $\omega \in \Omega$.

The above obstruction is present also when $\Omega = \mathbb{T}^N$ with $N > 1$. In this case, however, a special class of functions may be identified, for which condition (9.4) is easily tested: that is $C^\omega(\mathbb{T}^N)$, the class of *real analytic* functions on $\mathbb{T}^N$. They may be thought of as the real analytic functions on $\mathbb{R}^N$ which are 1-periodic in each variable. To get an element of $\mathcal{P}$, complete $\mathbb{T}^N$ with $\Psi(t) = \nu t$ for a nonresonant $\nu \in \mathbb{R}^N$.

Lemma 9.3. *Assume that $U \in C^\omega(\mathbb{T}^N)$. Then either $U = 0$ or condition (9.4) is fulfilled.*

In particular, this is true when U is a linear combination of characters of $\mathbb{T}^N$. Notice that the corresponding functions $U(\nu t)$ are trigonometric polynomials: in fact, all of them can be obtained by tuning the representation.

Proof of Lemma 9.3. Each function $u_\omega(t) = U(\nu t)$ is real analytic in the variable t. Thus either $u_\omega(t) = 0$ for a discrete set of t's or it is identically zero. If the last case occurs for a single ω, then U must be identically zero on Ω due to $\overline{\nu\mathbb{R}} = \mathbb{T}^N$. Consider now the case where $u_\omega^{-1}(0)$ is discrete set for every $\omega \in \Omega$: then is has Lebesgue measure zero, and Proposition 9.1 applies to show that condition (9.4) is satisfied. $\square$

Remark 9.4. It is well know that, if $\varphi \in C^{\omega}(\mathbb{R}^N)$ is not identically zero, then $\varphi^{-1}(0)$ has Lebesgue measure zero in $\mathbb{R}^N$: adding the obvious periodicity condition, this fact would have been given an alternative proof of Lemma 9.3. Notice however that, when $N = 1$, this property is exactly what we used in the proof of the lemma. Moreover, the same property is most easily proved for $N > 1$ by induction on the dimension N, taking into account that the Lebesgue measure on $\mathbb{R}^N$ is a product measure. This is not so different from the use we made of Proposition 9.1, based on the decomposition of λ.

It is probably worth to stress that, the analyticity assumption considered in Lemma 9.3 cannot be transferred from Ω to a single flow line. Consider for instance the real analytic function:

$$u(t) = \left\{2 - \sin(t) - \sin(\sqrt{2}t)\right\}^{3/2}.$$

We may represent it on $\mathbb{T}^2$ by means of:

$$U(\theta_1, \theta_2) = \left\{2 - \sin(2\pi\theta_1) - \sin(2\pi\theta_2)\right\}^{3/2}$$

which is continuous, but certainly not analytic. Notice however that, though Lemma 9.3 cannot apply, the condition (9.4) is satisfied. Indeed, U is nonanalytic only at $(1/2, 1/2) \in \mathbb{T}^2$ and the problems are then confined to the flow line trough it. But this line has Haar measure zero, due to Lemma 6.5, and hence Proposition 9.1 could have been used to conclude.

In fact, Proposition 9.1 may be helpful also when more complicate Ω are considered, possibly without any differentiable structure. To make an example, consider the function:

$$u(t) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \sin\left(\frac{2\pi t}{p^n}\right) \tag{9.6}$$

where p is any real number satisfying $p > 1$. This function is almost periodic, inasmuch it is the uniform limit on the real line of its partial sums, which are trigonometric polynomials. Consider now $\Omega = \mathcal{H}_u$ with its canonical flow Ψ_u. If $p \in \mathbb{N}$ then u is a limit periodic function and $\mathcal{H}_u$ is shown to be a p-adic solenoid S_p, which does not admit any differentiable structure (see [12]). The situation is even worst when $p \notin \mathbb{N}$ but the point is that, doesn't matter how complicate $\mathcal{H}_u$ may be, we may use Proposition 9.1 to guarantee that condition (9.4) is satisfied: it is sufficient to show that *every* function in $\mathcal{H}_u$ is real analytic, since nontriviality is obvious. To this aim, use a standard diagonal argument to prove that all the elements of $\mathcal{H}_h$ may be written as:

$$u_\theta(t) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \sin\left(2\pi \frac{t + \theta_n}{p^n}\right)$$

for some real sequence $\theta = (\theta_n)_n$. Indeed, only a restricted set of θ's gives rise to elements of the hull, but this is not relevant for the following. To prove that $u_\theta(t)$ is real analytic, we first extend it to the complex plane by means of:

$$u_\theta(z) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \sin\left(2\pi \frac{z + \theta_n}{p^n}\right)$$

where $z = t + is$. The single terms in the series are entire functions. Moreover, for every $\rho > 0$, the estimate:

$$\frac{1}{2^n}\left|\sin\left(2\pi \frac{z + \theta_n}{p^n}\right)\right| \leqslant \frac{e^{2\pi\rho/p^n}}{2^n} \leqslant \frac{e^{2\pi\rho}}{2^n}$$

holds in the strip $|\operatorname{Im} z| \leqslant \rho$. As a consequence, the series converges uniformly in all these strips. Montel's theorem then applies to show that $u_\theta(z)$ is an entire function, so that its trace $u_\theta(t)$ on the real axis must be real analytic too.

We conclude the section with the question of the genericity, in the sense of Baire, of condition (9.4). Here Ω is given, and we may distinguish two situations: the average of U is assigned or it is not. To assign it, by requiring for instance that:

$$\int_\Omega U\,d\lambda = 0 \tag{9.7}$$

may be relevant to avoid the case of U's with constant sign. The resulting subset of $C(\Omega)$ will be denoted by $C_0(\Omega)$. We need first a density lemma.

Lemma 9.5. *For every $V \in C(\Omega)$ and every $\varepsilon > 0$, there exists $U \in C(\Omega)$ such that condition* (9.4) *is verified and moreover*:

$$\|U - V\|_\infty < \varepsilon \quad \textit{and} \quad \int_\Omega U\,d\lambda = \int_\Omega V\,d\lambda.$$

The conclusion is quite evident when $\Omega = \mathbb{T}^N$. Indeed, $C^\omega(\mathbb{T}^N)$ is a dense subset of $C(\mathbb{T}^N)$ and $C_0(\mathbb{T}^N)$, due to the Stone–Weierstrass theorem: hence Lemma 9.3 implies the result. However, the general Ω does not admit any differentiable structure: this is for instance the case of solenoids, where to talk about analyticity does not seem to have so much sense.

Proof of Lemma 9.5. For every positive integer k, the set:

$$\mathcal{V}_k = \left\{c \in \mathbb{R}\colon\ \lambda\left(V^{-1}(c)\right) \geqslant 1/k\right\}$$

contains at most k elements. This follows from $\lambda(\Omega) = 1$. As a consequence, the set:

$$\mathcal{V} = \left\{c \in \mathbb{R}\colon\ \lambda\left(V^{-1}(c)\right) > 0\right\} = \bigcup_k \mathcal{V}_k$$

is at most countable and we can always find a sequence $c_n \to 0$ such that $c_n \notin \mathcal{V}$. Moreover, we may also assume that $0 \in \mathcal{V}$, since otherwise the conclusions of the lemma are true with $U = V$. Define now:

$$W_n(\omega) = V(\omega) - c_n.$$

We know that condition (9.4) is satisfied by W_n and that $\|W_n - V\|_\infty = |c_n| \to 0$. However W_n has not the same average of V, due to $0 \in \mathcal{V}$. To overcome the problem, we distinguish two cases.

Assume first $V(\omega_0) \neq 0$ for a suitable $\omega_0 \in \Omega$. By continuity, we can find $\delta > 0$ and an open neighborhood A of zero such that:

$$\left|V(\omega)\right| \geqslant \delta \quad \forall \omega \in \omega_0 + A.$$

Consider an Urysohn function F in Ω, satisfying:

$$F(\omega) = \begin{cases} 1 & \text{if } \omega = 0, \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Since F is continuous and nonnegative, clearly:

$$\rho = \int_{\Omega} F \, d\lambda > 0.$$

Consider then the continuous functions:

$$U_n(\omega) = W_n(\omega) + \frac{c_n}{\rho} F(\omega - \omega_0) = V(\omega) + c_n \left\{ \frac{1}{\rho} F(\omega - \omega_0) - 1 \right\}.$$

Due to the invariance property of λ, one has now:

$$\int_{\Omega} U_n \, d\lambda = \int_{\Omega} V \, d\lambda$$

while the approximation property $\|U_n - V\|_\infty \to 0$ is preserved. In particular, for large values of n we know that:

$$\left|U_n(\omega)\right| \geqslant \delta/2 \quad \forall \omega \in \omega_0 + A.$$

As a consequence, U_n can only vanish in $\omega_0 + A^C$, where the equality $U_n = W_n$ is satisfied. Thus, for the same values of n:

$$U_n^{-1}(0) \subset V^{-1}(c_n)$$

which has zero measure, due to $c_n \notin \mathcal{V}$.

Let us finally consider the case where V is identically zero in Ω. The conclusion follows from the previous case, if we are able to show that a sequence $V_n \in C(\Omega)$ exists, satisfying:

$$V_n \neq 0, \qquad \int_{\Omega} V_n \, d\lambda = 0, \qquad \|V_n - V\|_\infty \to 0.$$

To construct it, consider two points $\omega_0 \neq \omega_1$ in Ω, choose an open neighborhood A of zero such that:

$$(\omega_0 + \overline{A}) \cap (\omega_1 + \overline{A}) = \emptyset$$

and define the Urysohn function F as before. Finally take a sequence $\alpha_n \neq 0$ such that $\alpha_n \to 0$ and set:

$$V_n(\omega) = \alpha_n \left\{ F(\omega - \omega_0) - F(\omega - \omega_1) \right\}.$$

The condition:

$$\int_{\Omega} V_n \, d\lambda = 0$$

follows again from the invariance of λ, while the other required properties are totally obvious. □

Proposition 9.6. *The class of $U \in C(\Omega)$ which satisfy condition (9.4) is of second Baire category in $C(\Omega)$. The same result is true when $C(\Omega)$ is replaced by $C_0(\Omega)$.*

Proof. The proof for $C(\Omega)$ is essentially the same as for $C_0(\Omega)$, so that we will treat only the second case. For $\varepsilon > 0$, consider the set:

$$\mathcal{A}_\varepsilon = \{u \in C_0(\Omega)\colon\ \lambda(U^{-1}(0)) < \varepsilon\}.$$

We show that this set is open by proving that, if $U \in \mathcal{A}_\varepsilon$ and $U_n \to U$ in $C_0(\Omega)$, then $U_n \in \mathcal{A}_\varepsilon$ eventually. For writing convenience, set:

$$K = U^{-1}(0), \qquad K_n = U_n^{-1}(0)$$

and denote by χ_n the characteristic function of the set K_n^C. Since for every $\omega \notin K$ the equality:

$$\chi_n(\omega) = 1$$

holds eventually, the Lebesgue Dominated Convergence Theorem implies:

$$\lambda(K_n^C) \geqslant \lambda((K_n \cup K)^C) = \int_{K^C} \chi_n \, d\lambda \to \lambda(K^C) > 1 - \varepsilon.$$

As a consequence $\lambda(K_n) < \varepsilon$ for n large, showing that $U_n \in \mathcal{A}_\varepsilon$ and hence that $\mathcal{A}_\varepsilon$ is open. On the other hand, Lemma 9.5 implies that the set $\mathcal{A}_\varepsilon$ is dense in $C_0(\Omega)$. The conclusion then follows by noticing that the set of $U \in C_0(\Omega)$ which satisfy condition (9.4) writes as $\bigcap_{n\in\mathbb{N}_+} \mathcal{A}_{1/n}$. $\square$

Next we consider a different type of generic result: in some sense, the representing group Ω is now allowed to vary. More precisely, we will work directly in the Banach space $AP(\mathbb{R})$ of all the almost periodic functions, with the standard sup-norm, or in the subspace:

$$AP_0(\mathbb{R}) = \{u \in AP(\mathbb{R})\colon\ \bar{u} = 0\}.$$

Every $u \in AP(\mathbb{R})$ is extended by continuity, i.e. represented on the pair $(\mathcal{H}_u, \Psi_u)$ by the function $U_u(v) = v(0)$. Denote by λ_u the Haar measure on $\mathcal{H}_u$. Then look at the intrinsic condition:

$$\lambda_u(U_u^{-1}(0)) = 0. \tag{9.8}$$

Proposition 9.7. *The class of $u \in AP(\mathbb{R})$ which satisfy condition (9.8) is of second Baire category in $AP(\mathbb{R})$. The same result holds when $AP(\mathbb{R})$ is replaced by $AP_0(\mathbb{R})$.*

Proof. Set $K_u = U_u^{-1}(0) \subset \mathcal{H}_u$. We adapt the proof of Proposition 9.6, by restricting again to the case $AP_0(\mathbb{R})$ and showing that the set:

$$\mathcal{A}_\varepsilon = \{u \in AP_0(\mathbb{R})\colon\ \lambda_u(K_u) < \varepsilon\}$$

is open and dense in $AP_0(\mathbb{R})$. Density is no longer a problem here, since the trigonometric polynomials are dense in $AP_0(\mathbb{R})$ and are real analytic. To show that $\mathcal{A}_\varepsilon$ is open, for every integer n consider the set:

$$\mathcal{A}_\varepsilon^n = \left\{u \in AP_0(\mathbb{R})\colon\ \int_{\mathcal{H}_u} \frac{d\lambda_u}{1 + n|U_u|} < \varepsilon\right\}.$$

Since:

$$\int_{\mathcal{H}_u} \frac{d\lambda_u}{1+n|U_u|} = \lambda_u(K_u) + \int_{\mathcal{H}_u \setminus K_u} \frac{d\lambda_u}{1+n|U_u|} \geqslant \lambda_u(K_u) \tag{9.9}$$

it is clear that:

$$\mathcal{A}_\varepsilon^n \subset \mathcal{A}_\varepsilon \quad \forall n. \tag{9.10}$$

Moreover, from the explicit form of the Haar integral, see formula (6.3), one deduces that:

$$\int_{\mathcal{H}_u} \frac{d\lambda_u}{1+n|U_u|} = \overline{\frac{1}{1+n|u|}}.$$

Now, the right hand side is continuous in u and this shows that $\mathcal{A}_\varepsilon^n$ is open in $AP_0(\mathbb{R})$. To prove that also $\mathcal{A}_\varepsilon$ is open in $AP_0(\mathbb{R})$, it is now enough to show that:

$$\mathcal{A}_\varepsilon = \bigcup_{n \in \mathbb{N}_+} \mathcal{A}_\varepsilon^n.$$

One partial inclusion is given by (9.10). The other follows from (9.9), since:

$$\int_{\mathcal{H}_u \setminus K_u} \frac{d\lambda_u}{1+n|U_u|} \to 0$$

due to Lebesgue's Dominated Convergence Theorem. □

Remark 9.8. It is maybe worth pointing out that condition (9.8) respects the hull, in the following sense: given an arbitrary $u \in AP(\mathbb{R})$, either it is satisfied for all $v \in \mathcal{H}_u$ or it is by no one. Now, belonging to the same hull is an equivalence relation in $AP(\mathbb{R})$, which operates a partition of $AP(\mathbb{R})$ itself in equivalence classes: Proposition 9.7 then states that the generic equivalence class is made by functions which satisfy condition (9.8).

To see why condition (9.8) respects the hull, take $v \in \mathcal{H}_u$ and begin by noticing that $\mathcal{H}_u = \mathcal{H}_v$ as sets and also as metric spaces. They are not the same group, since the neutral elements are different. However, it is not too hard to check that the map:

$$u_\tau \in \mathcal{H}_u \mapsto v_\tau \in \mathcal{H}_v$$

is well defined and extends to a continuous homomorphism of groups. By the very construction, this map defines a morphism $(\mathcal{H}_u, \Psi_u) \to (\mathcal{H}_v, \Psi_v)$ in the category $\mathcal{P}$. In the footsteps of Lemma 8.3, one then finds that the equality:

$$\lambda_v\big(U_v^{-1}(0)\big) = \lambda_u\big(U_u^{-1}(0)\big)$$

is always satisfied.

Appendix A

In this section, we consider the differential equation:

$$\dot{x} = x + \frac{1}{c} g(cx) + h(t) \tag{A.1}$$

where:

$$g(\xi) = \begin{cases} 2 & \text{if } \xi \leqslant -1, \\ -2\xi & \text{if } -1 \leqslant \xi \leqslant 1, \\ -2 & \text{if } \xi \geqslant 1. \end{cases}$$

Up to the change of variable $u = cx$ introduced in Section 3, this equation is that considered in the Introduction, corresponding to the nonlinearity (1.4). The aim of this appendix is to estimate the threshold value for the parameter c, over which Theorem 8.1 applies, when the forcing term is given by:

$$h(t) = \sin(t) + \sin(\sqrt{2}t).$$

This term is representable on the pair $(\mathbb{T}^2, \Psi)$, where $\Psi(t) = (t/2\pi, \sqrt{2}t/2\pi)$ for every t, by the continuous function:

$$H(\theta_1, \theta_2) = \sin(2\pi\theta_1) + \sin(2\pi\theta_2).$$

Begin by noticing that $\|g\|_\infty = \|g'\|_\infty = 2$. Moreover, defining:

$$K(r) = \sup\left\{ \frac{|g(\xi_1) - g(\xi_2)|}{|\xi_1 - \xi_2|} \colon \xi_1 \neq \xi_2,\ |\xi_1| \geqslant r,\ |\xi_2| \geqslant r \right\}$$

as in Section 3, it is not difficult to check that:

$$K(r) = \begin{cases} 2 & \text{if } r \leqslant 1, \\ 2/r & \text{if } r \geqslant 1 \end{cases}$$

so that we may choose:

$$r_\varepsilon = \begin{cases} 2/\varepsilon & \text{if } 0 < \varepsilon < 2, \\ 0 & \text{if } \varepsilon \geqslant 2. \end{cases}$$

Following the proof of Theorem 8.1, the threshold value (8.10) then becomes:

$$c^* = \frac{2}{\delta}\left(1 + \frac{1}{\varepsilon}\right) \tag{A.2}$$

where $\delta > 0$ and $\varepsilon > 0$ are chosen according to:

$$\varepsilon + 2\lambda\bigl(Y_H^{-1}\bigl([-\delta, \delta]\bigr)\bigr) < 1. \tag{A.3}$$

Notice that this condition yields the automatic restriction $0 < \varepsilon < 1$, which has been used to write (A.2). Moreover, it is clear that condition (A.3) may be satisfied if and only if:

$$\lambda\bigl(Y_H^{-1}(0)\bigr) < 1/2.$$

Together with Lemma 8.2, this accounts for the assumption (1.7) of Theorem 1.1 in the Introduction. Of course, the same condition is obtained by inserting the concrete values of α and $\|g'\|_\infty$ directly into condition (8.4).

Observe that Y_H has a nice expression in terms of H, namely:

$$Y_H(\omega) = -\int_0^{+\infty} e^{-t} H(\omega \cdot t)\, dt$$

which permits to compute it in the case we are interested in. The result is:

$$Y_H(\theta_1, \theta_2) = -\frac{1}{2}\big\{\sin(2\pi\theta_1) + \cos(2\pi\theta_1)\big\} - \frac{1}{3}\big\{\sin(2\pi\theta_2) + \sqrt{2}\cos(2\pi\theta_2)\big\}.$$

Consider now the quantity:

$$\lambda_H(\delta) = \lambda\big(Y_H^{-1}\big([-\delta, \delta]\big)\big) \tag{A.4}$$

which appears in (A.3). Clearly it depends continuously on δ, vanishes at $\delta = 0$ and it is not difficult to check that it is unitary for every $\delta \geqslant \delta_H^2$, where:

$$\delta_H^2 = \|Y_H\|_\infty = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}.$$

Moreover, λ_H is strictly increasing in the interval $0 \leqslant \delta \leqslant \delta_H^2$.

Condition (A.3) yields a further restriction on δ, the admissible interval becoming:

$$0 < \delta < \delta_H \quad \text{where } \delta_H = \lambda_H^{-1}(1/2).$$

For such δ's, condition (A.3) is satisfied if and only if $0 < \varepsilon < 1 - 2\lambda_H(\delta)$. Inserting this information into (A.2), we finally obtain the threshold function:

$$c_H(\delta) = \frac{2}{\delta}\left\{1 + \frac{1}{1 - 2\lambda_H(\delta)}\right\}. \tag{A.5}$$

This is a continuous function which explodes at the boundary of the admissible interval. The best condition on the parameter c then becomes:

$$c > c_H^* := \min\big\{c_H(\delta)\colon\ 0 < \delta < \delta_H\big\}.$$

To compute c_H^* concretely, we need to know the explicit expression of the function $\lambda_H(\delta)$: as far as we know, this is possible only numerically. Before following this way, notice that the expression of the function Y_H may be simplified, without altering the result. Indeed, the function:

$$F_H(\theta_1, \theta_2) = \frac{1}{\sqrt{2}}\sin(2\pi\theta_1) + \frac{1}{\sqrt{3}}\sin(2\pi\theta_2)$$

is, up to the sign, a translated version of $Y_H(\theta_1, \theta_2)$ and hence the equality:

$$\lambda_H(\delta) = \lambda\big(F_H^{-1}\big([-\delta, \delta]\big)\big) = 1 - 2\lambda\big(F_H^{-1}\big((\delta, 1]\big)\big)$$

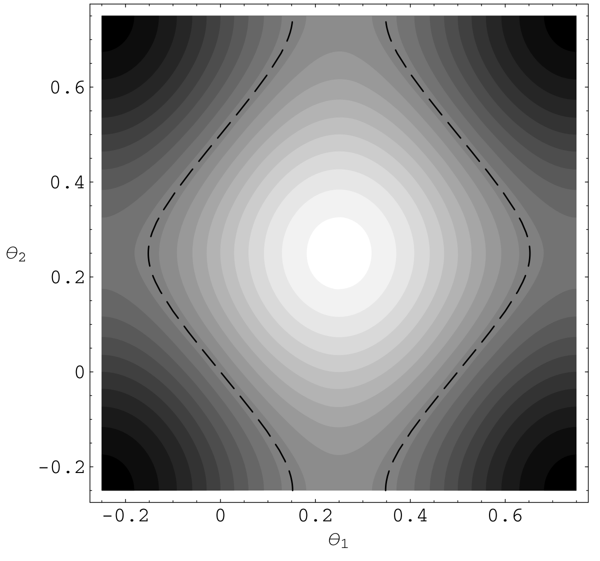

Fig. 1. Level sets of F_H.

follows from the invariance property of the Haar measure and symmetry arguments. In Fig. 1 we provide a picture of the level sets of the function F_H, as obtained by a numerical computation: the value of the level increases from black to white. The zero level is regular and it is shown as a dashed line.

As the picture suggests, there is a change in the topology of the level sets when δ varies in the range $[0, \delta_H^2]$. It is not difficult to check that the bifurcation value is:

$$\delta_H^1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}.$$

To see why, let us describe the involved lines by using the parameter θ_2. Up to the periodicity in θ_1, the level set $F_H^{-1}(\delta)$ is the union of two graphs, associated to the functions:

$$f_H^\delta(\theta_2) = \frac{1}{2\pi} \arcsin\left(\sqrt{2}\delta - \sqrt{\frac{2}{3}} \sin(2\pi\theta_2)\right),$$

$$g_H^\delta(\theta_2) = \frac{1}{2} - \frac{1}{2\pi} \arcsin\left(\sqrt{2}\delta - \sqrt{\frac{2}{3}} \sin(2\pi\theta_2)\right).$$

Moreover, it is clear that the region inside the graphs corresponds exactly to the points where $F_H > \delta$.

Now, when $0 \leqslant \delta < \delta_H^1$ the two functions are defined everywhere and don't cross each other: they correspond to the unbounded vertical lines which appear in the picture above. When on the contrary $\delta_H^1 < \delta \leqslant \delta_H^2$, the functions are defined in a restricted domain. Up to the periodicity in the variable θ_2, this domain is given by:

$$a_H^\delta \leqslant \theta_2 \leqslant b_H^\delta$$

where we set:

$$a_H^\delta = \frac{1}{2\pi} \arcsin\left(\sqrt{3}\delta - \sqrt{\frac{3}{2}}\right), \qquad b_H^\delta = \frac{1}{2} - \frac{1}{2\pi} \arcsin\left(\sqrt{3}\delta - \sqrt{\frac{3}{2}}\right).$$

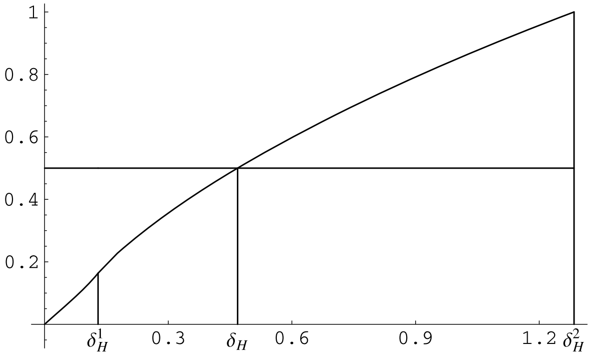

Fig. 2. Graph of the function λ_H.

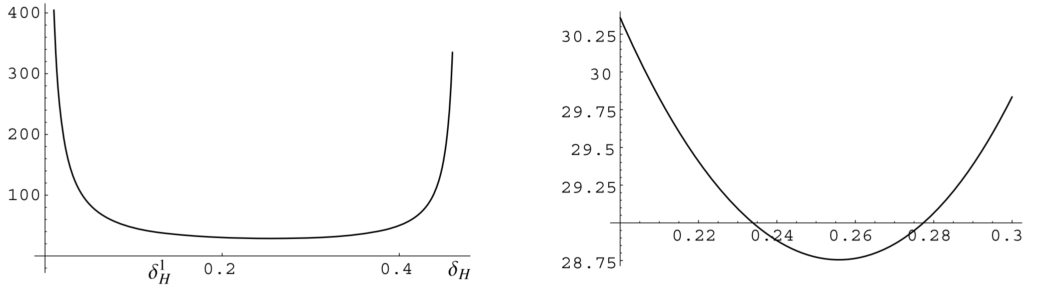

Fig. 3. Graph of the function c_H.

The two graphs intersect now exactly at the end points of the common domain, giving rise to a closed curve.

Notice that the above end points may be extended by continuity to the interval $[0, \delta_H^1]$ by setting:

$$a_H^\delta = -\frac{1}{4}, \qquad b_H^\delta = \frac{3}{4}$$

which defines a full periodicity interval in the variable θ_2. Thus, we may express the desired measure by the quite compact formula:

$$\lambda_H(\delta) = 1 - 2 \int_{a_H^\delta}^{b_H^\delta} \left\{ g_H^\delta(\theta_2) - f_H^\delta(\theta_2) \right\} d\theta_2 \tag{A.6}$$

which is valid in all the interval $0 \leqslant \delta \leqslant \delta_H^2$. Fig. 2 shows the graphs of the function λ_H, as obtained by a numerical computation.

A numerical computation shows that the bifurcation value δ_H^1 lies inside the admissible interval $(0, \delta_H)$. Inserting the corresponding numerical values of $\lambda_H(\delta)$ into (A.5), we finally obtain the graph of the threshold function. This is shown in Fig. 3: the right hand side is a magnified version of the left one, centered around the minimum point. The position of the corresponding minimum value justifies the estimate:

$$c_H^* \approx 28.75.$$

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