J. Differential Equations 254 (2013) 961–982

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

On the Cauchy problem for the modified Novikov equation with peakon solutions

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A R T I C L E   I N F O

Article history:
Received 28 April 2012
Revised 25 September 2012
Available online 25 October 2012

MSC:
35B30
35G25
35A10
35Q53

Keywords:
Analytic solutions
Besov space
Local well-posedness
Blow-up

A B S T R A C T

A new nonlinear dispersive partial differential equation with cubic nonlinearity, which includes the famous Novikov equation as special case, is investigated. We first establish the local well-posedness in a range of the Besov spaces $B^s_{p,r}$, $p, r \in [1,\infty]$, $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ but $s \neq 2+\frac{1}{p}$ (which generalize the Sobolev spaces H^s), well-posedness in H^s with $s > \frac{3}{2}$, is also established by applying Kato's semigroup theory. Then we give the precise blow-up scenario. Moreover, with analytic initial data, we show that its solutions are analytic in both variables, globally in space and locally in time. Finally, we prove that peakon solutions to the equation are global weak solutions.

1. Introduction

In this paper, we are concerned with the following Cauchy problem of the modified Novikov equation with peakon solutions

$$\begin{cases} u_t - u_{txx} + (b+1)u^2u_x = buu_xu_{xx} + u^2u_{xxx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where b is real parameter. By comparison with the Novikov equation ($b = 3$), it is easy to find that (1.1) is more general.

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.09.016

By comparison with the b-equation

$$\begin{cases} u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

it is easy to see that (1.1) has nonlinear terms that are cubic, rather than quadratic of b-equation. The b-equation can be derived as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, cf. [26,27]. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated [26,27]. The solutions of the b-equation were studied numerically for various values of b in [44,45], where b was taken as a bifurcation parameter. The symmetry conditions necessary for integrability of the b-equation were investigated in [50]. The KdV equation, the Camassa–Holm equation and the Degasperis–Procesi equation are the only three integrable equations, which were shown in [23,24] by using Painlevé analysis. The b-equation admits peakon solutions for any $b \in R$, cf. [23,44,45]. In [28], Escher and Yin established the local well-posedness for the b-equation, gave blow-up scenario; they also proved the uniqueness and existence of global weak solution to the equation provided the initial data satisfy certain sign conditions. Gui et al. showed that the blow-up phenomena occur in finite time for certain initial profiles and obtained a global existence result for the b-equation [33].

For $b = 2$, Eq. (1.2) becomes the Camassa–Holm equation

$$\begin{cases} u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

modeling the unidirectional propagation of shallow water waves over a flat bottom, $u(t,x)$ stands for the fluid velocity at time t in the spatial direction x. It is a well-known integrable equation describing the velocity dynamics of shallow water waves. This equation spontaneously exhibits emergence of singular solutions from smooth initial conditions. It has a bi-Hamilton structure [29] and is completely integrable [5,6]. In particular, it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. After the birth of the Camassa–Holm equation, many works have been carried out to probe its dynamic properties. Such as, Eq. (1.3) has traveling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the traveling waves of largest amplitude (see [7,12,19,13]). It is shown in [17,8,14] that the inverse spectral or scattering approach is a powerful tool to handle the Camassa–Holm equation and analyze its dynamics. It is worthwhile to mention that Eq. (1.3) gives rise to geodesic flow of a certain invariant metric on the Bott–Virasoro group [15,51], and this geometric illustration leads to a proof that the Least Action Principle holds. It is shown in [10] that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. Moreover, the Camassa–Holm equation has global conservative solutions [3,42] and dissipative solutions [4,43]. For other methods to handle the problems relating to various dynamic properties of the Camassa–Holm equation and other shallow water equations, the reader is referred to [2,20,16,9,35,30,36–41,11,10,18] and the references therein.

For $b = 3$, Eq. (1.1) becomes the Novikov equation

$$\begin{cases} u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.4}$$

which has been recently discovered by Vladimir Novikov in a symmetry classification of nonlocal PDEs with quadratic or cubic nonlinearity [53]. The perturbative symmetry approach yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate Eq. (1.4) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, then proved that the equation is integrable, which can be thought as a generalization of the Camassa–Holm equation. In [55], it is shown that the Novikov equation admits peakon solutions like the Camassa–Holm. Also, it has a Lax pair in matrix form and a bi-Hamiltonian structure. Furthermore, it has infinitely many conserved quantities. Like Camassa–Holm, the most important quantity conserved by a solution u to Novikov equation is its H^1-norm $\|u\|_{H^1}^2 = \int_R (u^2 + u_x^2)$, which plays an important role in the study of Eq. (1.4). In [34,52,59,60,62], the authors study well-posedness and dependence on initial data for the Cauchy problem for Novikov equation. Recently, in [47], a global existence result and conditions on the initial data were considered. Existence and uniqueness of global weak solution to Novikov equation with initial data under some conditions were proved in [61]. The Novikov equation with dissipative term was considered in [63]. Multipeakon solutions were studied in [55,46,31]. The Cauchy problem of the Novikov equation on the circle was investigated in [58].

Motivated by the references cited above, the goal of the present paper is to establish qualitative results for the initial value problem (1.1). We first study the local well-posedness for the strong solutions to the Cauchy problem (1.1). The proof of the local well-posedness is inspired by the argument of approximate solutions by Danchin [22] in the study of the local well-posedness to the Camassa–Holm equation. However, one problematic issue is that we here deal with a higher order nonlinearity in the Besov spaces, making the proof of several required nonlinear estimates somewhat delicate. These difficulties are nevertheless overcome by carefully estimates for each iterative approximation of solutions to (1.1). With the local well-posedness obtained in hand, we then present a precise blow-up scenario and a conservative property. We also prove the analyticity of its solutions $u = u(t,x)$ in both variables, with x in R and t in an interval around zero, provided that the initial profile u_0 is an analytic function on the real line. Hence, this analytic result can be viewed as a Cauchy–Kowalevski theorem for (1.1). Finally, we prove that peakon solutions to Eq. (1.1) are global weak solutions.

The rest of this paper is organized as follows. In Section 2, we prove the local well-posedness of the initial value problem (1.1) in the Besov space $B^s_{p,r}$, $p,r \in [1,\infty]$, $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$. In Section 3, local well-posedness is established in H^s for $s > \frac{3}{2}$. In Section 4, blow-up scenario and global existence result of (1.1) are derived. Section 5 is devoted to the study of the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Finally, we prove that peakon solutions to Eq. (1.1) are global weak solutions.

2. Local well-posedness in the Besov spaces

In this section, we shall establish local well-posedness for the Cauchy problem (1.1) in the Besov spaces.

Note that if $p(x) = \frac{1}{2}e^{-|x|}$, we have $(1-\partial_x^2)^{-1} f = p * f$ for all the $f \in L^2$, and $p * (u - u_{xx}) = u$, where we denote by $*$ the convolution. Then we can rewrite the Cauchy problem (1.1) as follows:

$$\begin{cases} u_t + u^2 u_x + p * \big(bu^2 u_x + (6-b)u u_x u_{xx} + 2u_x^3\big) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

or in the equivalent form

$$\begin{cases} u_t + u^2 u_x = -\big(1-\partial_x^2\big)^{-1}\left(\partial_x\left(\dfrac{6-b}{2} u u_x^2 + \dfrac{b}{3}u^3\right) + \dfrac{b-2}{2}u_x^3\right), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

Now we are in the position to state local well-posedness result for the Cauchy problem (1.1), the definition of Besov–Sobolev spaces $B^s_{p,r}$, $E^s_{p,r}(T)$ and $\mathcal{S}'$ is given in [22,21].

Theorem 2.1. *Let $p,r \in [1,\infty]$ and $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ but $s \neq 2+\frac{1}{p}$. Assume that $u_0 \in B^s_{p,r}$. There exist a time $T > 0$ and a unique solution $u \in E^s_{p,r}(T)$ to Eq. (2.1) such that the map $u_0 \mapsto u : B^s_{p,r} \mapsto C([0,T]; B^{s'}_{p,r}) \cap C^1([0,T]; B^{s'-1}_{p,r})$ is continuous for every $s' < s$ when $r = \infty$ and $s' = s$ when $r < \infty$.*

In the following, we denote $C>0$ a generic constant only depending on p, r, s, b, which may be different on different lines. Uniqueness and continuity with respect to the initial data are an immediate consequence of the following result.

Proposition 2.1. *Assume that $p, r \in [1,\infty]$ and $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ but $s \neq 2+\frac{1}{p}$. Let $u \in L^\infty([0,T]; B^s_{p,r}) \cap C([0,T]; \mathcal{S}')$ be two given solutions to Eq.* (2.1) *with the initial data $u_0 \in B^s_{p,r}$. Then for every $t \in [0,T)$, we have*

$$\left\|u(t)-v(t)\right\|_{B^{s-1}_{p,r}} \leqslant \|u_0-v_0\|_{B^{s-1}_{p,r}} \exp\left\{C\int_0^t \left(\left\|u(\tau)\right\|^2_{B^s_{p,r}} + \left\|v(\tau)\right\|^2_{B^s_{p,r}}\right) d\tau\right\}. \tag{2.2}$$

Proof. We firstly consider the case $s \neq 2+\frac{1}{p}$. Let $w = u - v$. It is obvious that w solves the transport equation

$$\begin{cases} w_t + u^2 w_x = -v_x(u+v)w + f + g, & x \in \mathbb{R},\ t>0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where

$$f = -\partial_x\left(1-\partial_x^2\right)\left[\frac{6-b}{2}u(u+v)_x w_x + \frac{6-b}{2}v_x^2 w + \frac{b}{3}\left(u^2+uv+v^2\right)w\right],$$
$$g = -\frac{b-2}{2}\left(1-\partial_x^2\right)^{-1}\left[\left(u_x^2+u_x v_x+v_x^2\right)w_x\right].$$

When $s-1 < 1+\frac{1}{p}$, applying Lemma 2.2 in [32] to (2.3) leads to

$$\left\|w(t)\right\|_{B^{s-1}_{p,r}} \leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|\partial_x v^2(\tau')\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} d\tau'} + \int_0^t e^{C\int_\tau^t \|\partial_x v^2(\tau')\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} d\tau'}$$
$$\times\left(\left\|u_x(u+v)w\right\|_{B^{s-1}_{p,r}} + \|f_1\|_{B^{s-1}_{p,r}} + \|f_2\|_{B^{s-1}_{p,r}}\right) d\tau. \tag{2.4}$$

From (2.4), if $s-1 < 1+\frac{1}{p}$, by using $B^{s-1}_{p,r} \hookrightarrow L^\infty$ with $s-1 > \frac{1}{p}$, we have

$$\left\|w(t)\right\|_{B^{s-1}_{p,r}} \leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|v^2(\tau')\|_{B^s_{p,r}} d\tau'} + \int_0^t e^{C\int_\tau^t \|v^2(\tau')\|_{B^s_{p,r}} d\tau'}$$
$$\times\left(\left\|u_x(u+v)w\right\|_{B^{s-1}_{p,r}} + \|f_1\|_{B^{s-1}_{p,r}} + \|f_2\|_{B^{s-1}_{p,r}}\right) d\tau. \tag{2.5}$$

When $s-1 > 1+\frac{1}{p}$, applying Lemma 2.2 in [32] to (2.3) leads to

$$\left\|w(t)\right\|_{B^{s-1}_{p,r}} \leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|\partial_x v^2(\tau')\|_{B^{s-2}_{p,r}} d\tau'} + \int_0^t e^{C\int_\tau^t \|\partial_x v^2(\tau')\|_{B^{s-2}_{p,r}} d\tau'}$$
$$\times\left(\left\|u_x(u+v)w\right\|_{B^{s-1}_{p,r}} + \|f_1\|_{B^{s-1}_{p,r}} + \|f_2\|_{B^{s-1}_{p,r}}\right) d\tau. \tag{2.6}$$

From (2.6), if $s-1>1+\frac{1}{p}$, by using $B^{s-1}_{p,r}\hookrightarrow L^\infty$ with $s-1>\frac{1}{p}$, we have

$$\big\|w(t)\big\|_{B^{s-1}_{p,r}}\leqslant\|w_0\|_{B^{s-1}_{p,r}}e^{C\int_0^t\|v^2(\tau')\|_{B^s_{p,r}}\,d\tau'}+\int_0^t e^{C\int_\tau^t\|v^2(\tau')\|_{B^s_{p,r}}\,d\tau'}\times\Big(\big\|u_x(u+v)w\big\|_{B^{s-1}_{p,r}}+\|f_1\|_{B^{s-1}_{p,r}}+\|f_2\|_{B^{s-1}_{p,r}}\Big)\,d\tau. \tag{2.7}$$

By using the definition of the Besov spaces $B^s_{p,r}$, S^{-2} multiplier property of $-(1-\partial_x^2)^{-1}$ and $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ which leads to the fact that $B^{s-1}_{p,r}$ is an algebra, we have

$$\big\|\partial_x v^2(\tau')\big\|_{B^{s-1}_{p,r}}\leqslant C\big\|v^2\big\|_{B^s_{p,r}}\leqslant C\|v\|^2_{B^s_{p,r}}, \tag{2.8}$$

$$\begin{aligned}\big\|(u+v)u_x w\big\|_{B^{s-1}_{p,r}}&\leqslant C\|u+v\|_{B^{s-1}_{p,r}}\|u_x\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-1}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}.\end{aligned} \tag{2.9}$$

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the S^{-2} multiplier property of $-(1-\partial_x^2)^{-1}$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, Proposition 2.2 in [32], the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\|f\|_{B^{s-1}_{p,r}}&\leqslant C\left\|\frac{6-b}{2}u(u+v)_x w_x+\frac{6-b}{2}v_x^2 w+\frac{b}{3}\big(u^2+uv+v^2\big)w\right\|_{B^{s-2}_{p,r}}\\&\leqslant C\big\|u(u+v)_x\big\|_{B^{s-1}_{p,r}}\|w_x\|_{B^{s-2}_{p,r}}+C\big\|v_x^2\big\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-2}_{p,r}}+C\big\|\big(u^2+uv+v^2\big)\big\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-2}_{p,r}}\\&\leqslant C\Big(\big\|u(u+v)_x\big\|_{B^{s-1}_{p,r}}+\|v\|_{B^s_{p,r}}+\big\|u^2+uv+v^2\big\|_{B^{s-1}_{p,r}}\Big)\|w\|_{B^{s-1}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\|g\|_{B^{s-1}_{p,r}}&\leqslant C\big\|\big(u_x^2+u_xv_x+v_x^2\big)w_x\big\|_{B^{s-2}_{p,r}}\\&\leqslant C\big\|u_x^2+u_xv_x+v_x^2\big\|_{B^{s-1}_{p,r}}\|w_x\|_{B^{s-2}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}\\&\leqslant C\big(\|u\|^2_{B^s_{p,r}}+\|v\|^2_{B^s_{p,r}}\big)\|w\|_{B^{s-1}_{p,r}}.\end{aligned} \tag{2.11}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^{s-1}_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (2.10) and (2.11). Inserting (2.8)–(2.11) into (2.5) or (2.7) and applying Gronwall's inequality yields (2.2). □

Now let us start the proof of Theorem 2.1, which is motivated by the proof of local existence theorem about the Camassa–Holm equation in [21]. Firstly, we shall use the classical Friedrichs regularization method to construct the approximate solutions to the Cauchy problem (2.1).

Lemma 2.1. *Assume that $u^0 = 0$. Let $1 \leqslant p, r \leqslant +\infty$, $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$ and $u_0 \in B^s_{p,r}$. Then there exists a sequence of smooth functions $\{u^n\}_{n\in N} \in C(\mathbb{R}^+; B^\infty_{p,r})$ solving the following linear transport equation by induction*:

$$\begin{cases} (\partial_t + (u^n)^2\partial_x)u^{n+1} = -(1-\partial_x^2)^{-1}\left(\partial_x\left(\frac{6-b}{2}u^n(u^n)_x^2 + \frac{b}{3}(u^n)^3\right) + \frac{b-2}{2}(u^n)_x^3\right), \\ u^{n+1}(x,0) = u_0^{n+1}(x) = S_{n+1}u_0. \end{cases} \tag{2.12}$$

Moreover, there is a maximal existence time $T > 0$ such that the solutions u^n satisfy the following conditions:

(i) *$\{u^{(n)}\}_{n\in N}$ is uniformly bounded in $E^s_{p,r}(T)$.*
(ii) *$\{u^{(n)}\}_{n\in N}$ is a Cauchy sequence in $C([0,T]; B^{s-1}_{p,r})$.*

Proof. Since all data $S_{n+1}u_0 \in B^\infty_{p,r}$, Theorem 3.3.1 in [22] enables us to show by induction that for all $n \in N$, Eq. (2.1) has a global solution which belongs to $C(\mathbb{R}^+; B^\infty_{p,r})$. Since $B^s_{p,r}$ is an algebra with $s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$, when $\max\{1+\frac{1}{p}, \frac{3}{2}\} < s \leqslant 2 + \frac{1}{p}$, by using the S^{-2} multiplier property of $(1-\partial_x^2)^{-1}$, the definition of the Besov spaces $B^s_{p,r}$, Lemma 2.2 in [32] and the fact that $B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ is an algebra, we have

$$\begin{aligned} &\left\| -(1-\partial_x^2)^{-1}\left(\partial_x\left(\frac{6-b}{2}u^n(u^n)_x^2 + \frac{b}{3}(u^n)^3\right) + \frac{b-2}{2}(u^n)_x^3\right)\right\|_{B^s_{p,r}} \\ &\quad \leqslant C\left\|u^n(u^n)_x^2 + (u^n)^3\right\|_{B^{s-1}_{p,r}} + C\left\|(u^n)_x\right\|^3_{B^s_{p,r}} \\ &\quad \leqslant C\left\|u^n\right\|^3_{B^s_{p,r}}. \end{aligned} \tag{2.13}$$

When $s > 2 + \frac{1}{p}$, by using the fact that $B^{s-1}_{p,r}$ is an algebra and the standard algebra properties of the Besov spaces used in the previous paragraphs, we also can obtain the estimate of (2.13).

For $s > \max\{\frac{3}{2}, 1 + \frac{1}{p}\}$ but $s \neq 2 + \frac{1}{p}$, by virtue of Lemma 2.2 in [32], we deduce

$$\begin{aligned} &e^{-C\int_0^t \|\partial_x(u^n)^2(\tau)\|_{B^{s-1}_{p,r}}\,d\tau}\left\|u^{(n+1)}(t)\right\|_{B^s_{p,r}} \\ &\quad \leqslant \|S_{n+1}u_0\|_{B^s_{p,r}} + C\int_0^t e^{-C\int_0^\tau \|\partial_x(u^n)^2(\tau')\|_{B^s_{p,r}}\,d\tau'}\left\|u^n(\tau)\right\|^3_{B^s_{p,r}}\,d\tau. \end{aligned} \tag{2.14}$$

Hence, we get

$$\begin{aligned} \left\|u^{(n+1)}(t)\right\|_{B^{s-1}_{p,r}} &\leqslant e^{C\int_0^t \|\partial_x(u^n)^{m+1}(\tau')\|_{B^{s-1}_{p,r}}\,d\tau'}\|u_0\|_{B^s_{p,r}} \\ &\quad + C\int_0^t e^{C\int_\tau^t \|\partial_x(u^n)^2(\tau)\|_{B^{s-1}_{p,r}}\,d\tau}\left\|u^n(\tau)\right\|^3_{B^s_{p,r}}\,d\tau. \end{aligned} \tag{2.15}$$

Let us choose a $T>0$ such that $4C\|u_0\|^2_{B^s_{p,r}}<1$, and suppose by induction that for all $t\in[0,T]$,

$$\left\|u^{(n)}(t)\right\|_{B^s_{p,r}}\leqslant\frac{\|u_0\|_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{2}}}.\tag{2.16}$$

Indeed, since $B^{s-1}_{p,r}$ is an algebra, one obtains from (2.16) that

$$\begin{aligned}C\int_\tau^t\left\|\partial_x u^n(\tau')\right\|^2_{B^{s-1}_{p,r}}d\tau'&\leqslant C\int_\tau^t\left\|u^n(\tau')\right\|^2_{B^s_{p,r}}d\tau'\\&\leqslant C\int_\tau^t\frac{\|u_0\|^2_{B^s_{p,r}}}{1-4C\|u_0\|^2_{B^s_{p,r}}t}\,d\tau\\&=\frac{1}{4}\ln\bigl(1-4C\|u_0\|^2_{B^s_{p,r}}\tau\bigr)-\frac{1}{4}\ln\bigl(1-4C\|u_0\|^2_{B^s_{p,r}}t\bigr).\end{aligned}\tag{2.17}$$

And then inserting the above inequalities (2.16), (2.17) into (2.15) leads to

$$\begin{aligned}\left\|u^{(n+1)}(t)\right\|_{B^s_{p,r}}&\leqslant\frac{\|u_0\|_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{4}}}+\frac{C}{(1-4c\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{4}}}\\&\quad\times\int_0^t\bigl(1-4C\|u_0\|^2_{B^s_{p,r}}\tau\bigr)^{\frac{1}{4}}\frac{\|u_0\|^3_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}\tau)^{\frac{3}{2}}}\,d\tau\\&\leqslant\frac{\|u_0\|_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{4}}}\left(1+C\int_0^t\frac{\|u_0\|^2_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{5}{4}}}\,d\tau\right)\\&=\frac{\|u_0\|_{B^s_{p,r}}}{(1-4C\|u_0\|^2_{B^s_{p,r}}t)^{\frac{1}{2}}}.\end{aligned}\tag{2.18}$$

Thus $\{u^{(n)}\}_{\in N}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. In view of Lemmas 2.1 and 2.3 in [32], we have

$$\left\|\bigl(u^{(n)}\bigr)^2\partial_x u^{(n+1)}\right\|_{B^{s-1}_{p,r}}\leqslant C\left\|u^{(n)}\right\|^2_{B^s_{p,r}}\left\|\partial_x u^{(n+1)}u_0\right\|_{B^{s-1}_{p,r}}\leqslant C\left\|u^{(n)}\right\|^2_{B^s_{p,r}}\left\|u^{(n+1)}u_0\right\|_{B^s_{p,r}}.\tag{2.19}$$

Combining (2.19) and (2.13) with Eq. (2.1) we deduce that $\partial_t u^{(n+1)}\in C([0,T];B^{s-1}_{p,r})$ is uniformly bounded. Thus we get (i).

Next we show (ii). By Eq. (2.1), for all $m,n\in N$, we obtain

$$\partial_t+\bigl(u^{(n+m)}\bigr)^2\partial_x\bigl(u^{(n+m+1)}-u^{(n+m)}\bigr)=-\bigl(\bigl(u^{(n+m)}\bigr)^2-\bigl(u^{(n)}\bigr)^2\bigr)u_x^{(n+1)}+f',\tag{2.20}$$

where

$$f' = -\partial_x\big(1-\partial_x^2\big)^{-1}\left[\frac{6-b}{2}\big(u^{(m+n)}\big(u^{(m+n)}\big)_x^2 - u^{(n)}\big(u^{(n)}\big)_x^2\big) + \frac{b}{3}\big(\big(u^{(m+n)}\big)^3 - \big(u^{(n)}\big)^3\big)\right]$$
$$- \frac{b-2}{2}\big(1-\partial_x^2\big)^{-1}\big[\big(u^{(m+n)}\big)_x^3 - \big(u^{(n)}\big)_x^3\big].$$

Similar to the proof of Proposition 2.1, $s > \max\{\frac{3}{2}, 1+\frac{1}{p}\}$ but $s \neq 2+\frac{1}{p}$, noting that

$$\begin{aligned}\big\|u_0^{(n+m+1)} - u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}} &= \|S_{n+m+1}u_0 - S_{n+1}u_0\|_{B^s_{p,r}} \\ &= \left\|\sum_{k=n+1}^{m+n}\Delta_k u_0\right\|_{B^s_{p,r}} \leqslant C2^{-n}\|u_0\|_{B^s_{p,r}},\end{aligned} \tag{2.21}$$

we obtain

$$\begin{aligned}&\big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|_{B^{s-1}_{p,r}} \\ &\leqslant e^{C\int_0^t \|u^{(n+m)}(\tau)\|^2_{B^s_{p,r}}\,d\tau}\left(\big\|u_0^{(n+m+1)} - u_0^{(n+1)}\big\|_{B^{s-1}_{p,r}} + \int_0^t e^{-C\int_0^\tau \|u^{(n+m)}(\tau')\|^2_{B^{s-1}_{p,r}}\,d\tau'}\right. \\ &\quad \left.\times\big(\big\|u^{(m+n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n)}\big\|^2_{B^s_{p,r}} + \big\|u^{(n+1)}\big\|^2_{B^s_{p,r}}\big)\big\|\big(u^{(m+n)} - u^{(m)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\right)d\tau \\ &\leqslant C\left(2^{-n} + \int_0^t \big\|\big(u^{(n+m)} - u^{(n+1)}\big)(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right).\end{aligned} \tag{2.22}$$

As $\|u^{(m)}\|_{B^s_{p,r}}$ and C are bounded independently of m, n, there exists constant C_1 independent of m, n such that

$$\big\|\big(u^{(n+m+1)} - u^{(n+1)}\big)(t)\big\|_{L^\infty([0,T);B^{s-1}_{p,r})} \leqslant C_1 2^{-n}.$$

Thus $\{u^{(n)}\}_{n\in N}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$. This completes the proof of Lemma 2.1. □

Proof of Theorem 2.1. Thanks to Lemma 2.1, we obtain that $\{u^{(n)}\}_{n\in N}$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, so it converges to some function $u \in C([0,T];C([0,T];B^{s-1}_{p,r}))$. We now have to check that u belongs to $E^s_{p,r}(T)$ and solves the Cauchy problem (2.1). Since $\{u^n\}_{n\in N}$ is uniformly bounded in $L^\infty([0,T];C([0,T];B^s_{p,r}))$ according to Lemma 2.1, the Fatou property for Besov spaces guarantees that u also belongs to $L^\infty([0,T];C([0,T];B^s_{p,r}))$.

On the other hand, as $\{u^{(n)}\}_{n\in N}$ converges to u in $C([0,T];B^{s-1}_{p,r})$, Proposition 2.1 combined with an obvious interpolation argument ensures that the convergence holds in $C([0,T];B^{s'}_{p,r})$, for any $s' < s$. It is then easy to pass to the limit in Eq. (2.1) and to conclude that u is indeed a solution to the Cauchy problem (2.1). Thanks to the fact that u belongs to $L^\infty([0,T];C([0,T];B^s_{p,r}))$, the right-hand side of the equation

$$u_t + u^2u_x = P(D)\big(bu^2u_x + (6-b)uu_xu_{xx} + 2u_x^3\big)$$

belongs to $L^\infty([0,T];C([0,T];B^{s-2}_{p,r}))$. In particular, for the case $r<\infty$, Theorem 3.3.1 in [22] enables us to conclude that $u \in C([0,T];B^{s-1}_{p,r})$ for any $s' < s$. Finally, using the equation again, we see that

$\partial_t u \in C([0,T]; B^{s'}_{p,r})$ if $r < \infty$, and in $L^\infty([0,T]; C([0,T]; B^{s-1}_{p,r}))$ otherwise. Moreover, a standard use of a sequence of viscosity approximate solutions $(u_\varepsilon)_\varepsilon > 0$ for the Cauchy problem (2.1) which converges uniformly in $C([0,T]; B^s_{p,r}) \cap C^1([0,T]; B^{s-1}_{p,r})$ gives the continuity of the solution u in $E^s_{p,r}$. □

3. Local well-posedness in H^s with $s > \frac{3}{2}$

In this section, by applying Kato's semigroup theory [49], we can obtain the local well-posedness in Sobolev spaces H^s with $s > \frac{3}{2}$.

Theorem 3.1. *Given $u(x,0) = u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, then there exist a maximal $T = T(\|u_0\|_s) > 0$ and a unique solution u to Eq.* (2.1) *such that*

$$u(\cdot, u_0) \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \to u(\cdot, u_0) : H^s \to C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ is continuous.

Remark 3.1. When $p = r = 2$, the Besov space $B^s_{p,r}(R)$ coincides with the Sobolev space $H^s(R)$, so Theorem 2.1 implies Theorem 3.1. But we still want to give a proof for Theorem 3.1 by a theorem due to Kato [49], since the estimates themselves are very interesting.

Set $A(u) = u^2\partial_x$, $f(u) = -G * (bu^2 u_x + (6-b)u^m u_x u_{xx} + 2u_x^3)$, $Y = H^s$, $X = H^{s-1}$ and $Q = \Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. Obviously, Q is an isomorphism of H^s onto H^{s-1}. In order to prove Theorem 3.1 by applying Kato's theorem [49], we only need to verify $A(u)$ and $f(u)$ which satisfy the conditions of Kato's theorem.

Lemma 3.1. *The operator $A(u) = u^2\partial_x$ with $u \in H^s$, $s > \frac{3}{2}$, belongs to $G(L^2, 1, \beta)$.*

Lemma 3.2. *The operator $A(u) = u^2\partial x$ with $u \in H^s$, $s > \frac{3}{2}$, belongs to $G(H^{s-1}, 1, \beta)$.*

Lemma 3.3. *Let the operator $A(u) = u^2\partial x$ with $u \in H^s$, $s > \frac{3}{2}$. The operator $A(u) \in L(H^s, H^{s-1})$. Moreover,*

$$\big\|\big(A(y) - A(z)\big)w\big\|_{s-1} \leqslant \mu_1 \|y - z\|_{s-1} \|w\|_s, \quad y, z, w \in H^s.$$

Lemma 3.4. *The operator $B(u) = [\Lambda, u^2]\partial_x \Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s$ with $s > \frac{3}{2}$. Moreover,*

$$\big\|\big(B(y) - B(z)\big)w\big\|_{s-1} \leqslant \mu_2 \|y - z\|_s \|w\|_{s-1}, \quad y, z \in H^s,\ w \in H^{s-1}.$$

Proofs of the above Lemmas 3.1–3.4 can be found in [52].

Lemma 3.5. *Let $f(u) = -(1-\partial_x^2)^{-1}(\partial_x(\frac{6-b}{2}uu_x^2 + \frac{b}{3}u^3) + \frac{b-2}{2}u_x^3)$, then $f(u)$ is bounded on bounded sets in H^s, and satisfies*

$$\big\|f(y) - f(z)\big\|_s \leqslant \mu_3 \|y - z\|_s, \quad y, z \in H^s,$$
$$\big\|f(y) - f(z)\big\|_{s-1} \leqslant \mu_4 \|y - z\|_{s-1}, \quad y, z \in H^s.$$

Proof. Let $y, z \in H^{s-1}$, $s > \frac{3}{2}$. Since H^s is a Banach algebra, it follows that

$$
\begin{aligned}
\big\| f(y)-f(z)\big\|_s \leqslant{}& \left\| -\partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{6-b}{2}yy_x^2+\frac{b}{3}y^3-\frac{6-b}{2}zz_x^2-\frac{b}{3}z^3\right)\right\|_s \\
&+\left\| -\big(1-\partial_x^2\big)^{-1}\left(\frac{b-2}{2}y_x^3-z_x^3\right)\right\|_s \\
\leqslant{}& \frac{|6-b|}{2}\big\| yy_x^2-zz_x^2\big\|_{s-1}+\frac{|b|}{3}\big\| y^3-z^3\big\|_{s-1}+\frac{|b-2|}{2}\big\| y_x^3-z_x^3\big\|_{s-1}. \qquad (3.1)
\end{aligned}
$$

In view of Lemma 2.4 in [65], $u \to g(u)-g(0)$ is a C^∞-map from H^{s-1} to H^{s-1}, where $g(u)=u$ or $g(u)=u^2$. From the mean value theorem [25], we infer that there is some $M>0$, depending only on $\max\{\|y\|_s, \|z\|_s\}$, such that

$$
\big\| g(y)-g(z)\big\|_{s-1} \leqslant M\|y-z\|_{s-1}.
$$

Hence

$$
\begin{aligned}
\big\| f(y)-f(z)\big\|_s &\leqslant \frac{|6-b|}{2}\left(\big\|(y-z)y_x^2\big\|_{s-1}+\big\|z\big(y_x^2-z_x^2\big)\big\|_{s-1}\right)+CM\|y-z\|_s \\
&\leqslant C\|y-z\|_{s-1}+c\|y-z\|_s+CM\|y-z\|_s \\
&\leqslant C\|y-z\|_s.
\end{aligned}
$$

Taking $z=0$ in the above inequality, we obtain that f is bounded on bounded set in H^s.

Next, let $y,z\in H^s$, $s>\frac{3}{2}$. Since H^{s-1} is a Banach algebra, we have

$$
\begin{aligned}
\big\| f(y)-f(z)\big\|_s \leqslant{}& \left\| -\partial_x\big(1-\partial_x^2\big)^{-1}\left(\frac{|6-b|}{2}yy_x^2+\frac{|b|}{3}y^3-\frac{6-b}{2}zz_x^2-\frac{|b|}{3}z^3\right)\right\|_{s-1} \\
&+\left\| -\big(1-\partial_x^2\big)^{-1}\left(\frac{b-2}{2}\big(y_x^3-z_x^3\big)\right)\right\|_{s-1} \\
\leqslant{}& \frac{|6-b|}{2}\big\| yy_x^2-zz_x^2\big\|_{s-2}+\frac{|b|}{3}\big\| y^3-z^3\big\|_{s-1}+\frac{b-2}{2}\big\| y_x^3-z_x^3\big\|_{s-2} \\
\leqslant{}& \frac{|6-b|}{2}\left(\big\|(y-z)y_x^2\big\|_{s-2}+\big\|z\big(y_x^2-z_x^2\big)\big\|_{s-2}\right)+CM\|y-z\|_{s-1} \\
\leqslant{}& C\|y-z\|_{s-1}+c\|y-z\|_s+CM\|y-z\|_{s-1} \\
\leqslant{}& C\|y-z\|_{s-1}.
\end{aligned}
$$

This completes the proof of Lemma 3.5. □

Proof of Theorem 3.1. Combining Kato's theory and Lemmas 3.1–3.5, we can get the statement of Theorem 3.1. □

Theorem 3.2. *Assume that $u_0\in H^s$, $s>\frac{3}{2}$. Then T in Theorem 3.1 may be chosen independent of s in the following sense. If $u=u(\cdot,u_0)\in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$ solves Eq. (1.2) (or Eq. (2.1)), and if $u_0\in H^{s_1}$ for some $s_1\neq s$, $s_1>\frac{3}{2}$, then $u\in C([0,T);H^{s_1})\cap C^1([0,T);H^{s_1-1})$ and with the same T. In particular, if $u_0\in H^\infty=\bigcap_{s\geqslant 0}H^s$, then $u\in C([0,T);H^\infty)$.*

Proof. It suffices to consider the case $s_1>s$, since the case $s_1<s$ is obvious from uniqueness which is guaranteed by Theorem 3.1. In order to prove Theorem 3.2 for $s_1>s$, let us return to Eq. (1.2). Set $y(t)=\Lambda^2u(t)$. Then we have

$$\frac{dy}{dt} + A(t)y + B(t)y = 0, \qquad y(0) = \Lambda^2 u(0), \tag{3.2}$$

where $A(t)y = \partial_x(u^2 y)$, $B(t)y = (b-2)uu_x y$.

Because $u \in C([0,T]; H^s)$ and $u_0 \in H^{s_1}$, we have $y \in C([0,T]; H^{s-2})$ and $y(0) = \Lambda^2 u(0) \in C([0,T]; H^{s_1-2})$. It is our purpose to deduce $y \in C([0,T]; H^{s_1-2})$, which implies $u \in C([0,T]; H^{s_1})$. This will complete the proof of Theorem 3.2. $\square$

Since $u \in C([0,T]; H^s)$, $u_x \in H^{s-1}$, and H^{s-1} is a Banach algebra, we obtain $B(t) \in L(H^{s-1})$.

Following the arguments in Lemmas 3.1–3.3 in [48], we first need to prove that the family $A(t)$ has a unique evolution operator $\{U(t,\tau)\}$ associated with the spaces $X = H^h$ and $Y = H^k$, where $-s \leqslant h \leqslant s-2$, $1-s \leqslant k \leqslant s-1$, and $k \geqslant h+1$. Therefore, according to the proof of Lemma 3.1 in [48], we need to verify the following three conditions.

(i) $A(t) \in G(H^h, 1, \beta)$, $\forall y \in H^s$.
(ii) $\Lambda^h \partial_x [\Lambda^{k-h}, u^{m+1}]\Lambda^{-k}$ is uniformly bounded on L^2.
(iii) $A(t) \in L(H^k, H^h)$ is strongly continuous in t.

Let us begin to verify (i). Due to H^h being a Hilbert space, $A(t) \in G(H^h, 1, \beta)$ if and only if there is a real number β such that [49]

(a) $(A(t)y, y)_h \geqslant -\beta\|y\|_h^2$,
(b) $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H^h.

First, we prove (a). Take $y \in H^h$. Note that

$$\begin{aligned}\Lambda^h \partial_x\big(u^2 y\big) = \Lambda^h \partial_x\big(-\big[\Lambda^{-h}, u^2\big]\Lambda^h y + \Lambda^{-h}\big(u^2 \Lambda^h y\big)\big) \\ - \Lambda^h \partial_x\big[\Lambda^{-h}, u^2\big]\Lambda^h y + \partial_x\big(u\Lambda^h y\big).\end{aligned}$$

Then we have

$$\begin{aligned}\big(A(t)y, y\big)_h &= -\big(\Lambda^h \partial_x\big[\Lambda^{-h}, u^2\big]\Lambda^h y, \Lambda^h y\big)_0 + \big(\partial_x\big(u^2\Lambda^h y\big), \Lambda^h y\big)_0 \\ &= \big(\Lambda^{h+1}\big[\Lambda^{-h}, u^2\big]\Lambda^h y, \Lambda^{h-1} y\big)_0 + \big(uu_x \Lambda^h y, \Lambda^h y\big)_0 \\ &\leqslant \big\|\Lambda^{h+1}\big[\Lambda^{-h}, u^2\big]\big\|_{L(L^2)}\big\|\Lambda^h y\big\|_0^2 + \|uu_x\|_{L^\infty}\big\|\Lambda^h y\big\|_0^2 \\ &\leqslant \big(c\|u\|_s + c\|u\|_s^2\big)\|y\|_h^2,\end{aligned}$$

where we applied Lemma 5.1 in [64] with $r = -(h+1), k = 0$. Setting $\beta = c\|u\|_s + c\|u\|_s^2$, we have $(A(t)y, y)_h \geqslant -\beta\|y\|_h^2$.

Secondly, we prove (b). Let $S = \Lambda^{s-1-h}$. Note that S is an isomorphism of H^{s-1} onto H^h and H^{s-1} is continuously and densely embedded in H^h as $-s \leqslant h \leqslant s-2$. Define

$$A_1(t) := SA(t)S^{-1} = \Lambda^{s-1-h}A(t)\Lambda^{h+1-s},$$

$$B_1(t) := A_1(t) - A(t) = \big[S, A(t)\big]S^{-1}.$$

Let $y \in H^h$ and $u \in H^s$, $s > \frac{3}{2}$. Then we have

$$\begin{aligned}\|B_1(t)y\|_h &= \|\Lambda^h\partial_x[\Lambda^{s-1-h},u^2]\Lambda^{h+1-s}y\|_0\\ &\leqslant \|\Lambda^h\partial_x[\Lambda^{s-1-h},u^2]\Lambda^{1-s}\|_{L(L^2)}\|\Lambda^h y\|_0\\ &\leqslant \|u\|_s\|u\|_0,\end{aligned}$$

where we applied Lemma 5.1 in [64] with $r=-(h+1)$, $k=s-1$. Therefore, we obtain $B_1(t)\in L(H^h)$. Note that

$$A(t)u=\partial_x(u^2y)=2u^2u_x+u^2\partial_x y \quad \text{and} \quad u_x\in L(H^{s-1}).$$

Applying Lemma 3.2 and a perturbation theorem for semigroups, we have H^{s-1} is $-A(t)$-admissible. Then by applying Lemma 5.3 in [64] with $Y=H^{s-1}$, $X=H^h$ and $S=\Lambda^{s-1-h}$, we obtain that $-A_1(t)$ is the infinitesimal generator of a C_0-semigroup on H^h. Due to $A_1(t)=A(t)+B_1(t)$ and $B_1(t)\in L(H^h)$, by a perturbation theorem for semigroups, we have that $-A(t)$ is the infinitesimal generator of a C_0-semigroup on H_h. This proves (b).

Next, we verify (ii). Take $y\in L^2$. Then we have

$$\|\Lambda^h\partial_x[\Lambda^{k-h},u^2]\Lambda^{-k}y\|\leqslant C\|u\|_s\|y\|_0,$$

where we applied Lemma 5.1 in [64] with $r=-(h+1)$, $k=k$.

Finally, we verify (iii). Take $y\in H^k$. Then

$$\begin{aligned}\|\partial_x(u^2(t+\tau)-u^2(t))y\| &\leqslant \|(u^2(t+\tau)-u^2(t))y\|_{h+1}\\ &\leqslant C\|u^2(t+\tau)-u^2(t)\|_{s-1}\|y\|_{h+1}\\ &\leqslant C\|u\|_s\|u(t+\tau)-u(t)\|_s\|y\|_k,\end{aligned}$$

where we applied Lemma 5.1 in [64] with $r=s-1$, $t=h+1$. By the continuity of u, we prove (iii). Thus the above three conditions imply the existence and uniqueness of evolution operator $U(t,\tau)$ for the family $A(t)$. In particular $U(t,\tau)$ maps H^r into itself for $-s\leqslant r\leqslant s-1$.

Next, we choose $Y=H^{s-2}$ and $X=H^{s-3}$. Note that

$$y\in C([0,T);H^{s-2})\cap C^1([0,T);H^{s-3}).$$

By the properties of evolution operator $U(t,\tau)$, we can obtain

$$\frac{d}{d\tau}(U(t,\tau))y(\tau)=U(t,\tau)(-B(\tau)y(\tau)).$$

Integrating the above equality in $\tau\in[0,t]$, we obtain

$$y(t)=U(t,0)y(0)-\int_0^t U(t,\tau)B(\tau)y(\tau)\,d\tau. \tag{3.3}$$

If $s<s_1\leqslant s+1$, then we have $B(t)\in L(H^{s_1-2})$ is strongly continuous in $[0,t)$, and $H^{s-1}H^{s_1-2}\subset H^{s_1-2}$ as $s-1>\frac{1}{2}$. Due to $-s<s-2<s_1-2\leqslant s-1$, the family $\{U(t,\tau)\}$ is strongly continuous on H^{s_1-2} to itself. Note that $y(0)\in H^{s_1-2}$. We regard Eq. (3.3) as an integral equation of Volterra type which can be solved for y by successive approximation. Then the result of Theorem 3.2 is obtained.

If $s_1>s+1$, we obtain the result of Theorem 3.2 by repeated application of the above argument. This completes the proof of Theorem 3.2. $\square$

4. Blow-up scenario and global conservative property

After establishing local well-posedness theory, a natural question is whether the corresponding solution exists globally or not. We establish criteria for the blow-up of solutions to Eq. (2.1).

Theorem 4.1. *Let $u_0 \in H^r$, $r > \frac{3}{2}$. If T is the maximal existence time of corresponding solution of the initial data u_0, then the H^r-norm of $u(t,x)$ to Eq.* (1.2) (*or* (2.1)) *blows up on* $[0,T)$ *if and only if*

$$\overline{\lim_{t\uparrow T}}\left\|u_x(t,x)\right\|_{L^\infty}=\infty.$$

Proof. Let $u(t,x)$ be the solution of Eq. (2.1) with the initial data $u_0 \in H^r$, $r > \frac{3}{2}$, which is guaranteed by Theorem 3.1.

If $\overline{\lim}_{t\uparrow T}\|u_x(t,x)\|_{L^\infty}=\infty$, by Sobolev's embedding theorem, we obtain the solution $u(t,x)$ will blow up in finite time.

Next, applying the operator Λ^r to Eq. (2.1), multiplying by $\Lambda^r u$, and integrating by parts on $\mathbb{R}$, we have

$$\frac{d}{dt}(u,u)_r=-2\big(u^2u_x,u\big)_r+2\big(f(u),u\big)_r, \tag{4.1}$$

where $f=-(1-\partial_x^2)^{-1}(\partial_x(\frac{6-b}{2}uu_x^2+\frac{b}{3}u^3)+\frac{b-2}{2}u_x^3)$. Assume there exists an $M>0$, such that $\overline{\lim}_{t\uparrow T}\|u_x(t,x)\|_{L^\infty}\leqslant M$. Then we have

$$\begin{aligned}\big|\big(u^2u_x,u\big)_r\big|&=\big|\big(\Lambda^r\big(u^2u_x\big),\Lambda^r u\big)_0\big|=\big|\big(\big[\Lambda^r,u^2\big]u_x,\Lambda^r u\big)_0+\big(u^2\Lambda^r u_x,\Lambda^r u\big)_0\big|\\&\leqslant C\|u\|_r\big(\|u\|_{L^\infty}\|u_x\|_{L^\infty}\|u\|_r+\|u_x\|_{L^\infty}\|u\|_{L^\infty}\|u\|_r\big)+\|u\|_{L^\infty}\|u_x\|_{L^\infty}\left\|\Lambda^r u\right\|_{L^2}^2\\&\leqslant C\|u\|_r^2\|u\|_{L^\infty}\|u_x\|_{L^\infty}\leqslant C\|u\|_r^2,\end{aligned} \tag{4.2}$$

where we applied Lemma 3.1 in [28] with $s=r$, Lemma 3.2 in [28] with $F(u)=u^2$ and $s=r$. Similarly,

$$\begin{aligned}\big\|uu_x^2\big\|_{r-1}&=\big\|\big[\Lambda^{r-1},u\big]u_x^2+u\Lambda^{r-1}u_x^2\big\|_{L^2}\\&\leqslant C\big(\|\partial_x u\|_{L^\infty}\big\|\Lambda^{r-1}u_x^2\big\|_{L^2}+\big\|\Lambda^{r-1}u\big\|_{L^2}\big\|u_x^2\big\|_{L^\infty}\big)\\&\leqslant C\big(M\big\|u_x^2\big\|_{r-1}+M^2\|u\|_{r-1}\big)\leqslant C\|u\|_r.\end{aligned}$$

On the other hand, we estimate the second term of the right-hand side of Eq. (2.1):

$$\begin{aligned}\big(f(u),u\big)_r&=\left(-\big(1-\partial_x^2\big)^{-1}\left(\partial_x\left(\frac{6-b}{2}uu_x^2+\frac{b}{3}u^3\right)+\frac{b-2}{2}u_x^3\right),u\right)\\&\leqslant C\|u\|_r\left(\frac{|6-b|}{2}\big\|uu_x^2\big\|_{r-1}+\frac{|b|}{3}\big\|u^3\big\|_{r-1}+\frac{|b-2|}{2}\big\|u_x^3\big\|_{r-2}\right)\\&\leqslant C\|u\|_r^2.\end{aligned} \tag{4.3}$$

From (4.1)–(4.3), we obtain

$$\frac{d}{dt}\|u\|_r^2\leqslant C\|u\|_r^2.$$

Thus using Gronwall's inequality, we get

$$\left\|u(t)\right\|_r^2 \leqslant \|u_0\|_r^2 \exp(Ct).$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Assume that $u_0 \in H^r(\mathbb{R})$, $r > \frac{3}{2}$. If $b = 1$, then every solution will exist globally in time. If $b > 1$, then the solution blows up in finite time if and only if the slope of uu_x becomes unbounded from below in finite time. If $b < 1$, then the solution blows up in finite time if and only if the slope of uu_x becomes unbounded from above in finite time.*

Proof. Applying Theorem 3.1 and a simple density argument, it suffices to consider the case $r = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq. (2.1) with initial data $u_0 \in H^3(\mathbb{R})$. From Theorem 3.1 we know that $u \in C([0, T); H^3(\mathbb{R})) \cap C^1([0, T); H^2(\mathbb{R}))$.

Set $y = u - u_{xx}$; by direct computation, one has

$$\|y\|_{L^2}^2 = \int_R (u - u_{xx})^2 \, dx = \int_R \left(u^2 + 2u_x^2 + u_{xx}^2\right) dx. \tag{4.4}$$

Hence,

$$\|u\|_{H^2}^2 \leqslant \|y\|_{L^2}^2 \leqslant 2\|u\|_{H^2}^2. \tag{4.5}$$

Multiplying Eq. (1.2) by $u - u_{xx}$, after integrating by parts, we get

$$\frac{d}{dt} \int_R y^2 \, dx = 2 \int_R y y_t \, dx = -2 \int_R u^2 y y_x \, dx - 2b \int_R u u_x y^2 \, dx = 2(b-1) \int_R u u_x y^2 \, dx. \tag{4.6}$$

From (4.6), we see that if $b = 1$, then we have

$$\left\|u_x(t, \cdot)\right\|_{L^\infty} \leqslant \left\|u(t, \cdot)\right\|_2 \leqslant \left\|y(t, \cdot)\right\|_{L^2} \leqslant \left\|y_0(t, \cdot)\right\|_{L^2} < \infty.$$

This implies, in view of Theorem 4.1, that every solution exists globally in time.

If $b > 1$ and the slope of the solution is bounded from below or if $b < 1$ and the slope of uu_x is bounded from above on $[0, T) \times \mathbb{R}$, from (4.6), one gets that

$$\int_R y^2 \, dx = 2(b-1) \int_0^t \int_R u u_x y^2 \, dx \, ds + \int_R y_0^2 \, dx \leqslant C_1 \int_0^t \int_R y^2 \, dx \, ds + C_2, \tag{4.7}$$

where $C_1, C_2 > 0$. Due to Gronwall's inequality, it is clear that

$$\int_R y^2 \, dx \leqslant C_2\left(1 + C_2 t e^{C_1 t}\right), \quad \text{a.e. } t \in [0, T).$$

So combining with (4.6), we obtain that when uu_x is bounded from below on $[0, T)$, then so does the H^2-norm of the solution.

On the other hand, because of $u = G * y$, we can get

$$u = G * y = \int_R G(x-\xi)y(\xi)\,d\xi \quad \text{and} \quad u_x = G * y = \int_R G_x(x-\xi)y(\xi)\,d\xi.$$

Therefore

$$\|uu_x\|_{L^\infty} \leqslant \|u\|_{L^\infty}\|u_x\|_{L^\infty} \leqslant \|G\|_{L^2}\|G_x\|_{L^2}\|y\|_{L^2}^2 \leqslant 2\|G\|_{L^2}\|G_x\|_{L^2}\|y\|_{H^2}^2, \tag{4.8}$$

where we used (4.6). Hence, (4.8) tells us if H^2-norm of the solution is bounded then the L^∞-norm of uu_x is bounded. This completes the proof of Theorem 4.1. □

In order to demonstrate a conservative property, consider the following differential equation

$$\begin{cases} \dfrac{dq(t,x)}{dt} = u^2\big(q(t,x),t\big), & t \in [0,t), \\ q(0,t) = x, & x \in R. \end{cases} \tag{4.9}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following useful result on the above initial value problem.

Theorem 4.3. *Let $u_0 \in H^s$, $s \geqslant 3$, and T be the maximal existence time of the corresponding solution $u(t,x)$ to Eq. (2.1). Then Eq. (4.9) has a unique solution $q \in C^1([0,T)\times\mathbb{R},\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t 2uu_x\big(q(s,x),s\big)\,ds\right), \qquad q_x(0,x) = 1, \quad x \in \mathbb{R},\ 0 \leqslant t < T.$$

Furthermore, setting $y = u - u_{xx}$, we obtain

$$y\big(q(x,t),t\big)q_x^{\frac{b}{2}}(t,x) = y_0(x) = u_0(x) - u_{0xx}(x), \quad x \in \mathbb{R},\ 0 \leqslant t < T.$$

Proof. First, for fixed $x \in \mathbb{R}$ we deal with an ordinary differential equation. By the Sobolev embedding theorem we have that $u \in C^1([0,T)\times\mathbb{R},\mathbb{R})$. Differentiating the first equation in (4.9) with respect to x, one has

$$\frac{d}{dx}q_t = q_{xt} = 2uu_xq_x, \quad t \in [0,T).$$

Hence

$$q_x(t,x) = \exp\left(\int_0^t 2uu_x\big(q(s,x),s\big)\,ds\right), \qquad q_x(0,x) = 1,$$

which is always positive before the blow-up time. Therefore, the function $q(x,t)$ is an increasing diffeomorphism of the line before blow-up.

For all $t > 0$, a simple computation shows that

$$\begin{aligned}\frac{d}{dt}\big(y(q)q_x^{\frac{b}{2}}\big) &= \big(y_t + y_x(q)q_t\big)q_x^{\frac{b}{2}} + \frac{b}{2}y(q)q_{xt}q_x^{\frac{b-2}{2}}\\ &= \big(y_t(q) + u^2(q)y_x(q) + buu_x(q)y(q)\big)q_x^{\frac{b}{2}}\\ &= 0.\end{aligned}$$

Therefore, $y(q)q_x^{\frac{b}{2}}$ is independent of the time variable t. That is

$$y\big(q(t,x),t\big)q_x^{\frac{b}{2}}(x,t) = y_0(x) = u_0(x) - u_{0xx}(x). \qquad \square$$

5. Analyticity of solutions

In this section, we shall study the analyticity of the Cauchy problem (1.1) based on a contraction type argument in a suitably chosen scale of the Banach spaces. Such an approach to analytic regularity of solutions to Cauchy problem (1.1) was initiated by Ovsjannikov [56,57] as an abstract Cauchy–Kowalevski theorem and later further developed by Nirenberg [54], Baouendi et al. [65] among others and subsequently applied to the Euler and Navier–Stokes equations.

In order to state the main result, we will need a suitable scale of Banach spaces as follows. For any $s > 0$, we set

$$E_s = \left\{u \in C^\infty(\mathbb{R}):\ |||u|||_s = \sup_{k\in N_0} \frac{s^k\|\partial^k u\|_{H^2}}{k!/(k+1)^2} < \infty\right\},$$

where $H^2(\mathbb{R})$ is the Sobolev space of order two on the real line and N_0 is the set of nonnegative integers. One can easily verify that E_s equipped with the norm $|||\cdot|||_s$ is a Banach space and that, for any $0 < s' < s$, E_s is continuously embedded in $E_{s'}$ with

$$|||u|||_{s'} \leqslant |||u|||_s.$$

Another simple consequence of the definition is that any u in E_s is a real analytic function on $\mathbb{R}$. Crucial for our purposes is the fact that each E_s forms an algebra under pointwise multiplication of functions.

Our main theorem is stated as follows.

Theorem 5.1. *If the initial data u_0 is analytic and belongs to a space E_{s_0}, for some $0 < s_0 \leqslant 1$, then there exist an $\varepsilon > 0$ and a unique solution $u(t,x)$ to the Cauchy problem (2.1) that is analytic on $(-\varepsilon,\varepsilon)\times\mathbb{R}$.*

For the proof of Theorem 5.1, we need the following theorem.

Theorem 5.2. *(See [1].) Let $\{X_s\}_{0<s<1}$ be a scale of decreasing Banach spaces, namely for any $s' < s$ we have $X_s \subset X_{s'}$ and $|||\cdot|||_{s'} \leqslant |||\cdot|||_s$. Consider the Cauchy problem*

$$\begin{cases}\dfrac{du}{dt} = F\big(t,u(t)\big),\\ u(0) = 0.\end{cases} \tag{5.1}$$

Let T, R and C be positive constants and assume that F satisfies the following conditions

(1) *If for* $0 < s' < s < 1$ *the function* $t \mapsto u(t)$ *is holomorphic in* $|t| < T$ *and continuous on* $|t| \leqslant T$ *with values in* X_s *and*

$$\sup_{|t|\leqslant T} \left|\left|\left| u(t) \right|\right|\right|_s < R,$$

then $t \mapsto F(t, u(t))$ *is a holomorphic function on* $|t| < T$ *with values in* $X_{s'}$*.*

(2) *For any* $0 < s' < s < 1$ *and any* $u, v \in X_s$ *with* $|||u|||_s < R$, $|||v|||_s < R$,

$$\sup_{|t|\leqslant T} \left|\left|\left| F(t,u) - F(t,v) \right|\right|\right|_{s'} \leqslant \frac{C}{s-s'} |||u - v|||_s.$$

(3) *There exists* $M > 0$ *such that for any* $0 < s < 1$,

$$\sup_{|t|\leqslant T} \left|\left|\left| F(t,0) \right|\right|\right|_s \leqslant \frac{M}{1-s}.$$

Then there exist a $T_0 \in (0, T)$ *and a unique function* $u(t)$*, which for every* $s \in (0,1)$ *is holomorphic in* $|t| < (1-s)T_0$ *with values in* X_s*, and is a solution to the Cauchy problem* (5.1)*.*

We restate the Cauchy problem (2.1) in a more convenient form. Let $v = u_x$, then the problem (2.1) can be written as a system for u and v

$$\begin{cases} u_t = -u^2 v - \left(1-\partial_x^2\right)^{-1}\left(bu^2 v + (6-b)uvv_x + 2v^3\right) = F(u,v), \\ v_t = -2uv^2 - u^2 v_x - \left(1-\partial_x^2\right)^{-1}\partial_x\left(bu^2 v + (6-b)uvv_x + 2v^3\right) = G(u,v), \\ u(x,0) = u_0(x), \quad v(x,0) = u_0'(x). \end{cases} \tag{5.2}$$

Proof of Theorem 5.1. Theorem 5.1 is a straightforward consequence of the abstract Cauchy–Kowalevski theorem [1]. We only need verify the conditions (1)–(3) in the statement of the abstract Cauchy–Kowalevski theorem (5.2) for both $F(u,v)$ and $G(u,v)$ in the system (5.2) since neither F nor G depend on t explicitly. We observe that, for $0 < s' < s < 1$, the estimates in Lemma 2.4 and Theorem 2.1 in [40] imply the following bounds

$$\left|\left|\left| F(u,v) \right|\right|\right|_{s'} \leqslant |b+1| |||u|||_s^2 |||v|||_s + 2|||v|||_s^3 + \frac{C}{s-s'} |||u|||_s |||v|||_s^2$$

and

$$\begin{aligned} \left|\left|\left| G(u,v) \right|\right|\right|_{s'} \leqslant{}& 2|||u|||_s |||v|||_s^2 + \frac{C}{s-s'} |||u|||_s^2 |||v|||_s \\ &+ |b| |||u|||_s^2 |||v|||_s + \frac{C}{s-s'} |||u|||_s |||v|||_s^2 + 2|||v|||_s^3, \end{aligned}$$

where the constant C depends only on R, hence condition (1) holds.

Note that to verify the second condition it suffices to estimate

$$\left|\left|\left| F(u_1,v) - F(u_2,v) \right|\right|\right|_{s'}, \ \left|\left|\left| F(u,v_1) - F(u,v_2) \right|\right|\right|_{s'},$$

and

$$\left|\left|\left| G(u_1,v) - G(u_2,v) \right|\right|\right|_{s'}, \ \left|\left|\left| G(u,v_1) - G(u,v_2) \right|\right|\right|_{s'}.$$

Since

$$\big|\!\big|\!\big|F(u_1,v_1)-F(u_2,v_2)\big|\!\big|\!\big|_{s'}\leqslant\big|\!\big|\!\big|F(u_1,v_1)-F(u_1,v_2)\big|\!\big|\!\big|_{s'}+\big|\!\big|\!\big|F(u_1,v_2)-F(u_2,v_2)\big|\!\big|\!\big|_{s'}$$

and

$$\big|\!\big|\!\big|G(u_1,v_1)-G(u_2,v_2)\big|\!\big|\!\big|_{s'}\leqslant\big|\!\big|\!\big|G(u_1,v_1)-G(u_1,v_2)\big|\!\big|\!\big|_{s'}+\big|\!\big|\!\big|G(u_1,v_2)-G(u_2,v_2)\big|\!\big|\!\big|_{s'},$$

using this together with Lemma 2.4 and Theorem 2.1 in [40], we get the following estimates

$$\begin{aligned}
\big|\!\big|\!\big|F(u_1,v)-F(u_2,v)\big|\!\big|\!\big|_{s'}&\leqslant C\big|\!\big|\!\big|u_1^2-u_2^2\big|\!\big|\!\big|_s|\!|\!|v|\!|\!|_s+\frac{C}{s-s'}|\!|\!|u_1-u_2|\!|\!|_s|\!|\!|v|\!|\!|_s^2\\
&\leqslant C|\!|\!|u_1-u_2|\!|\!|_s+\frac{C}{s-s'}|\!|\!|u_1-u_2|\!|\!|_s,\\
\big|\!\big|\!\big|F(u,v_1)-F(u,v_2)\big|\!\big|\!\big|_{s'}&\leqslant C|\!|\!|v_1-v_2|\!|\!|_s|\!|\!|u|\!|\!|_s^2+\frac{C}{s-s'}\big|\!\big|\!\big|v_1^2-v_2^2\big|\!\big|\!\big|_s|\!|\!|u|\!|\!|_s+2\big|\!\big|\!\big|v_1^3-v_2^3\big|\!\big|\!\big|_s\\
&\leqslant C|\!|\!|v_1-v_2|\!|\!|_s+\frac{C}{s-s'}|\!|\!|v_1-v_2|\!|\!|_s,\\
\big|\!\big|\!\big|G(u_1,v)-G(u_2,v)\big|\!\big|\!\big|_{s'}&\leqslant C|\!|\!|u_1-u_2|\!|\!|_s|\!|\!|v|\!|\!|_s^2+\frac{C}{s-s'}\big|\!\big|\!\big|u_1^2-u_2^2\big|\!\big|\!\big|_s|\!|\!|v|\!|\!|_s\\
&\quad+|b|\big|\!\big|\!\big|u_1^2-u_2^2\big|\!\big|\!\big|_s|\!|\!|v|\!|\!|_s+\frac{C}{s-s'}|\!|\!|u_1-u_2|\!|\!|_s|\!|\!|v|\!|\!|_s^2\\
&\leqslant C|\!|\!|u_1-u_2|\!|\!|_s+\frac{C}{s-s'}|\!|\!|u_1-u_2|\!|\!|_s,\\
\big|\!\big|\!\big|G(u,v_1)-G(u,v_2)\big|\!\big|\!\big|_{s'}&\leqslant C\big|\!\big|\!\big|v_1^2-v_2^2\big|\!\big|\!\big|_s|\!|\!|u|\!|\!|_s+\frac{C}{s-s'}|\!|\!|v_1-v_2|\!|\!|_s|\!|\!|u|\!|\!|_s^2+C|\!|\!|v_1-v_2|\!|\!|_s|\!|\!|u|\!|\!|_s^2\\
&\quad+\frac{C}{s-s'}\big|\!\big|\!\big|v_1^2-v_2^2\big|\!\big|\!\big|_s|\!|\!|u|\!|\!|_s+C\big|\!\big|\!\big|v_1^3-v_2^3\big|\!\big|\!\big|_s\\
&\leqslant C|\!|\!|v_1-v_2|\!|\!|_s+\frac{C}{s-s'}|\!|\!|v_1-v_2|\!|\!|_s,
\end{aligned}\tag{5.3}$$

where the constant C depends only on R, b.

Now, we verify the third condition. Note that u_0 is analytic by the assumption of Theorem 5.1. We can deduce that both $|\!|\!|u_0|\!|\!|_s$ and $|\!|\!|u_0'|\!|\!|_s$ are bounded. For $0<s'<s<1$, by Lemma 2.4 and Theorem 2.1 in [40], we have

$$\big|\!\big|\!\big|F(u_0,v_0)\big|\!\big|\!\big|_{s'}\leqslant|b+1||\!|\!|u_0|\!|\!|_s^2|\!|\!|v_0|\!|\!|_s+2|\!|\!|v_0|\!|\!|_s^3+\frac{C}{s-s'}|\!|\!|u_0|\!|\!|_s|\!|\!|v_0|\!|\!|_s^2$$

and

$$\begin{aligned}
\big|\!\big|\!\big|G(u_0,v_0)\big|\!\big|\!\big|_{s'}&\leqslant 2|\!|\!|u_0|\!|\!|_s|\!|\!|v_0|\!|\!|_s^2+\frac{C}{s-s'}|\!|\!|u_0|\!|\!|_s^2|\!|\!|v_0|\!|\!|_s\\
&\quad+|b||\!|\!|u_0|\!|\!|_s^2|\!|\!|v_0|\!|\!|_s+\frac{C}{s-s'}|\!|\!|u_0|\!|\!|_s|\!|\!|v_0|\!|\!|_s^2+2|\!|\!|v_0|\!|\!|_s^3,
\end{aligned}$$

where the constant C depends only on R, hence condition (3) holds. The conditions (1) through (3) above are now easily verified once our system (5.2) is transformed into a new system with zero initial data as in (5.1). The proof of Theorem 5.1 is complete. □

6. Peakon solutions

In this section we define strong solutions and weak solutions for Eq. (1.1). We also prove that its peakon solutions are weak solutions.

Note that Eq. (1.2) has the soliton waves with corner at its peak. Obviously, such solitons are not strong solutions to Eq. (2.1). In order to provide a mathematical framework for the study of solitons, we define the notion of weak solutions to Eq. (2.1). Let

$$F(u) = u^2 u_x + p * \big(bu^2 u_x + (6-b)uu_x u_{xx} + 2u_x^3\big).$$

Then Eq. (2.1) can be written as

$$u_t + F(u) = 0, \qquad u(0,x) = u_0. \tag{6.1}$$

Definition 6.1. Assume $u_0 \in H^s$, $s \in [0, \frac{3}{2}]$. If $u(x,t) \in L^\infty_{\mathrm{loc}}([0,T); H^s)$ and satisfies the following identity

$$\int_0^T \int_R \big(u\varphi_t - F(u)\varphi\big)\, dx\, dt + \int_R u_0 \varphi(0,x)\, dx = 0,$$

for all $\varphi \in \mathbb{C}_c^\infty([0,T) \times \mathbb{R})$. Let $\mathbb{C}_c^\infty([0,T) \times \mathbb{R})$ denote the space of all functions on $[0,T) \times \mathbb{R}$, which is restricted to $[0,T) \times R$ is a smooth function on $\mathbb{R}^2$ with compact support contained in $(-T,T) \times \mathbb{R}$. Then $u(t,x)$ is called a weak solution to Eq. (2.1). If $u(t,x)$ is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution to Eq. (2.1) (or (1.1)).

Theorem 6.1. *The peakon solitary*

$$u(t,x) = \pm c^{\frac{1}{2}} e^{-|x-ct-x_0|}, \quad c > 0,\ x_0 = constant,$$

is a global weak solution to Eq. (2.1)*. Moreover,* $\forall c > 0$, $u(t,x) \in L^\infty_{\mathrm{loc}}([0,T); H^1)$.

Proof. Since x_0 is constant, it is only to consider $u(t,x) = c^{\frac{1}{2}} e^{-|x-ct|}$. Note that

$$\begin{aligned}
&\int_0^T \int_R \big(u\varphi_t - F(u)\varphi\big)\, dx\, dt + \int_R u_0 \varphi(0,x)\, dx \\
&\quad = \int_0^T \int_R \big(u_t + F(u)\big)\varphi\, dx\, dt \\
&\quad = \int_0^T \int_R \big(u_t + u^{m+1}u_x + p * \big(bu^{m+1}u_x + (6-b)u^m u_x u_{xx} + 2u_x^3\big)\big).
\end{aligned} \tag{6.2}$$

Since

$$\begin{cases} u_t = c^{\frac{3}{2}} e^{-|x-ct|} \operatorname{sgn}(x-ct) = (cu) \operatorname{sgn}(x-ct), \\ u_x = -c^{\frac{1}{2}} e^{-|x-ct|} \operatorname{sgn}(x-ct) = -(u) \operatorname{sgn}(x-ct), \end{cases} \tag{6.3}$$

it follows that

$$u_t + u^2 u_x = \left(cu - u^3\right) \operatorname{sgn}(x - ct). \tag{6.4}$$

On the other hand, in view of (6.2), we have

$$\begin{aligned}
&p * \left(bu^2 u_x + (6-b)uu_x u_{xx} + 2u_x^3\right) \\
&\quad = \partial_x p * \left(\frac{6-b}{2} u^m u_x^2 + \frac{b}{3} u^3\right) + \frac{b-2}{2} p * \left(u_x^3\right) \\
&\quad = \int_R \partial_x \left(\frac{1}{2} e^{-|x-y|}\right) \left(\frac{6-b}{2} uu_x^2 + \frac{b}{3} u^3\right)(t, y)\, dy + \int_R \frac{b-2}{4} e^{-|x-y|} u_x^3\, dx \\
&\quad = \int_{-\infty}^{x} -\frac{1}{2} e^{y-x} \left(\frac{6-b}{2} + \frac{b}{3} + \frac{b-2}{2} \operatorname{sgn}(y - ct)\right) u^3(t, y)\, dy \\
&\qquad + \int_{x}^{\infty} \frac{1}{2} e^{y-x} \left(\frac{6-b}{2} + \frac{b}{3} - \frac{b-2}{2} \operatorname{sgn}(y - ct)\right) u^3(t, y)\, dy.
\end{aligned} \tag{6.5}$$

If $x < ct$, using $u(t, x) = c^{\frac{1}{2}} e^{-|x-ct|}$, we deduce from (6.5) that

$$\begin{aligned}
&p * \left(bu^2 u_x + (6-b)uu_x u_{xx} + 2u_x^3\right) \\
&\quad = \int_{-\infty}^{x} -\left(\frac{6-b}{2} + \frac{b}{3} - \frac{b-2}{2}\right) e^{y-x} u^3(t, y)\, dy + \int_{x}^{ct} \left(\frac{6-b}{2} + \frac{b}{3} + \frac{b-2}{2}\right) e^{x-y} u^3(x, y)\, dy \\
&\qquad + \int_{ct}^{\infty} \left(\frac{6-b}{2} + \frac{b}{3} - \frac{b-2}{2}\right) e^{x-y} u^3\, dy \\
&\quad = -\frac{1}{4} \left(\frac{6-b}{2} + \frac{b}{3} - \frac{b-2}{2}\right) c^{\frac{3}{2}} e^{3(x-ct)} + \frac{1}{2} \left(\frac{6-b}{2} + \frac{b}{3} + \frac{b-2}{2}\right) c^{\frac{3}{2}} \left(e^{x-ct} - e^{3(x-ct)}\right) \\
&\qquad + \frac{1}{4} \left(\frac{6-b}{2} + \frac{b}{3} - \frac{b-2}{2}\right) c^{\frac{3}{2}} e^{x-ct} \\
&\quad = -c^{\frac{3}{2}} e^{3(x-ct)} + c^{\frac{3}{2}} e^{x-ct} \\
&\quad = cu - u^3.
\end{aligned} \tag{6.6}$$

Similarly, if $x \geqslant ct$, we have

$$p * \left(bu^2 u_x + (6-b)uu_x u_{xx} + 2u_x^3\right) = -cu + u^3. \tag{6.7}$$

In view of (6.6) and (6.7), we obtain

$$p * \left(bu^2 u_x + (6-b)uu_x u_{xx} + 2u_x^3\right) = \left(cu - u^3\right) \operatorname{sgn}(x - ct). \tag{6.8}$$

Combining (6.1), (6.4), (6.8) with Definition 6.1, we deduce the desired result. This completes the proof of Theorem 6.1. □

Acknowledgments

The authors are grateful to the referees for their valuable suggestions and comments on the original manuscript and would like to say thanks to Professor Zhaoyang Yin for sending several of their papers to the authors. This work was partially supported by NSF of China (11071266), partially supported by Scholarship Award for Excellent Doctoral Student granted by Ministry of Education and partially supported by the Educational Science Foundation of Chongqing China (KJ121302).

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