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J. Differential Equations 258 (2015) 1015–1024

Journal of Differential Equations

www.elsevier.com/locate/jde

# Instability of internal equatorial water waves

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Received 3 July 2014; revised 18 August 2014

Available online 1 December 2014

**Abstract**

In the following paper we present criteria for the hydrodynamical instability of internal equatorial water waves. We show, by way of the short-wavelength perturbation approach, that certain geophysical waves propagating above the equatorial thermocline are linearly unstable when the wave steepness exceeds a given threshold.
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*MSC:* primary 76B15; secondary 76B47, 35B50, 26E05

*Keywords:* Geophysical flows; Instability; Internal waves; Thermocline; Currents

## 1. Introduction

In this paper we analyse the stability of some recently derived exact, explicit solutions to the geophysical governing equations in the equatorial $\beta$-plane using the short-wavelength perturbation method. The solutions we analyse prescribe steady, unidirectional, internal travelling waves which propagate above the thermocline, which is an interface separating two distinct vertical ocean layers of differing densities [9,11,17,33].

In [9] an explicit exact solution to the full geophysical governing equations was obtained which corresponds to the classical two-layer model describing oscillations of the thermocline in the equatorial region [17]. The solution in [9] is remarkable since, even in the setting where

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http://dx.doi.org/10.1016/j.jde.2014.08.019
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Coriolis effects are ignored, there are only a handful of explicit exact solutions to the full governing equations for water waves. Perhaps the most celebrated of these is Gerstner's wave [3,6,22,24], an exact solution that is explicit in the Lagrangian formulation [2,6]. Recently, quite a number of Gerstner-type exact and explicit solutions have been derived which model various physical and geophysical scenarios [3,4,7,9,11,25,28–31,36,38]. The formulation presented in [9] is quite unique in the sense that the fluid motion diminishes as one ascends from the thermocline towards the surface, as opposed to the Gerstner formulation for surface waves whereby motion decreases as one descends beneath the surface.

Recently, in [30], it was shown that the model describing internal waves propagating above the thermocline in [9] could be adapted to allow for a constant underlying current. This idea extends back to [37] when Mollo-Christensen introduced a current-like term into Gerstner's solution for gravity waves in order to describe billows between two fluids, and it was recently employed in the geophysical setting in [25]. The solutions presented in [9,30] represent explicit examples of exact equatorial waves whereby the fluid motion dies out at great depth, and we note that the recent papers [8,10,27] rigorously established the existence of exact equatorial surface waves which admit an underlying vorticity distribution.

In this paper we apply the short-wavelength perturbation method to derive instability criteria for the internal wave solutions which were obtained in [9,30]. These criteria are stated in Propositions 4.1 and 4.2 below. From a mathematical viewpoint, establishing the hydrodynamical stability or instability of a flow is difficult, given that the fully nonlinear governing equations for fluid motion are highly intractable [15,16,19,20]. Physically, the question of hydrodynamic stability is important since, for instance, unstable flows cannot be observed in practice since they are rapidly destroyed by any minor perturbations or disturbances. For certain solutions which have an explicit Lagrangian formulation, it transpires that the short-wavelength perturbation method of instability analysis, which was independently developed by the authors of [1,18,35], has a remarkably elegant formulation and application. This was first established for Gerstner's solution to the gravity water wave problem in [34], and for geophysical flows in [12]. Recent work [21,26,32] has applied the instability analysis to a variety of contexts. The current paper is, to the best of our knowledge, the first application of this approach to internal waves propagating above the equatorial thermocline.

## 2. Governing equations and model

We take the earth to be a perfect sphere of radius $R = 6378$ km with constant rotational speed of $\Omega = 73 \cdot 10^{-6}$ rad/s, and $g = 9.8\ \mathrm{m\,s^{-2}}$ is the gravitational acceleration at the surface of the earth. In a reference frame with the origin located at a point on earth's surface and rotating with the earth, we take the $x$-axis to be the longitudinal direction (horizontally due east), the $y$-axis to be the latitudinal direction (horizontally due north) and the $z$-axis to be vertically upwards. The governing equations for geophysical ocean waves [14] are given by

$$u_t + uu_x + vu_y + wu_z + 2\Omega w \cos\phi - 2\Omega v \sin\phi = -\frac{1}{\rho} P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u \sin\phi = -\frac{1}{\rho} P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u \cos\phi = -\frac{1}{\rho} P_z - g, \tag{2.1c}$$

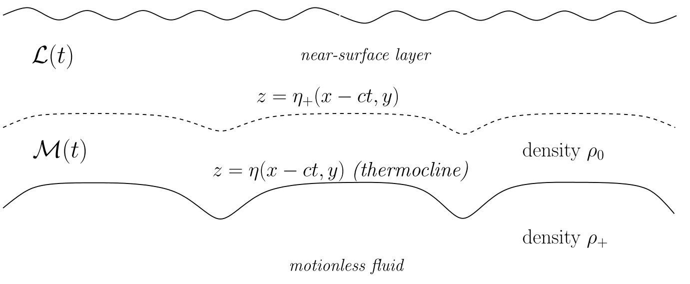


Fig. 1. Schematic of the two-layer model.

together with the equation of incompressibility

$$\nabla \cdot \mathbf{U} = 0, \tag{2.2}$$

and the equation of mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \tag{2.3}$$

Here $\mathbf{U} = (u, v, w)$ is the velocity field of the fluid, the variable $\phi$ represents the latitude, $\rho$ is the density of the fluid, and $P$ is the pressure of the fluid. In the equatorial region, where the latitude $\phi$ is relatively small, the full governing equations for geophysical water waves (2.1) may be rendered more tractable by approximating the Coriolis terms. When $\phi$ is small, but not constant, the $\beta$-plane approximation $\sin\phi \approx \phi$, $\cos\phi \approx 1$ may be employed [14] which is equivalent to locally approximating the earth's curved surface by a plane. Accordingly the governing equations reduce to the $\beta$-plane equations

$$u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\frac{1}{\rho} P_x, \tag{2.4a}$$

$$v_t + uv_x + vv_y + wv_z + \beta y u = -\frac{1}{\rho} P_y, \tag{2.4b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho} P_z - g, \tag{2.4c}$$

where $\beta = 2\Omega/R = 2.28 \cdot 10^{-11}\ \mathrm{m}^{-1}\,\mathrm{s}^{-1}$. In the recent paper [9], a formulation is presented for an exact, explicit solution of (2.4) which prescribes steady, unidirectional, internal travelling waves which propagate above the thermocline, which is an interface separating two distinct vertical ocean layers of differing densities. To finish this section we present an outline of the two-layer model that these solutions adhere to. (See Fig. 1.) The uppermost fluid layer, of density $\rho_0$, which lies above the thermocline is subdivided into two parts. The near-surface layer, to which wind effects are confined, is labelled $\mathcal{L}(t)$. Typical values for the mean-depth of $\mathcal{L}(t)$ are 80 m. Beneath $\mathcal{L}(t)$ is a layer where the fluid motion is entirely due to the propagation of equatorial internal waves, this layer is denoted $\mathcal{M}(t)$, and typical values for the mean-depth of $\mathcal{M}(t)$ are 40 m, cf. [9]. Finally, the thermocline lies at the boundary of $\mathcal{M}(t)$ and the deeper, motionless layer of fluid which has density $\rho_+ > \rho_0$. The thermocline is labelled $z = \eta(x - ct, y)$, while the interface separating $\mathcal{L}(t)$ and $\mathcal{M}(t)$ is denoted $z = \eta_+(x - ct, y)$, where $c$ is the constant wave phase-speed. The fluid is assumed motionless beneath the thermocline, and so $u \equiv v \equiv w \equiv 0$ for $z < \eta(x - ct, y)$.

## 3. Exact internal wave solutions

Recently in [9] an exact explicit solution of the governing equations (2.4) was derived which prescribes equatorially-trapped waves propagating in the layer $\mathcal{M}(t)$. These waves travel eastwards, and so $c > 0$, with a vanishing meridional velocity, and so $v \equiv 0$, reducing the governing equations to

$$u_t + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho_0} P_x, \tag{3.1a}$$

$$\beta yu = -\frac{1}{\rho_0} P_y, \tag{3.1b}$$

$$w_t + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho_0} P_z - g, \tag{3.1c}$$

together with $u_x + w_z = 0$ in $\eta(x - ct, y) < z < \eta_+(x - ct, y)$, and the boundary condition

$$P = P_0 - \rho_+ gz \quad \text{on } z = \eta(x - ct, y). \tag{3.1d}$$

The solution presented in [9] takes the following form, where the Eulerian coordinates of fluid particles $(x, y, z)$ are expressed as functions of the time $t$ and Lagrangian labelling variables $(q, r, s)$:

$$x = q - \frac{1}{k} e^{-k(r+f(s))} \sin\left[k(q - ct)\right], \tag{3.2a}$$

$$y = s, \tag{3.2b}$$

$$z = r - \frac{1}{k} e^{-k(r+f(s))} \cos\left[k(q - ct)\right]. \tag{3.2c}$$

Here $k$ is the wavenumber and $c > 0$ is the constant speed of propagation of the waves. The parameters $q$, $r$, $s$ vary as follows: $q \in \mathbb{R}$ and $s \in [-s_0, s_0]$, where $s_0 = \sqrt{c_0/\beta} \approx 250$ km is the equatorial radius of deformation [14]. For each fixed latitude $s$, we have $r \in [r_0(s), r_+(s)]$, where $r_0(s)$ determines the thermocline $\eta$ and $r_+(s)$ determines the interface $\eta_+$ separating $\mathcal{M}(t)$ and $\mathcal{L}(t)$. It is shown in [9] that $r_+(s) > r_0(s) > 0$ and furthermore the wave profiles $\eta$ and $\eta_+$ determined by $r_0$, $r_+$ respectively are trochoids, which are characteristically nonlinear and display a crest-trough asymmetry in the wave profile. The solutions in (3.2) represent equatorially trapped waves, and the function $f(s)$ which determines the exponential decay of the motion away from the equator is given by

$$f(s) = \frac{c\beta}{2\tilde{g}} s^2, \tag{3.3}$$

where $\tilde{g}$ is the reduced gravity, defined by

$$\tilde{g} = g\frac{\rho_+ - \rho_0}{\rho_0},$$

and the wave phase-speed is defined by the dispersion relation

$$c = \frac{\sqrt{\Omega^2 + k\tilde{g}} + \Omega}{k} > 0, \tag{3.4}$$

which we note is equivalent to the relation

$$\tilde{g} = kc^2 - 2\Omega c.$$

A typical value for the reduced gravity is $\tilde{g} = 6 \cdot 10^{-3}\ \mathrm{m\,s^{-2}}$ [17]. The waves described by (3.2) are symmetric about the equator, and the motion they induce diminishes the further one ascends above the thermocline. An additional feature of the form of the solution (3.2) is that the particle trajectories for the underlying flow are closed circles in a fixed latitudinal plane. The existence of closed particle paths is typical of Gerstner-type waves, and it is a phenomenon which does not apply to most irrotational water waves. In the case of Stokes waves of both finite depth [5,13], and infinite depth [23], the particle trajectories are in fact not closed. If we define

$$\chi = k\big(r + f(s)\big), \qquad \theta = k(q - ct),$$

the determinant of the Jacobian of the transformation (3.2) is given by $1 - e^{-2\chi}$, which is time independent [9]. Hence the flow defined by (3.2) is volume preserving, and so (2.2) holds in the Eulerian framework [2]. In order for the transformation (3.2) to be well-defined it is necessary that

$$r + f(s) \geq r^* > 0.$$

The velocity gradient tensor can be calculated to give

$$\nabla\mathbf{U} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{cke^{-\chi}}{1 - e^{-2\chi}} \begin{pmatrix} -\sin\theta & 0 & -\cos\theta - e^{-\chi} \\ \frac{\beta s(e^{-\chi} - \cos\theta)}{kc - 2\Omega} & 0 & \frac{\beta s\sin\theta}{kc - 2\Omega} \\ e^{-\chi} - \cos\theta & 0 & \sin\theta \end{pmatrix}, \tag{3.5}$$

and so the vorticity of the flow prescribed by (3.2) is $\omega = (w_y - v_z, u_z - w_x, v_x - u_y)$

$$= \left(s\frac{kc^2\beta}{\tilde{g}}\frac{e^{-\chi}\sin\theta}{1 - e^{-2\chi}}, \frac{2kce^{-2\chi}}{1 - e^{-2\chi}}, s\frac{kc^2\beta}{\tilde{g}}\frac{e^{-\chi}\cos\theta - e^{-2\chi}}{1 - e^{-2\chi}}\right). \tag{3.6}$$

Also, the steepness of the wave-profile of the thermocline, defined to be half the amplitude of the wave multiplied by the wavenumber, is given by

$$\tau(s) = e^{-\chi}, \tag{3.7}$$

which is maximum $\tau_0 = e^{-kr_0}$ at the equator.

## 4. Instability analysis

We now present the main result of this paper, which we prove below using the short-wavelength instability method to analyse the flow in the $\mathcal{M}(t)$ layer as determined by the solution (3.2) of the governing equations (2.4).

**Proposition 4.1.** *The internal equatorial waves propagating eastward above the thermocline, prescribed by (3.2), are unstable to short wavelength perturbations if the steepness of the wave-profile at the thermocline exceeds the threshold*

$$e^{-kr_0} > \frac{\sqrt{\Omega^2 + k\tilde{g}} - 3\Omega}{3\sqrt{\Omega^2 + k\tilde{g}} - \Omega} \left( \lessapprox \frac{1}{3} \right). \tag{4.1}$$

To prove this instability result we examine the evolution of a localised and rapidly-varying infinitesimal perturbation of the flow, as represented at time $t$ by the wave packet

$$\mathbf{u}(\mathbf{X}, t) = \varepsilon \mathbf{b}(\mathbf{X}, \boldsymbol{\xi_0}, \mathbf{b_0}, t) e^{i\Phi(\mathbf{X}, \boldsymbol{\xi_0}, \mathbf{b_0}, t)/\delta}. \tag{4.2}$$

Here $\mathbf{X} = (x, y, z)$, $\Phi$ is a scalar function, and at $t = 0$ we have

$$\Phi(\mathbf{X}, \boldsymbol{\xi_0}, \mathbf{b_0}, 0) = \mathbf{X} \cdot \boldsymbol{\xi_0}, \qquad \mathbf{b}(\mathbf{X}, \boldsymbol{\xi_0}, \mathbf{b_0}, 0) = \mathbf{b_0}(\mathbf{X}, \boldsymbol{\xi_0}).$$

The normalised wave vector $\boldsymbol{\xi_0}$ is subject to the transversality condition $\boldsymbol{\xi_0} \cdot \mathbf{b_0} = 0$, and $\mathbf{b_0}$ is the normalised amplitude of the short-wavelength perturbation of the flow which has the velocity field $\mathbf{U}(\mathbf{X}) \equiv (u\ v\ w)^T(x, y, z)$. Then the evolution in time of $\mathbf{X}$, of the perturbation amplitude $\mathbf{b}$, and of the wave vector $\boldsymbol{\xi} = \nabla\Phi$, is governed at the leading order in the small parameters $\epsilon$ and $\delta$ by the system of ODEs

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{U}(\mathbf{X}, t), \\ \dot{\boldsymbol{\xi}} = -(\nabla \mathbf{U})^T \boldsymbol{\xi}, \\ \dot{\mathbf{b}} = -L\mathbf{b} - \mathbf{b} \cdot (\nabla \mathbf{U}) + \left( \left[ L\mathbf{b} + 2\mathbf{b} \cdot (\nabla \mathbf{U}) \right] \cdot \boldsymbol{\xi} \right) \dfrac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}, \end{cases} \tag{4.3}$$

with initial conditions

$$\mathbf{X}(0) = \mathbf{X_0}, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi_0}, \qquad \mathbf{b}(0) = \mathbf{b_0}.$$

Here $(\nabla \mathbf{U})^T$ is the transpose of the velocity gradient tensor (3.5) and, for the system defined by (2.4), $L = L(\mathbf{X})$ is given by

$$L = \begin{pmatrix} 0 & -\beta y & 2\Omega \\ \beta y & 0 & 0 \\ -2\Omega & 0 & 0 \end{pmatrix},$$

cf. [12,21,26] for details. We note that systems of ODEs along the lines of (4.3), which describe the evolution of a rapidly-varying short-wavelength perturbation, were first derived independently by a number of authors [1,18,35].

To prove the instability of the internal waves prescribed by (3.2), it suffices to demonstrate by way of (4.3) an exponential growth-rate in amplitude $\mathbf{b}$ for some particular initial disturbance. While in (4.3) the second and third equations are linear, the first equation is usually nonlinear but decouples from the other two. The first equation in (4.3) provides the particle trajectory of the basic (undisturbed) flow, while the second and third equations govern to leading order the evolution along this trajectory of the local wave vector and of the amplitude of the perturbation, respectively. Choosing the latitudinal wave vector $\boldsymbol{\xi}_0 = (0\ 1\ 0)^T$, and it follows immediately from (3.5) and the second equation in (4.3) that $\boldsymbol{\xi}(t) = (0\ 1\ 0)^T$ for all $t \geq 0$. The third equation in (4.3) then implies that $\mathbf{b} = (b_1, b_2, b_3)$ is governed by

$$\begin{cases} \dot{b}_1 = \beta s b_2 - 2\Omega b_3 + \dfrac{kce^{-\chi}\sin\theta}{1-e^{-2\chi}} b_1 - \dfrac{kc\beta s e^{-\chi}(e^{-\chi}-\cos\theta)}{(kc-2\Omega)(1-e^{-2\chi})} b_2 - \dfrac{kce^{-\chi}(e^{-\chi}-\cos\theta)}{1-e^{-2\chi}} b_3, \\ \dot{b}_2 = 0, \\ \dot{b}_3 = 2\Omega b_1 + \dfrac{kce^{-\chi}(e^{-\chi}+\cos\theta)}{1-e^{-2\chi}} b_1 - \dfrac{kc\beta s e^{-\chi}\sin\theta}{(kc-2\Omega)(1-e^{-2\chi})} b_2 - \dfrac{kce^{-\chi}\sin\theta}{1-e^{-2\chi}} b_3. \end{cases} \tag{4.4}$$

Noting that the choice $b_2(0) = 0$ implies $b_2(t) = 0$ for all $t \geq 0$, and accordingly $\boldsymbol{\xi}(t) \cdot \mathbf{b}(t) = 0$, the system (4.4) reduces to the two-dimensional system

$$\dot{B} = \begin{pmatrix} \frac{kce^{-\chi}\sin\theta}{1-e^{-2\chi}} & -2\Omega - \frac{kce^{-\chi}(e^{-\chi}-\cos\theta)}{1-e^{-2\chi}} \\ 2\Omega + \frac{kce^{-\chi}(e^{-\chi}+\cos\theta)}{1-e^{-2\chi}} & -\frac{kce^{-\chi}\sin\theta}{1-e^{-2\chi}} \end{pmatrix} B, \tag{4.5}$$

where $B = \binom{b_1}{b_3}$. This system is nonautonomous, however the change of variables

$$P = \begin{pmatrix} \cos(kct/2) & -\sin(kct/2) \\ \sin(kct/2) & \cos(kct/2) \end{pmatrix}$$

transforms the planar system (4.5) to an autonomous system for $Q = P^{-1}B$,

$$\frac{d}{dt}Q(t) = DQ(t),$$

where

$$D = \begin{pmatrix} \frac{kce^{-\chi}}{1-e^{-2\chi}}\sin(kq) & \frac{kce^{-\chi}}{1-e^{-2\chi}}\cos(kq) - 2\Omega - \frac{kce^{-2\chi}}{1-e^{-2\chi}} + \frac{kc}{2} \\ \frac{kce^{-\chi}}{1-e^{-2\chi}}\cos(kq) + 2\Omega + \frac{kce^{-2\chi}}{1-e^{-2\chi}} - \frac{kc}{2} & -\frac{kce^{-\chi}}{1-e^{-2\chi}}\sin(kq) \end{pmatrix}.$$

Since $B = PQ$, and $P$ is periodic with time, we deduce that the short-wavelength rapidly-varying perturbation $\mathbf{u}$, defined in (4.2), grows exponentially with time if $D$ has a positive eigenvalue. The eigenvalues $\lambda$ for $D$ are given by the quadratic equation

$$\lambda^2 = \frac{-\mathfrak{a}^2(e^{-2\chi})^2 + \mathfrak{b}e^{-2\chi} - \mathfrak{c}^2}{4(1-e^{-2\chi})^2},$$

where $\mathfrak{a} = (4\Omega - 3kc)$, $\mathfrak{b} = 10k^2c^2 + 32\Omega(\Omega - kc)$, $\mathfrak{c} = (4\Omega - kc)$. Using the relation $\mathfrak{b} = \mathfrak{a}^2 + \mathfrak{c}^2$, we find that the quadratic equation involving $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ which appears in the numerator above has roots precisely for

$$e^{-2\chi} = \frac{\mathfrak{c}^2}{\mathfrak{a}^2}, 1,$$

and consequently we deduce that exponential growth occurs for $Q(t)$ if and only if

$$e^{-\chi} > \frac{kc - 4\Omega}{3kc - 4\Omega}. \tag{4.6}$$

Together with the dispersion relation (3.4), and (3.7), this proves Proposition 4.1.

*4.1. Constant underlying current*

Recently, one of the authors showed in [30] that the solution (3.2) could be modified to account for a constant underlying zonal current of strength $U$. The resulting solution is given by

$$x = q - Ut - \frac{1}{k} e^{-k(r+f(s))} \sin\left[k(q - ct)\right], \tag{4.7a}$$

$$y = s, \tag{4.7b}$$

$$z = r - \frac{1}{k} e^{-k(r+f(s))} \cos\left[k(q - ct)\right]. \tag{4.7c}$$

Considerations similar to those carried out above apply to the instability analysis of the solution with an underlying current, with the proviso that $\tilde{g}$ is replaced with $\mathfrak{g} = \tilde{g} - 2\Omega U$ throughout. This leads us to the following result.

**Proposition 4.2.** *The internal equatorial waves propagating eastward above the thermocline in the presence of a constant underlying current, as prescribed by (4.7), are unstable to short wavelength perturbations if the steepness of the wave-profile at the thermocline exceeds the threshold*

$$e^{-kr_0} > \frac{\sqrt{\Omega^2 + k(\tilde{g} - 2\Omega U)} - 3\Omega}{3\sqrt{\Omega^2 + k(\tilde{g} - 2\Omega U)} - \Omega} \left(\lessapprox \frac{1}{3}\right). \tag{4.8}$$

It follows immediately from relation (4.8) that the presence of an underlying current can either (slightly) increase, or diminish, the threshold for instability, depending on whether the current is adverse or following.

## 5. Final remarks

We note that the instability thresholds obtained in Propositions 4.1 and 4.2 are both (slightly) less than $1/3$. This is of note since it contrasts with the situation that holds for eastward-propagating equatorial surface water waves in the $\beta$-plane [12,21], where the threshold is slightly greater than $1/3$. This is presumably an artefact of the waves considered in this paper being internal waves. Interestingly, the threshold in relation (4.1) has strong similarities to that which pertains for *westward* propagating surface waves in the $f$-plane, cf. [26], with the obvious difference of reduced gravity $\tilde{g}$ appearing in relations for internal waves. Of course, when we ignore geophysical effects and omit Coriolis terms, the thresholds on the right-hand side of (4.1) and (4.8) (and those derived in [12,21,26]) become precisely 1/3, which matches the instability threshold first derived for Gerstner's gravity water wave in [34].

Quantitatively, the thresholds we derive in Propositions 4.1 and 4.2 differ considerably from those of surface water waves, since the Coriolis terms involving $\Omega$ are quite a bit closer, in terms of order of magnitude, to the reduced gravity $\tilde{g}$ than to the gravitational constant $g$. One consequence of this is that Coriolis effects play a greater role in the instability thresholds for internal waves, that we have derived here in Propositions 4.1 and 4.2, than they do for surface water waves in [12,21,26]. From the physical viewpoint of applying the instability thresholds we have derived, the reader should consult qualitative data pertaining to equatorial water waves which can be found in the papers [7,9,11] and the references cited therein.

## Acknowledgment

D.H. acknowledges the support of the Science Foundation Ireland (SFI) grant 13/CDA/2117.

## References

[1] B.J. Bayly, Three-dimensional instabilities in quasi-two dimensional inviscid flows, in: R.W. Miksad, et al. (Eds.), Nonlinear Wave Interactions in Fluids, ASME, New York, 1987, pp. 71–77.
[2] A. Bennett, Lagrangian Fluid Dynamics, Cambridge University Press, Cambridge, 2006.
[3] A. Constantin, On the deep water wave motion, J. Phys. A 34 (2001) 1405–1417.
[4] A. Constantin, Edge waves along a sloping beach, J. Phys. A 34 (2001) 9723–9731.
[5] A. Constantin, The trajectories of particles in Stokes waves, Invent. Math. 166 (2006) 523–535.
[6] A. Constantin, Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 81, SIAM, Philadelphia, 2011.
[7] A. Constantin, An exact solution for equatorially trapped waves, J. Geophys. Res. 117 (2012) C05029.
[8] A. Constantin, On the modelling of equatorial waves, Geophys. Res. Lett. 39 (2012) L05602.
[9] A. Constantin, Some three-dimensional nonlinear equatorial flows, J. Phys. Oceanogr. 43 (2013) 165–175.
[10] A. Constantin, On equatorial wind waves, Differential Integral Equations 26 (2013) 237–252.
[11] A. Constantin, Some nonlinear, equatorially trapped, nonhydrostatic internal geophysical waves, J. Phys. Oceanogr. 44 (2014) 781–789.
[12] A. Constantin, P. Germain, Instability of some equatorially trapped waves, J. Geophys. Res. Oceans 118 (2013) 2802–2810.
[13] A. Constantin, W. Strauss, Pressure beneath a Stokes wave, Comm. Pure Appl. Math. 63 (2010) 533–557.
[14] B. Cushman-Roisin, J.-M. Beckers, Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects, Academic Press, Waltham, MA, 2011.
[15] P.G. Drazin, Introduction to Hydrodynamic Stability, Cambridge University Press, Cambridge, 2002.
[16] P.G. Drazin, W.H. Reid, Hydrodynamic Stability, Cambridge University Press, Cambridge, 2004.
[17] A.V. Fedorov, J.N. Brown, Equatorial waves, in: J. Steele (Ed.), Encyclopedia of Ocean Sciences, Academic Press, San Diego, CA, 2009, pp. 3679–3695.
[18] S. Friedlander, M.M. Vishik, Instability criteria for the flow of an inviscid incompressible fluid, Phys. Rev. Lett. 66 (1991) 2204–2206.
[19] S. Friedlander, V. Yudovich, Instabilities in fluid motion, Notices Amer. Math. Soc. 46 (1999) 1358–1367.
[20] S. Friedlander, Lectures on stability and instability of an ideal fluid, in: Hyperbolic Equations and Frequency Interactions, in: IAS/Park City Math. Ser., vol. 5, Amer. Math. Soc., Providence, RI, 1999, pp. 227–304.
[21] F. Genoud, D. Henry, Instability of equatorial water waves with an underlying current, J. Math. Fluid Mech. 16 (2014) 661–667.
[22] F. Gerstner, Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile, Ann. Phys. 2 (1809) 412–445.
[23] D. Henry, On the deep-water Stokes flow, Int. Math. Res. Not. IMRN 2008 (2008), 7 pp.
[24] D. Henry, On Gerstner's water wave, J. Nonlinear Math. Phys. 15 (2008) 87–95.
[25] D. Henry, An exact solution for equatorial geophysical water waves with an underlying current, Eur. J. Mech. B Fluids 38 (2013) 18–21.

[26] D. Henry, H.-C. Hsu, Instability of equatorial water waves in the $f$-plane, Discrete Contin. Dyn. Syst. 35 (2015) 909–916.
[27] D. Henry, A. Matioc, On the existence of equatorial wind waves, Nonlinear Anal. 101 (2014) 113–123.
[28] H.-C. Hsu, An exact solution for nonlinear internal equatorial waves in the $f$-plane approximation, J. Math. Fluid Mech. 16 (2014) 463–471.
[29] H.-C. Hsu, Some nonlinear internal equatorial waves with a strong underlying current, Appl. Math. Lett. 34 (2014) 1–6.
[30] H.-C. Hsu, Some nonlinear internal equatorial flows, Nonlinear Anal. Real World Appl. 18 (2014) 69–74.
[31] H.-C. Hsu, An exact solution of equatorial waves, Monatsh. Math. (2014), http://dx.doi.org/10.1007/s00605-014-0618-2, in press.
[32] D. Ionescu-Kruse, Instability of edge waves along a sloping beach, J. Differential Equations 256 (2014) 3999–4012.
[33] T. Izumo, The equatorial current, meridional overturning circulation, and their roles in mass and heat exchanges during the El Niño events in the tropical Pacific Ocean, Ocean Dyn. 55 (2005) 110–123.
[34] S. Leblanc, Local stability of Gerstner's waves, J. Fluid Mech. 506 (2004) 245–254.
[35] A. Lifschitz, E. Hameiri, Local stability conditions in fluid dynamics, Phys. Fluids 3 (1991) 2644–2651.
[36] A.V. Matioc, An exact solution for geophysical equatorial edge waves over a sloping beach, J. Phys. A 45 (2012) 365501.
[37] E. Mollo-Christensen, Gravitational and geostrophic billows: some exact solutions, J. Atmospheric Sci. 35 (1978) 1395–1398.
[38] R. Stuhlmeier, On edge waves in stratified water along a sloping beach, J. Nonlinear Math. Phys. 18 (2011) 127–137.