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Global weak solutions for the three-dimensional chemotaxis-Navier–Stokes system with nonlinear diffusion ☆

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Abstract

We consider an initial–boundary value problem for the incompressible chemotaxis-Navier–Stokes equations generalizing the porous-medium-type diffusion model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n\chi(c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \end{cases}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$. Here $\kappa \in \mathbb{R}$, $\Phi \in W^{1,\infty}(\Omega)$, $0 < \chi \in C^2([0,\infty))$ and $0 \le f \in C^1([0,\infty))$ with $f(0) = 0$. It is proved that under appropriate structural assumptions on f and χ, for any choice of $m \ge \frac{2}{3}$ and all sufficiently smooth initial data (n_0, c_0, u_0) the model possesses at least one global weak solution.

MSC: 35Q92; 35K55; 35Q35; 76S05; 92C17

Keywords: Chemotaxis; Navier–Stokes equation; Nonlinear diffusion; Global existence

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1. Introduction

Chemotaxis is the directed movement of living cells under the effects of chemical gradients. Aerobic bacteria such as *Bacillus subtilis* often live in thin fluid layers near solid-air-water contact line, in which the swimming bacteria move towards higher concentration of oxygen according to mechanism of chemotaxis and meanwhile the movement of fluid is under the influence of gravitational force generated by bacteria themselves. Both the oxygen concentration and bacteria density are transported by the fluid and diffuse through the fluid [5,14,20].

To model such biological processes, Tuval et al. [22] proposed the following model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), & x \in \Omega,\ t > 0, \\ u_t + \kappa(u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0 \end{cases} \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^N$, where the scalar functions $n = n(x,t)$ and $c = c(x,t)$ denote bacterial density and the concentration of oxygen, respectively. The vector $u = (u_1(x,t), u_2(x,t), \cdots, u_N(x,t))$ is the fluid velocity field and the associated pressure is represented by $P = P(x,t)$. The function χ is called the chemotactic sensitivity, f is the oxygen consumption rate by the bacteria and $\kappa \in \mathbb{R}$ measures the strength of nonlinear fluid convection. The given function Φ stands for the gravitational potential produced by the action of physical forces on the cell.

The chemotaxis fluid system has been studied in the last few years and the main focus is on the solvability result. Under the assumption that $\chi(c) = \chi$ is a constant and f is monotonically increasing with $f(0) = 0$, Lorz [14] constructed local weak solutions in a bounded domain $\mathbb{R}^N$ $(N = 2, 3)$ with no-flux boundary condition and in $\mathbb{R}^2$ in the case of inhomogeneous Dirichlet conditions for oxygen. In bounded convex domains $\Omega \subset \mathbb{R}^2$, Winkler [28] proved that the initial–boundary value problem for (1.1) possesses a unique global classical solution. In [30] the same author showed that the global classical solutions obtained in [28] stabilize to the spatially uniform equilibrium $(\bar{n}_0, 0, 0)$ with $\bar{n}_0 := \frac{1}{|\Omega|}\int_\Omega n_0(x)dx$ as $t \to \infty$. Zhang and Li [32] proved that such solution converges to the equilibrium $(\bar{n}_0, 0, 0)$ exponentially in time. By deriving a new type of entropy–energy estimate, Jiang et al. [11] generalized the result of [30] to general bounded domains. For the well-posedness of the Cauchy problem to (1.1) in the whole space we refer the reader to [2,3,6,13,31,34].

When the nonlinear convective term is ignored ($\kappa = 0$ in (1.1)), which means the fluid motion is slow, the model is simplified to the chemotaxis-Stokes equation. In this modified version, global weak solutions are constructed for the two-dimensional Cauchy problem [6]. In a bounded convex domain $\Omega \subset \mathbb{R}^3$, the chemotaxis-Stokes system possesses at least one global weak solution [28].

The diffusion of bacteria may depend nonlinearly on their densities [9,19,20,23]. Introducing this into the model (1.1) leads to the chemotaxis-Navier–Stokes system with nonlinear diffusion [4]

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$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\Phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0. \end{cases} \tag{1.2}$$

Up to now, the main issue of investigation to (1.2) seems to concentrate on the chemotaxis-Stokes variant. Under the assumption $D(n) = mn^{m-1}$, Di Francesco et al. [4] proved that when $m \in (\frac{3}{2}, 2]$ the chemotaxis-Stokes system admits a global-in-time solution for general initial data in the bounded domain $\Omega \subset \mathbb{R}^2$. The same result holds in three-dimensional setting under the constraint $m \in (\frac{7+\sqrt{217}}{12}, 2]$. Tao and Winkler [19,20] extended the global existence result so as to cover the whole range $m \in (1,\infty)$ in the bounded domain $\Omega \subset \mathbb{R}^2$ and $m \in (\frac{8}{7},\infty)$ in the bounded convex domain $\Omega \subset \mathbb{R}^3$. In [13], global existence of weak solution to the Cauchy problem of chemotaxis-Stokes system is established with $m = \frac{4}{3}$ in $\Omega = \mathbb{R}^2$. Recently, Duan and Xiang [7] generalized the global existence result for all exponents $m \in [1,\infty)$.

In contrast to the chemotaxis-Stokes system, very few results of global solvability are available for the full nonlinear chemotaxis-Navier–Stokes system. In the case $\Omega \subseteqq \mathbb{R}^2$, global weak solutions are constructed by setting $D(n) = mn^{m-1}$ with $m \in [1,\infty)$ [7]. For the three-dimensional initial–boundary value problem, the only result we are aware of is that when $m > \frac{4}{3}$ the full system with nonlinear diffusion admits a global weak solution provided that $\Phi \in L^1_{loc}((0,\infty); L^1_{loc}(\Omega))$ with $\nabla\Phi \in L^2_{loc}((0,\infty); L^\infty(\Omega))$, and χ and f are continuous differentiable satisfying $\chi' \geq 0$, $f \geq 0$ and $f(0) = 0$ [24].

Recently, for sufficiently smooth initial data (n_0, c_0, u_0), Winkler [29] established global weak solutions of (1.1) in bounded convex domains $\Omega \subset \mathbb{R}^3$ under the assumptions $\chi \in C^2([0,\infty))$, $f \in C^1([0,\infty))$ with $f(0) = 0$ and $\Phi \in W^{1,\infty}(\Omega)$. Motivated by the work of [29], our purpose of the present paper is to consider the full chemotaxis-Navier–Stokes system with nonlinear diffusion. In order to formulate our result, we specify the precise mathematical setting: we shall subsequently consider (1.2) along with boundary conditions

$$D(n)\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0, \quad u = 0, \quad x\in\partial\Omega,\ t>0 \tag{1.3}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x\in\Omega \tag{1.4}$$

in a bounded convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary, where we assume

$$\begin{cases} n_0 \in L\log L(\Omega) \text{ is positive},\\ c_0 \in L^\infty(\Omega) \text{ is nonnegative and such that } \sqrt{c_0} \in W^{1,2}(\Omega),\\ u_0 \in L^2_\sigma(\Omega). \end{cases} \tag{1.5}$$

With respect to the parameter function in (1.2), we shall suppose throughout the paper that

$$D(s) \in C^{1+\gamma}_{loc}((0,\infty)) \quad \text{for some } \gamma > 0, \tag{1.6}$$

$$D_1 s^{m-1} \leq D(s) \leq D_2 s^{m-1} \quad \text{for all } s > 0 \tag{1.7}$$

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with $m \geq \frac{2}{3}$ and $D_2 \geq D_1 > 0$, and that

$$\begin{cases} \chi \in C^2([0,\infty)), \quad \chi > 0 \quad \text{in } [0,\infty), \\ f \in C^1([0,\infty)), \quad f(0) = 0, \quad f > 0 \quad \text{in } (0,\infty), \\ \Phi \in W^{1,\infty}(\Omega). \end{cases} \tag{1.8}$$

Moreover, we shall require the further technical assumptions

$$\left(\frac{f}{\chi}\right)' > 0, \quad \text{on } [0,\infty) \tag{1.9}$$

$$\left(\frac{f}{\chi}\right)'' \leq 0, \quad \text{on } [0,\infty) \tag{1.10}$$

and

$$(\chi \cdot f)' \geq 0, \quad \text{on } [0,\infty). \tag{1.11}$$

Our main result reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary and $\kappa \in \mathbb{R}$. Suppose that the assumptions (1.5)–(1.11) hold. Then there exists at least one global weak solution (in the sense of Definition 6.1 below) of (1.2)–(1.4) such that*

$$n^{\frac{m}{2}} \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \quad \textit{and} \quad c^{\frac{1}{4}} \in L^4_{loc}([0,\infty); W^{1,4}(\Omega)).$$

Remark 1.1. (i) If the diffusion function $D(u) \equiv 1$ in (1.2), this is consistent with the result of [29].

(ii) Theorem 1.1 shows that the model (1.2)–(1.4) possesses a global weak solution even when the diffusion effect is rather mild. However, we have to leave open here whether the lower bound of diffusion exponent $m = \frac{2}{3}$ is optimal to guarantee global weak solvability.

The rest of this paper is organized as follows. In Section 2, we introduce a family of regularized problems and give some preliminary properties. Based on an energy-type inequality, a priori estimates are given in Section 3. Section 4 is devoted to showing the global existence of the regularized problems. In Section 5, we further establish some ε-independent estimates. Finally, we give the proof of the main result in Section 6.

Notations. Throughout the paper, for any vectors $v \in \mathbb{R}^3$ and $w \in \mathbb{R}^3$, we denote by $v \otimes w$ the matrix $A_{3\times 3}$ with $a_{ij} = v_i w_j$ for $i, j \in \{1,2,3\}$. We set $L\log L(\Omega)$ is the standard Orlicz space and $L^2_\sigma(\Omega) := \left\{\varphi \in L^2(\Omega) | \nabla \cdot \varphi = 0\right\}$ denotes the Hilbert space of all solenoidal vectors in $L^2(\Omega)$. As usual $\mathcal{P}$ denotes the Helmholtz projection in $L^2(\Omega)$. We write $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ and $C^\infty_{0,\sigma}(\Omega) := C^\infty_0(\Omega) \cap L^2_\sigma(\Omega)$. We represent A as the realization of Stokes operator $-\mathcal{P}\Delta$ in $L^2_\sigma(\Omega)$ with domain $D(A) := W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega)$. Also $n(\cdot,t)$, $c(\cdot,t)$ and $u(\cdot,t)$ will be denoted sometimes by $n(t)$, $c(t)$ and $u(t)$.

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2. Regularized problem

Our intention is to construct a global weak solution as the limit of smooth solutions of appropriately regularized problems. According to the idea from [29] (see also [20]), let us first consider the approximate problems

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot (D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla \cdot (n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - F_\varepsilon(n_\varepsilon) f(c_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla \Phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_{0\varepsilon}(x), \quad c_\varepsilon(x,0) = c_{0\varepsilon}(x), \quad u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega \end{cases} \tag{2.1}$$

for $\varepsilon \in (0,1)$, where the approximate initial data $n_{0\varepsilon} \geq 0$, $c_{0\varepsilon} \geq 0$ and $u_{0\varepsilon}$ satisfy

$$\begin{cases} n_{0\varepsilon} \in C_0^\infty(\Omega), \\ \int_\Omega n_{0\varepsilon} = \int_\Omega n_0, \\ n_{0\varepsilon} \to n_0, \quad \varepsilon \to 0 \quad \text{in } L\log L(\Omega), \end{cases} \tag{2.2}$$

$$\begin{cases} \sqrt{c_{0\varepsilon}} \in C_0^\infty(\Omega), \\ \|c_{0\varepsilon}\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}, \\ \sqrt{c_{0\varepsilon}} \to \sqrt{c_0}, \quad \varepsilon \to 0 \quad \text{a.e. in } \Omega \text{ and in } W^{1,2}(\Omega), \end{cases} \tag{2.3}$$

and

$$\begin{cases} u_{0\varepsilon} \in C_{0,\sigma}^\infty(\Omega), \\ \|u_{0\varepsilon}\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}, \\ u_{0\varepsilon} \to u_0, \quad \varepsilon \to 0 \quad \text{in } L^2(\Omega). \end{cases} \tag{2.4}$$

The approximate functions in (2.1) can be chosen as

$$\begin{aligned} D_\varepsilon(s) &:= D(s+\varepsilon), \quad \text{for all } s \geq 0, \\ F_\varepsilon(s) &:= \frac{1}{\varepsilon}\ln(1+\varepsilon s), \quad \text{for all } s \geq 0, \end{aligned} \tag{2.5}$$

and the standard Yosida approximate Y_ε [17] is defined by

$$Y_\varepsilon v := (1+\varepsilon A)^{-1} v, \quad \text{for all } v \in L_\sigma^2(\Omega).$$

It is easy to verify our choice of F_ε above guarantees that for each $\varepsilon \in (0,1)$

$$0 \leq F_\varepsilon'(s) = \frac{1}{1+\varepsilon s} \leq 1, \quad \text{for all } s \geq 0, \tag{2.6}$$

$$sF_\varepsilon'(s) = \frac{s}{1+\varepsilon s} \leq \frac{1}{\varepsilon}, \quad \text{for all } s \geq 0, \tag{2.7}$$

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$$0 \le F_\varepsilon(s) \le s, \quad \text{for all } s \ge 0, \tag{2.8}$$

and

$$F_\varepsilon(s) \to 1, \quad F'_\varepsilon(s) \to 1, \quad \text{as } \varepsilon \to 0 \quad \text{for all } s \ge 0.$$

The first lemma concerns the local solvability of the approximate problems (2.1). The proof is based on well-established methods involving the Schauder fixed point theorem, the standard regularity theory of parabolic equations and the Stokes system (for details see [20,28,29], for instance).

Lemma 2.1. *For any $\varepsilon \in (0,1)$, there exist a maximal existence time $T_{\max,\varepsilon} \in (0,\infty]$ and functions $n_\varepsilon > 0$, $c_\varepsilon > 0$ and u_ε fulfilling*

$$\begin{aligned}
&n_\varepsilon \in C^0\left(\bar\Omega \times [0,T_{\max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0,T_{\max,\varepsilon})\right),\\
&c_\varepsilon \in C^0\left(\bar\Omega \times [0,T_{\max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0,T_{\max,\varepsilon})\right) \quad \text{and}\\
&u_\varepsilon \in C^0\left(\bar\Omega \times [0,T_{\max,\varepsilon})\right) \cap C^{2,1}\left(\bar\Omega \times (0,T_{\max,\varepsilon})\right)
\end{aligned}$$

such that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is a classical solution of (2.1) in $\Omega \times (0,T_{\max,\varepsilon})$. Moreover, if $T_{\max,\varepsilon} < \infty$, then

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|u_\varepsilon(\cdot,t)\|_{D(A^\alpha)} \to \infty, \quad t \to T_{\max,\varepsilon}$$

for all $q > 3$ and $\alpha > \frac{3}{4}$.

The following estimates of n_ε and c_ε are basic but important in the proof of our result.

Lemma 2.2. *For each $\varepsilon \in (0,1)$, we have*

$$\int_\Omega n_\varepsilon(\cdot,t) = \int_\Omega n_0 \quad \text{for all } t \in (0,T_{\max,\varepsilon}) \tag{2.9}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} =: M \quad \text{in } \Omega \times (0,T_{\max,\varepsilon}). \tag{2.10}$$

Proof. Integrating the first equation in (2.1) and using (2.2), we obtain (2.9). Since $f \ge 0$ by our assumption (1.8) and $F_\varepsilon \ge 0$ by (2.8), an application of the maximum principle to the second equation in (2.1) gives (2.10). □

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3. An energy-type inequality

In this section, we shall utilize an energy inequality associated with the first two equations in (2.1) to establish a priori estimates. The main idea of the proof is similar to the strategy introduced in [29, Section 3]. However, since the nonlinear diffusion case is involved in the computations, we prefer to give enough details for the convenience of the reader. The first inequality will play an important role in our proof.

Lemma 3.1. *Let (1.6)–(1.11) hold. There exists $K \geq 1$ such that for any $\varepsilon \in (0,1)$, the solution of (2.1) satisfies*

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2\right\} + \frac{1}{K}\left\{\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\right\}$$
$$\leq K\int_\Omega |\nabla u_\varepsilon|^2 \quad \textit{for all } t \in (0, T_{\max,\varepsilon}), \tag{3.1}$$

where $\Psi(s) := \int_1^s \frac{d\sigma}{\sqrt{g(\sigma)}}$ with $g(s) := \frac{f(s)}{\chi(s)}$.

Proof. The proof is based on the first two equations in (2.1) and integration by parts and detailed computations can be found in [29, Lemmas 3.1–3.4]. □

Based on Lemma 3.1, we can modify the above energy-type inequality (3.1) to contain all components of n_ε, c_ε and u_ε.

Lemma 3.2. *Let Ψ be as given by Lemma 3.1 and suppose that (1.6)–(1.11) hold. Then for any $\varepsilon \in (0,1)$, there exists $C > 0$ such that*

$$\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + K\int_\Omega |u_\varepsilon|^2\right\}$$
$$+ \frac{1}{2K}\left\{D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2\right\}$$
$$\leq C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}), \tag{3.2}$$

where D_1 and K are constants provided by (1.7) and Lemma 3.1, respectively.

Proof. Multiplying both sides of the third equation in (2.1) by u_ε and integrating by parts over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = -\kappa\int_\Omega (Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\cdot u_\varepsilon + \int_\Omega n_\varepsilon\nabla\Phi\cdot u_\varepsilon$$

for all $t \in (0, T_{\max,\varepsilon})$. Since $\nabla\cdot u_\varepsilon = 0$ implies $\nabla\cdot Y_\varepsilon u_\varepsilon = 0$, we thereby obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega n_\varepsilon\nabla\Phi\cdot u_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

Substituting this into (3.1), we get

$$\begin{aligned}
&\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + K\int_\Omega |u_\varepsilon|^2\right\} \\
&\quad + \frac{1}{K}\left\{\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\right\} + K\int_\Omega |\nabla u_\varepsilon|^2 \\
&\le 2K\int_\Omega n_\varepsilon \nabla\Phi\cdot u_\varepsilon \quad \text{for all } t\in(0,T_{\max,\varepsilon}).
\end{aligned} \tag{3.3}$$

Using (1.7) and (2.5), we have for each $\varepsilon\in(0,1)$

$$\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2 \ge D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 \quad \text{for all } t\in(0,T_{\max,\varepsilon}). \tag{3.4}$$

By (1.8), Hölder's inequality and the embedding $W^{1,2}(\Omega)\hookrightarrow L^6(\Omega)$ for $n=3$, we can find $C_1>0$ such that for each $\varepsilon\in(0,1)$

$$\begin{aligned}
2K\int_\Omega n_\varepsilon\nabla\Phi\cdot u_\varepsilon &\le 2K\|\Phi\|_{W^{1,\infty}}\|n_\varepsilon+\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)} \\
&\le C_1\|n_\varepsilon+\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla u_\varepsilon\|_{L^2(\Omega)} \quad \text{for all } t\in(0,T_{\max,\varepsilon}).
\end{aligned}$$

Note that

$$\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)} \le \left(\|n_0\|_{L^1(\Omega)}+|\Omega|\right)^{\frac{m}{2}} \quad \text{for all } t\in(0,T_{\max,\varepsilon}) \tag{3.5}$$

by (2.2) and (2.9). It follows from the Gagliardo–Nirenberg inequality [25] that

$$\begin{aligned}
\|n_\varepsilon+\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} &= \left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{12}{5m}}(\Omega)}^{\frac{2}{m}} \\
&\le C_2\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^2(\Omega)}^{\frac{1}{3m-1}}\cdot\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{5m-2}{m(3m-1)}} + \left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{m}}\right) \\
&\le C_3\left(\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2\right)^{\frac{1}{2(3m-1)}}+1\right) \quad \text{for all } t\in(0,T_{\max,\varepsilon})
\end{aligned}$$

with $C_2>0$ and $C_3>0$. Since $\frac{1}{3m-1}\le 1$ by our assumption $m\ge\frac{2}{3}$, Young's inequality yields $C_4>0$ such that

$$\begin{aligned}
2K\int_\Omega n_\varepsilon\nabla\Phi\cdot u_\varepsilon &\le C_1C_3\left(\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2\right)^{\frac{1}{2(3m-1)}}+1\right)\cdot\|\nabla u_\varepsilon\|_{L^2(\Omega)} \\
&\le \frac{1}{K}C_1C_3\left(\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2\right)^{\frac{1}{3m-1}}+1\right)+\frac{K}{2}\int_\Omega|\nabla u_\varepsilon|^2
\end{aligned}$$

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$$\leq \frac{D_1}{2K}\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \frac{K}{2}\int_\Omega |\nabla u_\varepsilon|^2 + C_4 \tag{3.6}$$

for all $t \in (0, T_{\max,\varepsilon})$. Inequality (3.2) then follows by combining (3.3), (3.4) and (3.6). □

We can now use Lemma 3.2 to establish a priori estimates for the solution of (2.1).

Lemma 3.3. *Let Ψ and K be as given by Lemma 3.1, and assume that the requirements of Lemma 3.2 are satisfied. Then there exists $C \geq 0$ such that for any $\varepsilon \in (0,1)$ we have*

$$\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + K\int_\Omega |u_\varepsilon|^2 \leq C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}) \tag{3.7}$$

and

$$D_1\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \int_0^T\int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_0^T\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_0^T\int_\Omega |\nabla u_\varepsilon|^2 \leq C(T+1) \tag{3.8}$$

for all $T \in (0, T_{\max,\varepsilon})$.

Proof. Set

$$y_\varepsilon(t) := \int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 + K\int_\Omega |u_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{3.9}$$

and

$$h_\varepsilon(t) := D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \int_\Omega \frac{|D^2 c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega |\nabla u_\varepsilon|^2$$

for all $t \in (0, T_{\max,\varepsilon})$. Then (3.2) implies

$$y_\varepsilon'(t) + \frac{1}{2K}h_\varepsilon(t) \leq C \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.10}$$

In order to introduce dissipative term in (3.10), we show that $y_\varepsilon(t)$ is dominated by $h_\varepsilon(t)$. Now using the inequality

$$z\ln z \leq \frac{3}{3m-1} z^{m+\frac{2}{3}} \quad \text{for all } z > 0$$

with $m \geq \frac{2}{3}$, we can find positive constants C_1, C_2 and C_3 fulfilling for each $\varepsilon \in (0,1)$

$$\begin{aligned}\int_\Omega n_\varepsilon \ln n_\varepsilon &\leq \frac{3}{3m-1}\int_\Omega (n_\varepsilon+\varepsilon)^{m+\frac{2}{3}} \\ &= \frac{3}{3m-1}\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2(3m+2)}{3m}}(\Omega)}^{\frac{2(3m+2)}{3m}}\end{aligned}$$

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$$
\begin{aligned}
&\le \frac{3C_1}{3m-1}\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^2(\Omega)}^{\frac{3m}{3m+2}}\cdot\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{3m+2}}+\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}\right)^{\frac{2(3m+2)}{3m}}\\
&\le C_2\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^2(\Omega)}^{2}\cdot\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{4}{3m}}+\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2(3m+2)}{3m}}\right)\\
&\le C_3\left(\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2+1\right)\quad \text{for all } t\in(0,T_{\max,\varepsilon}) \qquad (3.11)
\end{aligned}
$$

by the Gagliardo–Nirenberg inequality and (3.5). According to (1.8), we have

$$
g(s):=\frac{f(s)}{\chi(s)}\in C^1([0,M])\quad\text{and}\quad g(0)=0.
$$

Hence the mean value theorem yields $g(s)\ge \min_{\tau\in[0,M]} g'(\tau)s =: \bar{M}s$. We now apply Young's inequality and (2.10) to obtain

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla\Psi(c_\varepsilon)|^2 &= \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{g(c_\varepsilon)}\\
&\le \frac{1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{1}{4}\int_\Omega \frac{c_\varepsilon^3}{g^2(c_\varepsilon)}\\
&\le \frac{1}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{M|\Omega|}{4\bar{M}^2}\quad \text{for all } t\in(0,T_{\max,\varepsilon}). \qquad (3.12)
\end{aligned}
$$

From the Poincaré inequality, we have $C_4>0$ such that

$$
K\int_\Omega |u_\varepsilon|^2\le C_4\int_\Omega|\nabla u_\varepsilon|^2\quad\text{for all } t\in(0,T_{\max,\varepsilon}). \qquad (3.13)
$$

It follows easily from (3.11)–(3.13) that

$$
y_\varepsilon(t)\le C_5 h_\varepsilon(t)+C_6\quad\text{for all } t\in(0,T_{\max,\varepsilon})
$$

with $C_5:=\max\left\{\frac{D_1}{C_3},\frac{1}{4},C_4\right\}$ and $C_6:=C_3+\frac{M|\Omega|}{4\bar{M}^2}$. This, along with (3.10), yields

$$
y_\varepsilon'(t)+\frac{1}{4KC_5}y_\varepsilon(t)+\frac{1}{4K}h_\varepsilon(t)\le C_7:=C+\frac{C_6}{4KC_5}\quad\text{for all } t\in(0,T_{\max,\varepsilon}).
$$

Noting that $h_\varepsilon(t)\ge 0$, a standard ODE comparison argument implies

$$
y_\varepsilon(t)\le\max\left\{\sup_{\varepsilon\in(0,1)} y_\varepsilon(0),\quad 4KC_5C_7\right\}\quad\text{for all } t\in(0,T_{\max,\varepsilon}). \qquad (3.14)
$$

In view of (2.2)–(2.4) and [29, Lemma 3.7], we obtain (3.7). On the other hand, since $z\ln z\ge -\frac{1}{e}$ for all $z>0$, we have $y_\varepsilon(t)\ge -\frac{|\Omega|}{e}$. Therefore, a time integration of (3.14) directly leads to (3.8). □

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4. Global existence for the regularized problem (2.1)

With Lemma 3.3 at hand, we are now in the position to show the solution of the approximate problem (2.1) is actually global in time. The idea of the proof is based on the argument in [29, Section 3] for the linear diffusion case $m = 1$.

Lemma 4.1. *For each $\varepsilon \in (0, 1)$, the solutions of (2.1) are global in time.*

Proof. Assume for contradiction that $T_{\max,\varepsilon} < \infty$ for some $\varepsilon \in (0, 1)$. By Lemma 3.3, we obtain $C_1 > 0$ and $C_2 > 0$ such that

$$\int_\Omega |u_\varepsilon|^2 \le C_1 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{4.1}$$

and

$$\begin{aligned} \int_0^{T_{\max,\varepsilon}} \int_\Omega |\nabla c_\varepsilon|^4 &= \int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} c_\varepsilon^3 \\ &\le M^3 \int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \\ &\le C_2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \end{aligned} \tag{4.2}$$

Multiplying the first equation in (2.1) by $p(n_\varepsilon + \varepsilon)^{p-1}$ with $p \in [m + 1, 2(m + 1)]$ and using integration by parts we obtain

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^p + p(p-1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2 \\ &= p(p-1) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} n_\varepsilon F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot \nabla n_\varepsilon \end{aligned} \tag{4.3}$$

for all $t \in (0, T_{\max,\varepsilon})$. We deduce from (1.7), (2.7) and Young's inequality that

$$\begin{aligned} &\frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^p + p(p-1) D_1 \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 \\ &\le \frac{p(p-1)}{\varepsilon} \max_{s \in [0,M]} \chi(s) \int_\Omega (n_\varepsilon + \varepsilon)^{p-2} \nabla c_\varepsilon \cdot \nabla n_\varepsilon \\ &\le p(p-1) D_1 \int_\Omega (n_\varepsilon + \varepsilon)^{m+p-3} |\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon + \varepsilon)^{2(p-m-1)} + C_3 \int_\Omega |\nabla c_\varepsilon|^4 \end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ with some $C_3 > 0$. (Here and below in this section the constants may depend on ε.) Since $2(p - m - 1) \le p$ for $p \le 2(m + 1)$, applying Young's inequality again, we obtain

$$\frac{d}{dt} \int_\Omega (n_\varepsilon + \varepsilon)^p \le \int_\Omega (n_\varepsilon + \varepsilon)^p + C_3 \int_\Omega |\nabla c_\varepsilon|^4 + C_4 \quad \text{for all } t \in (0, T_{\max,\varepsilon})$$

with some $C_4 > 0$. Integrating this yields $C_5 > 0$ such that

$$\int_\Omega (n_\varepsilon + \varepsilon)^p \le C_5 \quad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{4.4}$$

where $p \in [1, 2(m+1)]$.

We now prove the boundedness of u_ε. From (4.1), we can find $C_6 > 0$ and $C_7 > 0$ fulfilling

$$\begin{aligned} \|Y_\varepsilon u_\varepsilon(t)\|_{L^\infty(\Omega)} &= \|(1+\varepsilon A)^{-1} u_\varepsilon(t)\|_{L^\infty(\Omega)} \\ &\le C_6 \|u_\varepsilon(t)\|_{L^2(\Omega)} \\ &\le C_7 \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \end{aligned} \tag{4.5}$$

due to the embedding $D(1+\varepsilon A) \hookrightarrow L^\infty(\Omega)$. We apply the Helmholtz projection $\mathcal{P}$ to the third equation in (2.1), test the resulting identity by Au_ε and integrate by parts over Ω to have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2 + \int_\Omega |Au_\varepsilon|^2 = \int_\Omega \mathcal{P}H_\varepsilon \cdot Au_\varepsilon \quad \text{for all } t \in (0, T_{\max,\varepsilon})$$

with $H_\varepsilon(x,t) := n_\varepsilon \nabla\Phi - \kappa(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon$, where we have used

$$\int_\Omega \phi \cdot A\phi = \int_\Omega |A^{\frac{1}{2}}\phi|^2 = \int_\Omega |\nabla \phi|^2 \quad \text{for all } \phi \in D(A). \tag{4.6}$$

Applying Young's inequality, $\|\mathcal{P}\phi\|_{L^2(\Omega)} \le \|\phi\|_{L^2(\Omega)}$ for all $\phi \in L^2(\Omega)$ [17, Lemma 2.5.2], (4.5) and (1.8), we can estimate

$$\begin{aligned} \int_\Omega \mathcal{P}H_\varepsilon \cdot Au_\varepsilon &\le \int_\Omega |Au_\varepsilon|^2 + \frac{1}{4}\int_\Omega |\mathcal{P}H_\varepsilon|^2 \\ &\le \int_\Omega |Au_\varepsilon|^2 + \frac{|\kappa|}{2}\int_\Omega |(Y_\varepsilon u_\varepsilon \cdot \nabla)u_\varepsilon|^2 + \frac{1}{2}\int_\Omega |n_\varepsilon \nabla\Phi|^2 \\ &\le \int_\Omega |Au_\varepsilon|^2 + C_8\left(\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega n_\varepsilon^2\right) \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \end{aligned}$$

with some $C_8 > 0$. Hence we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}} u_\varepsilon|^2 \le C_8\left(\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega n_\varepsilon^2\right) \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

This, along with (4.4) and (4.6), gives $C_9 > 0$ such that

$$\int_\Omega |\nabla u_\varepsilon|^2 \le C_9 \quad \text{for all } t \in (0, T_{\max,\varepsilon}).$$

We thereby obtain

$$\|\mathcal{P}H_\varepsilon(t)\|_{L^2(\Omega)} \le C_{10} \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{4.7}$$

with some $C_{10} > 0$. Applying the fractional power A^α with $\alpha \in (\frac{3}{4}, 1)$ to both sides of the variation-of-constants formula

$$u_\varepsilon(t) = e^{-tA}u_{0\varepsilon} + \int_0^t e^{-(t-s)A}\mathcal{P}H_\varepsilon(s)ds \quad \text{for all } t \in (0, T_{\max,\varepsilon}),$$

using the well-known smoothing estimate of the Stokes semigroup [8] and (4.7), we have $C_{11} > 0$ and $C_{12} > 0$ satisfying

$$\begin{aligned}\|A^\alpha u_\varepsilon(t)\|_{L^2(\Omega)} &\le \|A^\alpha e^{-tA}u_{0\varepsilon}\|_{L^2(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}H_\varepsilon(s)\|_{L^2(\Omega)}ds \\ &\le C_{11}t^{-\alpha}\|u_{0\varepsilon}\|_{L^2(\Omega)} + C_{11}\int_0^t (t-s)^{-\alpha}\|\mathcal{P}H_\varepsilon(s)\|_{L^2(\Omega)}ds \\ &\le C_{12} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon})\end{aligned}$$

with any $\tau \in (0, T_{\max,\varepsilon})$. In view of the embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ asserted by our choice of α [17, Lemma 2.4.3], we can find $C_{13} > 0$ and $C_{14} > 0$ such that

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le C_{13}\|A^\alpha u_\varepsilon(t)\|_{L^2(\Omega)} \le C_{14} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon}). \tag{4.8}$$

Let $r := \min\{2(m+1), 4\}$, then $r > 3$ due to $m \ge \frac{2}{3}$. Employing ∇ to both sides of the variation-of-constants formula for c_ε

$$c_\varepsilon(t) = e^{(t-\frac{\tau}{2})\Delta}c_\varepsilon(\frac{\tau}{2}) - \int_{\frac{\tau}{2}}^t e^{(t-s)\Delta}\left(F_\varepsilon(n_\varepsilon)f(c_\varepsilon) + u_\varepsilon \cdot \nabla c_\varepsilon\right)(s)ds \quad \text{for all } t \in (\frac{\tau}{2}, T_{\max,\varepsilon}),$$

recalling the standard smoothing estimates of Neumann heat semigroup ([27, Lemma 1.3], see also [15]), we have C_{15} such that

$$\begin{aligned}\|\nabla c_\varepsilon(t)\|_{L^r(\Omega)} &\le \left\|\nabla e^{(t-\frac{\tau}{2})\Delta}c_\varepsilon(\frac{\tau}{2})\right\|_{L^r(\Omega)} + \int_{\frac{\tau}{2}}^t \left\|\nabla e^{(t-s)\Delta}\left(F_\varepsilon(n_\varepsilon)f(c_\varepsilon) + u_\varepsilon \cdot \nabla c_\varepsilon\right)(s)\right\|_{L^r(\Omega)} ds \\ &\le C_{15}\left(t - \frac{\tau}{2}\right)^{-\frac{1}{2}}\left\|c_\varepsilon(\frac{\tau}{2})\right\|_{L^r(\Omega)} + C_{15}\int_{\frac{\tau}{2}}^t (t-s)^{-\frac{1}{2}}\|(F_\varepsilon(n_\varepsilon)f(c_\varepsilon))(s)\|_{L^r(\Omega)}ds \\ &\quad + C_{15}\int_{\frac{\tau}{2}}^t (t-s)^{-\frac{1}{2}}\|(u_\varepsilon \cdot \nabla c_\varepsilon)(s)\|_{L^r(\Omega)}ds \quad \text{for all } t \in (\frac{\tau}{2}, T_{\max,\varepsilon}).\end{aligned}$$

Hence, an application of (2.8), (2.10), (4.8), (4.4), (4.2) and the Hölder inequality yields positive constants C_{16}, C_{17}, C_{18}, C_{19} and C_{20} fulfilling

$$\begin{aligned}\|\nabla c_\varepsilon(t)\|_{L^r(\Omega)} &\le C_{16}\tau^{-\frac{1}{2}} + C_{17}\int_0^t (t-s)^{-\frac{1}{2}}ds + C_{18}\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla c_\varepsilon(s)\|_{L^4(\Omega)}ds \\ &\le C_{16}\tau^{-\frac{1}{2}} + 2C_{17}T_{\max,\varepsilon}^{\frac{1}{2}} + C_{18}\left(\int_0^{T_{\max,\varepsilon}}(t-s)^{-\frac{2}{3}}ds\right)^{\frac{3}{4}}\left(\int_0^{T_{\max,\varepsilon}}\int_\Omega |\nabla c_\varepsilon(s)|^4\right)^{\frac{1}{4}}\end{aligned}$$

$$
\begin{aligned}
&\le C_{16}\tau^{-\frac{1}{2}} + 2C_{17}T_{\max,\varepsilon}^{\frac{1}{2}} + C_{19}T_{\max,\varepsilon}^{\frac{1}{4}} \\
&\le C_{20} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon}).
\end{aligned} \tag{4.9}
$$

We next rewrite the variation-of-constants formula for c_ε in the form

$$
c_\varepsilon(t) = e^{t(\Delta-1)}c_{0,\varepsilon} + \int_0^t e^{(t-s)(\Delta-1)}\left(c_\varepsilon - F_\varepsilon(n_\varepsilon)f(c_\varepsilon) - u_\varepsilon\cdot\nabla c_\varepsilon\right)(s)ds
$$

for all $t \in (0, T_{\max,\varepsilon})$. Picking $\theta \in (\frac{1}{2} + \frac{3}{2r}, 1)$, then the domain of the fractional power $D((-\Delta + 1)^\theta) \hookrightarrow W^{1,\infty}(\Omega)$ [26,33]. Hence, by virtue of L^p-L^q estimates associated heat semigroup [26], (2.10), (2.8), (1.8), (4.4), (4.8) and (4.9), there exist positive constants C_{21}, C_{22}, C_{23}, C_{24} and C_{25} such that

$$
\begin{aligned}
\|c_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} &\le C_{21}\left\|(-\Delta+1)^\theta c_\varepsilon(t)\right\|_{L^r(\Omega)} \\
&\le C_{22}t^{-\theta}e^{-\nu t}\left\|c_{0,\varepsilon}\right\|_{L^r(\Omega)} \\
&\quad + C_{22}\int_0^t (t-s)^{-\theta}e^{-\nu(t-s)}\left\|\left(c_\varepsilon - F_\varepsilon(n_\varepsilon)f(c_\varepsilon) - u_\varepsilon\cdot\nabla c_\varepsilon\right)(s)\right\|_{L^r(\Omega)}ds \\
&\le C_{22}\tau^{-\theta}e^{-\nu t}\left\|c_{0,\varepsilon}\right\|_{L^r(\Omega)} + C_{23}\int_0^t (t-s)^{-\theta}e^{-\nu(t-s)}ds \\
&\quad + C_{23}\int_0^t (t-s)^{-\theta}e^{-\nu(t-s)}\|n_\varepsilon(s)\|_{L^r(\Omega)}ds \\
&\quad + C_{23}\int_0^t (t-s)^{-\theta}e^{-\nu(t-s)}\|\nabla c_\varepsilon(s)\|_{L^r(\Omega)}ds \\
&\le C_{22}\tau^{-\theta}e^{-\nu t}\left\|c_{0,\varepsilon}\right\|_{L^r(\Omega)} + C_{24}\Gamma(1-\theta) \\
&\le C_{25} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon})
\end{aligned} \tag{4.10}
$$

with some $\nu > 0$, where $\Gamma(\cdot)$ is the Gamma function. We may then apply (4.3) once more and Young's inequality to obtain positive constants C_{26}, C_{27} and C_{28} satisfying

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + p(p-1)D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \\
&\le C_{26}\int_\Omega (n_\varepsilon+\varepsilon)^{p-2}|\nabla n_\varepsilon| \\
&\le p(p-1)D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + C_{27}\int_\Omega (n_\varepsilon+\varepsilon)^{p-m-1} \\
&\le p(p-1)D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + \int_\Omega (n_\varepsilon+\varepsilon)^p + C_{28} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon}).
\end{aligned}
$$

Therefore, integrating with respect to t yields $C_{29} > 0$ such that

$$
\int_\Omega n_\varepsilon^p \le C_{29} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon})
$$

with any $p \geq 1$. Upon an application of the well-known Moser–Alikakos iteration procedure [1,18], we see that

$$\|n_\varepsilon(t)\|_{L^\infty(\Omega)} \leq C_{30} \quad \text{for all } t \in (\tau, T_{\max,\varepsilon}) \tag{4.11}$$

with some $C_{30} > 0$. In view of (4.8) and (4.11), we apply Lemma 2.1 to reach a contradiction. □

5. Further ε-independent estimates for (2.1)

In order to pass to limits in (2.1) with safety, we need some more ε-independent estimates for the solution.

Lemma 5.1. *Suppose that (1.6)–(1.11) hold. There exists $C > 0$ such that for all $\varepsilon \in (0, 1)$, the solutions of (2.1) satisfy*

$$\int_0^T \int_\Omega \left|\nabla n_\varepsilon^{\frac{m}{2}}\right|^2 \leq C(T+1), \quad \textit{for all } T > 0, \tag{5.1}$$

$$\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^p \leq C(T+1), \quad 1 \leq p \leq \frac{3m+2}{3} \quad \textit{for all } T > 0, \tag{5.2}$$

$$\int_0^T \int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \leq C(T+1) \quad \textit{for all } T > 0, \tag{5.3}$$

$$\int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon)^{\frac{3m+2}{3m+1}} \leq C(T+1) \quad \textit{for all } T > 0 \tag{5.4}$$

and

$$\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \leq C(T+1) \quad \textit{for all } T > 0. \tag{5.5}$$

Moreover, if $\frac{2}{3} \leq m \leq 2$, then we have

$$\int_0^T \int_\Omega |\nabla n_\varepsilon|^{\frac{3m+2}{4}} \leq C(T+1), \quad \textit{for all } T > 0. \tag{5.6}$$

Proof. From Lemma 3.3 we know that there exists $C_1 > 0$ such that

$$\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 \leq C_1(T+1) \quad \text{for all } T > 0. \tag{5.7}$$

Then, (5.1) is a direct consequence of (5.7). Due to the fact that Ω is bounded, we only need to prove (5.2) with $p = \frac{3m+2}{3}$. We employ the Gagliardo–Nirenberg inequality to find $C_2 > 0$ and

$C_3 > 0$ fulfilling

$$\begin{aligned}
&\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{3m+2}{3}} \\
&= \int_0^T \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2(3m+2)}{3m}}(\Omega)}^{\frac{2(3m+2)}{3m}} \\
&\le C_2 \left(\left\| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^2(\Omega)}^{\frac{3m}{3m+2}} \cdot \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{3m+2}} + \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)} \right)^{\frac{2(3m+2)}{3m}} \\
&= C_2 \left(\left\| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^2(\Omega)}^{2} \cdot \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{4}{3m}} + \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2(3m+2)}{3m}} \right) \\
&\le C_3(T+1) \quad \text{for all } T > 0.
\end{aligned}$$

Recalling the proof in Lemma 3.2, we obtain

$$y_\varepsilon'(t) + \frac{1}{2K} \int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \le K \quad \text{for all } T > 0,$$

where y_ε and K are provided by (3.9) and Lemma 3.1, respectively. Integrating this in time over $(0, T)$ yields (5.3). By Hölder's inequality, (5.2) and (5.3), we can find $C_4 > 0$ such that

$$\begin{aligned}
&\int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon) \nabla n_\varepsilon)^{\frac{3m+2}{3m+1}} \\
&\le \left(\int_0^T \int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \right)^{\frac{3m+2}{6m+2}} \left(\int_0^T \int_\Omega (D_\varepsilon(n_\varepsilon) n_\varepsilon)^{\frac{3m+2}{3m}} \right)^{\frac{3m}{6m+2}} \\
&\le D_2^{\frac{3m+2}{6m+2}} \left(\int_0^T \int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon} |\nabla n_\varepsilon|^2 \right)^{\frac{3m+2}{6m+2}} \left(\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{\frac{3m+2}{3}} \right)^{\frac{3m}{6m+2}} \\
&\le C_4(T+1) \quad \text{for all } T > 0.
\end{aligned}$$

Using the Gagliardo–Nirenberg inequality and Lemma 3.3 again we have

$$\begin{aligned}
\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} &= \int_0^T \|u_\varepsilon(t)\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} \\
&\le C_5 \int_0^T \left(\|\nabla u_\varepsilon(t)\|_{L^2(\Omega)}^2 \cdot \|u_\varepsilon(t)\|_{L^2(\Omega)}^{\frac{4}{3}} + \|u_\varepsilon(t)\|_{L^2(\Omega)}^{\frac{10}{3}} \right) \\
&\le C_6(T+1) \quad \text{for all } T > 0
\end{aligned}$$

with $C_5 > 0$ and $C_6 > 0$. Finally, we prove (5.6). Since $\frac{2}{3} \le m \le 2$, applying (5.7) and Young's inequality we get $C_7 > 0$ and $C_8 > 0$ such that

$$\begin{aligned}\int_0^T\int_\Omega |\nabla n_\varepsilon|^{\frac{3m+2}{4}} &= \int_0^T\int_\Omega n_\varepsilon^{\frac{(m-2)(3m+2)}{8}}|\nabla n_\varepsilon|^{\frac{3m+2}{4}} n_\varepsilon^{\frac{(2-m)(3m+2)}{8}}\\ &\le C_7\left(\int_0^T\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2+\int_0^T\int_\Omega n_\varepsilon^{\frac{3m+2}{3}}\right)\\ &\le C_8(T+1), \quad \text{for all } T>0.\end{aligned}$$

This completes the proof. □

We derive the L^p-bound for $n_\varepsilon+\varepsilon$ and the estimate of space–time integral $\int_0^T\int_\Omega |\nabla(n_\varepsilon+\varepsilon)^{\bar r}|^2$ as a supplement to the regularity property concerning n_ε in the case $m>2$.

Lemma 5.2. *Let $m>\frac{10}{9}$. For all $\varepsilon\in(0,1)$, there exists $C(T)>0$ and $C>0$ such that*

$$\int_\Omega (n_\varepsilon+\varepsilon)^p \le C(T) \quad \text{for all } t>0. \tag{5.8}$$

with $1\le p<9(m-1)$ and

$$\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\bar r}\right|^2 \le C(T+1) \quad \text{for all } T>0. \tag{5.9}$$

with $\frac{m}{2}<\bar r<5(m-1)$.

Proof. It is based on a bootstrap argument ([20], see also [10,21]). We multiply the first equation in (2.1) by $p(n_\varepsilon+\varepsilon)^{p-1}$ to deduce that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + p(p-1)\int_\Omega (n_\varepsilon+\varepsilon)^{p-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\\ &= p(p-1)\int_\Omega n_\varepsilon(n_\varepsilon+\varepsilon)^{p-2}F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla n_\varepsilon\end{aligned} \tag{5.10}$$

for all $t>0$. However, unlike the proof of Lemma 4.1 we deal with $n_\varepsilon F'_\varepsilon(n_\varepsilon)$ together, because our goal is to get an ε-independent bound (5.8). More precisely, from (5.10) and (2.6) we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + p(p-1)D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\ &\le p(p-1)\max_{s\in[0,M]}\chi(s)\int_\Omega (n_\varepsilon+\varepsilon)^{p-1}\nabla c_\varepsilon\cdot\nabla n_\varepsilon\end{aligned} \tag{5.11}$$

for all $t>0$. Applying the Hölder and Young inequalities in the right-hand side of (5.11), we have $C_1>0$ such that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + p(p-1)D_1\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2\\ &\le p(p-1)\max_{s\in[0,M]}\chi(s)\left\|(n_\varepsilon+\varepsilon)^{\frac{m+p-3}{2}}\nabla n_\varepsilon\right\|_{L^2(\Omega)}\cdot\left\|(n_\varepsilon+\varepsilon)^{\frac{p-m+1}{2}}\right\|_{L^4(\Omega)}\cdot\|\nabla c_\varepsilon\|_{L^4(\Omega)}\end{aligned}$$

$$\leq \frac{p(p-1)D_1}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + C_1\left\|(n_\varepsilon+\varepsilon)^{\frac{p-m+1}{2}}\right\|^2_{L^4(\Omega)}\cdot\|\nabla c_\varepsilon\|^2_{L^4(\Omega)} \tag{5.12}$$

for all $t>0$. Assume that

$$\int_\Omega (n_\varepsilon+\varepsilon)^{p_i} \leq C(T) \quad \text{for all } t>0$$

holds with some $p_i \geq 1$ (this is true for $p_1 := 1$ by (2.9) and $\varepsilon < 1$). The Gagliardo–Nirenberg inequality gives $C_2 > 0$ such that

$$\begin{aligned}\left\|(n_\varepsilon+\varepsilon)^{\frac{p-m+1}{2}}\right\|^2_{L^4(\Omega)} &= \left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|^{\frac{2(p-m+1)}{p+m-1}}_{L^{\frac{4(p-m+1)}{p+m-1}}(\Omega)}\\ &\leq C_2\Bigg(\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|^{\alpha}_{L^2(\Omega)}\cdot\left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|^{1-\alpha}_{L^{\frac{2p_i}{p+m-1}}(\Omega)}\\ &\quad + \left\|(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p_i}{p+m-1}}(\Omega)}\Bigg)^{\frac{2(p-m+1)}{p+m-1}}\end{aligned}$$

for all $t>0$, where

$$\alpha = \frac{1-\frac{p_i}{2(p-m+1)}}{1-\frac{p_i}{3(p+m-1)}} \in (0,1).$$

If

$$\frac{2(p-m+1)}{p+m-1}\alpha = 1,$$

which is equivalent to $p = 3(m-1)+\frac{2}{3}p_i$, and therefore by Young's inequality

$$\begin{aligned}&C_1\left\|(n_\varepsilon+\varepsilon)^{\frac{p-m+1}{2}}\right\|^2_{L^4(\Omega)}\cdot\|\nabla c_\varepsilon\|^2_{L^4(\Omega)}\\ &\leq C_3\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}+1\right)\|\nabla c_\varepsilon\|^2_{L^4(\Omega)}\\ &\leq \frac{p(p-1)D_1}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 + C_4\|\nabla c_\varepsilon\|^4_{L^4(\Omega)} + C_5\end{aligned}$$

for all $t>0$ with certain positive constants C_3, C_4 and C_5. Substituting this into (5.12), we obtain

$$\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^p + \frac{p(p-1)D_1}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m+p-3}|\nabla n_\varepsilon|^2 \leq C_4\|\nabla c_\varepsilon\|^4_{L^4(\Omega)} + C_5 \tag{5.13}$$

for all $t>0$. By integration, we finally get

$$\int_\Omega (n_\varepsilon+\varepsilon)^p \leq C(T) \quad \text{for all } t>0$$

with $p = 3(m-1) + \frac{2}{3}p_i$. By this iterative procedure, there exists a sequence $\{p_i\}$ such that

$$p_{i+1} = 3(m-1) + \frac{2}{3}p_i.$$

It is easy to check that the sequence $\{p_i\}$ is increasing and $p_i \to 9(m-1)$ as $i \to \infty$. Therefore, we can reach any $p < 9(m-1)$ by finite steps and (5.8) is thereby proved. Another integration of (5.13) yields

$$\begin{aligned}\int_0^T \int_\Omega \left|\nabla (n_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}}\right|^2 &\le C_6 \left(\int_\Omega (n_{0\varepsilon} + 1)^p + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 + 1 \right) \\ &\le C_7(T+1) \quad \text{for all } T > 0\end{aligned}$$

with $C_6 > 0$ and $C_7 > 0$. This proves (5.9). □

In order to derive strong compactness properties, we also need some estimates concerning the time derivative of the solution.

Lemma 5.3. *Let $\gamma := \max\{1, \frac{m}{2}\}$. There exists $C > 0$ such that for all $\varepsilon \in (0,1)$ we have*

$$\int_0^T \left\| \frac{\partial}{\partial t} n_\varepsilon^\gamma \right\|_{(W^{2,q}(\Omega))^*} dt \le C(T+1) \quad \textit{for all } T > 0 \tag{5.14}$$

with some $q > 3$, and

$$\int_0^T \left\| \frac{\partial}{\partial t} \sqrt{c_\varepsilon} \right\|_{(W^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{3}} dt \le C(T+1) \quad \textit{for all } T > 0 \tag{5.15}$$

as well as

$$\int_0^T \left\| \frac{\partial}{\partial t} u_\varepsilon \right\|_{(W_{0,\sigma}^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{4}} dt \le C(T+1) \quad \textit{for all } T > 0. \tag{5.16}$$

Proof. Multiplying the first equation in (2.1) by $\gamma n_\varepsilon^{\gamma-1}\varphi$ with $\varphi \in C^\infty(\bar{\Omega})$ and integrating by parts, we obtain

$$\begin{aligned}\left| \int_\Omega (n_\varepsilon^\gamma)_t \varphi \right| &= \Big| -\gamma(\gamma-1) \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\gamma-2} |\nabla n_\varepsilon|^2 \varphi - \gamma \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\gamma-1} \nabla n_\varepsilon \cdot \nabla\varphi \\ &\quad + \gamma(\gamma-1) \int_\Omega n_\varepsilon^{\gamma-1} F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot \nabla n_\varepsilon \varphi \\ &\quad + \gamma \int_\Omega n_\varepsilon^\gamma F_\varepsilon'(n_\varepsilon) \chi(c_\varepsilon) \nabla c_\varepsilon \cdot \nabla\varphi + \int_\Omega n_\varepsilon^\gamma u_\varepsilon \cdot \nabla\varphi \Big| \\ &\le \Big(\gamma(\gamma-1) \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\gamma-2} |\nabla n_\varepsilon|^2 + \gamma \int_\Omega D_\varepsilon(n_\varepsilon) n_\varepsilon^{\gamma-1} |\nabla n_\varepsilon|\end{aligned}$$

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$$+\gamma(\gamma-1)\int_\Omega |n_\varepsilon^{\gamma-1}F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla n_\varepsilon|$$
$$+\gamma\int_\Omega |n_\varepsilon^{\gamma}F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon|+\int_\Omega n_\varepsilon^\gamma|u_\varepsilon|\Bigg)\|\varphi\|_{W^{1,\infty}(\Omega)}$$

for all $t>0$. Due to the embedding $W^{2,q}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ for $q>3$, we deduce $C_1>0$ such that

$$\begin{aligned}
&\int_0^T \left\|(n_\varepsilon^\gamma)_t\right\|_{(W^{2,q}(\Omega))^*}dt\\
&\le C_1\Bigg(\gamma(\gamma-1)\int_0^T\int_\Omega D_\varepsilon(n_\varepsilon)n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2+\gamma\int_0^T\int_\Omega D_\varepsilon(n_\varepsilon)n_\varepsilon^{\gamma-1}|\nabla n_\varepsilon|\\
&\quad+\gamma(\gamma-1)\int_0^T\int_\Omega |n_\varepsilon^{\gamma-1}F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla n_\varepsilon|\\
&\quad+\gamma\int_0^T\int_\Omega |n_\varepsilon^{\gamma}F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon|+\int_0^T\int_\Omega n_\varepsilon^\gamma|u_\varepsilon|\Bigg)\\
&=: I_1+I_2+I_3+I_4+I_5
\end{aligned}\tag{5.17}$$

for all $T>0$. By (1.7), (5.9), (5.1) and Young's inequality, we can find $C_2>0$ such that

$$\begin{aligned}
I_1&\le D_2\gamma(\gamma-1)\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^m n_\varepsilon^{\gamma-1}|\nabla n_\varepsilon|^2\\
&\le \frac{D_2(\gamma-1)}{m+1}\int_0^T\int_\Omega \nabla(n_\varepsilon+\varepsilon)^{m+1}\cdot\nabla n_\varepsilon^\gamma\\
&\le \frac{D_2(\gamma-1)}{2(m+1)}\left(\int_0^T\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m+1}\right|^2+\int_0^T\int_\Omega |\nabla n_\varepsilon^\gamma|^2\right)\\
&\le C_2(T+1).
\end{aligned}\tag{5.18}$$

Employing (1.7), (5.3) and Young's inequality we have $C_3>0$ such that

$$\begin{aligned}
I_2&\le \frac{\gamma}{2}\int_0^T\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2+\frac{\gamma}{2}\int_0^T\int_\Omega D_\varepsilon(n_\varepsilon)n_\varepsilon^{2\gamma-1}\\
&\le \frac{\gamma}{2}\int_0^T\int_\Omega \frac{D_\varepsilon(n_\varepsilon)}{n_\varepsilon}|\nabla n_\varepsilon|^2+\frac{\gamma D_2}{2}\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\gamma-2}\\
&\le C_3(T+1),
\end{aligned}\tag{5.19}$$

where we have used when $1\le m\le 2$

$$\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^{m+2\gamma-2}=\int_0^T\int_\Omega (n_\varepsilon+\varepsilon)^m\le C_4(T+1)\quad\text{by (5.2)}$$

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with $C_4 > 0$ and in the case $m > 2$

$$\int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{m+2\gamma-2} = \int_0^T \int_\Omega (n_\varepsilon + \varepsilon)^{2(m-1)} \le C(T) \quad \text{by (5.8)}.$$

From (2.6), (2.10), (4.2) and (5.1), we estimate

$$\begin{aligned} I_3 &\le \gamma(\gamma-1)C_5 \int_0^T \int_\Omega |n_\varepsilon^{\gamma-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon| \\ &\le C_5(\gamma-1) \int_0^T \int_\Omega |\nabla n_\varepsilon^\gamma \cdot \nabla c_\varepsilon| \\ &\le \frac{C_5(\gamma-1)}{2} \left(\int_0^T \int_\Omega |\nabla n_\varepsilon^\gamma|^2 + \frac{1}{2} \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{2} \right) \\ &\le C_6(T+1) \end{aligned} \tag{5.20}$$

with $C_5 > 0$ and $C_6 > 0$. Applying (2.6), (2.10), (4.2) and (5.2), we find $C_7 > 0$ and $C_8 > 0$ such that

$$\begin{aligned} I_4 &\le C_7 \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{4\gamma}{3}} + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \right) \\ &\le C_8(T+1). \end{aligned} \tag{5.21}$$

Moreover, we use (5.2) and (5.5) to give $C_9 > 0$ and $C_{10} > 0$ fulfilling

$$\begin{aligned} I_5 &\le C_9 \left(\int_0^T \int_\Omega n_\varepsilon^{\frac{10\gamma}{7}} + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \right) \\ &\le C_{10}(T+1). \end{aligned} \tag{5.22}$$

Then, (5.14) follows by combining (5.17)–(5.22). Multiplying the second equation in (2.1) by $\frac{\varphi}{2\sqrt{c_\varepsilon}}$ with $\varphi \in C^\infty(\bar{\Omega})$ and the third by $\phi \in \left(C_{0,\sigma}^\infty(\bar{\Omega})\right)^3$, we obtain (5.15) and (5.16) in a completed similar manner (see [29] for details). □

6. Global weak solutions for (1.2)–(1.4)

We are now in the position to construct global weak solutions for (1.2)–(1.4). Before going into details, let us first give the definition of weak solution.

Definition 6.1. We call (n, c, u) a *global weak solution* of (1.2)–(1.4) if

$$n \in L^1_{loc}(\bar{\Omega} \times [0,\infty)), \quad c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad u \in \left(L^1_{loc}([0,\infty); W_0^{1,1}(\Omega))\right)^3$$

such that $n \ge 0$ and $c \ge 0$ *a.e.* in $\Omega \times (0,\infty)$, and that

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$$nf(c) \in L^1_{loc}([0,\infty); L^1(\Omega)),$$
$$D(n)\nabla n,\ n\chi(c)\nabla c,\ nu \text{ and } cu \text{ belong to } \left(L^1_{loc}([0,\infty); L^1(\Omega))\right)^3,$$
$$u\otimes u \in \left(L^1_{loc}([0,\infty); L^1(\Omega))\right)^{3\times 3}$$

and that

$$\int_0^\infty\int_\Omega n_t\phi_1 - \int_0^\infty\int_\Omega nu\cdot\nabla\phi_1 = -\int_0^\infty\int_\Omega D(n)\nabla n\cdot\nabla\phi_1 + \int_0^\infty\int_\Omega n\chi(c)\nabla c\cdot\nabla\phi_1,$$
$$\int_0^\infty\int_\Omega c_t\phi_2 - \int_0^\infty\int_\Omega cu\cdot\nabla\phi_2 = -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\phi_2 - \int_0^\infty\int_\Omega nf(c)\phi_2,$$
$$\int_0^\infty\int_\Omega u_t\cdot\phi_3 - \kappa\int_0^\infty\int_\Omega u\otimes u\cdot\nabla\phi_3 = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\phi_3 + \int_0^\infty\int_\Omega n\nabla\Phi\cdot\phi_3$$

hold for all $\phi_1 \in C_0^\infty(\bar\Omega\times[0,\infty))$, $\phi_2 \in C_0^\infty(\bar\Omega\times[0,\infty))$ and $\phi_3 \in \left(C_0^\infty(\Omega\times[0,\infty))\right)^3$ satisfying $\nabla\cdot\phi_3 = 0$.

We can now pass to the proof of our main result.

Proof of Theorem 1.1. Let γ be given by Lemma 5.1 and set

$$\beta := \begin{cases} \frac{3m+2}{4}, & 1\le m\le 2,\\ 2, & m>2. \end{cases}$$

By Lemma 3.3, Lemma 5.1 and Lemma 5.3, for some $C>0$ which is independent of ε, we have

$$\|n_\varepsilon^\gamma\|_{L^\beta_{loc}([0,\infty);W^{1,\beta}(\Omega))} \le C(T+1),$$
$$\|(n_\varepsilon^\gamma)_t\|_{L^1_{loc}([0,\infty);(W^{2,q}(\Omega))^*)} \le C(T+1) \quad \text{with some } q>3,$$
$$\|\sqrt{c_\varepsilon}\|_{L^2_{loc}([0,\infty);W^{2,2}(\Omega))} \le C(T+1),$$
$$\|(\sqrt{c_\varepsilon})_t\|_{L^{\frac53}_{loc}([0,\infty);(W^{1,\frac52}(\Omega))^*)} \le C(T+1),$$
$$\|u_\varepsilon\|_{L^2_{loc}([0,\infty);W^{1,2}(\Omega))} \le C(T+1), \quad \text{and}$$
$$\|u_{\varepsilon t}\|_{L^{\frac54}_{loc}([0,\infty);(W^{1,\frac52}_{0,\sigma}(\Omega))^*)} \le C(T+1)$$

for all $T>0$. Therefore, the Aubin–Lions lemma ([12], see [16] for the case involving the space L^p with $p=1$) asserts that

$$(n_\varepsilon^\gamma)_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^\beta_{loc}(\bar\Omega\times[0,\infty)),$$
$$(\sqrt{c_\varepsilon})_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^2_{loc}([0,\infty);W^{1,2}(\Omega)) \quad \text{and}$$
$$(u_\varepsilon)_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^2_{loc}(\bar\Omega\times[0,\infty)).$$

This yields a subsequence $\varepsilon := \varepsilon_j \in (0,1)$ $(j \in \mathbb{N})$ and the limit functions n, c and u such that

$$\begin{aligned} n_\varepsilon^\gamma &\to n^\gamma && \text{in } L^\beta_{loc}(\bar{\Omega} \times [0,\infty)), \text{ and a.e. in } \Omega \times (0,\infty), \\ n_\varepsilon &\rightharpoonup n && \text{in } L^{\frac{3m+2}{3}}_{loc}(\bar{\Omega} \times [0,\infty)), \\ D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon &\rightharpoonup D(n)\nabla n && \text{in } L^{\frac{3m+2}{3m+1}}_{loc}(\bar{\Omega} \times [0,\infty)) \end{aligned}$$

and

$$\begin{aligned} \sqrt{c_\varepsilon} &\to \sqrt{c} && \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \text{ and a.e. in } \Omega \times (0,\infty), \\ c_\varepsilon &\overset{*}{\rightharpoonup} c && \text{in } L^\infty(\Omega \times (0,\infty)), \\ \nabla c_\varepsilon^{\frac{1}{4}} &\rightharpoonup \nabla c^{\frac{1}{4}} && \text{in } L^4_{loc}(\bar{\Omega} \times [0,\infty)) \end{aligned}$$

as well as

$$\begin{aligned} u_\varepsilon &\to u && \text{in } L^2_{loc}(\bar{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \\ u_\varepsilon &\overset{*}{\rightharpoonup} u && \text{in } L^\infty([0,\infty); L^2_\sigma(\Omega)), \\ u_\varepsilon &\rightharpoonup u && \text{in } L^{\frac{10}{3}}_{loc}(\bar{\Omega} \times [0,\infty)), \\ \nabla u_\varepsilon &\rightharpoonup \nabla u && \text{in } L^2_{loc}(\bar{\Omega} \times [0,\infty)) \end{aligned}$$

as $\varepsilon \to 0$. Moreover, using interpolation inequality for L^p-norm, we have

$$n_\varepsilon \to n \quad \text{in } L^{\frac{10}{7}}_{loc}(\bar{\Omega} \times [0,\infty))$$

as $\varepsilon \to 0$. According to (2.10) and the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)c_\varepsilon^{\frac{3}{4}} &\to \chi(c)c^{\frac{3}{4}} && \text{in } L^{20}_{loc}(\bar{\Omega} \times [0,\infty)), \\ f(c_\varepsilon) &\to f(c) && \text{in } L^{\frac{10}{3}}_{loc}(\bar{\Omega} \times [0,\infty)), \\ c_\varepsilon &\to c && \text{in } L^2_{loc}(\bar{\Omega} \times [0,\infty)), \\ F_\varepsilon(n_\varepsilon) &\to n && \text{in } L^{\frac{10}{7}}_{loc}(\bar{\Omega} \times [0,\infty)), \\ Y_\varepsilon u_\varepsilon &\to u && \text{in } L^2_{loc}(\bar{\Omega} \times [0,\infty)) \end{aligned}$$

as $\varepsilon \to 0$. Therefore,

$$\begin{aligned} n_\varepsilon u_\varepsilon &\rightharpoonup nu && \text{in } L^1_{loc}(\bar{\Omega} \times [0,\infty)), \\ n_\varepsilon F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon &\rightharpoonup n\chi(c)\nabla c && \text{in } L^1_{loc}(\bar{\Omega} \times [0,\infty)), \\ F_\varepsilon(n_\varepsilon)f(c_\varepsilon) &\to nf(c) && \text{in } L^1_{loc}(\bar{\Omega} \times [0,\infty)), \end{aligned}$$

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$$c_\varepsilon u_\varepsilon \to cu \qquad \text{in } L^1_{loc}(\bar{\Omega} \times [0,\infty)),$$
$$Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u \qquad \text{in } L^1_{loc}(\bar{\Omega} \times [0,\infty))$$

as $\varepsilon \to 0$. Based on the above convergence properties, we can pass to the limit in each term of weak formulation for (2.1) to construct a global weak solution of (1.2)–(1.4) and thereby completes the proof. □

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References

[1] N.D. Alikakos, L^p bounds of solutions of reaction–diffusion equations, Comm. Partial Differential Equations 4 (1979) 827–868.
[2] M. Chae, K. Kang, J. Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. 33 (2013) 2271–2297.
[3] M. Chae, K. Kang, J. Lee, Global existence and temporal decay in Keller–Segel models coupled to fluid equations, Comm. Partial Differential Equations 39 (2014) 1205–1235.
[4] M. Di Francesco, A. Lorz, P. Markowich, Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior, Discrete Contin. Dyn. Syst. 28 (2010) 1437–1453.
[5] C. Dombrowski, L. Cisneros, S. Chatkaew, R.E. Goldstein, J.O. Kessler, Self-concentration and large-scale coherence in bacterial dynamics, Phys. Rev. Lett. 93 (2004) 098103.
[6] R. Duan, A. Lorz, P. Markowich, Global solutions to the coupled chemotaxis-fluid equations, Comm. Partial Differential Equations 35 (2010) 1635–1673.
[7] R. Duan, Z. Xiang, A note on global existence for the chemotaxis-Stokes model with nonlinear diffusion, Int. Math. Res. Not. IMRN (2014) 1833–1852.
[8] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system, J. Differential Equations 62 (1986) 186–212.
[9] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.
[10] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (2005) 52–107.
[11] J. Jiang, H. Wu, S. Zheng, Global existence and asymptotic behavior of solutions to a chemotaxis-fluid system on general bounded domain, arXiv:1409.0412, 2014.
[12] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Gauthier-Villars, Paris, 1969.
[13] J.-G. Liu, A. Lorz, A coupled chemotaxis-fluid model: global existence, Ann. Inst. H. Poincaré Anal. Non Linéaire 28 (2011) 643–652.
[14] A. Lorz, Coupled chemotaxis fluid model, Math. Models Methods Appl. Sci. 20 (2010) 987–1004.
[15] P. Quittner, P. Souplet, Superlinear parabolic problems, in: Blow-up, Global Existence and Steady States, in: Birkhäuser Adv. Texts. Basl. Lehrbücher (Birkhäuser Advanced Texts: Basel Textbooks), Birkhäuser Verlag, Basel, 2007.
[16] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4) 146 (1987) 65–96.
[17] H. Sohr, The Navier–Stokes equations, in: An Elementary Functional Analytic Approach, in: Birkhäuser Adv. Texts. Basl. Lehrbücher (Birkhäuser Advanced Texts: Basel Textbooks), Birkhäuser Verlag, Basel, 2001.
[18] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations 252 (2012) 692–715.
[19] Y. Tao, M. Winkler, Global existence and boundedness in a Keller–Segel–Stokes model with arbitrary porous medium diffusion, Discrete Contin. Dyn. Syst. 32 (2012) 1901–1914.
[20] Y. Tao, M. Winkler, Locally bounded global solutions in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013) 157–178.

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[21] J.I. Tello, M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations 32 (2007) 849–877.

[22] I. Tuval, L. Cisneros, C. Dombrowski, C.W. Wolgemuth, J.O. Kessler, R.E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (2005) 2277–2282.

[23] J.L. Vázquez, The porous medium equation, in: Mathematical Theory, in: Oxford Math. Monogr., The Clarendon Press, Oxford University Press, Oxford, 2007.

[24] D. Vorotnikov, Weak solutions for a bioconvection model related to Bacillus subtilis, Commun. Math. Sci. 12 (2014) 545–563.

[25] M. Winkler, A critical exponent in a degenerate parabolic equation, Math. Methods Appl. Sci. 25 (2002) 911–925.

[26] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, Math. Nachr. 283 (2010) 1664–1673.

[27] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differential Equations 248 (2010) 2889–2905.

[28] M. Winkler, Global large-data solutions in a chemotaxis-(Navier–)Stokes system modeling cellular swimming in fluid drops, Comm. Partial Differential Equations 37 (2012) 319–351.

[29] M. Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier–Stokes system, arXiv:1410.5929, 2014.

[30] M. Winkler, Stabilization in a two-dimensional chemotaxis-Navier–Stokes system, Arch. Ration. Mech. Anal. 211 (2014) 455–487.

[31] Q. Zhang, Local well-posedness for the chemotaxis-Navier–Stokes equations in Besov spaces, Nonlinear Anal. Real World Appl. 17 (2014) 89–100.

[32] Q. Zhang, Y. Li, Convergence rates of solutions for a two-dimensional chemotaxis-Navier–Stokes system, Discrete Contin. Dyn. Syst. Ser. B (2015), in press, paper ID: DCDSB1971.

[33] Q. Zhang, Y. Li, Global boundedness of solutions to a two-species chemotaxis system, Z. Angew. Math. Phys. 66 (2015) 83–93.

[34] Q. Zhang, X. Zheng, Global well-posedness for the two-dimensional incompressible chemotaxis-Navier–Stokes equations, SIAM J. Math. Anal. 46 (2014) 3078–3105.

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