Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 259 (2015) 7662–7680
Journal of Differential Equations
www.elsevier.com/locate/jde

Second order estimates for Hessian equations of parabolic type on Riemannian manifolds

Heming Jiao

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

Received 13 August 2014; revised 14 June 2015
Available online 6 September 2015

Abstract

In this paper, we establish the second order estimates for solutions of the first initial–boundary value problem for general Hessian type fully nonlinear parabolic equations on Riemannian manifolds. The techniques used in this article can work for a wide range of fully nonlinear PDEs under very general conditions.

Keywords: Fully nonlinear parabolic equations; Riemannian manifolds; *A priori* estimates; The first initial–boundary value problem

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\bar{M} := M \cup \partial M$. We shall study the equation

$$f(\lambda(\nabla^2 u + A[u])) - u_t = \psi(x, t, u, \nabla u) \tag{1.1}$$

in $M_T = M \times (0, T] \subset M \times \mathbb{R}$, where f is a symmetric smooth function of n variables, $\nabla^2 u$ denotes the Hessian of $u(x, t)$ with respect to $x \in M$, $A[u] = A(x, t, \nabla u)$ is a $(0, 2)$ tensor on $\bar{M}$ which may depend on $t \in [0, T]$ and ∇u and

$$\lambda(\nabla^2 u + A[u]) = (\lambda_1, \dots, \lambda_n)$$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric g.

E-mail address: jiao@hit.edu.cn.

http://dx.doi.org/10.1016/j.jde.2015.08.031
0022-0396/

In this work we are mainly concerned with the *a priori* C^2 estimates for solutions to (1.1) satisfying the boundary condition

$$u = \varphi \text{ on } \mathcal{P}M_T, \tag{1.2}$$

where $\varphi \in C^\infty(\overline{\mathcal{P}M_T})$ and $\mathcal{P}M_T = BM_T \cup SM_T$ is the parabolic boundary of M_T with $BM_T = M \times \{0\}$ and $SM_T = \partial M \times [0, T]$.

The idea of this paper is mainly from Guan and Jiao [7] where the authors studied the second order estimates for the elliptic counterpart of (1.1):

$$f(\lambda(\nabla^2 u + A(x, u, \nabla u))) = \psi(x, u, \nabla u). \tag{1.3}$$

Comparing with the elliptic case, the main difficulty in deriving the second order estimates for the parabolic equation (1.1) is from its degeneracy which is overcome by using the strict subsolution in this paper. Surprisingly, thanks to the strict subsolution, we are able to relax some restrictions to f. Again because of the degeneracy, we do not get the higher estimates and the existence of classical solution. It is useful to consider viscosity solutions to (1.1) which will be addressed in forthcoming papers.

The first initial–boundary value problem for equation of the form (1.1) in $\mathbb{R}^n$ with $A \equiv 0$ and $\psi = \psi(x, t)$ was studied by Ivochkina and Ladyzhenskaya in [9] (when $f = \sigma_n^{1/n}$) and [10]. Jiao and Sui [11] treated the case that $A \equiv \chi(x)$ and $\psi = \psi(x, t)$ on Riemannian manifolds using the techniques of [5] and [7]. For the elliptic Hessian equations on manifolds, we refer the reader to Li [12], Urbas [14], Guan [4–6], Guan and Jiao [7] and their references.

As in [2], where the authors studied the equations (1.3) with $A \equiv 0$ and $\psi = \psi(x)$ in a bounded domain of $\mathbb{R}^n$, $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ is assumed to be defined on Γ, where Γ is an open, convex, symmetric proper subcone of $\mathbb{R}^n$ with vertex at the origin and

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^n : \text{ each component } \lambda_i > 0\} \subseteq \Gamma,$$

and to satisfy the following structure conditions in this paper:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \le i \le n, \tag{1.4}$$

$$f \text{ is concave in } \Gamma, \tag{1.5}$$

and without loss of generality,

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma. \tag{1.6}$$

Typical examples are given by $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$, $1 \le l < k \le n$, defined in the cone $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0,\ j = 1, \ldots, k\}$, where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n.$$

We call a function $u(x, t)$ admissible if $\lambda(\nabla^2 u + A[u]) \in \Gamma$ in $M \times [0, T]$. It is shown in [2] that (1.4) ensures that equation (1.1) is parabolic for admissible solutions. (1.5) means that the function F defined by $F(A) = f(\lambda[A])$ is concave for $A \in \mathcal{S}^{n\times n}$ with $\lambda[A] \in \Gamma$, where $\mathcal{S}^{n\times n}$ is the set of $n \times n$ symmetric matrices.

Throughout the paper we assume that $A[u]$ is smooth on $\bar{M}_T$ for $u \in C^\infty(\bar{M}_T)$, $\psi \in C^\infty(T^*\bar{M} \times [0,T] \times \mathbb{R})$ (for convenience we shall write $\psi = \psi(x,t,z,p)$ for $(x,p) \in T^*\bar{M}$, $t \in [0,T]$ and $z \in \mathbb{R}$ though) and that $\lambda(\nabla^2\varphi(x,0) + A[\varphi(x,0)]) \in \Gamma$ for all $x \in \bar{M}$.

Note that for fixed $(x,t) \in \bar{M}_T$ and $p \in T_x^*M$,

$$A(x,t,p) : T_xM \times T_xM \to \mathbb{R}$$

is a symmetric bilinear map. We shall use the notations

$$A^{\xi\eta}(x,\cdot,\cdot) := A(x,\cdot,\cdot)(\xi,\eta), \;\; \xi,\eta \in T_xM$$

and, for a function $v \in C^{2,1}_{x,t}(M_T)$, $A[v] := A(x,t,\nabla v)$, $A^{\xi\eta}[v] := A^{\xi\eta}(x,t,\nabla v)$ (see [7]).

In this paper we assume that there exists an admissible function $\underline{u} \in C^2(\bar{M}_T)$ satisfying

$$f(\lambda(\nabla^2\underline{u} + A[\underline{u}])) - \underline{u}_t \geq \psi(x,t,\underline{u},\nabla\underline{u}) + \delta_0 \text{ in } M \times [0,T] \tag{1.7}$$

for some positive constant δ_0 with $\underline{u} = \varphi$ on $\partial M \times [0,T]$ and $\underline{u} \leq \varphi$ in $M \times \{0\}$.

We shall prove the following theorem.

Theorem 1.1. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.1). Suppose that (1.4)–(1.7) hold. Assume that*

$$-\psi(x,t,z,p) \text{ and } A^{\xi\xi}(x,t,p) \text{ are concave in } p, \;\; \forall \xi \in T_xM, \tag{1.8}$$

$$\psi_z \leq 0. \tag{1.9}$$

Then

$$\max_{\bar{M}_T} |\nabla^2 u| \leq C_1\big(1 + \max_{\mathcal{P}M_T} |\nabla^2 u|\big) \tag{1.10}$$

where $C_1 > 0$ depends on $|u|_{C^1_x(\bar{M}_T)}$ and $|\underline{u}|_{C^2(\bar{M}_T)}$. Suppose that u also satisfies the boundary condition (1.2) and, in addition, assume that

$$f(\lambda(\nabla^2\varphi(x,0) + A[\varphi(x,0)])) - \varphi_t(x,0) = \psi[\varphi(x,0)], \;\; \forall x \in \partial M, \tag{1.11}$$

and

$$\varphi_t(x,t) + \psi(x,t,z,p) > 0 \tag{1.12}$$

for each $(x,t) \in SM_T$, $p \in T_x^*\bar{M}$ and $z \in \mathbb{R}$. Then there exists $C_2 > 0$ depending on $|u|_{C^1_x(\bar{M}_T)}$, $|\underline{u}|_{C^2(\bar{M}_T)}$ and $|\varphi|_{C^4(\mathcal{P}M_T)}$ such that

$$\max_{\mathcal{P}M_T} |\nabla^2 u| \leq C_2. \tag{1.13}$$

Since u is admissible, we have, by (1.8),

$$\triangle u + \mathrm{tr}A_{p_k}(x,t,0)\nabla_k u + \mathrm{tr}A(x,t,0) \geq \triangle u + \mathrm{tr}A(x,t,\nabla u) > 0$$

and by the maximum principle it is easy to derive the estimates

$$\max_{\bar{M}_T} |u| + \max_{\mathcal{P}M_T} |\nabla u| \leq C. \tag{1.14}$$

Combining with the gradient estimates (Theorems 5.1–5.3), we can prove the following theorem immediately.

Theorem 1.2. *Let $u \in C^4(\bar{M}_T)$ be an admissible solution of (1.1) in M_T with $u \geq \underline{u}$ in M_T and $u = \varphi$ on $\mathcal{P}M_T$. Suppose that (1.4)–(1.9) and (1.11)–(1.12) hold. Then we have*

$$|u|_{C^{2,1}_{x,t}(\bar{M}_T)} \leq C, \tag{1.15}$$

where $C > 0$ depends on n, M and $|\underline{u}|_{C^2(\bar{M}_T)}$ under any of the following additional assumptions: (i) (5.1)–(5.3) hold for $\gamma_1 < 4$, $\gamma_2 = 2$ in (5.1); (ii) (M^n, g) has nonnegative sectional curvature and (5.1) hold for $\gamma_1, \gamma_2 < 2$; (iii) (5.1), (5.16)–(5.20) hold for $\gamma_1, \gamma_2 < 4$ in (5.1) and $\gamma < 2$ in (5.18)–(5.20).

The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and present a brief review of some elementary formulas. In Section 3 and Section 4, we establish the global and boundary estimates for second order derivatives respectively. The gradient estimates are derived in Section 5.

2. Preliminaries

Throughout the paper ∇ denotes the Levi-Civita connection of (M^n, g). The curvature tensor is defined by

$$R(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z.$$

Let $e_1, \ldots, e_n$ be local frames on M^n. We denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$. Define the Christoffel symbols Γ^k_{ij} by $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$ and the curvature coefficients

$$R_{ijkl} = g(R(e_k, e_l)e_j, e_i), \quad R^i_{jkl} = g^{im} R_{mjkl}.$$

We shall use the notations $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i\nabla_j - \Gamma^k_{ij}\nabla_k$, etc.

For a differentiable function v defined on M^n, we usually identify ∇v with the gradient of v, and use $\nabla^2 v$ to denote the Hessian of v which is locally given by $\nabla_{ij} v = \nabla_i(\nabla_j v) - \Gamma^k_{ij}\nabla_k v$. We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij}\nabla_l v, \tag{2.1}$$

$$\nabla_{ijkl}v - \nabla_{klij}v = R^m_{ljk}\nabla_{im}v + \nabla_i R^m_{ljk}\nabla_m v + R^m_{lik}\nabla_{jm}v$$
$$+ R^m_{jik}\nabla_{lm}v + R^m_{jil}\nabla_{km}v + \nabla_k R^m_{jil}\nabla_m v. \tag{2.2}$$

Let $u \in C^4(\bar{M}_T)$ be an admissible solution of equation (1.1). For simplicity we shall denote $U := \nabla^2 u + A(x,t,\nabla u)$ and, under a local frame $e_1, \dots, e_n$,

$$U_{ij} \equiv U(e_i, e_j) = \nabla_{ij}u + A^{ij}(x,t,\nabla u),$$
$$\nabla_k U_{ij} \equiv \nabla U(e_i, e_j, e_k) = \nabla_{kij}u + \nabla_k A^{ij}(x,t,\nabla u)$$
$$\equiv \nabla_{kij}u + A^{ij}_{x_k}(x,t,\nabla u) + A^{ij}_{p_l}(x,t,\nabla u)\nabla_{kl}u, \tag{2.3}$$

$$(U_{ij})_t \equiv (U(e_i,e_j))_t = (\nabla_{ij}u)_t + A^{ij}_t(x,t,\nabla u) + A^{ij}_{p_l}(x,t,\nabla u)(\nabla_l u)_t$$
$$\equiv \nabla_{ij}u_t + A^{ij}_t(x,t,\nabla u) + A^{ij}_{p_l}(x,t,\nabla u)\nabla_l u_t, \tag{2.4}$$

where $A^{ij} = A^{e_i e_j}$ and $A^{ij}_{x_k}$ denotes the *partial* covariant derivative of A when viewed as depending on $x \in M$ only, while the meanings of A^{ij}_t and $A^{ij}_{p_l}$, etc. are obvious. Similarly we can calculate $\nabla_{kl}U_{ij} = \nabla_k\nabla_l U_{ij} - \Gamma^m_{kl}\nabla_m U_{ij}$, etc.

Let F be the function defined by

$$F(h) = f(\lambda(h))$$

for a $(0,2)$ tensor h on M.

Following the literature we denote throughout this paper

$$F^{ij} = \frac{\partial F}{\partial h_{ij}}(U), \quad F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij}\partial h_{kl}}(U)$$

under an orthonormal local frame $e_1, \dots, e_n$. The matrix $\{F^{ij}\}$ has eigenvalues $f_1, \dots, f_n$ and is positive definite by assumption (1.4), while (1.5) implies that F is a concave function of U_{ij} (see [2]). Moreover, when $\{U_{ij}\}$ is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij}U_{ij} = \sum f_i\lambda_i, \quad F^{ij}U_{ik}U_{kj} = \sum f_i\lambda_i^2, \quad \lambda(U) = (\lambda_1, \dots, \lambda_n).$$

Define the linear operator $\mathcal{L}$ locally by

$$\mathcal{L}v = F^{ij}\nabla_{ij}v + (F^{ij}A^{ij}_{p_k} - \psi_{p_k})\nabla_k v - v_t,$$

for $v \in C^{2,1}_{x,t}(M_T)$. We can prove

Theorem 2.1. *Let u be an admissible solution to (1.1) with $u \geq \underline{u}$ in M_T. Assume that (1.4), (1.5), (1.8) and (1.9) hold. Then there exists a constant $\theta > 0$ depending only on δ_0 and $\underline{u}$ such that*

$$\mathcal{L}(\underline{u} - u) \geq \theta\Big(1 + \sum F^{ii}\Big). \tag{2.5}$$

Proof. Since $\underline{u}$ is admissible satisfying (1.7), there exists a constant $\varepsilon_0 > 0$ such that $\{\lambda(\nabla^2\underline{u} + A[\underline{u}] - \varepsilon_0 g) : x \in \bar{M}_T\}$ is a compact subset of Γ and

$$f(\lambda(\nabla^2\underline{u} + A[\underline{u}] - \varepsilon_0 g)) - \underline{u}_t \geq \psi[\underline{u}] + \frac{\delta_0}{2} \text{ in } M_T.$$

Let $\theta = \min\{\frac{\delta_0}{2}, \varepsilon_0\}$. For each $(x,t) \in M_T$, we may assume $\{U_{ij}\} = \{\nabla_{ij}u + A^{ij}\}$ is diagonal at (x,t). From (1.8), (1.9) and the concavity of F, we see, at (x,t),

$$\begin{aligned} F^{ii}(\underline{U}_{ii} - \varepsilon_0 g_{ii} - U_{ii}) - (\underline{u} - u)_t &\geq \psi(x,t,\underline{u},\nabla\underline{u}) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \\ &\geq \psi(x,t,u,\nabla\underline{u}) - \psi(x,t,u,\nabla u) + \frac{\delta_0}{2} \\ &\geq \psi_{p_k}\nabla_k(\underline{u} - u) + \frac{\delta_0}{2}. \end{aligned} \tag{2.6}$$

By (1.8) again, we have

$$\begin{aligned} F^{ii}(\underline{U}_{ii} - U_{ii}) &= F^{ii}\nabla_{ii}(\underline{u} - u) + F^{ii}(A^{ii}(x,t,\nabla\underline{u}) - A^{ii}(x,t,\nabla u)) \\ &\leq F^{ii}\nabla_{ii}(\underline{u} - u) + F^{ii}A^{ii}_{p_k}\nabla_k(\underline{u} - u). \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we get

$$\mathcal{L}(\underline{u} - u) \geq \varepsilon_0 \sum F^{ii} + \frac{\delta_0}{2} \geq \theta\Big(1 + \sum F^{ii}\Big). \qquad \square$$

3. Global estimates for second order derivatives

In this section, we prove (1.10) in Theorem 1.1 for which we set

$$W = \max_{(x,t)\in\bar{M}_T} \max_{\xi\in T_xM, |\xi|=1} (\nabla_{\xi\xi}u + A^{\xi\xi}(x,t,\nabla u)e^{\phi},$$

as in [7], where ϕ is a function to be determined. It suffices to estimate W. We may assume W is achieved at $(x_0,t_0) \in \bar{M}_T - \mathcal{P}M_T$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\nabla_i e_j = 0$, and U is diagonal at (x_0,t_0). We may assume $U_{11}(x_0,t_0) \geq \ldots \geq U_{nn}(x_0,t_0)$. Thus, $W = U_{11}(x_0,t_0)e^{\phi(x_0,t_0)}$.

At the point (x_0,t_0) where the function $\log U_{11} + \phi$ attains its maximum, we have

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i\phi = 0 \text{ for each } i = 1, \ldots, n, \tag{3.1}$$

$$\frac{(U_{11})_t}{U_{11}} + \phi_t \geq 0, \tag{3.2}$$

and

$$0 \geq \sum_i F^{ii}\{\frac{\nabla_{ii}U_{11}}{U_{11}} - \frac{(\nabla_i U_{11})^2}{U_{11}^2} + \nabla_{ii}\phi\}. \tag{3.3}$$

Differentiating equation (1.1) twice, we find

$$F^{ii}\nabla_k U_{ii} - \nabla_k u_t = \psi_{x_k} + \psi_u \nabla_k u + \psi_{p_j}\nabla_{kj} u, \text{ for all } k, \tag{3.4}$$

and

$$\begin{aligned} &F^{ii}\nabla_{11}U_{ii} + F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} - \nabla_{11}u_t \\ &\quad\geq \psi_{p_j}\nabla_{11j}u + \psi_{p_l p_k}\nabla_{1k}u\nabla_{1l}u - CU_{11} \\ &\quad\geq \psi_{p_j}\nabla_j U_{11} + \psi_{p_1 p_1}U_{11}^2 - CU_{11} \\ &\quad= -U_{11}\psi_{p_j}\nabla_j\phi + \psi_{p_1 p_1}U_{11}^2 - CU_{11}. \end{aligned} \tag{3.5}$$

Next, by (3.1) and (3.4),

$$\begin{aligned} F^{ii}(\nabla_{ii}A^{11} - \nabla_{11}A^{ii}) &\geq F^{ii}(A^{11}_{p_j}\nabla_{iij}u - A^{ii}_{p_j}\nabla_{11j}u) \\ &\quad + F^{ii}(A^{11}_{p_i p_i}U_{ii}^2 - A^{ii}_{p_1 p_1}U_{11}^2) - CU_{11}\sum F^{ii} \\ &\geq U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi + A^{11}_{p_j}\nabla_j u_t - CU_{11}\sum F^{ii} - CU_{11} \\ &\quad - C\sum_{i\geq 2}F^{ii}U_{ii}^2 - U_{11}^2\sum_{i\geq 2}F^{ii}A^{ii}_{p_1 p_1}. \end{aligned} \tag{3.6}$$

Note that

$$\nabla_{ii}U_{11} \geq \nabla_{11}U_{ii} + \nabla_{ii}A^{11} - \nabla_{11}A^{ii} - CU_{11}. \tag{3.7}$$

Thus, by (3.5), (3.6) and (3.2), we have, at (x_0, t_0),

$$\begin{aligned} F^{ii}\nabla_{ii}U_{11} &\geq F^{ii}\nabla_{11}U_{ii} - CU_{11}(1 + \sum F^{ii}) + A^{11}_{p_j}\nabla_j u_t \\ &\quad - C\sum_{i\geq 2}F^{ii}U_{ii}^2 - U_{11}^2\sum_{i\geq 2}F^{ii}A^{ii}_{p_1 p_1} + U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi \\ &\geq U_{11}\mathcal{L}\phi - U_{11}F^{ii}\nabla_{ii}\phi - F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} + \psi_{p_1 p_1}U_{11}^2 \\ &\quad - CU_{11}(1 + \sum F^{ii}) - CF^{ii}U_{ii}^2 - U_{11}^2\sum_{i\geq 2}F^{ii}A^{ii}_{p_1 p_1}. \end{aligned} \tag{3.8}$$

It follows that, by (3.3),

$$\mathcal{L}\phi \leq U_{11}\sum_{i\geq 2}F^{ii}A^{ii}_{p_1 p_1} - \psi_{p_1 p_1}U_{11} + C(1 + \sum F^{ii}) + \frac{C}{U_{11}}F^{ii}U_{ii}^2 + E, \tag{3.9}$$

where

$$E = \frac{1}{U_{11}^2}F^{ii}(\nabla_i U_{11})^2 + \frac{1}{U_{11}}F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl}.$$

Let

$$\phi = \frac{\delta |\nabla u|^2}{2} + b\eta,$$

where $0 < \delta \ll 1 \ll b$ are undetermined constants and η is a C^2 function which may depend on u but not on its derivatives. We calculate, at (x_0, t_0),

$$\nabla_i \phi = \delta \nabla_j u \nabla_{ij} u + b \nabla_i \eta = \delta \nabla_i u U_{ii} - \delta \nabla_j u A^{ij} + b \nabla_i \eta, \tag{3.10}$$

$$\phi_t = \delta \nabla_j u (\nabla_j u)_t + b\eta_t \tag{3.11}$$

and

$$\nabla_{ii} \phi \geq \frac{\delta}{2} U_{ii}^2 - C\delta + \delta \nabla_j u \nabla_{iij} u + b \nabla_{ii} \eta. \tag{3.12}$$

From (2.1) and (3.4), we derive

$$\begin{aligned} F^{ii} \nabla_j u \nabla_{iij} u &\geq F^{ii} \nabla_j u (\nabla_j U_{ii} - \nabla_j A^{ii}) - C|\nabla u|^2 \sum F^{ii} \\ &\geq (\psi_{p_l} - F^{ii} A^{ii}_{p_l}) \nabla_j u \nabla_{jl} u + \nabla_j u \nabla_j (u_t) - C(1 + \sum F^{ii}). \end{aligned} \tag{3.13}$$

Therefore,

$$\mathcal{L}\phi \geq b\mathcal{L}\eta + \frac{\delta}{2} F^{ii} U_{ii}^2 - C \sum F^{ii} - C. \tag{3.14}$$

Let $\eta = \underline{u} - u$. We get from (3.10) that

$$(\nabla_i \phi)^2 \leq C\delta^2 (1 + U_{ii}^2) + 2b^2 (\nabla_i \eta)^2 \leq C\delta^2 U_{ii}^2 + Cb^2. \tag{3.15}$$

For fixed $0 < s \leq 1/3$ let

$$J = \{i : U_{ii} \leq -sU_{11}\}, \quad K = \{i : U_{ii} > -sU_{11}\}.$$

Using a result of Andrews [1] and Gerhardt [3] as in [5] and [7] (see [14] also), we have

$$E \leq Cb^2 \sum_{i \in J} F^{ii} + C\delta^2 \sum F^{ii} U_{ii}^2 + C \sum F^{ii} + C(\delta^2 U_{11}^2 + b^2) F^{11}. \tag{3.16}$$

Therefore, by (3.9), (3.14) and (3.16), we have

$$\begin{aligned} b\mathcal{L}\eta &\leq \Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\Big) F^{ii} U_{ii}^2 + Cb^2 \sum_{i \in J} F^{ii} + C \sum F^{ii} \\ &\quad + C(\delta^2 U_{11}^2 + b^2) F^{11} + C \\ &\leq \Big(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\Big) F^{ii} U_{ii}^2 + Cb^2 \sum_{i \in J} F^{ii} + C \sum F^{ii} \\ &\quad + Cb^2 F^{11} + C. \end{aligned} \tag{3.17}$$

Choose δ sufficiently small such that $C\delta^2 - \frac{\delta}{2}$ is negative and let

$$c_1 := -\frac{1}{2}\Big(C\delta^2 - \frac{\delta}{2}\Big) > 0.$$

We may assume

$$C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}} \leq -c_1$$

for otherwise we have $U_{11} \leq \frac{C}{c_1}$ and we are done. Thus, by (2.5), choosing b sufficiently large, we derive from (3.17) that

$$c_1 F^{ii} U_{ii}^2 - Cb^2 F^{11} - Cb^2 \sum_{i\in J} F^{ii} \leq 0.$$

Then we can get a bound $U_{11}(x_0, t_0) \leq C$ since $|U_{ii}| \geq sU_{11}$ for $i \in J$. The proof of (1.10) is completed.

4. Boundary estimates for second order derivatives

In this section, we consider the estimates for second order derivatives on the parabolic boundary $\mathcal{P}M_T$. We may assume $\varphi \in C^4(\bar{M}_T)$.

Fix a point $(x_0, t_0) \in SM_T$. We shall choose smooth orthonormal local frames $e_1, \ldots, e_n$ around x_0 such that when restricted to ∂M, e_n is normal to ∂M. Since $u - \underline{u} = 0$ on SM_T we have

$$\nabla_{\alpha\beta}(u - \underline{u}) = -\nabla_n(u - \underline{u})\Pi(e_\alpha, e_\beta), \ \ \forall\, 1 \leq \alpha, \beta < n \text{ on } SM_T, \tag{4.1}$$

where Π denotes the second fundamental form of ∂M. Therefore,

$$|\nabla_{\alpha\beta} u| \leq C, \ \ \forall\, 1 \leq \alpha, \beta < n \text{ on } SM_T. \tag{4.2}$$

Let $\rho(x)$ denote the distance from $x \in M$ to x_0,

$$\rho(x) \equiv \mathrm{dist}_{M^n}(x, x_0),$$

and set

$$M_\delta = \{X = (x,t) \in M \times (0, T] : \rho(x) < \delta, t \leq t_0 + \delta\}.$$

For the mixed tangential–normal and pure normal second derivatives at (x_0, t_0), we shall use the following barrier function as in [5],

$$\Psi = A_1 v + A_2 \rho^2 - A_3 \sum_{l<n} |\nabla_l (u - \varphi)|^2 \tag{4.3}$$

where $v = u - \underline{u}$. By differentiating the equation (1.1) and straightforward calculation, we obtain

$$\mathcal{L}(\nabla_k(u-\varphi)) \le C\Big(1 + \sum f_i|\lambda_i| + \sum f_i\Big), \ \forall\, 1 \le k \le n. \tag{4.4}$$

Similar to [5] (see [7] also), using Proposition 2.19 and Corollary 2.21 of [5] and Theorem 2.1, we can prove that there exist uniform positive constants δ sufficiently small, and A_1, A_2, A_3 sufficiently large such that

$$\mathcal{L}(\Psi \pm \nabla_\alpha(u-\varphi)) \le 0 \text{ in } M_\delta \tag{4.5}$$

and $\Psi \pm \nabla_\alpha(u-\varphi) \ge 0$ on $\mathcal{P}M_\delta$. Thus, by the maximum principle, we see $\Psi \pm \nabla_\alpha(u-\varphi) \ge 0$ in M_δ. Then we get

$$|\nabla_{n\alpha}u(x_0,t_0)| \le \nabla_n\Psi(x_0,t_0) \le C, \ \ \forall\, \alpha < n. \tag{4.6}$$

Remark 4.1. We remark that in the proof of Corollary 2.21 in [5], the following condition is needed

$$\sum f_i(\lambda)\lambda_i \ge 0, \ \forall\, \lambda \in \Gamma \tag{4.7}$$

which can be derived from (1.4)–(1.6).

It remains to derive

$$\nabla_{nn}u(x_0,t_0) \le C \tag{4.8}$$

since $\triangle u \ge -C$. We shall use an idea of Trudinger [13] as [5] and [7] to prove that there exist uniform positive constants c_0, R_0 such that for all $R > R_0$, $(\lambda'[U], R) \in \Gamma$ and

$$f(\lambda'[U], R) - u_t \ge \psi[u] + c_0 \text{ on } \overline{SM_T} \tag{4.9}$$

which implies (4.8) by Lemma 1.2 in [2], where $\lambda'[U] = (\lambda'_1, \dots, \lambda'_{n-1})$ denote the eigenvalues of the $(n-1)\times(n-1)$ matrix $\{U_{\alpha\beta}\}_{1\le\alpha,\beta\le(n-1)}$ and $\psi[u] = \psi(\cdot,\cdot,u,\nabla u)$. For $R > 0$ and a symmetric $(n-1)^2$ matrix $\{r_{\alpha\beta}\}$ with $(\lambda'(\{r_{\alpha\beta}\}), R) \in \Gamma$, define

$$G[r_{\alpha\beta}] \equiv f(\lambda'[\{r_{\alpha\beta}\}], R)$$

and consider

$$m_R \equiv \min_{(x,t)\in\overline{SM_T}} G[U_{\alpha\beta}(x,t)] - u_t(x,t) - \psi[u].$$

Note that G is concave and m_R is increasing in R by (1.4), and that

$$c_R \equiv \inf_{\overline{SM_T}} (G[\underline{U}_{\alpha\beta}] - \underline{u}_t - \psi[\underline{u}]) > 0$$

when R is sufficiently large.

We wish to show $m_R > 0$ for R sufficiently large. Without loss of generality we assume $m_R < c_R/2$ (otherwise we are done) and suppose m_R is achieved at a point $(x_0, t_0) \in \overline{SM_T}$. Choose local orthonormal frames around x_0 as before and assume $\nabla_{nn}u(x_0, t_0) \geq \nabla_{nn}\underline{u}(x_0, t_0)$. Let $\sigma_{\alpha\beta} = \langle \nabla_\alpha e_\beta, e_n \rangle$ and

$$G_0^{\alpha\beta} = \frac{\partial G}{\partial r_{\alpha\beta}}[U_{\alpha\beta}(x_0, t_0)].$$

Note that $\sigma_{\alpha\beta} = \Pi(e_\alpha, e_\beta)$ on ∂M and that

$$G_0^{\alpha\beta}(r_{\alpha\beta} - U_{\alpha\beta}(x_0, t_0)) \geq G[r_{\alpha\beta}] - G[U_{\alpha\beta}(x_0, t_0)] \tag{4.10}$$

for any symmetric matrix $\{r_{\alpha\beta}\}$ with $(\lambda'[\{r_{\alpha\beta}\}], R) \in \Gamma$ by the concavity of G.

In particular, since $u_t = \underline{u}_t = \varphi_t$ on $\overline{SM_T}$, we have

$$\begin{aligned} &G_0^{\alpha\beta}U_{\alpha\beta} - \psi[u] - \underline{u}_t - G_0^{\alpha\beta}U_{\alpha\beta}(x_0, t_0) + \psi[u](x_0, t_0) + u_t(x_0, t_0) \\ &\geq G[U_{\alpha\beta}] - \psi[u] - u_t - m_R \geq 0 \end{aligned} \tag{4.11}$$

on $\overline{SM_T}$.

From (4.1) we see that

$$U_{\alpha\beta} = \underline{U}_{\alpha\beta} - \nabla_n(u - \underline{u})\sigma_{\alpha\beta} + A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}] \text{ on } \overline{SM_T}. \tag{4.12}$$

Note that at (x_0, t_0), we have

$$\begin{aligned} \nabla_n(u - \underline{u})G_0^{\alpha\beta}\sigma_{\alpha\beta} &= G_0^{\alpha\beta}(\underline{U}_{\alpha\beta} - U_{\alpha\beta}) + G_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\ &\geq G[\underline{U}_{\alpha\beta}] - G[U_{\alpha\beta}] + G_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\ &= G[\underline{U}_{\alpha\beta}] - \psi[u] - u_t - m_R + G_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\ &\geq c_R - m_R + \psi[\underline{u}] + \underline{u}_t - \psi[u] - u_t \\ &\quad + G_0^{\alpha\beta}(A^{\alpha\beta}[u] - A^{\alpha\beta}[\underline{u}]) \\ &\geq \frac{c_R}{2} + H[u] - H[\underline{u}] \end{aligned} \tag{4.13}$$

where $H[u] = G_0^{\alpha\beta}A^{\alpha\beta}[u] - \psi[u]$.

Define

$$\Phi = -\eta\nabla_n(u - \underline{u}) + H[u] - \underline{u}_t + Q$$

where $\eta = G_0^{\alpha\beta}\sigma_{\alpha\beta}$ and

$$Q \equiv G_0^{\alpha\beta}\nabla_{\alpha\beta}\underline{u} - G_0^{\alpha\beta}U_{\alpha\beta}(x_0, t_0) + \psi[u](x_0, t_0) + u_t(x_0, t_0).$$

By virtue of (4.11) and (4.12) we see that $\Phi \geq 0$ on $\overline{SM_T}$ and $\Phi(x_0, t_0) = 0$.

Next, by (4.4) and (1.8),

$$\begin{aligned}\mathcal{L}H &\le H_z[u]\mathcal{L}u + H_{p_k}[u]\mathcal{L}\nabla_k u + F^{ij}H_{p_k p_l}[u]\nabla_{ki}u\nabla_{lj}u \\ &\quad + C\Big(\sum F^{ii} + \sum f_i|\lambda_i| + 1\Big) \\ &\le C\Big(\sum F^{ii} + \sum f_i|\lambda_i| + 1\Big) + H_z[u]\mathcal{L}u.\end{aligned}$$

Since $H_z[u] \ge 0$, by Theorem 2.1, we have

$$\mathcal{L}u = \mathcal{L}(u - \underline{u}) + \mathcal{L}\underline{u} \le C\Big(1 + \sum F^{ii}\Big).$$

It follows that

$$\mathcal{L}H \le C\Big(\sum F^{ii} + \sum f_i|\lambda_i| + 1\Big).$$

Therefore,

$$\mathcal{L}\Phi \le C\Big(\sum F^{ii} + \sum f_i|\lambda_i| + 1\Big). \tag{4.14}$$

By the compatibility condition (1.11), we find

$$c'_R \equiv \inf_{x\in\partial M} G(\nabla_{\alpha\beta}\varphi + A[\varphi])(x,0) - \psi[\varphi](x,0) - \varphi_t(x,0) > 0$$

when R is sufficiently large. We may assume $m_R < \frac{c'_R}{2}$ (otherwise we are done). For $x \in \bar{M}$, by the concavity of G again, we have

$$\begin{aligned}\Phi(x_0,0) &= G_0^{\alpha\beta}(U_{\alpha\beta}(x_0,0) - U_{\alpha\beta}(x_0,t_0)) \\ &\quad - \psi[u](x_0,0) - \underline{u}_t(x_0,0) + \psi[u](x_0,t_0) + u_t(x_0,t_0) \\ &= G_0^{\alpha\beta}(\nabla_{\alpha\beta}\varphi + A[\varphi](x_0,0) - U_{\alpha\beta}(x_0,t_0)) \\ &\quad - \varphi_t(x_0,0) + u_t(x_0,t_0) + \psi[u](x_0,t_0) - \psi[\varphi](x_0,0) \\ &\ge G(\nabla_{\alpha\beta}\varphi + A[\varphi])(x_0,0) - G(U_{\alpha\beta}(x_0,t_0)) \\ &\quad - \varphi_t(x_0,0) + u_t(x_0,t_0) + \psi[u](x_0,t_0) - \psi[\varphi](x_0,0) \\ &\ge c'_R - m_R > \frac{c'_R}{2}.\end{aligned}$$

Note that on BM_T,

$$\Phi = -\eta\nabla_n(\varphi - \underline{u}) + H[\varphi] - \underline{u}_t + Q$$

is a known function independent of the solution u. It follows that $\Phi \ge 0$ on BM_δ provided δ is sufficiently small. Thus, we get $\Phi \ge 0$ on $\mathcal{P}M_\delta$.

Consider the function Ψ defined in (4.3) as before. Similarly, there exist another group of constants $A_1 \gg A_2 \gg A_3 \gg 1$ such that

$$\begin{cases} \mathcal{L}(\Psi + \Phi) \le 0 & \text{in } M_\delta, \\ \Psi + \Phi \ge 0 & \text{on } \mathcal{P}M_\delta. \end{cases} \tag{4.15}$$

By the maximum principle we find $\Psi + \Phi \ge 0$ in M_δ. It follows that $\nabla_n \Phi(x_0, t_0) \ge -\nabla_n \Psi(x_0, t_0) \ge -C$.

Following [7], we write $u^s = su + (1-s)\underline{u}$ and

$$H[u^s] = G_0^{\alpha\beta} A^{\alpha\beta}[u^s] - \psi[u^s].$$

We have

$$\begin{aligned} H[u] - H[\underline{u}] &= \int_0^1 \frac{dH[u^s]}{dt} ds \\ &= (u - \underline{u}) \int_0^1 H_z[u^s] ds + \sum \nabla_k (u - \underline{u}) \int_0^1 H_{p_k}[u^s] ds. \end{aligned}$$

Therefore, at (x_0, t_0),

$$H[u] - H[\underline{u}] = \nabla_n (u - \underline{u}) \int_0^1 H_{p_n}[u^s] ds \tag{4.16}$$

and

$$\begin{aligned} \nabla_n H[u] &= \nabla_n H[\underline{u}] + \sum \nabla_{kn}(u - \underline{u}) \int_0^1 H_{p_k}[u^s] ds \\ &\quad + \nabla_n (u - \underline{u}) \int_0^1 (H_z[u^s] + H_{x_n p_n}[u^s] + H_{z p_n}[u^s] \nabla_n u^s) ds \\ &\quad + \nabla_n (u - \underline{u}) \sum \int_0^1 H_{p_n p_l}[u^s] \nabla_{ln} u^s ds \\ &\le \nabla_{nn}(u - \underline{u}) \int_0^1 (H_{p_n}[u^s] + s H_{p_n p_n}[u^s] \nabla_n (u - \underline{u})) ds + C \\ &\le \nabla_{nn}(u - \underline{u}) \int_0^1 H_{p_n}[u^s] ds + C \end{aligned} \tag{4.17}$$

since $H_{p_n p_n} \le 0$, $\nabla_{nn}(u - \underline{u}) \ge 0$ and $\nabla_n(u - \underline{u}) \ge 0$. It follows that

$$\begin{aligned}\nabla_n \Phi(x_0, t_0) &\le -\eta(x_0, t_0)\nabla_{nn}(x_0, t_0) + \nabla_n H[u](x_0, t_0) + C \\ &\le \Big(-\eta(x_0, t_0) + \int_0^1 H_{p_n}[u^s](x_0, t_0)ds \Big)\nabla_{nn}u(x_0, t_0) + C. \end{aligned} \tag{4.18}$$

By (4.13) and (4.16),

$$\eta(x_0, t_0) - \int_0^1 H_{p_n}[u^s](x_0, t_0)ds \ge \frac{c_R}{2\nabla_n(u - \underline{u})(x_0, t_0)} \ge \epsilon_1 c_R > 0 \tag{4.19}$$

for some uniform $\epsilon_1 > 0$ independent of R. This gives

$$\nabla_{nn}u(x_0, t_0) \le \frac{C}{\epsilon_1 c_R}. \tag{4.20}$$

So we obtain an *a priori* upper bound for all eigenvalues of $\{U_{ij}(x_0, t_0)\}$. Now by (1.12), there exists a constant $\nu_0 > 0$ such that

$$\inf_{(x,t)\in \overline{SM_T}} \varphi_t(x, t) + \psi(x, t, u, \nabla u) \ge \nu_0.$$

It follows that $\lambda[\{U_{ij}(x_0, t_0)\}]$ is contained in a compact subset of Γ by (1.6), and therefore

$$m_R = G[U_{\alpha\beta}(x_0, t_0)] - u_t(x_0, t_0) - \psi[u](x_0, t_0) > 0$$

when R is sufficiently large. Then (4.9) is valid and the proof of (1.13) is completed.

5. Gradient estimates

In this section we establish the gradient estimates to prove Theorems 5.1–5.3 below. Throughout the section, we assume (1.4)–(1.5), (1.8) and the following growth conditions

$$\begin{cases} p \cdot \nabla_x A^{\xi\xi}(x, t, p) \le \bar\psi_1(x, t, z)|\xi|^2(1 + |p|^{\gamma_1}) \\ p \cdot \nabla_x \psi(x, t, z, p) + |p|^2\psi_z(x, t, z, p) \ge -\bar\psi_2(x, t, z)(1 + |p|^{\gamma_2}) \end{cases} \tag{5.1}$$

for some functions $\bar\psi_1, \bar\psi_2 \ge 0$ and constants $\gamma_1, \gamma_2 > 0$.

Since the proofs of Theorems 5.1–5.3 are similar to those in the elliptic case (see [8]), we only provide a sketch here.

Theorem 5.1. *Let $u \in C^3(\bar M_T)$ be an admissible solution of (1.1). Assume, in addition, that*

$$\lim_{\sigma\to\infty} f(\sigma \mathbf{1}) = +\infty \tag{5.2}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$ *and there exists a constant* $c_0 > 0$ *such that*

$$A^{\xi\xi}_{p_k p_l}(x, t, p)\eta_k\eta_l \le -c_0|\xi|^2|\eta|^2 + c_0|g(\xi, \eta)|^2, \ \forall \xi, \eta \in T_x M. \tag{5.3}$$

Suppose that $\gamma_1 < 4$, $\gamma_2 = 2$ *in* (5.1), *and that there is an admissible function* $\underline{u} \in C^2(\bar{M}_T)$. *Then*

$$\max_{\bar{M}_T} |\nabla u| \le C_3\big(1 + \max_{\mathcal{P}M_T} |\nabla u|\big) \tag{5.4}$$

where C_3 *is a positive constant depending on* $|u|_{C^0(\bar{M}_T)}$ *and* $|\underline{u}|_{C^1_x(\bar{M}_T)}$.

Proof. Let $w = |\nabla u|$ and ϕ be a positive function to be determined. Suppose the function $w\phi^{-a}$ achieves a positive maximum at an interior point $(x_0, t_0) \in M_T - \mathcal{P}M_T$ where $a < 1$ is a positive constant. Choose a smooth orthonormal local frame $e_1, \dots, e_n$ about x_0 such that $\nabla_{e_i} e_j = 0$ at x_0 and $\{U_{ij}(x_0, t_0)\}$ is diagonal.

The function $\log w - a\log\phi$ attains its maximum at (x_0, t_0) where for $i = 1, \dots, n$,

$$\frac{\nabla_i w}{w} - \frac{a\nabla_i\phi}{\phi} = 0, \tag{5.5}$$

$$\frac{w_t}{w} - \frac{a\phi_t}{\phi} \ge 0 \tag{5.6}$$

and

$$\frac{\nabla_{ii} w}{w} + \frac{(a - a^2)|\nabla_i\phi|^2}{\phi^2} - \frac{a\nabla_{ii}\phi}{\phi} \le 0. \tag{5.7}$$

Note that

$$w\nabla_i w = \nabla_l u\nabla_{il} u, \ \ ww_t = \nabla_l u(\nabla_l u)_t.$$

By (2.1), (5.5) and (3.4),

$$\begin{aligned} w\nabla_{ii} w &= \nabla_l u\nabla_{iil} u + \nabla_{il} u\nabla_{il} u - \nabla_i w\nabla_i w \\ &= (\nabla_{lii} u + R^k_{iil}\nabla_k u)\nabla_l u + \Big(\delta_{kl} - \frac{\nabla_k u\nabla_l u}{w^2}\Big)\nabla_{ik} u\nabla_{il} u \\ &\ge (\nabla_l U_{ii} - A^{ii}_{p_k}\nabla_{lk} u - A^{ii}_{x_l})\nabla_l u - C|\nabla u|^2 \\ &= \nabla_l u\nabla_l U_{ii} - \frac{aw^2}{\phi}A^{ii}_{p_k}\nabla_k\phi - \nabla_l u A^{ii}_{x_l} - Cw^2. \end{aligned} \tag{5.8}$$

By (3.4), (5.5) and (5.6),

$$\begin{aligned} F^{ii}\nabla_l u\nabla_l U_{ii} &= \nabla_l u\psi_{x_l} + \psi_u|\nabla u|^2 + \psi_{p_k}\nabla_l u\nabla_{lk} u + \nabla_l u\nabla_l u_t \\ &\ge \nabla_l u\psi_{x_l} + \psi_u|\nabla u|^2 + \frac{aw^2}{\phi}\psi_{p_k}\nabla_k\phi + \frac{aw^2}{\phi}\phi_t. \end{aligned} \tag{5.9}$$

Let $\phi = (u - \underline{u}) + b > 0$, where $b = 1 + \sup_{M_T}(\underline{u} - u)$.

By the condition (5.3) we have

$$
\begin{aligned}
-A^{ii}_{p_k}\nabla_k\phi &= A^{ii}_{p_k}(x,t,\nabla u)\nabla_k(\underline{u}-u)\\
&\geq A^{ii}(x,t,\nabla\underline{u}) - A^{ii}(x,t,\nabla u) + \frac{c_0}{2}(|\nabla\phi|^2 - |\nabla_i\phi|^2).
\end{aligned}
\tag{5.10}
$$

We may assume that c_0 is sufficiently small and that

$$
\frac{2a-2a^2-c_0a\phi}{2\phi^2} > 0
$$

by choosing a sufficiently small.

Thus, by (5.7), (5.8), (5.9) and (5.10), we find

$$
\begin{aligned}
0 \geq\ & \frac{a}{\phi}F^{ii}(\underline{U}_{ii}-U_{ii}) + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii} + \frac{2a-2a^2-c_0a\phi}{2\phi^2}F^{ii}|\nabla_i\phi|^2\\
& - \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u + \frac{a}{\phi}\psi_{p_k}\nabla_k\phi + \frac{a}{\phi}\phi_t - C\sum F^{ii}\\
\geq\ & \frac{a}{\phi}F^{ii}(\underline{U}_{ii}-U_{ii}) + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii} - C\sum F^{ii}\\
& + \frac{a}{\phi}(\psi(x,t,u,\nabla u) - \psi(x,t,u,\nabla\underline{u}))\\
& - \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u + \frac{a}{\phi}(u-\underline{u})_t.
\end{aligned}
\tag{5.11}
$$

Choose $B>0$ sufficiently large such that

$$
F(2Bg+\underline{U}) \geq F(Bg) \ \text{ in } \bar{M}_T.
$$

Therefore, by the concavity of F,

$$
\begin{aligned}
F^{ii}(\underline{U}_{ii}-U_{ii}) &\geq F(2Bg+\underline{U}) - F(U) - 2B\sum F^{ii}\\
&\geq F(Bg) - 2B\sum F^{ii} - \psi(x,t,u,\nabla u) - u_t.
\end{aligned}
\tag{5.12}
$$

It follows from (5.1), (5.2), (5.11) and (5.12) that

$$
\begin{aligned}
0 \geq\ & \frac{a}{\phi}F(Bg) - C - (C+2B)\sum F^{ii} + \frac{ac_0|\nabla\phi|^2}{2\phi}\sum F^{ii}\\
& - \frac{1}{w^2}F^{ii}A^{ii}_{x_l}\nabla_l u + \frac{1}{w^2}\psi_{x_l}\nabla_l u + \psi_u\\
\geq\ & (\frac{ac_0|\nabla\phi|^2}{2\phi} - 3B - C|\nabla u|^{\gamma_1-2})\sum F^{ii}
\end{aligned}
\tag{5.13}
$$

provided B is chosen sufficiently large. Thus, we get a bound $|\nabla u(x_0, t_0)| \le C$ and so the proof of Theorem 5.1 is completed. □

Theorem 5.2. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1) with $u \ge \underline{u}$ in M_T. Assume, in addition, that (1.7), (1.9) and (5.1) hold for $\gamma_1, \gamma_2 < 2$ in (5.1) and that (M^n, g) has nonnegative sectional curvature. Then (5.4) holds.*

Proof. Since (M^n, g) has nonnegative sectional curvature, in an orthonormal local frame,

$$R^k_{iil}\nabla_k u \nabla_l u \ge 0.$$

In the proof of Theorem 5.1, similar to (5.8), we have

$$w\nabla_{ii}w \ge \nabla_l u \nabla_l U_{ii} - \frac{aw^2}{\phi} A^{ii}_{p_k}\nabla_k \phi - \nabla_l u A^{ii}_{x_l}. \tag{5.14}$$

It follows from (2.5), (5.1), (5.7), (5.9) and (5.14) that

$$\begin{aligned} 0 &\ge \frac{a}{\phi}\mathcal{L}(\underline{u} - u) + \frac{1}{w^2}\nabla_l u \psi_{x_l} + \psi_u - \frac{\nabla_l u}{w^2} F^{ii} A^{ii}_{x_l} + \frac{a - a^2}{\phi^2} F^{ii}|\nabla_i \phi|^2 \\ &\ge \frac{a}{\phi}\theta(1 + \sum F^{ii}) - C|\nabla u|^{\gamma_1 - 2}\sum F^{ii} - C|\nabla u|^{\gamma_2 - 2} + \frac{a - a^2}{\phi^2} F^{ii}|\nabla_i \phi|^2 \end{aligned} \tag{5.15}$$

provided $|\nabla u|$ is sufficiently large. Choosing a sufficiently small, we can obtain a bound $|\nabla u(x_0, t_0)| \le C$ and (5.4) holds. □

Theorem 5.3. *Let $u \in C^3(\bar{M}_T)$ be an admissible solution of (1.1) in M_T. Assume, in addition, that (5.1) hold for $\gamma_1, \gamma_2 < 4$,*

$$f \text{ is homogeneous of degree one}, \tag{5.16}$$

$$f_j(\lambda) \ge \nu_1\Big(1 + \sum f_i(\lambda)\Big) \text{ for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{5.17}$$

where ν_1 is a uniform positive constant and there exist a continuous function $\bar{\psi} \ge 0$ and a positive constant $\gamma < 2$ such that when $|p|$ is sufficiently large,

$$p \cdot D_p\psi(x, t, z, p),\ -p \cdot D_p A^{\xi\xi}(x, t, p)/|\xi|^2 \le \bar{\psi}(x, t, z)(1 + |p|^\gamma), \tag{5.18}$$

$$-\psi(x, t, z, p) \le \bar{\psi}(x, t, z)(1 + |p|^\gamma), \tag{5.19}$$

$$|A^{\xi\eta}(x, t, p)| \le \bar{\psi}(x, t, z)|\xi||\eta|(1 + |p|^\gamma),\ \ \forall \xi, \eta \in T_x\bar{M}; \xi \perp \eta. \tag{5.20}$$

Then (5.4) holds.

Proof. In the proof of Theorem 5.1, we take $\phi = -u + \sup_{M_T} u + 1$. By the concavity of A^{ii} with respect to p,

$$A^{ii} = A^{ii}(x, t, \nabla u) \le A^{ii}(x, t, 0) + A^{ii}_{p_k}(x, t, 0)\nabla_k u \tag{5.21}$$

Thus, from (5.16), (5.19) and (5.21), we find

$$\begin{aligned} -F^{ii}\nabla_{ii}\phi = F^{ii}\nabla_{ii}u = F^{ii}U_{ii} - F^{ii}A^{ii} &= u_t + \psi - F^{ii}A^{ii} \\ &\geq u_t + \psi - C(1+|\nabla u|)\sum F^{ii} \\ &\geq u_t - C(1+|\nabla u|)\sum F^{ii} - C|\nabla u|^{\gamma}. \end{aligned} \tag{5.22}$$

By virtue of (5.7), (5.8), (5.9), (5.1), (5.18) and (5.22), we see that for $a < 1$,

$$\begin{aligned} 0 \geq{}& \frac{(a-a^2)}{\phi^2}F^{ii}|\nabla_i u|^2 + \frac{\nabla_l u\psi_{x_l}}{w^2} + \psi_u - \frac{a}{\phi}\psi_{p_k}\nabla_k u - \frac{a}{\phi}u_t \\ &+ \frac{a}{\phi}F^{ii}A^{ii}_{p_k}\nabla_k u - F^{ii}\frac{\nabla_l u A^{ii}_{x_l}}{w^2} + \frac{a}{\phi}u_t \\ &- C|\nabla u|^{\gamma} - C(1+|\nabla u|)\sum F^{ii} \\ \geq{}& c_1 F^{ii}|\nabla_i u|^2 - C(|\nabla u|^{\gamma_2-2} + |\nabla u|^{\gamma}) \\ &- C(1+|\nabla u| + |\nabla u|^{\gamma_1-2} + |\nabla u|^{\gamma})\sum F^{ii} \end{aligned} \tag{5.23}$$

provided $|\nabla u|$ is sufficiently large.

Without loss of generality we assume $\nabla_1 u(x_0,t_0) \geq \frac{1}{n}|\nabla u(x_0,t_0)| > 0$. Recall that $U_{ij}(x_0,t_0)$ is diagonal. By (5.5), (5.21) and (5.20), we have

$$\begin{aligned} U_{11} &= -\frac{a}{\phi}|\nabla u|^2 + A^{11} + \frac{1}{\nabla_1 u}\sum_{k\geq 2}\nabla_k u A^{1k} \\ &\leq -\frac{a}{\phi}|\nabla u|^2 + C(1+|\nabla u| + |\nabla u|^{\gamma-2}) < 0 \end{aligned} \tag{5.24}$$

provided $|\nabla u|$ is sufficiently large. Therefore, by (5.16),

$$f_1 \geq \nu_1\Big(1 + \sum_{i=1}^n f_i\Big)$$

and a bound $|\nabla u(x_0,t_0)| \leq C$ follows from (5.23). □

Acknowledgment

This is an improvement of part of my thesis. I wish to thank my adviser Professor Bo Guan for leading me to this problem and many useful suggestions and comments.

References

[1] B. Andrews, Contraction of convex hypersurfaces in Euclidean space, Calc. Var. Partial Differential Equations 2 (1994) 151–171.

[2] L. Caffarelli, L. Nirenberg, J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III: functions of the eigenvalues of the Hessian, Acta Math. 155 (1985) 261–301.

[3] C. Gerhardt, Closed Weingarten hypersurfaces in Riemannian manifolds, J. Differential Geom. 43 (1999) 612–641.

[4] B. Guan, The Dirichlet problem for Hessian equations on Riemannian manifolds, Calc. Var. Partial Differential Equations 8 (1999) 45–69.

[5] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, Duke Math. J. 163 (2014) 1491–1524.

[6] B. Guan, The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, arXiv:1403.2133.

[7] B. Guan, H.-M. Jiao, Second order estimates for Hessian type fully nonlinear elliptic equations on Riemannian manifolds, Calc. Var. Partial Differential Equations (2015), http://dx.doi.org/10.1007/s00526-015-0880-8.

[8] B. Guan, H.-M. Jiao, The Dirichlet problem for Hessian type fully nonlinear elliptic equations on Riemannian manifolds, Discrete Contin. Dyn. Syst. Ser. A 36 (2016) 701–714.

[9] N.M. Ivochkina, O.A. Ladyzhenskaya, On parabolic equations generated by symmetric functions of the principal curvatures of the evolving surface or of the eigenvalues of the Hessian. Part I: Monge–Ampère equations, St. Petersburg Math. J. 6 (1995) 575–594.

[10] N.M. Ivochkina, O.A. Ladyzhenskaya, Flows generated by symmetric functions of the eigenvalues of the Hessian, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 221 (1995) 127–144 (in Russian); English transl. in: J. Math. Sci. 87 (1997) 3353–3365.

[11] H.-M. Jiao, Z.-N. Sui, The first initial–boundary value problem for a class of fully nonlinear parabolic equations on Riemannian manifolds, Int. Math. Res. Not. IMRN 2015 (9) (2015) 2576–2595.

[12] Y.-Y. Li, Some existence results of fully nonlinear elliptic equations of Monge–Ampère type, Comm. Pure Appl. Math. 43 (1990) 233–271.

[13] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995) 151–164.

[14] J.I.E. Urbas, Hessian equations on compact Riemannian manifolds, in: Nonlinear Problems in Mathematical Physics and Related Topics, II, Kluwer/Plenum, New York, 2002, pp. 367–377.