ARTICLE IN PRESS YJDEQ:8783

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Journal of Differential Equations

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Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise ☆

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Received 26 June 2016; revised 27 March 2017

Abstract

We first establish some sufficient conditions for constructing a random exponential attractor for a continuous cocycle on a separable Banach space and weighted spaces of infinite sequences. Then we apply our abstract result to study the existence of random exponential attractors for non-autonomous first order dissipative lattice dynamical systems with multiplicative white noise.

MSC: 37L55; 60H15; 35B40

Keywords: Random exponential attractor; Cocycle; Stochastic lattice system; Fractal dimension; Multiplicative white noise

1. Introduction

It is well known that the theory of attractors (including the global attractor, pullback attractor or kernel sections, uniform attractor, exponential attractor, pullback and uniform exponential attractor) for deterministic autonomous and non-autonomous dynamical systems or evolution

☆ This work is supported by the National Natural Science Foundation of China under Grant Number 11471290.
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http://dx.doi.org/10.1016/j.jde.2017.03.044

equations has been developed intensively since late seventies of the last century, see [2,4,11,12,14,20,21,24,25,30,31,34,37,38,40–44,46].

When dealing with effects of uncertainty or noise from natural phenomena, the study of stochastic evolution equations has attracted lots of interests from both mathematicians and physicists [1,16,17]. The random attractor, was first studied by Ruelle [32], is an important concept to describe asymptotic behavior for a random dynamical system and to capture the essential dynamics with possibly extremely wide fluctuations. Since the mid-90s of the last century, there have been many publications concerning the theory of random attractors (mainly on existence, semicontinuity and bound of Hausdorff/fractal dimensions) and applications to stochastic evolution equations (such as Navier–Stokes equation, reaction–diffusion equations, wave equations and lattice systems with noise), see [3,5,6,8,9,16–19,23,26,27,29,33,36,39,47,48] and the references wherein.

However, there is an intrinsic drawback that random attractor sometimes attracts orbits at a relatively slow rate so that it takes an unexpected long time to reach it. Moreover, in general, it is usually difficult to estimate the attracting rate in terms of physical parameters of the studied system. And the attractor is possible sensitive to perturbations which makes it unobservable in experiments and numerical simulations. To overcome this drawback, Shirikyan and Zelik in [33] introduced the concept of random exponential attractor, which has finite fractal dimension and attracts exponentially any trajectory and is positively invariant, then it contains random attractor and become an appropriate alternative to study the asymptotic behavior of random dynamical systems. And [33] presents some sufficient conditions for constructing a random exponential attractor for an autonomous random dynamical system and application to nonlinear reaction–diffusion system with a random perturbation. But the method or conditions given in [33] is not easy to be verified for some stochastic partial differential equations and lattice systems driven by white noises.

We notice that the evolution mode of states in a random system is, in some sense, similar to the deterministic non-autonomous one and there were several construction methods to obtain a pullback exponential attractor for a (deterministic) process, see [11,12,20,21,25,30,43]. We also notice that there is a fundamental difference between random system and deterministic one. In contrast to the deterministic case, a trajectory of a random system is often unbounded in time (explicitly, along the path of sample point). Thus, if no imposition some “strongly” restriction on the system, then the constants in appropriate squeezing property (playing a key role in the construction of an exponential attractor) will depend on time (hence, be unbounded), and so, a trivial straightforward extension from deterministic system to random system does not work. Fortunately, some time averages of these quantities can be bounded and possibly controlled, which provides a useful way for constructing an exponential attractor for a random system.

In this article, motivated by ideas of [25,33,43,47], we first establish some sufficient conditions for the existence and construction of a random exponential attractor for a continuous cocycle on a separable Banach space and weighted spaces of infinite sequences. Here it is worth mentioning that our conditions just need to check the boundedness of some random variables in the mean and can be easily verified for some stochastic evolution equations.

Recently, lattice dynamical systems (LDSs) (or ordinary differential equations on infinite lattices) have drawn much attention from researchers because of their wide range of applications in various areas (e.g. [13,15]). Since Bates et al. [4] in 2001 presented a framework on the existence and upper semicontinuity of a global attractor associated with autonomous first-order LDSs, there have been a lot of publications concerning various attractors (including global attractor, uniform attractor, pullback attractor or kernel section, exponential attractor, pullback and uniform exponential attractor) for deterministic autonomous and non-autonomous LDSs, see [4,7–9,37,38,40–44,46]. And there are many publications concerning the existence, upper semi-continuity and bound of Kolmogorov entropy of random attractors for autonomous and non-autonomous stochastic LDSs driven by additive and multiplicative white noises, see [3,6,28,36,39] and so on. But until now, as we know, there is no result concerning the dimension of random attractor and existence of random exponential attractors for stochastic LDSs.

As an application to our abstract result, we study the existence of a random exponential attractor for the following first order non-autonomous lattice systems with multiplicative white noise:

$$\begin{cases} du_i = (-\lambda_i u_i - (Au)_i + f_i(u_i,t) + g_i(t))\,dt + \epsilon u_i \circ dw(t), \quad t > \tau, \\ u_i(\tau) = u_{i,\tau}, \quad i \in \mathbb{Z}, \quad \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\epsilon \in \mathbb{R}$; for $i \in \mathbb{Z}$, $\lambda_i > 0$, u_i, $g_i(t)$, $f_i(u_i,t) \in \mathbb{R}$; $u = (u_i)_{i\in\mathbb{Z}}$, $w(t)$ is a two-sided real-valued Wiener process on a probability space and $\circ$ denotes the Stratonovich sense of the stochastic term; A is a linear coupled operator.

For non-autonomous stochastic system (1.1), Bates et al. [6] studied the existence of its random attractor in a weighted space of sequences and proved the existence of an infinite dimensional random attractor of (1.1) when the function $f_i(s,t)$ is taken a special form.

Here we will prove the existence of a random exponential attractor in weighted spaces of sequences for (1.1) under some dissipative conditions, and hence, (1.1) possess a random attractor with finite fractal dimension, which implies that the asymptotic behavior of (1.1) can be described by finite independent parameters.

The rest of paper is organized as follows. In section 2, we present some sufficient conditions for the existence of a random exponential attractor for a continuous cocycle on separable Banach space and weighted spaces of sequences. In section 3, we study the existence of a random exponential attractor for system (1.1) in weighted spaces of sequences.

2. Random exponential attractor for cocycle

In this section, we first establish some sufficient conditions for constructing a random exponential attractor for a continuous cocycle, and then give some special conditions which can be verified for some stochastic evolution equations.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\omega\}_{t\in\mathbb{R}})$ be *an ergodic metric dynamical system* on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ is a family of measure preserving transformations such that $(t,\omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_s\theta_t$ for all s, $t \in \mathbb{R}$. In addition, if for any $F \in \mathcal{F}$, provided $\mathbb{P}(\theta_t^{-1}F \triangle F) = 0$, it holds that $\mathbb{P}(F) = 0$ or 1 for all $t \in \mathbb{R}$.

Let X be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a *continuous cocycle* on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ if for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(\cdot,\tau,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(ii) $\Phi(0,\tau,\omega,\cdot)$ is the identity on X;
(iii) $\Phi(t+s,\tau,\omega,\cdot) = \Phi(t,\tau+s,\theta_s\omega,\Phi(s,\tau,\omega,\cdot))$;
(iv) $\Phi(t,\tau,\omega) = \Phi(t,\tau,\omega,\cdot)$: $X \to X$ is continuous.

Denote $\mathcal{D}(X)$ the collection of all tempered families of nonempty bounded subsets of X, that is, for any family $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$, it holds that for every $\varsigma > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\lim_{t\to-\infty} e^{\varsigma t}\|D(\tau + t, \theta_t\omega)\|_X = 0$, where $\|D\|_X = \sup_{u\in D}\|u\|$. For any $u_\tau \in X$ and $\omega \in \Omega$, the subset $\{\Phi(t,\tau,\omega)u_\tau : t \in [\tau,+\infty)\} \subset X$ is called a random trajectory starting from u_τ at initial time $\tau \in \mathbb{R}$ for $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$. Recall that the distance between a point $u \in X$ and a subset $F \subset X$ is given by $\mathrm{d}(u,F) = \inf_{v\in F}||u - v||_X$. The Hausdorff and symmetric distances between two subsets are defined by, respectively,

$$\mathrm{d}_h(F_1,F_2) = \sup_{u\in F_1}\mathrm{d}(u,F_2), \quad \mathrm{d}_s(F_1,F_2) = \max\{\mathrm{d}_h(F_1,F_2), \mathrm{d}_h(F_2,F_1)\}, \quad \forall F_1,F_2 \subset X.$$

Definition 2.1. A family $\{\mathcal{A}(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ of subsets of X is called *a random exponential attractor* in $\mathcal{D}(X)$ for the continuous cocycle $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$ over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\omega\}_{t\in\mathbb{R}})$ if there is a set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that for any $\tau \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$, it holds that

(i) Compactness: $\mathcal{A}(\tau,\omega)$ is compact in X and measurable in ω.
(ii) Finite-dimensionality: there exists a random variable ζ_ω $(< \infty)$ such that $\sup_{\tau\in\mathbb{R}}\dim_f\mathcal{A}(\tau,\omega) \le \zeta_\omega < \infty$, where $\dim_f\mathcal{A}(\tau,\omega) = \limsup_{\varepsilon\to0+}\frac{\ln N_\varepsilon(\mathcal{A}(\tau,\omega))}{-\ln\varepsilon}$ is the fractal dimension of $\mathcal{A}(\tau,\omega)$ and $N_\varepsilon(\mathcal{A}(\tau,\omega))$ is the minimal number of balls with radius ε covering $\mathcal{A}(\tau,\omega)$ in X.
(iii) Positive invariance: $\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{A}(\tau-t,\theta_{-t}\omega) \subseteq \mathcal{A}(\tau,\omega)$ for all $t \ge 0$.
(iv) Exponential attraction: there exists a constant $a > 0$ such that for any $B \in \mathcal{D}(X)$, there exist random variables $t_B(\tau,\omega) \ge 0$, $Q(\tau,\omega,||B||_X) > 0$ satisfying

$$\mathrm{d}_h(\Phi(t,\tau-t,\theta_{-t}\omega)B(\tau-t,\theta_{-t}\omega),\mathcal{A}(\tau,\omega)) \le Q(\tau,\omega,||B||_X)e^{-at}, \quad t \ge t_B(\tau,\omega).$$

Remark 2.1. By Definition 2.1, existence of a random exponential attractor $\{\mathcal{A}(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ for $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$ with $\sup_{\tau\in\mathbb{R}}\dim_f\mathcal{A}(\tau,\omega) \le \tilde{\zeta}$ (positive constant) implies the existence of a random attractor of $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$ with finite fractal dimension.

Let $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\omega\}_{t\in\mathbb{R}})$. Assume that there exist a family of tempered closed random subsets $\{\chi(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ of X satisfying the following conditions: for any fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(H1) *there exists a tempered random variable R_ω (independent of τ) such that $\sup_{\tau\in\mathbb{R}}\sup_{u,v\in\chi(\tau,\omega)}||u-v||_X \le R_\omega < \infty$, and $R_{\theta_t\omega}$ is continuous in t for all $t \in \mathbb{R}$;*
(H2) *positive invariance: $\Phi(t,\tau-t,\theta_{-t}\omega)\chi(\tau-t,\theta_{-t}\omega) \subseteq \chi(\tau,\omega)$ for all $t \ge 0$;*
(H3) *$\{\chi(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ is pullback absorbing in the sense that for any set $B \in \mathcal{D}(X)$, there exists $T_B = T_B(\tau,\omega) \ge 0$ such that*

$$\Phi(t,\tau-t,\theta_{-t}\omega)B(\tau-t,\theta_{-t}\omega) \subseteq \chi(\tau,\omega), \quad \forall t \ge T_B; \tag{2.1}$$

(H4) *there exist a positive constant $T > 0$ and a random variable $\tilde{L}_\omega = \tilde{L}_\omega(T) > 0$ such that*

$$||\Phi(t,\tau,\omega)u - \Phi(t,\tau,\omega)v||_X \le \tilde{L}_\omega||u-v||_X, \quad \forall u,v \in \chi(\tau,\omega), t \in [0,T]; \tag{2.2}$$

(H5) *there are random variables* $\delta_\omega = \delta_\omega(T) \geq 0$, $L_\omega = L_\omega(T) > 0$ *such that*

$$||\Phi(T,\tau,\omega)u - \Phi(T,\tau,\omega)v||_X \leq \delta_\omega ||u-v||_X + ||K(T,\tau,\omega)u - K(T,\tau,\omega)v||_X, \quad \forall u,v \in \chi(\tau,\omega), \tag{2.3}$$

where $K(T,\tau,\omega)\colon \chi(\tau,\omega) \to Z$ *satisfies*

$$||K(T,\tau,\omega)u - K(T,\tau,\omega)v||_Z \leq L_\omega ||u-v||_X, \quad \forall u,v \in \chi(\tau,\omega), \tag{2.4}$$

and Z *is another Banach space which is embedded compactly in* X*;*

(H6) $\ln \tilde{L}_\omega \in L^1(\Omega,\mathbb{P})$ *and there exists* $\mu_\omega > 0$ *such that*

$$\ln N_{\mu_\omega} \in L^1(\Omega,\mathbb{P}), \quad \ln a_\omega = \ln[2(\delta_\omega + \mu_\omega L_\omega)] \in L^1(\Omega,\mathbb{P}), \tag{2.5}$$

and

$$0 \leq \mathbf{E}(\ln N_{\mu_\omega}) < \infty, \quad 0 \leq \mathbf{E}(\ln \tilde{L}_\omega) < \infty, \quad -\infty < \mathbf{E}(\ln a_\omega) < 0, \tag{2.6}$$

where N_μ *is the minimal number of closed balls of* X *with radius* μ *which cover the closed unit ball* $\bar{B}^Z(0;1)$ *of* Z *centered at* 0*; "*$\mathbf{E}$*" denotes the expectation.*

Theorem 2.1. *Assume that conditions (H1)–(H6) are satisfied. Then there exists a random exponential attractor* $\{\mathcal{A}(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ *for the continuous cocycle* $\{\Phi(t,\tau,\omega)\}_{t\geq 0,\tau\in\mathbb{R},\omega\in\Omega}$ *with the following properties: for any* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$*,*

(i) $\mathcal{A}(\tau,\omega) \subseteq \chi(\tau,\omega)$ *is a compact set of* X*;*
(ii) $\Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{A}(\tau-t,\theta_{-t}\omega) \subseteq \mathcal{A}(\tau,\omega)$ *for all* $t \geq 0$*;*
(iii) $\dim_f \mathcal{A}(\tau,\omega) \leq -\frac{8\mathbf{E}[\ln N_\omega]}{\mathbf{E}[\ln a_\omega]} < \infty$*;*
(iv) for any set $B \in \mathcal{D}(E)$*, there exist random variables* $\hat{T}_\omega \geq 0$, $\hat{b}_{\omega,B} > 0$ *such that*

$$\mathrm{d}_h(\Phi(t,\tau-t,\theta_{-t}\omega)B(\tau-t,\theta_{-t}\omega),\mathcal{A}(\tau,\omega)) \leq \hat{b}_{\omega,B} e^{\frac{\mathbf{E}[\ln a_\omega]}{8T}t}, \quad t \geq T_B + \hat{T}_\omega. \tag{2.7}$$

Proof. For any fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let

$$a_\omega = 2(\delta_\omega + \mu_\omega L_\omega) > 0. \tag{2.8}$$

For any $m \in \mathbb{Z}$, $n, j \in \mathbb{N}$, write

$$\Phi(n,\tau,m-j,\omega) = \Phi(nT,\tau+mT-jT,\theta_{mT-jT}\omega),$$
$$\chi(\tau,m-j,\omega) = \chi(\tau+mT-jT,\theta_{mT-jT}\omega).$$

1) Covering of $\Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega)$.

Consider a discrete cocycle $\{\Phi(n,\tau,m,\omega)\}_{m\in\mathbb{Z},n\in\mathbb{N},\tau\in\mathbb{R},\omega\in\Omega}$ in X over $\mathbb{R}$ and $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\omega\}_{t\in\mathbb{R}})$. Similar to the proof of Theorem 2.1 in [47], $\Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega)$ has a covering of closed balls of X with radius $r_{1\sim n,m,\omega} = a_{\theta_{mT-T}\omega} a_{\theta_{mT-2T}\omega} \cdots a_{\theta_{mT-nT}\omega} R_{\theta_{mT-nT}\omega}$ and centers in itself by induction on n:

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$$\begin{cases} \Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega) \subset \bigcup_{i=1}^{N_{1\sim n,m,\omega}} \bar{B}(u_{-n,m,i}; r_{1\sim n,m,\omega}), \\ u_{-n,m,i} \in \Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega), \quad 1 \le i \le N_{m,n,\omega}, \\ N_{1\sim n,m,\omega} = N_{\mu_{\theta_{mT-T}\omega}} N_{\mu_{\theta_{mT-2T}\omega}} \cdots N_{\mu_{\theta_{mT-nT}\omega}}, \end{cases} \tag{2.9}$$

where $\bar{B}(u; r)$ denotes the closed ball of X centered at u with radius r.

If $n = 0$, then by the identity of $\Phi(0,\tau,\omega)$ on X, $\Phi(0,\tau,m,\omega)\chi(\tau,m,\omega) = \chi(\tau,m,\omega)$, thus, we can take $u_{0,m,1} \in \chi(\tau,m,\omega) \subset \bar{B}(u_{0,m,1}; R_{\theta_{mT}\omega})$ arbitrarily by condition (H1).

Suppose $n = k$, (2.9) holds. Consider $n = k+1$, then by the cocycle property of Φ, we have

$$\begin{cases} \Phi(k+1,\tau,m-k-1,\omega)\chi(\tau,m-k-1,\omega) \\ = \Phi(1,\tau,m,\omega)\Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega) \\ \subseteq \bigcup_{i=1}^{N_{2\sim(k+1),m,\omega}} \Phi(1,\tau,m,\omega)[\bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \\ \qquad\qquad \cap\, \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega)], \\ N_{2\sim(k+1),m,\omega} = N_{\mu_{\theta_{mT-(k+1)T}\omega}} N_{\mu_{\theta_{mT-kT}\omega}} \cdots N_{\mu_{\theta_{mT-2T}\omega}}, \\ r_{2\sim(k+1),m,\omega} = a_{mT-2T\omega} a_{\theta_{mT-3T}\omega} \cdots a_{\theta_{mT-(k+1)T}\omega} R_{\theta_{mT-(k+1)T}\omega}. \end{cases} \tag{2.10}$$

By (2.4),

$$\begin{aligned} &K(1,\tau,m,\omega)[\bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \cap \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega)] \\ &\quad \subset \bar{B}^Z\left(K(1,\tau,m,\omega)u_{-(k+1),m-1,i}; L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}\right) \\ &\quad \subset \bigcup_{j=1}^{N_{\mu_{\theta_{mT-T}\omega}}} \bar{B}\left(\tilde{V}_{-(k+1),m,i,j}; \mu_{\theta_{mT-T}\omega} L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}\right) \end{aligned} \tag{2.11}$$

with centers $\tilde{V}_{-(k+1),m,i,j} \in X$, $1 \le j \le N_{\mu_{\theta_{mT-T}\omega}}$ and radius $\mu_{\theta_{mT-T}\omega} L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}$. Then for every j, we can assume that

$$\begin{aligned} &K(1,\tau,m,\omega)[\bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \cap \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m,\omega,(k+1))] \\ &\quad \cap \bar{B}\left(\tilde{V}_{-(k+1),m-1,i,j}; \mu_{\theta_{mT-T}\omega} L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}\right) \end{aligned} \neq \emptyset.$$

So, for each j, we can choose a point $V_{-(k+1),m-1,i,j}$ such that

$$\begin{cases} V_{-(k+1),m-1,i,j} \in \bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \\ \qquad \cap\, \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega), \\ K(1,\tau,m,\omega)V_{-(k+1),m-1,i,j} \in \bar{B}\left(\tilde{V}_{-(k+1),m-1,i,j}; \mu_{\theta_{mT-T}\omega} L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}\right). \end{cases} \tag{2.12}$$

By (2.11),

$$\begin{aligned} &K(1,\tau,m,\omega)[\bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \cap \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega)] \\ &\quad \subset \bigcup_{j=1}^{N_{\mu_{\theta_{mT-T}\omega}}} \bar{B}\left(K(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}; 2\mu_{\theta_{mT-T}\omega} L_{\theta_{mT-T}\omega} r_{2\sim(k+1),m,\omega}\right). \end{aligned} \tag{2.13}$$

For any

$$u \in \bar{B}(u_{-(k+1),m-1,i}; r_{2\sim(k+1),m,\omega}) \cap \Phi(k,\tau,m-(k+1),\omega)\chi(\tau,m-(k+1),\omega),$$

there is a $j \in \{1,\dots,N_{\mu_{\theta_{mT-T}\omega}}\}$ such that

$$K(1,\tau,m,\omega)u \in \bar{B}\left(V_{-(k+1),m,i,j}; 2\mu_{\theta_{mT-T}\omega}L_{\theta_{mT-T}\omega}r_{2\sim(k+1),m,\omega}\right). \tag{2.14}$$

It then follows from (2.3) and (2.14) that

$$\begin{aligned} &||\Phi(1,\tau,m,\omega)u - \Phi(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}||_X \\ &\le \delta_{\theta_{mT-T}\omega}||u - V_{-(k+1),m-1,i,j}||_X + ||K(1,\tau,m,\omega)u - K(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}||_X \\ &\le \delta_{\theta_{mT-T}\omega}||u - V_{-(k+1),m-1,i,j}||_X + 2\mu_{\theta_{mT-T}\omega}L_{\theta_{mT-T}\omega}r_{2\sim(k+1),m,\omega}. \end{aligned} \tag{2.15}$$

By (2.12),

$$\begin{aligned} ||u - V_{-(k+1),m-1,i,j}||_X &\le ||u - u_{-(k+1),m-1,i}||_X + ||u_{-(k+1),m-1,i} - V_{-(k+1),m-1,i,j}||_X \\ &\le 2r_{2\sim(k+1),m,\omega}. \end{aligned}$$

Thus,

$$\begin{aligned} &||\Phi(1,\tau,m,\omega)u - \Phi(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}||_X \\ &\le 2(\delta_{\theta_{mT-T}\omega} + \mu_{\theta_{mT-T}\omega}L_{\theta_{mT-T}\omega})r_{2\sim(k+1),m,\omega} \\ &= r_{1\sim(k+1),m,\omega}. \end{aligned} \tag{2.16}$$

Hence, by (2.10), (2.13) and (2.16),

$$\begin{aligned} &\Phi(k+1,\tau,m-k-1,\omega)\chi(\tau,m-k-1,\omega) \\ &\subset \bigcup_{i=1}^{N_{2\sim(k+1),m,\omega}} \bigcup_{j=1}^{N_{\mu_{\theta_{mT-T}\omega}}} \bar{B}\left(\Phi(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}; r_{1\sim(k+1),m,\omega}\right) \\ &\subset \bigcup_{i=1}^{N_{1\sim(k+1),m,\omega}} \bar{B}\left(u_{-(k+1),m,i}; r_{1\sim(k+1),m,\omega}\right), \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} &\{u_{-(k+1),m,i}; 1 \le i \le N_{1\sim(k+1),m,\omega}\} \\ &= \{\Phi(1,\tau,m,\omega)V_{-(k+1),m-1,i,j}; 1 \le i \le N_{2\sim(k+1),m,\omega}, 1 \le j \le N_{\mu_{\theta_{mT-T}\omega}}\} \\ &\subset \Phi(k+1,\tau,m-k-1,\omega)\chi(\tau,m-k-1,\omega). \end{aligned} \tag{2.18}$$

By induction, we obtain (2.9).

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2) Construction of random exponential attractor for $\{\Phi(n,\tau,m,\omega)\}_{m\in\mathbb{Z},n\in\mathbb{N},\tau\in\mathbb{R},\omega\in\Omega}$.

For fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$, $m\in\mathbb{Z}$ and any $n\in\mathbb{N}$, put

$$\begin{aligned}\mathcal{A}_{-n}(\tau+mT,\theta_{mT}\omega)&=\{u_{-n,m,i}:1\le i\le N_{1\sim n,m,\omega}\}\\&\subseteq\Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega)\\&\subseteq\chi(\tau,m,\omega).\end{aligned}\tag{2.19}$$

Then

$$\Phi(p,\tau,m,\omega)\mathcal{A}_{-n}(\tau+mT-pT,\theta_{mT-pT}\omega)\subseteq\mathcal{A}_{-n-p}(\tau+mT,\theta_{mT}\omega),\quad\forall p\in\mathbb{N},\tag{2.20}$$

which implies that the number of element of $\mathcal{A}_{-n}(\tau+mT,\theta_{mT}\omega)$ satisfies:

$$\#\mathcal{A}_{-n_1}(\tau+mT,\theta_{mT}\omega)\le\#\mathcal{A}_{-n_2}(\tau+mT,\theta_{mT}\omega)\quad\text{for}\quad n_1\le n_2.\tag{2.21}$$

Set

$$\mathcal{A}(\tau+mT,\theta_{mT}\omega)=\overline{\cup_{n=0}^{\infty}\mathcal{A}_{-n}(\tau+mT,\theta_{mT}\omega)}\subseteq\chi(\tau+mT,\theta_{mT}\omega).\tag{2.22}$$

Then we can show that $\{\mathcal{A}(\tau+mT,\theta_{mT}\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ is a random exponential attractor for the discrete cocycle $\{\Phi(n,\tau,m,\omega)\}_{m\in\mathbb{Z},n\in\mathbb{N},\tau\in\mathbb{R},\omega\in\Omega}$. In fact, since $\{\theta_t\omega\}_{t\in R}$ is ergodic, by Birkhoff ergodic Theorem [35] and (H6), it follows that

$$\begin{aligned}&\frac{\ln a_{\theta_{mT-T}\omega}+\ln a_{\theta_{mT-2T}\omega}+\cdots+\ln a_{\theta_{mT-nT}\omega}}{n}\overset{n\to\infty}{\to}\mathbf{E}[\ln a_\omega]<0,\\&\frac{\ln N_{\mu_{\theta_{mT-T}\omega}}+\ln N_{\mu_{\theta_{mT-2T}\omega}}+\cdots+\ln N_{\mu_{\theta_{mT-nT}\omega}}}{n}\overset{n\to\infty}{\to}\mathbf{E}[\ln N_\omega],\\&\frac{\ln \tilde{L}_{\mu_{\theta_{mT-T}\omega}}+\ln \tilde{L}_{\mu_{\theta_{mT-2T}\omega}}+\cdots+\ln \tilde{L}_{\mu_{\theta_{mT-nT}\omega}}}{n}\overset{n\to\infty}{\to}\mathbf{E}[\ln \tilde{L}_\omega].\end{aligned}\tag{2.23}$$

Thus, for $\omega\in\Omega$, there exists a large integer $n_0=n_0(\omega)\ge 0$ such that for $n\ge n_0$,

$$\ln a_{\theta_{mT-T}\omega}+\ln a_{\theta_{mT-2T}\omega}+\cdots+\ln a_{\theta_{mT-nT}\omega}\le\frac{n}{2}\mathbf{E}[\ln a_\omega]<0,\tag{2.24}$$

$$\ln N_{\mu_{\theta_{mT-T}\omega}}+\ln N_{\mu_{\theta_{mT-2T}\omega}}+\cdots+\ln N_{\mu_{\theta_{mT-nT}\omega}}\le 2n\mathbf{E}[\ln N_\omega].\tag{2.25}$$

Let $\varepsilon_{n,m}(\omega)=a_{\theta_{mT-T}\omega}a_{\theta_{mT-2T}\omega}\cdots a_{\theta_{mT-nT}\omega}$. By (2.24),

$$\varepsilon_{n,m}(\omega)\ =e^{\ln a_{\theta_{mT-T}\omega}+\ln a_{\theta_{mT-2T}\omega}+\cdots+\ln a_{\theta_{mT-nT}\omega}}\le e^{\frac{n}{2}\mathbf{E}[\ln a_\omega]},\quad\forall n\ge n_0.$$

By (H1) and [1], there exists a tempered random variable $b_{\omega,m}$ (>0) such that

$$R_{\theta_{mT-nT}\omega}\le b_{m,\omega}e^{-\frac{n}{4}\mathbf{E}[\ln a_\omega]},\quad\forall n\in\mathbb{N}.$$

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Then for $n \geq n_0$, we have

$$0 < r_{1\sim n,m,\omega} = \varepsilon_{n,m}(\omega) R_{\theta_{mT-nT}\omega} \leq b_{m,\omega} e^{\frac{n}{4}\mathbf{E}[\ln a_\omega]} \overset{n\to+\infty}{\to} 0. \tag{2.26}$$

(1) *Compactness*. Let $0 < \varepsilon < 1$ be a given number. By (2.21), there exists an integer $n_\varepsilon = n_\varepsilon(\omega) \in \mathbb{N}$ such that $r_{1\sim n_\varepsilon,m,\omega} \leq \varepsilon < r_{1\sim(n_\varepsilon-1),m,\omega}$. By (2.21) and (2.24), for all $n \geq n_\varepsilon$,

$$\begin{aligned}
&\mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega)\\
&\subseteq \Phi(n, \tau, m-n, \omega)\chi(\tau, m-n, \omega)\\
&\subseteq \Phi(n_\varepsilon, \tau, m-n_\varepsilon, \omega)\Phi(n-n_\varepsilon, \tau, m-n, \omega)\chi(\tau, m-n, \omega)\\
&\subseteq \Phi(n_\varepsilon, \tau, m, \omega)\chi(\tau, m, \omega)\\
&\subseteq \bigcup_{i=1}^{N_{1\sim n_\varepsilon,m,\omega}} \bar{B}(u_{-n_\varepsilon,m,i}; r_{1\sim n_\varepsilon,m,\omega}) \subseteq \bigcup_{i=1}^{N_{1\sim n_\varepsilon,m,\omega}} \bar{B}(u_{-n_\varepsilon,m,i}; \varepsilon).
\end{aligned}$$

Thus,

$$\bigcup_{n=n_\varepsilon}^{+\infty} \mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega) \subseteq \bigcup_{i=1}^{N_{1\sim n_\varepsilon,m,\omega}} \bar{B}(u_{-n_\varepsilon,m,i}; \varepsilon), \tag{2.27}$$

that is, $\bigcup_{n=n_\varepsilon}^{+\infty} \mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega)$ is covered by $N_{1\sim n_\varepsilon,m,\omega}$ closed balls with radius ε. On the other hand, $\bigcup_{n=0}^{n_\varepsilon-1} \mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega) = \bigcup_{n=0}^{n_\varepsilon-1} \{u_{-n,m,i} : 1 \leq i \leq N_{1\sim n,m,\omega}\}$ is a finite set. So, by definition (2.22),

$$\mathcal{A}(\tau + mT, \theta_{mT}\omega) = \overline{\left(\bigcup_{n=0}^{n_\varepsilon-1} \mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega)\right) \cup \left(\bigcup_{n=n_\varepsilon}^{+\infty} \mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega)\right)} \tag{2.28}$$

is a compact set of X.

(2) *Finite fractal dimension*: $\dim_f \mathcal{A}(\tau + mT, \theta_{mT}\omega) = \limsup_{\varepsilon\to 0+} \frac{\ln N_\varepsilon(\mathcal{A}(\tau+mT,\theta_{mT}\omega))}{-\ln\varepsilon}$, where $N_\varepsilon(\mathcal{A}(\tau + mT, \theta_{mT}\omega))$ is the minimal number of balls with radius ε covering $\mathcal{A}(\tau + mT, \theta_{mT}\omega)$ in X. By (2.26), we have $n_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and $\ln\varepsilon \leq \ln r_{1\sim(n_\varepsilon-1),m,\omega}$, that is,

$$\frac{1}{-\ln\varepsilon} \leq \frac{1}{-\ln r_{1\sim(n_\varepsilon-1),m,\omega}} \leq \frac{1}{-\frac{n_\varepsilon-1}{4}\mathbf{E}[\ln a_\omega] - \ln b_{\omega,m}}. \tag{2.29}$$

Taking ε sufficient small such that $n_\varepsilon - 1 \geq n_0$, then by (2.27) and (2.28),

$$\begin{aligned}
N_\varepsilon(\mathcal{A}(\tau + mT, \theta_{mT}\omega)) &\leq \#\mathcal{A}_{-n_\varepsilon}(\tau + mT, \theta_{mT}\omega) + \sum_{n=0}^{n_\varepsilon-1} \#\mathcal{A}_{-n}(\tau + mT, \theta_{mT}\omega)\\
&\leq (n_\varepsilon + 1) N_{\mu_{\theta_{mT-n_\varepsilon T}\omega}} N_{\mu_{\theta_{mT-(n_\varepsilon-1)T}\omega}} \cdots N_{\mu_{\theta_{mT-T}\omega}}.
\end{aligned} \tag{2.30}$$

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By (2.25) and (2.30),

$$\begin{aligned}\ln N_\varepsilon(\mathcal{A}(\tau+mT,\theta_{mT}\omega)) &\le \ln(n_\varepsilon+1)+\ln N_{\mu_{\theta_{mT-T}\omega}}+\cdots+\ln N_{\mu_{\theta_{mT-n_\varepsilon T}\omega}}\\ &\le \ln(n_\varepsilon+1)+n_\varepsilon\cdot 2\mathbf{E}[\ln N_\omega]. \end{aligned}\tag{2.31}$$

It follows from (2.29) and (2.31) that $\dim_f \mathcal{A}(\tau+mT,\theta_{mT}\omega)$ has an upper bound (constant):

$$\begin{aligned}\dim_f \mathcal{A}(\tau+mT,\theta_{mT}\omega) &\le \lim_{n_\varepsilon\to+\infty}\sup \frac{\ln(n_\varepsilon+1)+n_\varepsilon\cdot 2\mathbf{E}[\ln N_\omega]}{-\frac{n_\varepsilon-1}{4}\mathbf{E}[\ln a_\omega]-\ln b_{\omega,m}}\\ &= -\frac{8\mathbf{E}[\ln N_\omega]}{\mathbf{E}[\ln a_\omega]}<\infty, \quad \forall \tau\in\mathbb{R},\omega\in\Omega,m\in\mathbb{Z}. \end{aligned}\tag{2.32}$$

(3) *Positive invariance*: for $\tau\in\mathbb{R}$, $\omega\in\Omega$, $m\in\mathbb{Z}$ and $p\in\mathbb{N}$, by (2.20),

$$\begin{aligned}&\Phi(pT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega)\\ &\subseteq \overline{\cup_{n=0}^{\infty}\Phi(pT,\tau+mT,\theta_{mT}\omega)\mathcal{A}_{-n}(\tau+mT,\theta_{mT}\omega)}\\ &\subseteq \overline{\cup_{n=0}^{\infty}\mathcal{A}_{-n}(\tau+mT+pT,\theta_{mT+pT}\omega)}\\ &=\mathcal{A}(\tau+mT+pT,\theta_{mT+pT}\omega), \quad p\in\mathbb{N}. \end{aligned}\tag{2.33}$$

(4) *Exponential attraction*: since

$$u_{-n,m,i}\in\mathcal{A}(\tau+mT,\theta_{mT}\omega)\cap\Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega), \quad \forall 1\le i\le N_{1\sim n,m,\omega},$$

by (2.9) and (2.26), for $n\ge n_0$,

$$\mathrm{d}_h(\Phi(n,\tau,m-n,\omega)\chi(\tau,m-n,\omega),\mathcal{A}(\tau+mT,\theta_{mT}\omega))\le r_{1\sim n,m,\omega}\le b_{\omega,m}e^{\frac{n}{4}\mathbf{E}[\ln a_\omega]}. \tag{2.34}$$

3) Construction of random exponential attractor for continuous cocycle $\{\Phi(t,\tau,\omega)\}_{t\ge0,\tau\in\mathbb{R},\omega\in\Omega}$.

For any given τ, $s\in\mathbb{R}$ and $\omega\in\Omega$, let $m\in\mathbb{Z}$ be the (fixed) integer such that $mT\le s<(m+1)T$, set

$$\mathcal{A}(\tau+s,\theta_s\omega)=\Phi(s-mT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega), \quad 0\le s-mT<T. \tag{2.35}$$

(1) *Compactness and finite fractal dimension*. By (H4), $\Phi(s-mT,\tau+mT,\theta_{mT}\omega)$ is Lipschitz continuous from $\chi(\tau+mT,\theta_{mT}\omega)$ into $\chi(\tau+s,\theta_s\omega)$ and $\mathcal{A}(\tau+mT,\theta_{mT}\omega)$ is compact, thus, $\mathcal{A}(\tau+s,\theta_s\omega)$ is compact, $\mathcal{A}(\tau+s,\theta_s\omega)\subseteq\chi(\tau+s,\theta_s\omega)$ and

$$\dim_f \mathcal{A}(\tau+s,\theta_s\omega)\le\dim_f\mathcal{A}(\tau+mT,\theta_{mT}\omega)\le-\frac{8\mathbf{E}[\ln N_\omega]}{\mathbf{E}[\ln a_\omega]}. \tag{2.36}$$

(2) *Positive invariance*. For $t\ge0$, let $m\in\mathbb{Z}$, $n\in\mathbb{N}$ be integers such that $(m+n)T\le s+t<(m+n+1)T$. When $n=0$, by definition (2.35), then

$$\begin{aligned}
&\Phi(t,\tau+s,\theta_s\omega)\mathcal{A}(\tau+s,\theta_s\omega)\\
&=\Phi(t,\tau+s,\theta_s\omega)\Phi(s-mT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega)\\
&=\Phi(t+s-mT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega)\\
&=\mathcal{A}(\tau+s+t,\theta_{s+t}\omega).
\end{aligned}$$

When $n>0$, then by the cocycle property of Φ and (2.20),

$$\begin{aligned}
&\Phi(t,\tau+s,\theta_s\omega)\mathcal{A}(\tau+s,\theta_s\omega)\\
&=\Phi(s+t-(m+n)T,\tau+(m+n)T,\theta_{(m+n)T}\omega)\Phi((m+n)T-s,\tau+s,\theta_s\omega)\\
&\quad\circ\Phi(s-mT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega)\\
&\subseteq\Phi(t+s-mT,\tau+mT,\theta_{mT}\omega)\mathcal{A}(\tau+mT,\theta_{mT}\omega)\\
&=\mathcal{A}(\tau+s+t,\theta_{s+t}\omega).
\end{aligned}\tag{2.37}$$

(3) *Exponential attraction*. Put $u_s\in\chi(\tau+s,\theta_s\omega)$, then by (H2),

$$u_{(m+1)T}=\Phi((m+1)T-s,\tau+s,\theta_s\omega)u_s\in\chi(\tau+(m+1)T,\theta_{(m+1)T}\omega).$$

Write

$$\Phi(s+t,m+n,\tau,\omega)=\Phi(s+t-(m+n)T,\tau+(m+n)T,\theta_{(m+n)T}\omega)$$

and by (H3), (2.34), for $n\ge n_0$,

$$\begin{aligned}
&\mathrm{d}(\Phi(t,\tau+s,\theta_s\omega)u_s,\mathcal{A}(\tau+s+t,\theta_{s+t}\omega))\\
&=\mathrm{d}(\Phi(s+t,m+n,\tau,\omega)\Phi((n-1)T,\tau+(m+1)T,\theta_{(m+1)T}\omega)u_{(m+n)T},\\
&\qquad\Phi(s+t,m+n,\tau,\omega)\mathcal{A}(\tau+(m+n)T,\theta_{(m+n)T}\omega))\\
&\le\tilde{L}_{\theta_{(m+n)T}\omega}\mathrm{d}(\Phi((n-1)T,\tau+(m+1)T,\theta_{(m+1)T}\omega)u_{(m+n)T},\mathcal{A}(\tau+(m+n)T,\theta_{(m+n)T}\omega))\\
&\le\tilde{L}_{\theta_{(m+n)T}\omega}b_{\omega,m+1}e^{\frac{n-1}{4}\mathbf{E}[\ln a_\omega]}.
\end{aligned}$$

So, for $n\ge n_0$,

$$\mathrm{d}_h(\Phi(t,\tau+s,\theta_s\omega)\chi(\tau+s,\theta_s\omega),\mathcal{A}(\tau+s+t,\theta_{s+t}\omega))\le b_{\omega,m+1}e^{\frac{n-1}{4}\mathbf{E}[\ln a_\omega]+\ln[\tilde{L}_{\theta_{(m+n)T}\omega}]}.$$

By (2.23), $\frac{\ln[\tilde{L}_{\theta_{(m+n)T}\omega}]}{n}\overset{n\to\infty}{\to}0$, then there exists $n_1=n_1(\omega)\in\mathbb{N}$ such that

$$\ln[\tilde{L}_{\theta_{(m+n)T}\omega}]\le-\frac{1}{8}n\mathbf{E}[\ln a_\omega],\quad\forall n\ge n_1.$$

Thus, for $n\ge\max\{n_0,n_1\}$,

$$\mathrm{d}_h(\Phi(t,\tau+s,\theta_s\omega)\chi(\tau+s,\theta_s\omega),\mathcal{A}(\tau+s+t,\theta_{s+t}\omega))\le\bar{b}_{\omega,m}e^{\frac{\mathbf{E}[\ln a_\omega]}{8T}t},$$

where

$$\bar{b}_{\omega,m} = b_{\omega,m+1} e^{-\frac{3}{8}\mathbf{E}[\ln a_\omega]} > 0.$$

In particular, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq \max\{n_0, n_1\}T$, it holds that

$$\mathrm{d}_h(\Phi(t, \tau - t, \theta_{-t}\omega)\chi(\tau - t, \theta_{-t}\omega), \mathcal{A}(\tau, \omega)) \leq \check{b}_\omega e^{\frac{\mathbf{E}[\ln a_\omega]}{8T} t}. \tag{2.38}$$

for some tempered random variable $\check{b}_\omega$. By (H3) and (2.38), the attracting property (iv) holds. The proof is completed. □

Theorem 2.2. *Suppose (H1)–(H6) hold. If $\{\Phi(t, \tau, \omega)\}_{t\geq 0, \tau\in\mathbb{R}, \omega\in\Omega}$ satisfies:*

$$\begin{cases} \lim_{t\searrow 0} \sup_{u\in\chi(\tau,\omega)} ||\Phi(t, \tau, \omega)u - u||_X = 0, \\ \lim_{t\searrow 0} \sup_{u\in\chi(\tau-t,\theta_{-t}\omega)} ||\Phi(0, \tau - t, \theta_{-t}\omega)u - u||_X = 0, \end{cases} \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \tag{2.39}$$

then the random exponential attractor $\{\mathcal{A}(\tau, \omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ in Theorem 2.1 has the following continuity: for any $\tau \in R$, $\omega \in \Omega$,

$$\lim_{t\searrow 0} \mathrm{d}_s(\mathcal{A}(\tau + t, \theta_t\omega), \mathcal{A}(\tau, \omega)) = 0, \ \lim_{t\searrow 0} \mathrm{d}_h(\mathcal{A}(\tau - t, \theta_{-t}\omega), \mathcal{A}(\tau, \omega)) = 0. \tag{2.40}$$

Proof. The proof is similar to that of Theorem 3.1 in [25]. □

Theorem 2.3. *Assume that conditions (H1)–(H3) and (H5)–(H6) with $\tilde{L}_\omega = L_\omega$ hold and the following condition holds:*

(H7) *there exist a positive constant $T > 0$, a random variable $\delta_\omega \geq 0$ and a N-dimensional subspace X_N of X such that the bounded projection P_N: $X \to X_N$ satisfies*

$$||(I - P_N)[\Phi(T, \tau, \omega)u - \Phi(T, \tau, \omega)v]||_X \leq \delta_\omega ||u - v||_X, \quad \forall u, v \in \chi(\tau, \omega), \tag{2.41}$$

where δ_ω, $N \in \mathbb{N}$ depend on (T, ω) but independent of τ.

Then $\{\Phi(t, \tau, \omega)\}_{t\geq 0,\tau\in\mathbb{R},\omega\in\Omega}$ possesses a random exponential attractor $\{\mathcal{A}(\tau, \omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ with properties stated as in Theorem 2.1.

Proof. By (2.41) and (2.2), for $u, v \in \chi(\tau, \omega)$,

$$\begin{aligned} &||\Phi(T, \tau, \omega)u - \Phi(T, \tau, \omega)v||_X \\ &\leq \delta_\omega ||u - v||_X + ||P_N \circ \Phi(T, \tau, \omega)u - P_N \circ \Phi(T, \tau, \omega)v||_X, \end{aligned} \tag{2.42}$$

$$\begin{aligned} &||P_N \circ \Phi(T, \tau, \omega)u - P_N \circ \Phi(T, \tau, \omega)v||_{X_N} \\ &\leq ||\Phi(T, \tau, \omega)u - \Phi(T, \tau, \omega)v||_X \leq \tilde{L}_\omega ||u - v||_X, \end{aligned} \tag{2.43}$$

and $P_N \circ \Phi(T, \tau, \omega)$: $\chi(\tau, \omega) \to X_N$, where $\dim X_N = N$ and X_N is embedded compactly in X. By Theorem 2.1, the proof is completed. □

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In practical applications to stochastic evolution equations by using Theorems 2.1–2.3, it is not easy to check conditions (H5)–(H6). Motivated by the idea of [22,47,48], the proof of Theorem 2.1 and Theorem 2.3, we next present some special conditions which can be verified to some stochastic evolution equations including partial differential equations on bounded domain and ordinary differential equations on infinite lattices.

Assume that

(H8) *there exist positive numbers* t_0, δ, *random variables* $C_0(\omega)$, $C_1(\omega) \geq 0$ *and* N*-dimensional projector* P_N: $X \to P_N X$ $(\dim(P_N X) = N)$ *such that for any* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and any* u, $v \in \chi(\tau, \omega)$,

$$||P_N \Phi(t_0, \tau, \omega)u - P_N \Phi(t_0, \tau, \omega)v||_X \leq e^{\int_0^{t_0} C_0(\theta_s \omega) ds} ||u - v||_X, \tag{2.44}$$

$$\begin{aligned} &||(I - P_N)\Phi(t_0, \tau, \omega)u - (I - P_N)\Phi(t_0, \tau, \omega)v||_X \\ &\leq \left(e^{\int_0^{t_0} C_1(\theta_s \omega) ds} + \delta e^{\int_0^{t_0} C_0(\theta_s \omega) ds}\right) ||u - v||_X, \end{aligned} \tag{2.45}$$

where t_0, δ, N *are independent of* (τ, ω) *but* δ, N *maybe depend on* t_0.

(H9) t_0, δ, $C_0(\omega)$, $C_1(\omega)$ *satisfy:*

$$\begin{cases} -\infty \leq \mathbf{E}[C_1(\omega)] < 0, \quad t_0 \geq \frac{2 \ln \frac{3}{16}}{\mathbf{E}[C_1(\omega)]} > 0, \\ 0 \leq \mathbf{E}[C_i^2(\omega)] < \infty, \quad i = 0, 1, \\ 0 < \delta \leq \min \left\{ \frac{1}{8}, e^{-\frac{t_0^2}{\ln \frac{3}{2}} (3\mathbf{E}[C_0^2(\omega)] + \mathbf{E}[C_1^2(\omega)])} \right\}. \end{cases} \tag{2.46}$$

As a consequence of the proof of Theorem 2.1, Theorem 2.3 and Theorem 2.8 in [48], we have the following Theorem.

Theorem 2.4. *Suppose conditions (H1)–(H3) and (H8)–(H9) hold. Then* $\{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$ *possesses a random exponential attractor* $\{\mathcal{A}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ *with properties: for any* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

(i) $\mathcal{A}(\tau, \omega)$ $(\subseteq \chi(\tau, \omega))$ *is a compact set of* X;
(ii) $\Phi(t, \tau, \omega)\mathcal{A}(\tau, \omega) \subseteq \mathcal{A}(t + \tau, \theta_t \omega)$ *for all* $t \geq 0$;
(iii) $\dim_f \mathcal{A}(\tau, \omega) \leq \frac{2N \ln\left(\frac{\sqrt{N}}{\delta} + 1\right)}{\ln \frac{4}{3}} < \infty$;
(iv) for any set $B \in \mathcal{D}(X)$, *there exist a random variable* $\tilde{T}_\omega \geq 0$ *and a tempered random variable* $\breve{b}_\omega > 0$ *such that*

$$\mathrm{d}_h(\Phi(t, \tau, \omega)B(\tau, \omega), \mathcal{A}(t + \tau, \theta_t \omega)) \leq \breve{b}_\omega e^{-\frac{\ln \frac{4}{3}}{4t_0} t}, \quad t \geq T_B + \tilde{T}_\omega.$$

Proof. It is proved by combing the proofs of Theorem 2.8 in [48], Theorem 2.1 and Theorem 2.3. □

Remark 2.2. If there exists a uniformly (with respect to $\tau \in \mathbb{R}$) tempered closed measurable $\mathcal{D}(X)$-pullback absorbing set $B_0 = \{B_0(\tau,\omega) = B_0(\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ (independent of τ) for Φ, that is, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B(\tau,\omega) \in B \in \mathcal{D}(X)$, there exists a $t_B(\tau,\omega) \geq 0$ such that $\Phi(t,\tau-t,\omega)B(\tau-t,\theta_{-t}\omega) \subseteq B_0(\omega)$ for all $t \geq t_B(\tau,\omega)$. Particularly, there exists a $t_{B_0}(\omega) \geq 0$ (independent of τ) such that $\Phi(t,\tau-t,\omega)B_0(\theta_{-t}\omega) \subseteq B_0(\omega)$ for all $t \geq t_{B_0}(\omega)$. Then for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, set

$$\tilde{\chi}(\tau,\omega,B_0) = \cup_{t\geq t_{B_0}(\omega)}\Phi(t,\tau-t,\theta_{-t}\omega)B_0(\theta_{-t}\omega), \quad \overline{\tilde{\chi}(\tau,\omega,B_0)} \subseteq B_0(\omega). \tag{2.47}$$

Thus, by the cocycle property and continuity of Φ, $\{\overline{\tilde{\chi}(\tau,\omega,B_0)}\}_{\tau\in\mathbb{R},\omega\in\Omega}$ satisfies (H1)–(H3).

To study the existence of a random exponential attractor for non-autonomous stochastic lattice dynamical systems in weighted spaces of infinite sequences, next we reformulate condition (H8) in Theorem 2.4 as a possibly checked one.

Let $\rho : \mathbb{Z} \to (0,+\infty)$ be a positive-valued function. Denote $\rho(i) = \rho_i$ for $i \in \mathbb{Z}$. Set

$$l^2_\rho = \{u = (u_i)_{i\in\mathbb{Z}} : u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}} \rho_i|u_i|^2 < +\infty\} \tag{2.48}$$

equipped with the inner product and norm

$$(u,v)_\rho = \sum_{i\in\mathbb{Z}} \rho_i u_i v_i, \quad \|u\|_\rho = \sqrt{\sum_{i\in\mathbb{Z}} \rho_i u_i^2}, \quad u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}} \in l^2_\rho, \tag{2.49}$$

then $(l^2_\rho, (\cdot,\cdot)_\rho)$ is a separable Hilbert space. When $\rho \equiv 1$, l^2_ρ reduces to the standard space l^2 with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$.

Write

$$\begin{aligned} l^2_{\rho,k} &= \{u = (u_i)_{i\in\mathbb{Z}} \in l^2_\rho : u_i = 0 \text{ if } |i| > k\}, \\ \bar{l}^2_{\rho,k} &= \{u = (u_i)_{i\in\mathbb{Z}} \in l^2_\rho : u_i = 0 \text{ if } |i| \leq k\}, \end{aligned} \tag{2.50}$$

then $l^2_{\rho,k}$ is a $(2k+1)$-dimensional subspace of l^2_ρ and $l^2_\rho = l^2_{\rho,k} \oplus \bar{l}^2_{\rho,k}$. Define a $(2k+1)$-dimensional bounded projection $\tilde{P}_k : l^2_\rho \to l^2_{\rho,k} \subset l^2_\rho$ by $(\tilde{P}_k u)_i = \begin{cases} u_i, & |i| \leq k, \\ 0, & |i| > k. \end{cases}$ Then for a continuous cocycle $\{\Phi(t,\tau,\omega)\}_{t\geq 0,\tau\in\mathbb{R},\omega\in\Omega}$ defining on l^2_ρ $(= X)$, condition (H8) can be reformulated as follows.

(H10) *There exist positive numbers t_0, δ, random variables $C_0(\omega)$, $C_1(\omega) \geq 0$ and $k \in \mathbb{N}$ such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any u, $v \in \chi(\tau,\omega) \subset l^2_\rho$, it holds that*

$$\begin{aligned} ||\tilde{P}_k\Phi(t_0,\tau,\omega)u - \tilde{P}_k\Phi(t_0,\tau,\omega)v||^2_\rho &= \sum_{|i|\leq k} \rho_i\,(\Phi(t_0,\tau,\omega)u - \Phi(t_0,\tau,\omega)v)^2_i \\ &\leq e^{2\int_0^{t_0} C_0(\theta_s\omega)ds}||u-v||^2_\rho, \end{aligned} \tag{2.51}$$

$$
\begin{aligned}
& ||(I-\tilde{P}_k)\Phi(t_0,\tau,\omega)u-(I-\tilde{P}_k)\Phi(t_0,\tau,\omega)v||_\rho^2 \\
=& \sum_{|i|>k}\rho_i\left(\Phi(t_0,\tau,\omega)u-\Phi(t_0,\tau,\omega)v\right)_i^2 \\
\leq & \left(e^{\int_0^{t_0}C_1(\theta_s\omega)ds}+\delta e^{\int_0^{t_0}C_0(\theta_s\omega)ds}\right)^2||u-v||_\rho^2,
\end{aligned}
\tag{2.52}
$$

where t_0, δ, k are independent of τ and ω.

3. Random exponential attractor for first order nonautonomous stochastic lattice systems

Let $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathbb{P}$ is the Wiener measure on $(\Omega,\mathcal{F})$. For $\omega\in\Omega$, $t\in\mathbb{R}$, define $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$, then $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\omega\}_{t\in\mathbb{R}})$ is an ergodic metric dynamical system (see [1]).

Consider the first order nonautonomous lattice system (1.1) with multiplicative white noise which can be written as the following vector form:

$$
\begin{cases} du=(-\lambda u-Au+f(u,t)+g(t))dt+\epsilon u\circ dw(t), \quad t>\tau, \\ u(\tau)=(u_{i,\tau})_{i\in\mathbb{Z}}=u_\tau, \quad \tau\in\mathbb{R}, \end{cases} \tag{3.1}
$$

where $\epsilon\in\mathbb{R}$; $u=(u_i)_{i\in\mathbb{Z}}$, $\lambda u=(\lambda_i u_i)_{i\in\mathbb{Z}}$, $f(u,t)=(f_i(u_i,t))_{i\in\mathbb{Z}}$, $g(t)=(g_i(t))_{i\in\mathbb{Z}}$; $w(t)$ is a two-sided real-valued Wiener process on probability space $(\Omega,\mathcal{F},\mathbb{P})$.

We choose a weight function $\rho:\mathbb{Z}\to\mathbb{R}_+$ with the following properties:

(A0) $\forall i\in\mathbb{Z}$, $0<\rho_i=\rho(i)\leq\hat{\rho}<+\infty$, $\rho(i)\leq c_0\rho(i\pm1)$, $|\rho(i\pm1)-\rho(i)|\leq a_0\rho(i)$ *for positive constants $\hat{\rho}$, c_0 and a_0.*

We make the following assumptions on A, f_i, g_i, λ_i:

(A1) *the coupled operator $-A$ is a non-negative and self-adjoint linear operator with decomposition $A=\bar{D}D=D\bar{D}$, where D is defined by*

$$
(Du)_i=\sum_{j=-m_0}^{j=m_0}d_ju_{i+j},\ \forall u=(u_i)_{i\in\mathbb{Z}},\ |d_j|\leq c_1\ \text{(constant)}, \quad -m_0\leq j\leq m_0, \tag{3.2}
$$

and $\bar{D}$ is the adjoint of D in l^2, i.e., $(Du,v)_{l^2}=(u,\bar{D}v)_{l^2}$ for $u,v\in l^2$. For example, a special case of A is

$$
(Au)_i=2u_i-u_{i-1}-u_{i+1}\ \text{and}\ (Du)_i=u_{i+1}-u_i. \tag{3.3}
$$

(A2) $\forall i\in\mathbb{Z}$, f_i, $f'_{i,s}\in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $f_i(0,t)=0$ *and there exist* $c_2\geq0$, $h_i\in C(\mathbb{R},\mathbb{R})$ *and* $\tilde{R}\in C(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$ *such that*

$$
\begin{cases} f'_{i,s}(s,t)\leq c_2, \quad sf_i(s,t)\leq h_i^2(t), \quad \forall s,\ t\in\mathbb{R},\ i\in\mathbb{Z}, \\ \sup_{i\in\mathbb{Z}}\max_{s\in[-r,r]}|f'_{i,s}(s,t)|\leq\tilde{R}(r,t),\ \forall r\in\mathbb{R}_+,\ \forall t\in\mathbb{R}. \end{cases} \tag{3.4}
$$

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(A3) $g(t) = (g_i(t))_{i\in\mathbb{Z}}$, $h(t) = (h_i(t))_{i\in\mathbb{Z}} \in \mathcal{G}$, *where*

$$\mathcal{G} = \{\eta \in C_b(\mathbb{R}, l^2_\rho) : \forall \varepsilon > 0, \text{ there exists } I(\varepsilon) \in \mathbb{N} \text{ such that } \sup_{t\in\mathbb{R}} \sum_{|i|>I(\varepsilon)} \rho_i \eta_i^2(t) < \varepsilon\} \tag{3.5}$$

and $C_b(\mathbb{R}, l^2_\rho)$ *is the space of all continuous bounded functions from* R *into* l^2_ρ.

(A4) *There exist two positive constants* α, $\bar{\alpha}$ *such that*

$$0 < \alpha \le \lambda_i \le \bar{\alpha} < +\infty, \quad \forall i \in \mathbb{Z}. \tag{3.6}$$

(A5) a_0 *in (A0) satisfies*

$$a_0 \le \min\left\{\frac{2}{c_1c_3}, \frac{\alpha}{2c_1c_3c_4}\right\}, \tag{3.7}$$

where

$$c_3 = c_0^{m_0-1} + \cdots + c_0 + 1, \quad c_4 = (2m_0+1)^2 c_0^{m_0}. \tag{3.8}$$

(A6) *There exist positive constants* $c_5 > 0$ *and* $\bar{\rho} > 0$ *such that*

$$f'_{i,s}(s,t) \le c_5|s|(1+|s|^p), \quad \forall t, s \in \mathbb{R}; \quad 0 < \bar{\rho} \le \rho_i, \quad \forall i \in \mathbb{Z}. \tag{3.9}$$

For example, $f_i(s,t) = b_{2\tilde{p}+1}s^{2\tilde{p}+1} + b_{2\tilde{p}}s^{2\tilde{p}} + \cdots + b_2 s^2$, $\tilde{p} \in \mathbb{N}$, $\forall s \in \mathbb{R}$, $\forall i \in \mathbb{Z}$, where $b_{2\tilde{p}+1} > 0$, satisfy conditions (A2) and (A6).

When $-A$ is defined by (3.3) and the weight function $\rho(s) = \frac{1}{(1+\kappa^2 s^2)^\iota}$ ($\iota > \frac{1}{2}$) (satisfies (A0)) and $f_i(s,t)$ satisfies some dissipative conditions similar to (A2), Bates et al. [6] studied the existence of a random attractor for non-autonomous stochastic system (3.1); in addition, [6] proved the existence of an infinite dimensional random attractor of (3.1) when $g_i(t) = 0$, $f_i(s,t) = -\beta(t)s^3$ with smooth function $\beta \in C^1(\mathbb{R}, \mathbb{R})$ and $0 < \beta_1 \le \beta(t) \le \beta_2$ for all $t \in \mathbb{R}$.

In this section, we will consider the existence of a random attractor for system (3.1) under conditions (A0)–(A5) based on Theorem 3.6 in [6] and prove the existence of a random exponential attractor for (3.1) under conditions (A0)–(A6) based on Theorem 2.4.

Write $z(\theta_t\omega) = -\alpha\int_{-\infty}^0 e^{\alpha s}(\theta_t\omega)(s)ds$ for $t \in \mathbb{R}$ and $\omega \in \Omega$, which is an Ornstein–Uhlenbeck stationary process solving the Itô equation $dz(\theta_t\omega) + \alpha z(\theta_t\omega)dt = dw(t)$, where $w(t)(\omega) = \omega(t)$ for $\omega \in \Omega$ [1,8]. From [1,10], it follows that $|z(\omega)|$ is tempered and for *a.e.* $\omega \in \Omega$, $t \mapsto z(\theta_t\omega)$ is continuous in t, moreover,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0. \tag{3.10}$$

In the following, for convenience, we identify "*a.e.* $\omega \in \Omega$" and "$\omega \in \Omega$".

Let $v_i(t,\omega) = e^{-\epsilon z(\theta_t\omega)}u_i(t,\omega)$, $i \in \mathbb{Z}$, where $u(t,\omega) = (u_i(t,\omega))_{i\in\mathbb{Z}}$ is a solution of (3.1). Then system (3.1) can be written as the following random system without white noise:

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$$\begin{cases} \frac{dv}{dt} + \lambda v + Av - \epsilon\alpha z(\theta_t\omega)v = e^{-\epsilon z(\theta_t\omega)} f(e^{\epsilon z(\theta_t\omega)}v, t) + e^{-\epsilon z(\theta_t\omega)} g(t), \quad t > \tau, \\ v(\tau,\omega) = v_\tau(\omega) = e^{-\epsilon z(\theta_\tau\omega)} u_\tau, \quad \tau \in \mathbb{R}. \end{cases} \tag{3.11}$$

Similar to Theorem 3.4 in [6], we have the following theorem.

Theorem 3.1. *Let (A0)–(A5) hold. Then for every $\tau \in R$, $\omega \in \Omega$ and any initial data $u_\tau \in l^2_\rho$ (or $v_\tau(\omega) \in l^2_\rho$), problem (3.11) admits a unique solution $v(\cdot, \tau, \omega, v_\tau(\omega)) \in C([\tau, \tau + T), l^2_\rho)$ for any $T > 0$ with $v(\tau, \tau, \omega, v_\tau(\omega)) = v_\tau(\omega)$, and $v(t, \tau, \omega, v_\tau)$ is continuous in v_τ. Moreover, the solution $v(t, \tau, \omega, v_\tau(\omega))$ of (3.11) generates a continuous cocycle Ψ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space l^2_ρ defined by*

$$\Psi(t, \tau, \omega, v_\tau(\omega)) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau(\theta_{-\tau}\omega)), \quad \forall t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega. \tag{3.12}$$

Obviously, for $t \in \mathbb{R}^+, \tau \in \mathbb{R}$ and $\omega \in \Omega$, $\tilde{\Psi}(t, \tau, \omega, u_\tau) = e^{-\epsilon z(\theta_t\omega)}\Psi(t, \tau, \omega, e^{-\epsilon z(\theta_\tau\omega)}u_\tau)$ defines a continuous cocycle $\tilde{\Psi}$ on l^2_ρ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ associated with (3.1). Therefore, cocycle $\tilde{\Psi}$ and Ψ (or (3.1) and (3.11)) have the same dynamics. In the following, we just consider the cocycle Ψ.

3.1. Existence of random attractor

First, we have the existence of a uniformly tempered measurable $\mathcal{D}(l^2_\rho)$-pullback absorbing set for Ψ.

Lemma 3.2. *Suppose (A0)–(A5) hold. Then there exists a tempered random variable $M_0(\omega) > 0$ (independent of τ)*

$$M_0^2(\omega) = 4\left(||h||^2_\rho + \frac{2}{\alpha}||g||^2_\rho\right) K_0(\omega) < \infty, \quad \text{where } K_0(\omega) = \int_{-\infty}^{0} e^{\alpha s + 2\epsilon\int_s^0 z(\theta_l\omega))dl - 2\epsilon z(\theta_s\omega)} ds, \tag{3.13}$$

where $||h||^2_\rho = \sup_{r\in\mathbb{R}} ||h(r)||^2_\rho$, $||g||^2_\rho = \sup_{r\in\mathbb{R}} ||g(r)||^2_\rho$, such that $M_0^2(\theta_t\omega)$ is continuous in t and the family of balls centered at 0 with radius $M_0(\omega)$

$$B_0 = \{B_0(\tau,\omega) = B_0(0, M_0(\omega)) = \{v \in l^2_\rho : ||v||_\rho \le M_0(\omega)\}|\tau \in \mathbb{R}, \omega \in \Omega\}, \tag{3.14}$$

is a measurable $\mathcal{D}(l^2_\rho)$-pullback absorbing set for Ψ, that is, for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B \in \mathcal{D}(l^2_\rho)$, there exists a $T_B(\tau, \omega) \ge 0$ such that the solution $v(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ of (3.11) with $v_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau - t, \theta_{-t}\omega)$ satisfies:

$$||v(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))||_\rho \le M_0(\omega), \quad \forall t \ge T_B(\tau, \omega), \tag{3.15}$$

where $T_B(\tau, \omega)$ is uniform for ϵ in a bounded interval of $\mathbb{R}$. Particular, there exists a $T_{B_0}(\omega) \ge 0$ (independent of τ) such that

$$v(r, \tau - t, \theta_{-\tau}\omega, B_0(\theta_{-t}\omega)) \in B_0(\theta_{r-\tau}\omega), \quad \forall t \ge T_{B_0}(\omega), \quad r \ge \tau - t. \tag{3.16}$$

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Proof. Taking the inner product of (3.11) with $v(r) = v(r, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))$ in l^2_ρ, we find that

$$\frac{d}{dt}||v||^2_\rho + 2(\lambda v, v)_\rho + 2(Av, v)_\rho - 2\alpha\epsilon z(\theta_{r-\tau}\omega)||v||^2_\rho$$
$$= 2e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(e^{\epsilon z(\theta_{r-\tau}\omega)}v, r), v)_\rho + 2e^{-\epsilon z(\theta_{r-\tau}\omega)}(g(r), v)_\rho, \quad r \geq \tau - t. \tag{3.17}$$

By [44,40] and (3.7),

$$2(Av, v)_\rho \geq (2 - a_0c_1c_3)||Dv||^2_\rho - a_0c_1c_3c_4||v||^2_\rho \geq -a_0c_1c_3c_4||v||^2_\rho \geq -\frac{\alpha}{2}||v||^2_\rho. \tag{3.18}$$

By (A2)–(A4), it follows that

$$2(f(e^{\epsilon z(\theta_{r-\tau}\omega)}v, r), v)_\rho = 2e^{-\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}}\rho_i f(r, e^{\epsilon z(\theta_{r-\tau}\omega)}v_i)e^{\epsilon z(\theta_{r-\tau}\omega)}v_i \leq 2e^{-\epsilon z(\theta_{r-\tau}\omega)}||h||^2_\rho, \tag{3.19}$$

$$2e^{-\epsilon z(\theta_{r-\tau}\omega)}(g(r), v)_\rho \leq \frac{\alpha}{4}||v||^2_\rho + \frac{4}{\alpha}e^{-2\epsilon z(\theta_{r-\tau}\omega)}||g||^2_\rho, \quad 2(\lambda v, v)_\rho \geq 2\alpha||v||^2_\rho, \tag{3.20}$$

Thus, putting (3.18)–(3.20) into (3.17), we have

$$\frac{d}{dt}||v||^2_\rho + \alpha(1 - 2\epsilon z(\theta_{r-\tau}\omega))||v||^2_\rho + \frac{\alpha}{4}||v||^2_\rho$$
$$\leq 2e^{-2\epsilon z(\theta_{r-\tau}\omega)}\left(||h||^2_\rho + \frac{1}{\alpha}||g||^2_\rho\right), \quad r \geq \tau - t. \tag{3.21}$$

By applying Gronwall inequality to (3.21) on $[\tau - t, \tau]$, we have

$$||v(\tau)||^2_\rho + \frac{\alpha}{4}\int_{\tau-t}^{\tau} e^{-\int_s^\tau \alpha(1-2\epsilon z(\theta_{l-\tau}\omega))dl}||v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}(\theta_{-\tau}\omega))||^2_\rho ds$$
$$\leq e^{-\int_{-t}^0 \alpha(1-2\epsilon z(\theta_l\omega))dl}||v_{\tau-t}(\theta_{-\tau}\omega)||^2_\rho + \frac{1}{2}M_0^2(\omega). \tag{3.22}$$

By (3.10), $M_0(\omega)$ is a tempered random variable and $M_0(\theta_t\omega)$ is continuous in t. By (3.10) and $v_{\tau-t}(\theta_{-\tau}\omega) \in B(\tau - t, \theta_{-t}\omega)$, $e^{-\int_{-t}^0 \alpha(1-2\epsilon z(\theta_l\omega))dl}||v_{\tau-t}(\theta_{-\tau}\omega)||^2_\rho \overset{t\to+\infty}{\to} 0$. By (3.22), the proof is completed. □

Now we consider an estimate on the tails of solutions of (3.11) with initial data in B_0 for large t, which implies the asymptotic compactness of solutions.

Lemma 3.3. *Suppose (A0)–(A5) hold. Then for every $\tau \in R$, $\omega \in \Omega$ and $\varepsilon > 0$, there exist $T(\tau, \omega, \varepsilon) > 0$ and $M(\tau, \omega, \varepsilon) \in \mathbb{N}$ such that the solution $v(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ ($r \geq \tau - t$, $t \geq 0$) of (3.11) with $v_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega)$ satisfies*

$$\sum_{|i|>M(\tau,\omega,\varepsilon)} \rho_i |v_i(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))|^2 \leqslant \varepsilon, \quad t \geq T(\tau, \omega, \varepsilon). \tag{3.23}$$

Proof. Choose a smooth increasing function $\theta \in C^1(\mathbb{R}_+, \mathbb{R})$ satisfying

$$\begin{cases} \theta(s) = 0, \ \forall s \in [0, 1]; \\ 0 \leq \theta(s) \leq 1, \ \forall s \in [1, 2]; \\ \theta(s) = 1, \ \forall s \in [2, \infty); \\ |\theta'(s)| \leq C_1, \ \forall s \in \mathbb{R}_+ \quad \text{for some positive constant } C_1. \end{cases} \tag{3.24}$$

Let $M \in \mathbb{N}$ be a positive integer. Set $x_i = \theta\left(\frac{|i|}{M}\right) v_i$, $x = (x_i)_{i\in\mathbb{Z}}$. Taking the inner product of (3.11) with x in l^2_ρ, we have that for $r \geq \tau - s$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i v_i^2 + (\lambda v, x)_\rho + (Av, x)_\rho - \alpha\epsilon z(\theta_{r-\tau}\omega)\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i v_i^2 \\ &= e^{-\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i f_i(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_i) v_i + e^{-\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i g_i(r) v_i. \end{aligned} \tag{3.25}$$

Similar to (3.18)–(3.20),

$$\begin{aligned} 2(\lambda v, x)_\rho + 2(Av, x)_\rho &\geq \sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{M})\rho_i(2 - a_0c_1c_3)(Du)_i^2 - \frac{c_6}{M}||v||^2_\rho \\ &\quad + (2\alpha - a_0c_1c_3c_4)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{M})\rho_i v_i^2 \\ &\geq (2\alpha - a_0c_1c_3c_4)\sum_{i\in\mathbb{Z}} \theta(\frac{|i|}{M})\rho_i v_i^2 - \frac{c_6}{M}||v||^2_\rho, \end{aligned}$$

$$\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i f_i(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_i) v_i \leq e^{-\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i h_i^2(r),$$

$$2e^{-\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i g_i(r) v_i \leq \frac{\alpha}{4}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i v_i^2 + \frac{4}{\alpha}e^{-2\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right)\rho_i g_i^2(r),$$

where

$$c_6 = 2C_1 m_0 c_0^{m_0/2} c_1[1 + (2m_0 + 1)^2].$$

Thus, we obtain

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i v_i^2+\alpha(1-2\epsilon z(\theta_{r-\tau}\omega))\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i v_i^2$$
$$\leq \frac{c_6}{M}||v||_\rho^2+2e^{-2\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i\left(h_i^2(r)+\frac{2}{\alpha}g_i^2(r)\right),\quad r\geq\tau-t. \tag{3.26}$$

By Gronwall inequality on $[\tau-t,\tau]$, by (3.22), we have

$$\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i v_i^2(\tau)$$
$$\leq e^{-\int_{-t}^0\alpha(1-2\epsilon z(\theta_l\omega))dl}||v_{\tau-t}(\theta_{-\tau}\omega)||_\rho^2$$
$$+\frac{c_6}{M}\int_{\tau-t}^{\tau}e^{-\int_s^\tau\alpha(1-2\epsilon z(\theta_{l-\tau}\omega))dl}||v(s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega))||_\rho^2ds$$
$$+\int_{-\infty}^{0}e^{\alpha s+2\epsilon\int_s^0 z(\theta_l\omega))dl-2\epsilon z(\theta_s\omega)}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i\left(h_i^2(s+\tau)+\frac{2}{\alpha}g_i^2(s+\tau)\right)ds$$
$$\leq\left(1+\frac{4c_6}{\alpha}\right)e^{-\int_{-t}^0\alpha(1-2\epsilon z(\theta_l\omega))dl}M_0^2(\theta_{-t}\omega)+\left(\frac{c_7}{M}+\gamma_M\right)K_0(\omega), \tag{3.27}$$

where

$$c_7=\frac{8c_6}{\alpha}\left(||h||_\rho^2+\frac{2}{\alpha}||g||_\rho^2\right),\quad \gamma_M=\sup_{r\in\mathbb{R}}\sum_{|i|\geq M}\rho_i\left(h_i^2(r)+\frac{2}{\alpha}g_i^2(r)\right). \tag{3.28}$$

By $h(t), g(t)\in\mathcal{G}$, (3.10) and

$$e^{-\int_{-t}^0\alpha(1-2\epsilon z(\theta_l\omega))dl}M_0^2(\theta_{-t}\omega)\overset{t\to+\infty}{\to}0,\quad \frac{c_7}{M\alpha}K_0(\omega)\overset{M\to+\infty}{\to}0, \tag{3.29}$$

it follows that for any $\varepsilon>0$, $\tau\in\mathbb{R}$ and $\omega\in\Omega$, there exist $T(\tau,\omega,\varepsilon)>0$ and $M(\tau,\omega,\varepsilon)\in\mathbb{N}$ such that for $t\geq T(\tau,\omega,\varepsilon)$, $M\geq M(\tau,\omega,\varepsilon)$,

$$\sum_{|i|>2M}\rho_i|v_i(\tau,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))|^2\leq\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i v_i^2(\tau)\leqslant\varepsilon. \tag{3.30}$$

The proof is completed. □

According to Theorem 3.6 in [6] and Lemmas 3.2–3.3, we have the existence of a random attractor in $\mathcal{D}(l_\rho^2)$ for Ψ.

Please cite this article in press as: S. Zhou, Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.03.044

Theorem 3.4. *Suppose (A0)–(A5) hold. Then the continuous cocycle Ψ associated with (3.11) has a unique random attractor $\mathcal{R} = \{\mathcal{R}(\tau, \omega)\}_{\tau\in\mathbb{R},\omega\in\Omega} \in \mathcal{D}(l^2_\rho)$ and $\mathcal{R}(\tau, \omega) \subset B_0(\omega)$ for any $\tau \in \mathbb{R}$, $\omega \in \Omega$.*

As a direct consequence of (3.27) and (3.29), we have

Corollary 3.5. *Suppose (A0)–(A5) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\nu > 0$ and $I \in \mathbb{N}$, there exists $\tilde{T}_\nu(\omega) > 0$ (independent of τ) such that the solution $v(r, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$ ($r \geq \tau - t$) of (3.11) with $v_{\tau-t}(\theta_{-\tau}\omega) \in B_0(\theta_{-t}\omega)$ satisfies*

$$\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{I}\right) \rho_i v_i^2(\tau, \tau - t, \theta_{-\tau}\omega, \varphi_{\tau-t}(\theta_{-\tau}\omega))$$
$$\leq \nu + \left(\frac{c_7}{I} + \gamma_I\right) K_0(\omega), \quad \forall t \geq \tilde{T}_\nu(\omega), \quad I \in \mathbb{N}. \tag{3.31}$$

3.2. Random exponential attractor

From Theorem 3.4, we see that under conditions (A0)–(A5), the cocycle Ψ associated with (3.11) has a random attractor $\mathcal{R} = \{\mathcal{R}(\tau, \omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ in $\mathcal{D}(l^2_\rho)$. However, in this case, for $\tau \in \mathbb{R}$, $\omega \in \Omega$, the fractal dimension of $\mathcal{R}(\tau, \omega)$ maybe is infinite, for example, see section 7 in [6].

In this subsection, we prove the existence of a random exponential attractor for Ψ based on Theorem 2.4 under conditions (A0)–(A6), which implies the finiteness of fractal dimension of $\mathcal{R}(\tau, \omega)$.

Choosing a fixed positive number $\nu = \nu_0 > 0$ in (3.31) small enough such that

$$\left(\frac{16c_5^2}{\alpha\bar{\rho}}\nu_0 + \frac{2^{p+4}c_5^2}{\alpha\bar{\rho}^{p+1}}\nu_0^{p+1}\right)\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}} \leq \frac{\alpha}{8}. \tag{3.32}$$

For any $\tau \in \mathbb{R}$ and $\omega \in \Omega$, set

$$\chi_1(\tau, \omega) \ = \cup_{s\geq \max\{t_{B_0}(\omega), \tilde{T}_{\nu_0}(\omega)\}} v(\tau, \tau - s, \theta_{-\tau}\omega, B_0(\theta_{-s}\omega)) \subseteq B_0(\omega), \tag{3.33}$$

then by Lemma 3.2, Remark 2.2 and (3.16), the family of closed sets $\{\overline{\chi_1(\tau, \omega)}\}_{\tau\in\mathbb{R},\omega\in\Omega}$ satisfies (H1)–(H3), that is, for any fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, it holds that

(i) boundedness: $\sup_{\tau\in\mathbb{R}} \sup_{u,v\in\overline{\chi_1(\tau,\omega)}} ||u - v||_X \leq 2M_0(\omega) < \infty$ and $M_0(\theta_t\omega)$ is continuous in t for all $t \in \mathbb{R}$;
(ii) positive invariance: $\Psi(t, \tau - t, \theta_{-t}\omega)\overline{\chi_1(\tau - t, \theta_{-t}\omega)} \subseteq \overline{\chi_1(\tau, \omega)}$ for all $t \geq 0$;
(iii) pullback absorbing property: for any set $B \in \mathcal{D}(l^2_\rho)$, there exists $T_B = T_B(\tau, \omega) \geq 0$ such that $\Psi(t, \tau - t, \theta_{-t}\omega)B(\tau - t, \theta_{-t}\omega) \subseteq \overline{\chi_1(\tau, \omega)}$ for all $t \geq T_B$.

Moreover, by Corollary 3.5 and definition (3.33) of $\chi_1(\tau, \omega)$, it holds that for any $\tau \in \mathbb{R}\omega \in \Omega$ and $\tilde{v} = (\tilde{v}_i)_{i\in\mathbb{Z}} \in \chi_1(\tau, \omega)$,

$$\sum_{|i|\geq 2I} \rho_i \tilde{v}_i^2 \leq \nu_0 + \left(\frac{c_7}{I} + \gamma_I\right) K_0(\omega), \quad I \in \mathbb{N}. \tag{3.34}$$

Next, we show that $\{\overline{\chi_1(\tau,\omega)}\}_{\tau\in\mathbb{R},\omega\in\Omega}$ satisfies (H9)–(H10). Then we obtain the existence of a random exponential attractor for Ψ by Theorem 2.4. To do this, by the cocycle property and continuity of Ψ, it is sufficient to prove that $\{\chi_1(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ satisfies (H9)–(H10).

Let $v(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega))$ $(r\geq\tau-t)$ be a solution of (3.11) with $v_{\tau-t}(\theta_{-\tau}\omega)\in\chi_1(\tau-t,\theta_{-t}\omega)$. Then by the cocycle property of Ψ (or v), (3.16) and (3.34), we have that

$$||v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t}(\theta_{-\tau}\omega))||_\rho\leq M_0(\theta_{r-\tau}\omega),\quad \forall r\geq\tau-t, \tag{3.35}$$

and

$$\sum_{|i|\geq 2I}\rho_i v_i^2(r,\tau-t,\theta_{-\tau}\omega,\varphi_{\tau-t}(\theta_{-\tau}\omega)) \\ \leq \nu_0+\left(\frac{c_7}{I}+\gamma_I\right)K_0(\theta_{r-\tau}\omega),\quad \forall r\geq\tau-t,\quad I\in\mathbb{N}. \tag{3.36}$$

We first prove that Ψ has the Lipschitz property on $\chi_1(\tau,\omega)$.

Lemma 3.6. *Suppose (A0)–(A5) hold. Then for every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $t\geq 0$ and $v_{j,\tau-t}(\theta_{-\tau}\omega)\in\chi_1(\tau-t,\theta_{-t}\omega)$, $j=1,2$, there exists a random variable $C_2(\omega)>0$ such that*

$$||v(\tau,\tau-t,\theta_{-\tau}\omega,v_{1,\tau-t}(\theta_{-\tau}\omega))-v(\tau,\tau-t,\theta_{-\tau}\omega,v_{2,\tau-t}(\theta_{-\tau}\omega))||_\rho \\ \leq e^{\int_{-t}^0 C_2(\theta_s\omega)ds}||v_{1,\tau-t}(\theta_{-\tau}\omega)-v_{2,\tau-t}(\theta_{-\tau}\omega)||_\rho. \tag{3.37}$$

Proof. For every $\tau\in\mathbb{R}$, $\omega\in\Omega$, $t\geq 0$ and $j=1,2$, $v_{j,\tau-t}(\theta_{-\tau}\omega)\in\chi_1(\tau-t,\theta_{-t}\omega)\subseteq B_0(\theta_{-t}\omega)$, let

$$v_j(r)=v(r,\tau-t,\theta_{-\tau}\omega,v_{j,\tau-t}(\theta_{-\tau}\omega)),\quad y(r)=v_1(r)-v_2(r),\quad r\geq\tau-t, \tag{3.38}$$

then for $r\geq\tau-t$,

$$\begin{cases}\frac{dy}{dt}+\lambda y+Ay-\epsilon\alpha z(\theta_{r-\tau}\omega)y=e^{-\epsilon z(\theta_{r-\tau}\omega)}[f(r,e^{bz(\theta_{r-\tau}\omega)}v_1(r))-f(r,e^{bz(\theta_{r-\tau}\omega)}v_2(r))],\\ y(\tau,\omega)=v_{1\tau}(\omega)-v_{2\tau}(\omega).\end{cases} \tag{3.39}$$

By (3.35), for $r\geq\tau-t$, it holds that

$$||v_1(r)||_\rho\leq M_0(\theta_{r-\tau}\omega),\quad ||v_2(r)||_\rho\leq M_0(\theta_{r-\tau}\omega),\quad \forall r\geq\tau-t. \tag{3.40}$$

Taking the inner product $(\cdot,\cdot)_\rho$ of (3.39) with $y(r)$, we find that

$$\frac{1}{2}\frac{d}{dt}\|y\|_\rho^2+(\lambda y,y)_\rho+(Ay,y)_\rho \\ =e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(r,e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r))-f(r,e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r)),y)_\rho+\epsilon\alpha z(\theta_{r-\tau}\omega)||y||_\rho^2. \tag{3.41}$$

By (A2) and (A5),

$$(\lambda y, y)_\rho \geq \alpha ||y||_\rho^2 \geq 0, \quad 2(Ay, y)_\rho \geq -a_0 c_1 c_3 c_4 ||y||_\rho^2,$$

$$\begin{aligned} & e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_1(r)) - f(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_2(r)), y)_\rho \\ &= e^{-\epsilon z(\theta_{r-\tau}\omega)} \sum_{i\in\mathbb{Z}} \rho_i [f_i(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_{1,i}(r)) - f_i(r, e^{\epsilon z(\theta_{r-\tau}\omega)} v_{2,i}(r)) y_i \\ &= \sum_{i\in\mathbb{Z}} \rho_i f'_{i,s}(r, e^{\epsilon z(\theta_{r-\tau}\omega)}((1-\vartheta_i) v_{1,i}(r) + \vartheta_i v_{2,i}(r)) y_i^2(r), \quad \vartheta_i \in (0,1) \\ &\leq c_2 ||y||_\rho^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} \frac{d}{dt}||y(r)||_\rho^2 &\leq 2[\epsilon\alpha z(\theta_{r-\tau}\omega) + c_2 + a_0 c_1 c_3 c_4]||y(r)||^2 \\ &\leq 2C_2(\theta_{r-\tau}\omega)||y(r)||^2, \quad \forall r \geq \tau - t, \end{aligned} \tag{3.42}$$

where

$$C_2(\omega) = |\epsilon\alpha z(\omega)| + c_2 + a_0 c_1 c_3 c_4. \tag{3.43}$$

By applying Gronwall inequality to (3.42) on $[\tau - t, r]$ $(r \geq \tau - t)$, we have

$$\begin{aligned} & ||y(r, \tau - t, \theta_{-\tau}\omega, v_{1,\tau-t}(\theta_{-\tau}\omega))||_\rho^2 \\ &\leq e^{2\int_{\tau-t}^r C_2(\theta_{s-\tau}\omega)ds} ||y(\tau - t, \tau - t, \theta_{-\tau}\omega, v_{1,\tau-t}(\theta_{-\tau}\omega))||_\rho^2. \end{aligned} \tag{3.44}$$

Set $r = \tau$, (3.37) holds. The proof is completed. □

Lemma 3.7. *Suppose (A0)–(A6) hold. Then for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq 0$, there exist random variables $C_3(\omega) > 0$, $C_4(\omega) > 0$ such that for any $I \in \mathbb{N}$ and $v_{j,\tau-t}(\theta_{-\tau}\omega) \in \chi_1(\tau - t, \theta_{-t}\omega)$, $j = 1, 2$,*

$$\begin{aligned} & \sum_{|i|\leq 4I} \rho_i (v_i(\tau, \tau - t, \theta_{-\tau}\omega, v_{1,\tau-t}(\theta_{-\tau}\omega)) - v_i(\tau, \tau - t, \theta_{-\tau}\omega, v_{2,\tau-t}(\theta_{-\tau}\omega)))^2 \\ &\leq e^{\int_{-t}^0 C_3(\theta_s\omega)ds} ||v_{1,\tau-t}(\theta_{-\tau}\omega) - v_{2,\tau-t}(\theta_{-\tau}\omega)||_\rho \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} & \sum_{|i|>4I} \rho_i (v_i(\tau, \tau - t, \theta_{-\tau}\omega, v_{1,\tau-t}(\theta_{-\tau}\omega)) - v_i(\tau, \tau - t, \theta_{-\tau}\omega, v_{2,\tau-t}(\theta_{-\tau}\omega)))^2 \\ &\leq \left(e^{-\frac{\alpha}{2}t + \int_{-t}^0 C_4(\theta_s\omega)ds} + \delta_I e^{\int_{-t}^0 C_3(\theta_s\omega)ds}\right) ||v_{1,\tau-t}(\theta_{-\tau}\omega) - v_{2,\tau-t}(\theta_{-\tau}\omega)||_\rho, \end{aligned} \tag{3.46}$$

where

$$\delta_I = \frac{1}{\sqrt[4]{2\alpha}}\sqrt{\frac{1}{I} + \gamma_I + \frac{1}{I^{p+1}} + \gamma_I^{p+1}}. \tag{3.47}$$

Proof. Let v_j, $j = 1, 2$, and y be as in the proof of Lemma 3.6. For $i \in \mathbb{Z}$, set $q_i = \theta\left(\frac{|i|}{M}\right) y_i$, $q = (q_i)_{i\in\mathbb{Z}}$. Taking the inner product of (3.39) with q in l^2_ρ, we have that for $r \geq \tau - t$,

$$\begin{aligned}&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2 + 2(\lambda y, q)_\rho + 2(Ay, q)_\rho - 2\epsilon\alpha z(\theta_{r-\tau}\omega)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2 \\ &= 2e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(r, e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r)) - f(r, e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r)), q)_\rho.\end{aligned} \tag{3.48}$$

By computation, we have

$$\begin{aligned}&2(\lambda y, q)_\rho + 2(Ay, q)_\rho - \epsilon\alpha z(\theta_{r-\tau}\omega)\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2 \\ &\geq (2\alpha - a_0c_1c_3c_4 - \epsilon\alpha z(\theta_{r-\tau}\omega))\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2 - \frac{c_6}{M}||y||_\rho^2 \\ &\geq (\frac{5\alpha}{4} - \epsilon\alpha z(\theta_{r-\tau}\omega))\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2 - \frac{c_6}{M}||y||_\rho^2\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}&2e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(r, e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r)) - f(r, e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r)), q)_\rho \\ &= 2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i f'_{i,s}(r, e^{\epsilon z(\theta_{r-\tau}\omega)}((1-\vartheta_i)v_{1,i}(r) + \vartheta_i v_{2,i}(r))y_i^2(r), \quad \vartheta_i \in (0,1) \\ &\leq 2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i c_5(e^{\epsilon z(\theta_{r-\tau}\omega)}|v_{1,i}(r)| + e^{\epsilon z(\theta_{r-\tau}\omega)}|v_{2,i}(r)|)y_i^2(r) \\ &\quad + 2\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i c_5(|e^{\epsilon z(\theta_{r-\tau}\omega)}v_{1,i}(r)|^{p+1} + |e^{\epsilon z(\theta_{r-\tau}\omega)}v_{2,i}(r)|^{p+1})y_i^2(r) \\ &\leq \frac{\alpha}{4}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2 + \frac{16c_5^2}{\alpha}e^{2\epsilon z(\theta_{r-\tau}\omega)}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i[v_{1,i}^2(r) + v_{2,i}^2(r)]y_i^2(r) \\ &\quad + \frac{16c_5^2}{\alpha}\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}[v_{1,i}^{2p+2}(r) + v_{2,i}^{2p+2}(r)]y_i^2(r).\end{aligned} \tag{3.50}$$

By (3.36), for $I \in \mathbb{N}$,

$$\sum_{|i|\geq 2I}[v_{1,i}^2(r) + v_{2,i}^2(r)] \leq \frac{2}{\bar{\rho}}\nu_0 + \frac{2}{\bar{\rho}}\left(\frac{c_7}{I} + \gamma_I\right)K_0(\theta_{r-\tau}\omega)$$

and

$$\sum_{|i|\geq 2I}[v_{1,i}^{2p+2}(r)+v_{2,i}^{2p+2}(r)]\leq \frac{2^{p+1}}{\bar\rho^{p+1}}\nu_0^{p+1}+\frac{2^{p+1}}{\bar\rho^{p+1}}\left(\frac{c_7}{I}+\gamma_I\right)^{p+1}K_0^{p+1}(\theta_{r-\tau}\omega).$$

Thus, for $M\geq 2I$, we obtain

$$\begin{aligned}
&2e^{-\epsilon z(\theta_{r-\tau}\omega)}(f(r,e^{\epsilon z(\theta_{r-\tau}\omega)}v_1(r))-f(r,e^{\epsilon z(\theta_{r-\tau}\omega)}v_2(r)),q)_\rho\\
&\leq\left(\frac{\alpha}{4}+\frac{32c_5^2\nu_0}{\bar\rho\alpha}e^{2\epsilon z(\theta_{r-\tau}\omega)}+\frac{2^{p+5}c_5^2}{\alpha\bar\rho^{p+1}}\nu_0^{p+1}e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}\right)\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2\\
&\quad+\frac{32c_5^2\nu_0}{\bar\rho\alpha}\left(\frac{c_7}{I}+\gamma_I\right)e^{2\epsilon z(\theta_{r-\tau}\omega)}K_0(\theta_{r-\tau}\omega)\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2(r)\\
&\quad+\frac{2^{p+5}c_5^2}{\alpha\bar\rho^{p+1}}\left(\frac{c_7}{I}+\gamma_I\right)^{p+1}e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}K_0^{p+1}(\theta_{r-\tau}\omega)\sum_{i\in\mathbb{Z}}\theta(\frac{|i|}{M})\rho_i y_i^2(r). \qquad (3.51)
\end{aligned}$$

So, by (3.48)–(3.51), we have that for $M\geq 2I$,

$$\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2(r)+(\alpha-2\epsilon\alpha z(\theta_{r-\tau}\omega)-2K_1(\theta_{r-\tau}\omega))\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2(r)\\
&\leq\left(\frac{c_6}{2I}+\frac{32c_5^2\nu_0}{\bar\rho\alpha}\left(\frac{c_7}{I}+\gamma_I\right)e^{2\epsilon z(\theta_{r-\tau}\omega)}K_0(\theta_{r-\tau}\omega)\right)||y(r)||_\rho^2\\
&\quad+\frac{2^{p+5}c_5^2}{\alpha\bar\rho^{p+1}}\left(\frac{c_7}{I}+\gamma_I\right)^{p+1}e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}K_0^{p+1}(\theta_{r-\tau}\omega)||y(r)||_\rho^2\\
&\leq c_8\left(\frac{1}{I}+\gamma_I\right)\left(1+e^{2\epsilon z(\theta_{r-\tau}\omega)}K_0(\theta_{r-\tau}\omega)\right)e^{\int_{\tau-t}^r C_2(\theta_{s-\tau}\omega)ds}||y(\tau-t)||_\rho^2\\
&\quad+c_9\left(\frac{1}{I^{p+1}}+\gamma_I^{p+1}\right)e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}K_0^{p+1}(\theta_{r-\tau}\omega)e^{\int_{\tau-t}^r C_2(\theta_{s-\tau}\omega)ds}||y(\tau-t)||_\rho^2\\
&\leq c_{10}\left(\frac{1}{I}+\gamma_I+\frac{1}{I^{p+1}}+\gamma_I^{p+1}\right)\\
&\quad\times\left(1+e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}K_0^{p+1}(\theta_{r-\tau}\omega)\right)e^{2\int_{\tau-t}^r C_2(\theta_{s-\tau}\omega)ds}||y(\tau-t)||_\rho^2, \qquad (3.52)
\end{aligned}$$

where

$$c_8=\max\left\{\left(\frac{c_6}{2}+\frac{32c_5^2\nu_0}{\bar\rho\alpha}\right)c_7,\frac{32c_5^2\nu_0}{\bar\rho\alpha}\right\},\quad c_9=\max\left\{\frac{2^{3p+5}c_5^2}{\alpha\bar\rho^{p+1}}c_7^{p+1},\frac{2^{3p+5}c_5^2}{\alpha\bar\rho^{p+1}}\right\},$$
$$c_{10}=\max\left\{c_8,c_9,\frac{2p+1}{p},\frac{p+2}{p+1}\right\},$$

$$K_1(\omega) = \frac{16c_5^2\nu_0}{\bar\rho\alpha}e^{2\epsilon z(\omega)} + \frac{2^{p+4}c_5^2}{\alpha\bar\rho^{p+1}}\nu_0^{p+1}e^{2(p+1)\epsilon z(\omega)}. \tag{3.53}$$

By applying Gronwall inequality to (3.52) on $[\tau - t, \tau]$ ($t \geq 0$), we have that for $M \geq 2I$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))) \\
&\leq \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2(\tau - t, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega)))e^{\int_{\tau-t}^{\tau}(-\alpha+2\epsilon\alpha z(\theta_{s-\tau}\omega)+2K_1(\theta_{s-\tau}\omega))ds} \\
&+\left(\frac{1}{I} + \gamma_I + \frac{1}{I^{p+1}} + \gamma_I^{p+1}\right)||y(\tau - t)||_\rho^2 \\
&\times\int_{\tau-t}^{\tau} c_{10}\Big(1+e^{2(p+1)\epsilon z(\theta_{r-\tau}\omega)}K_0^{p+1}(\theta_{r-\tau}\omega)\Big)\, e^{2\int_{\tau-t}^{r} C_2(\theta_{s-\tau}\omega)ds}e^{\int_r^{\tau}(-\alpha+2\epsilon\alpha z(\theta_{s-\tau}\omega)+2K_1(\theta_{s-\tau}\omega))ds}dr \\
&\leq e^{\int_{-t}^{0}(-\alpha+2|\epsilon\alpha z(\theta_s\omega)|+2K_1(\theta_s\omega))ds}||y(\tau - t)||_\rho^2 \\
&+\left(\frac{1}{I} + \gamma_I + \frac{1}{I^{p+1}} + \gamma_I^{p+1}\right)||y(\tau - t)||_\rho^2\, e^{\int_{-t}^{0}(2C_2(\theta_s\omega)+2|\epsilon\alpha z(\theta_s\omega)|+2K_1(\theta_s\omega))ds} \\
&\times\int_{-t}^{0} c_{10}\left(1 + e^{2(p+1)\epsilon z(\theta_r\omega)}K_0^{p+1}(\theta_r\omega)\right)e^{\alpha r}dr.
\end{aligned} \tag{3.54}$$

Since $\sqrt{x} \leq e^x$ for all $x \geq 0$, it follows that

$$\begin{aligned}
&\int_{-t}^{0} c_{10}\left(1 + e^{2(p+1)\epsilon z(\theta_r\omega)}K_0^{p+1}(\theta_r\omega)\right)e^{\alpha r}dr \\
&\leq \left(\int_{-t}^{0} c_{10}^2\left(1 + e^{2(p+1)\epsilon z(\theta_r\omega)}K_0^{p+1}(\theta_r\omega)\right)^2 dr\right)^{\frac{1}{2}}\left(\int_{-t}^{0} e^{2\alpha r}dr\right)^{\frac{1}{2}} \\
&\leq \frac{1}{\sqrt{2\alpha}}e^{\int_{-t}^{0} c_{10}^2\left(1+e^{2(p+1)\epsilon z(\theta_r\omega)}K_0^{p+1}(\theta_r\omega)\right)^2 dr}.
\end{aligned}$$

By (3.54), for $M \geq 2I$,

$$\begin{aligned}
&\sum_{|i|>4I}\rho_i y_i^2(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))) \\
&\leq \sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{M}\right)\rho_i y_i^2(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))) \\
&\leq \left(e^{\int_{-t}^{0}(-\alpha+2C_4(\theta_s\omega))ds} + \delta_I^2 e^{\int_{-t}^{0} 2C_3(\theta_s\omega)dr}\right)||y(\tau - t)||_\rho^2,
\end{aligned} \tag{3.55}$$

Please cite this article in press as: S. Zhou, Random exponential attractor for cocycle and application to non-autonomous stochastic lattice systems with multiplicative white noise, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.03.044

where

$$C_3(\omega) = C_2(\omega) + |\epsilon\alpha z(\omega)| + K_1(\omega) + \frac{1}{2}c_{10}^2\left(1 + e^{2(p+1)\epsilon z(\omega)}K_0^{p+1}(\omega)\right)^2, \tag{3.56}$$

$$C_4(\omega) = |\epsilon\alpha z(\omega)| + K_1(\omega), \tag{3.57}$$

i.e., (3.46) holds. From (3.37) and $C_3(\omega) > C_2(\omega) \geq 0$, it follows that (3.45) holds. The proof is completed. □

Now we estimate the expectation of $C_4(\omega)$, $C_4^2(\omega)$ and $C_3^2(\omega)$. For this purpose, we have recourse to the following result in [26,45].

Lemma 3.8. *(see [26,45]) The Ornstein–Uhlenbeck process $z(\theta_t\omega)$ satisfies*

$$\mathbf{E}[|z(\theta_s\omega)|^r] = \frac{\Gamma(\frac{1+r}{2})}{\sqrt{\pi\alpha^r}}, \quad \forall r > 0, s \in \mathbb{R}, \tag{3.58}$$

$$\mathbf{E}[e^{\varepsilon z(\theta_s\omega)}] \leq \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}, \quad \forall s \in \mathbb{R}, \quad |\varepsilon| \leq 1, \tag{3.59}$$

and

$$E\left[e^{\varepsilon\int_\tau^{\tau+t}|z(\theta_s\omega)|ds}\right] \leq e^{\frac{\varepsilon}{\sqrt{\alpha}}t}, \quad \text{for } \alpha^3 \geq \varepsilon^2 \geq 0, \ \tau \in R, t \geq 0; \tag{3.60}$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 3.9. *If (3.32) holds and the coefficient ϵ of the random term in (3.1) is small such that*

$$|\epsilon| \leq \min\left\{\frac{1}{16(p+1)}, \frac{\sqrt{\pi\alpha}}{4}, \frac{\alpha\sqrt{\alpha}}{9}\right\}, \tag{3.61}$$

then

$$0 < \mathbf{E}[C_4(\omega)] \leq \frac{3\alpha}{8}, \quad \mathbf{E}[C_j^2(\omega)] < \infty, \quad j = 3, 4. \tag{3.62}$$

Proof. It follows from (3.59) and (3.61) that

$$\mathbf{E}[e^{\varepsilon z(\theta_s\omega)}] \leq \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}, \quad \mathbf{E}[e^{2(p+1)\epsilon z(\omega)}] \leq \frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}}, \tag{3.63}$$

then by (3.32), (3.53) and (3.63),

$$\mathbf{E}[K_1(\omega)] \leq \left(\frac{16c_5^2}{\alpha\bar{\rho}}\nu_0 + \frac{2^{p+4}c_5^2}{\alpha\bar{\rho}^{p+1}}\nu_0^{p+1}\right)\frac{4\sqrt{\pi} + 3e}{3\sqrt{\pi}} \leq \frac{\alpha}{8}. \tag{3.64}$$

By (3.57), (3.58) and (3.64),

$$\mathbf{E}[C_4(\omega)] = \alpha|\epsilon|\mathbf{E}[|z(\omega)|] + \mathbf{E}[K_1(\omega)] = |\epsilon|\sqrt{\frac{\alpha}{\pi}} + \frac{\alpha}{8} \le \frac{3\alpha}{8}. \tag{3.65}$$

By (3.61) and (3.66), we have that

$$C_4^2(\omega) \le 2\alpha^2\epsilon^2|z(\omega)|^2 + 2K_1^2(\omega),$$

$$\begin{aligned}\mathbf{E}[K_1^2(\omega)] &\le 2\left(\frac{16c_5^2\nu_0}{\alpha\bar{\rho}}\right)^2 \mathbf{E}[e^{4\epsilon z(\omega)}] + 2\left(\frac{2^{p+4}c_5^2}{\alpha\bar{\rho}^{p+1}}\nu_0^{p+1}\right)^2 \mathbf{E}[e^{4(p+1)\epsilon z(\omega)}] \\ &\le \left(\frac{512c_5^4\nu_0^2}{\alpha^2\bar{\rho}^2} + \frac{2^{2p+8}c_5^4}{\alpha^2\bar{\rho}^{2p+2}}\nu_0^{2p+2}\right)\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}} \doteq c_{12}.\end{aligned}$$

Thus,

$$\mathbf{E}[C_4^2(\omega)] \le \alpha\epsilon^2 + 2c_{12} < \infty. \tag{3.66}$$

By (3.56),

$$C_3^2(\omega) \le c_{11}\left(1 + \alpha^2\epsilon^2|z(\omega)|^2 + K_1^2(\omega) + e^{8(p+1)\epsilon z(\omega)} + K_0^{4p+4}(\omega)\right). \tag{3.67}$$

By (3.13), (3.60), (3.61) and Hölder inequality,

$$\begin{aligned}\mathbf{E}[K_0^{4p+4}(\omega)] &= \mathbf{E}\left(\int_{-\infty}^{0} e^{\alpha s + 2\epsilon\int_s^0 z(\theta_l\omega))dl - 2\epsilon z(\theta_s\omega)}ds\right)^{4p+4} \\ &\le \left(\int_{-\infty}^{0} e^{\frac{4p+4}{4p+3}\alpha s}ds\right)^{4p+3}\int_{-\infty}^{0}\mathbf{E}e^{(2p+2)\alpha s + 2|\epsilon|(4p+4)\int_s^0 z(\theta_l\omega))dl + 2|\epsilon|(4p+4)z(\theta_s\omega)}ds \\ &\le 2\left(\frac{4p+3}{4p+4}\right)^{4p+3}\int_{-\infty}^{0} e^{(2p+2)\alpha s}\left(\mathbf{E}e^{16|\epsilon|(p+1)\int_s^0 z(\theta_l\omega))dl} + \mathbf{E}e^{16|\epsilon|(p+1)z(\theta_s\omega)}\right)ds \\ &\le 2\left(\frac{4p+3}{4p+4}\right)^{4p+3}\int_{-\infty}^{0} e^{(2p+2)\alpha s}\left(e^{-\frac{16|\epsilon|(p+1)}{\sqrt{\alpha}}s} + \frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}}\right)ds \\ &\le \frac{1}{(p+1)}\left(\frac{4p+3}{4p+4}\right)^{4p+3}\left(\frac{1}{\alpha - \frac{8|\epsilon|}{\sqrt{\alpha}}} + \frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}\alpha}\right) \\ &\doteq c_{13}.\end{aligned}$$

By (3.67),

$$\mathbf{E}[C_3^2(\omega)] \le c_{11}\left(1+\alpha\epsilon^2+c_{12}+\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}}+c_{13}\right)<\infty. \tag{3.68}$$

The proof is completed. □

Lemma 3.10. *Suppose (A0)–(A5) hold. Then*

$$\begin{cases} \lim_{t\searrow 0}\sup_{v\in\chi_1(\tau,\omega)}||\Psi(t,\tau,\omega)v-v||_\rho=0, \\ \lim_{t\searrow 0}\sup_{u\in\chi_1(\tau-t,\theta_{-t}\omega)}||\ \Psi(0,\tau-t,\theta_{-t}\omega)u-u||_\rho=0, \end{cases} \quad \forall \tau\in\mathbb{R},\omega\in\Omega. \tag{3.69}$$

Proof. It follows from the following estimate that for $v\in\chi_1(\tau,\omega)$ and $t\ge 0$,

$$\begin{aligned} &||\Psi(t,\tau,\omega)v-v||_\rho^2 \\ =&\ ||v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau(\theta_{-\tau}\omega))-v_\tau(\theta_{-\tau}\omega)||_\rho^2 \\ \le&\ 5t\int_\tau^{\tau+t}\left(||\lambda v(r)||_\rho^2+||Av(r)||_\rho^2+||\epsilon\alpha z(\theta_{r-\tau}\omega)v(r)||_\rho^2\right)dr \\ &+5t\int_\tau^{\tau+t}\left(||e^{-\epsilon z(\theta_{r-\tau}\omega)}f(e^{\epsilon z(\theta_{r-\tau}\omega)}v(r),r)||_\rho^2+||e^{-\epsilon z(\theta_{r-\tau}\omega)}g(r)||_\rho^2\right)dr \\ \le&\ 5t\int_0^t\left(\bar{\alpha}^2+(2m_0+1)^4c_1^4c_0^{2m_0}+|\epsilon\alpha z(\theta_r\omega)|^2\right)M_0^2(\theta_r\omega)dr \\ &+5t\int_\tau^{\tau+t}\left(\sum_{i\in\mathbb{Z}}\rho_i[f'_{i,s}(\vartheta_i e^{\epsilon z(\theta_{r-\tau}\omega)}v_i(r),r)]^2v_i^2(r)+e^{-2\epsilon z(\theta_{r-\tau}\omega)}||g||_\rho^2\right)dr \\ \le&\ 5t\int_0^t\left(\bar{\alpha}^2+(2m_0+1)^4c_1^4c_0^{2m_0}+|\epsilon\alpha z(\theta_r\omega)|^2\right)M_0^2(\theta_r\omega)dr \\ &+5t\int_0^t\left(\tilde{R}(e^{\epsilon z(\theta_r\omega)}M_0(\theta_r\omega),r)M_0^2(\theta_r\omega)+e^{-\epsilon z(\theta_r\omega)}||g||_\rho^2\right)dr \overset{t\searrow 0}{\rightarrow} 0, \end{aligned}$$

that is, $\lim_{t\searrow 0}\sup_{v\in\chi_1(\tau,\omega)}||\Psi(t,\tau,\omega)v-v||_\rho=0$. Similarly, $\lim_{t\searrow 0}\sup_{u\in\chi(\tau-t,\theta_{-t}\omega)}||\Psi(0,\tau-t,\theta_{-t}\omega)u-u||_\rho=0$. The proof is completed. □

As a consequence of Lemma 3.7, Lemmas 3.9–3.10, Theorem 2.2 and Theorem 2.4, we have our main result.

Theorem 3.11. *Suppose (A0)–(A6) and (3.32), (3.61) hold. Then* $\{\Psi(t,\tau,\omega)\}_{t\geq 0,\tau\in\mathbb{R},\omega\in\Omega}$ *possesses a random exponential attractor* $\{\mathcal{K}(\tau,\omega)\}_{\tau\in\mathbb{R},\omega\in\Omega}$ *with properties: for any* $\tau\in\mathbb{R}$ *and* $\omega\in\Omega$,

(i) $\mathcal{R}(\tau,\omega)\subseteq\mathcal{K}(\tau,\omega)\subseteq\overline{\chi_1(\tau,\omega)}$ *and* $\mathcal{K}(\tau,\omega)$ *is a compact set of* l_ρ^2;
(ii) $\Psi(t,\tau,\omega)\mathcal{K}(\tau,\omega)\subseteq\mathcal{K}(t+\tau,\theta_t\omega)$ *for all* $t\geq 0$;
(iii) there exists a finite integer $I_0\in\mathbb{N}$ *such that*

$$\dim_f\mathcal{R}(\tau,\omega)\leq\dim_f\mathcal{K}(\tau,\omega)\leq\frac{2(8I_0+1)\ln\left(\frac{\sqrt{(8I_0+1)}}{\delta_{I_0}}+1\right)}{\ln\frac{4}{3}}<\infty; \tag{3.70}$$

(iv) for any set $B\in\mathcal{D}(l_\rho^2)$, *there exist a random variable* $\tilde{T}_\omega\geq 0$ *and a tempered random variable* $\breve{b}_\omega>0$ *such that*

$$\mathrm{d}_h(\Psi(t,\tau,\omega)B(\tau,\omega),\mathcal{K}(t+\tau,\theta_t\omega))\leq\breve{b}_\omega e^{-\frac{\alpha\ln\frac{4}{3}}{64\ln\frac{16}{3}}t},\quad t\geq T_B+\tilde{T}_\omega; \tag{3.71}$$

(v) for any $\tau\in\mathbb{R}$, $\omega\in\Omega$, $\lim_{t\searrow 0}\mathrm{d}_s(\mathcal{K}(\tau+t,\theta_t\omega),\mathcal{K}(\tau,\omega))=0$ *and* $\lim_{t\searrow 0}\mathrm{d}_h(\mathcal{K}(\tau-t,\theta_{-t}\omega),\mathcal{K}(\tau,\omega))=0$.

Proof. From (3.65),

$$-\frac{\alpha}{2}<\mathbf{E}[-\frac{\alpha}{2}+C_4(\omega)]\leq-\frac{\alpha}{2}+\frac{3\alpha}{8}=-\frac{\alpha}{8}<0. \tag{3.72}$$

Take $t=t_0$ in (3.45) and (3.46) by

$$0<\frac{4\ln\frac{16}{3}}{\alpha}\leq t_0=\frac{2\ln\frac{3}{16}}{\mathbf{E}[-\frac{\alpha}{2}+C_4(\omega)]}<\frac{16\ln\frac{16}{3}}{\alpha}<+\infty. \tag{3.73}$$

Then

$$-\frac{\alpha}{16\ln\frac{16}{3}}\leq-\frac{1}{4t_0}<-\frac{\alpha}{64\ln\frac{16}{3}}<0.$$

From (3.62),

$$0<3\mathbf{E}[C_3^2(\omega)]+\mathbf{E}[C_4^2(\omega)]<\infty. \tag{3.74}$$

By (3.73) and (3.74),

$$0<e^{-\frac{1}{\ln\frac{3}{2}}t_0^2\left(3\mathbf{E}[C_3^2(\omega)]+\mathbf{E}[C_4^2(\omega)]\right)}<+\infty. \tag{3.75}$$

Comparing (2.52) and (3.46), we see that

$$0<\delta=2\delta_I=\frac{2}{\sqrt[4]{2\alpha}}\sqrt{\frac{1}{I}+\gamma_I+\frac{1}{I^{p+1}}+\gamma_I^{p+1}}, \tag{3.76}$$

where

$$\gamma_I = \sup_{r\in\mathbb{R}} \sum_{|i|\geq I} \rho_i \left(h_i^2(r) + \frac{2}{\alpha} g_i^2(r) \right). \tag{3.77}$$

Let

$$\tilde{\gamma} = \min\left\{ \frac{1}{8}, e^{-\frac{1}{\ln\frac{3}{2}} t_0^2 \left(3\mathbf{E}[C_3^2(\omega)] + \mathbf{E}[C_4^2(\omega)]\right)} \right\} \in (0, +\infty) \tag{3.78}$$

be a bounded fixed positive number. By $h(t), g(t) \in \mathcal{G}$ and $\lim_{I\to+\infty} \frac{1}{I} = \lim_{I\to+\infty} \frac{1}{I^{p+1}} = 0$, it then follows from (3.76) that there exists a finite integer $I_0 \in \mathbb{N}$ such that $0 < 2\delta_{I_0} \leq \tilde{\gamma}$. Then Theorem 2.4 and Theorem 2.2 implies the statements in Theorem 3.11. The proof is completed. □

Remark 3.1. Suppose (A0)–(A6) hold. If the positive number c_2 in (3.4) (in (A2)) and $|\epsilon|$ in (3.1) are both suitable small such that

$$|\epsilon|\sqrt{\frac{\alpha}{\pi}} + c_2 < \frac{\alpha}{2}. \tag{3.79}$$

Then the statements in Theorem 3.11 are still valid. In fact, in this case, (3.52) can be written as

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right) \rho_i y_i^2(r) + (\frac{5}{4}\alpha - 2\epsilon\alpha z(\theta_{r-\tau}\omega) - 2c_2) \sum_{i\in\mathbb{Z}} \theta\left(\frac{|i|}{M}\right) \rho_i y_i^2(r) \leq \frac{c_6}{M} ||y(r)||_\rho^2,$$

thus,

$$\begin{aligned} &\sum_{|i|>2M} \rho_i y_i^2(\tau, \tau - t, \theta_{-\tau}\omega, y_{\tau-t}(\theta_{-\tau}\omega))) \\ &\leq \left(e^{\int_{-t}^0 (-\frac{\alpha}{2} + C_6(\theta_s\omega))ds} + \sqrt{\frac{c_6}{\alpha M}} e^{\int_{-t}^0 C_5(\theta_s\omega)ds} \right)^2 ||y(\tau - t)||_\rho^2, \end{aligned}$$

where

$$C_5(\omega) = 2|\epsilon\alpha z(\omega)| + 2c_2 + a_0c_1c_3c_4, \qquad C_6(\omega) = |\epsilon\alpha z(\omega)| + c_2,$$
$$\mathbf{E}[C_5^2(\omega)] \leq 3\left(2\alpha\epsilon^2 + 4c_2^2 + a_0^2c_1^2c_3^2c_4^2\right) < \infty,$$

and

$$\mathbf{E}[C_6(\omega)] = |\epsilon|\sqrt{\frac{\alpha}{\pi}} + c_2, \quad \mathbf{E}[C_6^2(\omega)] \leq \alpha\epsilon^2 + 2c_2^2 < \infty.$$

Acknowledgments

The author would like to thank the referee for carefully reading the manuscript and making very useful suggestions and comments.

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