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New classes of non-convolution integral equations arising from Lie symmetry analysis of hyperbolic PDEs

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Abstract

In this paper we consider some new classes of integral equations that arise from Lie symmetry analysis. Specifically, we consider the task of obtaining solutions of a Cauchy problem for some classes of second order hyperbolic partial differential equations. Our analysis leads to new integral equations of non-convolution type, which can be solved by classical methods. We derive solutions of these integral equations, which in turn lead to solutions of the associated Cauchy problems.
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1. Introduction

The theory of continuous symmetry groups of systems of partial differential equations was developed by Lie in the last decades of the nineteenth century. A symmetry is a transformation which maps solutions to other solutions. Lie developed a method for systematically computing all continuous symmetries of a given system of differential equations. Excellent modern accounts may be found in the books by Olver [17] and those of Bluman and his coauthors, such as the volume [2].

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Symmetries are powerful tools and they allow us to solve a wide variety of problems. For example, we may compute fundamental solutions of parabolic equations from a knowledge of their symmetries. See the papers [10], [11], [6], [9] for details, together with many examples and applications. Boundary value problems may also be solved by the method of group invariant solutions. The aforementioned book [2] contains an extensive discussion of this topic. Symmetries are also essential to the study of conservation laws. A chapter of [17] is devoted to this.

A new way of applying Lie symmetries was introduced in [6] and developed extensively in [7]. The idea is to construct an integral operator that maps test functions to solutions of the PDE by integration against a symmetry solution. The purpose of this paper is to show how the method can lead to new classes of non-convolution integral equations, which may be solved by a combination of classical techniques.

We first provide an example to illustrate how the method works. We then turn to a class of second order linear hyperbolic equations and derive some new integral equations which arise in the solution of Cauchy problems associated to these equations. We then solve these equations using a novel combination of integral transform methods.

The outline of the paper is as follows. In Section 2 we introduce the method that forms the basis of our analysis. In Section 3 we determine the symmetries of a class of hyperbolic PDEs of the form $u_{tt} = u_{xx} + f(x)u$. We show that there are nontrivial symmetries when f satisfies one of three families of Riccati equations. In Section 4 we study the equation $u_{tt} = y^2 u_{yy} + y u_y + y^2 u$ (which is equivalent to $u_{tt} = u_{xx} + e^{2x}u$, with $y = e^x$) and show how the symmetries lead to a solution of the Cauchy problem for this equation via new non-convolution integral equations, which we are able so solve. See Theorems 4.2 and 4.3. In Section 5 we set up the integrals for the Cauchy problem for the more general equation $u_{tt} = u_{xx} + (Ae^x + Be^{-x})^{-2}u$. In Section 6 we solve the integral equation in the case $A = B = 1/2$ and obtain a solution of the problem $u_{tt} = u_{xx} + (\operatorname{sech}^2 x)u$, $x > 0$, subject to $u(x,0) = f(x)$, $u_t(x,0) = 0$. (See Theorem 6.6.) A brief conclusion follows. We believe that the methods of this paper may prove to be of use in the study of various types of linear PDEs.

2. Introduction to the method

We illustrate the approach by solving the problem

$$u_{tt} = u_{xx} - \frac{A}{x^2}u, \ u(x,0) = f(x), u_t(x,0) = g(x), \ x > 0, t \geq 0.$$

This problem was solved using different symmetries in [7]. Here we use a simpler approach. We suppose that $f, g \in \mathcal{S}(\mathbb{R}_+)$ the Schwartz space, consisting of smooth functions whose derivatives (including the function itself) decay at infinity faster than any power, i.e. Schwartz functions are rapidly decreasing. It is known that the Fourier Transform and Mellin transforms are automorphisms of the Schwartz space, see [22]. This space is a topological vector space of functions φ such that $\varphi \in C^\infty(\mathbb{R}_+)$ and $x^\alpha \varphi^{(\beta)}(x) \to 0$, $x \to \infty$, $\alpha, \beta \in \mathbb{N}_0$. We note here that throughout the paper $L_1(\sigma)$ will denote the usual Lebesgue space of measurable functions f such that $\int_\sigma |f(s)|ds < \infty, \sigma = \{s \in \mathbb{C} : s = \frac{1}{2} + i\tau, \tau \in \mathbb{R}\}$.

It is elementary that the scaling transformation $u(x,t) \to u(\lambda x, \lambda t)$ is a symmetry. The idea is to introduce a new solution by setting

$$U(x,t) = \int_0^\infty \varphi(\lambda) u(\lambda x, \lambda t) d\lambda. \tag{2.1}$$

With separation constant equal to one half, separation of variables leads to the solution $u_1(x,t) = \sqrt{x} I_\nu(x) e^t$, where $\nu = \frac{1}{2}\sqrt{4A+1}$ and $I_\nu(x)$ denotes the usual modified Bessel function (see 9.6.10 of [1]). We denote the Bessel function of the first kind by $J_\nu(z)$ (9.1.10 of [1]). Applying the symmetry and replacing λ with $i\lambda$, we obtain a solution

$$u(x,t) = \int_0^\infty \varphi(\lambda)\sqrt{\lambda x} J_\nu(\lambda x)\cos(\lambda t) d\lambda + \int_0^\infty \psi(\lambda)\sqrt{\lambda x} J_\nu(\lambda x)\sin(\lambda t) d\lambda.$$

We absorbed the terms in i into φ and ψ. The requirement that $u(x,0) = f(x)$ leads to the equation

$$f(x) = \int_0^\infty \varphi(\lambda)\sqrt{\lambda x} J_\nu(\lambda x) d\lambda.$$

We immediately recognize that f must be the Hankel transform of φ. The inversion theorem for the Hankel transform (see [23]) yields

$$\varphi(\lambda) = \int_0^\infty f(y)\sqrt{\lambda y} J_\nu(\lambda y) dy.$$

For the condition $u_t(x,0) = g(x)$ we find

$$\int_0^\infty \lambda\psi(\lambda)\sqrt{\lambda x} J_\nu(\lambda x) d\lambda = g(x), \tag{2.2}$$

which gives

$$\psi(\lambda) = \frac{1}{\lambda}\int_0^\infty g(y)\sqrt{\lambda y} J_\nu(\lambda y) dy. \tag{2.3}$$

Hence

$$u(x,t) = \int_0^\infty\int_0^\infty f(y)\sqrt{xy}\lambda J_\nu(\lambda x) J_\nu(\lambda y)\cos(\lambda t) dy d\lambda$$
$$+ \int_0^\infty\int_0^\infty g(y)\sqrt{xy} J_\nu(\lambda x) J_\nu(\lambda y)\sin(\lambda t) dy d\lambda,$$

solves our initial value problem. Thus we have obtained a solution of the initial value problem from a separable solution and a scaling symmetry. This solution can be simplified by making use of the result

$$\begin{aligned}
&\int_0^\infty \sin(cx) J_\mu(ax) J_\mu(bx) dx \\
&\quad = 0, \qquad\qquad 0 < c < b - a, 0 < a < b \\
&\quad = \frac{1}{2\sqrt{ab}} P_{\mu-1/2}\left(\frac{b^2 + a^2 - c^2}{2ab}\right), \quad b - a < c < b + a, 0 < a < b \\
&\quad = \frac{\cos(\mu\pi)}{\pi\sqrt{ab}} Q_{\mu-1/2}\left(-\frac{b^2 + a^2 - c^2}{2ab}\right), \ b + a < c, 0 < a < b,
\end{aligned}$$

which is valid for $\mu > -1$. (See formula 6.672.1 of [13].) Here P_ν and Q_ν are Legendre functions, [16]. We define

$$p(x, y, t) = \int_0^\infty J_\nu(\lambda x) J_\nu(\lambda y) \sin(\lambda t) d\lambda, \tag{2.4}$$

via the preceding integral and the solution can be written

$$u(x, t) = \int_0^\infty \int_0^\infty f(y) \sqrt{xy} \lambda J_\nu(\lambda x) J_\nu(\lambda y) \cos(\lambda t) dy d\lambda + \int_0^\infty \sqrt{xy} g(y) p(x, y, t) dy. \tag{2.5}$$

The absolute convergence of the second integral justifies our use of Fubini's Theorem.

The integral $\int_0^\infty \lambda J_\nu(\lambda x) J_\nu(\lambda y) \cos(\lambda t) d\lambda$ is divergent, so we cannot reverse the order of integration in the first integral in (2.5). However, we can define a solution in the distributional sense by noting that for each n

$$\int_0^n \lambda J_\nu(\lambda x) J_\nu(\lambda y) \cos(\lambda t) d\lambda = \frac{\partial}{\partial t} \int_0^n J_\nu(\lambda x) J_\nu(\lambda y) \sin(\lambda t) d\lambda.$$

Taking the limit as $n \to \infty$ we define the distribution Λ on $\mathcal{S}(\mathbb{R}_+)$ by setting

$$\Lambda(f) = \int_0^\infty f(y) \frac{\partial}{\partial t} \int_0^\infty J_\nu(\lambda x) J_\nu(\lambda y) \sin(\lambda t) d\lambda dy. \tag{2.6}$$

This leads to a solution in the distributional sense given by

$$u(x, t) = \frac{\partial}{\partial t} \int_0^\infty f(y) \sqrt{xy} p(x, y, t) dy + \int_0^\infty \sqrt{xy} g(y) p(x, y, t) dy. \tag{2.7}$$

In general, this procedure requires us to solve one or more integral equations if we are to satisfy the initial data. Often this equation turns out to be a familiar integral transform with a known inversion formula, as happened with our example. In this paper we will show how it can lead to entirely new solvable integral equations.

This method hints at a deep connection between Lie symmetries and classical harmonic analysis and indeed this is the case. The relationship between Lie symmetry analysis and harmonic analysis and representation theory has been developed over the years, beginning with [4], [3] and [5]. See also [8], [9], [21], [20] and [12].

The basic idea of this work is that for linear PDEs, it is often possible to realize the Lie symmetries as global representations of the underlying Lie group in the following sense. One has a Lie group $\mathcal{G}$, a representation (σ, V) of $\mathcal{G}$ and a mapping $A : V \to H$ where H is a solution space of the equation. So A maps $f \in V$ to some solution of the PDE. Then if the action of the symmetries on solutions $u = Af$ is denoted by $\rho(g)u$ for $g \in \mathcal{G}$, the following relationship holds

$$(\rho(g)Af)(x) = (A\sigma(g)f)(x), \ g \in \mathcal{G}, \tag{2.8}$$

where $f \in V$.

In fact this relationship has been established for many classes of PDEs and the previous references detail some of the work done in this area.

A consequence of all this is that the operator

$$v(x) = \int_{\Omega} \varphi(\epsilon)\rho(\exp \epsilon \mathbf{v})u(x)d\epsilon,$$

is essentially a group theoretic Fourier transform, which inherits properties of the representations equivalent to ρ. Here we have assumed that g comes from a one parameter group of symmetries which is generated by the vector field $\mathbf{v}$, so that $(\rho(g)u)(x) = \rho(\exp(\epsilon \mathbf{v})u)(x), \epsilon \in \Omega \subseteq \mathbb{R}$. It is therefore not surprising that this technique so often leads to the problem of inverting a standard integral transform. In this paper we consider some problems where the integral equations do not appear in the existing literature.

3. The equation $u_{tt} = u_{xx} + f(x)u$

We will determine the forms of the potential function f which allow nontrivial symmetries. For an in depth discussion of the computation of Lie symmetries, see the book by Olver [17]. We look for a vector field $\mathbf{v} = \xi(x,t)\partial_x + \tau(x,t)\partial_t + \phi(x,t,u)\partial_u$ which generates the Lie symmetries of the PDE. An application of Lie's Theorem (Theorem 2.31, p. 104, [17]) leads to the defining equations

$$\phi^{tt} = \phi^{xx} + f(x)\phi + f'(x)\xi u, \tag{3.1}$$

where

$$\phi^{xx} = \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + \phi_{uu}u_x^2 + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt}$$
$$\phi^{tt} = \phi_{tt} - \xi_{tt}u_x + (2\phi_{tu} - \tau_{tt})u_t + \phi_{uu}u_t^2 - 2\xi_t u_{xt} + (\phi_u - 2\tau_t)u_{tt}.$$

From the defining equations (3.1), we find the conditions

$$\begin{aligned}\phi = \alpha u + \beta,\ \xi_t = \tau_x, -\xi_{tt} = 2\phi_{xu} - \xi_{xx},\\ 2\phi_{ut} = \tau_{tt} - \tau_{xx}, \phi_u - 2\tau_t = \phi_u - 2\xi_x\\ \phi_{tt} + (\phi_u - 2\tau_t) f(x) u = \phi_{xx} + f(x)\phi + f'(x)u\xi.\end{aligned}$$

The function β will be an arbitrary solution of the PDE. Nothing further can be said about it. From the first and last of these equations we have

$$\alpha_{tt} = \alpha_{xx} + f'(x)\xi + 2\tau_t f(x). \tag{3.2}$$

It is also easy to see that $\xi_{xx} = \xi_{tt}$ and $\tau_{tt} = \tau_{xx}$. Consequently $2\phi_{ut} = 0$, and this means α is independent of t. Similarly we see that α is also independent of x. It is thus a constant. From (3.2) we see that

$$2\xi_x + \frac{f'}{f}\xi = 0. \tag{3.3}$$

This means that $\xi(x,t) = e^{-\frac{1}{2}\ln f} c(t) = \frac{c(t)}{\sqrt{f}}$, where c has yet to be fixed. With $G(x) = \frac{c(t)}{\sqrt{f}}$, we now require

$$c''(t)G(x) = G''(x)c(t). \tag{3.4}$$

This is only possible if $G''/G = \lambda$ is a constant. We then have three possibilities for c:

$$c(t) = \begin{cases} c_1 t + c_2 \\ c_1 \sin(\omega t) + c_2 \cos(\omega t) \\ c_1 e^{\omega t} + c_2 e^{-\omega t}. \end{cases} \tag{3.5}$$

Clearly the three possible choices for G are

$$G(x) = \begin{cases} Ax + B \\ Ae^{\omega x} + Be^{-\omega x} \\ A\cos\omega x + B\sin\omega x \end{cases} \tag{3.6}$$

where A, B are arbitrary constants. Correspondingly, the possible choices for f are

$$f(x) = \begin{cases} \frac{1}{(Ax+B)^2} \\ \frac{1}{(Ae^{\omega x} + Be^{-\omega x})^2} \\ \frac{1}{(A\cos\omega x + B\sin\omega x)^2}. \end{cases} \tag{3.7}$$

We can now write down a basis for the Lie algebra of symmetries for each choice of f. In each case, the Lie algebra is isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{r}$, where $\mathfrak{r}$ is the Lie algebra generating the additive Lie group $\mathbb{R}$.

We are interested in the second case. The first case is obtainable from our illustrative example by a linear change of variables. If

$$f(x) = \frac{1}{(Ae^{\omega x} + Be^{-\omega x})^2},$$

a basis for the Lie symmetry algebra is

$$\mathbf{v}_1 = e^{\omega t}(Ae^{\omega x} + Be^{-\omega x})\partial_x + e^{\omega t}(Ae^{\omega x} - Be^{-\omega x})\partial_t,$$
$$\mathbf{v}_2 = e^{-\omega t}(Ae^{\omega x} + Be^{-\omega x})\partial_x - e^{-\omega t}(Ae^{\omega x} - Be^{-\omega x})\partial_t,$$
$$\mathbf{v}_3 = \partial_t, \mathbf{v}_4 = u\partial_u, \mathbf{v}_\beta = \beta\partial_u.$$

The vector field $\mathbf{v}_3$ clearly induces the symmetry $u(x,t) \to u(x,t+\epsilon)$. Let us obtain the symmetry arising from the first vector field. We have to solve

$$\frac{d\tilde{x}}{d\epsilon} = e^{\omega\tilde{t}}(Ae^{\omega\tilde{x}} + Be^{-\omega\tilde{x}}),\ \tilde{x}(0) = x, \tag{3.8}$$
$$\frac{d\tilde{t}}{d\epsilon} = e^{\omega\tilde{t}}(Ae^{\omega\tilde{x}} - Be^{-\omega\tilde{x}}),\ \tilde{t}(0) = t. \tag{3.9}$$

Adding these equations gives

$$\frac{d(\tilde{x}+\tilde{t})}{d\epsilon} = 2Ae^{\omega(\tilde{x}+\tilde{t})}.$$

Now setting $Z = \tilde{x} + \tilde{t}$, we solve this first order DE and get

$$e^{\omega(\tilde{x}+\tilde{t})} = \frac{e^{\omega(x+t)}}{1 - 2A\epsilon\omega e^{\omega(x+t)}}. \tag{3.10}$$

Next we solve

$$\frac{d(\tilde{x}-\tilde{t})}{d\epsilon} = 2Be^{-\omega(\tilde{x}-\tilde{t})}. \tag{3.11}$$

With $z = \tilde{x} - \tilde{t}$, $z(0) = x - t$ we obtain the expression

$$e^{\omega(\tilde{x}-\tilde{t})} = e^{\omega(x-t)} + 2B\epsilon\omega. \tag{3.12}$$

Taking logs we have a pair of simultaneous equations for $\tilde{x}$ and $\tilde{t}$. This gives us

$$e^{\omega\tilde{x}} = \left(\frac{e^{2\omega x} + 2B\epsilon\omega e^{\omega(x+t)}}{1 - 2A\epsilon\omega e^{\omega(x+t)}}\right)^{\frac{1}{2}} \tag{3.13}$$

and

$$e^{\omega\tilde{t}} = \frac{e^{\omega(x+t)}}{\sqrt{(e^{2\omega x} + 2B\epsilon\omega e^{\omega(x+t)})(1 - 2A\epsilon\omega e^{\omega(x+t)})}}.$$

From these we obtain expressions for $\tilde{x}$ and $\tilde{t}$. From this we conclude that if u is a solution of

$$u_{tt} = u_{xx} + \frac{1}{(Ae^{\omega x} + Be^{-\omega x})^2}u, \tag{3.14}$$

then so is

$$\tilde{u}(x,t;\epsilon) = u\left(\frac{1}{2}\ln\left(\frac{e^{2\omega x} + 2B\epsilon\omega e^{\omega(x+t)}}{1 - 2A\epsilon\omega e^{\omega(x+t)}}\right), \ln\left(\frac{e^{\omega(x+t)}}{\sqrt{(e^{2\omega x} + 2B\epsilon\omega e^{\omega(x+t)})(1 - 2A\epsilon\omega e^{\omega(x+t)})}}\right)\right). \tag{3.15}$$

The vector field $\mathbf{v}_2$ can be exponentiated by similar means. We have

$$e^{2\omega\tilde{t}} = \frac{2B\epsilon\omega + e^{\omega(x+t)}}{2A\epsilon\omega + e^{\omega(x-t)}},\ e^{2\omega\tilde{x}} = (2B\epsilon\omega + e^{\omega(x+t)})(2A\epsilon\omega + e^{\omega(x-t)}).$$

The details are left to the interested reader.

4. The case of one zero constant

To begin our analysis we will first consider the special case when $A = 0$, $B = 1$. The case B arbitrary can be obtained from this by an obvious change of variables. This leads to the equation

$$u_{tt} = u_{xx} + e^{2x}u, \quad x \in \mathbb{R}. \tag{4.1}$$

For convenience we will make the change of variables $y = e^x$. This leads to the equation

$$u_{tt} = y^2u_{yy} + yu_y + y^2u, \quad y > 0. \tag{4.2}$$

We will solve this subject to the initial conditions $u(y,0) = f(x)$, and $u_t(y,0) = g(y)$ where f, g are suitable functions. It is straightforward to see that a stationary solution is $u_0(y) = J_0(y)$, where $J_0(y)$ is the usual zeroth order Bessel function of the first kind, [1]. Application of the symmetry (3.15) under the change of variables shows that

$$\tilde{u}_1(y,t;\lambda) = J_0(\sqrt{y^2 + 2\lambda ye^t}), \tag{4.3}$$

is also a solution. Since $t \to -t$ is a symmetry, it is clear that

$$\tilde{u}_2(y,t;\lambda) = J_0(\sqrt{y^2 + 2\lambda ye^{-t}}) \tag{4.4}$$

is again a solution. We now consider

$$u(y,t)=\frac{1}{2}\int_0^\infty \varphi(\lambda)\left[J_0(\sqrt{y^2+2\lambda y e^t})+J_0(\sqrt{y^2+2\lambda y e^{-t}})\right]d\lambda, \tag{4.5}$$

where φ has suitable decay to guarantee convergence of the integral. This is easily seen to satisfy the PDE (4.2) and the conditions

$$u(y,0)=\int_0^\infty \varphi(\lambda)J_0(\sqrt{y^2+2\lambda y})d\lambda \tag{4.6}$$

and $u_t(y,0)=0$. Our first task is then to solve the integral equation

$$\int_0^\infty \varphi(\lambda)J_0(\sqrt{y^2+2\lambda y})d\lambda = f(y). \tag{4.7}$$

We proceed by taking the Laplace transform in y of both sides. We suppose that $\varphi\in L_1(\mathbb{R}_+)$. Via the known inequality for the Bessel function $\sqrt{x}|J_\nu(x)|<C, x>0$, where C is an absolutely positive constant, we have from (4.7) the estimate

$$|f(y)|\le C\int_0^\infty \frac{|\varphi(\lambda)|}{(y^2+2\lambda y)^{1/4}}d\lambda \le \frac{C}{\sqrt{y}}\|\varphi\|_{L_1(\mathbb{R}_+)}. \tag{4.8}$$

Thus

$$\left|\int_0^\infty f(y)e^{-sy}dy\right| \le C\|\varphi\|_{L_1(\mathbb{R}_+)}\int_0^\infty \frac{e^{-sy}}{\sqrt{y}}dy = \frac{\sqrt{\pi}C\|\varphi\|_{L_1(\mathbb{R}_+)}}{\sqrt{s}}.$$

Now 10.2.6, p. 59 of [19] gives

$$\int_0^\infty J_0(\sqrt{y^2+2\lambda y})e^{-sy}dy = \frac{e^{-\lambda(\sqrt{s^2+1}-s)}}{\sqrt{s^2+1}}. \tag{4.9}$$

If $F(s)=\int_0^\infty f(y)e^{-sy}dy$ then an application of the Laplace transform and Fubini's Theorem (justified by (4.8)), gives

$$\int_0^\infty \varphi(\lambda)\frac{e^{-\lambda(\sqrt{s^2+1}-s)}}{\sqrt{s^2+1}}d\lambda = F(s). \tag{4.10}$$

Letting Φ denote the Laplace transform of φ, we have

$$\frac{\Phi(\sqrt{s^2+1}-s)}{\sqrt{s^2+1}} = F(s). \tag{4.11}$$

The substitution $s = \sinh k$ reduces this to

$$\Phi(e^{-k}) = \cosh k F(\sinh k). \tag{4.12}$$

Putting $k = -\ln \xi$ gives

$$\Phi(\xi) = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right) F\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right). \tag{4.13}$$

Thus

$$\varphi(\lambda) = \mathcal{L}_{\xi}^{-1}\left[\frac{1}{2}\left(\xi + \frac{1}{\xi}\right) F\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right)\right]. \tag{4.14}$$

We have thus established the following.

Proposition 4.1. *The equation $u_{tt} = y^2 u_{yy} + y u_y + y^2 u$ with $u(y,0) = f(y)$ and $u_t(y,0) = 0$ has a solution*

$$u(y,t) = \frac{1}{2}\int_0^{\infty} \mathcal{L}_{\xi}^{-1}\left[\frac{1}{2}\left(\xi + \frac{1}{\xi}\right) F\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right)\right](\lambda) \times \left[J_0(\sqrt{y^2 + 2\lambda y e^t}) + J_0(\sqrt{y^2 + 2\lambda y e^{-t}})\right] d\lambda, \tag{4.15}$$

provided the integral converges. Here F is the Laplace transform of f.

Our next task is to determine an expression for the inverse Laplace transform. An elementary approach is to assume that f is analytic. That is

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) y^n. \tag{4.16}$$

In this case

$$F(s) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{1}{s^{n+1}}, \tag{4.17}$$

so that

$$\frac{1}{2}\left(\xi + \frac{1}{\xi}\right) F\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{(\xi^2 + 1)(2\xi)^n}{(1 - \xi^2)^{n+1}}. \tag{4.18}$$

Thus

$$\varphi(\lambda) = \sum_{n=0}^{\infty} f^{(n)}(0) g_n(\lambda), \tag{4.19}$$

where

$$g_n(\lambda) = \mathcal{L}_\xi^{-1}\left[\frac{(\xi^2+1)(2\xi)^n}{(1-\xi^2)^{n+1}}\right], \tag{4.20}$$

which can be explicitly computed for any n.

We can also obtain an explicit solution via a double integral as follows. Making the substitution $s = \sinh(\ln\xi) = \frac{1}{2}\left(\xi - \frac{1}{\xi}\right)$, $\xi > 1$ in the equation

$$\Phi(\sqrt{s^2+1} - s) = \sqrt{s^2+1}F(s), \tag{4.21}$$

we obtain

$$\frac{1}{\xi}\Phi\left(\frac{1}{\xi}\right) = \left(\frac{1}{\xi^2}+1\right)\int_0^\infty e^{-y\left(\xi-\frac{1}{\xi}\right)} f(2y)dy, \ \xi > 1. \tag{4.22}$$

To proceed, we use two operational relations for the Laplace transform, which may be found in [19]. Specifically, if $g(s) = (\mathcal{L}h)(s)$, then via the uniqueness property for the Laplace transform

$$\frac{1}{s}g\left(\frac{1}{s}\right) = \mathcal{L}\left[\int_0^\infty J_0\left(2\sqrt{\lambda y}\right)h(\lambda)d\lambda\right](s), \tag{4.23}$$

which is (21) on page 171 of [19]. Further, with $I_{\pm1}(z)$ standard modified Bessel functions, [1], we have

$$\left(\frac{1}{s^2}+1\right)g\left(s-\frac{1}{s}\right) = \mathcal{L}\left[\int_0^y (y-\lambda)^{1/2}I_1\left(2\sqrt{\lambda(y-\lambda)}\right)h(\lambda)\frac{d\lambda}{\sqrt{\lambda}}\right.$$
$$\left. - \int_0^y (y-\lambda)^{-1/2}I_{-1}\left(2\sqrt{\lambda(y-\lambda)}\right)h(\lambda)\sqrt{\lambda}\,d\lambda\right](s), \tag{4.24}$$

from formula (27), again p. 171 of [19]. Hence, returning to (4.22), we cancel the Laplace transform via the uniqueness property and take into account that $I_1(z) = I_{-1}(z)$ to obtain for $y > 0$

$$\int_0^\infty J_0\left(2\sqrt{\lambda y}\right)\varphi(\lambda)d\lambda = \int_0^y (y-2\lambda)\frac{I_1\left(2\sqrt{\lambda(y-\lambda)}\right)}{\sqrt{\lambda(y-\lambda)}}f(2\lambda)d\lambda. \tag{4.25}$$

However the left-hand side of (4.25) represents a variant of the Hankel transform of index zero, which admits the symmetric inversion formula for φ whose Mellin transform $\varphi^*(s) \in L_1(\sigma)$. (See the details in [25], Ch. 2, Section 2.2, Example 2.2.) We remark here that the Mellin transform is defined by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx. \tag{4.26}$$

Its inverse transform under certain conditions is given by the integral

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) x^{-s} ds. \tag{4.27}$$

See [23], [25] for details of the Mellin transform.

Thus, inverting the Hankel transform, we arrive at the unique and explicit solution of the integral equation (4.7), namely,

$$\varphi(\lambda) = \int_0^\infty \int_0^{2y} J_0\left(2\sqrt{\lambda y}\right) \frac{I_1\left(\sqrt{u(2y-u)}\right)}{\sqrt{u(2y-u)}} (y-u) f(u)\, du dy, \tag{4.28}$$

$\lambda > 0$. We note, that the interchange of the order of integration in (4.28) is impossible because the corresponding integral with respect to y will be divergent.

Now we observe that the inequality

$$1 < \Gamma(\nu+1) \left(\frac{2}{x}\right)^\nu I_\nu(x) < \cosh x < e^x \tag{4.29}$$

(see [14]) leads to

$$\left| \int_0^{2y} \frac{I_1\left(\sqrt{u(2y-u)}\right)}{\sqrt{u(2y-u)}} (y-u) f(u)\, du \right| \le \frac{y}{2} e^y \int_0^{2y} |f(u)| du. \tag{4.30}$$

Zemanian constructed a Frechet space $\mathfrak{H}_\mu(\mathbb{R}_+)$ characterized by three properties: Every $\phi \in \mathfrak{H}_\mu(\mathbb{R}_+)$ is a rapidly decreasing smooth function on $\mathbb{R}_+$, with an expansion of the form

$$\phi(x) = x^{\mu+1/2} \left(a_0 + a_2 x^2 + \cdots + a_{2k} x^{2k} + R_{2k}(x)\right), \tag{4.31}$$

where

$$a_{2k} = \frac{1}{k! 2^k} \lim_{x \to 0} (x^{-1} D)^k (x^{-\mu-1/2} \phi(x)), \tag{4.32}$$

and the remainder satisfies $(x^{-1}D)^k R_{2k}(x) = o(1)$ as $x \to 0^+$. (See Lemma 5.2.1 of [26].) Zemanian proves that the Hankel transform of index μ is an automorphism on $\mathfrak{H}_\mu(\mathbb{R}_+)$, [26]. We see then that if $ye^y \int_0^{2y} |f(u)| du \in \mathfrak{H}_0(\mathbb{R}_+)$, then $\varphi \in \mathfrak{H}_0(\mathbb{R}_+)$.

We summarize our results by the following result.

Theorem 4.2. *Let f be a given continuous function on $\mathbb{R}_+$ such that $f(y) = O(y^{-1/2})$, $y \to \infty$. Let $\varphi \in L_1(\mathbb{R}_+)$ and its Mellin transform $\varphi^*(s) \in L_1(\sigma)$. Then φ given by formula (4.28) represents the unique solution of the integral equation (4.7). Moreover if $ye^y F(y) \in \mathfrak{H}_0(\mathbb{R}_+)$, where $F(y) = \int_0^{2y} |f(u)|du$, and*

$$\tilde{f}(\lambda) = \int_0^\infty \int_0^{2y} J_0\left(2\sqrt{\lambda y}\right) \frac{I_1\left(\sqrt{z(2y-z)}\right)}{\sqrt{z(2y-z)}} (y-z) f(z) dz dy, \tag{4.33}$$

then

$$u(y,t) = \frac{1}{2} \int_0^\infty \tilde{f}(\lambda) \left[J_0(\sqrt{y^2 + 2\lambda y e^t}) + J_0(\sqrt{y^2 + 2\lambda y e^{-t}}) \right] d\lambda, \tag{4.34}$$

is a solution of

$$u_{tt} = y^2 u_{yy} + y u_y + y^2 u, \tag{4.35}$$

satisfying the initial conditions $u(y,0) = f(y)$, $u_t(y,0) = 0$.

To obtain a solution satisfying $u(y,0) = 0$ and $u_t(y,0) = g(y)$ let

$$w(y,t) = \frac{1}{2} \int_0^\infty \psi(\lambda) \left[J_0(\sqrt{y^2 + 2\lambda y e^t}) - J_0(\sqrt{y^2 + 2\lambda y e^{-t}}) \right] d\lambda. \tag{4.36}$$

Then $w(y,0) = 0$ and

$$w_t(y,0) = -\int_0^\infty \psi(\lambda) \frac{\lambda y J_1\left(\sqrt{y^2 + 2\lambda y}\right)}{\sqrt{y^2 + 2\lambda y}} d\lambda. \tag{4.37}$$

We therefore seek to solve

$$-\int_0^\infty \lambda \psi(\lambda) \frac{J_1\left(\sqrt{y^2 + 2\lambda y}\right)}{\sqrt{y^2 + 2\lambda y}} d\lambda = \frac{1}{y} g(y) = \tilde{g}(y). \tag{4.38}$$

This can actually be reduced to the case covered by Theorem 4.2. In fact, assuming that $\lambda\psi(\lambda)$ is absolutely continuous on $\mathbb{R}_+$ and its derivative $\varphi(\lambda) = [\lambda\psi(\lambda)]'$ satisfies conditions of Theorem 4.2, we integrate by parts in (4.38), employing the equality

$$\frac{y}{\sqrt{y^2 + 2\lambda y}} J_1(\sqrt{y^2 + 2\lambda y}) = -\frac{d}{d\lambda} J_0(\sqrt{y^2 + 2\lambda y})$$

and obtain the integral equation

$$\int_0^\infty [\lambda\psi(\lambda)]' J_0(\sqrt{y^2+2\lambda y})d\lambda = -g(y).$$

Hence our previous result gives

$$\psi(\lambda) = \frac{1}{\lambda}\int_0^\lambda \int_0^\infty \int_0^{2y} J_0\left(2\sqrt{yv}\right) \frac{I_1\left(\sqrt{u(2y-u)}\right)}{\sqrt{u(2y-u)}}(u-y)g(u)\,dudydv, \tag{4.39}$$

for $\lambda > 0$.

Theorem 4.3. *Suppose that $ye^y G(y) \in \mathfrak{H}_0(\mathbb{R}_+)$, $G(y) = \int_0^{2y} |g(z)|dz$. Then the equation $u_{tt} = y^2 u_{yy} + yu_y + y^2 u$ with $u(y,0) = 0$ and $u_t(y,0) = g(y)$ has a solution*

$$u(y,t) = \frac{1}{2}\int_0^\infty \tilde{g}(\lambda)\left[J_0(\sqrt{y^2+2\lambda ye^t}) - J_0(\sqrt{y^2+2\lambda ye^{-t}})\right]d\lambda, \tag{4.40}$$

where

$$\tilde{g}(\lambda) = \frac{1}{\lambda}\int_0^\lambda \int_0^\infty \int_0^{2y} J_0\left(2\sqrt{yv}\right) \frac{I_1\left(\sqrt{z(2y-z)}\right)}{\sqrt{z(2y-z)}}(z-y)g(z)dzdydv. \tag{4.41}$$

Hence, provided that f and g satisfy the conditions of Theorems 4.2 and 4.3, a solution of our original problem may be written

$$u(y,t) = \frac{1}{2}\left(\int_0^\infty \tilde{f}(\lambda)\left[J_0(\sqrt{y^2+2\lambda ye^t}) + J_0(\sqrt{y^2+2\lambda ye^{-t}})\right]d\lambda \right.$$
$$\left. + \int_0^\infty \tilde{g}(\lambda)\left[J_0(\sqrt{y^2+2\lambda ye^t}) - J_0(\sqrt{y^2+2\lambda ye^{-t}})\right]d\lambda\right). \tag{4.42}$$

4.0.1. The direct Laplace transform approach

It is worth showing how the Laplace transform method can be applied to (4.38). This approach is computationally useful. On page 60 of [19] we find

$$\int_0^\infty \frac{J_1(\sqrt{y^2+2\lambda y})}{\sqrt{y^2+2\lambda y}} e^{-sy} dy = \frac{1 - e^{-\lambda(\sqrt{s^2+1}-s)}}{\lambda}. \tag{4.43}$$

If the Laplace transform of $g(y)/y$ is G, then taking Laplace transform of both sides of (4.38), we obtain

$$-\int_0^\infty \psi(\lambda)(1-e^{-\lambda(\sqrt{s^2+1}-s)})d\lambda = G(s). \tag{4.44}$$

Which is the same as

$$\Psi(\sqrt{s^2+1}-s) - \Psi(0) = G(s). \tag{4.45}$$

We have denoted the Laplace transform of ψ by Ψ. Putting $s = \sinh k$ and $k = -\ln \xi$ gives

$$\Psi(\xi) - \Psi(0) = G\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right). \tag{4.46}$$

So that

$$\psi(\lambda) = \Psi(0)\delta(\lambda) + \mathcal{L}_\xi^{-1}\left[G\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right)\right](\lambda). \tag{4.47}$$

The term $\Psi(0)\delta(\lambda)$ is in principle arbitrary, but making a choice for ψ will fix it, and in any case, it actually makes no contribution to the solution since the two terms involving Bessel functions cancel when evaluated at $\lambda = 0$.

Once more, if

$$g(y) = \sum_{n=0}^\infty \frac{1}{n!} g^{(n)}(0) y^n \tag{4.48}$$

then

$$G\left(\frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\right) = \sum_{n=0}^\infty g^{(n)}(0) \frac{(2\xi)^{n+1}}{(1-\xi^2)^{n+1}} \tag{4.49}$$

and so

$$\psi(\lambda) = \Psi(0)\delta(\lambda) + \sum_{n=0}^\infty g^{(n)}(0) k_n(\lambda) \tag{4.50}$$

where

$$k_n(\lambda) = \mathcal{L}_\xi^{-1}\left[\frac{(2\xi)^{n+1}}{(1-\xi^2)^{n+1}}\right]. \tag{4.51}$$

Since the delta function terms make no contribution, the solution can be written

$$w(y,t) = \frac{1}{2}\int_0^\infty \widehat{g}(\lambda)\left[J_0(\sqrt{y^2 + 2\lambda y e^t}) - J_0(\sqrt{y^2 + 2\lambda y e^{-t}})\right] d\lambda, \tag{4.52}$$

with $\widehat{g}(\lambda) = \sum_{n=0}^\infty g^{(n)}(0) k_n(\lambda)$.

Finding examples using the direct Laplace transform approach is not difficult.

Example 4.1. We consider the solution in the case when $f(y) = 0$, $g(y) = \frac{1}{4}y^2 {}_1F_2(\frac{1}{2}; \frac{3}{2}, 2; -\frac{y^2}{4}) - \frac{1}{2}y$, with ${}_1F_2$ a hypergeometric function. We find that

$$\int_0^\infty \frac{g(y)}{y} e^{-sy} dy = \frac{-1}{1+s+\sqrt{s^2+1}} = \Psi(\sqrt{s^2+1} - s) - \Psi(0). \tag{4.53}$$

Putting $s = \sinh k$ gives

$$\frac{-1}{1+e^k} = \Psi(e^{-k}) - \Psi(0) \tag{4.54}$$

and $\xi = e^{-k}$ leads to

$$\frac{-\xi}{1+\xi} = \frac{1}{1+\xi} - 1 = \Psi(\xi) - \Psi(0), \tag{4.55}$$

and so

$$\Psi(\xi) = \frac{1}{1+\xi} + \Psi(0) - 1. \tag{4.56}$$

The inverse Laplace transform of both sides gives

$$\psi(\lambda) = e^{-\lambda} + (\Psi(0) - 1)\delta(\lambda). \tag{4.57}$$

The natural choice here is to take $\psi(\lambda) = e^{-\lambda}$ and then $\Psi(0) - 1 = 0$. The solution our method gives for the initial value problem is then

$$u(y,t) = \frac{1}{2}\int_0^\infty e^{-\lambda} \left[J_0(\sqrt{y^2 + 2\lambda y e^t}) - J_0(\sqrt{y^2 + 2\lambda y e^{-t}}) \right] d\lambda. \tag{4.58}$$

This integral does not seem to be previously known. Although it is easy to evaluate numerically, it is worthwhile to give an evaluation.

In order to calculate (4.58) explicitly, we substitute the series expansion for the Bessel function (see 9.1.10 of [1]) inside the integral and change the order of integration and summation via the absolute and uniform convergence. Thus we obtain

$$\begin{aligned} u_1(y,t) &= \int_0^\infty e^{-\lambda} J_0(\sqrt{y^2 + 2\lambda y e^t}) d\lambda \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{4^k (k!)^2} \int_0^\infty e^{-\lambda} \left(y^2 + 2\lambda y e^t \right)^k d\lambda. \end{aligned} \tag{4.59}$$

Hence, the latter series becomes

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k(k!)^2}\sum_{m=0}^{k}m!\binom{k}{m}y^{2k-m}(2e^t)^m=\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k}}{4^k k!}\sum_{m=0}^{k}\frac{(2e^t)^m}{(k-m)!y^m}. \tag{4.60}$$

Making a straightforward substitution in the inner sum and then changing the order of summation, we get

$$\sum_{k=0}^{\infty}\frac{(-1)^k y^{2k}}{4^k k!}\sum_{m=0}^{k}\frac{(2e^t)^m}{(k-m)!y^m}=\sum_{m=0}^{\infty}\sum_{k=m}^{\infty}\left(-\frac{ye^t}{2}\right)^{k-m}\left(-\frac{y^2}{4}\right)^m\frac{1}{m!k!}$$
$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(-\frac{ye^t}{2}\right)^{k}\left(-\frac{y^2}{4}\right)^m\frac{1}{m!(k+m)!}.$$

The latter double series can be rewritten, recalling the definition of the Bessel function. Hence the value of the integral $u_1(y,t)$ is

$$u_1(y,t)=\sum_{k=0}^{\infty}(-1)^k e^{kt}J_k(y). \tag{4.61}$$

The series (4.61) has no simple expression, but can be written in terms of the Lommel function of two variables [18],

$$U_\nu(x,y)=\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{y}\right)^{2k+\nu}J_{2k+\nu}(y),\ x,y>0. \tag{4.62}$$

Thus appealing to relation (5.7.5.1) in [18], we write (4.61) in the form

$$u_1(y,t)=U_0(-e^t iy,y)+iU_1(-e^t iy,y),$$

where $i=\sqrt{-1}$ and, correspondingly, the value of the integral (4.58) as

$$u(y,t)=\frac{\left[U_0(-e^t iy,y)-U_0(-e^{-t}iy,y)\right]+\left[U_1(-e^t iy,y)-U_1(-e^{-t}iy,y)\right]}{2}. \tag{4.63}$$

The interested reader can produce many examples. For instance, the choice $f(y)=0$, $g(y)=yJ_0(y)$ leads to $\psi(\lambda)=2\cos(\lambda)$, for which $\Psi(0)=0$.

4.0.2. The case $A\neq 0$ and $B=0$

Now let us briefly consider the equation

$$u_{tt}=u_{xx}+\frac{1}{A^2e^{2x}}u,\ x\in\mathbb{R}. \tag{4.64}$$

Without loss of generality we take $A=1$. Now set $y=e^{-x}$. This again produces the equation

$$u_{tt}=y^2u_{yy}+yu_y+y^2u,\ y>0. \tag{4.65}$$

The analysis then proceeds exactly as in the previous case.

5. The case when A and B are both nonzero

Now consider the equation $u''(x) + \frac{1}{(Ae^{\omega x}+Be^{-\omega x})^2}u = 0$. Suppose that ω, $AB > 0$ and let $\alpha = \frac{\sqrt{AB(1+AB)}}{AB}$. Then with $\beta = \frac{1-\alpha}{2}$,

$$u(x) = \left(1 + \frac{A}{B}e^{2\omega x}\right)^{\beta} {}_2F_1\left(\beta, \beta; 2\beta; -\left(1 + \frac{A}{B}e^{2\omega x}\right)\right),$$

is a stationary solution of the PDE $u_{tt} = u_{xx} + \frac{1}{(Ae^{\omega x}+Be^{-\omega x})^2}u$. Here ${}_2F_1$ is Gauss' hypergeometric function (Chapter 15 of [1]). Applying the symmetry we obtain the new, non-stationary solution

$$K(x,t;\epsilon) = \left(\frac{1 + \frac{A}{B}e^{2\omega x}}{1 + 2A\epsilon\omega e^{\omega(x+t)}}\right)^{\beta} {}_2F_1\left(\beta, \beta; 2\beta; \frac{-(1 + \frac{A}{B}e^{2\omega x})}{1 + 2A\epsilon\omega e^{\omega(x+t)}}\right).$$

We assume that β is not a negative integer here. We can immediately construct an integral operator mapping functions to solutions by setting

$$u(x,t) = \frac{1}{2}\int_0^{\infty} \varphi(\epsilon)\,[K(x,t;\epsilon) + K(x,-t;\epsilon)]\,d\epsilon \tag{5.1}$$

for suitable φ. This will satisfy $u_t(y,0) = 0$ and

$$u(x,0) = \int_0^{\infty} \varphi(\epsilon)\left(\frac{1 + \frac{A}{B}e^{2\omega x}}{1 + 2A\epsilon\omega e^{\omega x}}\right)^{\beta} {}_2F_1\left(\beta, \beta; 2\beta; \frac{-(1 + \frac{A}{B}e^{2\omega x})}{1 + 2A\epsilon\omega e^{\omega x}}\right)d\epsilon. \tag{5.2}$$

If we take

$$u(x,t) = \frac{1}{2}\int_0^{\infty} \psi(\epsilon)[K(x,t;\epsilon) - K(x,-t;\epsilon)]d\epsilon, \tag{5.3}$$

we obtain a solution satisfying $u(x,0) = 0$ and

$$u_t(x,0) = \int_0^{\infty} \psi(\epsilon)\frac{\partial}{\partial t}K(x,t;\epsilon)|_{t=0}d\epsilon. \tag{5.4}$$

If we require $u_t(x,0) = g(x)$, we have an integral equation for ψ.

We will not attempt to solve these difficult integral equations here. Rather we look at the special case $\omega = 1$ and $A = B = 1/2$. Then we obtain the solution

$$u(x,t) = c_1 P_{\nu}\left(\frac{2\epsilon e^{t+x} + e^{2x} - 1}{e^{2x} + 1}\right) + c_2 Q_{\nu}\left(\frac{2\epsilon e^{t+x} + e^{2x} - 1}{e^{2x} + 1}\right),$$

for the equation

$$u_{tt} = u_{xx} + (\operatorname{sech}^2 x)u. \tag{5.5}$$

In this $\nu = \frac{1}{2}\left(\sqrt{5} - 1\right)$. Suppose we set $c_1 = 0$, $c_2 = 1$. Then we can form a solution of the PDE by setting

$$u(x,t) = \int_0^\infty \varphi(\epsilon) Q_\nu \left(\frac{2\epsilon e^{t+x} + e^{2x} - 1}{e^{2x} + 1} \right) d\epsilon. \tag{5.6}$$

Different domains of integration can also be considered. The operator defined by (5.6) maps functions φ to solutions of (5.5). We will show how this operator can be used to solve the initial value problem

$$u_{tt} = u_{xx} + (\operatorname{sech}^2 x)u \tag{5.7}$$

$$u(x,0) = f(x), \tag{5.8}$$

$$u_t(x,0) = 0. \tag{5.9}$$

We suppose that f lies in some suitable function space, which will be discussed below. Now we remark that standard Laplace transform results and formula 23(3) on page 337 of [19] gives the result

$$\int_0^\infty \frac{\sqrt{\frac{\pi}{2}} e^{-e^{-t}\xi \sinh x - \frac{t}{2}} I_{\nu+\frac{1}{2}}\left(e^{-t}\xi \cosh x\right)}{\sqrt{\xi \operatorname{sech} x}} e^{-\epsilon\xi} d\xi = Q_\nu \left(\frac{2\epsilon e^{t+x} + e^{2x} - 1}{e^{2x} + 1} \right).$$

Thus formula (5.6) can be rewritten

$$\begin{aligned} w(x,t) &= \int_0^\infty \int_0^\infty \varphi(\epsilon) \frac{\sqrt{\frac{\pi}{2}} e^{-e^{-t}\xi \sinh x - \frac{t}{2}} I_{\nu+\frac{1}{2}}\left(e^{-t}\xi \cosh x\right)}{\sqrt{\xi \operatorname{sech} x}} e^{-\epsilon\xi} d\xi d\epsilon \\ &= \int_0^\infty \Phi(\xi) \frac{\sqrt{\frac{\pi}{2}} e^{-e^{-t}\xi \sinh x - \frac{t}{2}} I_{\nu+\frac{1}{2}}\left(e^{-t}\xi \cosh x\right)}{\sqrt{\xi \operatorname{sech} x}} d\xi, \end{aligned} \tag{5.10}$$

in which Φ is the Laplace transform of φ. We obviously require suitable decay for φ in order to justify the use of Fubini's Theorem. Thus from the symmetry solution we have obtained an operator which maps functions Φ from a suitable test space to solutions of (5.5).

If we let $\xi \to i\xi$ and use linearity we obtain the solutions

$$w_1(x,t) = \frac{\cos(e^{-t}\xi \sinh x) e^{-\frac{t}{2}} J_{\nu+\frac{1}{2}}\left(e^{-t}\xi \cosh x\right)}{\sqrt{\xi \operatorname{sech} x}} \tag{5.11}$$

$$w_2(x,t) = \frac{\sin(e^{-t}\xi \sinh x) e^{-\frac{t}{2}} J_{\nu+\frac{1}{2}}\left(e^{-t}\xi \cosh x\right)}{\sqrt{\xi \operatorname{sech} x}}. \tag{5.12}$$

From these we obtain the solution of (5.5) given by

$$u(x,t)=\sqrt{\frac{\pi}{2}}\int_0^\infty \Phi(\xi)\frac{\cos(e^{-t}\xi\sinh x)e^{-\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{-t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}d\xi$$
$$+\sqrt{\frac{\pi}{2}}\int_0^\infty \Psi(\xi)\frac{\sin(e^{-t}\xi\sinh x)e^{-\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{-t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}d\xi.$$

Notice that time reversal is also a symmetry and so

$$\bar{u}(x,t)=\sqrt{\frac{\pi}{2}}\int_0^\infty \Phi(\xi)\frac{\cos(e^{t}\xi\sinh x)e^{\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}d\xi$$
$$+\sqrt{\frac{\pi}{2}}\int_0^\infty \Psi(\xi)\frac{\sin(e^{t}\xi\sinh x)e^{\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}d\xi,$$

is again a solution of the PDE. We will therefore let

$$u(x,t)=\frac{1}{2}\sqrt{\frac{\pi}{2}}\int_0^\infty \Phi(\xi)\left[\frac{\cos(e^{t}\xi\sinh x)e^{\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}\right.$$
$$\left.+\frac{\cos(e^{-t}\xi\sinh x)e^{-\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{-t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}\right]d\xi.$$

It is straightforward to see that

$$u_{tt}=u_{xx}+(\operatorname{sech}^2 x)u, \tag{5.13}$$

$$u(x,0)=\sqrt{\frac{\pi}{2}}\int_0^\infty \Phi(\xi)\frac{\cos(\xi\sinh x)J_{\nu+\frac{1}{2}}(\xi\cosh x)}{\sqrt{\xi\operatorname{sech}x}}d\xi \tag{5.14}$$

and

$$u_t(x,0)=0. \tag{5.15}$$

If we absorb the $\sqrt{\xi}$ into Φ, which is after all arbitrary, we see that if we can solve the integral equation

$$f(x)\sqrt{\operatorname{sech}x}=\sqrt{\frac{\pi}{2}}\int_0^\infty \Phi(\xi)\cos(\xi\sinh x)J_\mu(\xi\cosh x)\,d\xi, \tag{5.16}$$

for Φ, with $\mu = \nu + \frac{1}{2}$, then we have a solution of our Cauchy problem. This is not a standard integral equation and does not seem to appear anywhere in the existing literature. The solution is more involved than the previous integral equations we solved. Consequently, we will discuss its solution in the following section.

6. Solving the integral equation

Let $x \in \mathbb{R}_+$ and consider the following non-convolution integral equations of the first kind

$$\sqrt{\frac{\pi}{2}} \int_0^\infty \varphi(\xi) \cos(\xi \sinh x) J_\nu(\xi \cosh x) d\xi = f(x), \tag{6.1}$$

$$\sqrt{\frac{\pi}{2}} \int_0^\infty \varphi(\xi) \sin(\xi \sinh x) J_\nu(\xi \cosh x) d\xi = f(x), \tag{6.2}$$

where $f(x)$ is a given function and $\varphi(\xi)$ is to be determined and $J_\nu(z)$, $\nu \in \mathbb{C}$ is the Bessel function of the first kind [1]. Though only the solution of the first equation is needed for our analysis, we present the solution of both.

The key ingredients to solve integral equations (6.1) and (6.2) are the following integrals (see [13], 6.699(2) and 6.699(1), p. 723, with $\lambda = s - 1$),

$$\int_0^\infty x^{s-1} \cos(bx) J_\nu(cx) dx = \frac{2^{s-1}\Gamma\left(\frac{s+\nu}{2}\right)}{c^s \Gamma(1 + \frac{(\nu - s)}{2})} {}_2F_1\left(\frac{s+\nu}{2}, \frac{s-\nu}{2}; \frac{1}{2}; \frac{b^2}{c^2}\right), \tag{6.3}$$

where $c > b$, $-\mathrm{Re}(\nu) < \mathrm{Re}(s) < 3/2$, $\Gamma(z)$ is the Euler gamma function, see Chapter 6 of [1]. We also have

$$\int_0^\infty x^{s-1} \sin(bx) J_\nu(cx) dx = \frac{2^s b \Gamma\left(\frac{s+\nu+1}{2}\right)}{c^{s+1}\Gamma\left(\frac{\nu - s + 1}{2}\right)} {}_2F_1\left(\frac{s+\nu+1}{2}, \frac{s+1-\nu}{2}; \frac{3}{2}; \frac{b^2}{c^2}\right). \tag{6.4}$$

As we see the left-hand sides of (6.3) and (6.4) represent the Mellin transforms of the kernels in equations (6.1) and (6.2). Recall that for two functions f, g the generalized Mellin–Parseval equality holds (see [23], [25] for the details)

$$\int_0^\infty f(xy) g(y) dy = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) g^*(1-s) x^{-s} ds, \ x > 0. \tag{6.5}$$

The Fourier cosine and sine transforms of the integrable function f are defined by the formulas

$$(F_{\{\substack{c\\s}\}} f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{\begin{matrix}\cos(xy)\\ \sin(xy)\end{matrix}\right\} f(y) dy, \tag{6.6}$$

see [23].

This equation does not have a unique solution. However if we impose certain extra conditions on φ, uniqueness can be obtained. We present the general case first.

Theorem 6.1. *Let* $\mathrm{Re}(\nu) > -1/2$, $\varphi(\xi) \in S(\mathbb{R}_+)$, $e^{3x/2} f(x) \in L_1(\mathbb{R}_+)$, *and* $y^{-\nu-1/2}(F_c g)(y) \in S(\mathbb{R}_+)$, *where* $g(x) = (x^2+1)^{-\nu/2} f\left(\sinh^{-1} x\right)$. *Let also the Mellin transform* φ^* *of* φ *satisfy the condition*

$$\varphi^*(s)|s|^{-\nu} e^{(\pi/2-\delta)|s|} \in L_1(\sigma), \ \delta \in [0, \pi/2),$$

where

$$\begin{aligned} h_c(\tau) &= \varphi^*(1/2+i\tau)\Gamma(1/2-i\tau+\nu) + \varphi^*(1/2-i\tau)\Gamma(1/2+i\tau+\nu) \\ &= O(\tau^2), \end{aligned} \tag{6.7}$$

$\tau \to 0$. *Then the integral equation (6.1) has the following solutions*

$$\begin{aligned} \varphi(\xi) = &-\frac{1}{4\pi i}\int_\sigma \frac{\rho(s)}{\Gamma(1-s+\nu)}\,\xi^{-s} ds + \frac{2\xi^{-(1+\nu)}}{\pi\sqrt{\pi}} \times \\ &\int_{-\infty}^{\infty} e^{(\nu+1/2)u}\left[\left(\frac{1}{\sqrt{2}}+\sqrt{2}\,\nu\right)\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)-e^u\xi^{-1}\right)+\cos(\pi\nu/2)\right. \\ &\times\left[\frac{e^u}{\sqrt{2}\xi}\left(\cos\left(e^u\xi^{-1}\right)-\frac{1}{\sqrt{2}}\sin\left(e^u\xi^{-1}\right)\right)-\sin\left(e^u\xi^{-1}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right] \\ &\left.+\sin\left(\frac{\pi\nu}{2}\right)\sin\left(e^u\xi^{-1}\right)\left(1-\frac{e^u}{2\xi}\right)\right] \\ &\times\int_0^\infty y^{-1/2-\nu}\cos(y\sinh u)\int_0^\infty \cos(y\sinh x)(\cosh x)^{1-\nu} f(x)\,dxdydu, \end{aligned} \tag{6.8}$$

where $\xi > 0$ *and depending on an arbitrary function* $\rho(s)$, *which is odd on* σ, *i.e.*

$$\rho(s) = -\rho(1-s), s = 1/2+i\tau. \tag{6.9}$$

Proof. Since via integration by parts in the integral (4.26) it is not difficult to verify that the Mellin transform $\varphi^*(s)$ of $\varphi \in S(\mathbb{R}_+)$ belongs to $L_1(\sigma)$, we have by the inversion formula (4.27)

$$\varphi(\xi) = \frac{1}{2\pi i}\int_\sigma \varphi^*(s)\xi^{-s} ds. \tag{6.10}$$

Substituting this integral into (6.1), we can proceed if we may change the order of integration. But since (see 9.1.7 of [1]) $J_\nu(x) = O(x^\nu)$, $x \to 0$ and $\mathrm{Re}(\nu) > -1/2$, this will be possible if we show (see, for instance, in [25], Section 2.1) that there exists a positive absolute constant C such that any fixed $x > 0$ and for almost all $E_1, E_2 > 1$ and all $\tau \in \mathbb{R}$

$$\left| \int_{E_1}^{E_2} \xi^{-1/2-i\tau} \cos(\xi \sinh x) J_\nu(\xi \cosh x) d\xi \right| < C.$$

The latter inequality follows, employing the asymptotic behavior of the Bessel function at infinity (see 9.2.1 of [1])

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi\nu}{2}\right), \ x \to \infty$$

and from the uniform convergence with respect to τ of the following integrals

$$\int_{E_1}^{E_2} \xi^{-1-i\tau} \cos\left(\xi(\sinh x \pm \cosh x) \mp \frac{\pi\nu}{2}\right) d\xi. \tag{6.11}$$

This fact can be verified with the use of the mean value theorems and integration by parts, and we omit the details. Hence with the use of the Boltz formula,

$${}_2F_1(a,b,c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{1-z}\right) \tag{6.12}$$

(see 15.3.4 of [1]), in the right-hand side of (6.3), the equation (6.1) takes the form

$$\sqrt{\frac{\pi}{2}} \frac{\cosh^\nu x}{2\pi i} \int_\sigma \frac{\Gamma\left(\frac{1-s+\nu}{2}\right)}{2^s \Gamma\left(\frac{1+\nu+s}{2}\right)} {}_2F_1\left(\frac{1-s+\nu}{2}, \frac{\nu+s}{2}; \frac{1}{2}; -\sinh^2 x\right) \varphi^*(s) ds$$
$$= f(x). \tag{6.13}$$

To proceed, we make the substitution $s = 1/2 + i\tau$ in the integral (6.13) to obtain

$$\sqrt{\frac{\pi}{2}} \frac{\cosh^\nu x}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1/2-i\tau+\nu}{2}\right) {}_2F_1\left(\frac{1/2-i\tau+\nu}{2}, \frac{1/2+\nu+i\tau}{2}; \frac{1}{2}; -\sinh^2 x\right)}{2^{1/2+i\tau} \Gamma\left(\frac{3/2+\nu+i\tau}{2}\right)}$$
$$\times \varphi^*(1/2 + i\tau) d\tau = f(x). \tag{6.14}$$

Next we split the integral as $\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_0^{\infty}$ and note the symmetry ${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)$. The substitution $\tau \to -\tau$ in the first integral and the duplication formula for the gamma function,

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z-1/2} \Gamma(z) \Gamma(z + 1/2),$$

(6.1.18 of [1]) leads to the new form of (6.1)

$$\int_0^\infty h_c(\tau)\frac{{}_2F_1\left(\frac{1/2+\nu+i\tau}{2},\frac{1/2+\nu-i\tau}{2};\frac{1}{2};-\sinh^2 x\right)}{\Gamma\left(\frac{3/2+\nu+i\tau}{2}\right)\Gamma\left(\frac{3/2+\nu-i\tau}{2}\right)}d\tau = 2^{3/2+\nu}\frac{f(x)}{\cosh^\nu(x)}. \tag{6.15}$$

Moreover, the integral (6.15) converges absolutely by virtue of the Stirling asymptotic formula for the gamma function (see 6.1.37 of [1]) and the behavior at infinity in τ of the hypergeometric function (cf. [24], Theorem 1.12)

$${}_2F_1\left(\frac{1/2+\nu+i\tau}{2},\frac{1/2+\nu-i\tau}{2};\frac{1}{2};-\sinh^2 x\right) = O(1),\ \tau\to\infty.$$

In the mean time, the latter hypergeometric function has the representation (cf. [18], Vol. 2, relation (2.16.21.1), [24], formula (1.101))

$$\begin{aligned}&{}_2F_1\left(\frac{1/2+\nu+i\tau}{2},\frac{1/2+\nu-i\tau}{2};\frac{1}{2};-\sinh^2 x\right) =\\ &\times\frac{2^{3/2-\nu}}{\Gamma\left(\frac{1/2+\nu+i\tau}{2}\right)\Gamma\left(\frac{1/2+\nu-i\tau}{2}\right)}\int_0^\infty y^{\nu-1/2}\cos(y\sinh x)K_{i\tau}(y)dy,\end{aligned} \tag{6.16}$$

$\mathrm{Re}(\nu) > -1/2$, where $K_{i\tau}(x)$ is the modified Bessel function of the third kind or the Macdonald function [1], and it is the kernel of the Kontorovich–Lebedev transform [24].

Substituting the right-hand side of (6.16) into the left-hand side of (6.15), we change the order of integration by the Fubini theorem, which is justified via the absolute convergence of the iterated integrals. Hence after the use of the duplication formula for the gamma-function and the substitution $u = \sinh x$, we arrive at the following integral equation

$$\int_0^\infty\int_0^\infty \frac{h_c(\tau)\ K_{i\tau}(y)y^{\nu-1/2}\cos(uy)}{\left|\Gamma\left(\frac{1}{2}+\nu+i\tau\right)\right|^2}d\tau dy = 2\pi\frac{f\left(\sinh^{-1}\ u\right)}{(u^2+1)^{\nu/2}}, \tag{6.17}$$

since $\Gamma(z)\Gamma(\bar{z}) = \Gamma(z)\overline{\Gamma(z)} = |\Gamma(z)|^2$.

Further, by the definition of $h_c(\tau)$, the Stirling asymptotic formula for the Gamma function, the uniform inequality for the Macdonald function [24]

$$|K_{i\tau}(y)| \le e^{-\delta\tau/2}K_0(y\cos\delta),\ \delta\in\left[0,\frac{\pi}{2}\right),$$

and our assumption $\varphi^*(s)|s|^{-\nu}e^{(\pi/2-\delta)|s|}\in L_1(\sigma)$, one can apply the Fourier cosine transform to both sides of the equality (6.17). Hence, returning to the original variable, we arrive at the equation

$$\int_0^\infty \frac{h_c(\tau)\ K_{i\tau}(y)}{\left|\Gamma\left(\frac{1}{2}+\nu+i\tau\right)\right|^2}d\tau = 4\int_0^\infty\frac{\cos(y\sinh x)}{y^{-1/2+\nu}}(\cosh x)^{1-\nu}f(x)dx. \tag{6.18}$$

The left-hand side of (6.18) is related to the Kontorovich–Lebedev transform [24]. Recall that for suitable functions g, we have

$$g(x) = \frac{2}{\pi^2 x} \int_0^\infty \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}(y) g(y) dy d\tau. \tag{6.19}$$

If we denote the right hand side of (6.18) by $\phi(y)$ and introduce

$$H_c(\tau) = \frac{\pi^2 h_c(\tau)}{2\tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} + \nu + i\tau\right)\right|^2}, \tag{6.20}$$

then

$$\phi(y) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) H_c(\tau) K_{i\tau}(y) d\tau. \tag{6.21}$$

But the conditions of the theorem permit us to invert the Kontorovich–Lebedev transform, so by (6.19) we have

$$H_c(\tau) = \int_0^\infty \frac{\phi(y)}{y} K_{i\tau}(y) dy. \tag{6.22}$$

Thus we obtain

$$h_c(\tau) = \frac{8}{\pi^2} \tau \sinh(\pi\tau) \left|\Gamma\left(\frac{1}{2} + \nu + i\tau\right)\right|^2 \int_0^\infty y^{-1/2-\nu} K_{i\tau}(y) \times \int_0^\infty \frac{\cos(y \sinh x)}{(\cosh x)^{\nu-1}} f(x) dx dy. \tag{6.23}$$

In the right-hand side of (6.23), we employ the Parseval equality for the Fourier cosine transform [23], which is allowed via conditions of the theorem. Then using relation (2.16.14.1) in [18], Vol. 2 and simple substitutions, the latter equality (6.23) becomes

$$h_c(\tau) = \frac{8\sqrt{2}}{\pi\sqrt{\pi}} \tau \sinh\left(\frac{\pi\tau}{2}\right) \left|\Gamma\left(\frac{1}{2} + \nu + i\tau\right)\right|^2 \int_0^\infty \cos(\tau u) \times \int_0^\infty \frac{\cos(y \sinh u)}{y^{\nu-1/2}} \int_0^\infty \frac{\cos(y \sinh x)}{(\cosh x)^{\nu-1}} f(x)\, dx dy du. \tag{6.24}$$

Now, recalling the definition of $h_c(\tau)$ in terms of φ^*, let us write the equation (6.24) with respect to the variable $s = 1/2 + i\tau$, dividing first both of its sides by $\Gamma(1/2 + \nu + i\tau)$ and

appealing to the reflection formula for the Gamma function. After making the change of variable $t = e^u$, we have

$$\varphi^*(s)\frac{\Gamma(1-s+\nu)}{\Gamma(s+\nu)} + \varphi^*(1-s) = \frac{8\sqrt{2}}{\sqrt{\pi}}\frac{\Gamma(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}\times$$
$$\int_1^\infty \left[t^{s-\frac{3}{2}} + \frac{1}{t^{s+\frac{1}{2}}}\right]\int_0^\infty \frac{\cos(\frac{1}{2}y(t-\frac{1}{t}))}{y^{1/2+\nu}}\int_0^\infty \frac{\cos(y\sinh x)}{(\cosh x)^{\nu-1}} f(x)dxdydt, \tag{6.25}$$

where $s = 1/2 + i\tau$, $\mathrm{Re}(\nu) > -1/2$. Further, denoting the iterated integral in the right-hand side of (6.25) by

$$F(s) = \frac{8\sqrt{2}}{\sqrt{\pi}}\int_1^\infty \left[t^{s-\frac{3}{2}} + \frac{1}{t^{s+\frac{1}{2}}}\right]\int_0^\infty \frac{\cos(\frac{1}{2}y(t-\frac{1}{t}))}{y^{1/2+\nu}}\int_0^\infty \frac{\cos(y\sinh x)}{(\cosh x)^{\nu-1}} f(x)dxdydt, \tag{6.26}$$

equation (6.25) becomes

$$\varphi^*(s)\frac{\Gamma(1-s+\nu)}{\Gamma(s+\nu)} + \varphi^*(1-s) = \frac{F(s)\,\Gamma(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}. \tag{6.27}$$

Note that $F(s) = F(1-s)$.

Hence,

$$\varphi^*(1-s)\frac{\Gamma(s+\nu)}{\Gamma(1-s+\nu)} + \varphi^*(s) = \frac{F(1-s)\,\Gamma(s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}, \tag{6.28}$$

or

$$\varphi^*(1-s) + \varphi^*(s)\frac{\Gamma(1-s+\nu)}{\Gamma(s+\nu)} = \frac{F(1-s)\,\Gamma(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}. \tag{6.29}$$

Therefore, adding equations (6.27) and (6.29), we find

$$2\varphi^*(1-s) + 2\varphi^*(s)\frac{\Gamma(1-s+\nu)}{\Gamma(s+\nu)} = \frac{[F(1-s)+F(s)]\,\Gamma(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}. \tag{6.30}$$

Hence

$$\left[2\varphi^*(1-s) - \frac{F(s)\,\Gamma(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}\right]\Gamma(s+\nu) = -\Gamma(1-s+\nu)$$
$$\times\left[2\varphi^*(s) - \frac{F(1-s)\Gamma(s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}\right].$$

If we denote the right hand side of the latter algebraic equation by $\rho(s)$, we see that it satisfies the parity property (6.9). Choice of either $\varphi^*(s)$ or $\varphi^*(1-s)$ fixes the other. Hence final solutions of the equation (6.1) can be expressed in terms of the inverse Mellin transform (4.27)

$$\varphi(\xi) = -\frac{1}{4\pi i}\int\limits_{\sigma} \frac{\rho(s)\xi^{-s}}{\Gamma(1-s+\nu)}\,ds + \frac{1}{4\pi i}\int\limits_{\sigma} \frac{F(1-s)\,\Gamma\,(1-s+\nu)}{\Gamma\left(\frac{s-1/2}{2}\right)\Gamma\left(\frac{1/2-s}{2}\right)}\xi^{-s}ds, \tag{6.31}$$

where $\xi > 0$ and $\rho(s)$ is an arbitrary function, which is odd on the line σ, i.e. satisfies (6.9). The integrals converge via the relation (6.25) and properties of $\varphi^*(s)$ that we have assumed. Our final goal is to calculate the second integral in the right-hand side of (6.31). To do this, we will employ the generalized Parseval equality of type (6.5) (see [15], Chapter 7, Theorem 23), because the gamma-ratio behaves on σ as

$$\frac{\Gamma(1-s+\nu)}{\Gamma((s-1/2)/2)\Gamma((1/2-s)/2)} = O(|s|^{1+\mathrm{Re}(\nu)}),\ |\mathrm{Im}s| \to \infty,$$

and the inverse Mellin transform generally does not exist since the corresponding integral (4.27) may diverge. So, recalling (6.26), we observe that according to conditions of the theorem, the iterated integral

$$\int\limits_0^{\infty} y^{-1/2-\nu}\cos(y(t-1/t)/2)\int\limits_0^{\infty}\cos(y\sinh x)(\cosh x)^{1-\nu}f(x)\,dxdy$$

is a function of $(t-1/t)/2$ from the Schwartz space. Therefore, due to the absolute and uniform convergence of the corresponding integral (6.26), $F(1-s)$ is analytic in the right half-plane $\mathrm{Re}(s) > 1/2$. Hence, returning to (6.31) and using the duplication formula for the gamma function $\Gamma(1-s+\nu)$ in the latter integral, we change the contour σ on the right-hand infinite loop L_+, encircling the right-hand simple poles of the obtained gamma functions in the numerator. Precisely, with a simple substitution, we find

$$\begin{aligned}&\frac{1}{4\pi i}\int\limits_{\sigma} \frac{F(1-s)\,\Gamma(1-s+\nu)}{\Gamma((s-1/2)/2)\Gamma((1/2-s)/2)}\xi^{-s}ds\\ &= -\frac{2^{\nu}}{2\pi i\sqrt{\pi}}\int\limits_{L_+} \frac{F(1-2s)\,\Gamma((1+\nu)/2-s)\Gamma(1+\nu/2-s)}{\Gamma(s-1/4)\Gamma(1/4-s)}\left(4\xi^2\right)^{-s}ds.\end{aligned} \tag{6.32}$$

Meanwhile, the integral

$$-\frac{1}{2\pi i}\int\limits_{L_+} \frac{\Gamma((1+\nu)/2-s)\Gamma(1+\nu/2-s)}{\Gamma(s-1/4)\Gamma(1/4-s)}z^{-s}ds \tag{6.33}$$

can be calculated, using the Slater theorem [18], Vol. 3, and it can be expressed in terms of the hypergeometric functions ${}_1F_2$. Consequently, we obtain

$$-\frac{1}{2\pi i}\int\limits_{L_+}\frac{\Gamma((1+\nu)/2-s)\Gamma(1+\nu/2-s)}{\Gamma(s-1/4)\Gamma(1/4-s)}z^{-s}ds=\frac{\sin(\frac{\pi}{2}(\nu+\frac{3}{2}))}{\sqrt{\pi}z^{1+\nu/2}}$$
$$\times\left(\frac{3}{2}+\nu\right){}_1F_2\left(\frac{7}{4}+\frac{\nu}{2};\frac{3}{4}+\frac{\nu}{2},\frac{3}{2};-\frac{1}{z}\right)-\frac{\sin(\frac{\pi}{2}(\nu+\frac{1}{2}))}{2\sqrt{\pi}z^{(1+\nu)/2}}$$
$$\times\left(\frac{1}{2}+\nu\right){}_1F_2\left(\frac{5}{4}+\frac{\nu}{2};\frac{1}{4}+\frac{\nu}{2},\frac{1}{2};-\frac{1}{z}\right). \tag{6.34}$$

However, the right-hand side of (6.34) can be simplified and written in terms of the elementary trigonometric functions, appealing to relation (7.14.1.1) in [18], Vol. 3 and keeping in mind particular cases of the modified Bessel function of the first kind. Thus we, finally, derive

$$-\frac{1}{2\pi i}\int\limits_{L_+}\frac{\Gamma((1+\nu)/2-s)\Gamma(1+\nu/2-s)}{\Gamma(s-1/4)\Gamma(1/4-s)}z^{-s}ds=$$
$$\frac{z^{-(1+\nu)/2}}{2\sqrt{\pi}}\left[\cos\left(\frac{2}{\sqrt{z}}\right)\left[\left(\frac{1}{2}+\nu\right)\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right)+\frac{\cos(\pi\nu/2)}{\sqrt{z}}\right]\right.$$
$$-\sin\left(\frac{2}{\sqrt{z}}\right)\left[\left(\frac{3}{2}+\nu\right)\cos\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right)+\frac{1}{2}\cos\left(\frac{\pi\nu}{2}\right)\right.$$
$$\left.\left.+\frac{1}{\sqrt{z}}\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)\right)\right]\right]. \tag{6.35}$$

Hence, combining with (6.26), (6.31), (6.32), after straightforward manipulations we arrive at the final form (6.8) of solutions for the integral equation (6.1), completing the proof of Theorem 1. □

Some simplification may be made to our solution.

Corollary 6.2. *When* $\mathrm{Re}(\nu)>-1/2$, $\nu\neq 2n+1/2$, $n\in\mathbb{N}_0$, *the solutions (6.8) take the form*

$$\varphi(\xi)=-\frac{1}{4\pi i}\int\limits_{\sigma}\frac{\rho(s)}{\Gamma(1-s+\nu)}\,\xi^{-s}ds+\frac{\Gamma(\frac{1}{2}-\nu)}{\xi^{(1+\nu)}}\sin\left(\frac{\pi}{2}(\nu+\frac{1}{2})\right)\times$$
$$\frac{1}{\pi\sqrt{\pi}}\int\limits_{-\infty}^{\infty}e^{(\nu+\frac{1}{2})u}\left[\left(\frac{1}{\sqrt{2}}+\sqrt{2}\,\nu\right)\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)-\frac{e^u}{\xi}\right)+\cos\left(\frac{\pi\nu}{2}\right)\right.$$
$$\times\left[\frac{e^u}{\sqrt{2}\xi}\left(\cos\left(\frac{e^u}{\xi}\right)-\frac{1}{\sqrt{2}}\sin\left(\frac{e^u}{\xi}\right)\right)-\sin\left(\frac{e^u}{\xi}\right)\left(1+\frac{1}{\sqrt{2}}\right)\right]$$
$$\left.+\sin\left(\frac{\pi\nu}{2}\right)\sin\left(e^u\xi^{-1}\right)\left(1-\frac{e^u}{2\xi}\right)\right]\times$$
$$\int\limits_0^{\infty}\left[(\sinh u+\sinh x)^{\nu-1/2}+|\sinh u-\sinh x|^{\nu-1/2}\right]\frac{f(x)}{(\cosh x)^{\nu-1}}\,dxdu, \tag{6.36}$$

for $\xi>0$.

Proof. The proof will be completed if we compute the inner integral with respect to y in (6.8) after the corresponding interchange of the order of integration. In fact, let $|\mathrm{Re}(\nu)| < 1/2$. Then since $e^{3x/2}\ f(x) \in L_1(\mathbb{R}_+)$, the integral with respect to x in (6.8) converges absolutely and uniformly for $y \geq 0$. Hence the interchange is possible, where the relatively convergent integral by y can be calculated via relation (2.5.3.10) in [18], Vol. 1. This leads to (6.36). Moreover, the obtained integral with respect to x in the right-hand side of (6.36) converges absolutely and uniformly for ν, $\mathrm{Re}(\nu) \geq \nu_0 > -1/2$ under the condition $e^{3x/2} f(x) \in L_1(\mathbb{R}_+)$, representing an analytic function of ν in the domain $D = \{\nu \in \mathbb{C} : \mathrm{Re}(\nu) > -1/2,\ \nu \neq 2n + 1/2,\ n \in \mathbb{N}_0\}$. Hence, by analytic continuation, equality (6.36) holds for all $\nu \in D$. □

6.1. Conditions for uniqueness

We now turn to the problem of uniqueness of solutions for the integral equation (6.1). One way to obtain this is to assume the Mellin transform $\varphi^*(s)$ satisfies a particular condition on σ.

Corollary 6.3. *Suppose that f and φ satisfy the conditions of Theorem 6.1 and the Mellin transform of φ has the additional property that*

$$\varphi^*(s)\Gamma(1-s+\nu) = \varphi^*(1-s)\Gamma(s+\nu),\ s \in \sigma \tag{6.37}$$

i.e. this product is even on σ. Then the integral equation (6.1) has a unique solution

$$\varphi(\xi) = \frac{1}{2\pi i}\int_\sigma \frac{F(s)\Gamma(s+\nu)\Gamma(1-s+\nu)}{2\Gamma\left(\frac{1/2-s}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)}\xi^{-s}ds, \tag{6.38}$$

where F is given by (6.26).

Remark 6.4. Equality (6.37) can be rewritten, for instance, in terms of the modified Laplace transforms of the function φ. Indeed, recalling the Mellin–Parseval identity (6.5), equality (6.37) can be written in the following equivalent form

$$x^{-1-\nu}\int_0^\infty \exp(-t/x)\varphi(t)t^\nu dt = x^\nu \int_0^\infty \exp(-xt)\varphi(t)t^\nu dt,\ x > 0.$$

Proof. Using the condition on the Mellin transform φ^*, we find from equation (6.30) that

$$\begin{aligned}\varphi^*(s) &= \frac{[F(1-s)+F(s)]\Gamma(s+\nu)\Gamma(1-s+\nu)}{4\Gamma\left(\frac{1/2-s}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)}\\ &= \frac{F(s)\Gamma(s+\nu)\Gamma(1-s+\nu)}{2\Gamma\left(\frac{1/2-s}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)},\end{aligned} \tag{6.39}$$

since $F(s) = F(1-s)$ on σ. Inverting the Mellin transform we end up with the unique solution in the form,

$$\varphi(\xi) = \frac{1}{2\pi i} \int\limits_{\sigma} \frac{F(s)\Gamma(s+\nu)\Gamma(1-s+\nu)}{2\Gamma\left(\frac{1/2-s}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)} \xi^{-s} ds. \quad \square \tag{6.40}$$

Next we determine when $\psi^*(s) = \varphi^*(s)\Gamma(1-s+\nu) = \psi^*(1-s)$ on σ.

Lemma 6.5. *Let ψ be the inverse Mellin transform of*

$$\psi^*(s) = \varphi^*(s)\Gamma(1-s+\nu) = \psi^*(1-s).$$

If $g(y) = \psi(e^y)e^{1/2y}$ is even, then $\psi^*(s) = \psi^*(1-s)$ for $s \in \sigma$.

Proof. If $s \in \sigma$, then $s = \frac{1}{2} + i\tau$. Clearly we require

$$\int\limits_0^\infty x^{-1/2+i\tau}\psi(x)dx = \int\limits_0^\infty x^{-1/2-i\tau}\psi(x)dx, \tag{6.41}$$

for every $\tau \in \mathbb{R}$. We convert this to a Fourier transform by setting $x = e^y$. This gives

$$\int\limits_{-\infty}^\infty e^{i\tau y}\psi(e^y)e^{1/2y}dy = \int\limits_{-\infty}^\infty e^{-i\tau y}\psi(e^y)e^{1/2y}dy. \tag{6.42}$$

So the Fourier transform of $g(y) = \psi(e^y)e^{1/2y}$ is even. This happens precisely when g is even. $\square$

We are then able to write down a solution of our Cauchy problem.

Theorem 6.6. *Let* $\mathrm{Re}(\nu) > -1/2$, $\varphi(\xi) \in S(\mathbb{R}_+)$, $e^{3x/2} f(x) \in L_1(\mathbb{R}_+)$, *and* $y^{-\nu-1/2}(F_c g)(y) \in \mathcal{S}(\mathbb{R}_+)$, *where* $g(x) = (x^2+1)^{-\nu/2} f\left(\sinh^{-1} x\right)$. *Let also the Mellin transform φ^* of φ satisfy the conditions*

$$\varphi^*(s)|s|^{-\nu}e^{(\pi/2-\delta)|s|} \in L_1(\sigma),\ \delta \in [0, \pi/2),$$

and

$$\varphi^*(s)\Gamma(1-s+\nu) = \varphi^*(1-s)\Gamma(s+\nu),\ s \in \sigma \tag{6.43}$$

where

$$\begin{aligned} h_c(\tau) &= \varphi^*(1/2+i\tau)\Gamma(1/2-i\tau+\nu) + \varphi^*(1/2-i\tau)\Gamma(1/2+i\tau+\nu) \\ &= O(\tau^2), \end{aligned} \tag{6.44}$$

$\tau \to 0$. *Define*

$$F(s)=\int_1^\infty\left[\zeta^{s-\frac{3}{2}}+\frac{1}{\zeta^{s+\frac{1}{2}}}\right]\int_0^\infty\frac{\cos(\frac{1}{2}\eta(\zeta-\frac{1}{\zeta}))}{\eta^{1/2+\nu}}\int_0^\infty\frac{\cos(\eta\sinh\xi)}{(\cosh\xi)^{\nu-1/2}}f(\xi)d\xi d\eta d\zeta. \quad (6.45)$$

Then the problem

$$u_{tt}=u_{xx}+(\operatorname{sech}^2 x)u,$$
$$u(x,0)=f(x),\ u_t(x,0)=0,$$

has a solution

$$u(x,t)=\frac{1}{\pi i}\int_0^\infty\int_\sigma K(t,x,\xi)\frac{F(s)\Gamma(s+\nu)\Gamma(1-s+\nu)}{\Gamma\left(\frac{1/2-s}{2}\right)\Gamma\left(\frac{s-1/2}{2}\right)}\xi^{-s}dsd\xi, \quad (6.46)$$

where

$$K(t,x,\xi)=\frac{\cos(e^t\xi\sinh x)e^{\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^t\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}+\frac{\cos(e^{-t}\xi\sinh x)e^{-\frac{t}{2}}J_{\nu+\frac{1}{2}}\left(e^{-t}\xi\cosh x\right)}{\sqrt{\xi\operatorname{sech}x}}. \quad (6.47)$$

Proof. We only need to establish convergence of the final integral and this follows from the conditions of the theorem. Specifically, since the Fourier cosine transform maps Schwartz functions to Schwartz functions, and the Mellin transform in s of a Schwartz function, with $\operatorname{Im}(s)=t$, is Schwartz in t, the function F has rapid decay on the line σ and hence its inverse Mellin transform has rapid decay. This guarantees the convergence of the final integral. □

Simplification of this result may be possible, but we will not discuss this question here.

6.2. The second integral equation

We finish by turning now to the solution of the integral equation (6.2). This can be established in the same manner as was used for the proof of Theorem 6.1 with the use of the integral (6.4) and is given by our next result.

Theorem 6.7. *Let* $\operatorname{Re}(\nu)>-\frac{1}{2}$, $\varphi(\xi)\in\mathcal{S}(\mathbb{R}_+)$, $e^{3x/2}\ f(x)\in L_1(\mathbb{R}_+)$, *and* $y^{-\nu-1/2}(F_s g)(y)\in\mathcal{S}(\mathbb{R}_+)$, *where* $g(x)=(x^2+1)^{-\nu/2}\ f\left(\sinh^{-1}(x)\right)$. *Let also the Mellin transform* φ^* *of* φ *satisfy the condition*

$$\varphi^*(s)|s|^{-\nu}e^{(\pi/2-\delta)|s|}\in L_1(\sigma),\ \delta\in[0,\pi/2), \quad (6.48)$$

where

$$h_s(\tau)=\varphi^*(1/2+i\tau)\Gamma(3/2-i\tau+\nu)+\varphi^*(1/2-i\tau)\Gamma(3/2+i\tau+\nu)$$
$$=O(\tau^2),\ \tau\to 0. \quad (6.49)$$

Then the integral equation (6.2) has the following solutions

$$\varphi(\xi) = -\frac{1}{4\pi i}\int\limits_{\sigma}\frac{\rho(s)}{\Gamma(2-s+\nu)}\,\xi^{-s}ds + \frac{\sqrt{2}\,\xi^{-(2+\nu)}}{\pi\sqrt{\pi}}$$
$$\times\int\limits_{-\infty}^{\infty} e^{(\nu+3/2)u}\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)-\frac{e^u}{\xi}\right)\cosh u$$
$$\times\int\limits_{0}^{\infty} y^{1/2-\nu}\cos(y\sinh u)\int\limits_{0}^{\infty}\sin(y\sinh x)(\cosh x)^{1-\nu}f(x)\,dxdydu, \tag{6.50}$$

for $\xi > 0$, depending on an arbitrary function $\rho(s)$, which satisfies relation (6.9) on σ.

Proof. In fact, an analog of equation (6.13) will be the equality

$$\sqrt{\frac{\pi}{2}}\frac{\sinh x\,\cosh^{\nu}x}{2\pi i}\int\limits_{\sigma}\varphi^*(s)\frac{2\Gamma\left(\frac{2-s+\nu}{2}\right)}{2^s\Gamma\left(\frac{2+\nu+s}{2}\right)}\times$$
$${}_2F_1\left(\frac{2-s+\nu}{2},\frac{1+\nu+s}{2};\frac{3}{2};-\sinh^2 x\right)ds = f(x). \tag{6.51}$$

Hence, by a similar calculation to that leading to (6.15), we get

$$\int\limits_0^{\infty} h_s(\tau)\frac{{}_2F_1\left(\frac{3/2+\nu+i\tau}{2},\frac{3/2+\nu-i\tau}{2};\frac{3}{2};-\sinh^2 x\right)}{\left[\Gamma\left(\frac{5/2+\nu+i\tau}{2}\right)\Gamma\left(\frac{5/2+\nu-i\tau}{2}\right)\right]}\,d\tau = 2^{3/2+\nu}\frac{\cosh^{-\nu}x}{\sinh x}\,f(x) \tag{6.52}$$

and since

$$\sinh x\;{}_2F_1\left(\frac{3/2+\nu+i\tau}{2},\frac{3/2+\nu-i\tau}{2};\frac{3}{2};-\sinh^2 x\right) =$$
$$\frac{2^{1/2-\nu}}{\Gamma((3/2+\nu+i\tau)/2)\Gamma((3/2+\nu-i\tau)/2)}\int\limits_0^{\infty} y^{\nu-1/2}\sin(y\sinh x)K_{i\tau}(y)dy, \tag{6.53}$$

$\mathrm{Re}(\nu) > -1/2$, we derive an analog of the equation (6.17)

$$\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{y^{\nu-1/2}h_s(\tau)\sin(uy)\;K_{i\tau}(y)}{\Gamma(\frac{3}{2}+\nu+i\tau)\Gamma(\frac{3}{2}+\nu-i\tau)}d\tau dy = \pi(u^2+1)^{-\nu/2}\;f\left(\sinh^{-1}(u)\right). \tag{6.54}$$

Further, according to conditions of the theorem we may take the Fourier sine transform and then the Kontorovich–Lebedev transform, to the function $h_s(\tau)$ in the form

$$h_s(\tau) = \frac{4}{\pi^2}\tau \sinh(\pi\tau)\left|\Gamma\left(\frac{3}{2}+\nu+i\tau\right)\right|^2 \int_0^\infty y^{-1/2-\nu} K_{i\tau}(y)$$

$$\times \int_0^\infty \sin(y\sinh x)(\cosh x)^{1-\nu} f(x)\, dxdy. \tag{6.55}$$

Moreover, after the use of the Parseval equality for the Fourier sine transform [23], relation (2.16.14.1) in [18], Vol. 2, integration by parts and differentiation under the integral sign in the integral with respect to u, which is permitted under conditions of the theorem, we obtain

$$h_s(\tau) = \frac{4\sqrt{2}}{\pi\sqrt{\pi}}\cosh\left(\frac{\pi\tau}{2}\right)\left|\Gamma\left(\frac{3}{2}+\nu+i\tau\right)\right|^2 \int_0^\infty\int_0^\infty\int_0^\infty \cos(\tau u)\cosh u$$

$$\times\ y^{1/2-\nu}\cos(y\sinh u)\sin(y\sinh x)(\cosh x)^{1-\nu} f(x)\, dxdydu. \tag{6.56}$$

Then similarly to equations (6.25), (6.26) and (6.27) we derive from (6.56)

$$\varphi^*(s)\frac{\Gamma(2-s+\nu)}{\Gamma(1+s+\nu)} + \varphi^*(1-s) = \frac{G(s)\,\Gamma(2-s+\nu)}{\Gamma((s+1/2)/2)\Gamma((3/2-s)/2)},\ s\in\sigma, \tag{6.57}$$

where

$$G(s) = \frac{2\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^\infty\int_0^\infty\int_0^\infty e^{u(1/2-s)}\cosh u\ y^{1/2-\nu}\cos(y\sinh u)$$

$$\times \sin(y\sinh x)(\cosh x)^{1-\nu} f(x)\, dxdydu. \tag{6.58}$$

Hence as in the proof of Theorem 6.1, the final solutions of the equation (6.2) can be expressed in terms of the inverse Mellin transform (4.27)

$$\varphi(\xi) = -\frac{1}{4\pi i}\int_\sigma \frac{\rho(s)\xi^{-s}}{\Gamma(2-s+\nu)}\, ds + \frac{1}{4\pi i}\int_\sigma \frac{G(1-s)\,\Gamma(2-s+\nu)\xi^{-s}}{\Gamma\left(\frac{s+\frac{1}{2}}{2}\right)\Gamma\left(\frac{\frac{3}{2}-s}{2}\right)} ds. \tag{6.59}$$

Then as above,

$$\frac{1}{4\pi i}\int_\sigma \frac{G(1-s)\,\Gamma(2-s+\nu)}{\Gamma((s+1/2)/2)\Gamma((3/2-s)/2)}\xi^{-s} ds =$$

$$-\frac{2^{\nu+1}}{2\pi i\sqrt{\pi}}\int_{L_+} \frac{G(1-2s)\,\Gamma(1+\nu/2-s)\Gamma(3/2+\nu/2-s)}{\Gamma(s+1/4)\Gamma(3/4-s)}\left(4\xi^2\right)^{-s} ds. \tag{6.60}$$

Meanwhile, the integral

$$-\frac{1}{2\pi i}\int\limits_{L_+}\frac{\Gamma(1+\nu/2-s)\Gamma(3/2+\nu/2-s)}{\Gamma(s+1/4)\Gamma(3/4-s)}z^{-s}ds$$

can be calculated in terms of the elementary functions. Precisely, we derive

$$-\frac{1}{2\pi i}\int\limits_{L_+}\frac{\Gamma(1+\frac{\nu}{2}-s)\Gamma(\frac{3}{2}+\frac{\nu}{2}-s)}{\Gamma(s+\frac{1}{4})\Gamma(\frac{3}{4}-s)}z^{-s}ds=\frac{\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)-\frac{2}{\sqrt{z}}\right)}{\sqrt{\pi}z^{1+\frac{\nu}{2}}}. \tag{6.61}$$

Hence, combining with (6.58), (6.59) and (6.60), we arrive at the final form (6.50) of solutions for the integral equation (6.2), completing the proof of Theorem 6.7. □

The corresponding corollary for the values $\nu\in\mathbb{C}:\ \mathrm{Re}(\nu)>-1/2,\ \nu\neq 2n+7/2,\ n\in\mathbb{N}_0$ can be formulated as follows.

Corollary 6.8. *When* $\mathrm{Re}(\nu)>-\frac{1}{2},\ \nu\neq 2n+7/2,\ n\in\mathbb{N}_0$, *the solutions (6.50) take the form*

$$\begin{aligned}\varphi(\xi)=&-\frac{1}{4\pi i}\int\limits_{\sigma}\frac{\rho(s)}{\Gamma(2-s+\nu)}\,\xi^{-s}ds+\xi^{-(2+\nu)}\Gamma(3/2-\nu)\sin(\pi(3/2-\nu)/2)\\&\times\frac{1}{\pi\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}e^{(\nu+3/2)u}\sin\left(\frac{\pi}{2}\left(\nu+\frac{1}{2}\right)-e^{u}\xi^{-1}\right)\cosh u\\&\times\left[\int\limits_{0}^{\infty}(\sinh u+\sinh x)^{\nu-3/2}(\cosh x)^{1-\nu}f(x)\,dxdu\right.\\&-\int\limits_{0}^{u}(\sinh u-\sinh x)^{\nu-3/2}(\cosh x)^{1-\nu}f(x)\,dxdu\\&\left.+\int\limits_{u}^{\infty}(\sinh x-\sinh u)^{\nu-3/2}(\cosh x)^{1-\nu}f(x)\,dxdu\right],\ \xi>0.\end{aligned}$$

We conclude with a condition that will guarantee unique solutions for the second integral equation (6.2). The proof is similar to that for Corollary 6.3 and we omit it.

Corollary 6.9. *Let* f *and* φ *satisfy the conditions of Theorem 6.7 and suppose further that*

$$\varphi^*(s)\Gamma(2-s+\nu)=\varphi^*(1-s)\Gamma(1+s+\nu),\ s\in\sigma.$$

Then equation (6.2) has a unique solution given by

$$\varphi(\xi) = \frac{1}{4\pi i} \int_{\sigma} \frac{G(1-s)\Gamma(1+s+\nu)\Gamma(2-s+\nu)}{\Gamma\left(\frac{3/2-s}{2}\right)\Gamma\left(\frac{s+1/2}{2}\right)} \xi^{-s} ds, \tag{6.62}$$

where G is given by (6.58).

7. Conclusion

The methods introduced in [7] and extended in the current work lead to many interesting problems involving integral transforms and integral equations. There are still very many open questions, even in terms of the focus of this paper. For example, one would like to be able to solve the general integral equation (5.2). When applied to other PDEs, the method also generates integral equations that do not seem to have been studied in the literature and many interesting questions arise from the investigation of these problems. We hope that this work stimulates further research in this area.

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