ARTICLE IN PRESS YJDEQ:9046

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model

Chunyin Jin

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

Received 30 April 2017; revised 23 August 2017

Abstract

In this paper, we develop a unified framework that can be used to establish the well-posedness of kinetic Cucker–Smale model with or without noise, for general initial data regardless of the supports; meanwhile we rigorously justify the vanishing noise limit. Our proof is based on weighted energy estimates and the velocity averaging lemma in kinetic theory.

MSC: 35B35; 35D35; 35Q83; 35Q92

Keywords: Well-posedness; Strong solution; Kinetic Cucker–Smale model; Velocity averaging lemma

1. Introduction

In this paper, we study the well-posedness of weak and strong solutions to the following kinetic Cucker–Smale model with or without noise:

$$\begin{cases} f_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}} \cdot (L[f]f) = \sigma \Delta_{\boldsymbol{v}} f, \\ f|_{t=0} = f_0(\boldsymbol{x}, \boldsymbol{v}), \end{cases} \tag{1.1}$$

where $L[f]$ is given by

E-mail address: jinchunyin@pku.edu.cn.

https://doi.org/10.1016/j.jde.2017.10.001

$$L[f](t,\boldsymbol{x},\boldsymbol{v}) = \int_{\mathbb{R}^{2d}} \varphi(|\boldsymbol{x}-\boldsymbol{y}|) f(t,\boldsymbol{y},\boldsymbol{v}^*)(\boldsymbol{v}^*-\boldsymbol{v})d\boldsymbol{y}d\boldsymbol{v}^*.$$

Here $f(t,\boldsymbol{x},\boldsymbol{v})$ is the density distribution function in $\mathbb{R}^+ \times \mathbb{R}^{2d}$ $(d \geq 1)$. $\varphi(\cdot)$ is a positive non-increasing function denoting the interaction kernel, and σ represents the noise strength. For convenience, we suppose $\varphi \in C^\infty$. If not, we mollify it by convolution. In fact, we only need $\varphi \in C^2$. Without loss of generality, we postulate that

$$\max\{|\varphi|, |\varphi'|, |\varphi''|\} \leq 1 \quad \text{and} \quad 0 \leq \sigma \leq 1$$

in the sequel.

Recently, the Cucker–Smale model and related models have attracted much attention from researchers in diverse fields, including biology, physics and mathematics. People wish to understand the mechanisms that lead to the collective behaviors, such as flocking of birds, schooling of fish and swarming of bacteria, by modeling, simulation and mathematical analysis. Among them, the Cucker–Smale model is a basic one used to describe flocking, which was put forward in 2007. Motivated by their pioneer work [5], Ha–Liu [11] presented a complete analysis for flocking of the Cucker–Smale model by using the Lyapunov functional approach. Then they rigorously justified the mean-field limit from the particle model to the kinetic Cucker–Smale model. Later, Carrillo et al. [2] give an elegant proof for the mean-field limit by employing the modern theory of optimal transport. In [3], they further refined the results in [11] and proved the unconditional flocking theorem for the measure-valued solutions to the kinetic Cucker–Smale model. Nowadays, studies of the Cucker–Smale model from particle to kinetic and hydrodynamic description have been launched. We refer the interested readers to [8–10,14] and the references therein for the results related to hydrodynamic Cucker–Smale models and the review paper [4] for the state of the art in this research topic.

However, most mathematical models in this territory are just derived formally. The rigorous limits and stabilities of many models are still unknown. Even though the stability for the kinetic Cucker–Smale model has been established in measure space in [2] and [11], however the general results in regular function space are still lacking. As far as we know, there is only existence theory for weak solutions in function space; see [15]. The proofs of [2] and [11] are both based on the analysis to the characteristics, under the condition that the initial data have compact support. In fact, this method can only deal with the kinetic Cucker–Smale model without noise. Now in the present paper, we will provide a unified framework that can be used to establish the well-posedness of solutions to the kinetic Cucker–Smale model with or without noise, no matter whether the initial data have compact support or not; see Section 3 and 4.

It is well-known that we can construct the admissible weak solutions to the hyperbolic conservation laws by using the vanishing viscosity approach; see [1,13]. Similarly, can we recover weak and strong solutions to the kinetic Cucker–Smale model by the vanishing noise limit? This is another problem we are concerned with. By using the velocity averaging lemma and subtle mathematical analysis, we give a positive answer to this question. The reader can also refer to [6,7] for further application of the velocity averaging lemma in kinetic theory.

The rest of the paper is organized as follows. In Section 2, we prove the well-posedness of weak and strong solutions to the kinetic Cucker–Smale model in the Sobolev space. In Section 3, we mainly study the kinetic Cucker–Smale model with noise by introducing two weighted Hilbert spaces. Section 4 is devoted to the study of vanishing noise limit. In the last section, we summarize our paper and make a brief comment on it.

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

Notation. Throughout the paper, C represents a general positive constant that may depend on the initial data. We write C_* to emphasize that C depends on $*$. $c(t)$ denotes a general positive function continuous in $[0, \infty)$. C, C_* and $c(t)$ may take different values in different expressions.

2. Kinetic Cucker–Smale model without noise

In this section, we prove the well-posedness of weak and strong solutions to the kinetic Cucker–Smale without noise, i.e., $\sigma = 0$ in (1.1). Then the equation (1.1) reduces to

$$\begin{cases} f_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}} \cdot (L[f]f) = 0, \\ f|_{t=0} = f_0(\boldsymbol{x}, \boldsymbol{v}). \end{cases} \tag{2.1}$$

Most previous results about this equation were obtained under the condition that the initial data $f_0(\boldsymbol{x}, \boldsymbol{v})$ have compact support with respect to $\boldsymbol{x}$ and $\boldsymbol{v}$; see [2](Theorem 3.10), [12](Theorem 3.3) and [11](Theorem 6.2). In Section 2, we manage to establish our theorems under minimum restrictions on initial data, by using the traditional characteristics method. However, the initial data are still required to be compactly supported in $\boldsymbol{v}$-variable. In order to deal with the general non-compactly supported initial data, we are forced to develop a framework by adding a noise term $\sigma \Delta_{\boldsymbol{v}} f$ to the right-hand side of (2.1). To our delight, it turns out that this framework is a unified one in that it applies to both the kinetic Cucker–Smale model with and without noise, regardless of supports of the initial data; see Section 3 and 4. Next we present the definition and results in this section.

Definition 2.1. Let $0 \le f(t, \boldsymbol{x}, \boldsymbol{v}) \in C([0, +\infty), L^1(\mathbb{R}^{2d}))$. $f(t, \boldsymbol{x}, \boldsymbol{v})$ is a weak solution to (2.1) if

$$f_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}} \cdot (L[f]f) = 0 \quad \text{in } \mathcal{D}'([0, +\infty) \times \mathbb{R}^{2d}).$$

We say $f(t, \boldsymbol{x}, \boldsymbol{v})$ is a strong solution if $f(t, \boldsymbol{x}, \boldsymbol{v})$ is a weak solution and $f(t, \boldsymbol{x}, \boldsymbol{v}) \in C([0, +\infty), W^{1,1}(\mathbb{R}^{2d}))$.

Denote

$$R(t) = \sup\{|\boldsymbol{v}| : (\boldsymbol{x}, \boldsymbol{v}) \in \operatorname{supp} f(t, \cdot, \cdot)\}.$$

We have the following theorems.

Theorem 2.1. *Assume $R_0 > 0$ and $0 \le f_0(\boldsymbol{x}, \boldsymbol{v}) \in W^{1,1}(\mathbb{R}^{2d})$, $supp_{\boldsymbol{v}} f_0(\boldsymbol{x}, \cdot) \subseteq B(R_0)$. Then (2.1) admits a unique weak solution $f(t, \boldsymbol{x}, \boldsymbol{v}) \in C([0, +\infty), L^1(\mathbb{R}^{2d}))$ in the sense of Definition 2.1, with the bound of its $\boldsymbol{v}$-support $R(t)$ satisfying*

$$R(t) \le R_0 + \|f_0\|_{L^1(\mathbb{R}^{2d})} R_0 t.$$

Moreover, there exists $c(t) \le C(1 + t)e^{C(t+t^2)}$ such that

$$\sup_{0\le t\le T}\|f(t)-g(t)\|_{L^1(\mathbb{R}^{2d})}\le\|f_0-g_0\|_{L^1(\mathbb{R}^{2d})}\exp\left(\int_0^T c(t)dt\right),\quad \forall T\ge 0,$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are weak solutions with initial data f_0 and g_0 satisfying the above conditions, respectively.

Theorem 2.2. *Assume $R_0>0$ and $0\le f_0(\boldsymbol{x},\boldsymbol{v})\in W^{2,1}(\mathbb{R}^{2d})$, $supp_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot)\subseteq B(R_0)$. Then (2.1) admits a unique strong solution $f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,+\infty),W^{1,1}(\mathbb{R}^{2d}))$ in the sense of Definition 2.1, with the bound of its $\boldsymbol{v}$-support $R(t)$ satisfying*

$$R(t)\le R_0+\|f_0\|_{L^1(\mathbb{R}^{2d})}R_0 t.$$

Moreover, there exists $c(t)\le C(1+t)e^{C(t+t^2)}$ such that

$$\sup_{0\le t\le T}\|f(t)-g(t)\|_{W^{1,1}(\mathbb{R}^{2d})}\le\|f_0-g_0\|_{W^{1,1}(\mathbb{R}^{2d})}\exp\left(\int_0^T c(t)dt\right),\quad \forall T\ge 0,$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are strong solutions with initial data f_0 and g_0 satisfying the above conditions, respectively.

Remark 2.1. Compared with the initial data, the solutions in Theorem 2.1 and 2.2 lose one order regularity, this is due to the fact that the L^1 type Sobolev space is not reflexive and the bounded sequence cannot guarantee a weakly convergent subsequence.

Remark 2.2. We can refine the estimate of $R(t)$ by using the particle method as in [3]. In fact, it is uniformly bounded in time. If we further require $f_0(\boldsymbol{x},\boldsymbol{v})\in C^1(\mathbb{R}^{2d})$, then the weak solution in Theorem 2.1 becomes the classical one. Thus we improve the conditions of Theorem 3.3 in [12].

Remark 2.3. We can also establish the well-posedness of classical solutions by using the same method and the Sobolev embedding if we improve the regularity of initial data.

In the following subsection, we derive some a priori estimates that are needed in our proof.

2.1. A priori estimates

Lemma 2.1. *Assume $R_0>0$ and $f_0(\boldsymbol{x},\boldsymbol{v})\ge 0$, $supp_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot)\subseteq B(R_0)$. If $f(t,\boldsymbol{x},\boldsymbol{v})$ is a smooth solution to (2.1), then*

$$(1)\ \|f(t)\|_{L^1(\mathbb{R}^{2d})}=\|f_0\|_{L^1(\mathbb{R}^{2d})},\quad \forall t\ge 0;$$

$$(2)\ \int_{\mathbb{R}^{2d}} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v}\le\int_{\mathbb{R}^{2d}} f_0(\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v},\quad \forall t\ge 0;$$

$$(3)\ R(t)\le R_0+\|f_0\|_{L^1(\mathbb{R}^{2d})}R_0 t,\quad \forall t\ge 0.$$

Proof. (1) Direct integrating (2.1)-1 over $[0,t]\times\mathbb{R}^{2d}$ gives the conclusion.

(2) Multiplying (2.1)-1 by $\boldsymbol{v}^2$, we obtain

$$\frac{\partial}{\partial t}(f\boldsymbol{v}^2)+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(f\boldsymbol{v}^2)+\nabla_{\boldsymbol{v}}\cdot(L[f](f\boldsymbol{v}^2))=2fL[f]\cdot\boldsymbol{v}. \tag{2.2}$$

Integrating (2.2) over $[0,t]\times\mathbb{R}^{2d}$ yields

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v}+\int_0^t\int_{\mathbb{R}^{4d}}\varphi(|\boldsymbol{x}-\boldsymbol{y}|)f(s,\boldsymbol{y},\boldsymbol{v}^*)f(s,\boldsymbol{x},\boldsymbol{v})(\boldsymbol{v}^*-\boldsymbol{v})^2 d\boldsymbol{y}d\boldsymbol{v}^*d\boldsymbol{x}d\boldsymbol{v}ds\\ =\int_{\mathbb{R}^{2d}} f_0(\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v}.\end{aligned} \tag{2.3}$$

Then we prove $f\geq 0$ by the method of characteristics. Define $(X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))$ as the characteristic issuing from $(\boldsymbol{x}_0,\boldsymbol{v}_0)$. It satisfies

$$\begin{cases}\dfrac{dX}{dt}=V,\\ \dfrac{dV}{dt}=\displaystyle\int_{\mathbb{R}^{2d}}\varphi(|X-\boldsymbol{y}|)f(t,\boldsymbol{y},\boldsymbol{v}^*)(\boldsymbol{v}^*-V)d\boldsymbol{y}d\boldsymbol{v}^*.\end{cases} \tag{2.4}$$

Define

$$a(t,\boldsymbol{x}):=\int_{\mathbb{R}^{2d}}\varphi(|\boldsymbol{x}-\boldsymbol{y}|)f(t,\boldsymbol{y},\boldsymbol{v}^*)d\boldsymbol{y}d\boldsymbol{v}^*,$$

$$\mathbf{b}(t,\boldsymbol{x}):=\int_{\mathbb{R}^{2d}}\varphi(|\boldsymbol{x}-\boldsymbol{y}|)f(t,\boldsymbol{y},\boldsymbol{v}^*)\boldsymbol{v}^*d\boldsymbol{y}d\boldsymbol{v}^*.$$

Solving the equation (2.1) gives

$$f(t,X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))=f_0(\boldsymbol{x}_0,\boldsymbol{v}_0)\exp\left(d\int_0^t a(\tau,X(\tau))d\tau\right)\geq 0.$$

This together with (2.3) leads to

$$\int_{\mathbb{R}^{2d}} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v}\leq\int_{\mathbb{R}^{2d}} f_0(\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}^2 d\boldsymbol{x}d\boldsymbol{v},\quad \forall t\geq 0.$$

(3) It follows from the characteristic equation (2.4) that

$$V(t)=V_0e^{-\int_0^t a(\tau,X(\tau))d\tau}+e^{-\int_0^t a(\tau,X(\tau))d\tau}\int_0^t\mathbf{b}(\tau,X(\tau))e^{\int_0^\tau a(s,X(s))ds}d\tau. \tag{2.5}$$

Using Cauchy's inequality, we have

$$|\mathbf{b}(t,\boldsymbol{x})| \le \left(\int_{\mathbb{R}^{2d}} f(t,\boldsymbol{y},\boldsymbol{v}^*)d\boldsymbol{y}d\boldsymbol{v}^*\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2d}} f(t,\boldsymbol{y},\boldsymbol{v}^*)|\boldsymbol{v}^*|^2 d\boldsymbol{y}d\boldsymbol{v}^*\right)^{\frac{1}{2}} \le \|f_0\|_{L^1(\mathbb{R}^{2d})}R_0. \tag{2.6}$$

From (2.5), we deduce that

$$R(t) \le R_0 + \|f_0\|_{L^1(\mathbb{R}^{2d})}R_0 t, \quad \forall t \ge 0.$$

This completes the proof. □

Lemma 2.2. *Assume $R_0 > 0$ and $0 \le f_0(\boldsymbol{x},\boldsymbol{v}) \in W^{1,1}(\mathbb{R}^{2d})$, $supp_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot) \subseteq B(R_0)$. If $f(t,\boldsymbol{x},\boldsymbol{v})$ is a smooth solution to (2.1), then there exists $c(t) \le C(1+t)e^{C(t+t^2)}$ such that*

(1) $\|f(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le \|f_0\|_{W^{1,1}(\mathbb{R}^{2d})}e^{C(t+t^2)}, \quad \forall t \ge 0;$

(2) $\sup\limits_{0\le t\le T} \|f(t)-g(t)\|_{L^1(\mathbb{R}^{2d})} \le \|f_0-g_0\|_{L^1(\mathbb{R}^{2d})} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are smooth solutions with initial data f_0 and g_0 satisfying the above conditions, respectively.

Proof. (1) Applying $\nabla_{\boldsymbol{x}}$ to (2.1)-1, we have

$$(\nabla_{\boldsymbol{x}} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{x}} f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\nabla_{\boldsymbol{x}} f) = -\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f - f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]. \tag{2.7}$$

Multiplying (2.7) by $(sgn(\partial_{x_1} f), sgn(\partial_{x_2} f), \cdots, sgn(\partial_{x_d} f))$, and integrating the resulting equation over $\mathbb{R}^{2d}$, we obtain

$$\frac{d}{dt}\|\nabla_{\boldsymbol{x}} f\|_{L^1(\mathbb{R}^{2d})} \le 2R(t)\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} + d\|f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})}. \tag{2.8}$$

Applying $\nabla_{\boldsymbol{v}}$ to (2.1)-1 gives

$$(\nabla_{\boldsymbol{v}} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{v}} f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\nabla_{\boldsymbol{v}} f) = -\nabla_{\boldsymbol{v}} L[f]\cdot\nabla_{\boldsymbol{v}} f - \nabla_{\boldsymbol{x}} f. \tag{2.9}$$

Similarly, we derive

$$\frac{d}{dt}\|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} \le \|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} + \|\nabla_{\boldsymbol{x}} f\|_{L^1(\mathbb{R}^{2d})}. \tag{2.10}$$

Adding (2.8) to (2.10) and using the fact that $\frac{d}{dt}\|f\|_{L^1(\mathbb{R}^{2d})} = 0$, we arrive at

$$\frac{d}{dt}\|f(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le \big((2R(t)+d+1)\|f(t)\|_{L^1(\mathbb{R}^{2d})}+1\big)\,\|f(t)\|_{W^{1,1}(\mathbb{R}^{2d})}. \tag{2.11}$$

Using Lemma 2.1 and solving the above Gronwall's inequality lead to

$$\|f(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le \|f_0\|_{W^{1,1}(\mathbb{R}^{2d})} e^{C(t+t^2)}, \quad \forall t \ge 0. \tag{2.12}$$

(2) For two smooth solutions $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ with initial data f_0 and g_0, respectively, we define

$$\bar{h} := f - g.$$

It follows from the equation (2.1) that

$$\bar{h}_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\bar{h} + \nabla_{\boldsymbol{v}}\cdot(L[f]\bar{h}) = -g\nabla_{\boldsymbol{v}}\cdot L[\bar{h}] - L[\bar{h}]\cdot\nabla_{\boldsymbol{v}}g. \tag{2.13}$$

Multiplying (2.13) by $sgn(\bar{h})$ and integrating the resulting equation over $\mathbb{R}^{2d}$, we obtain

$$\frac{d}{dt}\|\bar{h}(t)\|_{L^1(\mathbb{R}^{2d})} \le d\|g(t)\|_{L^1(\mathbb{R}^{2d})}\|\bar{h}(t)\|_{L^1(\mathbb{R}^{2d})} + 2R(t)\|\nabla_{\boldsymbol{v}}g(t)\|_{L^1(\mathbb{R}^{2d})}\|\bar{h}(t)\|_{L^1(\mathbb{R}^{2d})}. \tag{2.14}$$

Using (2.12) and solving the above Gronwall's inequality, we deduce that there exists $c(t) \le C(1+t)e^{C(t+t^2)}$ such that

$$\sup_{0\le t\le T}\|f(t)-g(t)\|_{L^1(\mathbb{R}^{2d})} \le \|f_0-g_0\|_{L^1(\mathbb{R}^{2d})}\exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0. \tag{2.15}$$

This completes the proof. □

Lemma 2.3. *Assume $R_0 > 0$ and $0 \le f_0(\boldsymbol{x},\boldsymbol{v}) \in W^{2,1}(\mathbb{R}^{2d})$, $supp_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot) \subseteq B(R_0)$. If $f(t,\boldsymbol{x},\boldsymbol{v})$ is a smooth solution to (2.1), then there exists $c(t) \le C(1+t)e^{C(t+t^2)}$ such that*

(1) $\|f(t)\|_{W^{2,1}(\mathbb{R}^{2d})} \le \|f_0\|_{W^{2,1}(\mathbb{R}^{2d})} e^{C(t+t^2)}, \quad \forall t \ge 0;$

(2) $\displaystyle\sup_{0\le t\le T}\|f(t)-g(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le \|f_0-g_0\|_{W^{1,1}(\mathbb{R}^{2d})}\exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are smooth solutions with initial data f_0 and g_0 satisfying the above conditions, respectively.

Proof. Based on Lemma 2.2, we only need to estimate the second-order derivatives. Applying $\partial_{x_i}\partial_{x_j}$ to (2.1)-1, we obtain

$$\begin{aligned}(\partial_{x_i}\partial_{x_j}f)_t &+ \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\partial_{x_i}\partial_{x_j}f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\partial_{x_i}\partial_{x_j}f) \\ &= -\partial_{x_j}L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{x_i}f - \partial_{x_i}f\partial_{x_j}\nabla_{\boldsymbol{v}}\cdot L[f] - \partial_{x_i}\partial_{x_j}L[f]\cdot\nabla_{\boldsymbol{v}}f \\ &\quad - \partial_{x_i}L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{x_j}f - \partial_{x_j}f\partial_{x_i}\nabla_{\boldsymbol{v}}\cdot L[f] - f\partial_{x_i}\partial_{x_j}\nabla_{\boldsymbol{v}}\cdot L[f].\end{aligned} \tag{2.16}$$

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

Multiplying (2.16) by $sgn(\partial_{x_i}\partial_{x_j} f)$ and integrating the resulting equation over $\mathbb{R}^{2d}$, we get

$$\begin{aligned}\frac{d}{dt}\|\partial_{x_i}\partial_{x_j} f\|_{L^1(\mathbb{R}^{2d})} \le{}& 2R(t)\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}}\partial_{x_i} f\|_{L^1(\mathbb{R}^{2d})} + d\|\partial_{x_i} f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})}\\ &+ 2R(t)\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} + 2R(t)\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}}\partial_{x_j} f\|_{L^1(\mathbb{R}^{2d})}\\ &+ d\|\partial_{x_j} f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})} + d\|f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})}.\end{aligned} \tag{2.17}$$

Applying $\partial_{x_i}\partial_{v_j}$ to (2.1)-1 yields

$$\begin{aligned}&(\partial_{x_i}\partial_{v_j} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\partial_{x_i}\partial_{v_j} f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\partial_{x_i}\partial_{v_j} f)\\ &\quad= -\partial_{v_j} L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{x_i} f - \partial_{x_i}\partial_{v_j} L[f]\cdot\nabla_{\boldsymbol{v}} f\\ &\qquad - \partial_{x_i} L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{v_j} f - \partial_{v_j} f\,\partial_{x_i}\nabla_{\boldsymbol{v}}\cdot L[f] - \partial_{x_i}\partial_{x_j} f.\end{aligned} \tag{2.18}$$

Using the same method as above, we deduce that

$$\begin{aligned}\frac{d}{dt}\|\partial_{x_i}\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})} \le{}& \|f\|_{L^1(\mathbb{R}^{2d})}\|\partial_{v_j}\partial_{x_i} f\|_{L^1(\mathbb{R}^{2d})}\\ &+ \|f\|_{L^1(\mathbb{R}^{2d})}\|\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})} + 2R(t)\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}}\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})}\\ &+ d\|\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})} + \|\partial_{x_i}\partial_{x_j} f\|_{L^1(\mathbb{R}^{2d})}.\end{aligned} \tag{2.19}$$

Applying $\partial_{v_i}\partial_{v_j}$ to (2.1)-1 leads to

$$\begin{aligned}&(\partial_{v_i}\partial_{v_j} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\partial_{v_i}\partial_{v_j} f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\partial_{v_i}\partial_{v_j} f)\\ &\quad= -\partial_{v_j} L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{v_i} f\\ &\qquad - \partial_{v_i} L[f]\cdot\nabla_{\boldsymbol{v}}\partial_{v_j} f - \partial_{x_i}\partial_{v_j} f.\end{aligned} \tag{2.20}$$

Similarly, we deduce

$$\begin{aligned}\frac{d}{dt}\|\partial_{v_i}\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})} \le{}& \|f\|_{L^1(\mathbb{R}^{2d})}\|\partial_{v_j}\partial_{v_i} f\|_{L^1(\mathbb{R}^{2d})}\\ &+ \|f\|_{L^1(\mathbb{R}^{2d})}\|\partial_{v_i}\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})} + \|\partial_{x_i}\partial_{v_j} f\|_{L^1(\mathbb{R}^{2d})}.\end{aligned} \tag{2.21}$$

Adding (2.17), (2.19), (2.21) together, summing over all $1 \le i, j \le d$ and then combining (2.11), by Lemma 2.1 we arrive at

$$\frac{d}{dt}\|f(t)\|_{W^{2,1}(\mathbb{R}^{2d})} \le C(1+t)\|f(t)\|_{W^{2,1}(\mathbb{R}^{2d})}. \tag{2.22}$$

Solving the above Gronwall's inequality gives

$$\|f(t)\|_{W^{2,1}(\mathbb{R}^{2d})} \le \|f_0\|_{W^{2,1}(\mathbb{R}^{2d})} e^{C(t+t^2)}, \quad \forall t \ge 0. \tag{2.23}$$

(2) For two smooth solutions $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ with initial data f_0 and g_0 satisfying the initial condition in Lemma 2.3, respectively, we define

$$\bar{h} := f - g, \quad \overline{F} := \nabla_{\boldsymbol{x}} f - \nabla_{\boldsymbol{x}} g \quad \text{and} \quad \overline{G} := \nabla_{\boldsymbol{v}} f - \nabla_{\boldsymbol{v}} g.$$

It follows from (2.7) that

$$\begin{aligned} &\overline{F}_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \overline{F} + \nabla_{\boldsymbol{v}} \cdot (L[f] \otimes \overline{F}) \\ &= - \nabla_{\boldsymbol{v}} \cdot (L[\bar{h}] \otimes \nabla_{\boldsymbol{x}} g) - \nabla_{\boldsymbol{x}} L[\bar{h}] \cdot \nabla_{\boldsymbol{v}} f \\ &\quad - \nabla_{\boldsymbol{x}} L[g] \cdot \overline{G} - \bar{h} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[f] - g \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[\bar{h}]. \end{aligned} \tag{2.24}$$

Multiplying (2.24) by $sgn(\overline{F})$ with each component being the signal function of the corresponding one of $\overline{F}$, and then integrating the resulting equation over $\mathbb{R}^{2d}$, we obtain

$$\begin{aligned} \frac{d}{dt} \|\overline{F}\|_{L^1(\mathbb{R}^{2d})} \le\ & d \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} \|\nabla_{\boldsymbol{x}} g\|_{L^1(\mathbb{R}^{2d})} + 2R(t) \|\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} g\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} \\ & + 2R(t) \|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} + 2R(t) \|g\|_{L^1(\mathbb{R}^{2d})} \|\overline{G}\|_{L^1(\mathbb{R}^{2d})} \\ & + d \|f\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} + d \|g\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})}. \end{aligned} \tag{2.25}$$

From (2.9), we deduce that

$$\begin{aligned} &\overline{G}_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \overline{G} + \nabla_{\boldsymbol{v}} \cdot (L[f] \otimes \overline{G}) \\ &= - \nabla_{\boldsymbol{v}} \cdot (L[\bar{h}] \otimes \nabla_{\boldsymbol{v}} g) - \nabla_{\boldsymbol{v}} L[\bar{h}] \cdot \nabla_{\boldsymbol{v}} f - \nabla_{\boldsymbol{v}} L[g] \cdot \overline{G} - \overline{F}. \end{aligned} \tag{2.26}$$

Similarly, we have

$$\begin{aligned} \frac{d}{dt} \|\overline{G}\|_{L^1(\mathbb{R}^{2d})} \le\ & d \|\nabla_{\boldsymbol{v}} g\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} + 2R(t) \|\nabla_{\boldsymbol{v}}^2 g\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} \\ & + \|\nabla_{\boldsymbol{v}} f\|_{L^1(\mathbb{R}^{2d})} \|\bar{h}\|_{L^1(\mathbb{R}^{2d})} + \|g\|_{L^1(\mathbb{R}^{2d})} \|\overline{G}\|_{L^1(\mathbb{R}^{2d})} + \|\overline{F}\|_{L^1(\mathbb{R}^{2d})}. \end{aligned} \tag{2.27}$$

Adding (2.25) to (2.27) and combining (2.14) lead to

$$\frac{d}{dt} \|f(t) - g(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le C(1+t) e^{C(t+t^2)} \|f(t) - g(t)\|_{W^{1,1}(\mathbb{R}^{2d})}, \tag{2.28}$$

where we have used Lemma 2.1 and (2.23). Solving the above Gronwall's inequality, we deduce that there exists $c(t) \le C(1+t) e^{C(t+t^2)}$ such that

$$\sup_{0 \le t \le T} \|f(t) - g(t)\|_{W^{1,1}(\mathbb{R}^{2d})} \le \|f_0 - g_0\|_{W^{1,1}(\mathbb{R}^{2d})} \exp\left(\int_0^T c(t) dt \right), \quad \forall T \ge 0. \tag{2.29}$$

This completes the proof. $\square$

Higher order estimates can also be obtained with the same method. Next we present the proof of Theorem 2.1 and Theorem 2.2.

2.2. *Proof of Theorem 2.1 and Theorem 2.2*

We first mollify the initial data by convolution, i.e.,

$$f_0^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = f_0 * j_\varepsilon(\boldsymbol{x},\boldsymbol{v}),$$

where j_ε is the standard mollifier. Using the contraction principle, we can obtain the local smooth solution by the standard procedure. Combining with the a priori estimate in Lemma 2.2(1), one can extend the local smooth solution to be global-in-time.

Then using the stability estimate in Lemma 2.2(2), we infer that

$$\sup_{0\le t\le T} \|f^{\varepsilon_i}(t) - f^{\varepsilon_j}(t)\|_{L^1(\mathbb{R}^{2d})} \le \|f_0^{\varepsilon_i} - f_0^{\varepsilon_j}\|_{L^1(\mathbb{R}^{2d})} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0, \tag{2.30}$$

where $f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})$ and $f^{\varepsilon_j}(t,\boldsymbol{x},\boldsymbol{v})$ are smooth solutions with initial data $f_0^{\varepsilon_i}$ and $f_0^{\varepsilon_j}$, respectively. From (2.30), we know there exists $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T], L^1(\mathbb{R}^{2d}))$ such that

$$f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v}) \to f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{in } C([0,T], L^1(\mathbb{R}^{2d})), \text{ as } \varepsilon_i \to 0.$$

It is easy to see that $f(t,\boldsymbol{x},\boldsymbol{v})$ verifies (2.1) in the sense of distributions.

Take smooth initial data $f_0^{\varepsilon_i}$ and $g_0^{\varepsilon_i}$. We also have

$$\sup_{0\le t\le T} \|f^{\varepsilon_i}(t) - g^{\varepsilon_i}(t)\|_{L^1(\mathbb{R}^{2d})} \le \|f_0^{\varepsilon_i} - g_0^{\varepsilon_i}\|_{L^1(\mathbb{R}^{2d})} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0. \tag{2.31}$$

Letting $\varepsilon_i \to 0$, we obtain the stability estimate for weak solutions to (2.1), which amounts to uniqueness of the weak solution. Due to the arbitrariness of T, we know the unique weak solution $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), L^1(\mathbb{R}^{2d}))$.

Theorem 2.2 can be proved in the same way. We omit its proof for brevity. Thus we complete the proof.

3. Kinetic Cucker–Smale model with noise

In this section, we study the kinetic Cucker–Smale model with noise, i.e.,

$$\begin{cases} f_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}}\cdot(L[f]f) = \sigma\Delta_{\boldsymbol{v}} f, \quad 0<\sigma\le 1 \\ f|_{t=0} = f_0(\boldsymbol{x},\boldsymbol{v}). \end{cases} \tag{3.1}$$

Unlike (2.1), the $\boldsymbol{v}$-support of the solution $f(t,\boldsymbol{x},\boldsymbol{v})$ to this equation may be unbounded, even if the initial data have compact support. Thus, the method in section 2 is not valid. In order to circumvent this difficulty, we introduce two weighted Hilbert spaces to establish the well-posedness of weak and strong solutions to (3.1). Define

$$\|f\|_{L^2(\omega)} = \left(\int_{\mathbb{R}^{2d}} f^2(\boldsymbol{x},\boldsymbol{v})\omega(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} \right)^{\frac{1}{2}},$$

$$\|f\|_{L^2(\nu)} = \left(\int_{\mathbb{R}^{2d}} f^2(\boldsymbol{x},\boldsymbol{v})\nu(\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} \right)^{\frac{1}{2}},$$

$$X = \{f: \ f \in L^2(\omega), \ \nabla_{\boldsymbol{x}} f \in L^2(\nu), \ \nabla_{\boldsymbol{v}} f \in L^2(\mathbb{R}^{2d})\},$$

$$\|f\|_X^2 = \|f\|_{L^2(\omega)}^2 + \|\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2 + \|\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})}^2,$$

where $\omega(\boldsymbol{x},\boldsymbol{v}) = (1+\boldsymbol{v}^2)(1+\boldsymbol{x}^2+\boldsymbol{v}^2)^\alpha$, $\alpha > d$ and $\nu(\boldsymbol{v}) = 1+\boldsymbol{v}^2$. The readers will understand why we introduce such weights $\omega(\boldsymbol{x},\boldsymbol{v})$ and $\nu(\boldsymbol{v})$ from the proof of Lemma 3.2 and 3.3. Next we present the definition and results in this section.

Definition 3.1. Let $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), L^2(\mathbb{R}^{2d}))$. $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution to (3.1) if

$$f_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}}\cdot(L[f]f) = \sigma\Delta_{\boldsymbol{v}} f, \quad \text{in } \mathcal{D}'([0,+\infty)\times\mathbb{R}^{2d}).$$

We say $f(t,\boldsymbol{x},\boldsymbol{v})$ is a strong solution if $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution and $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), H^1(\mathbb{R}^{2d})) \cap L^2((0,T)\times\mathbb{R}^d, H^2(\mathbb{R}^d_{\boldsymbol{v}}))$, $\forall T > 0$.

Remark 3.1. Since the strong solution means that a solution satisfies the equation almost everywhere, thus $f(t,\boldsymbol{x},\boldsymbol{v})$ is still a strong solution to (2.1) if $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution and $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), H^1(\mathbb{R}^{2d}))$.

Theorem 3.1. *Assume* $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in L^2(\omega)$. *Then* (3.1) *admits a unique weak solution* $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), L^2(\omega))$ *in the sense of Definition 3.1. Moreover, there exists* $c(t) \le C\exp(e^{Ct})$ *such that*

$$(1)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^2(\omega)}^2 + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f(\tau)\|_{L^2(\omega)}^2 d\tau$$
$$\le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)}^2 \exp\left(e^{Ct}-1\right), \quad a.e.\ t \ge 0;$$

$$(2)\ \sup_{0\le t\le T} \|f(t)-g(t)\|_{L^2(\omega)} \le \|f_0-g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ *and* $g(t,\boldsymbol{x},\boldsymbol{v})$ *are weak solutions with initial data* f_0 *and* g_0 *satisfying the above condition, respectively.*

Theorem 3.2. *Assume* $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in X$. *Then* (3.1) *admits a unique strong solution* $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), X)$ *in the sense of Definition 3.1. Moreover, there exists* $c(t) \le C\exp(e^{Ct})$ *such that*

$$
\begin{aligned}
&(1)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X^2 + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f(\tau)\|_X^2 d\tau \\
&\quad \leq \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X^2 \exp\left(e^{Ct}-1\right), \quad a.e.\ t \geq 0; \\
&(2)\ \sup_{0\leq t\leq T} \|f(t)-g(t)\|_X \leq \|f_0-g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T \geq 0,
\end{aligned}
$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are strong solutions with initial data f_0 and g_0 satisfying the above condition, respectively.

Remark 3.2. We can also establish the well-posedness of classical solutions to (3.1) by using the same method and the Sobolev embedding if we improve the regularity of initial data.

In the following subsection, we derive some a priori estimates that are needed in our proof.

3.1. A priori estimates

Lemma 3.1. *Assume $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in L^2(\omega)$. If $f(t,\boldsymbol{x},\boldsymbol{v})$ is a smooth solution to (3.1), then*

$$
\begin{aligned}
&(1)\ \|f(t)\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}, \quad \forall t \geq 0; \\
&(2)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^1(\mathbb{R}^{2d})} \leq \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^1(\mathbb{R}^{2d})} e^{Ct}, \quad \forall t \geq 0; \\
&(3)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^2(\omega)}^2 + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f(\tau)\|_{L^2(\omega)}^2 d\tau \\
&\quad \leq \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)}^2 \exp\left(e^{Ct}-1\right), \quad \forall t \geq 0.
\end{aligned}
$$

Proof. (1) Since $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in L^2(\omega)$, it is easy to see that

$$
\begin{aligned}
\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^1(\mathbb{R}^{2d})} &= \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0 \omega^{\frac{1}{2}} \omega^{-\frac{1}{2}} d\boldsymbol{x} d\boldsymbol{v} \\
&\leq \|\omega^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^{2d})} \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)} \\
&\leq C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)}.
\end{aligned}
\tag{3.2}
$$

Direct integrating (3.1) over $[0,t] \times \mathbb{R}^{2d}$ yields

$$
\|f(t)\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}, \quad \forall t \geq 0.
$$

(2) Multiplying (3.1) by $(1+\boldsymbol{v}^2)^{\frac{1}{2}}$, we deduce that

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$
\begin{aligned}
&((1+\boldsymbol{v}^2)^{\frac{1}{2}} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}((1+\boldsymbol{v}^2)^{\frac{1}{2}} f) + \nabla_{\boldsymbol{v}}\cdot(L[f](1+\boldsymbol{v}^2)^{\frac{1}{2}} f) \\
&= \sigma(1+\boldsymbol{v}^2)^{\frac{1}{2}}\Delta_{\boldsymbol{v}} f + fL[f]\cdot\nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2)^{\frac{1}{2}}.
\end{aligned}
\tag{3.3}
$$

Integrating (3.3) over $\mathbb{R}^{2d}$ and performing integration by parts, we obtain

$$
\begin{aligned}
\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})} &\le \sigma\|f\|_{L^1(\mathbb{R}^{2d})} + \|f\boldsymbol{v}\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^1(\mathbb{R}^{2d})} + \|f\|_{L^1(\mathbb{R}^{2d})}\|f\boldsymbol{v}\|_{L^1(\mathbb{R}^{2d})} \\
&\le (2\|f_0\|_{L^1(\mathbb{R}^{2d})}+1)\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}.
\end{aligned}
\tag{3.4}
$$

Solving the above Gronwall's inequality gives

$$
\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^1(\mathbb{R}^{2d})} \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^1(\mathbb{R}^{2d})} e^{Ct}, \quad \forall t\ge 0.
\tag{3.5}
$$

(3) Multiplying (3.1) by $2f(1+\boldsymbol{v}^2)\omega$, we get

$$
\begin{aligned}
&((1+\boldsymbol{v}^2)\omega f^2)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}((1+\boldsymbol{v}^2)\omega f^2) + \nabla_{\boldsymbol{v}}\cdot(L[f](1+\boldsymbol{v}^2)\omega f^2) \\
=& f^2 L[f]\cdot\nabla_{\boldsymbol{v}}((1+\boldsymbol{v}^2)\omega) + (1+\boldsymbol{v}^2) f^2 \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\omega \\
&+ 2\sigma(1+\boldsymbol{v}^2)\omega f\Delta_{\boldsymbol{v}} f - (1+\boldsymbol{v}^2)\omega f^2\nabla_{\boldsymbol{v}}\cdot L[f].
\end{aligned}
\tag{3.6}
$$

Integrating (3.6) over $\mathbb{R}^{2d}$ leads to

$$
\begin{aligned}
&\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)} \\
=& \int_{\mathbb{R}^{2d}} f^2 L[f]\cdot\nabla_{\boldsymbol{v}}((1+\boldsymbol{v}^2)\omega)d\boldsymbol{x}d\boldsymbol{v} + \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2) f^2\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\omega d\boldsymbol{x}d\boldsymbol{v} \\
&+ \int_{\mathbb{R}^{2d}} 2\sigma(1+\boldsymbol{v}^2)\omega f\Delta_{\boldsymbol{v}} f d\boldsymbol{x}d\boldsymbol{v} - \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)\omega f^2\nabla_{\boldsymbol{v}}\cdot L[f] d\boldsymbol{x}d\boldsymbol{v}.
\end{aligned}
\tag{3.7}
$$

We estimate each term of the R.H.S. of (3.7) as follows.

$$
\int_{\mathbb{R}^{2d}} f^2 L[f]\cdot\nabla_{\boldsymbol{v}}((1+\boldsymbol{v}^2)\omega)d\boldsymbol{x}d\boldsymbol{v} \le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)},
\tag{3.8}
$$

$$
\int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2) f^2\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\omega d\boldsymbol{x}d\boldsymbol{v} \le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)},
\tag{3.9}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^{2d}} 2\sigma(1+\boldsymbol{v}^2)\omega f\Delta_{\boldsymbol{v}} f d\boldsymbol{x}d\boldsymbol{v} &= -2\sigma\int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)|\nabla_{\boldsymbol{v}} f|^2\omega d\boldsymbol{x}d\boldsymbol{v} \\
&\quad + \sigma\int_{\mathbb{R}^{2d}} f^2\Delta_{\boldsymbol{v}}((1+\boldsymbol{v}^2)\omega)d\boldsymbol{x}d\boldsymbol{v},
\end{aligned}
\tag{3.10}
$$

$$-\int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)\omega f^2 \nabla_{\boldsymbol{v}} \cdot L[f] d\boldsymbol{x} d\boldsymbol{v} \le d\|f\|_{L^1(\mathbb{R}^{2d})} \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)}. \tag{3.11}$$

Substituting (3.8)–(3.11) into (3.7) yields

$$\begin{aligned} &\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)} + \sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f\|^2_{L^2(\omega)} \\ &\le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)} + C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)} \\ &\quad + d\|f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)}. \end{aligned} \tag{3.12}$$

Using (3.5) and solving the above Gronwall's inequality, we deduce that

$$\begin{aligned} &\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|^2_{L^2(\omega)} + \sigma\int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f(\tau)\|^2_{L^2(\omega)} d\tau \\ &\le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|^2_{L^2(\omega)} \exp\left(e^{Ct}-1\right), \quad \forall t \ge 0. \end{aligned} \tag{3.13}$$

This completes the proof. □

Lemma 3.2. *If $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are two smooth solutions with initial data f_0 and g_0 satisfying the condition in Lemma 3.1, respectively, then there exists $c(t) \le C\exp(e^{Ct})$ such that*

$$\sup_{0\le t\le T} \|f(t)-g(t)\|_{L^2(\omega)} \le \|f_0-g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0.$$

Proof. Define $\bar{h} := f-g$. It follows from the equation (3.1) that

$$\bar{h}_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\bar{h} + \nabla_{\boldsymbol{v}}\cdot(L[f]\bar{h} + L[\bar{h}]g) = \sigma\Delta_{\boldsymbol{v}}\bar{h}. \tag{3.14}$$

Multiplying (3.14) by $2\bar{h}\omega$, we deduce that

$$\begin{aligned} &(\bar{h}^2\omega)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\bar{h}^2\omega) + \nabla_{\boldsymbol{v}}\cdot(L[f]\bar{h}^2\omega) \\ &= 2\sigma\omega\bar{h}\Delta_{\boldsymbol{v}}\bar{h} + \bar{h}^2\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\omega - \bar{h}^2\omega\nabla_{\boldsymbol{v}}\cdot L[f] \\ &\quad + \bar{h}^2 L[f]\cdot\nabla_{\boldsymbol{v}}\omega - 2\bar{h}\omega\nabla_{\boldsymbol{v}}\cdot(L[\bar{h}]g). \end{aligned} \tag{3.15}$$

Integrating (3.15) over $\mathbb{R}^{2d}$ gives

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$
\begin{aligned}
&\frac{d}{dt}\|\bar{h}\|^2_{L^2(\omega)}\\
&=2\sigma\int_{\mathbb{R}^{2d}}\omega\bar{h}\Delta_{\boldsymbol{v}}\bar{h}d\boldsymbol{x}d\boldsymbol{v}+\int_{\mathbb{R}^{2d}}\bar{h}^2\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\omega d\boldsymbol{x}d\boldsymbol{v}-\int_{\mathbb{R}^{2d}}\bar{h}^2\omega\nabla_{\boldsymbol{v}}\cdot L[f]d\boldsymbol{x}d\boldsymbol{v}\\
&\quad+\int_{\mathbb{R}^{2d}}\bar{h}^2L[f]\cdot\nabla_{\boldsymbol{v}}\omega d\boldsymbol{x}d\boldsymbol{v}-2\int_{\mathbb{R}^{2d}}\bar{h}\omega\nabla_{\boldsymbol{v}}\cdot(L[\bar{h}]g)d\boldsymbol{x}d\boldsymbol{v}\\
&=\sum_{k=1}^{5}I_k.
\end{aligned}
\tag{3.16}
$$

We estimate each I_k $(1\le k\le 5)$ as follows.

$$
\begin{aligned}
I_1&=-2\sigma\|\nabla_{\boldsymbol{v}}\bar{h}\|^2_{L^2(\omega)}+\sigma\int_{\mathbb{R}^{2d}}\bar{h}^2\Delta_{\boldsymbol{v}}\omega d\boldsymbol{x}d\boldsymbol{v};\\
I_2&\le C\int_{\mathbb{R}^{2d}}\bar{h}^2\omega d\boldsymbol{x}d\boldsymbol{v}=C\|\bar{h}\|^2_{L^2(\omega)};\\
I_3&\le C\|f\|_{L^1(\mathbb{R}^{2d})}\|\bar{h}\|^2_{L^2(\omega)};\\
I_4&\le C\|(1+\boldsymbol{v}^2)^{\frac12}f\|_{L^1(\mathbb{R}^{2d})}\|\bar{h}\|^2_{L^2(\omega)};\\
I_5&=2\int_{\mathbb{R}^{2d}}gL[\bar{h}]\cdot(\nabla_{\boldsymbol{v}}\bar{h}\omega+\nabla_{\boldsymbol{v}}\omega\bar{h})d\boldsymbol{x}d\boldsymbol{v}\\
&\le C\|\bar{h}\|_{L^2(\omega)}\|(1+\boldsymbol{v}^2)^{\frac12}g\|_{L^2(\omega)}\|\nabla_{\boldsymbol{v}}\bar{h}\|_{L^2(\omega)}+C\|g\|_{L^2(\omega)}\|\bar{h}\|^2_{L^2(\omega)}\\
&\le\sigma\|\nabla_{\boldsymbol{v}}\bar{h}\|^2_{L^2(\omega)}+C\|(1+\boldsymbol{v}^2)^{\frac12}g\|^2_{L^2(\omega)}\|\bar{h}\|^2_{L^2(\omega)}+C\|g\|_{L^2(\omega)}\|\bar{h}\|^2_{L^2(\omega)}.
\end{aligned}
$$

In the estimate of I_5, we have used the weighted Hölder inequality, Young's inequality and the following fact that

$$
\begin{aligned}
|L[\bar{h}]|&\le C(1+\boldsymbol{v}^2)^{\frac12}\int_{\mathbb{R}^{2d}}|\bar{h}|(1+|\boldsymbol{v}^*|^2)^{\frac12}(1+|\boldsymbol{y}|^2+|\boldsymbol{v}^*|^2)^{\frac{\alpha}{2}}(1+|\boldsymbol{y}|^2+|\boldsymbol{v}^*|^2)^{-\frac{\alpha}{2}}d\boldsymbol{y}d\boldsymbol{v}^*\\
&\le C(1+\boldsymbol{v}^2)^{\frac12}\|(1+|\boldsymbol{y}|^2+|\boldsymbol{v}^*|^2)^{-\frac{\alpha}{2}}\|_{L^2(\mathbb{R}^{2d})}\|\bar{h}\|_{L^2(\omega)}\\
&\le C(1+\boldsymbol{v}^2)^{\frac12}\|\bar{h}\|_{L^2(\omega)}.
\end{aligned}
$$

Substituting these estimates into (3.16), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\bar{h}\|^2_{L^2(\omega)}+\sigma\|\nabla_{\boldsymbol{v}}\bar{h}\|^2_{L^2(\omega)}\\
&\le C\left(1+\|(1+\boldsymbol{v}^2)^{\frac12}f\|_{L^1(\mathbb{R}^{2d})}+\|(1+\boldsymbol{v}^2)^{\frac12}g\|^2_{L^2(\omega)}\right)\|\bar{h}\|^2_{L^2(\omega)}.
\end{aligned}
\tag{3.17}
$$

Using Lemma 3.1 and solving the above Gronwall's inequality, we infer that there exists $c(t) \leq C \exp(e^{Ct})$ such that

$$\sup_{0 \leq t \leq T} \|f(t) - g(t)\|_{L^2(\omega)} \leq \|f_0 - g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \geq 0.$$

This completes the proof. □

Lemma 3.3. *Assume* $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x}, \boldsymbol{v}) \in X$. *If* $f(t, \boldsymbol{x}, \boldsymbol{v})$ *is a smooth solution to* (3.1), *then there exists* $c(t) \leq C \exp(e^{Ct})$ *such that*

$$\begin{aligned}
&(1)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X^2 + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f(\tau)\|_X^2 d\tau \\
&\qquad \leq \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X^2 \exp\left(e^{Ct} - 1\right), \quad \forall t \geq 0; \\
&(2)\ \sup_{0 \leq t \leq T} \|f(t) - g(t)\|_X \leq \|f_0 - g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T \geq 0,
\end{aligned}$$

where $f(t, \boldsymbol{x}, \boldsymbol{v})$ *and* $g(t, \boldsymbol{x}, \boldsymbol{v})$ *are smooth solutions with initial data* f_0 *and* g_0 *satisfying the above condition, respectively.*

Proof. Based on Lemma 3.1 and 3.2, we only need to estimate the first order derivatives. Applying $\nabla_{\boldsymbol{x}}$ to (3.1) yields

$$(\nabla_{\boldsymbol{x}} f)_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{x}} f) + \nabla_{\boldsymbol{v}} \cdot (L[f] \otimes \nabla_{\boldsymbol{x}} f) = \sigma \Delta_{\boldsymbol{v}} \nabla_{\boldsymbol{x}} f - \nabla_{\boldsymbol{x}} L[f] \cdot \nabla_{\boldsymbol{v}} f - f \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[f]. \tag{3.18}$$

Multiplying (3.18) by $2(1+\boldsymbol{v}^2)^2 \nabla_{\boldsymbol{x}} f$, we have

$$\begin{aligned}
&(|\nabla_{\boldsymbol{x}} f|^2 (1+\boldsymbol{v}^2)^2)_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}(|\nabla_{\boldsymbol{x}} f|^2 (1+\boldsymbol{v}^2)^2) + \nabla_{\boldsymbol{v}} \cdot (L[f] |\nabla_{\boldsymbol{x}} f|^2 (1+\boldsymbol{v}^2)^2) \\
&= 2\sigma (1+\boldsymbol{v}^2)^2 \Delta_{\boldsymbol{v}} \nabla_{\boldsymbol{x}} f \cdot \nabla_{\boldsymbol{x}} f - (1+\boldsymbol{v}^2)^2 |\nabla_{\boldsymbol{x}} f|^2 \nabla_{\boldsymbol{v}} \cdot L[f] \\
&\quad + |\nabla_{\boldsymbol{x}} f|^2 L[f] \cdot \nabla_{\boldsymbol{v}} (1+\boldsymbol{v}^2)^2 - 2(1+\boldsymbol{v}^2)^2 \nabla_{\boldsymbol{x}} f \cdot (\nabla_{\boldsymbol{x}} L[f] \cdot \nabla_{\boldsymbol{v}} f + f \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[f]).
\end{aligned} \tag{3.19}$$

Integrating (3.19) over $\mathbb{R}^{2d}$ leads to

$$\begin{aligned}
\frac{d}{dt} \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2 \leq 2\sigma & \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)^2 \Delta_{\boldsymbol{v}} \nabla_{\boldsymbol{x}} f \cdot \nabla_{\boldsymbol{x}} f \, d\boldsymbol{x} d\boldsymbol{v} \\
& - \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)^2 |\nabla_{\boldsymbol{x}} f|^2 \nabla_{\boldsymbol{v}} \cdot L[f] d\boldsymbol{x} d\boldsymbol{v}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{2d}} |\nabla_{\boldsymbol{x}} f|^2 L[f]\cdot\nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2)^2 d\boldsymbol{x}d\boldsymbol{v}\\
&-\int_{\mathbb{R}^{2d}} 2(1+\boldsymbol{v}^2)^2\nabla_{\boldsymbol{x}} f\cdot(\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f+f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f])d\boldsymbol{x}d\boldsymbol{v}.
\end{aligned}
\tag{3.20}
$$

We estimate the right-hand side of (3.20) term by term.

$$
\begin{aligned}
&2\sigma\int_{\mathbb{R}^{2d}}(1+\boldsymbol{v}^2)^2\Delta_{\boldsymbol{v}}\nabla_{\boldsymbol{x}} f\cdot\nabla_{\boldsymbol{x}} f d\boldsymbol{x}d\boldsymbol{v}\\
&=-2\sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{x}} f\|^2_{L^2(\nu)}+\sigma\int_{\mathbb{R}^{2d}}|\nabla_{\boldsymbol{x}} f|^2\Delta_{\boldsymbol{v}}(1+\boldsymbol{v}^2)^2 d\boldsymbol{x}d\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2d}}(1+\boldsymbol{v}^2)^2|\nabla_{\boldsymbol{x}} f|^2\nabla_{\boldsymbol{v}}\cdot L[f]d\boldsymbol{x}d\boldsymbol{v}\\
&\le C\|f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|^2_{L^2(\nu)},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}}|\nabla_{\boldsymbol{x}} f|^2 L[f]\cdot\nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2)^2 d\boldsymbol{x}d\boldsymbol{v}\\
&\le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|^2_{L^2(\nu)},
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^{2d}}2(1+\boldsymbol{v}^2)^2\nabla_{\boldsymbol{x}} f\cdot\Big(\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f+f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]\Big)d\boldsymbol{x}d\boldsymbol{v}\\
&=\int_{\mathbb{R}^{2d}}2f\nabla_{\boldsymbol{x}} L[f]:\Big[(1+\boldsymbol{v}^2)^2\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}} f+\nabla_{\boldsymbol{x}} f\otimes\nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2)^2\Big]d\boldsymbol{x}d\boldsymbol{v}\\
&\quad-\int_{\mathbb{R}^{2d}}2(1+\boldsymbol{v}^2)^2 f\nabla_{\boldsymbol{x}} f\cdot\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]d\boldsymbol{x}d\boldsymbol{v}\\
&\le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^2(\omega)}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}\\
&\quad+C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^2(\omega)}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}\\
&\le\sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{x}} f\|^2_{L^2(\nu)}+C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)}\\
&\quad+\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|^2_{L^2(\nu)}.
\end{aligned}
$$

Substituting these estimates into (3.20), we obtain

$$\begin{aligned}
&\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2+\sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2\\
\leq& C\left(1+\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\right)\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2\\
&+C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}^2\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^2(\omega)}^2.
\end{aligned} \tag{3.21}$$

Applying $\nabla_{\boldsymbol{v}}$ to (3.1), we deduce that

$$(\nabla_{\boldsymbol{v}} f)_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{v}} f)+\nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\nabla_{\boldsymbol{v}} f)=\sigma\Delta_{\boldsymbol{v}}\nabla_{\boldsymbol{v}} f-\nabla_{\boldsymbol{v}} L[f]\cdot\nabla_{\boldsymbol{v}} f-\nabla_{\boldsymbol{x}} f. \tag{3.22}$$

Multiplying (3.22) by $2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{v}} f$ leads to

$$\begin{aligned}
&(|\nabla_{\boldsymbol{v}} f|^2(1+\boldsymbol{v}^2))_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(|\nabla_{\boldsymbol{v}} f|^2(1+\boldsymbol{v}^2))+\nabla_{\boldsymbol{v}}\cdot(L[f]|\nabla_{\boldsymbol{v}} f|^2(1+\boldsymbol{v}^2))\\
=&2\sigma(1+\boldsymbol{v}^2)\Delta_{\boldsymbol{v}}\nabla_{\boldsymbol{v}} f\cdot\nabla_{\boldsymbol{v}} f+|\nabla_{\boldsymbol{v}} f|^2 L[f]\cdot\nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2)-2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{v}} f\cdot\nabla_{\boldsymbol{x}} f\\
&-(1+\boldsymbol{v}^2)|\nabla_{\boldsymbol{v}} f|^2\nabla_{\boldsymbol{v}}\cdot L[f]-2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{v}} f\cdot\nabla_{\boldsymbol{v}} L[f]\cdot\nabla_{\boldsymbol{v}} f.
\end{aligned} \tag{3.23}$$

Similarly, we have

$$\begin{aligned}
&\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})}^2+\sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})}^2\\
\leq& C\left(1+\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\right)\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})}^2\\
&+\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} f\|_{L^2(\nu)}^2.
\end{aligned} \tag{3.24}$$

Combining (3.12), (3.21), (3.24) and using Lemma 3.1, we deduce that there exists $c(t)\leq Ce^{Ct}$ such that

$$\frac{d}{dt}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_X^2+\sigma\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f\|_X^2\leq c(t)\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_X^2. \tag{3.25}$$

Solving the above Gronwall's inequality yields

$$\begin{aligned}
&\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X^2+\sigma\int_0^t\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f(\tau)\|_X^2 d\tau\\
\leq&\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X^2\exp\left(e^{Ct}-1\right),\quad\forall t\geq 0.
\end{aligned} \tag{3.26}$$

(2) For two smooth solutions $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ with initial data f_0 and g_0 satisfying the initial condition in Lemma 3.3, respectively, we define

$$\bar{h}:=f-g,\quad\overline{F}:=\nabla_{\boldsymbol{x}} f-\nabla_{\boldsymbol{x}} g\quad\text{and}\quad\overline{G}:=\nabla_{\boldsymbol{v}} f-\nabla_{\boldsymbol{v}} g.$$

It follows from (3.18) that

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$\begin{aligned}
&\overline{F}_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \overline{F} + \nabla_{\boldsymbol{v}} \cdot (L[f] \otimes \overline{F}) \\
&= \sigma \Delta_{\boldsymbol{v}} \overline{F} - \nabla_{\boldsymbol{v}} \cdot (L[\bar{h}] \otimes \nabla_{\boldsymbol{x}} g) - \nabla_{\boldsymbol{x}} L[\bar{h}] \cdot \nabla_{\boldsymbol{v}} f \\
&\quad - \nabla_{\boldsymbol{x}} L[g] \cdot \nabla_{\boldsymbol{v}} \bar{h} - \bar{h} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[g] - f \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[\bar{h}].
\end{aligned} \tag{3.27}$$

Multiplying (3.27) by $2(1+\boldsymbol{v}^2)\overline{F}$, we obtain

$$\begin{aligned}
&((1+\boldsymbol{v}^2)\overline{F}^2)_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}}((1+\boldsymbol{v}^2)\overline{F}^2) + \nabla_{\boldsymbol{v}} \cdot (L[f](1+\boldsymbol{v}^2)\overline{F}^2) \\
&= 2\sigma(1+\boldsymbol{v}^2)\Delta_{\boldsymbol{v}} \overline{F} \cdot \overline{F} - 2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{v}} \cdot (L[\bar{h}] \otimes \nabla_{\boldsymbol{x}} g) \cdot \overline{F} \\
&\quad + \overline{F}^2 L[f] \cdot \nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2) - (1+\boldsymbol{v}^2)\overline{F}^2 \nabla_{\boldsymbol{v}} \cdot L[f] \\
&\quad - 2(1+\boldsymbol{v}^2)\overline{F} \cdot \Big(\nabla_{\boldsymbol{x}} L[\bar{h}] \cdot \nabla_{\boldsymbol{v}} f + f \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[\bar{h}] + \nabla_{\boldsymbol{x}} L[g] \cdot \nabla_{\boldsymbol{v}} \bar{h} + \bar{h} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[g]\Big).
\end{aligned} \tag{3.28}$$

Integrating (3.28) over $\mathbb{R}^{2d}$ gives

$$\begin{aligned}
\frac{d}{dt}\|\overline{F}\|^2_{L^2(\nu)} &= 2\sigma \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)\Delta_{\boldsymbol{v}} \overline{F} \cdot \overline{F} \, d\boldsymbol{x} d\boldsymbol{v} \\
&\quad - \int_{\mathbb{R}^{2d}} 2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{v}} \cdot (L[\bar{h}] \otimes \nabla_{\boldsymbol{x}} g) \cdot \overline{F} \, d\boldsymbol{x} d\boldsymbol{v} \\
&\quad + \int_{\mathbb{R}^{2d}} \Big(\overline{F}^2 L[f] \cdot \nabla_{\boldsymbol{v}}(1+\boldsymbol{v}^2) - (1+\boldsymbol{v}^2)\overline{F}^2 \nabla_{\boldsymbol{v}} \cdot L[f]\Big) d\boldsymbol{x} d\boldsymbol{v} \\
&\quad - 2\int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)\overline{F} \cdot \Big(\nabla_{\boldsymbol{x}} L[\bar{h}] \cdot \nabla_{\boldsymbol{v}} f + f \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[\bar{h}] \\
&\qquad + \nabla_{\boldsymbol{x}} L[g] \cdot \nabla_{\boldsymbol{v}} \bar{h} + \bar{h} \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{v}} \cdot L[g]\Big) d\boldsymbol{x} d\boldsymbol{v} \\
&= \sum_{i=1}^{4} J_i.
\end{aligned} \tag{3.29}$$

We estimate each J_i as follows.

$$J_1 = -2\sigma \|\nabla_{\boldsymbol{v}} \overline{F}\|^2_{L^2(\nu)} + \sigma \int_{\mathbb{R}^{2d}} \overline{F}^2 \Delta_{\boldsymbol{v}}(1+\boldsymbol{v}^2) d\boldsymbol{x} d\boldsymbol{v},$$

$$\begin{aligned}
J_2 &= 2\int_{\mathbb{R}^{2d}} (L[\bar{h}] \otimes \nabla_{\boldsymbol{x}} g) : \nabla_{\boldsymbol{v}}((1+\boldsymbol{v}^2)\overline{F}) d\boldsymbol{x} d\boldsymbol{v} \\
&\le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} g\|_{L^2(\nu)} \|\nabla_{\boldsymbol{v}} \overline{F}\|_{L^2(\nu)} \|\bar{h}\|_{L^2(\omega)} + C\|\nabla_{\boldsymbol{x}} g\|_{L^2(\nu)} \|\bar{h}\|_{L^2(\omega)} \|\overline{F}\|_{L^2(\nu)} \\
&\le \sigma \|\nabla_{\boldsymbol{v}} \overline{F}\|^2_{L^2(\nu)} + C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} g\|^2_{L^2(\nu)} \|\bar{h}\|^2_{L^2(\omega)} + \|\overline{F}\|^2_{L^2(\nu)},
\end{aligned}$$

$$J_3 \le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\|\overline{F}\|^2_{L^2(\nu)} + C\|f\|_{L^1(\mathbb{R}^{2d})}\|\overline{F}\|^2_{L^2(\nu)},$$

$$\begin{aligned}
J_4 &= 2\int_{\mathbb{R}^{2d}} \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}(\bar{h}(1+\boldsymbol{v}^2)) : \Big(\bar{h}\nabla_{\boldsymbol{x}} L[g] + f\nabla_{\boldsymbol{x}} L[\bar{h}]\Big) d\boldsymbol{x}d\boldsymbol{v} \\
&= \int_{\mathbb{R}^{2d}} \Big(2(1+\boldsymbol{v}^2)\nabla_{\boldsymbol{x}}\overline{G} + 4\overline{F}\otimes\boldsymbol{v}\Big) : \Big(\bar{h}\nabla_{\boldsymbol{x}} L[g] + f\nabla_{\boldsymbol{x}} L[\bar{h}]\Big) d\boldsymbol{x}d\boldsymbol{v} \\
&\le C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} g\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{v}}\overline{F}\|_{L^2(\nu)}\|\bar{h}\|_{L^2(\omega)} + C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} g\|_{L^1(\mathbb{R}^{2d})}\|\bar{h}\|_{L^2(\omega)}\|\overline{F}\|_{L^2(\nu)} \\
&\quad + C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^2(\omega)}\|\nabla_{\boldsymbol{v}}\overline{F}\|_{L^2(\nu)}\|\bar{h}\|_{L^2(\omega)} + C\|f\|_{L^2(\omega)}\|\bar{h}\|_{L^2(\omega)}\|\overline{F}\|_{L^2(\nu)} \\
&\le \sigma\|\nabla_{\boldsymbol{v}}\overline{F}\|^2_{L^2(\nu)} + C\Big(\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} g\|^2_{L^1(\mathbb{R}^{2d})} + \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)}\Big)\|\bar{h}\|^2_{L^2(\omega)} + C\|\overline{F}\|^2_{L^2(\nu)}.
\end{aligned}$$

Substituting these estimates into (3.29), we obtain

$$\begin{aligned}
\frac{d}{dt}\|\overline{F}\|^2_{L^2(\nu)} \le\, & C\Big(1 + \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|_{L^1(\mathbb{R}^{2d})}\Big)\|\overline{F}\|^2_{L^2(\nu)} + C\Big(\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} g\|^2_{L^1(\mathbb{R}^{2d})} \\
& + \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f\|^2_{L^2(\omega)} + \|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{x}} g\|^2_{L^2(\nu)}\Big)\|\bar{h}\|^2_{L^2(\omega)}.
\end{aligned} \tag{3.30}$$

It follows from (3.22) that

$$\begin{aligned}
& \overline{G}_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\overline{G} + \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\overline{G}) \\
=\, & \sigma\Delta_{\boldsymbol{v}}\overline{G} - \nabla_{\boldsymbol{v}}\cdot(L[\bar{h}]\otimes\nabla_{\boldsymbol{v}} g) - \nabla_{\boldsymbol{v}} L[\bar{h}]\cdot\nabla_{\boldsymbol{v}} f - \nabla_{\boldsymbol{v}} L[g]\cdot\overline{G} - \overline{F}.
\end{aligned} \tag{3.31}$$

Multiplying (3.31) by $2\overline{G}$ yields

$$\begin{aligned}
& (\overline{G}^2)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\overline{G}^2) + \nabla_{\boldsymbol{v}}\cdot(L[f]\overline{G}^2) \\
=\, & -\overline{G}^2\nabla_{\boldsymbol{v}}\cdot L[f] - 2\sigma\Delta_{\boldsymbol{v}}\overline{G}\cdot\overline{G} - 2\nabla_{\boldsymbol{v}}\cdot(L[\bar{h}]\otimes\nabla_{\boldsymbol{v}} g)\cdot\overline{G} \\
& - 2\overline{G}\cdot\nabla_{\boldsymbol{v}} L[\bar{h}]\cdot\nabla_{\boldsymbol{v}} f - 2\overline{G}\cdot\nabla_{\boldsymbol{v}} L[g]\cdot\overline{G} - 2\overline{F}\cdot\overline{G}.
\end{aligned} \tag{3.32}$$

Integrating (3.32) over $\mathbb{R}^{2d}$ and performing integration by parts, we have

$$\begin{aligned}
\frac{d}{dt}\|\overline{G}\|^2_{L^2(\mathbb{R}^{2d})} \le\, & C(1 + \|f\|_{L^1(\mathbb{R}^{2d})} + \|g\|_{L^1(\mathbb{R}^{2d})})\|\overline{G}\|^2_{L^2(\mathbb{R}^{2d})} + \|\overline{F}\|^2_{L^2(\nu)} \\
& + C\Big(\|\nabla_{\boldsymbol{v}} f\|^2_{L^2(\mathbb{R}^{2d})} + \|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} g\|^2_{L^2(\mathbb{R}^{2d})}\Big)\|\bar{h}\|^2_{L^2(\omega)}.
\end{aligned} \tag{3.33}$$

Combining (3.17), (3.30), (3.33) and using (3.26), Lemma 3.1, we deduce that there exists $c(t) \le C\exp(e^{Ct})$

$$\frac{d}{dt}\|f(t) - g(t)\|^2_X \le c(t)\|f(t) - g(t)\|^2_X, \tag{3.34}$$

which implies

$$\sup_{0\le t\le T}\|f(t)-g(t)\|_X \le \|f_0-g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T\ge 0. \tag{3.35}$$

This completes the proof. □

3.2. Proof of Theorem 3.1 and 3.2

We first mollify the initial data by convolution, i.e.,

$$f_0^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = f_0 * j_\varepsilon(\boldsymbol{x},\boldsymbol{v}),$$

where j_ε is the standard mollifier. Using the contraction principle, we can obtain the local smooth solution by the standard procedure. Combining with the a priori estimate in Lemma 3.1(3), one can extend the local smooth solution to be global-in-time.

Then using the stability estimate in Lemma 3.2, we infer that

$$\sup_{0\le t\le T}\|f^{\varepsilon_i}(t)-f^{\varepsilon_j}(t)\|_{L^2(\omega)} \le \|f_0^{\varepsilon_i}-f_0^{\varepsilon_j}\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T\ge 0, \tag{3.36}$$

where $f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})$ and $f^{\varepsilon_j}(t,\boldsymbol{x},\boldsymbol{v})$ are smooth solutions with initial data $f_0^{\varepsilon_i}$ and $f_0^{\varepsilon_j}$, respectively. From (3.36), we know there exists $f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,T],L^2(\omega))$ such that

$$f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v}) \to f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{in } C([0,T],L^2(\omega))\ \forall T>0, \text{ as } \varepsilon_i\to 0. \tag{3.37}$$

It is easy to see that $f(t,\boldsymbol{x},\boldsymbol{v})$ verifies (3.1) in the sense of distributions.

Take smooth initial data $f_0^{\varepsilon_i}$ and $g_0^{\varepsilon_i}$. We also have

$$\sup_{0\le t\le T}\|f^{\varepsilon_i}(t)-g^{\varepsilon_i}(t)\|_{L^2(\omega)} \le \|f_0^{\varepsilon_i}-g_0^{\varepsilon_i}\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T\ge 0. \tag{3.38}$$

Letting $\varepsilon_i\to 0$, we obtain the stability estimate for weak solutions to (3.1), which also amounts to uniqueness of the weak solution. Due to the arbitrariness of T, we know the unique weak solution $f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,+\infty),L^2(\omega))$. It follows from 3.1(3) that for all f^{ε_i},

$$\begin{aligned}
&\|(1+\boldsymbol{v}^2)^{\frac12} f^{\varepsilon_i}(t)\|_{L^2(\omega)}^2 + \sigma\int_0^t \|(1+\boldsymbol{v}^2)^{\frac12}\nabla_{\boldsymbol{v}} f^{\varepsilon_i}(\tau)\|_{L^2(\omega)}^2 d\tau\\
&\le \|(1+\boldsymbol{v}^2)^{\frac12} f_0^{\varepsilon_i}\|_{L^2(\omega)}^2 \exp\left(e^{Ct}-1\right)\\
&\le C\exp(e^{Ct}), \quad \forall t\ge 0.
\end{aligned} \tag{3.39}$$

Using (3.37), we infer that for any $t > 0$,

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\varepsilon_i} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} f, \quad \text{weakly-}\star \text{ in } L^\infty((0,t), L^2(\omega)), \text{ as } \varepsilon_i \to 0, \tag{3.40}$$

and

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} f^{\varepsilon_i} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} f, \quad \text{weakly in } L^2((0,t), L^2(\omega)), \text{ as } \varepsilon_i \to 0. \tag{3.41}$$

Combining (3.39)–(3.41) and letting $\varepsilon_i \to 0$ lead to

$$\begin{aligned} &\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|^2_{L^2(\omega)} + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f(\tau)\|^2_{L^2(\omega)} d\tau \\ &\le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|^2_{L^2(\omega)} \exp\left(e^{Ct}-1\right), \quad \text{a.e. } t \ge 0. \end{aligned}$$

Theorem 3.2 can be proved in the same way. We omit its proof for brevity. Thus we complete the proof.

4. Vanishing noise limit

In this section, we study the vanishing noise limit as σ tends to 0. In fact, we can pass to the limits of both weak and strong solution sequences to (3.1). Our results are as follows.

Definition 4.1. Let $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), L^2(\mathbb{R}^{2d}))$. $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution to (2.1) if

$$f_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}}\cdot(L[f]f) = 0, \quad \text{in } \mathcal{D}'([0,+\infty)\times\mathbb{R}^{2d}).$$

We say $f(t,\boldsymbol{x},\boldsymbol{v})$ is a strong solution if $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution and $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), H^1(\mathbb{R}^{2d}))$.

Theorem 4.1. *Assume $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in L^2(\omega)$. Then (2.1) admits a unique weak solution $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty), L^2(\omega))$ in the sense of Definition 4.1. Moreover, there exists $c(t) \le C\exp(e^{Ct})$ such that*

$$(1)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^2(\omega)} \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)} \exp\left(e^{Ct}-1\right), \quad \textit{a.e. } t \ge 0;$$

$$(2)\ \sup_{0\le t\le T} \|f(t)-g(t)\|_{L^2(\omega)} \le \|f_0-g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ and $g(t,\boldsymbol{x},\boldsymbol{v})$ are weak solutions with initial data f_0 and g_0 satisfying the above condition, respectively.

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

Theorem 4.2. *Assume* $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0(\boldsymbol{x},\boldsymbol{v}) \in X$. *Then* (2.1) *admits a unique strong solution* $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,+\infty),X)$ *in the sense of Definition 4.1. Moreover, there exists* $c(t) \le C\exp(e^{Ct})$ *such that*

$$(1)\ \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X \exp\left(e^{Ct}-1\right), \quad a.e.\ t \ge 0;$$

$$(2)\ \sup_{0\le t\le T} \|f(t)-g(t)\|_X \le \|f_0-g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$$

where $f(t,\boldsymbol{x},\boldsymbol{v})$ *and* $g(t,\boldsymbol{x},\boldsymbol{v})$ *are strong solutions with initial data* f_0 *and* g_0 *satisfying the above condition, respectively.*

In fact, Remark 3.2 still holds as $\sigma \to 0$. The following velocity averaging lemma is due to [6](Theorem 5 and Remark 3 of Theorem 3). It plays an import role in the proof of Theorem 4.1.

Lemma 4.1 *(DiPerna and Lions). Let* $m \ge 0$, $f,g \in L^2(\mathbb{R}\times\mathbb{R}^{2d})$ *and* $f(t,\boldsymbol{x},\boldsymbol{v}), g(t,\boldsymbol{x},\boldsymbol{v})$ *satisfy*

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f = \nabla_{\boldsymbol{v}}^{\xi} g \quad in\ \mathcal{D}',$$

where $\nabla_{\boldsymbol{v}}^{\xi} = \partial_{\boldsymbol{v}^1}^{\xi^1}\partial_{\boldsymbol{v}^2}^{\xi^2}\cdots\partial_{\boldsymbol{v}^d}^{\xi^d}$ *and* $|\xi| = \sum_{i=1}^d \xi^i = m$. *Then for any* $\phi(\boldsymbol{v}) \in C_c^\infty(\mathbb{R}^d)$, *it holds that*

$$\left\| \int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\phi(\boldsymbol{v})d\boldsymbol{v} \right\|_{H^s(\mathbb{R}\times\mathbb{R}^d)} \le C_\phi\left(\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^{2d})} + \|g\|_{L^2(\mathbb{R}\times\mathbb{R}^{2d})}\right),$$

where $s = \frac{1}{2(1+m)}$ *and* C_ϕ *is a positive constant.*

Denote the solution to (3.1) by $f^\sigma(t,\boldsymbol{x},\boldsymbol{v})$. Then we present the proof of the above two theorems.

Proof of Theorem 4.1. According to Theorem 3.1 (1), we know

$$\begin{aligned} &\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^\sigma(t)\|_{L^2(\omega)}^2 + \sigma\int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}}\nabla_{\boldsymbol{v}} f^\sigma(\tau)\|_{L^2(\omega)}^2 d\tau \\ &\le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)}^2 \exp\left(e^{Ct}-1\right), \quad \text{a.e. } t \ge 0. \end{aligned} \tag{4.1}$$

Thus there exists a sequence $\{f^{\sigma_j}(t,\boldsymbol{x},\boldsymbol{v})\}$ such that

$$f^{\sigma_j}(t,\boldsymbol{x},\boldsymbol{v}) \rightharpoonup f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{weakly-}\star \text{ in } L^\infty((0,T),L^2(\mathbb{R}^{2d}))\ \forall T>0,\ \text{as } \sigma_j \to 0. \tag{4.2}$$

This also leads to

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$\int_0^T \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f^{\sigma_j} \psi \, d\boldsymbol{x} d\boldsymbol{v} dt \to \int_0^T \int_{\mathbb{R}^{2d}} (1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f \psi \, d\boldsymbol{x} d\boldsymbol{v} dt, \quad \text{as } \sigma_j \to 0, \tag{4.3}$$

for any $\psi(t, \boldsymbol{x}, \boldsymbol{v}) \in C_c^\infty((0, T) \times \mathbb{R}^{2d})$, which implies that

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f^{\sigma_j} \to (1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f \quad \text{in } \mathcal{D}', \text{ as } \sigma_j \to 0. \tag{4.4}$$

It follows from (4.1) that the sequence $\{(1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f^{\sigma_j}\}$ admits a weak-$\star$ limit in $L^\infty((0, T), L^2(\mathbb{R}^{2d}))$, $\forall T > 0$. Using uniqueness of the limit in the sense of distributions, we also have

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f^{\sigma_j} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} \omega^{\frac{1}{2}} f, \tag{4.5}$$

weakly-$\star$ in $L^\infty((0, T), L^2(\mathbb{R}^{2d}))$ $\forall T > 0$, as $\sigma_j \to 0$. Therefore, we deduce that

$$\operatorname*{ess\,sup}_{0 \le t \le T} \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|^2_{L^2(\omega)} \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|^2_{L^2(\omega)} \exp\left(e^{CT} - 1\right), \quad \forall T > 0. \tag{4.6}$$

Next, we prove

$$L[f^{\sigma_j}] f^{\sigma_j} \to L[f] f \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{R}^{2d}), \text{ as } \sigma_j \to 0.$$

Rewrite (3.1) in the form of

$$\frac{\partial f^{\sigma_j}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f^{\sigma_j} = \nabla_{\boldsymbol{v}} \cdot \left(\sigma_j \nabla_{\boldsymbol{v}} f^{\sigma_j} - L[f^{\sigma_j}] f^{\sigma_j}\right). \tag{4.7}$$

Using Lemma 4.1 and (4.1), we infer that for all f^{σ_j},

$$\left\| \int_{\mathbb{R}^d} f^{\sigma_j}(t, \boldsymbol{x}, \boldsymbol{v}) \phi(\boldsymbol{v}) d\boldsymbol{v} \right\|_{H^{\frac{1}{4}}([0,T] \times \mathbb{R}^d)} \le C_\phi (1+T) \exp(Ce^{CT}), \quad \forall \phi(\boldsymbol{v}) \in C_c^\infty(\mathbb{R}^d).$$

The reader is referred to [6] (step 1 in Section 4, p. 748) for the details in applying Lemma 4.1. Since

$$H^{\frac{1}{4}}([0, T] \times K) \hookrightarrow\hookrightarrow L^1([0, T] \times K) \quad \text{for any compact } K \text{ in } \mathbb{R}^d,$$

there exists a subsequence, still denoted by $\{f^{\sigma_j}\}$, such that

$$\int_{\mathbb{R}^d} f^{\sigma_j} \phi(\boldsymbol{v}) d\boldsymbol{v} \to \int_{\mathbb{R}^d} f \phi(\boldsymbol{v}) d\boldsymbol{v} \quad \text{in } L^1_{loc}((0, T) \times \mathbb{R}^d), \text{ as } \sigma_j \to 0. \tag{4.8}$$

For any $\varepsilon > 0$, if we choose R suitably large, it follows from (4.1) that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \left| \int_{|\boldsymbol{v}|>R} (1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j} d\boldsymbol{v} \right| d\boldsymbol{x} dt \\
&\le \left(\int_0^T \int_{\mathbb{R}^d} \int_{|\boldsymbol{v}|>R} \omega^{-1} d\boldsymbol{v} d\boldsymbol{x} dt \right)^{\frac{1}{2}} \left(\int_0^T \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j}(t)\|^2_{L^2(\omega)} dt \right)^{\frac{1}{2}} \\
&\le CT \exp(Ce^{CT})\varepsilon
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
&\int_0^T \int_{|\boldsymbol{x}|>R} \left| \int_{\mathbb{R}^d} (1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j} d\boldsymbol{v} \right| d\boldsymbol{x} dt \\
&\le \left(\int_0^T \int_{|\boldsymbol{x}|>R} \int_{\mathbb{R}^d} \omega^{-1} d\boldsymbol{v} d\boldsymbol{x} dt \right)^{\frac{1}{2}} \left(\int_0^T \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j}(t)\|^2_{L^2(\omega)} dt \right)^{\frac{1}{2}} \\
&\le CT \exp(Ce^{CT})\varepsilon.
\end{aligned} \tag{4.10}$$

Combining (4.8), (4.9) and (4.10), we infer that

$$\int_{\mathbb{R}^d} (1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j} d\boldsymbol{v} \to \int_{\mathbb{R}^d} (1+\boldsymbol{v}^2)^{\frac{1}{2}} f d\boldsymbol{v} \quad \text{in } L^1((0,T)\times\mathbb{R}^d), \text{ as } \sigma_j \to 0. \tag{4.11}$$

It follows from (4.11) that there exists a subsequence, still denoted by $\{f^{\sigma_j}\}$, such that

$$L[f^{\sigma_j}] \to L[f], \quad \text{a.e. } (t, \boldsymbol{x}, \boldsymbol{v}) \in [0,T]\times\mathbb{R}^{2d}, \text{ as } \sigma_j \to 0. \tag{4.12}$$

From (4.1), it is easy to see that

$$\left| \int_{\mathbb{R}^{2d}} \varphi(|\boldsymbol{x}-\boldsymbol{y}|) f^{\sigma_j}(t, \boldsymbol{y}, \boldsymbol{v}^*)(1+|\boldsymbol{v}^*|^2)^{\frac{1}{2}} d\boldsymbol{y} d\boldsymbol{v}^* \right| \le C \exp\left(\frac{1}{2}e^{CT}\right), \quad \forall t \in [0,T]. \tag{4.13}$$

Combining (4.8), (4.12) and (4.13), we deduce that

$$\int_0^T \int_{\mathbb{R}^{2d}} L[f^{\sigma_j}] f^{\sigma_j} \phi_1(\boldsymbol{v})\phi_2(t,\boldsymbol{x}) d\boldsymbol{v} d\boldsymbol{x} dt \to \int_0^T \int_{\mathbb{R}^{2d}} L[f] f \phi_1(\boldsymbol{v})\phi_2(t,\boldsymbol{x}) d\boldsymbol{v} d\boldsymbol{x} dt, \tag{4.14}$$

for any $\phi_1(\boldsymbol{v}) \in C_c^\infty(\mathbb{R}^d)$, $\phi_2(t,\boldsymbol{x}) \in C_c^\infty((0,T)\times\mathbb{R}^d)$, as $\sigma_j \to 0$. Using the density of the sums of the function with the form $\phi_1(\boldsymbol{v})\phi_2(t,\boldsymbol{x})$ in $C_c^\infty((0,T)\times\mathbb{R}^{2d})$, we can show that

$$L[f^{\sigma_j}]f^{\sigma_j} \to L[f]f \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}),\ \text{as } \sigma_j \to 0.$$

Therefore,

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f + \nabla_{\boldsymbol{v}}\cdot(L[f]f) = 0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}). \tag{4.15}$$

Then we prove $f \in C([0,T], L^2(\omega))$. Since $(1+\boldsymbol{v}^2)^{\frac{1}{2}} f \in L^\infty((0,T), L^2(\omega))$, by interpolation, it is sufficient to prove $f \in C([0,T], L^2(\mathbb{R}^{2d}))$. From (4.6) and (4.15), we know

$$f_t \in L^\infty((0,T), H^{-1}(\mathbb{R}^{2d})). \tag{4.16}$$

This together with the fact that $f \in L^\infty((0,T), L^2(\mathbb{R}^{2d}))$ gives

$$f \in C([0,T], L^2(\mathbb{R}^{2d}) - W), \tag{4.17}$$

which means that f is continuous in $[0,T]$ with respect to the weak topology in $L^2(\mathbb{R}^{2d})$. In the following, we prove $\|f(t)\|_{L^2(\mathbb{R}^{2d})} \in C[0,T]$.

Take the standard mollifier $j_\varepsilon(\boldsymbol{x}-\cdot, \boldsymbol{v}-\cdot)$ as the test function in (4.15). Denoting $f * j_\varepsilon(\boldsymbol{x},\boldsymbol{v})$ by $\langle f\rangle_\varepsilon$, we have

$$\begin{aligned}&(\langle f\rangle_\varepsilon)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle f\rangle_\varepsilon + \nabla_{\boldsymbol{v}}\cdot(L[f]\langle f\rangle_\varepsilon)\\ =&-\nabla_{\boldsymbol{x}}\cdot\langle \boldsymbol{v} f\rangle_\varepsilon + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle f\rangle_\varepsilon - \nabla_{\boldsymbol{v}}\cdot\langle L[f]f\rangle_\varepsilon + \nabla_{\boldsymbol{v}}\cdot(L[f]\langle f\rangle_\varepsilon).\end{aligned} \tag{4.18}$$

Multiplying (4.18) by $2\langle f\rangle_\varepsilon$ yields

$$\begin{aligned}&\frac{\partial}{\partial t}\langle f\rangle_\varepsilon^2 + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle f\rangle_\varepsilon^2 + \nabla_{\boldsymbol{v}}\cdot\Big(L[f]\langle f\rangle_\varepsilon^2\Big)\\ =&-\langle f\rangle_\varepsilon^2\nabla_{\boldsymbol{v}}\cdot L[f] - 2\left[\nabla_{\boldsymbol{x}}\cdot\langle \boldsymbol{v} f\rangle_\varepsilon - \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle f\rangle_\varepsilon\right]\cdot\langle f\rangle_\varepsilon\\ &-2\left[\nabla_{\boldsymbol{v}}\cdot\langle L[f]f\rangle_\varepsilon - \nabla_{\boldsymbol{v}}\cdot(L[f]\langle f\rangle_\varepsilon)\right]\cdot\langle f\rangle_\varepsilon.\end{aligned} \tag{4.19}$$

Integrating (4.19) over $\mathbb{R}^{2d}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\langle f\rangle_\varepsilon\|_{L^2(\mathbb{R}^{2d})}^2 =& -\int_{\mathbb{R}^{2d}}\langle f\rangle_\varepsilon^2\nabla_{\boldsymbol{v}}\cdot L[f]d\boldsymbol{x}d\boldsymbol{v}\\ &-\int_{\mathbb{R}^{2d}} 2\left[\nabla_{\boldsymbol{x}}\cdot\langle \boldsymbol{v} f\rangle_\varepsilon - \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle f\rangle_\varepsilon\right]\cdot\langle f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\\ &-\int_{\mathbb{R}^{2d}} 2\left[\nabla_{\boldsymbol{v}}\cdot\langle L[f]f\rangle_\varepsilon - \nabla_{\boldsymbol{v}}\cdot(L[f]\langle f\rangle_\varepsilon)\right]\cdot\langle f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\\ =&\sum_{i=1}^3 K_i.\end{aligned} \tag{4.20}$$

We estimate each K_i $(1 \le i \le 3)$ as follows.

$$|K_1| \le C\|f\|_{L^1(\mathbb{R}^{2d})}\|f\|^2_{L^2(\mathbb{R}^{2d})}.$$

$$\begin{aligned}|K_2| &= 2\left|\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}(\boldsymbol{w}-\boldsymbol{v})\cdot\nabla_{\boldsymbol{x}} j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})f(t,\boldsymbol{z},\boldsymbol{w})d\boldsymbol{z}d\boldsymbol{w}\langle f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\right|\\ &\le 2C\|f\|^2_{L^2(\mathbb{R}^{2d})},\end{aligned}$$

where we have used the fact that

$$|\boldsymbol{w}-\boldsymbol{v}| \le \varepsilon \quad \text{and} \quad \|\nabla_{\boldsymbol{x}} j_\varepsilon\|_{L^1(\mathbb{R}^{2d})} \le \frac{C}{\varepsilon}.$$

$$\begin{aligned}|K_3| &= 2\left|\int_{\mathbb{R}^{2d}}\nabla_{\boldsymbol{v}}\cdot\int_{\mathbb{R}^{2d}}\Big(L[f](\boldsymbol{z},\boldsymbol{w})-L[f](\boldsymbol{x},\boldsymbol{v})\Big)j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})f(t,\boldsymbol{z},\boldsymbol{w})d\boldsymbol{z}d\boldsymbol{w}\langle f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\right|\\ &\le 2C\|f\|_{L^1(\mathbb{R}^{2d})}\|f\|^2_{L^2(\mathbb{R}^{2d})}\\ &\quad + 2\left|\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}}\Big(L[f](\boldsymbol{z},\boldsymbol{w})-L[f](\boldsymbol{x},\boldsymbol{v})\Big)\cdot\nabla_{\boldsymbol{v}} j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})f(t,\boldsymbol{z},\boldsymbol{w})d\boldsymbol{z}d\boldsymbol{w}\langle f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\right|\\ &\le 2C\|f\|_{L^1(\mathbb{R}^{2d})}\|f\|^2_{L^2(\mathbb{R}^{2d})} + 2C\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac{1}{2}}f\|_{L^2(\mathbb{R}^{2d})}\|f\|_{L^2(\mathbb{R}^{2d})}.\end{aligned}$$

Substituting these estimates into (4.20) and integrating the resulting inequality over $[t_1, t_2]$, $\forall t_1, t_2 \in [0, T]$, we obtain

$$\left|\|\langle f(t_2)\rangle_\varepsilon\|^2_{L^2(\mathbb{R}^{2d})} - \|\langle f(t_1)\rangle_\varepsilon\|^2_{L^2(\mathbb{R}^{2d})}\right| \le C\exp(Ce^{CT})|t_2-t_1|, \quad \forall t_1, t_2 \in [0, T],$$

where we have used (4.6). Letting $\varepsilon \to 0$ yields

$$\left|\|f(t_2)\|^2_{L^2(\mathbb{R}^{2d})} - \|f(t_1)\|^2_{L^2(\mathbb{R}^{2d})}\right| \le C\exp(Ce^{CT})|t_2-t_1|, \quad \forall t_1, t_2 \in [0, T]. \tag{4.21}$$

Combining (4.17) with (4.21), we deduce that

$$f \in C([0, T], L^2(\mathbb{R}^{2d})), \quad \forall T \ge 0. \tag{4.22}$$

Similarly to (4.5), we can prove

$$\omega^{\frac{1}{2}} f^{\sigma_j} \rightharpoonup \omega^{\frac{1}{2}} f, \quad \text{weakly-}\star \text{ in } L^\infty((0,T), L^2(\mathbb{R}^{2d}))\ \forall T > 0, \text{ as } \sigma_j \to 0 \tag{4.23}$$

and

$$\omega^{\frac{1}{2}} g^{\sigma_j} \rightharpoonup \omega^{\frac{1}{2}} g, \quad \text{weakly-}\star \text{ in } L^\infty((0,T), L^2(\mathbb{R}^{2d}))\ \forall T>0, \text{ as } \sigma_j \to 0, \tag{4.24}$$

for solution sequences $\{f^{\sigma_j}\}$ and $\{g^{\sigma_j}\}$ with initial data f_0 and g_0, respectively. It follows from Theorem 3.1(2) that

$$\sup_{0\le t\le T} \|f^{\sigma_j}(t) - g^{\sigma_j}(t)\|_{L^2(\omega)} \le \|f_0 - g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0. \tag{4.25}$$

Combining (4.23)–(4.25) and letting $\sigma_j \to 0$ give

$$\sup_{0\le t\le T} \|f(t) - g(t)\|_{L^2(\omega)} \le \|f_0 - g_0\|_{L^2(\omega)} \exp\left(\int_0^T c(t)dt\right), \quad \forall T \ge 0,$$

which implies uniqueness of the weak solution. Due to the arbitrariness of T, it is easily shown that the unique weak solution $f \in C([0,\infty), L^2(\omega))$. From (4.6) we infer that

$$\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_{L^2(\omega)} \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_{L^2(\omega)} \exp\left(e^{Ct}-1\right), \quad \text{a.e. } t \ge 0.$$

This completes the proof. □

Proof of Theorem 4.2. According to Theorem 3.2, we have

$$\begin{aligned} &\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^\sigma(t)\|_X^2 + \sigma \int_0^t \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f^\sigma(\tau)\|_X^2 d\tau \\ &\le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X^2 \exp\left(e^{Ct}-1\right), \quad \text{a.e. } t \ge 0. \end{aligned} \tag{4.26}$$

From (4.26) and the equation (3.1), we deduce that f^σ are uniformly bounded in $L^\infty((0,T), H^1(\mathbb{R}^{2d}))$, and $\frac{\partial f^\sigma}{\partial t}$ are uniformly bounded in $L^2((0,T), L^2(\mathbb{R}^{2d}))$, $\forall T > 0$. It follows from the Ascoli–Arzela theorem and (4.26) that there exists a sequence $\{f^{\sigma_j}\}$ such that

$$f^{\sigma_j}(t,\boldsymbol{x},\boldsymbol{v}) \to f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{in } C([0,T], L^2(\mathbb{R}^{2d})), \text{ as } \sigma_j \to 0. \tag{4.27}$$

It is easy to see that $f(t,\boldsymbol{x},\boldsymbol{v})$ verifies (2.1) in the sense of distributions. Similarly to the proof in Theorem 4.1, we can show that

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} f^{\sigma_j} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} f, \quad \text{weakly-}\star \text{ in } L^\infty((0,T), L^2(\omega)),$$
$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} f^{\sigma_j} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{x}} f, \quad \text{weakly-}\star \text{ in } L^\infty((0,T), L^2(\nu)),$$

and

$$(1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f^{\sigma_j} \rightharpoonup (1+\boldsymbol{v}^2)^{\frac{1}{2}} \nabla_{\boldsymbol{v}} f, \quad \text{weakly-}\star \text{ in } L^\infty((0,T), L^2(\mathbb{R}^{2d}))$$

for any $T > 0$, as $\sigma_j \to 0$. Thus we have

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$\operatorname*{ess\ sup}_{0\le t\le T} \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X \exp\left(e^{CT}-1\right), \quad \forall T>0. \tag{4.28}$$

Next, we prove $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T],X)$, $\forall T>0$. In fact, by using (4.28) and the interpolation, it suffices to prove

$$f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T],W^{1,2}(\mathbb{R}^{2d})), \quad \forall T>0. \tag{4.29}$$

Similarly to the proof in Theorem 4.1, we can easily show that

$$f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T],W^{1,2}(\mathbb{R}^{2d})-W), \quad \forall T>0, \tag{4.30}$$

which means that f is continuous in $[0,T]$ with respect to the weak topology in $W^{1,2}(\mathbb{R}^{2d})$. The following proof is devoted to demonstrating that

$$\|f(t)\|_{W^{1,2}(\mathbb{R}^{2d})} \in C[0,T], \quad \forall T>0.$$

Based on Theorem 4.1, we just show that $\|\nabla_{\boldsymbol{x}} f\|_{L^2(\mathbb{R}^{2d})}$ and $\|\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})}$ are in $C[0,T]$, $\forall T>0$. Since

$$\begin{aligned}&(\nabla_{\boldsymbol{x}} f)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(\nabla_{\boldsymbol{x}} f) + \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\nabla_{\boldsymbol{x}} f) = -\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f - f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]\\ &\quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}),\end{aligned}$$

taking the standard mollifier $j_\varepsilon(\boldsymbol{x}-\cdot,\boldsymbol{v}-\cdot)$ as the test function yields

$$\begin{aligned}&(\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon)_t + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon + \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon)\\ &= \langle -\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f - f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]\rangle_\varepsilon\\ &\quad - [\nabla_{\boldsymbol{x}}\cdot\langle\boldsymbol{v}\otimes\nabla_{\boldsymbol{x}} f\rangle_\varepsilon - \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon]\\ &\quad - [\nabla_{\boldsymbol{v}}\cdot\langle L[f]\otimes\nabla_{\boldsymbol{x}} f\rangle_\varepsilon - \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon)].\end{aligned} \tag{4.31}$$

Multiplying (4.31) by $2\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon$ and integrating the resulting equation over $\mathbb{R}^{2d}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon\|^2_{L^2(\mathbb{R}^{2d})} &= -\int_{\mathbb{R}^{2d}} \langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon^2 \nabla_{\boldsymbol{v}}\cdot L[f]\,d\boldsymbol{x}d\boldsymbol{v}\\ &\quad -2\int_{\mathbb{R}^{2d}} \langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon\cdot\langle\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f + f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\\ &\quad -2\int_{\mathbb{R}^{2d}} [\nabla_{\boldsymbol{x}}\cdot\langle\boldsymbol{v}\otimes\nabla_{\boldsymbol{x}} f\rangle_\varepsilon - \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon]\cdot\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\\ &\quad -2\int_{\mathbb{R}^{2d}} [\nabla_{\boldsymbol{v}}\cdot\langle L[f]\otimes\nabla_{\boldsymbol{x}} f\rangle_\varepsilon - \nabla_{\boldsymbol{v}}\cdot(L[f]\otimes\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon)]\cdot\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon d\boldsymbol{x}d\boldsymbol{v}\\ &= \sum_{i=1}^4 H_i.\end{aligned} \tag{4.32}$$

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

We estimate each H_i $(1 \le i \le 4)$ as follows.

$$|H_1| \le \|\nabla_{\boldsymbol{v}} \cdot L[f]\|_{L^\infty(\mathbb{R}^{2d})} \|\langle \nabla_{\boldsymbol{x}} f\rangle_\varepsilon\|^2_{L^2(\mathbb{R}^{2d})} \le C\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{x}} f\|^2_{L^2(\mathbb{R}^{2d})};$$

$$\begin{aligned} |H_2| &\le \|\nabla_{\boldsymbol{x}} f\|_{L^2(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{x}} L[f]\cdot\nabla_{\boldsymbol{v}} f + f\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{v}}\cdot L[f]\|_{L^2(\mathbb{R}^{2d})} \\ &\le C\|\nabla_{\boldsymbol{x}} f\|_{L^2(\mathbb{R}^{2d})}\Big(\|(1+\boldsymbol{v}^2)^{\frac12} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac12}\nabla_{\boldsymbol{v}} f\|_{L^2(\mathbb{R}^{2d})} \\ &\quad + \|f\|_{L^1(\mathbb{R}^{2d})}\|f\|_{L^2(\mathbb{R}^{2d})}\Big); \end{aligned}$$

$$\begin{aligned} |H_3| &\le 2\int_{\mathbb{R}^{2d}}\left|\int_{\mathbb{R}^{2d}}(\boldsymbol{w}-\boldsymbol{v})\cdot\nabla_{\boldsymbol{x}} j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})(\nabla_{\boldsymbol{x}} f)(t,\boldsymbol{z},\boldsymbol{w})d\boldsymbol{z}d\boldsymbol{w}\right||\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon|d\boldsymbol{x}d\boldsymbol{v} \\ &\le C\|\nabla_{\boldsymbol{x}} f\|^2_{L^2(\mathbb{R}^{2d})}; \end{aligned}$$

$$\begin{aligned} |H_4| \le\, & 2\int_{\mathbb{R}^{2d}}\left|\int_{\mathbb{R}^{2d}}\nabla_{\boldsymbol{v}}\cdot L[f](\nabla_{\boldsymbol{x}} f)(t,\boldsymbol{z},\boldsymbol{w})j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})d\boldsymbol{z}d\boldsymbol{w}\cdot\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon\right|d\boldsymbol{x}d\boldsymbol{v} \\ & + 2\int_{\mathbb{R}^{2d}}\Bigg|\int_{\mathbb{R}^{2d}}\nabla_{\boldsymbol{v}} j_\varepsilon(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{v}-\boldsymbol{w})\cdot\Big[\Big(L[f](\boldsymbol{z},\boldsymbol{w})-L[f](\boldsymbol{x},\boldsymbol{v})\Big)\otimes(\nabla_{\boldsymbol{x}} f)(t,\boldsymbol{z},\boldsymbol{w})\Big] \\ & \qquad d\boldsymbol{z}d\boldsymbol{w}\cdot\langle\nabla_{\boldsymbol{x}} f\rangle_\varepsilon\Bigg|d\boldsymbol{x}d\boldsymbol{v} \\ \le\, & C\|f\|_{L^1(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{x}} f\|^2_{L^2(\mathbb{R}^{2d})} \\ & + C\|(1+\boldsymbol{v}^2)^{\frac12} f\|_{L^1(\mathbb{R}^{2d})}\|(1+\boldsymbol{v}^2)^{\frac12}\nabla_{\boldsymbol{x}} f\|_{L^2(\mathbb{R}^{2d})}\|\nabla_{\boldsymbol{x}} f\|_{L^2(\mathbb{R}^{2d})}, \end{aligned}$$

where $(\nabla_{\boldsymbol{x}} f)(t,\boldsymbol{z},\boldsymbol{w})$ denotes the value of $\nabla_{\boldsymbol{x}} f$ at $(t,\boldsymbol{z},\boldsymbol{w})$. Substituting these estimates into (4.32) and integrating the resulting inequality over (t_1,t_2), $\forall t_1,t_2\in[0,T]$, then letting $\varepsilon\to 0$, we obtain

$$\left|\|\nabla_{\boldsymbol{x}} f(t_2)\|^2_{L^2(\mathbb{R}^{2d})} - \|\nabla_{\boldsymbol{x}} f(t_1)\|^2_{L^2(\mathbb{R}^{2d})}\right| \le C\exp(Ce^{CT})|t_2-t_1|, \tag{4.33}$$

where we have used (4.28). Combining (4.30) and (4.33), we know

$$\nabla_{\boldsymbol{x}} f \in C([0,T],L^2(\mathbb{R}^{2d})), \quad \forall T>0.$$

Similarly, we can prove $\nabla_{\boldsymbol{v}} f\in C([0,T],L^2(\mathbb{R}^{2d}), \forall T>0$.

As for the solution sequences $\{f^{\sigma_j}\}$ and $\{g^{\sigma_j}\}$ with initial data f_0 and g_0, respectively, it follows from (4.26) that

$$f^{\sigma_j} \rightharpoonup f, \quad \text{weakly-}\star \text{ in } L^\infty((0,T),X)\ \forall T>0, \text{ as } \sigma_j\to 0 \tag{4.34}$$

and

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001

$$g^{\sigma_j} \rightharpoonup g, \quad \text{weakly-}\star \text{ in } L^\infty((0,T),X)\ \forall T>0, \text{ as } \sigma_j \to 0. \tag{4.35}$$

From Theorem 3.2(2), we deduce that

$$\sup_{0\le t\le T} \|f^{\sigma_j}(t)-g^{\sigma_j}(t)\|_X \le \|f_0-g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T\ge 0. \tag{4.36}$$

Combining (4.34)–(4.36) and letting $\sigma_j \to 0$ lead to

$$\sup_{0\le t\le T} \|f(t)-g(t)\|_X \le \|f_0-g_0\|_X \exp\left(\int_0^T c(t)dt\right), \quad \forall T\ge 0,$$

which amounts to uniqueness of the strong solution. Due to the arbitrariness of T, we can easily prove the unique strong solution $f \in C([0,+\infty),X)$. From (4.28) we infer that

$$\|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f(t)\|_X \le \|(1+\boldsymbol{v}^2)^{\frac{1}{2}} f_0\|_X \exp\left(e^{Ct}-1\right), \quad \text{a.e. } t\ge 0.$$

This completes the proof. □

5. Conclusion

In this paper, we have developed a framework that can be used to establish the well-posedness of weak, strong and classical solutions to the kinetic Cucker–Smale model with or without noise, no matter whether the initial data have compact support or not. Besides, we also rigorously justify the vanishing noise limit, which can be as a counterpart result to the vanishing viscosity method in hyperbolic conservation laws. Therefore we present complete theory for the kinetic Cucker–Smale model except for the large-time behavior of the solution.

Our proof is based on weighted energy estimates and subtle compact analysis. The two weighted Hilbert spaces we introduced and the velocity averaging lemma in kinetic theory play important roles in our analysis. However, the time-asymptotic behavior of the solution is difficult to analyze. Maybe we can begin with some special situations. As for the kinetic Cucker–Smale model with noise, we guess the solution will tend to its steady state, if the initial perturbations are suitably small. This problem will be pursued in our future.

Acknowledgments

The author would like to thank Prof. Seung-Yeal Ha for his comments on the manuscript. He is also indebted to the anonymous referee for valuable advice, which improves the presentation of this paper significantly.

References

[1] Stefano Bianchini, Alberto Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, Ann. of Math. 161 (2005) 223–342.

[2] José Alfredo Canizo, José Antonio Carrillo, Jesús Rosado, A well-posedness theory in measures for some kinetic models of collective motion, Math. Models Methods Appl. Sci. 21 (03) (2011) 515–539.

[3] José Antonio Carrillo, Massimo Fornasier, Jesús Rosado, Giuseppe Toscani, Asymptotic flocking dynamics for the kinetic Cucker–Smale model, SIAM J. Math. Anal. 42 (1) (2010) 218–236.

[4] José Antonio Carrillo, Massimo Fornasier, Giuseppe Toscani, Francesco Vecil, Particle, kinetic, and hydrodynamic models of swarming, in: Mathematical Modeling of Collective Behavior in Socio-economic and Life Sciences, Springer, 2010, pp. 297–336.

[5] Felipe Cucker, Steve Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control 52 (5) (2007) 852–862.

[6] Ronald J. DiPerna, Pierre-Louis Lions, Global weak solutions of Vlasov–Maxwell systems, Comm. Pure Appl. Math. 42 (6) (1989) 729–757.

[7] Ronald J. DiPerna, Pierre-Louis Lions, On the Cauchy problem for Boltzmann equations: global existence and weak stability, Ann. of Math. 130 (2) (1989) 321–366.

[8] Seung-Yeal Ha, Feimin Huang, Yi Wang, A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation, J. Differential Equations 257 (5) (2014) 1333–1371.

[9] Seung-Yeal Ha, Moon-Jin Kang, Bongsuk Kwon, A hydrodynamic model for the interaction of Cucker–Smale particles and incompressible fluid, Math. Models Methods Appl. Sci. 24 (11) (2014) 2311–2359.

[10] Seung-Yeal Ha, Moon-Jin Kang, Bongsuk Kwon, Emergent dynamics for the hydrodynamic Cucker–Smale system in a moving domain, SIAM J. Math. Anal. 47 (5) (2015) 3813–3831.

[11] Seung-Yeal Ha, Jian-Guo Liu, A simple proof of the Cucker–Smale flocking dynamics and mean-field limit, Commun. Math. Sci. 7 (2) (2009) 297–325.

[12] Seung-Yeal Ha, Eitan Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, Kinet. Relat. Models 1 (3) (2008) 415–435.

[13] Eberhard Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (3) (1950) 201–230.

[14] Chunyin Jin, Well posedness for pressureless Euler system with a flocking dissipation in Wasserstein space, Nonlinear Anal. 128 (2015) 412–422.

[15] Trygve K. Karper, Antoine Mellet, Konstantina Trivisa, Existence of weak solutions to kinetic flocking models, SIAM J. Math. Anal. 45 (1) (2013) 215–243.

Please cite this article in press as: C. Jin, Well-posedness of weak and strong solutions to the kinetic Cucker–Smale model, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.10.001