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Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups ☆

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Abstract

In this paper we study the Cauchy problem for the semilinear damped wave equation for the sub-Laplacian on the Heisenberg group. In the case of the positive mass, we show the global in time well-posedness for small data for power like nonlinearities. We also obtain similar well-posedness results for the wave equations for Rockland operators on general graded Lie groups. In particular, this includes higher order operators on $\mathbb{R}^n$ and on the Heisenberg group, such as powers of the Laplacian or the sub-Laplacian. In addition, we establish a new family of Gagliardo–Nirenberg inequalities on a graded Lie groups that play a crucial role in the proof, but which are also of interest on their own: if $\mathbb{G}$ is a graded Lie group of homogeneous dimension Q and $a > 0$, $1 < r < \frac{Q}{a}$, and $1 \leq p \leq q \leq \frac{rQ}{Q-ar}$, then we have the following Gagliardo–Nirenberg type inequality

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^{s}_{\dot{L}^r_a(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})}$$

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for $s=(\frac{1}{p}-\frac{1}{q})(\frac{a}{Q}+\frac{1}{p}-\frac{1}{r})^{-1}\in[0,1]$ provided that $\frac{a}{Q}+\frac{1}{p}-\frac{1}{r}\neq 0$, where $\dot{L}^r_a$ is the homogeneous Sobolev space of order a over L^r. If $\frac{a}{Q}+\frac{1}{p}-\frac{1}{r}=0$, we have $p=q=\frac{rQ}{Q-ar}$, and then the above inequality holds for any $0\leq s\leq 1$.

MSC: 35L71; 58J45; 22E30; 35L75

Keywords: Nonlinear damped wave equation; Heisenberg group; Sub-Laplacian; Rockland operator; Graded group; Gagliardo–Nirenberg inequality

Contents

1. Introduction

In this paper we investigate the global in time well-posedness for the damped wave equation for the sub-Laplacian on the Heisenberg group. Strichartz estimates for the wave equation for the sub-Laplacian on the Heisenberg group have been analysed by Bahouri, Gérard and Xu in [5] where a weak decay rate in dispersive estimates was established. Recently, such results were extended to step 2 stratified Lie groups by Bahouri, Fermanian-Kammerer and Gallagher [4] where it was shown that the decay rate of solution may depend on the dimension of the centre of the group. Wave equations for the full Laplacian on the Heisenberg group have been investigated in [8], [23] where better decay rates have been obtained. We also mention recently published papers [21,22] where attractors were used to study damped hyperbolic equations.

One purpose of this paper is to investigate the global in time well-posedness of the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} \partial_t^2 u(t)-\mathcal{L}u(t)+b\partial_t u(t)+mu(t)=f(u), \quad t>0, \\ u(0)=u_0\in L^2(\mathbb{H}^n), \\ \partial_t u(0)=u_1\in L^2(\mathbb{H}^n), \end{cases} \tag{1.1}$$

with the damping term determined by $b>0$ and with the mass $m>0$, where $\mathbb{H}^n$ is the Heisenberg group and $\mathcal{L}$ is the sub-Laplacian. Consequently, we also establish similar results for a more general setting, namely, when the Heisenberg group $\mathbb{H}^n$ is replaced by a general graded Lie

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group $\mathbb{G}$, and the sub-Laplacian $\mathcal{L}$ is replaced by an arbitrary Rockland operator $\mathcal{R}$, i.e. by an arbitrary left-invariant homogeneous hypoelliptic differential operator.

The nonlinearity f in this paper will be assumed to satisfy, for some $p > 1$, the conditions

$$\begin{cases} f(0) = 0, \\ |f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|. \end{cases} \tag{1.2}$$

In particular, this includes the cases

$$f(u) = \mu |u|^{p-1} u, \quad \text{for some } p > 1, \quad \mu \in \mathbb{C}, \tag{1.3}$$

as well as the more general case of differentiable functions f satisfying

$$|f'(u)| \leq C|u|^{p-1}. \tag{1.4}$$

To fix the notation concerning the equation (1.1), for $n \in \mathbb{N}$, the Heisenberg group $\mathbb{H}^n$ is the manifold $\mathbb{R}^{2n+1}$ endowed with the group structure

$$(x, y, t) \circ (x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)),$$

where (x, y, t) and (x', y', t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}^n$. The sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ is given by

$$\mathcal{L} := \sum_{j=1}^{n} (X_j^2 + Y_j^2), \quad \text{with } X_j := \partial_{x_j} - \frac{y_j}{2}\partial_t, \;\; Y_j := \partial_{y_j} + \frac{x_j}{2}\partial_t. \tag{1.5}$$

In this case, in Theorem 3.2 we will show the global in time well-posedness of the Cauchy problem (1.1):

- for small data $(u_0, u_1) \in H^1_{\mathcal{L}}(\mathbb{H}^n) \times L^2(\mathbb{H}^n)$,
- and for nonlinearities $f(u)$ satisfying (1.2) for $1 < p \leq 1 + 1/n$.

Here $H^1_{\mathcal{L}}(\mathbb{H}^n)$ denotes the sub-Laplacian Sobolev space, analysed by Folland [9]. Consequently, we extend this result beyond the setting of the Heisenberg group and second order operators, in a way that we now describe.

Following Folland and Stein [15], we recall that $\mathbb{G}$ is a graded Lie group if there is a gradation on its Lie algebra $\mathfrak{g}$, i.e. a vector space decomposition

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} V_j \;\text{ such that } [V_i, V_j] \subset V_{i+j},$$

will all but finitely many of V_j being zero, see Section 4.1 for a precise definition. This leads to a family of dilations on it with rational weights, compatible with the group structure. If V_1 generates $\mathfrak{g}$ as an algebra, the group is said to be stratified and the sum of squares of a basis of vector fields in V_1 yields a sub-Laplacian on $\mathbb{G}$. However, non-stratified graded $\mathbb{G}$ may not have a homogeneous sub-Laplacian or Laplacian but they always have so-called Rockland operators. Such operators appeared in the hypoellipticity considerations by Rockland [28] defined by the condition that their infinitesimal representations are injective on smooth vectors. Suitable partial reformulations of these conditions were further proposed by Rockland [28] and Beals [1], until the resolution in [18] by Helffer and Nourrigat of what has become known as the Rockland conjecture, and what we can adopt as the definition here:

Rockland operators are left-invariant homogeneous hypoelliptic differential operators on $\mathbb{G}$.

In fact, the existence of such operators on nilpotent Lie groups singles out the class of graded groups, see [24], [39]. In the realm of homogeneous Lie groups, the graded groups can be also characterised by dilations having rational weights, see [14, Section 4.1.1].

Thus, in our extension of the obtained result from the Heisenberg group to graded Lie groups, we will work with positive Rockland operators $\mathcal{R}$. To give some examples, this setting includes:

- for $\mathbb{G}=\mathbb{R}^n$, $\mathcal{R}$ may be any positive homogeneous elliptic differential operator with constant coefficients. For example, we can take
$$\mathcal{R}=(-\Delta)^m \text{ or } \mathcal{R}=(-1)^m\sum_{j=1}^{n} a_j\left(\frac{\partial}{\partial x_j}\right)^{2m}, \quad a_j>0,\ m\in\mathbb{N};$$
- for $\mathbb{G}=\mathbb{H}^n$ the Heisenberg group, we can take
$$\mathcal{R}=(-\mathcal{L})^m \text{ or } \mathcal{R}=(-1)^m\sum_{j=1}^{n}(a_jX_j^{2m}+b_jY_j^{2m}), \quad a_j,b_j>0,\ m\in\mathbb{N},$$
where $\mathcal{L}$ is the sub-Laplacian and X_j, Y_j are the left-invariant vector fields in (1.5).
- for any stratified Lie group (or homogeneous Carnot group) with vectors $X_1,\dots,X_k$ spanning the first stratum, we can take
$$\mathcal{R}=(-1)^m\sum_{j=1}^{k} a_jX_j^{2m}, \quad a_j>0,$$
so that, in particular, for $m=1$, $\mathcal{R}$ is a positive sub-Laplacian;
- for any graded Lie group $\mathbb{G}\sim\mathbb{R}^n$ with dilation weights $\nu_1,\dots,\nu_n$ let us fix the basis $X_1,\dots,X_n$ of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ satisfying
$$D_rX_j=r^{\nu_j}X_j, \quad j=1,\dots,n,\ r>0,$$
where D_r denote the dilations on the Lie algebra. If ν_0 is any common multiple of $\nu_1,\dots,\nu_n$, the operator
$$\mathcal{R}=\sum_{j=1}^{n}(-1)^{\frac{\nu_0}{\nu_j}}a_jX_j^{2\frac{\nu_0}{\nu_j}}, \quad a_j>0,$$
is a Rockland operator of homogeneous degree $2\nu_0$. The Rockland operator can be also adapted to a special selection of vector fields generating the Lie algebra in a suitable way,

such as the vector fields from the first stratum on the stratified Lie groups. We refer to [14, Section 4.1.2] for many other examples and a discussion of Rockland operators.

In the setting of a general graded Lie group $\mathbb{G}$ of homogeneous dimension Q, which is defined by

$$Q = \nu_1 + \ldots + \nu_n,$$

we consider the nonlinear damped wave equation for a positive Rockland operator $\mathcal{R}$ of homogeneous degree ν,

$$\begin{cases} \partial_t^2 u(t) + \mathcal{R}u(t) + b\partial_t u(t) + mu(t) = F(u, \{\mathcal{R}^{j/\nu}u\}_{j=1}^{[\frac{\nu}{2}]-1}), \quad t > 0, \\ u(0) = u_0 \in L^2(\mathbb{G}), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{G}), \end{cases} \tag{1.6}$$

with the damping term determined by $b > 0$ and with the mass $m > 0$. Here $[\frac{\nu}{2}]$ stands for the integer part of $\frac{\nu}{2}$. In this case, in Theorem 4.6 and Theorem 4.7 we will show the global in time well-posedness of the Cauchy problem (1.6):

- for small data $(u_0, u_1) \in H^{\nu/2}(\mathbb{G}) \times L^2(\mathbb{G})$,
- and for nonlinearities $F : \mathbb{C}^{[\nu/2]} \to \mathbb{C}$ with the following property:

$$\begin{cases} F(0) = 0, \\ |F(U) - F(V)| \leq C(|U|^{p-1} + |V|^{p-1})|U - V|, \end{cases} \tag{1.7}$$

where $U = (\{\mathcal{R}^{j/\nu}u\}_{j=0}^{[\frac{\nu}{2}]-1})$, for $1 < p \leq 1 + \frac{2}{Q-2}$.

Here $H^{\nu/2}(\mathbb{H}^n)$ denotes the Sobolev space of order $\nu/2$ associated to $\mathcal{R}$, analysed in [11] and in [14, Section 4.4].

In the case of the Heisenberg group $\mathbb{G} = \mathbb{H}^n$ and $\mathcal{R} = -\mathcal{L}$, we have $\nu = 2$ and $Q = 2n + 2$, and this result recaptures the first result in Theorem 3.2 in this setting. Moreover, on stratified groups, i.e. with $\nu = 2$, this gives the class of semilinear equations in Theorem 4.6.

However, to simplify the exposition, we give a detailed proof in the case of the sub-Laplacian on the Heisenberg group, and then indicate the necessary modifications for the case of general positive Rockland operators on general graded Lie groups.

In both cases of $\mathbb{H}^n$ and more general graded Lie groups $\mathbb{G}$, our proof relies on the group Fourier analysis on $\mathbb{G}$ to obtain the exponential time decay for solutions to the linear problem. This is possible due to the inclusion of positive mass term $m > 0$ leading to the separation of the spectrum of $\mathcal{R}$ and of its infinitesimal representations from zero. Consequently, the nonlinear analysis relies on the application of the Gagliardo–Nirenberg inequality on $\mathbb{G}$. While such inequality is well-known on the Heisenberg group $\mathbb{H}^n$, the known graded group versions in [3] or on [14] are not suitable for our analysis. Thus, in Theorem 4.2 we derive the necessary version of the Gagliardo–Nirenberg inequality based on the graded group version of Sobolev inequality

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established in [14]. More generally, we show that if $\mathbb{G}$ is a graded Lie group of homogeneous dimension Q and

$$a > 0, \quad 1 < r < \frac{Q}{a} \text{ and } 1 \leq p \leq q \leq \frac{rQ}{Q - ar},$$

then we have the following Gagliardo–Nirenberg type inequality

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^s_{\dot{L}^r_a(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})} \simeq \|\mathcal{R}^{a/\nu} u\|^s_{L^r(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})}, \tag{1.8}$$

for $s = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{a}{Q} + \frac{1}{p} - \frac{1}{r}\right)^{-1} \in [0, 1]$ provided that $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} \neq 0$. Here $\dot{L}^r_a(\mathbb{G})$ is the homogeneous Sobolev space of order a over $L^r(\mathbb{G})$, and we refer to [11] and [14, Section 4.4] for an extensive analysis of these spaces and their properties in the setting of general graded Lie groups.

If $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} = 0$, we have $p = q = \frac{rQ}{Q-ar}$, and then (1.8) holds for any $0 \leq s \leq 1$.

The Fourier analysis we use follows the pseudo-differential analysis as described, for example, in [10], [2] or [13] in the case of the Heisenberg group, and in [14] on general graded Lie groups, see also [12].

The similar strategy of obtaining L^2-estimates for solutions of linear problems has been used in [16] in the analysis of weakly hyperbolic wave equations for the sub-Laplacians on compact Lie groups. Some techniques of similar type also appear in the analysis of general operators with discrete spectrum and with time-dependent coefficients in [34–36] by using nonharmonic analysis developed in [32,7,33,20,37]. Estimates in L^p for solution of the wave equation for the sub-Laplacian on the Heisenberg group were considered in [25], and on groups of Heisenberg type in [26]. The potential theory and functional estimates in the setting of stratified groups have been recently analysed in [30,31,29].

Throughout this paper we will often use the notation $\lesssim$ instead of $\leq$ to avoid repeating the constants which are not dependent on the main parameters, especially, on functions appearing in the estimates.

2. Linear damped wave equation on the Heisenberg group

In what follows, we will need some elements of the analysis on the Heisenberg group $\mathbb{H}^n$. It will be convenient for us to follow the notations from [14, Chapter 6] to which we refer for further details. We start by recalling the definition of the group Fourier transform on $\mathbb{H}^n$. For $f \in \mathcal{S}(\mathbb{H}^n)$ we denote its group Fourier transform by

$$\widehat{f}(\lambda) := \int_{\mathbb{H}^n} f(x)\pi_\lambda(x)^* dx \tag{2.1}$$

with the Schrödinger representations

$$\pi_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{2.2}$$

for all $\lambda \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$. The Fourier inversion formula then takes the form

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$$f(x)=\int\limits_{\lambda\in\mathbb{R}^*}\mathrm{Tr}[\widehat{f}(\lambda)\pi_\lambda(x)]\,|\lambda|^n d\lambda, \tag{2.3}$$

where Tr is the trace operator. The Plancherel formula becomes

$$\|f\|^2_{L^2(\mathbb{H}^n)}=\int\limits_{\lambda\in\mathbb{R}^*}\|\widehat{f}(\lambda)\|^2_{\mathrm{HS}[L^2(\mathbb{R}^n)]}\,|\lambda|^n d\lambda, \tag{2.4}$$

where $\|\cdot\|_{\mathrm{HS}[L^2(\mathbb{R}^n)]}$ is the Hilbert–Schmidt norm on $L^2(\mathbb{R}^n)$.

Now, we deal with the linear case of the Cauchy problem (1.1), that is,

$$\begin{cases} \partial_t^2 u(t,z)-\mathcal{L}u(t)+b\partial_t u(t,z)+mu(t,z)=0, \\ \hfill u(0,z)=u_0(z), \\ \hfill \partial_t u(0,z)=u_1(z), \\ \hfill \text{for all} \quad t>0 \quad \text{and} \quad z\in\mathbb{H}^n. \end{cases} \tag{2.5}$$

By acting by the group Fourier transform on this equation, we obtain

$$\begin{cases} \partial_t^2\widehat{u}(t,\lambda)+\sigma_{\mathcal{L}}(\lambda)\widehat{u}(t,\lambda)+b\partial_t\widehat{u}(t,\lambda)+m\widehat{u}(t,\lambda)=0, \quad t>0, \\ \hfill \widehat{u}(0,\lambda)=\widehat{u}_0(\lambda), \\ \hfill \partial_t\widehat{u}(0,\lambda)=\widehat{u}_1(\lambda), \end{cases} \tag{2.6}$$

where $\sigma_{\mathcal{L}}(\lambda)$ is the symbol of $-\mathcal{L}$. It takes the form

$$\sigma_{\mathcal{L}}(\lambda)=|\lambda|\mathrm{H}_w\equiv|\lambda|\sum_{j=1}^n(-\partial^2_{w_j}+w_j^2), \tag{2.7}$$

where $\mathrm{H}_w:=\sum_{j=1}^n(-\partial^2_{w_j}+w_j^2)$ is the harmonic operator acting on $L^2(\mathbb{R}^n)$, see e.g. [14, Section 6.2.1].

Since the harmonic oscillator H_w is essentially self-adjoint in $L^2(\mathbb{R}^n)$ and, its system of eigenfunctions $\{\psi_k\}_{k=1}^\infty$ is a basis in $L^2(\mathbb{R}^n)$, we have an ordered set of positive numbers $\{\mu_k\}_{k=1}^\infty$ such that

$$\mathrm{H}_w\psi_k(w)=\mu_k\psi_k(w), \quad w\in\mathbb{R}^n,$$

for all $k\in\mathbb{N}$. More precisely, H_w has eigenvalues

$$\lambda_k=\sum_{j=1}^n(2k_j+1), \quad k=(k_1,\dots,k_n)\in\mathbb{N}^n,$$

with corresponding eigenfunctions

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$$e_k(w) = \prod_{j=1}^{n} P_{k_j}(w_j) \mathrm{e}^{-\frac{|w|^2}{2}},$$

which form an orthogonal system in $L^2(\mathbb{R}^n)$. Here, $P_m(\cdot)$ is the m–th order Hermite polynomial and

$$P_m(t) = c_m \mathrm{e}^{\frac{|t|^2}{2}} \left(t - \frac{d}{dt}\right)^m \mathrm{e}^{-\frac{|t|^2}{2}}, \; t > 0, \; c_m = 2^{-m/2}(m!)^{-1/2}\pi^{-1/4}.$$

For more details on these see e.g. [27].

Consequently, for $(k, l) \in \mathbb{N} \times \mathbb{N}$, denoting

$$\widehat{u}(t,\lambda)_{kl} := (\widehat{u}(t,\lambda)\psi_l, \psi_k)_{L^2(\mathbb{R}^n)}, \tag{2.8}$$

we see that the equation (2.6) is reduced to the system

$$\begin{cases} \partial_t^2 \widehat{u}(t,\lambda)_{kl} + b\partial_t \widehat{u}(t,\lambda)_{kl} + (|\lambda|\mu_k + m)\widehat{u}(t,\lambda)_{kl} = 0, \quad t > 0, \\ \widehat{u}(0,\lambda)_{kl} = \widehat{u}_0(\lambda)_{kl} \in L^2(\mathbb{R}^n), \\ \partial_t \widehat{u}(0,\lambda)_{kl} = \widehat{u}_1(\lambda)_{kl} \in L^2(\mathbb{R}^n), \end{cases} \tag{2.9}$$

for each $\lambda \in \mathbb{R}^*$.

Now, we fix $\lambda \in \mathbb{R}^*$ and $(k, l) \in \mathbb{N} \times \mathbb{N}$. By solving the second order ordinary differential equation (2.9) with constant coefficients, we get the estimates

$$|\widehat{u}(t,\lambda)_{kl}| \lesssim \mathrm{e}^{-\frac{b}{2}t} \left[|\widehat{u}_0(\lambda)_{kl}| + (b^2/4 - |\lambda|\mu_k - m)^{-1/2} |\widehat{u}_1(\lambda)_{kl}| \right], \tag{2.10}$$

for $2\sqrt{|\lambda|\mu_k + m} < b$, and

$$|\widehat{u}(t,\lambda)_{kl}| \lesssim \mathrm{e}^{-\frac{b}{2}t} \left[\left(1 + \frac{b}{2}t\right) |\widehat{u}_0(\lambda)_{kl}| + t|\widehat{u}_1(\lambda)_{kl}| \right], \tag{2.11}$$

for $2\sqrt{|\lambda|\mu_k + m} = b$, and

$$|\widehat{u}(t,\lambda)_{kl}| \lesssim \mathrm{e}^{-(\frac{b}{2} - \sqrt{\frac{b^2}{4} - |\lambda|\mu_k - m})t} \left[|\widehat{u}_0(\lambda)_{kl}| + (|\lambda|\mu_k + m - b^2/4)^{-1/2} |\widehat{u}_1(\lambda)_{kl}| \right], \tag{2.12}$$

for $b < 2\sqrt{|\lambda|\mu_k + m}$. Thus, there exists a positive constant $\delta > 0$ such that in all the cases we have

$$\begin{aligned} &| \, |\lambda|\mu_k + m - b^2/4|^{1/2} |\widehat{u}(t,\lambda)_{kl}| \\ &\quad \lesssim \mathrm{e}^{-\delta t} \left[| \, |\lambda|\mu_k + m - b^2/4|^{1/2} |\widehat{u}_0(\lambda)_{kl}| + |\widehat{u}_1(\lambda)_{kl}| \right]. \end{aligned} \tag{2.13}$$

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Consequently, we obtain

$$\begin{aligned}\|(1-\sigma_{\mathcal{L}}(\lambda))^{1/2}|\widehat{u}(t,\lambda)\|^2_{\mathrm{HS}} &= \sum_{k,l=1}^{\infty}(1+|\lambda|\mu_k)\widehat{u}(t,\lambda)_{kl}|^2 \\ &\lesssim \mathrm{e}^{-\delta t}\left[\sum_{k,l=1}^{\infty}(1+|\lambda|\mu_k)|\widehat{u}_0(\lambda)_{kl}|^2+\sum_{k,l=1}^{\infty}|\widehat{u}_1(\lambda)_{kl}|^2\right] \\ &\lesssim \mathrm{e}^{-\delta t}\left[\|(1-\sigma_{\mathcal{L}}(\lambda))^{1/2}\widehat{u}_0(\lambda)\|^2_{\mathrm{HS}}+\|\widehat{u}_0(\lambda)\|^2_{\mathrm{HS}}\right].\end{aligned} \tag{2.14}$$

The same estimates work if we multiply the equation (2.9) by powers of the spectral decomposition of the symbol of the sub-Laplacian.

The Sobolev spaces $H^s_{\mathcal{L}}$, $s\in\mathbb{R}$, associated to $\mathcal{L}$, is defined as

$$H^s_{\mathcal{L}}(\mathbb{H}^n):=\left\{f\in\mathcal{D}'(\mathbb{H}^n):(I-\mathcal{L})^{s/2}f\in L^2(\mathbb{H}^n)\right\},$$

with the norm $\|f\|_{H^s_{\mathcal{L}}(\mathbb{H}^n)}:=\|(I-\mathcal{L})^{s/2}f\|_{L^2(\mathbb{H}^n)}$. We refer to Folland [9] for a thorough analysis of these spaces and their properties.

For the solution of the system (2.9), for each $\widehat{u}(t,\lambda)_{kl}$ for fixed $(k,l)\in\mathbb{N}\times\mathbb{N}$, we obtain an explicit formula

$$\begin{aligned}\widehat{u}(t,\lambda)_{kl} =&[\left(\frac{b}{4i\sqrt{|\lambda|\mu_k+m-b^2/4}}+\frac{1}{2}\right)\mathrm{e}^{(-b/2+i\sqrt{|\lambda|\mu_k+m-b^2/4})t} \\ &+\left(\frac{ib}{4\sqrt{|\lambda|\mu_k+m-b^2/4}}+\frac{1}{2}\right)\mathrm{e}^{(-b/2-i\sqrt{|\lambda|\mu_k+m-b^2/4})t}]\widehat{u}_0(\lambda)_{kl} \\ &+[\frac{1}{2i\sqrt{|\lambda|\mu_k+m-b^2/4}}\mathrm{e}^{(-b/2+i\sqrt{|\lambda|\mu_k+m-b^2/4})t} \\ &+\frac{i}{2\sqrt{|\lambda|\mu_k+m-b^2/4}}\mathrm{e}^{(-b/2-i\sqrt{|\lambda|\mu_k+m-b^2/4})t}]\widehat{u}_1(\lambda)_{kl}.\end{aligned} \tag{2.15}$$

To obtain similar Sobolev estimates for negative s we consider cases $b<2\sqrt{|\lambda|\mu_k+m}$ and $2\sqrt{|\lambda|\mu_k+m}<b$, and then the case $2\sqrt{|\lambda|\mu_k+m}\approx b$.

When $b<2\sqrt{|\lambda|\mu_k+m}$ let us denote $a_k:=\sqrt{|\lambda|\mu_k+m-b^2/4}$. Then by the direct calculations we get

$$\widehat{u}(t,\lambda)_{kl} =\mathrm{e}^{-b/2t}\left[\frac{b\sin(a_kt)}{2a_k}\right]\widehat{u}_0(\lambda)_{kl}+\mathrm{e}^{-b/2t}\left[\frac{\sin(a_kt)}{a_k}\right]\widehat{u}_1(\lambda)_{kl}. \tag{2.16}$$

When $b>2\sqrt{|\lambda|\mu_k+m}$ we denote $c_k:=\sqrt{b^2/4-|\lambda|\mu_k-m}$. Then we obtain

$$\widehat{u}(t,\lambda)_{kl} =\mathrm{e}^{-b/2t}\left[\frac{b\sinh(c_kt)}{2c_k}+\cosh(c_kt)\right]\widehat{u}_0(\lambda)_{kl}+\mathrm{e}^{-b/2t}\left[\frac{\sinh(c_kt)}{c_k}\right]\widehat{u}_1(\lambda)_{kl}. \tag{2.17}$$

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We observe that

$$\begin{aligned} \frac{\sin(a_k t)}{a_k} &= t + o(1), \text{ as } a_k \sim 0; \\ \frac{b\sinh(c_k t)}{2c_k} + \cosh(c_k t) &= \frac{bt}{2} + 1 + o(1), \text{ as } c_k \sim 0; \\ \frac{\sinh(c_k t)}{c_k} &= t + o(1), \text{ as } c_k \sim 0. \end{aligned} \tag{2.18}$$

Let us now define a characteristic function $\chi \in C_0^\infty([0,\infty))$ as

$$\chi(t) = \begin{cases} 1, & |t - b^2/4 + m| < 1; \\ 0, & |t - b^2/4 + m| > 2. \end{cases}$$

Then for any $s \in \mathbb{R}$ we have

$$\|w\|_{H^s_{\mathcal{L}}(\mathbb{H}^n)} \simeq \|\chi(\mathcal{L})w\|_{H^s_{\mathcal{L}}(\mathbb{H}^n)} + \|(1-\chi(\mathcal{L}))w\|_{H^s_{\mathcal{L}}(\mathbb{H}^n)}. \tag{2.19}$$

Since for any $s_1, s_2 \in \mathbb{R}$ we have

$$\|\chi(\mathcal{L})w\|_{H^{s_1}_{\mathcal{L}}(\mathbb{H}^n)} \cong \|\chi(\mathcal{L})w\|_{H^{s_2}_{\mathcal{L}}(\mathbb{H}^n)}, \tag{2.20}$$

in view of (2.18), the estimate (2.11) extends to the estimate in Sobolev spaces. The estimate for $\|(1-\chi(\mathcal{L}))w\|_{H^s_{\mathcal{L}}(\mathbb{H}^n)}$ for any s works in the same way as for $s \geq 1$. Therefore, summarising the arguments above, we obtain

Proposition 2.1. *Let $s \in \mathbb{R}$ and assume that $u_0 \in H^s_{\mathcal{L}}(\mathbb{H}^n)$ and $u_1 \in H^{s-1}_{\mathcal{L}}(\mathbb{H}^n)$. Then there exists a positive constant $\delta > 0$ such that*

$$\begin{aligned} \|u(t,z)\|^2_{H^s_{\mathcal{L}}(\mathbb{H}^n)} &= \|(1-\sigma_{\mathcal{L}}(\lambda))^{s/2}\widehat{u}(t,\lambda)\|^2_{L^2(\widehat{\mathbb{H}^n})} \\ &= \int_{\mathbb{R}^*} \|(1-\sigma_{\mathcal{L}}(\lambda))^{s/2}\widehat{u}(t,\lambda)\|^2_{\mathrm{HS}} |\lambda|^n d\lambda \\ &\lesssim e^{-2\delta t}(\|u_0\|^2_{H^s_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|^2_{H^{s-1}_{\mathcal{L}}(\mathbb{H}^n)}) \end{aligned} \tag{2.21}$$

holds for all $t > 0$.

Moreover, for any $\alpha \in \mathbb{N}_0$ we have

$$\begin{aligned} \|\partial_t^\alpha u(t,z)\|^2_{H^s_{\mathcal{L}}(\mathbb{H}^n)} &= \|(1-\sigma_{\mathcal{L}}(\lambda))^{s/2}\partial_t^\alpha\widehat{u}(t,\lambda)\|^2_{L^2(\widehat{\mathbb{H}^n})} \\ &= \int_{\mathbb{R}^*} \|(1-\sigma_{\mathcal{L}}(\lambda))^{(\alpha+s)/2}\widehat{u}(t,\lambda)\|^2_{\mathrm{HS}} |\lambda|^n d\lambda \\ &\lesssim e^{-2\delta t}(\|u_0\|^2_{H^{\alpha+s}_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|^2_{H^{\alpha+s-1}_{\mathcal{L}}(\mathbb{H}^n)}) \end{aligned}$$

for all $t > 0$.

3. Semilinear damped wave equations on the Heisenberg group

In this section we consider the semilinear wave equation for the sub-Laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$:

$$\begin{cases} \partial_t^2 u(t) - \mathcal{L}u(t) + b\partial_t u(t) + mu(t) = f(u), \quad t > 0, \\ u(0) = u_0 \in L^2(\mathbb{H}^n), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{H}^n). \end{cases} \tag{3.1}$$

The main case of interest may be of

$$f(u) = \mu |u|^{p-1} u, \tag{3.2}$$

for $p > 1$ and $\mu \in \mathbb{C}$, but we can treat a more general situation of f satisfying conditions (1.2), see also (3.4).

We now recall the Gagliardo–Nirenberg inequality on the Heisenberg group $\mathbb{H}^n$, see e.g. Folland [9] and Varopoulos [40], and also [6] for derivation of the best constants there:

Proposition 3.1. *Let $n \geq 1$, $2 \leq q \leq 2 + 2/n$, and let $Q := 2n + 2$ be the homogeneous dimension of $\mathbb{H}^n$. Then for $\theta = \frac{Q(q-2)}{2q}$ the following Gagliardo–Nirenberg inequality is true*

$$\|u\|_{L^q(\mathbb{H}^n)} \lesssim \|\nabla_{\mathbb{H}^n} u\|^{\theta}_{L^2(\mathbb{H}^n)} \|u\|^{1-\theta}_{L^2(\mathbb{H}^n)}, \tag{3.3}$$

where $\nabla_{\mathbb{H}^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$ is the horizontal gradient on $\mathbb{H}^n$.

We now formulate our main result for the Heisenberg group $\mathbb{H}^n$.

Theorem 3.2. *Let $b > 0$ and $m > 0$. Assume that f satisfies the properties*

$$\begin{cases} f(0) = 0, \\ |f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|, \end{cases} \tag{3.4}$$

for some $1 < p \leq 1 + 1/n$. Assume that the Cauchy data $u_0 \in H^1_{\mathcal{L}}(\mathbb{H}^n)$ and $u_1 \in L^2(\mathbb{H}^n)$ satisfy

$$\|u_0\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|_{L^2(\mathbb{H}^n)} \leq \varepsilon. \tag{3.5}$$

Then, there exists a small positive constant $\varepsilon_0 > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t^2 u(t) - \mathcal{L}u(t) + b\partial_t u(t) + mu(t) = f(u), \quad t > 0, \\ u(0) = u_0 \in H^1_{\mathcal{L}}(\mathbb{H}^n), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{H}^n), \end{cases}$$

has a unique global solution $u \in C(\mathbb{R}_+; H^1_{\mathcal{L}}(\mathbb{H}^n)) \cap C^1(\mathbb{R}_+; L^2(\mathbb{H}^n))$ for all $0 < \varepsilon \leq \varepsilon_0$.

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Moreover, there is a positive number $\delta_0 > 0$ *such that*

$$\|\partial_t^\alpha \mathcal{L}^\beta u(t)\|_{L^2(\mathbb{H}^n)} \lesssim e^{-\delta_0 t}, \tag{3.6}$$

for $(\alpha, \beta) = (0, 0)$, *or* $(\alpha, \beta) = (0, 1/2)$, *or* $(\alpha, \beta) = (1, 0)$.

As noted in the introduction, an example of f satisfying (3.4) is given by (3.2) or, more generally, by differentiable functions f such that

$$|f'(u)| \leq C|u|^{p-1}.$$

Proof of Theorem 3.2. Let us consider the closed subset Z of the space $C^1(\mathbb{R}_+;\ H^1_{\mathcal{L}}(\mathbb{H}^n))$ defined as

$$Z := \{u \in C^1(\mathbb{R}_+;\ H^1_{\mathcal{L}}(\mathbb{H}^n)) :\ \|u\|_Z \leq L\},$$

with

$$\|u\|_Z := \sup_{t\geq 0}\{(1+t)^{-1/2} e^{\delta t}(\|u(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|\mathcal{L}^{1/2} u(t,\cdot)\|_{L^2(\mathbb{H}^n)})\},$$

where $L > 0$ will be specified later. Now we define the mapping Γ on Z by

$$\Gamma[u](t) := u_{\mathrm{lin}}(t) + \int_0^t K[f(u)](t-\tau)d\tau, \tag{3.7}$$

where u_{lin} is the solution of the linear equation, and $K[f]$ is the solution of the following linear problem:

$$\begin{cases} \partial_t^2 w(t) - \mathcal{L}w(t) + b\partial_t w(t) + mw(t) = 0, \quad t > 0, \\ w(0) = 0, \\ \partial_t w(0) = f. \end{cases}$$

We claim that

$$\|\Gamma[u]\|_Z \leq L \tag{3.8}$$

for all $u \in Z$ and

$$\|\Gamma[u] - \Gamma[v]\|_Z \leq \frac{1}{r}\|u - v\|_Z \tag{3.9}$$

for all $u, v \in Z$ with $r > 1$. Once we proved inequalities (3.8) and (3.9), it follows that Γ is a contraction mapping on Z. The Banach fixed point theorem then implies that Γ has a unique fixed point in Z. It means that there exists a unique global solution u of the equation

$$u = \Gamma[u] \text{ in } Z,$$

which also gives the solution to (3.1). So, we now concentrate on proving (3.8) and (3.9).

Recalling the second assumption in (3.4) on f, namely,

$$|f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|,$$

applying it to functions $u = u(t)$ and $v = v(t)$ we get

$$\|(f(u) - f(v))(t,\cdot)\|^2_{L^2(\mathbb{H}^n)} \leq C \int_{\mathbb{H}^n} (|u(t,z)|^{p-1} + |v(t,z)|^{p-1})^2 |u(t,z) - v(t,z)|^2 dz.$$

Consequently, by the Hölder inequality, we get

$$\|(f(u) - f(v))(t,\cdot)\|^2_{L^2(\mathbb{H}^n)} \leq C(\|u(t,\cdot)\|^{p-1}_{L^{2p}(\mathbb{H}^n)} + \|v(t,\cdot)\|^{p-1}_{L^{2p}(\mathbb{H}^n)})^2 \|(u-v)(t,\cdot)\|^2_{L^{2p}(\mathbb{H}^n)}$$

since $\frac{1}{\frac{p}{p-1}} + \frac{1}{p} = 1$. By the Gagliardo–Nirenberg inequality (3.3), and by Young's inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b$$

for $0 \leq \theta \leq 1$, $a, b \geq 0$, we obtain

$$\begin{aligned} \|(f(u) - f(v))(t,\cdot)\|_{L^2(\mathbb{H}^n)} \leq C\Big[&\left(\|\mathcal{L}^{1/2} u(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|u(t,\cdot)\|_{L^2(\mathbb{H}^n)}\right)^{p-1} \\ &+ \left(\|\mathcal{L}^{1/2} v(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|v(t,\cdot)\|_{L^2(\mathbb{H}^n)}\right)^{p-1} \Big] \\ &\times \left(\|\mathcal{L}^{1/2}(u-v)(t,\cdot)\|_{L^2(\mathbb{H}^n)} + \|(u-v)(t,\cdot)\|_{L^2(\mathbb{H}^n)}\right). \end{aligned} \tag{3.10}$$

Recalling that $\|u\|_Z \leq L$ and $\|v\|_Z \leq L$, from (3.10) we get

$$\|(f(u) - f(v))(t,\cdot)\|_{L^2(\mathbb{H}^n)} \leq C(1+t)^{p/2} \mathrm{e}^{-\delta p t} L^{p-1} \|u - v\|_Z. \tag{3.11}$$

By putting $v = 0$ in (3.11), and using that $f(0) = 0$, we also have

$$\|f(u)(t,\cdot)\|_{L^2(\mathbb{H}^n)} \leq C(1+t)^{p/2} \mathrm{e}^{-\delta p t} L^p. \tag{3.12}$$

Now, let us estimate the integral operator

$$J[u](t,z) := \int_0^t K[f(u(\tau,z))](t-\tau) d\tau. \tag{3.13}$$

More precisely, for $\alpha = 0, 1$ and for all $\beta \geq 0$ we have

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$$|\partial_t^\alpha \mathcal{L}^\beta J[u](t,z)|^2 \leq \left| \int_0^t \partial_t^\alpha \mathcal{L}^\beta K[f(u(\tau,z))](t-\tau)d\tau \right|^2$$
$$\leq \left(\int_0^t \left| \partial_t^\alpha \mathcal{L}^\beta K[f(u(\tau,z))](t-\tau) \right| d\tau \right)^2$$
$$\leq t \int_0^t \left| \partial_t^\alpha \mathcal{L}^\beta K[f(u(\tau,z))](t-\tau) \right|^2 d\tau.$$

Then by using Proposition 2.1, for $(\alpha,\beta) = (0,0)$, $(\alpha,\beta) = (0,1/2)$ and $(\alpha,\beta) = (1,0)$ we get

$$\|\partial_t^\alpha \mathcal{L}^\beta J[u](t,\cdot)\|^2_{L^2(\mathbb{H}^n)} \leq t \int_0^t \|\partial_t^\alpha \mathcal{L}^\beta K[f(u(\tau,z))](t-\tau)\|^2_{L^2(\mathbb{H}^n)} d\tau$$
$$\leq Ct \int_0^t \mathrm{e}^{-2\delta(t-\tau)} \|f(u(\tau,\cdot))\|^2_{L^2(\mathbb{H}^n)} d\tau \tag{3.14}$$
$$= Ct\mathrm{e}^{-2\delta t} \int_0^t \mathrm{e}^{2\delta\tau} \|f(u(\tau,\cdot))\|^2_{L^2(\mathbb{H}^n)} d\tau.$$

Thus, using (3.11) and (3.12), we obtain from (3.14) that

$$\|\partial_t^\alpha \mathcal{L}^\beta (J[u]-J[v])(t,\cdot)\|_{L^2(\mathbb{H}^n)} \leq Ct^{1/2}\mathrm{e}^{-\delta t} L^{p-1}\|u-v\|_Z, \tag{3.15}$$

and

$$\|\partial_t^\alpha \mathcal{L}^\beta J[u](t,\cdot)\|_{L^2(\mathbb{H}^n)} \leq Ct^{1/2}\mathrm{e}^{-\delta t} L^p, \tag{3.16}$$

with the estimates (3.15)–(3.16) holding for $(\alpha,\beta) = (0,0)$, $(\alpha,\beta) = (0,1/2)$ and $(\alpha,\beta) = (1,0)$.

Consequently, by the definition of $\Gamma[u]$ in (3.7) and using Proposition 2.1 for the first term and estimates for $\|J[u]\|_Z$ for the second term below, we obtain

$$\begin{aligned} \|\Gamma[u]\|_Z &\leq \|u_{\text{lin}}\|_Z + \|J[u]\|_Z \\ &\leq C_1(\|u_0\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|_{L^2(\mathbb{H}^n)}) + C_2L^p, \end{aligned} \tag{3.17}$$

for some $C_1 > 0$ and $C_2 > 0$.

Moreover, in the similar way, we can estimate

$$\|\Gamma[u]-\Gamma[v]\|_Z \leq \|J[u]-J[v]\|_Z \leq C_3L^{p-1}\|u-v\|_Z, \tag{3.18}$$

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for some $C_3 > 0$. Taking some $r > 1$, we choose $L := rC_1(\|u_0\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|_{L^2(\mathbb{H}^n)})$ with sufficiently small $\|u_0\|_{H^1_{\mathcal{L}}(\mathbb{H}^n)} + \|u_1\|_{L^2(\mathbb{H}^n)} < \varepsilon$ so that

$$C_2 L^p \le \frac{1}{r} L, \quad C_3 L^{p-1} \le \frac{1}{r}. \tag{3.19}$$

Then estimates (3.17)–(3.19) imply the desired estimates (3.8) and (3.9). This means that we can apply the fixed point theorem for the existence of solutions.

The estimate (3.6) follows from (3.14). Theorem 3.2 is now proved. □

4. Nonlinear damped wave equations on graded Lie groups

In this section for a positive Rockland operator $\mathcal{R}$ we will derive the well-posedness results for the semilinear and then for nonlinear wave equation for small Cauchy data. At first, we start by recalling some definitions and notations following Folland and Stein [15] or [14, Section 3.1]. We also establish a new family of Gagliardo–Nirenberg inequalities on graded Lie groups.

4.1. *Gagliardo–Nirenberg inequalities*

A Lie algebra $\mathfrak{g}$ is called graded when it is endowed with a vector space decomposition

$$\mathfrak{g} = \bigoplus_{j=1}^{\infty} V_j \ \text{ such that } [V_i, V_j] \subset V_{i+j},$$

and where all but finitely many of V_j's are zero. Consequently, a connected simply connected Lie group $\mathbb{G}$ is called graded if its Lie algebra $\mathfrak{g}$ is graded. A special case of stratified $\mathbb{G}$ arises when the first stratum V_1 generates $\mathfrak{g}$ as an algebra.

Graded Lie groups are necessarily nilpotent. Moreover, they are also homogeneous Lie groups with a canonical choice of dilations. Namely, let us define the operator A by setting $AX = jX$ for $X \in V_j$. Then the dilations on $\mathfrak{g}$ are defined by

$$D_r := \operatorname{Exp}(A \ln r), \ r > 0.$$

The *homogeneous dimension* Q of $\mathbb{G}$ is defined by

$$Q := \nu_1 + \ldots + \nu_n = \operatorname{Tr} A.$$

From now on let $\mathbb{G}$ be a graded Lie group. Rockland operators have been originally defined in [28] through the representation theoretic language. Following [14, Definition 4.1.1], we say that $\mathcal{R}$ is a Rockland operator on $\mathbb{G}$ if $\mathcal{R}$ is a left-invariant differential operator which is homogeneous of a positive order $\nu \in \mathbb{N}$ and satisfies the following Rockland condition:

- for all representations $\pi \in \widehat{\mathbb{G}}$, excluding the trivial one, the operator $\pi(\mathcal{R})$ is injective on $\mathcal{H}_\pi^\infty$, namely, from

$$\pi(\mathcal{R})v = 0$$

it follows that $v = 0$ for all $v \in \mathcal{H}_\pi^\infty$.

Here $\widehat{\mathbb{G}}$ denotes the unitary dual of the graded Lie group $\mathbb{G}$, $\mathcal{H}_\pi^\infty$ is the space of smooth vectors of the representation $\pi \in \widehat{\mathbb{G}}$, and $\pi(\mathcal{R})$ is the infinitesimal representation (or the symbol) of $\mathcal{R}$ as an element of the universal enveloping algebra of $\mathbb{G}$, see [14, Definition 1.7.2]. For more information on graded Lie groups and Rockland operators we refer to [14, Chapter 4].

It has been shown by Helffer and Nourrigat in [18] that a left-invariant differential operator $\mathcal{R}$ of homogeneous positive degree $\nu \in \mathbb{N}$ satisfies the Rockland condition if and only if it is hypoelliptic. Such operators are called Rockland operators.

So, *a left-invariant differential operator is a Rockland operator if and only if it is homogeneous and hypoelliptic*.

The Sobolev spaces $H^s_{\mathcal{R}}(\mathbb{G})$, $s \in \mathbb{R}$, associated to positive Rockland operators $\mathcal{R}$ have been analysed in [11] using heat kernel methods, see also [14]. The positivity (of an operator) refers to the positivity in the operator sense. One of the equivalent definitions of Sobolev spaces is

$$H^s(\mathbb{G}) := H^s_{\mathcal{R}}(\mathbb{G}) := \left\{ f \in \mathcal{D}'(\mathbb{G}) : (I+\mathcal{R})^{s/\nu} f \in L^2(\mathbb{G}) \right\},$$

with the norm $\|f\|_{H^s_{\mathcal{R}}(\mathbb{G})} := \|(I+\mathcal{R})^{s/\nu} f\|_{L^2(\mathbb{G})}$, for a positive Rockland operator of homogeneous degree ν. Among other things, it has been shown that these Sobolev spaces are independent of a particular choice of the Rockland operator $\mathcal{R}$, so we may omit writing the subscript $\mathcal{R}$.

We now establish a version of the Gagliardo–Nirenberg inequality on graded Lie groups. Some version of such inequality was shown in [3], and also in [14, Theorem 4.4.28, (7)], namely, for $q, r \in (1, \infty)$ and $0 < \sigma < s$ there exists $C > 0$ such that

$$\|f\|_{\dot{L}^p_\sigma} \le C \|f\|^\theta_{L^q} \|f\|^{1-\theta}_{\dot{L}^r_s}, \tag{4.1}$$

where $\theta = 1 - \frac{\sigma}{s}$ and $p \in (1, \infty)$ is given via $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}$. Here $\dot{L}^p_\sigma$ is the homogeneous Sobolev space defined as the space of all $f \in \mathcal{D}'(\mathbb{G})$ such that $\mathcal{R}^{\sigma/\nu} f \in L^p(\mathbb{G})$, where $\mathcal{R}$ is a positive Rockland operator of homogeneous degree ν. Again, these spaces are independent of a particular choice of $\mathcal{R}$, see [14, Section 4.4]. We also note results on the best constants, for example, see [38,19]. However, this inequality (4.1) will not be suitable for our purpose, and we establish another version as a consequence of the following Sobolev inequality on $\mathbb{G}$:

Proposition 4.1 *([14, Proposition 4.4.13, (5)]). Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q. Let $a > 0$ and $1 < p < q < \infty$ be such that*

$$Q\left(\frac{1}{p} - \frac{1}{q}\right) = a.$$

Then we have the following Sobolev inequality

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|_{\dot{L}^p_a(\mathbb{G})} \simeq \|\mathcal{R}^{a/\nu} u\|_{L^p(\mathbb{G})}, \tag{4.2}$$

for all $u \in \dot{L}^p_a(\mathbb{G})$, and where $\mathcal{R}$ is any positive Rockland operator of homogeneous degree ν.

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If $\mathbb{G}$ is a stratified Lie group, $\mathcal{R}$ is a sub-Laplacian and $\nu = 2$, the estimate (4.2) was established by Folland [9]. We refer to [14, Proposition 4.4.13] for other embedding theorems on graded Lie groups.

We now show that the Sobolev inequality implies a family of the Gagliardo–Nirenberg inequalities, one of which is needed for our analysis:

Theorem 4.2. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Assume that*

$$a > 0, \quad 1 < r < \frac{Q}{a} \text{ and } 1 \le p \le q \le \frac{rQ}{Q - ar}. \tag{4.3}$$

Then we have the following Gagliardo–Nirenberg type inequality,

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^s_{\dot{L}^r_a(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})} \simeq \|\mathcal{R}^{a/\nu} u\|^s_{L^r(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})}, \tag{4.4}$$

for $s = \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{a}{Q} + \frac{1}{p} - \frac{1}{r}\right)^{-1} \in [0, 1]$, provided that $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} \neq 0$.

If $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} = 0$, we have $p = q = \frac{rQ}{Q-ar}$, in which case (4.4) holds for any $0 \le s \le 1$.

Proof. By the Hölder inequality, we have

$$\int_{\mathbb{G}} |u|^q dx = \int_{\mathbb{G}} |u|^{qs} |u|^{q(1-s)} dx \le \left(\int_{\mathbb{G}} |u|^{p^*} dx\right)^{\frac{qs}{p^*}} \left(\int_{\mathbb{G}} |u|^p dx\right)^{\frac{q(1-s)}{p}},$$

for any $s \in [0, 1]$ such that

$$\frac{qs}{p^*} + \frac{q(1-s)}{p} = 1. \tag{4.5}$$

Then by using Corollary 4.1, for $1 < r < p^* < \infty$ we obtain

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^s_{\dot{L}^r_a(\mathbb{G})} \|u\|^{1-s}_{L^p(\mathbb{G})},$$

where

$$Q\left(\frac{1}{r} - \frac{1}{p^*}\right) = a, \tag{4.6}$$

yielding (4.4). We only have to check that conditions (4.3) imply that $r < p^*$ and that $s \in [0, 1]$. Indeed, the relation (4.6) implies that $\frac{1}{p^*} = \frac{1}{r} - \frac{a}{Q} > 0$, and so (4.5) gives

$$s\left(\frac{a}{Q} + \frac{1}{p} - \frac{1}{r}\right) = \frac{1}{p} - \frac{1}{q} \ge 0.$$

The condition $q \le \frac{rQ}{Q-ar}$ guarantees that $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} \ge \frac{1}{p} - \frac{1}{q}$, so that s is uniquely determined for $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} \neq 0$. We also note that then automatically $s \in [0, 1]$.

Assume now that $\frac{a}{Q} + \frac{1}{p} - \frac{1}{r} = 0$. This implies that $p = \frac{rQ}{Q-ar}$, so that the conditions (4.3) imply that

$$p = q = \frac{rQ}{Q - ar}. \tag{4.7}$$

If $s = 0$, (4.4) trivially holds for $p = q$, so we may assume that $1 \ge s > 0$. Moreover, we can assume that $\|u\|_{L^q} \neq 0$ since otherwise there is noting to prove. Consequently, using that $s > 0$, $p = q$ and $\|u\|_{L^q} \neq 0$, inequality (4.4) reduces to the Sobolev inequality in Proposition 4.1 since we have $Q(\frac{1}{r} - \frac{1}{q}) = a$ under conditions (4.7), and since $r < q$ in view of $\frac{1}{r} - \frac{1}{q} = \frac{a}{Q}$ in this case. $\square$

A special case of Theorem 4.2 important for our further analysis is that of $p = r = 2$ and $a = 1$, in which case we obtain a more classical Gagliardo–Nirenberg inequality:

Corollary 4.3. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension $Q \ge 3$ and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Then for any*

$$2 \le q \le \frac{2Q}{Q-2} = 2 + \frac{4}{Q-2}$$

we have the following Gagliardo–Nirenberg type inequality,

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^s_{\dot{H}^1(\mathbb{G})} \|u\|^{1-s}_{L^2(\mathbb{G})} \simeq \|\mathcal{R}^{1/\nu} u\|^s_{L^2(\mathbb{G})} \|u\|^{1-s}_{L^2(\mathbb{G})}, \tag{4.8}$$

for $s = s(q) = \frac{Q(q-2)}{2q} \in [0, 1]$.

We also record another more general special case of Theorem 4.2 with $p = r = 2$, but with any $a > 0$:

Corollary 4.4. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Then for any*

$$0 < a < \frac{Q}{2} \text{ and } 2 \le q \le \frac{2Q}{Q-2a} = 2 + \frac{4a}{Q-2a}$$

we have the following Gagliardo–Nirenberg type inequality,

$$\|u\|_{L^q(\mathbb{G})} \lesssim \|u\|^s_{\dot{H}^a(\mathbb{G})} \|u\|^{1-s}_{L^2(\mathbb{G})} \simeq \|\mathcal{R}^{a/\nu} u\|^s_{L^2(\mathbb{G})} \|u\|^{1-s}_{L^2(\mathbb{G})}, \tag{4.9}$$

for $s = \frac{Q}{a}(\frac{1}{2} - \frac{1}{q}) \in [0, 1]$.

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4.2. Linear equation

Now, we are ready to deal with the linear equation

$$\begin{cases} \partial_t^2 u(t) + \mathcal{R}u(t) + b\partial_t u(t) + mu(t) = 0, \quad t > 0, \\ u(0) = u_0 \in L^2(\mathbb{G}), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{G}), \end{cases} \tag{4.10}$$

with the damping term determined by $b > 0$ and with the mass $m > 0$.

Following [14], we briefly recall some definitions related to the Fourier analysis on a graded Lie group $\mathbb{G}$. For $f \in \mathcal{S}(\mathbb{G})$ its group Fourier transform is given by

$$\widehat{f}(\pi) := \int_{\mathbb{G}} f(x)\pi(x)^* dx \tag{4.11}$$

with the representation $\pi \in \widehat{\mathbb{G}}$ realised as the mapping

$$\pi : \mathcal{H}_\pi \to \mathcal{H}_\pi, \tag{4.12}$$

where $\mathcal{H}_\pi$ is the representation space of π, and where we routinely identify π with its equivalence class. The Fourier inversion formula takes the form

$$f(x) = \int_{\widehat{\mathbb{G}}} \mathrm{Tr}[\widehat{f}(\pi)\pi(x)]d\mu(\pi), \tag{4.13}$$

where Tr is the trace operator, and $d\mu(\pi)$ is the Plancherel measure on $\widehat{\mathbb{G}}$. The Plancherel theorem says that

$$\|f\|^2_{L^2(\mathbb{G})} = \int_{\widehat{\mathbb{G}}} \|\widehat{f}(\pi)\|^2_{\mathrm{HS}[\mathcal{H}_\pi]} d\mu(\pi), \tag{4.14}$$

where $\|\cdot\|_{\mathrm{HS}[\mathcal{H}_\pi]}$ is the Hilbert–Schmidt norm on $\mathcal{H}_\pi$. We refer to [14] for details of the Fourier analysis on graded Lie groups.

Now, the group Fourier transform applied to (4.10) gives

$$\begin{cases} \partial_t^2 \widehat{u}(t,\pi) + \sigma_{\mathcal{R}}(\pi)\widehat{u}(t,\pi) + b\partial_t \widehat{u}(t,\pi) + m\widehat{u}(t,\pi) = 0, \quad t > 0, \\ \widehat{u}(0,\pi) = \widehat{u}_0(\pi), \\ \partial_t \widehat{u}(0,\pi) = \widehat{u}_1(\pi), \end{cases} \tag{4.15}$$

where $\sigma_{\mathcal{R}}(\pi) = \pi(\mathcal{R})$ is the symbol of $\mathcal{R}$ given by its infinitesimal representation. It is known that $\sigma_{\mathcal{R}}(\pi)$ has a discrete spectrum in $(0, \infty)$ for any non-trivial representation $\pi \in \widehat{\mathbb{G}}$, see [17], [39], and also [14, Remark 4.2.8, (4)]. Therefore, we can decompose (4.15) with respect to the basis of eigenvectors of $\pi(\mathcal{R})$. Repeating discussions of Section 2 we obtain

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Proposition 4.5. *Let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν on the graded Lie group $\mathbb{G}$. Suppose that $s \in \mathbb{R}$. Assume that $u_0 \in H^s(\mathbb{G})$ and $u_1 \in H^{s-\nu/2}(\mathbb{G})$. Then there exists a positive constant $\delta_1 > 0$ such that*

$$\|u(t,z)\|^2_{H^s(\mathbb{G})} \lesssim e^{-2\delta_1 t}(\|u_0\|^2_{H^s(\mathbb{G})} + \|u_1\|^2_{H^{s-\nu/2}(\mathbb{G})}) \tag{4.16}$$

for all $t > 0$. Moreover, for all $\alpha \in \mathbb{N}_0$ we obtain

$$\|\partial_t^\alpha u(t,z)\|^2_{H^s(\mathbb{G})} \lesssim e^{-2\delta_1 t}(\|u_0\|^2_{H^{s+\nu\alpha/2}(\mathbb{G})} + \|u_1\|^2_{H^{s+(\alpha-1)\nu/2}(\mathbb{G})})$$

for any $t > 0$.

4.3. Semilinear equations

From now on we assume that $\mathbb{G}$ is a graded Lie group of homogeneous dimension $Q \geq 3$. We now consider the semilinear equation associated to the positive Rockland operator $\mathcal{R}$ of homogeneous degree ν.

Theorem 4.6. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension $Q \geq 3$, and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $b > 0$ and $m > 0$. Assume that $1 < p \leq 1 + \frac{2}{Q-2}$ and that f satisfies the properties*

$$\begin{cases} f(0) = 0, \\ |f(u) - f(v)| \leq C(|u|^{p-1} + |v|^{p-1})|u - v|. \end{cases} \tag{4.17}$$

Assume that the Cauchy data $u_0 \in H^{\nu/2}(\mathbb{G})$ and $u_1 \in L^2(\mathbb{G})$ satisfy

$$\|u_0\|_{H^{\nu/2}(\mathbb{G})} + \|u_1\|_{L^2(\mathbb{G})} \leq \varepsilon. \tag{4.18}$$

Then, there exists a small positive constant $\varepsilon_0 > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t^2 u(t) + \mathcal{R}u(t) + b\partial_t u(t) + mu(t) = f(u), \quad t > 0, \\ u(0) = u_0 \in H^{\nu/2}(\mathbb{G}), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{G}), \end{cases} \tag{4.19}$$

has a unique global solution $u \in C(\mathbb{R}_+; H^{\nu/2}(\mathbb{G})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{G}))$ for all $0 < \varepsilon \leq \varepsilon_0$.

Moreover, there is a positive number $\delta_2 > 0$ such that

$$\|\partial_t^\alpha \mathcal{R}^\beta u(t)\|_{L^2(\mathbb{G})} \lesssim e^{-\delta_2 t}, \tag{4.20}$$

for all $(\alpha, \beta) \in \mathbb{N}_0 \times \frac{1}{\nu}\mathbb{N}_0$ and $\alpha + \nu\beta \leq \nu/2$.

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Proof. Similarly to the Heisenberg group case, we introduce the closed subsets $Z_{\mathcal{R}}$ of the space $C(\mathbb{R}_+;\ H^1(\mathbb{G}))$ defined by

$$Z_{\mathcal{R}} := \{u \in C^1(\mathbb{R}_+; H^1(\mathbb{G})) : \ \|u\|_{Z_{\mathcal{R}}} \le L_{\mathcal{R}}\},$$

with

$$\|u\|_{Z_{\mathcal{R}}} := \sup_{t\ge 0} \left\{ (1+t)^{-1/2} \mathrm{e}^{\delta_1 t} \left(\sum_{(\alpha,\beta)\in\mathbb{N}_0\times\frac{1}{\nu}\mathbb{N}_0}^{\alpha+\nu\beta\le\nu/2} \|\partial_t^\alpha \mathcal{R}^\beta u(t,\cdot)\|_{L^2(\mathbb{G})} \right) \right\},$$

where $L_{\mathcal{R}} > 0$ will be specified later. Indeed, the last sum is taken over terms $(\alpha,\beta) = \{(0,0),(1,0),(0,1/\nu),\dots,(0,\left[\frac{\nu}{2}\right]1/\nu)\}$.

Now we define the mapping $\Gamma_{\mathcal{R}}$ on $Z_{\mathcal{R}}$ by

$$\Gamma_{\mathcal{R}}[u](t) := u_{\mathrm{lin}}(t) + \int_0^t K_{\mathcal{R}}[f(u)](t-\tau)d\tau, \tag{4.21}$$

where u_{lin} is the solution of the linear equation, and $K_{\mathcal{R}}[f]$ is the solution of the following linear problem:

$$\begin{cases} \partial_t^2 w(t) + \mathcal{R}w(t) + b\partial_t w(t) + mw(t) = 0, & t>0, \\ w(0)=0, \\ \partial_t w(0) = f. \end{cases}$$

Now, we repeat the discussions of the proof of Theorem 3.2, namely, by the Hölder inequality, we obtain

$$\|(f(u)-f(v))(t,\cdot)\|^2_{L^2(\mathbb{G})} \le C(\|u(t,\cdot)\|^{p-1}_{L^{2p}(\mathbb{G})} + \|v(t,\cdot)\|^{p-1}_{L^{2p}(\mathbb{G})})^2 \|(u-v)(t,\cdot)\|^2_{L^{2p}(\mathbb{G})},$$

where $\frac{1}{\frac{p}{p-1}} + \frac{1}{p} = 1$. Then by taking into account the Gagliardo–Nirenberg inequality (4.8) of Corollary 4.3, and by Young's inequality, we get

$$\begin{aligned} \|(f(u)-f(v))(t,\cdot)\|_{L^2(\mathbb{G})} &\le C\Big[\Big(\|\mathcal{R}^{1/\nu}u(t,\cdot)\|_{L^2(\mathbb{G})} + \|u(t,\cdot)\|_{L^2(\mathbb{G})} \Big)^{p-1} \\ &+ \Big(\|\mathcal{R}^{1/\nu}v(t,\cdot)\|_{L^2(\mathbb{G})} + \|v(t,\cdot)\|_{L^2(\mathbb{G})} \Big)^{p-1} \Big] \\ &\times \Big(\|\mathcal{R}^{1/\nu}(u-v)(t,\cdot)\|_{L^2(\mathbb{G})} + \|(u-v)(t,\cdot)\|_{L^2(\mathbb{G})} \Big). \end{aligned} \tag{4.22}$$

Recalling that $\|u\|_{Z_{\mathcal{R}}} \le L_{\mathcal{R}}$ and $\|v\|_{Z_{\mathcal{R}}} \le L_{\mathcal{R}}$, from (4.22) we obtain

$$\|(f(u)-f(v))(t,\cdot)\|_{L^2(\mathbb{G})} \le C(1+t)^{p/2} \mathrm{e}^{-\delta_1 pt} L_{\mathcal{R}}^{p-1} \|u-v\|_{Z_{\mathcal{R}}}. \tag{4.23}$$

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Put $v = 0$ in (4.23). Then by using that $f(0) = 0$, we have

$$\|f(u)(t,\cdot)\|_{L^2(\mathbb{G})} \le C(1+t)^{p/2} \mathrm{e}^{-\delta_1 pt} L^p_{\mathcal{R}}. \tag{4.24}$$

Now, we estimate the following integral operator

$$J_{\mathcal{R}}[u](t,z) := \int_0^t K_{\mathcal{R}}[f(u(\tau,z))](t-\tau)d\tau. \tag{4.25}$$

Since

$$|\partial_t^\alpha \mathcal{R}^\beta J_{\mathcal{R}}[u](t,z)|^2 \le \left| \int_0^t \partial_t^\alpha \mathcal{R}^\beta K_{\mathcal{R}}[f(u(\tau,z))](t-\tau)d\tau \right|^2$$
$$\le \left(\int_0^t \left| \partial_t^\alpha \mathcal{R}^\beta K_{\mathcal{R}}[f(u(\tau,z))](t-\tau) \right| d\tau \right)^2$$
$$\le t \int_0^t \left| \partial_t^\alpha \mathcal{R}^\beta K_{\mathcal{R}}[f(u(\tau,z))](t-\tau) \right|^2 d\tau,$$

by Proposition 4.5, for all $(\alpha, \beta) = \{(0,0), (1,0), (0,1/\nu), \dots, (0, \left[\frac{\nu}{2}\right] 1/\nu)\}$, i.e. for $\alpha + \nu\beta \le \nu/2$, we obtain

$$\|\partial_t^\alpha \mathcal{R}^\beta J_{\mathcal{R}}[u](t,\cdot)\|^2_{L^2(\mathbb{G})} \le t \int_0^t \|\partial_t^\alpha \mathcal{R}^\beta K_{\mathcal{R}}[f(u(\tau,z))](t-\tau)\|^2_{L^2(\mathbb{G})} d\tau$$
$$\lesssim t \int_0^t \mathrm{e}^{-2\delta_1(t-\tau)} \|f(u(\tau,\cdot))\|^2_{L^2(\mathbb{G})} d\tau \tag{4.26}$$
$$= t\mathrm{e}^{-2\delta_1 t} \int_0^t \mathrm{e}^{2\delta_1 \tau} \|f(u(\tau,\cdot))\|^2_{L^2(\mathbb{G})} d\tau.$$

Thus, using (4.23) and (4.24), from (4.26) we get

$$\|\partial_t^\alpha \mathcal{R}^\beta (J_{\mathcal{R}}[u] - J_{\mathcal{R}}[v])(t,\cdot)\|_{L^2(\mathbb{G})} \lesssim t^{1/2} \mathrm{e}^{-\delta_1 t} L^{p-1}_{\mathcal{R}} \|u - v\|_{Z_{\mathcal{R}}}, \tag{4.27}$$

and

$$\|\partial_t^\alpha \mathcal{R}^\beta J_{\mathcal{R}}[u](t,\cdot)\|_{L^2(\mathbb{G})} \lesssim t^{1/2} \mathrm{e}^{-\delta_1 t} L^p_{\mathcal{R}}, \tag{4.28}$$

with the estimates (4.27)–(4.28).

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Finally, continuing to discuss as in the above Heisenberg case we obtain the statement of Theorem 4.6. □

4.4. Nonlinear equations

We note that the techniques of the proof of Theorem 4.6 allow us to consider the nonlinear equation (4.19) with more general nonlinearities. Namely, instead of f satisfying (4.17) we can deal with the function $F : \mathbb{C}^{[\nu/2]} \to \mathbb{C}$ with the following property:

$$\begin{cases} F(0) = 0, \\ |F(U) - F(V)| \leq C(|U|^{p-1} + |V|^{p-1})|U - V|, \end{cases} \tag{4.29}$$

where $U = (\{\mathcal{R}^{j/\nu}u\}_{j=0}^{[\frac{\nu}{2}]-1})$. Here $[\frac{\nu}{2}]$ stands for the integer part of $\frac{\nu}{2}$.

Theorem 4.7. *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension $Q \geq 3$, and let $\mathcal{R}$ be a positive Rockland operator of homogeneous degree ν. Let $b > 0$ and $m > 0$. Assume that $1 < p \leq 1 + \frac{2}{Q-2}$ and that F satisfies the properties (4.29). Assume that the Cauchy data $u_0 \in H^{\nu/2}(\mathbb{G})$ and $u_1 \in L^2(\mathbb{G})$ satisfy*

$$\|u_0\|_{H^{\nu/2}(\mathbb{G})} + \|u_1\|_{L^2(\mathbb{G})} \leq \varepsilon. \tag{4.30}$$

Then, there exists a small positive constant $\varepsilon_0 > 0$ such that the Cauchy problem

$$\begin{cases} \partial_t^2 u(t) + \mathcal{R}u(t) + b\partial_t u(t) + mu(t) = F(u, \{\mathcal{R}^{j/\nu}u\}_{j=1}^{[\frac{\nu}{2}]-1}), \quad t > 0, \\ u(0) = u_0 \in H^{\nu/2}(\mathbb{G}), \\ \partial_t u(0) = u_1 \in L^2(\mathbb{G}), \end{cases}$$

has a unique global solution $u \in C(\mathbb{R}_+; H^{\nu/2}(\mathbb{G})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{G}))$ for all $0 < \varepsilon \leq \varepsilon_0$.

Moreover, there is a positive number $\delta_3 > 0$ such that

$$\|u(t)\|_{L^2(\mathbb{G})} + \|\mathcal{R}^{1/\nu}u(t)\|_{L^2(\mathbb{G})} \lesssim e^{-\delta_3 t}. \tag{4.31}$$

Proof. As in the proof of Theorem 4.6, we consider the closed subset $Z_{1,\mathcal{R}}$:

$$Z_{1,\mathcal{R}} := \{u \in C(\mathbb{R}_+; H^1(\mathbb{G})) : \ \|u\|_{Z_{1,\mathcal{R}}} \leq L_{1,\mathcal{R}}\},$$

with the norm

$$\|u\|_{Z_{1,\mathcal{R}}} := \sup_{t\geq 0}\{(1+t)^{-1/2}e^{\delta_1 t}\left(\|u(t,\cdot)\|_{L^2(\mathbb{G})} + \|\mathcal{R}^{1/\nu}u(t,\cdot)\|_{L^2(\mathbb{G})}\right)\}.$$

Then similarly to the inequality (4.26), we have

$$\|\mathcal{R}^{\beta} J_{\mathcal{R}}[u](t,\cdot)\|^2_{L^2(\mathbb{G})} \lesssim t e^{-2\delta_1 t} \int_0^t e^{2\delta_1 \tau} \|F(u, \{\mathcal{R}^{j/\nu} u\}_{j=1}^{[\frac{\nu}{2}]-1})\|^2_{H^{\frac{\nu}{2}(2\beta-1)}(\mathbb{G})} d\tau. \tag{4.32}$$

Here we need to control $\|F(u, \{\mathcal{R}^{j/\nu} u\}_{j=1}^{[\frac{\nu}{2}]-1})\|_{H^{\frac{\nu}{2}(2\beta-1)}(\mathbb{G})}$ with $\frac{\nu}{2}(2\beta-1) \leq 0$. By using the Gagliardo–Nirenberg inequality (4.8) of Corollary 4.3, and by Young's inequality, we obtain

$$\begin{aligned} &\|F(u, \{\mathcal{R}^{j/\nu} u\}_{j=1}^{[\frac{\nu}{2}]-1}) - F(v, \{\mathcal{R}^{j/\nu} v\}_{j=1}^{[\frac{\nu}{2}]-1})\|_{H^{\frac{\nu}{2}(2\beta-1)}(\mathbb{G})} \\ &\lesssim \Big[\Big(\|\mathcal{R}^{1/\nu} u(t,\cdot)\|_{L^2(\mathbb{G})} + \|u(t,\cdot)\|_{L^2(\mathbb{G})} \Big)^{p-1} \\ &+ \Big(\|\mathcal{R}^{1/\nu} v(t,\cdot)\|_{L^2(\mathbb{G})} + \|v(t,\cdot)\|_{L^2(\mathbb{G})} \Big)^{p-1} \Big] \\ &\times \Big(\|\mathcal{R}^{1/\nu} (u-v)(t,\cdot)\|_{L^2(\mathbb{G})} + \|(u-v)(t,\cdot)\|_{L^2(\mathbb{G})} \Big) \end{aligned} \tag{4.33}$$

for $\beta = 0$ and $\beta = 1/\nu$. Consequently, repeating the rest of the proof as in the previous proofs, we obtain the statement of Theorem 4.7. □

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