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Stackelberg–Nash exact controllability for the Kuramoto–Sivashinsky equation

N. Carreño [a,*], M.C. Santos [b]

[a] *Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile*
[b] *Departamento de Matemática, Universidade Federal da Paraíba, 58051-085, João Pessoa - PB, Brazil*

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Abstract

This paper deals with a hierarchical control problem for the Kuramoto–Sivashinsky equation following a Stackelberg–Nash strategy. We assume that there is a main control, called the leader, and two secondary controls, called the followers. The leader tries to drive the solution to a prescribed target and the followers intend to be a Nash equilibrium for given functionals. It is known that this problem is equivalent to a null controllability result for an optimality system consisting of three non-linear equations. One of the novelties is a new Carleman estimate for a fourth-order equation with right-hand sides in Sobolev spaces of negative order, which allows to relax some geometric conditions for the observation sets for the followers.

MSC: 35K41; 35K55; 93C10; 93B05

Keywords: Kuramoto–Sivashinsky equation; Null controllability; Carleman inequalities; Hierarchical controls

1. Introduction

The Kuramoto–Sivashinsky (KS) equation is a fourth-order parabolic equation that serves as a model for phase turbulence in reaction-diffusion systems (see [17,18]) and for plane flame propagation (see [24]). The controllability properties of this one-dimensional partial differential

* Corresponding author.
E-mail addresses: nicolas.carrenog@usm.cl (N. Carreño), mcsantoso@mat.ufpb.br (M.C. Santos).

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equation has gained the attention of several researchers the last few years. We can mention, for instance, the works [6,4,7,5,14].

In this paper, we propose to study a multi-objective control problem for the KS equation following a hierarchical strategy. Let us be more precise: we consider a distributed system governed by the KS equation with a control v distributed over a sub domain $\mathcal{O}$. We assume that there are two (or more) goals we would like to achieve. The main one being of "controllability" type and the others expressing that the state of the system does not move too far from a given state. We divide the control v into two (or more) parts, say f, $v^1, v^2, \ldots$ where f is the main control, usually called the "Leader" and the controls v^i are the secondary controls, called "Followers". We will see that the leader is mainly responsible for the "controllability" property while the followers for some secondary objectives to be described later. This concept was mainly introduced by J.-L. Lions (see [19,20]) where some techniques are presented. These works motivated the study of this subject and a lot of other results appeared, see for instance [8,9,15,22,23]. We remark that all these previous works combine the concepts of multi-criteria optimization and approximate controllability. In the context of null controllability, few results are known and there are a lot of open questions related to this subject. Recently, for the heat equation, the authors in [2] improved the results known so far and proved a null controllability result, instead of an approximate controllability under some suitable geometric conditions. Later, in [3], the authors improved the result of [2] for, in some sense, less restrictive geometric conditions.

Consider the KS equation

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} + yy_x = f\mathbb{1}_{\mathcal{O}} + v^1\mathbb{1}_{\mathcal{O}_1} + v^2\mathbb{1}_{\mathcal{O}_2} & (x,t) \in (0,L)\times(0,T), \\ y(0,t) = y(L,t) = 0 & t \in (0,T), \\ y_x(0,t) = y_x(L,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L), \end{cases} \tag{1.1}$$

where $y = y(x,t)$ is the state and y^0 is a prescribed initial condition. In (1.1), the set $\mathcal{O} \subset (0,L)$ is the *main control domain* and $\mathcal{O}_1, \mathcal{O}_2 \subset (0,L)$ are the *secondary control domains* (all them are supposed to be small). We will assume that these sets are bounded intervals of the form $\mathcal{O} = (a,b)$ and $\mathcal{O}_i = (a_i, b_i)$ $(i = 1,2)$, moreover $\mathbb{1}_{\mathcal{O}}$, $\mathbb{1}_{\mathcal{O}_1}$ and $\mathbb{1}_{\mathcal{O}_2}$ are the characteristic functions of $\mathcal{O}$, $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively; the controls are f, v^1 and v^2, where f is the *leader* and v^1 and v^2 are the *followers*.

Let $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d} \subset (0,L)$ be open sets of the form $O_{i,d} = (a_{i,d}, b_{i,d})$, representing observation domains for the followers. We will consider the (secondary) functionals

$$J_i(f;v^1,v^2) := \frac{\alpha_i}{2}\int_0^T\int_{\mathcal{O}_{i,d}} \left|y - y_{i,d}\right|^2 dx\,dt + \frac{\mu_i}{2}\int_0^T\int_{\mathcal{O}_i} |v^i|^2\,dx\,dt, \quad i = 1,2 \tag{1.2}$$

and the main functional

$$J(f) := \frac{1}{2}\int_0^T\int_{\mathcal{O}} |f|^2\,dx\,dt, \tag{1.3}$$

where the $\alpha_i > 0$, $\mu_i > 0$ are constants and the $y_{i,d} = y_{i,d}(x,t)$ are given functions. The structure of the control process can be described as follows:

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1. The followers v^1 and v^2 assume that the leader f has made a choice and intend to be a *Nash equilibrium* for the costs J_i $(i = 1, 2)$, that is, once f has been fixed, we look for controls $v^i \in L^2(\mathcal{O}_i \times (0, T))$ that satisfy

$$J_1(f; v^1, v^2) = \min_{\hat{v}^1} J_1(f; \hat{v}^1, v^2), \quad J_2\left(f; v^1, v^2\right) = \min_{\hat{v}^2} J_2(f; v^1, \hat{v}^2). \tag{1.4}$$

2. Let us fix an uncontrolled trajectory of (1.1), that is, a sufficiently regular solution to the system

$$\begin{cases} \bar{y}_t + \bar{y}_{xxxx} + \nu \bar{y}_{xx} + \bar{y}\bar{y}_x = 0 & (t, x) \in [0, T] \times [0, L], \\ \bar{y}(t, 0) = \bar{y}(t, L) = 0 & t \in [0, T], \\ \bar{y}_x(t, 0) = \bar{y}_x(t, L) = 0 & t \in [0, T], \\ \bar{y}(x, 0) = \bar{y}^0(x) & x \in [0, L]. \end{cases} \tag{1.5}$$

   Once the Nash equilibrium has been identified and fixed for each f, we look for an optimal control $\hat{f} \in L^2(\mathcal{O} \times (0, T))$ such that

$$J(\hat{f}) = \min_f J(f), \tag{1.6}$$

   subject to the restriction of exact controllability

$$y(x, T) = \overline{y}(x, T) \text{ in } (0, L). \tag{1.7}$$

Note that, if the functionals J_i $(i = 1, 2)$ are convex, then (v^1, v^2) is a Nash equilibrium if and only if

$$J_1'(f; v^1, v^2)(\hat{v}^1, 0) = 0, \quad \forall \hat{v}^1 \in L^2\left(\mathcal{O}_1 \times (0, T)\right), \quad v^1 \in L^2(\mathcal{O}_1 \times (0, T)) \tag{1.8}$$

and

$$J_2'(f; v^1, v^2)(0, \hat{v}^2) = 0, \quad \forall \hat{v}^2 \in L^2\left(\mathcal{O}_2 \times (0, T)\right), \quad v^2 \in L^2(\mathcal{O}_2 \times (0, T)). \tag{1.9}$$

We will prove that if μ_1 and μ_2 are sufficiently large, then the functionals (1.2) are indeed convex for small data. More precisely, we have the following result.

Proposition 1.1. *There exists $r > 0$ (independent of μ_1 and μ_2) such that if*

$$\|f\|_{L^2(\mathcal{O}\times(0,T))} + \|y^0 - \bar{y}^0\|_{L^2(0,L)} \le r,$$

then, if (v^1, v^2) is a pair such that conditions (1.8)–(1.9) hold, there exists $C > 0$, independent of μ_1 and μ_2, such that

$$\langle D_i^2 J_i(f; v^1, v^2), (w^i, w^i)\rangle \ge C \int_0^T \int_{\mathcal{O}_i} |w^i|^2 \, dx \, dt, \quad \forall w^i \in L^2(\mathcal{O}_i \times (0, T)), \ i = 1, 2,$$

for μ_1 and μ_2 sufficiently large. In particular, the functionals J_1 and J_2 are convex in (v^1, v^2).

Therefore, for now on we will use (1.8) and (1.9) as an existence criterion for the Nash equilibrium.

We now state the main result of this paper.

Theorem 1.2. *Suppose*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \quad (i = 1, 2) \tag{1.10}$$

and $\mu_i > 0$ $(i = 1, 2)$ are sufficiently large. Assume that one of the following two conditions holds:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{1.11}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{1.12}$$

Let $\bar{y} \in L^\infty(0, T; W^{1,\infty}((0, L)))$ a given trajectory of the uncontrolled equation (1.5). Then there exist $\delta > 0$ and positive functions $\hat{\rho}_i = \hat{\rho}_i(t)$ blowing up at $t = T$ such that if y^0 and $\bar{y}$ satisfy

$$\|y^0 - \bar{y}^0\|^2_{L^2(0,L)} + \sum_{i=1,2} \int_0^T \int_{\mathcal{O}_{i,d}} \hat{\rho}_i^2 |\bar{y} - y_{i,d}|^2 \, dx \, dt < \delta, \tag{1.13}$$

there exist controls $f \in L^2(\mathcal{O} \times (0, T))$ and associated Nash equilibria (v^1, v^2) such that the corresponding solution to equation (1.1) satisfy the control condition (1.7).

Let $z := y - \bar{y}$. It is clear that property (1.7) is equivalent to a null controllability property for z, that is,

$$z(x, T) = 0 \text{ in } (0, L), \tag{1.14}$$

where z is the solution of the equation

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + z z_x + (\bar{y} z)_x = f \mathbb{1}_{\mathcal{O}} + v^1 \mathbb{1}_{\mathcal{O}_1} + v^2 \mathbb{1}_{\mathcal{O}_2} & (x, t) \in (0, L) \times (0, T), \\ z(0, t) = z(L, t) = 0 & t \in (0, T), \\ z_x(0, t) = z_x(L, t) = 0 & t \in (0, T), \\ z(x, 0) = y^0(x) - \bar{y}^0(x) & x \in (0, L). \end{cases} \tag{1.15}$$

On the other hand, the functionals J_i can be rewritten as

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2} \int_0^T \int_{\mathcal{O}_{i,d}} \left| z - z_{i,d} \right|^2 dx \, dt + \frac{\mu_i}{2} \int_0^T \int_{\mathcal{O}_i} |v^i|^2 \, dx \, dt \quad i = 1, 2, \tag{1.16}$$

where $z_{i,d} := y_{i,d} - \bar{y}$. Following the arguments in [2], we can prove that a pair (v^1, v^2) satisfying (1.8)–(1.9) is characterized by

$$v_i = -\frac{1}{\mu_i}\phi^i \mathbb{1}_{\mathcal{O}_i} \quad i = 1, 2, \tag{1.17}$$

where (z, ϕ^1, ϕ^2) is the solution of the optimality system

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + z z_x + (\bar{y} z)_x = f\,\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1 \mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2 \mathbb{1}_{\mathcal{O}_2} & (x,t) \in (0,L) \times (0,T), \\ -\phi^i_t + \phi^i_{xxxx} + \nu \phi^i_{xx} - (z + \bar{y})\phi^i_x = \alpha_i (z - z_{i,d}) \mathbb{1}_{\mathcal{O}_{i,d}} \quad i = 1, 2 & (x,t) \in (0,L) \times (0,T), \\ z(0,t) = z(L,t) = \phi^i(0,t) = \phi^i(L,t) = 0 \quad i = 1, 2 & t \in (0,T), \\ z_x(0,t) = z_x(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1, 2 & t \in (0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad i = 1, 2 & x \in (0,L). \end{cases} \tag{1.18}$$

Indeed, if (v_1, v_2) is a pair of followers satisfying conditions (1.8)–(1.9), then we have

$$\alpha_i \int_0^T \int_{\mathcal{O}_{i,d}} (z - z_{i,d}) p^i \, dx\, dt + \mu_i \int_0^T \int_{\mathcal{O}_i} v^i \hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)) \quad i = 1, 2, \tag{1.19}$$

where p^i is the derivative of z with respect to v^i in the direction $\hat{v}^i$. In fact, p^i is the solution of

$$\begin{cases} p^i_t + p^i_{xxxx} + \nu p^i_{xx} + (z p^i)_x + (\bar{y} p^i)_x = \hat{v}^i \mathbb{1}_{\mathcal{O}_i} & (x,t) \in (0,L) \times (0,T), \\ p^i(0,t) = p^i(L,t) = 0 & t \in (0,T), \\ p^i_x(0,t) = p^i_x(L,t) = 0 & t \in (0,T), \\ p^i(x,0) = 0 & x \in (0,L). \end{cases}$$

Now, let ϕ^i, $i = 1, 2$, be the solution of the (adjoint) system

$$\begin{cases} -\phi^i_t + \phi^i_{xxxx} + \nu \phi^i_{xx} - (z + \bar{y})\phi^i_x = \alpha_i (z - z_{i,d}) \mathbb{1}_{\mathcal{O}_{i,d}} & (x,t) \in (0,L) \times (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = 0 & t \in (0,T), \\ \phi^i_x(0,t) = \phi^i_x(L,t) = 0 & t \in (0,T), \\ \phi^i(x,T) = 0 & x \in (0,L). \end{cases}$$

Then, using the expression for $\alpha_i (z - z_{i,d}) \mathbb{1}_{\mathcal{O}_{i,d}}$ in (1.19), after integration by parts we obtain

$$\int_0^T \int_{\mathcal{O}_i} \left(\phi^i + \mu_i v^i\right) \hat{v}^i \, dx\, dt = 0 \quad \forall \hat{v}^i \in L^2(\mathcal{O}_i \times (0,T)) \quad i = 1, 2,$$

from where (1.17)–(1.18) follows.

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Notice that the existence of solution for (1.18) implies the existence of a Nash equilibria in the sense of (1.4). The well-posedness of (1.18) is given by Theorem 2.1, and the proof can be found in Appendix A. Therefore, the result given by Theorem 1.2 is equivalent to the (local) null controllability of (1.18), provided that condition (1.11) or (1.12) holds. Actually, since we only need to control the state z, we can talk about *partial null controllability* of system (1.18). Let us give some guidelines of the proof. First, we prove the null controllability of the linearized system around zero

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + (\bar{y}z)_x = f^0 + f\,\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2\mathbb{1}_{\mathcal{O}_2} & (x,t) \in (0,L)\times(0,T), \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} - \bar{y}\phi^i_x = f^i + \alpha_i z\mathbb{1}_{\mathcal{O}_{i,d}} \quad i=1,2 & (x,t) \in (0,L)\times(0,T), \\ z(0,t) = z(L,t) = \phi^i(0,t) = \phi^i(L,t) = 0 \quad i=1,2 & t\in(0,T), \\ z_x(0,t) = z_x(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i=1,2 & t\in(0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 & x\in(0,L), \end{cases} \tag{1.20}$$

where f^0 and f^i are (arbitrary) L^2-functions decaying exponentially to zero at $t = T$. It is well known by now that, with the help of a classical duality argument, the null controllability of system (1.20) can be deduced from an observability inequality for the solutions of the so called adjoint system, which in this case is given by

$$\begin{cases} -\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x = g^0 + \alpha_1\gamma^1\mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} & (x,t)\in(0,L)\times(0,T), \\ \gamma^i_t + \gamma^i_{xxxx} + \nu\gamma^i_{xx} + (\bar{y}\gamma^i)_x = g^i - \dfrac{1}{\mu_i}\psi\mathbb{1}_{\mathcal{O}_i} \quad i=1,2 & (x,t)\in(0,L)\times(0,T), \\ \psi(0,t) = \psi(L,t) = \gamma^i(0,t) = \gamma^i(L,t) = 0 \quad i=1,2 & t\in(0,T), \\ \psi_x(0,t) = \psi_x(L,t) = \gamma^i_x(0,t) = \gamma^i_x(L,t) = 0 \quad i=1,2 & t\in(0,T), \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 \quad i=1,2 & x\in(0,L), \end{cases} \tag{1.21}$$

where g^0 and g^i are (arbitrary) L^2-functions. Being more precise, the observability inequality that we prove in this case looks like

$$\int_0^L |\psi(x,0)|^2 dx + \int_0^T\int_0^L \bar{\rho}_0^2|\psi|^2\,dx\,dt + \sum_{i=1}^2\int_0^T\int_0^L \bar{\rho}_i^2|\gamma^i|^2\,dx\,dt \\ \le C\sum_{i=0}^2\int_0^T\int_0^L \widetilde{\rho}_i^2|g^i|^2\,dx\,dt + C\int_0^T\int_{\mathcal{O}} \widetilde{\rho}_3^2|\psi|^2\,dx\,dt \tag{1.22}$$

where $\bar{\rho}_i$ $(i = 0,1,2)$ and $\widetilde{\rho}_i$ $(i = 0,\dots,3)$ are suitable positive functions and C is a positive constant independent of ψ, γ^1 and γ^2. The presence of weights in an observability inequality is not always needed to deduce controllability results. However, they will provide useful decaying properties of the solution of system (1.21) to deal with the non-linearities.

The proof of (1.22) relies on Carleman estimates for the KS equation. They have the form

$$\int_0^T\int_0^L \rho_1|\varphi|^2\,dx\,dt \le C\int_0^T\int_0^L \rho_2|\varphi_t+\varphi_{xxxx}|^2\,dx\,dt + C\int_0^T\int_{\mathcal{O}} \rho_3|\varphi|^2\,dx\,dt$$

where φ is a smooth function such that $\varphi(0,t)=\varphi(L,t)=\varphi_x(0,t)=\varphi_x(L,t)=0$ in $t\in[0,T]$. Inequalities of this kind have proven to be powerful tools to deduce observability of parabolic equations. They were introduced in [13] and have been extensively used ever since. Notice that it does not suffice to combine Carleman estimates for the solutions of the equations in system (1.21) to obtain inequality (1.22). The idea is to use the couplings of the equations to estimate the unwanted local terms. Of course, the location of the observation sets $\mathcal{O}_{i,d}$ (see assumptions (1.10)–(1.12)) will play a fundamental role in our analysis. In particular, the weight functions will strongly depend on them.

The work is divided as follows: In Section 2 we prove that, if μ_i are sufficiently large, then the J_i are convex and then the criteria (1.8) and (1.9) are equivalent to (1.4). In Section 3 we will be concerned with Carleman estimates, the most technical part of the paper. We introduce the weight functions, recall known results and prove a new Carleman inequality for the KS equation needed to treat the case of assumption (1.12). Then, we deduce a Carleman estimate for the adjoint system. Section 4 establishes the observability inequality (1.22) and deals with the null controllability of the linear system (1.20). Finally, in Section 5, it is shown that the optimality system is locally null controllable using a local inversion argument and thus completing the proof of Theorem 1.2. In Appendix A, we give a proof of the well-posedness of the optimality system (1.18).

2. On the convexity of J_1 and J_2

In this section we prove Proposition 1.1. Before that, let us state a result that will be useful in the following. The proof can be found in Appendix A.

Theorem 2.1. *There exists $r>0$ such that if $\|f\|_{L^2(\mathcal{O}\times(0,T))}+\|z_0\|_{L^2(0,L)}\le r$, then there exists $\mu_0>0$ (depending on α_i, ν, r, $\|\bar{y}_x\|_\infty$, $\|z_{i,d}\|_{L^2(\mathcal{O}_{i,d}\times(0,T))}$) such that if $\mu_1,\mu_2\ge\mu_0$ the system (1.18) possesses a unique solution $(z,\phi^1,\phi^2)\in\left(L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H_0^2(0,L))\right)^3$. Furthermore, the solution is such that*

$$\|(\hat{z}(t),\hat{\phi}^1(t),\hat{\phi}^2(t))\|_{L^2(0,L)^3}+\|(\hat{z},\hat{\phi}^1,\hat{\phi}^2)\|_{L^2(0,T;H_0^2(0,L))^3}\le C \tag{2.1}$$

with C depending on α_i, ν, r, $\|\bar{y}_x\|_\infty$, $\|z_{i,d}\|_{L^2((0,L)\times(0,T))}$, but independent of μ_1 and μ_2.

Proof of Proposition 1.1. Let $f\in L^2(\mathcal{O}\times(0,T))$ be given and let (v^1,v^2) be such that conditions (1.8) and (1.9) hold. Note that, for any $s\in\mathbb{R}$ and $(w^1,w^2)\in L^2(\mathcal{O}_1\times(0,T))\times L^2(\mathcal{O}_2\times(0,T))$

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$$\langle D_1 J_1(f; v^1 + sw^1, v^2), w^2\rangle - \langle D_1 J_1(f; v^1, v^2), w^2\rangle = s\mu_1 \int_0^T \int_{\mathcal{O}_1} w^1 w^2 \, dx\, dt \\ + \alpha_1 \int_0^T \int_{\mathcal{O}_{1,d}} (z^s - z_{1,d}) p^s \, dx\, dt - \alpha_1 \int_0^T \int_{\mathcal{O}_{1,d}} (z - z_{1,d}) p \, dx\, dt, \tag{2.2}$$

where

$$\begin{cases} z^s_t + z^s_{xxxx} + \nu z^s_{xx} + z^s z^s_x + (\bar{y} z^s)_x & \\ \quad = f \mathbb{1}_{\mathcal{O}} + (v^1 + sw^1)\mathbb{1}_{\mathcal{O}_1} + v^2 \mathbb{1}_{\mathcal{O}_2} & (x,t) \in (0,L) \times (0,T), \\ z^s(0,t) = z^s(L,t) = z^s_x(0,t) = z^s_x(L,t) = 0 & t \in (0,T), \\ z^s(x,0) = y_0(x) - \bar{y}_0(x) & x \in (0,L), \end{cases}$$

p^s is the derivative of z^s with respect to v^1 in the direction w^2, i.e. the solution to

$$\begin{cases} p^s_t + p^s_{xxxx} + \nu p^s_{xx} + (z^s p^s + \bar{y} p^s)_x = w^2 \mathbb{1}_{\mathcal{O}_1} & (x,t) \in (0,L) \times (0,T), \\ p^s(0,t) = p^s(L,t) = p^s_x(0,t) = p^s_x(L,t) = 0 & t \in (0,T), \\ p^s(x,0) = 0 & x \in (0,L) \end{cases} \tag{2.3}$$

and we have used the notation $z = z^s|_{s=0}$ and $p = p^s|_{s=0}$.

Let us introduce the adjoint of (2.3):

$$\begin{cases} -\phi^s_t + \phi^s_{xxxx} + \nu \phi^s_{xx} - (z^s + \bar{y})\phi^s_x = \alpha_1 (z^s - z_{1,d}) \mathbb{1}_{\mathcal{O}_{1,d}} & (x,t) \in (0,L) \times (0,T) Q, \\ \phi^s(0,t) = \phi^s(L,t) = \phi^s_x(0,t) = p^s_x(L,t) = 0 & t \in (0,T), \\ \phi^s(x,T) = 0 & x \in (0,L) \end{cases} \tag{2.4}$$

and let us also set $\phi = \phi^s|_{s=0}$.

Replacing (2.4) into (2.2) and using integration by parts, we obtain the following identity:

$$\langle D_1 J_1(f; v^1 + sw^1, v^2), w^2\rangle - \langle D_1 J_1(f; v^1, v^2), w^2\rangle = s\mu_1 \int_0^T \int_{\mathcal{O}_1} w^1 w^2 \, dx\, dt \\ + \int_0^T \int_{\mathcal{O}_1} (\phi^s - \phi) w^2 \, dx\, dt.$$

Notice that

$$-(\phi^s - \phi)_t + (\phi^s - \phi)_{xxxx} + \nu(\phi^s - \phi)_{xx} - (z^s - z)\phi^s_x - (z + \bar{y})(\phi^s - \phi)_x = \alpha_1 (z^s - z) \mathbb{1}_{\mathcal{O}_{1,d}}$$

and

$$(z^s - z)_t + (z^s - z)_{xxxx} + \nu(z^s - z)_{xx} + (z^s - z) z^s_x + z(z^s - z)_x + (\bar{y}(z^s - z))_x = sw^1 \mathbb{1}_{\mathcal{O}_1}.$$

Consequently, the limits

$$\eta = \lim_{s\to 0} \frac{1}{s}(\phi^s - \phi) \text{ and } h = \lim_{s\to 0} \frac{1}{s}(z^s - z)$$

exist and satisfy

$$\begin{cases} -\eta_t + \eta_{xxxx} + \nu\eta_{xx} - h\phi_x - (z+\bar{y})\eta_x = \alpha_1 h \mathbb{1}_{\mathcal{O}_{1,d}} & (x,t)\in(0,L)\times(0,T), \\ h_t + h_{xxxx} + \nu h_{xx} + (hz + h\bar{y})_x = w^1 \mathbb{1}_{\mathcal{O}_1} & (x,t)\in(0,L)\times(0,T), \\ \eta(0,t) = \eta(L,t) = \eta_x(0,t) = \eta_x(L,t) = 0 & t\in(0,T), \\ h(0,t) = h(L,t) = h_x(0,t) = h_x(L,t) = 0 & t\in(0,T), \\ \eta(\cdot,T) = h(\cdot,0) = 0 & x\in(0,L). \end{cases} \tag{2.5}$$

Thus, from (2.5), we deduce that

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^2)\rangle = \mu_1 \int_0^T \int_{\mathcal{O}_1} w^1 w^2 \, dx\, dt + \int_0^T \int_{\mathcal{O}_1} \eta w^2 \, dx\, dt.$$

In particular, for all $w^1 \in L^2(\mathcal{O}_1 \times (0,T))$, one has

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1)\rangle = \mu_1 \int_0^T \int_{\mathcal{O}_1} |w^1|^2 \, dx\, dt + \int_0^T \int_{\mathcal{O}_1} \eta w^1 \, dx\, dt. \tag{2.6}$$

Let us show that, for some C only depending on L, $\mathcal{O}$, T, $\mathcal{O}_i$, $\mathcal{O}_{i,d}$, α_i, $\|\bar{y}_x\|_\infty$ and $\|y_0\|_{L^2(0,L)}$, we have

$$\left| \int_0^T \int_{\mathcal{O}_1} \eta w^1 \, dx\, dt \right| \le C(1 + \|f\|_{L^2(\mathcal{O}\times(0,T))}) \|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))} \quad \forall w^1 \in L^2(\mathcal{O}_1 \times (0,T)). \tag{2.7}$$

In fact, from standard energy estimates,

$$\frac{1}{2}\frac{d}{dt}\int_0^L |h|^2 \, dx + \int_0^L |h_{xx}|^2 \, dx = -\nu \int_0^L h_{xx} h \, dx + \int_0^L (z+\bar{y}) h h_x \, dx + \int_{\mathcal{O}_1} h w^1 \, dx$$

and, using Young's inequality,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L |h|^2 \, dx + \frac{1}{2}\int_0^L |h_{xx}|^2 \, dx \\ &\le \frac{1}{2}\int_{\mathcal{O}_1} |w^1|^2 \, dx + \left(\frac{1+\nu^2}{2} + \|(z(t)+\bar{y}(t))_x\|_{L^\infty(0,L)}\right) \int_0^L |h|^2 \, dx. \end{aligned}$$

By Gronwall's inequality we get that

$$\int_0^L |h(t)|^2\,dx + \int_0^T\int_0^L |h_{xx}|^2\,dx$$
$$\leq C_\nu \exp\Big(\int_0^T \big(\|z_x(t)\|^2_{L^\infty(0,L)} + \|\bar{y}_x(t)\|^2_{L^\infty(0,L)}\big)\,dt\Big)\|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))}$$

Now, notice that since (v^1, v^2) satisfy conditions (1.8)–(1.9), z is, together with (ϕ^1, ϕ^2), the solution of the optimality system (1.18) and (v^1, v^2) are given by (1.17), provided that μ_1 and μ_2 are sufficiently large. Therefore, from (2.1), the previous inequality becomes

$$\int_0^L |h(t)|^2\,dx + \int_0^T\int_0^L |h_{xx}|^2\,dx \leq C\|w^1\|^2_{L^2(\mathcal{O}_1\times(0,T))} \tag{2.8}$$

where C is independent of μ_1 and μ_2. Furthermore, notice that $\phi = \phi^1$.

Using the PDEs in (2.5), we also get the following:

$$\begin{aligned}
\int_0^T\int_{\mathcal{O}_1} \eta w^1\,dx\,dt &= \int_0^T\int_0^L (h_t + h_{xxxx} + \nu h_{xx} + (hz + h\bar{y})_x)\eta\,dx\,dt \\
&= \int_0^T\int_0^L h(-\eta_t + \eta_{xxxx} + \nu\eta_{xx} - (z+\bar{y})\eta_x)\,dx\,dt \\
&= \int_0^T\int_0^L (h\phi^1_x + \alpha_1 h \mathbb{1}_{\mathcal{O}_{1,d}})h\,dx\,dt \\
&= \int_0^T\int_0^L (|h|^2\phi^1_x + \alpha_1|h|^2\mathbb{1}_{\mathcal{O}_{1,d}})\,dx\,dt.
\end{aligned}$$

Then,

$$\begin{aligned}
\left|\int_0^T\int_{\mathcal{O}_1} \eta w^1\,dx\,dt\right| &\leq \int_0^T \Big(\|\phi^1_x\|_{L^\infty(0,L)}\int_0^L |h|^2\,dx\Big)dt + \alpha_1\int_0^T\int_0^L |h|^2\,dx\,dt \\
&\leq \|\phi^1_x\|_{L^2(0,T;L^\infty(0,L))}\|h\|^2_{L^4(0,T;L^2(0,L))} + \alpha_1\|h\|^2_{L^2((0,L)\times(0,T))}.
\end{aligned} \tag{2.9}$$

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Using (2.1) and (2.8) in (2.9), we obtain

$$\left| \int_0^T \int_{\mathcal{O}_1} \eta w^1 \, dx \, dt \right| \leq C \| w^1 \|^2_{L^2(\mathcal{O}_1 \times (0,T))},$$

where C does not depend on μ_1 nor μ_2.

We can use (2.7) in (2.6) and we get

$$\langle D_1^2 J_1(f; v^1, v^2), (w^1, w^1) \rangle \geq (\mu_1 - C) \int_0^T \int_{\mathcal{O}_1} |w^1|^2 \, dx \, dt, \tag{2.10}$$

for all $w^1 \in L^2(\mathcal{O}_1 \times (0,T))$.

Of course, the same can be done for $D_2^2 J_2(f; v^1, v^2)$ to obtain

$$\langle D_2^2 J_2(f; v^1, v^2), (w^2, w^2) \rangle \geq (\mu_2 - C) \int_0^T \int_{\mathcal{O}_2} |w^2|^2 \, dx \, dt, \tag{2.11}$$

for all $w^2 \in L^2(\mathcal{O}_2 \times (0,T))$. The proof is complete taking μ_1 and μ_2 large enough in (2.10) and (2.11). $\square$

3. Carleman estimates

This section is devoted to Carleman estimates. In a first step we prove a Carleman estimate for the linear fourth-order parabolic equation

$$\begin{cases} u_t + u_{xxxx} = F + \sum_{i=1}^4 \partial_x^i F^i & (x,t) \in (0,L) \times (0,T), \\ u(0,t) = u(L,t) = 0 & t \in (0,T), \\ u_x(0,t) = u_x(L,t) = 0 & t \in (0,T), \\ u(x,0) = u^0(x) & x \in (0,L), \end{cases} \tag{3.1}$$

where $u^0 \in L^2(0,L)$ and, F and F^i $(i = 1, \dots, 4)$ are functions belonging to $L^2((0,L) \times (0,T))$ and, in a second step, for the adjoint system (1.21). We will see how this result depends on the position of the observation sets $\mathcal{O}_{i,d}$.

3.1. Preliminaries

Let us begin by introducing the weight functions. Let ω and ω_0 be non-empty open subsets of $(0,L)$ such that $\omega_0 \subset\subset \omega$. Let η_0 be a $C^4([0,L])$-function such that

$$\begin{cases} \eta_0 > 0 \text{ in } (0,L), \ \eta_0(0) = \eta_0(L) = 0, \\ |\nabla \eta_0| > 0 \text{ in } [0,L] \setminus \overline{\omega}_0. \end{cases}$$

The existence of such a function in dimension higher than one is proved in [13, Lemma 1.1].

Next, we introduce the (positive) weight functions

$$\sigma(x,t) := \frac{\exp\left(4\lambda\|\eta_0\|_\infty\right) - \exp\left(\lambda(\|\eta_0\|_\infty + \eta_0(x))\right)}{t^{1/3}(T-t)^{1/3}}, \quad \xi(x,t) := \frac{\exp\left(\lambda(\|\eta_0\|_\infty + \eta_0(x))\right)}{t^{1/3}(T-t)^{1/3}}, \tag{3.2}$$

where $\lambda > 1$. Consider the following notations:

$$\sigma^*(t) = \max_{x\in[0,L]} \sigma(x,t), \quad \hat{\sigma}(t) = \min_{x\in[0,L]} \sigma(x,t),$$
$$\xi^*(t) = \min_{x\in[0,L]} \xi(x,t), \quad \hat{\xi}(t) = \max_{x\in[0,L]} \xi(x,t),$$

Notice that we have

$$\begin{aligned} \sigma^*(t) &= \frac{\exp\left(4\lambda\|\eta_0\|_\infty\right) - \exp\left(\lambda\|\eta_0\|_\infty\right)}{t^{1/3}(T-t)^{1/3}}, \quad \xi^*(t) = \frac{\exp\left(\lambda\|\eta_0\|_\infty\right)}{t^{1/3}(T-t)^{1/3}}, \\ \hat{\sigma}(t) &= \frac{\exp\left(4\lambda\|\eta_0\|_\infty\right) - \exp\left(2\lambda\|\eta_0\|_\infty\right)}{t^{1/3}(T-t)^{1/3}}, \quad \hat{\xi}(t) = \frac{\exp\left(2\lambda\|\eta_0\|_\infty\right)}{t^{1/3}(T-t)^{1/3}}. \end{aligned} \tag{3.3}$$

A simple computation shows that, for any $m, n \in \mathbb{N}$ we have

$$1 \le T^{2/3}\xi, \quad |\partial_t^m \partial_x^n \xi| \le C T^m \lambda^n \xi^{1+3m}, \quad |\partial_t^m \partial_x^n \sigma| \le C T^m \lambda^n \xi^{1+3m} \tag{3.4}$$

for every $\lambda > 1$. These properties will be used several times in this paper.

Finally, before stating the first Carleman estimate, let us keep in mind the following notation for the weighted energy:

$$\begin{aligned} I(u) :=& \int_0^T \int_0^L e^{-2s\sigma}\left(s^{-1}\xi^{-1}(|u_t|^2 + |u_{xxxx}|^2) + s\lambda^2\xi|u_{xxx}|^2\right) dx\,dt \\ &+ \int_0^T \int_0^L e^{-2s\sigma}\left(s^3\lambda^4\xi^3|u_{xx}|^2 + s^5\lambda^6\xi^5|u_x|^2 + s^7\lambda^8\xi^7|u|^2\right) dx\,dt. \end{aligned} \tag{3.5}$$

For the simpler case where the right-hand side of (3.1) belongs to $L^2((0,L)\times(0,T))$, we have the following result.

Proposition 3.1. *Let of $F \in L^2((0,L)\times(0,T))$, $F^i = 0$ for all $i = 1,\dots,4$ and $\omega \subset (0,L)$. Then, there exists $C(L,\omega) > 0$ such that for every $s \ge C(T^{2/3} + T^{1/3})$ and every $\lambda \ge C$ we have*

$$I(u) \le C\left(s^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma}\xi^7|u|^2\,dx\,dt + \int_0^T \int_0^L e^{-2s\sigma}|F|^2\,dx\,dt\right), \tag{3.6}$$

where u is the solution of (3.1).

The proof of Proposition 3.1 can be found in [7, Theorem 3.3]. In the case where the observation set are identical (see assumption (1.11)), we will see that, following the arguments in [2], Carleman estimate (3.6) suffices to prove the wanted observability for system (1.21). However, when this is not the case (see assumption (1.12)), we need a Carleman estimate with a right-hand side in weaker spaces.

3.2. A new Carleman estimate

Before stating our Carleman estimate, let us recall the notion of solution by transposition of equation (3.1). To this end, let $G \in L^2((0, L) \times (0, T))$ and consider the (adjoint) equation

$$\begin{cases} -\varphi_t + \varphi_{xxxx} = G & (x,t) \in (0,L) \times (0,T), \\ \varphi(0,t) = \varphi(L,t) = 0 & t \in (0,T), \\ \varphi_x(0,t) = \varphi_x(L,t) = 0 & t \in (0,T), \\ \varphi(x,T) = 0 & x \in (0,L). \end{cases} \tag{3.7}$$

The following lemma, corresponding to [6, Proposition 2.1], establishes the well-posedness of this equation.

Lemma 3.2. *For every $G \in L^2((0, L) \times (0, T))$, there exist a unique solution $\varphi \in L^2(0, T; H^4(0, L)) \cap C([0, T]; H_0^2)$ and a positive constant such that*

$$\|\varphi\|_{L^2(0,T;H^4(0,L))\cap C([0,T];H_0^2)} \le C\|G\|_{L^2((0,L)\times(0,T))}.$$

Now, let $H = H^4(0, L) \cap H_0^2(0, L)$ and assume that the right-hand side of equation (3.1) satisfies

$$F, F^i \in L^2((0,L) \times (0,T))(i = 1, \dots, 4) \quad \text{and} \quad F^3_{xxx}, F^4_{xxxx} \in L^2(0,T;H'), \tag{3.8}$$

where H' denotes de dual space of H. Notice that under (3.8), we can consider the dual product of the distribution F^4_{xxxx} with an element of $L^2(0, T; H)$ in the sense that there exist some functions g_1, g_2, h_1 and h_2 belonging to $L^2(0, T)$ such that

$$\begin{aligned} \langle F^4_{xxxx}, u\rangle = &\int_0^T\int_0^L F^4 u_{xxxx}\,dx\,dt + \int_0^T \big(g_2(t)u_{xx}(L,t) - g_1(t)u_{xx}(0,t)\big)\,dt \\ &- \int_0^T \big(h_2(t)u_{xxx}(L,t) - h_1(t)u_{xxx}(0,t)\big)\,dt, \quad \forall u \in L^2(0,T;H). \end{aligned}$$

Notice that if F^4 were regular enough, we would have

$$g_1(t) = F^4_x(0,t), \quad g_2(t) = F^4_x(L,t), \quad h_1(t) = F^4(0,t) \quad \text{and} \quad h_2(t) = F^4(L,t).$$

Therefore, in the following we will use this notation. Of course, the same argument can be adapted for the distribution F^3_{xxx}.

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With the previous considerations, we establish the notion of solution of equation (3.1).

Definition 3.3. Given $u^0 \in L^2(0,L)$, $F, F^i \in L^2((0,L)\times(0,T))$ $(i=1,\dots,4)$, we say that $u \in L^2((0,L)\times(0,T))$ is a *solution by transposition* of (3.1) if

$$\begin{aligned}\int_0^T\int_0^L uG\,dx\,dt &= \int_0^L u^0(x)\varphi(x,0)\,dx \\ &+ \int_0^T\int_0^L \left(F\varphi - F^1\varphi_x + F^2\varphi_{xx} - F^3\varphi_{xxx} + F^4\varphi_{xxxx}\right)dx\,dt \\ &+ \int_0^T \left(F^3\varphi_{xx}\right)\Big|_{x=0}^{x=L} dt + \int_0^T \left(F^4_x\varphi_{xx}\right)\Big|_{x=0}^{x=L} dt - \int_0^T \left(F^4\varphi_{xxx}\right)\Big|_{x=0}^{x=L} dt \qquad (3.9)\end{aligned}$$

for every $G \in L^2((0,L)\times(0,T))$, where φ is the unique solution of equation (3.7).

Notice that the existence and uniqueness of a solution of equation (3.1) is a direct consequence of Lemma 3.2 and Riesz's representation theorem.

We have the following theorem, which is one of the main results of the paper.

Theorem 3.4. *Let F, F^i $(i=1,\dots,4)$ satisfy (3.8). Then, there exists $C(L,\omega)>0$ such that for every $s \geq C(T^{2/3}+T^{1/3})$ and every $\lambda \geq C$ we have*

$$\begin{aligned}s^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma}\xi^7|u|^2\,dx\,dt &\leq C\Bigg(s^7\lambda^8\int_0^T\int_\omega e^{-2s\sigma}\xi^7|u|^2\,dx\,dt + \int_0^T\int_0^L e^{-2s\sigma}|F|^2\,dx\,dt \\ &+ \int_0^T\int_0^L e^{-2s\sigma}\left(s^2\lambda^2\xi^2|F^1|^2 + s^4\lambda^4\xi^4|F^2|^2 + s^6\lambda^6\xi^6|F^3|^2 + s^8\lambda^8\xi^8|F^4|^2\right)dx\,dt \\ &+ s^5\lambda^5\int_0^T e^{-2s\sigma^*}(\xi^*)^5\left(\left|F^3(L,t)\right|^2 + \left|F^3(0,t)\right|^2 + \left|F^4_x(L,t)\right|^2 + \left|F^4_x(0,t)\right|^2\right)dt \\ &+ s^7\lambda^7\int_0^T e^{-2s\sigma^*}(\xi^*)^7\left(\left|F^4(L,t)\right|^2 + \left|F^4(0,t)\right|^2\right)dt\Bigg), \qquad (3.10)\end{aligned}$$

where u is the solution of (3.1).

Remark 3.5. Carleman estimate (3.10) will be a key step for the purposes of this paper. Nevertheless, it has an interest on its own: it allows to obtain a null controllability result for equations like

$$\begin{cases} u_t + u_{xxxx} + a_0 u + a_1 u_x + a_2 u_{xx} + a_3 u_{xxx} = v\mathbb{1}_\omega & (x,t) \in (0,L)\times(0,T), \\ u(0,t) = u(L,t) = 0 & t \in (0,T), \\ u_x(0,t) = u_x(L,t) = 0 & t \in (0,T), \\ u(x,0) = u^0(x) & x \in (0,L), \end{cases}$$

where $a_i \in L^\infty((0,L)\times(0,T))$ for $i = 0,\dots,3$. Indeed, the corresponding adjoint equation is given by

$$\begin{cases} -\varphi_t + \varphi_{xxxx} = -a_0\varphi + (a_1\varphi)_x - (a_2\varphi)_{xx} + (a_3\varphi)_{xxx} & (x,t) \in (0,L)\times(0,T), \\ \varphi(0,t) = \varphi(L,t) = 0 & t \in (0,T), \\ \varphi_x(0,t) = \varphi_x(L,t) = 0 & t \in (0,T), \\ \varphi(x,T) = \varphi^T(x) & x \in (0,L). \end{cases}$$

Using estimate (3.10) with $F = -a_0\varphi$, $F^1 = a_1\varphi$, $F^2 = -a_2\varphi$, $F^3 = a_3\varphi$ and $F^4 = 0$, and taking s sufficiently large with respect to T and a_i $(i = 0,\dots,3)$, we get

$$\int_0^T\int_0^L e^{-2s\sigma}\xi^7|\varphi|^2\,dx\,dt \le C\int_0^T\int_\omega e^{-2s\sigma}(\xi)^7|\varphi|^2\,dx\,dt.$$

Proof. To prove inequality (3.10) we follow a duality argument introduced in [16] for the heat equation with right-hand side in $L^2(0,T;H^{-1}(0,L))$. Since then, this method have been used in different contexts (see [10,11,4,14]). In particular, several arguments remain unchanged with respect to the ones used in [4, Theorem 3.5], so we will use some estimates from there. We start with the null controllability problem of finding (φ, h) such that

$$\begin{cases} -\varphi_t + \varphi_{xxxx} = s^7\lambda^8\xi^7 e^{-2s\sigma}u + h\mathbb{1}_\omega & (x,t) \in (0,L)\times(0,T), \\ \varphi(0,t) = \varphi_x(0,t) = 0, \quad \varphi(L,t) = \varphi_x(L,t) = 0 & t \in (0,T), \\ \varphi(x,0) = 0, \quad \varphi(x,T) = 0 & x \in (0,L), \end{cases} \tag{3.11}$$

where u is the solution (by transposition) of equation (3.1). From the proof of Theorem 3.5 in [4], there exists a pair (φ, h) that solves this control problem. Furthermore, they satisfy

$$\begin{aligned} &\int_0^T\int_0^L e^{2s\sigma}\big(|\varphi|^2 + s^{-2}\lambda^{-2}\xi^{-2}|\varphi_x|^2 + s^{-4}\lambda^{-4}\xi^{-4}|\varphi_{xx}|^2 + s^{-6}\lambda^{-6}\xi^{-6}|\varphi_{xxx}|^2 \\ &\quad + s^{-8}\lambda^{-8}\xi^{-8}|\varphi_{xxxx}|^2\big)\,dx\,dt \\ &\quad + s^{-7}\lambda^{-8}\int_0^T\int_\omega e^{2s\sigma}\xi^{-7}|h|^2\,dx\,dt \le Cs^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma}\xi^7|u|^2\,dx\,dt \end{aligned} \tag{3.12}$$

and

$$s^{-5}\lambda^{-5}\int_0^T e^{2s\sigma^*}(\xi^*)^{-5}\big(|\varphi_{xx}(0,t)|^2+|\varphi_{xx}(L,t)|^2\big)\,dt$$
$$+s^{-7}\lambda^{-7}\int_0^T e^{2s\sigma^*}(\xi^*)^{-7}\big(|\varphi_{xxx}(0,t)|^2+|\varphi_{xxx}(L,t)|^2\big)\,dt\le Cs^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma}\xi^7|u|^2\,dx\,dt. \tag{3.13}$$

We now take $G=s^7\lambda^8\xi^7e^{-2s\sigma}u+h\mathbb{1}_\omega$ in equality (3.9). We obtain

$$\int_0^T\int_0^L s^7\lambda^8e^{-2s\sigma}\xi^7|u|^2\,dx\,dt$$
$$=-\int_0^T\int_\omega uh\,dx\,dt+\int_0^T\int_0^L\big(F\varphi-F^1\varphi_x+F^2\varphi_{xx}-F^3\varphi_{xxx}+F^4\varphi_{xxxx}\big)\,dx\,dt$$
$$+\int_0^T\big(F^3\varphi_{xx}\big)\big|_{x=0}^{x=L}\,dt+\int_0^T\big(F_x^4\varphi_{xx}\big)\big|_{x=0}^{x=L}\,dt-\int_0^T\big(F^4\varphi_{xxx}\big)\big|_{x=0}^{x=L}\,dt.$$

From this identity, we conclude using estimates (3.12), (3.13) and Young's inequality. □

3.3. *Carleman estimates for the adjoint system*

Here we prove some Carleman estimates for the adjoint system (1.21), provided that assumptions of Theorem 1.2 hold. In particular, we will deal with conditions (1.11) and (1.12) separately.

3.3.1. *Case $\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$*

We start assuming that condition (1.11) is satisfied. It is convenient then to denote $\mathcal{O}_d=\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$. Notice that system (1.21) now reads

$$\begin{cases}-\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x=g^0+\big(\alpha_1\gamma^1+\alpha_2\gamma^2\big)\mathbb{1}_{\mathcal{O}_d} & (x,t)\in(0,L)\times(0,T),\\ \gamma_t^i+\gamma_{xxxx}^i+\nu\gamma_{xx}^i+(\bar{y}\gamma^i)_x=g^i-\dfrac{1}{\mu_i}\psi\mathbb{1}_{\mathcal{O}_i}\quad i=1,2 & (x,t)\in(0,L)\times(0,T),\\ \psi(0,t)=\psi(L,t)=\gamma^i(0,t)=\gamma^i(L,t)=0\quad i=1,2 & t\in(0,T),\\ \psi_x(0,t)=\psi_x(L,t)=\gamma_x^i(0,t)=\gamma_x^i(L,t)=0\quad i=1,2 & t\in(0,T),\\ \psi(x,T)=\psi^T(x),\quad \gamma^i(x,0)=0\quad i=1,2 & x\in(0,L).\end{cases}$$

Following the arguments in [2], let $h:=\alpha_1\gamma^1+\alpha_2\gamma^2$. Of course, by linearity, the pair (ψ,h) is solution to

$$\begin{cases} -\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x=g^0+h\mathbb{1}_{\mathcal{O}_d} & (x,t)\in(0,L)\times(0,T),\\ h_t+h_{xxxx}+\nu h_{xx}+(\bar{y}h)_x=\alpha_1 g^1+\alpha_2 g^2-\dfrac{\alpha_1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1}-\dfrac{\alpha_2}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2} & (x,t)\in(0,L)\times(0,T),\\ \psi(0,t)=\psi(L,t)=h(0,t)=h(L,t)=0 & t\in(0,T),\\ \psi_x(0,t)=\psi_x(L,t)=h_x(0,t)=h_x(L,t)=0 & t\in(0,T),\\ \psi(x,T)=\psi^T(x),\quad h(x,0)=0 & x\in(0,L). \end{cases} \tag{3.14}$$

For the solutions of this system, we have the following result.

Proposition 3.6. *Assume that conditions (1.10) and (1.11) hold, and $\bar{y}\in L^\infty(0,T;W^{1,\infty}((0,L)))$. Then, there exists $C=C(L,\mathcal{O},\mathcal{O}_d,\alpha_i,\mu_i,\|\bar{y}\|_\infty,\|\bar{y}_x\|_\infty)>0$ such that for every $s\ge C(T^{2/3}+T^{1/3})$ and every $\lambda\ge C$ we have*

$$\begin{aligned} I(\psi)+I(h)\le C\Bigg(&\int_0^T\int_0^L e^{-2s\sigma}\left(s^7\lambda^8\xi^7|g^0|^2+|g^1|^2+|g^2|^2\right)dx\,dt\\ &+s^{15}\lambda^{16}\int_0^T\int_{\mathcal{O}} e^{-2s\sigma}\xi^{15}|\psi|^2\,dx\,dt\Bigg), \end{aligned} \tag{3.15}$$

where (ψ,h) is the solution of (3.14).

Proof. Since we are assuming condition (1.10), there exists a non-empty open set ω such that $\overline{\omega}\subset\mathcal{O}_d\cap\mathcal{O}$. From Proposition 3.1 applied to the first equation in (3.14), we have (recall the definition of $I(\cdot)$ from (3.5))

$$\begin{aligned} I(\psi)\le C\Bigg(&\int_0^T\int_0^L e^{-2s\sigma}\left(|\psi_{xx}|^2+\|\bar{y}\|_\infty^2|\psi_x|^2+|g^0|^2+|h|^2\right)dx\,dt\\ &+s^7\lambda^8\int_0^T\int_\omega e^{-2s\sigma}\xi^7|\psi|^2\,dx\,dt\Bigg) \end{aligned}$$

for every $s\ge C(T^{2/3}+T^{1/3})$ and every $\lambda\ge C$. From the properties of the weight functions (3.4), we obtain

$$I(\psi)\le C\left(\int_0^T\int_0^L e^{-2s\sigma}\left(|g^0|^2+|h|^2\right)dx\,dt+s^7\lambda^8\int_0^T\int_\omega e^{-2s\sigma}\xi^7|\psi|^2\,dx\,dt\right) \tag{3.16}$$

for every $s\ge C(T^{2/3}+T^{1/3})$ and every $\lambda\ge C$.

Similarly, applying Proposition 3.1 to the second equation in (3.14), we get

$$I(h) \leq C\Big(\int_0^T \int_0^L e^{-2s\sigma} \big(|h_{xx}|^2 + \|\bar{y}\|_\infty^2 |h_x|^2 + \|\bar{y}_x\|_\infty^2 |h|^2 + |g^1|^2 + |g^2|^2 + |\psi|^2 \big)\, dx\, dt \\ + s^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma} \xi^7 |h|^2\, dx\, dt \Big)$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$. Again, properties (3.4) yield

$$I(h) \leq C\Big(\int_0^T \int_0^L e^{-2s\sigma} \big(|g^1|^2 + |g^2|^2 + |\psi|^2 \big)\, dx\, dt + s^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma} \xi^7 |h|^2\, dx\, dt \Big) \quad (3.17)$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Now, we combine inequalities (3.16) and (3.17). Notice that the global terms of ψ and h in the right-hand side, can be absorbed by the left-hand side taking s and λ sufficiently large as before. Therefore, we end up with

$$I(\psi) + I(h) \leq C\Big(\int_0^T \int_0^L e^{-2s\sigma} \big(|g^0|^2 + |g^1|^2 + |g^2|^2 \big)\, dx\, dt \\ + s^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma} \xi^7 \big(|\psi|^2 + |h|^2 \big)\, dx\, dt \Big) \quad (3.18)$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

To finish the proof, we only need to eliminate the local term of h in the previous inequality. Let $\theta \in C_0^4(\mathcal{O}_d \cap \mathcal{O})$ such that $\theta \equiv 1$ in ω. From the equation satisfied by h in (3.14), we have that

$$h = -\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x - g^0 \quad \text{in } (\mathcal{O}_d \cap \mathcal{O}) \times (0, T).$$

Then,

$$Cs^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma} \xi^7 |h|^2\, dx\, dt \leq Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}_d \cap \mathcal{O}} e^{-2s\sigma} \xi^7 \theta |h|^2\, dx\, dt \\ = Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}_d \cap \mathcal{O}} e^{-2s\sigma} \xi^7 \theta h \big(-\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x - g^0 \big)\, dx\, dt. \quad (3.19)$$

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Let us treat each one of the terms in the last integral, which will be denoted by $A_1, \dots, A_5$. Integration by parts in time for the first term gives

$$A_1 = Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} \left(e^{-2s\sigma}\xi^7\right)_t \theta h\psi \, dx\, dt + Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^7 \theta h_t \psi \, dx\, dt.$$

Using weight properties (3.4) and Young's inequality, we obtain

$$A_1 \leq \frac{1}{10} I(h) + Cs^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^{15}|\psi|^2 \, dx\, dt$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Similarly for A_2, we integrate by parts in space. Again from properties (3.4) and Young's inequality we get

$$A_2 \leq \frac{1}{10} I(h) + Cs^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^{15}|\psi|^2 \, dx\, dt$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Terms A_3 and A_4 are treated analogously. Keeping in mind the regularity of $\bar{y}$, we have

$$A_3 \leq \frac{1}{10} I(h) + Cs^{11}\lambda^{12} \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^{11}|\psi|^2 \, dx\, dt$$

and

$$A_4 \leq \frac{1}{10} I(h) + Cs^{9}\lambda^{10} \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^{9}|\psi|^2 \, dx\, dt$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$. Notice that the constant in A_4 depends on the norms of $\bar{y}$.

For the last term, we obtain directly

$$A_5 \leq \frac{1}{10} I(h) + Cs^{7}\lambda^{8} \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^{7}|g^0|^2 \, dx\, dt$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Finally, adding the integrals $A_1, \dots, A_5$ in (3.19) we get

$$\begin{aligned}
&Cs^7\lambda^8 \int_0^T \int_\omega e^{-2s\sigma}\xi^7 |h|^2\, dx\, dt \\
&\quad \le \frac{1}{2} I(h) + Cs^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}_d \cap \mathcal{O}} e^{-2s\sigma}\xi^{15}|\psi|^2\, dx\, dt + Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}_d\cap\mathcal{O}} e^{-2s\sigma}\xi^7 |g^0|^2\, dx\, dt.
\end{aligned} \tag{3.20}$$

Plugging this inequality in (3.18) and using properties (3.4), we deduce estimate (3.15). □

3.3.2. Case $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$

Now we turn to assumption (1.12). Notice that the argument presented to prove Carleman estimate (3.15) is no longer viable in this case. Therefore, a new approach is needed. Let ω_1 and ω_2 be two disjoint non-empty open subsets of $(0, L)$ such that

$$\overline{\omega}_1 \subset \mathcal{O}_{1,d} \cap \mathcal{O} \quad \text{and} \quad \overline{\omega}_2 \subset \mathcal{O}_{2,d} \cap \mathcal{O}, \tag{3.21}$$

and let $\mathcal{O}_0$ be a non-empty subset of $\mathcal{O}$ such that $\overline{\mathcal{O}_0} \subset \mathcal{O}$ and

$$\overline{\omega}_1 \subset \mathcal{O}_0 \quad \text{and} \quad \overline{\omega}_2 \subset \mathcal{O}_0 \tag{3.22}$$

For $i = 1, 2$, let η_i be $C^4([0, L])$-functions such that

$$\begin{cases} \eta_i > 0 \quad \text{in } (0, L), \quad \eta_i(0) = \eta_i(L) = 0, \\ |\nabla \eta_i| > 0 \quad \text{in } [0, L] \setminus \overline{\omega}_i, \\ \eta_1 \equiv \eta_2 \quad \text{in } [0, L] \setminus \mathcal{O}_0. \end{cases} \tag{3.23}$$

Conditions in (3.23) say that these functions have their critical points in disjoint sets, but coincide outside a set containing these sets. This property will be very useful in our argument. The existence of such functions in this case (one dimension in space) does not require much discussion. However, a proof in dimension higher than one can be found in [3].

Analogously as in (3.2), for $i = 1, 2$, we define the weight functions

$$\sigma_i(x,t) := \frac{\exp\left(4\lambda \|\eta_i\|_\infty\right) - \exp\left(\lambda(\|\eta_i\|_\infty + \eta_i(x))\right)}{t^{1/3}(T-t)^{1/3}}, \quad \xi_i(x,t) := \frac{\exp\left(\lambda(\|\eta_i\|_\infty + \eta_i(x))\right)}{t^{1/3}(T-t)^{1/3}}, \tag{3.24}$$

where $\lambda > 1$, and $\hat{\sigma}_i$, σ_i^*, $\hat{\xi}_i$ and ξ_i^* as in (3.3). We denote by $I_i(\cdot)$ the corresponding weighted energy as in (3.5).

Remark 3.7. Notice that properties (3.4) are still valid for these new functions. Therefore, Proposition 3.1 and Theorem 3.4 are still valid for weight functions (3.24) instead of (3.2).

Before introducing the Carleman estimate, we distinguish two cases for $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ under assumption (1.12). If

$$(\mathcal{O}_{1,d} \cap \mathcal{O}_{2,d}) \cap \mathcal{O} = \emptyset, \tag{3.25}$$

then we can choose ω_1 and ω_2 satisfying (3.21) and

$$\overline{\omega}_1 \cap (\mathcal{O}_{2,d} \cap \mathcal{O}) = \emptyset \quad \text{and} \quad \overline{\omega}_2 \cap (\mathcal{O}_{1,d} \cap \mathcal{O}) = \emptyset. \tag{3.26}$$

On the other hand, if (3.25) does not hold, it means that $(\mathcal{O}_{1,d} \cap \mathcal{O}) \cap (\mathcal{O}_{2,d} \cap \mathcal{O}) \neq \emptyset$. Furthermore, since we are assuming (1.12), the symmetric difference between $\mathcal{O}_{1,d} \cap \mathcal{O}$ and $\mathcal{O}_{2,d} \cap \mathcal{O}$ is not the empty set. Therefore, there are $i, j \in \{1,2\}$, $i \neq j$, such that ω_1 and ω_2 can be chosen to satisfy (3.21) and

$$\overline{\omega}_i \cap (\mathcal{O}_{j,d} \cap \mathcal{O}) = \emptyset \quad \text{and} \quad \overline{\omega}_j \subset (\mathcal{O}_{i,d} \cap \mathcal{O}). \tag{3.27}$$

Remark 3.8. Without loss of generality, we will fix $i = 1$ and $j = 2$ in (3.27) for the rest of the paper, that is,

$$\omega_1 \cap (\mathcal{O}_{2,d} \cap \mathcal{O}) = \emptyset \quad \text{and} \quad \overline{\omega}_2 \subset (\mathcal{O}_{1,d} \cap \mathcal{O}).$$

The other case is completely analogous.

These two cases yield to slightly different Carleman estimates, which are presented in the following proposition.

Proposition 3.9. *Assume that conditions (1.10) and (1.12) hold, and $\bar{y} \in L^\infty(0,T; W^{1,\infty}((0,L)))$. Furthermore, let ω_1 and ω_2 be non-empty disjoint open set such that (3.21) and (3.22). Then,*

(i) *if (3.25) holds, there exists $C = C(L, \mathcal{O}, \mathcal{O}_{i,d}, \alpha_i, \mu_i, \|\bar{y}\|_\infty, \|\bar{y}_x\|_\infty) > 0$ such that for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$ we have*

$$\begin{aligned}
&s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1} \xi_1^7 |\psi|^2 \, dx\, dt + I_1(\gamma^1) + I_2(\gamma^2) \\
&\leq C\Bigg(s^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}} (e^{-2s\sigma_1}\xi_1^{15} + e^{-2s\sigma_2}\xi_2^{15})|\psi|^2 \, dx\, dt \\
&\quad + s^7\lambda^8 \int_0^T \int_0^L (e^{-2s\sigma_1}\xi_1^7 + e^{-2s\sigma_2}\xi_2^7)|g^0|^2 \, dx\, dt \\
&\quad + \int_0^T \int_0^L (e^{-2s\sigma_1}|g^1|^2 + e^{-2s\sigma_2}|g^2|^2) \, dx\, dt \Bigg)
\end{aligned} \tag{3.28}$$

or

(ii) *if* (3.27) *holds for* $(i, j) = (1, 2)$, *there exists* $C = C(L, \mathcal{O}, \mathcal{O}_{i,d}, \alpha_i, \mu_i, \|\bar{y}\|_\infty, \|\bar{y}_x\|_\infty) > 0$ *such that for every* $s \geq C(T^{2/3} + T^{1/3})$ *and every* $\lambda \geq C$ *we have*

$$\begin{aligned} & s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt + I_1(\gamma^1) + I_2(\alpha_1\gamma^1 + \alpha_2\gamma^2) \\ & \leq C\Big(s^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}} \big(e^{-2s\sigma_1}\xi_1^{15} + e^{-2s\sigma_2}\xi_2^{15}\big)|\psi|^2\,dx\,dt \\ & \quad + s^7\lambda^8 \int_0^T \int_0^L \big(e^{-2s\sigma_1}\xi_1^7 + e^{-2s\sigma_2}\xi_2^7\big)|g^0|^2\,dx\,dt \\ & \quad + \int_0^T \int_0^L \big(e^{-2s\sigma_1} + e^{-2s\sigma_2}\big)\big(|g^1|^2 + |g^2|^2\big)\,dx\,dt\Big) \end{aligned} \tag{3.29}$$

for every solution $(\psi, \gamma^1, \gamma^2)$ *of system* (1.21).

Proof. We start by considering a function $\Lambda \in C^4([0, L])$ such that

$$\begin{cases} \Lambda(x) = 0 \ \text{ for } x \in \mathcal{O}_0, \\ \Lambda(x) = 1 \ \text{ for } x \in [0, L] \setminus \overline{\mathcal{O}}. \end{cases} \tag{3.30}$$

It is straightforward to check that

$$-(\Lambda\psi)_t + (\Lambda\psi)_{xxxx} = \Lambda g^0 + \alpha_1\Lambda\gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\Lambda\gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} + R(\psi) \tag{3.31}$$

where

$$\begin{aligned} R(\psi) = & -2\nu\Lambda''\psi - \bar{y}\Lambda'\psi - \bar{y}_x\Lambda\psi - \Lambda''''\psi + (2\nu\Lambda'\psi + \bar{y}\Lambda\psi + 4\Lambda'''\psi)_x \\ & - (\nu\Lambda\psi + 6\Lambda''\psi)_{xx} + 4(\Lambda'\psi)_{xxx}. \end{aligned}$$

Notice that the function $\Lambda\psi$, together with equation (3.31), fulfills the conditions of Theorem 3.4 for weight functions (3.24) with $i = 1$ (see Remark 3.7). Thus, we obtain from inequality (3.10) for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$

$$\begin{aligned} & s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^7|\Lambda\psi|^2\,dx\,dt \\ & \leq C\Big(\int_0^T \int_0^L e^{-2s\sigma_1}|g^0|^2\,dx\,dt + \int_0^T \int_0^L e^{-2s\sigma_1}\big(|\Lambda\gamma^1|^2 + |\Lambda\gamma^2|^2\big)\,dx\,dt \end{aligned}$$

$$+ s^7\lambda^8 \int_0^T \int_{\omega_1} e^{-2s\sigma_1}\xi_1^7|\Lambda\psi|^2\,dx\,dt$$

$$+ \left(1 + \|\bar{y}\|_\infty^2 + \|\bar{y}_x\|_\infty^2\right)s^6\lambda^6 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^6|\psi|^2\,dx\,dt\Big), \tag{3.32}$$

where we have used properties (3.4). To estimate the last term in (3.32), notice that from the properties (3.30) of Λ, we have

$$s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt - s^7\lambda^8 \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt$$

$$\leq s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^7|\Lambda\psi|^2\,dx\,dt$$

and, since the weight functions are equal outside $\mathcal{O}_0$ (see (3.23)),

$$\int_0^T \int_0^L e^{-2s\sigma_1}\left(|\Lambda\gamma^1|^2 + |\Lambda\gamma^2|^2\right)dx\,dt = \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1}|\Lambda\gamma^1|^2\,dx\,dt$$

$$+ \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2}|\Lambda\gamma^2|^2\,dx\,dt.$$

Going back to (3.32), we get using (3.4) the estimate

$$s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt \leq C\Big(\int_0^T \int_0^L e^{-2s\sigma_1}|g^0|^2\,dx\,dt + \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1}|\gamma^1|^2\,dx\,dt$$

$$+ \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2}|\gamma^2|^2\,dx\,dt + s^7\lambda^8 \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt\Big) \tag{3.33}$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Now, we first prove Carleman estimate (3.28) which is somewhat simpler and then we turn to (3.29).

• Proof of (i). For $i = 1, 2$, we apply Proposition (3.1) for the equations satisfied by γ^i in system (1.21). We have

$$I_i(\gamma^i) \le C\Big(\int_0^T \int_0^L e^{-2s\sigma_i} \big(|\gamma^i_{xx}|^2 + \|\bar{y}\|^2_\infty |\gamma^i_x|^2 + \|\bar{y}_x\|^2_\infty |\gamma^i|^2 + |g^i|^2 + |\psi|^2\big)\, dx\, dt$$
$$+ s^7\lambda^8 \int_0^T \int_{\omega_i} e^{-2s\sigma_i} \xi_i^7 |\gamma^i|^2\, dx\, dt \Big), \quad i = 1, 2$$

for every $s \ge C(T^{2/3} + T^{1/3})$ and every $\lambda \ge C$. From properties (3.4), we get

$$I_i(\gamma^i) \le C\Big(\int_0^T \int_0^L e^{-2s\sigma_i} \big(|g^i|^2 + |\psi|^2\big)\, dx\, dt + s^7\lambda^8 \int_0^T \int_{\omega_i} e^{-2s\sigma_i} \xi_i^7 |\gamma^i|^2\, dx\, dt \Big), \quad i = 1, 2 \tag{3.34}$$

for every $s \ge C(T^{2/3} + T^{1/3})$ and every $\lambda \ge C$. From (3.23), we have

$$\int_0^T \int_0^L e^{-2s\sigma_i} |\psi|^2\, dx\, dt = \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_i} |\psi|^2\, dx\, dt + \int_0^T \int_{(0,L)\setminus\mathcal{O}} e^{-2s\sigma_1} |\psi|^2\, dx\, dt, \quad i = 1, 2.$$

Thus, summing inequalities (3.33) and (3.34) we obtain

$$s^7\lambda^8 \int_0^T \int_0^L e^{-2s\sigma_1} \xi_1^7 |\psi|^2\, dx\, dt + I_1(\gamma^1) + I_2(\gamma^2) \le C\Big(\int_0^T \int_{\mathcal{O}} \big(s^7\lambda^8 e^{-2s\sigma_1}\xi_1^7 + e^{-2s\sigma_2}\big)|\psi|^2\, dx\, dt$$
$$+ \int_0^T \int_0^L \big(e^{-2s\sigma_1}(|g^0|^2 + |g^1|^2) + e^{-2s\sigma_2}|g^2|^2\big)\, dx\, dt + s^7\lambda^8 \sum_{k=1,2} \int_0^T \int_{\omega_k} e^{-2s\sigma_k} \xi_k^7 |\gamma^k|^2\, dx\, dt \Big) \tag{3.35}$$

for every $s \ge C(T^{2/3} + T^{1/3})$ and every $\lambda \ge C$, where we have used again (3.4).

It remains to estimate the local terms corresponding to γ^1 and γ^2. For this, we will take advantage of the fact that we can choose ω_1 and ω_2 to satisfy (3.26) since we are assuming that (3.25) holds. Similarly to what we did in the proof of Proposition 3.6 (see (3.17) and (3.18)), we use the equations satisfied by γ^1 and γ^2 in system (1.21).

For $i = 1, 2$, let $\theta_i \in C_0^4(\mathcal{O}_{i,d} \cap \mathcal{O})$ such that $\theta \equiv 1$ in ω_i. Since (3.25) holds, observe that

$$\gamma^i = \frac{1}{\alpha_i}(-\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x - g^0) \quad \text{in } (\mathcal{O}_{i,d} \cap \mathcal{O}) \times (0, T), \quad i = 1, 2,$$

and thus we have

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$$Cs^7\lambda^8\int_0^T\int_{\omega_i} e^{-2s\sigma_i}\xi_i^7|\gamma^i|^2\,dx\,dt \le Cs^7\lambda^8\int_0^T\int_{\mathcal{O}_{i,d}\cap\mathcal{O}} e^{-2s\sigma_i}\xi_i^7\theta|\gamma^i|^2\,dx\,dt$$

$$=\frac{C}{\alpha_i}s^7\lambda^8\int_0^T\int_{\mathcal{O}_{i,d}\cap\mathcal{O}} e^{-2s\sigma_i}\xi_i^7\theta_i\gamma^i\big(-\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x-g^0\big)\,dx\,dt,\quad i=1,2.$$

Here we find the same (up to the constant C) expression as (3.19), therefore we obtain for free the estimate (compare with (3.20))

$$Cs^7\lambda^8\int_0^T\int_{\omega_i} e^{-2s\sigma_i}\xi_i^7|\gamma^i|^2\,dx\,dt \le \frac{1}{2}I_i(\gamma^i)+Cs^{15}\lambda^{16}\int_0^T\int_{\mathcal{O}} e^{-2s\sigma_i}\xi_i^{15}|\psi|^2\,dx\,dt$$
$$+Cs^7\lambda^8\int_0^T\int_{\mathcal{O}} e^{-2s\sigma_i}\xi_i^7|g^0|^2\,dx\,dt,\quad i=1,2 \qquad (3.36)$$

for every $s\ge C(T^{2/3}+T^{1/3})$ and every $\lambda\ge C$. Using this last inequality in (3.35) and properties (3.4), we find estimate (3.28).

• Proof of (ii). We now assume that (3.27) holds with $i=1$ and $j=2$ (see Remark 3.8). From (3.21) and (3.27), we choose ω_1 and ω_2 such that

$$\overline{\omega}_1\subset(\mathcal{O}_{1,d}\cap\mathcal{O})\setminus\mathcal{O}_{2,d}\quad\text{and}\quad\overline{\omega}_2\subset\mathcal{O}_{1,d}\cap\mathcal{O}_{2,d}\cap\mathcal{O}.$$

Notice that from the first equation in system (1.21) we have

$$\gamma^1=\frac{1}{\alpha_1}(-\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x-g^0)\quad\text{in }\big((\mathcal{O}_{1,d}\cap\mathcal{O})\setminus\mathcal{O}_{2,d}\big)\times(0,T),$$

thus we can combine estimates (3.33), (3.34) and (3.36) with $i=1$ to get

$$s^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7|\psi|^2\,dx\,dt+I_1(\gamma^1)\le C\bigg(s^{15}\lambda^{16}\int_0^T\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{15}|\psi|^2\,dx\,dt$$
$$+s^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7|g^0|^2\,dx\,dt+\int_0^T\int_0^L e^{-2s\sigma_1}|g^1|^2\,dx\,dt+\int_0^T\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2}|\gamma^2|^2\,dx\,dt\bigg)$$
$$(3.37)$$

for every $s\ge C(T^{2/3}+T^{1/3})$ and every $\lambda\ge C$.

To estimate the last term in (3.37), we borrow the idea from the proof of Proposition 3.6. Let $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$. From Proposition 3.1, we obtain an estimate similar to (3.17). Namely,

$$I_2(h) \leq C\left(\int_0^T \int_0^L e^{-2s\sigma_2}\left(|g^1|^2 + |g^2|^2 + |\psi|^2\right) dx\,dt + s^7\lambda^8 \int_0^T \int_{\omega_2} e^{-2s\sigma_2}\xi_2^7 |h|^2\,dx\,dt \right) \tag{3.38}$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$. Since we have

$$h = -\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x - g^0 \quad \text{in } (\mathcal{O}_{1,d} \cap \mathcal{O}_{2,d} \cap \mathcal{O}) \times (0,T),$$

we repeat the computations to obtain (3.20) (with $\theta \in C_0^4(\mathcal{O}_{1,d} \cap \mathcal{O}_{2,d} \cap \mathcal{O})$ such that $\theta \equiv 1$ in ω_2). This time we get

$$\begin{aligned} Cs^7\lambda^8 \int_0^T \int_{\omega_2} e^{-2s\sigma_2}\xi_2^7 |h|^2\,dx\,dt &\leq \frac{1}{2} I_2(h) + Cs^{15}\lambda^{16} \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_2}\xi_2^{15} |\psi|^2\,dx\,dt \\ &\quad + Cs^7\lambda^8 \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_2}\xi_2^7 |g^0|^2\,dx\,dt \end{aligned} \tag{3.39}$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

From (3.23) and (3.4), we have

$$\begin{aligned} &C \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2} |\gamma^2|^2\,dx\,dt \\ &\leq C\left(\int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1} |\gamma^1|^2\,dx\,dt + \int_0^T \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2} |h|^2\,dx\,dt \right) \\ &\leq \frac{1}{2} I_1(\gamma^1) + \frac{1}{4} I_2(h) \end{aligned} \tag{3.40}$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Putting together estimates (3.37)–(3.40) we finally get (3.29). □

4. Analysis of the linear system

This section is dedicated to prove an observability inequality from Carleman estimates in Propositions 3.6 and 3.9. Then, we will be able to establish the null controllability of linear system (1.20).

Before going into the details, let us make some remarks about the Carleman inequalities from Propositions 3.6 and 3.9.

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In the following, we will ask the functions defined in (3.23) to satisfy also that

$$\|\eta_1\|_\infty = \|\eta_2\|_\infty. \tag{4.1}$$

As it is stated in Remark 7 in [3], this can be possible even in higher dimension.
Let us unify Propositions 3.6 and 3.9.

Lemma 4.1. *Assume that condition (1.10) holds and $\bar{y} \in L^\infty(0,T;W^{1,\infty}((0,L)))$. If the functions defined by (3.23) satisfy (4.1), then*

(i) *if (1.11) holds, there exists $C>0$ such that for every $s \geq C(T^{2/3}+T^{1/3})$ and every $\lambda \geq C$ we have*

$$\begin{aligned} &s^7\lambda^8 \int_0^T\int_0^L e^{-2s\sigma^*}(\xi^*)^7|\psi|^2\,dx\,dt + s^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma^*}(\xi^*)^7|\alpha_1\gamma^1+\alpha_2\gamma^2|^2\,dx\,dt \\ &\leq C\Big(\int_0^T\int_0^L e^{-2s\hat{\sigma}}\big(s^7\lambda^8\hat{\xi}^7|g^0|^2+|g^1|^2+|g^2|^2\big)\,dx\,dt + s^{15}\lambda^{16}\int_0^T\int_{\mathcal{O}} e^{-2s\hat{\sigma}}\hat{\xi}^{15}|\psi|^2\,dx\,dt\Big), \end{aligned} \tag{4.2}$$

or

(ii) *if (1.12) holds, there exists $C_k>0$ such that for every $s \geq C_k(T^{2/3}+T^{1/3})$ and every $\lambda \geq C_k$ we have*

$$\begin{aligned} &s^7\lambda^8 \int_0^T\int_0^L e^{-2s\sigma_1^*}(\xi_1^*)^7|\psi|^2\,dx\,dt + s^7\lambda^8\int_0^T\int_0^L e^{-2s\sigma_1^*}(\xi_1^*)^7\big(|\gamma^1|^2+|\gamma^2|^2\big)\,dx\,dt \\ &\leq C_k\Big(\int_0^T\int_0^L e^{-2s\hat{\sigma}_1}\big(s^7\lambda^8\hat{\xi}_1^7|g^0|^2+|g^1|^2+|g^2|^2\big)\,dx\,dt + s^{15}\lambda^{16}\int_0^T\int_{\mathcal{O}} e^{-2s\hat{\sigma}_1}\hat{\xi}_1^{15}|\psi|^2\,dx\,dt\Big), \end{aligned} \tag{4.3}$$

where C_k may be different constants C_1 and C_2 depending if condition (3.25) holds or not, respectively, for every solution (ψ,γ^1,γ^2) of system (1.21).

Proof. The only case that is not direct from Propositions 3.6 and 3.9 is the one coming from inequality (3.29). However, it is enough to notice that $|\gamma^2| \leq \alpha_1\alpha_2^{-1}|\gamma^1| + \alpha_2^{-1}|\alpha_1\gamma^1+\alpha_2\gamma^2|$ and that under (4.1) the weights coincide when we take the minimum. □

4.1. Observability inequality

Let us now prove the observability inequality for the solutions of the adjoint system (1.21). We start by proving an energy estimate.

Lemma 4.2. *Assume that $\bar{y} \in L^\infty((0,L)\times(0,T))$. Then, there exist $\mu_0 > 0$ such that for every $\mu_1, \mu_2 \geq \mu_0$ and every $g^0, g^1, g^2 \in L^2((0,L)\times(0,T))$ and every $\psi^T \in L^2(0,L)$, the solution $(\psi, \gamma^1, \gamma^2)$ of system (1.21) satisfies*

$$\begin{aligned}
&\|\psi\|^2_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))} + \|\gamma^1\|^2_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))} \\
&\quad + \|\gamma^2\|^2_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))} + \|\alpha_1\gamma^1 + \alpha_2\gamma^2\|^2_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))} \\
&\leq C\Big(\|g^0\|^2_{L^2((0,L)\times(0,T))} + \|g^1\|^2_{L^2((0,L)\times(0,T))} + \|g^2\|^2_{L^2((0,L)\times(0,T))} + \|\psi^T\|^2_{L^2((0,L))}\Big), \qquad (4.4)
\end{aligned}$$

where $C > 0$ is a constant independent of μ_1 and μ_2.

Proof. Let us multiply the first equations of system (1.21) by ψ and integrate in space. Integration by parts yields

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\int_0^L |\psi|^2\,dx + \int_0^L |\psi_{xx}|^2\,dx \\
&\qquad = -\nu\int_0^L \psi_{xx}\psi\,dx + \int_0^L \bar{y}\psi_x\psi\,dx + \int_0^L g^0\psi\,dx + \alpha_1\int_{\mathcal{O}_{1,d}} \gamma^1\psi\,dx + \alpha_2\int_{\mathcal{O}_{2,d}} \gamma^2\psi\,dx.
\end{aligned}$$

Using Poincaré's and Young's inequalities, we have

$$-\frac{1}{2}\frac{d}{dt}\int_0^L |\psi|^2\,dx + \frac{1}{2}\int_0^L |\psi_{xx}|^2\,dx \leq C\int_0^L |\psi|^2\,dx + \int_0^L |g^0|^2\,dx + \int_0^L \big(|\gamma^1_{xx}|^2 + |\gamma^2_{xx}|^2\big)\,dx,$$

where C is a constant depending on $\|\bar{y}\|_\infty, \nu, \alpha_1$ and α_2, but independent of μ_1 and μ_2. Then, by Gronwall's inequality, we obtain for all $t \in (0,T)$

$$\begin{aligned}
&\int_0^L |\psi(t)|^2\,dx + \int_0^T\int_0^L |\psi_{xx}|^2\,dx\,dt \\
&\qquad \leq C\Big(\int_0^L |\psi^T|^2\,dx + \int_0^T\int_0^L |g^0|^2\,dx\,dt + \int_0^T\int_0^L \big(|\gamma^1_{xx}|^2 + |\gamma^2_{xx}|^2\big)\,dx\,dt\Big). \qquad (4.5)
\end{aligned}$$

Similar computations for the equations satisfied by γ^1 and γ^2 show that, for $i = 1, 2$,

$$\frac{1}{2}\frac{d}{dt}\int_0^L |\gamma^i|^2\,dx + \int_0^L |\gamma^i_{xx}|^2\,dx = -\nu\int_0^L \gamma^i_{xx}\gamma^i\,dx + \int_0^L \bar{y}\gamma^i\gamma^i_x\,dx + \int_0^L g^i\gamma^i\,dx + \frac{1}{\mu_i}\int_{\mathcal{O}_i} \psi\gamma^i\,dx$$

and then

$$\frac{1}{2}\frac{d}{dt}\int_0^L |\gamma^i|^2\,dx + \frac{1}{2}\int_0^L |\gamma^i_{xx}|^2\,dx \le C\int_0^L |\gamma^i|^2\,dx + \int_0^L |g^i|^2\,dx + \frac{1}{\mu_i^2}\int_0^L |\psi|^2\,dx,$$

where C is a constant depending on $\|\bar{y}\|_\infty$ and ν, but independent of μ_1 and μ_2. Since $\gamma^i(x,0)=0$, $i=1,2$, we get from Gronwall's inequality

$$\int_0^L |\gamma^i(t)|^2\,dx + \int_0^T\int_0^L |\gamma^i_{xx}|^2\,dx\,dt \le C\left(\int_0^T\int_0^L |g^i|^2\,dx\,dt + \frac{1}{\mu_i^2}\int_0^L |\psi(t)|^2\,dx\right), \quad i=1,2. \tag{4.6}$$

Now, let $h := \alpha_1\gamma^1 + \alpha^2\gamma^2$. Since h satisfies the second equation in system (3.14), we can obtain similarly as before the estimate

$$\begin{aligned}&\int_0^L |h(t)|^2\,dx + \int_0^T\int_0^L |h_{xx}|^2\,dx\,dt \\ &\le C\left(\int_0^T\int_0^L \left(|g^1|^2 + |g^2|^2\right)dx\,dt + \left(\frac{1}{\mu_1^2} + \frac{1}{\mu_2^2}\right)\int_0^L |\psi(t)|^2\,dx\right).\end{aligned} \tag{4.7}$$

Adding estimates (4.5)–(4.7) (with a rescaling of the constants if necessary), we obtain (4.4) by taking μ_1 and μ_2 sufficiently large. □

We are ready to prove the observability inequality (1.22). To do this, we need to introduce some new weight functions similar to the ones in (3.2) and (3.24) that do not degenerate at $t=0$. Let $\tau(t) \in C^1([0,T])$ be defined by

$$\tau(t) = \begin{cases} (T/2)^{2/3} & t \in [0, T/2), \\ t^{1/3}(T-t)^{1/3} & t \in [T/2, T], \end{cases}$$

and

$$\begin{aligned}\beta_i(x,t) &:= \frac{\exp\left(4\lambda\|\eta_i\|_\infty\right) - \exp\left(\lambda(\|\eta_i\|_\infty + \eta_i(x))\right)}{\tau(t)}, \\ \zeta_i(x,t) &:= \frac{\exp\left(\lambda(\|\eta_i\|_\infty + \eta_i(x))\right)}{\tau(t)}, \quad i=1,2,\end{aligned} \tag{4.8}$$

where $\lambda > 1$. Similarly, we define (β, ζ) corresponding to (σ, ξ) in (3.2).

Notice that properties (3.4) are valid for these new functions and we keep the notation introduced at the beginning of the section.

The observability inequality is given by the following proposition.

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Proposition 4.3. *Assume that condition* (1.10) *holds and* $\bar{y} \in L^\infty(0,T;W^{1,\infty}((0,L)))$. *Also, let the functions defined by* (3.23) *satisfy* (4.1) *and, s and λ be constants such that Lemma 4.1 is verified. If either* (1.11) *or* (1.12) *hold, then there exists a constant $C>0$ such that every solution* (ψ,γ^1,γ^2) *to system* (1.21) *satisfies*

$$\int_0^L |\psi(x,0)|^2\,dx + \int_0^T\int_0^L e^{-2s\beta_1^*}(\zeta_1^*)^7|\psi|^2\,dx\,dt + \int_0^T\int_0^L e^{-2s\beta_1^*}\big(|\gamma^1|^2+|\gamma^2|^2\big)\,dx\,dt$$
$$\leq C\Big(\int_0^T\int_0^L e^{-2s\hat{\beta}_1}\big(\hat{\zeta}_1^7|g^0|^2+|g^1|^2+|g^2|^2\big)\,dx\,dt + \int_0^T\int_{\mathcal{O}} e^{-2s\hat{\beta}_1}\hat{\zeta}_1^{15}|\psi|^2\,dx\,dt\Big). \quad (4.9)$$

Remark 4.4. In the spirit of unify both cases (1.11) and (1.12) is that we have considered only the weight functions (β_1,ζ_1). Of course, when (1.11) holds, it suffices to replace (β,ζ) by (β_1,ζ_1) and take ω_1 to only satisfy $\overline{\omega}_1 \subset \mathcal{O}_{1,d}\cap\mathcal{O}$.

Proof. It is classical to prove (4.9) from (4.2), (4.3), (4.4), and the fact that

$$(\zeta_1^*)^7 \geq \frac{4^{7/3}}{T^{14/3}}.$$

For details, please see [12, Lemma 1] or [4, Proposition 4.1], for instance. On the other hand, if (1.11) holds, we can get from (4.2) and (4.4) the estimate

$$\int_0^L |\psi(x,0)|^2\,dx + \int_0^T\int_0^L e^{-2s\beta^*}(\zeta^*)^7|\psi|^2\,dx\,dt + \int_0^T\int_0^L e^{-2s\beta^*}\big(|\alpha_1\gamma^1+\alpha_2\gamma^2|^2\big)\,dx\,dt$$
$$\leq C\Big(\int_0^T\int_0^L e^{-2s\hat{\beta}}\big(\hat{\zeta}^7|g^0|^2+|g^1|^2+|g^2|^2\big)\,dx\,dt + \int_0^T\int_{\mathcal{O}} e^{-2s\hat{\beta}}\hat{\zeta}^{15}|\psi|^2\,dx\,dt\Big).$$

To add the weighted integrals of γ^1 and γ^2, we do the following. Let $\rho(t)=e^{-s\beta^*}$. Notice that $\rho(t)$ is a positive non-increasing function in $(0,T)$. If we denote $\widetilde{\gamma}^i := \rho(t)\gamma^i$, we obtain from system (1.21) that

$$\widetilde{\gamma}^i_t + \widetilde{\gamma}^i_{xxxx} + \nu\widetilde{\gamma}^i_{xx} + (\bar{y}\widetilde{\gamma}^i)_x = \rho(t)g^i - \frac{1}{\mu_i}\rho(t)\psi\mathbb{1}_{\mathcal{O}_i} + \rho'(t)\gamma^i,$$
$$(x,t)\in(0,L)\times(0,T), \quad i=1,2.$$

As in the proof of Lemma 4.2, we multiply this equation by $\widetilde{\gamma}^i$ and integrate in $(0,L)$. We have for $i=1,2$

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$$\frac{1}{2}\frac{d}{dt}\int_0^L |\widetilde{\gamma}^i|^2\,dx + \frac{1}{2}\int_0^L |\widetilde{\gamma}^i_{xx}|^2\,dx \le C\int_0^L |\widetilde{\gamma}^i|^2\,dx + \int_0^L |\rho(t)g^i|^2\,dx + \frac{1}{\mu_i^2}\int_0^L |\rho(t)\psi|^2\,dx + \int_0^L \rho'(t)\rho(t)|\gamma^i|^2\,dx.$$

Since the last term is negative and $\widetilde{\gamma}^i(x,0)=0$, we obtain

$$\int_0^T\int_0^L e^{-2s\beta^*}|\gamma^i|^2\,dx\,dt \le C\left(\int_0^T\int_0^L e^{-2s\beta^*}|g^i|^2\,dx\,dt + \frac{1}{\mu_i^2}\int_0^T\int_0^L e^{-2s\beta^*}|\psi|^2\,dx\,dt\right), \quad i=1,2.$$

From here, it is easy to obtain (4.9) from properties (3.4) and the fact that $e^{-2s\beta^*}\le e^{-2s\hat{\beta}}$. □

4.2. Null controllability of the linear system

Let us denote by $\mathcal{L}$ the linear operator

$$\mathcal{L}u := u_t + u_{xxxx} + \nu u_{xx} + (\bar{y}u)_x$$

and by $\mathcal{L}^*$ its formal adjoint

$$\mathcal{L}^*u := -u_t + u_{xxxx} + \nu u_{xx} - \bar{y}u_x.$$

Also, consider the functional space

$$\begin{aligned}\mathcal{S} = \Big\{(z,\phi_1,\phi_2,f) :\ & e^{s\hat{\beta}_1}\hat{\zeta}_1^{-7/2}z \in L^2((0,L)\times(0,T)),\ e^{s\hat{\beta}_1}\phi^i \in L^2((0,L)\times(0,T))),\ i=1,2,\\ & e^{s\hat{\beta}_1}\hat{\zeta}_1^{-15/2}f \in L^2(\mathcal{O}\times(0,T)),\\ & e^{s\beta_1^*}(\zeta_1^*)^{-7/2}(\mathcal{L}z - f\mathbb{1}_{\mathcal{O}} + \mu_1^{-1}\phi^1\mathbb{1}_{\mathcal{O}_1} + \mu_2^{-1}\phi^2\mathbb{1}_{\mathcal{O}_2})\\ & \quad \in L^2((0,L)\times(0,T)),\\ & e^{s\beta_1^*}(\mathcal{L}^*\phi^i - \alpha_i z\mathbb{1}_{\mathcal{O}_{i,d}}) \in L^2(((0,L)\times(0,T)),\ i=1,2,\\ & e^{s\hat{\beta}_1}\hat{\zeta}_1^{-15/2}z \in L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L)),\\ & e^{s\hat{\beta}_1}\hat{\zeta}_1^{-15/2}\phi^i \in L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L)),\ i=1,2\Big\},\end{aligned}$$

which is a Banach space endowed with its natural norm.

We have the following result.

Proposition 4.5. *Let the assumptions of Proposition 4.3 be satisfied. Then, for any $z^0\in L^2(0,L)$, and any triplet (f^0,f^1,f^2) such that*

$$\int_0^T\int_0^L e^{2s\beta_1^*}(\zeta_1^*)^{-7}|f^0|^2\,dx\,dt < +\infty, \qquad \int_0^T\int_0^L e^{2s\beta_1^*}|f^i|^2\,dx\,dt < +\infty, \quad i=1,2, \tag{4.10}$$

there exists a control $f \in L^2((0,L)\times(0,T))$ *such that the solution* (z,ϕ^1,ϕ^2) *of system* (1.20) *satisfies* $(z,\phi_1,\phi_2,f)\in\mathcal{S}$. *In particular,* $z(x,T)=0$ *in* $(0,L)$.

Proof. We follow a classical strategy (see [12,4], for instance). Consider the space

$$\begin{aligned}\mathcal{P}_0=\big\{&(p,q^1,q^2)\in C^4([0,L]\times[0,T]):\\ &p(0,t)=q^i(0,t)=p(L,t)=q^i(L,t)=0 \quad \forall t\in(0,T), i=1,2,\\ &p_x(0,t)=q^i_x(0,t)=p_x(L,t)=q^i_x(L,t)=0 \quad \forall t\in(0,T), i=1,2,\\ &q^i(x,0)=0 \quad \forall x\in(0,L), i=1,2\big\}.\end{aligned}$$

Now, let $b:\mathcal{P}_0\times\mathcal{P}_0\to\mathbb{R}$ be the bilinear functional

$$\begin{aligned}&b\big((p,q^1,q^2),(r,w^1,w^2)\big)\\ &=\int_0^T\int_0^L e^{-2s\hat\beta_1}\hat\zeta_1^7\big(\mathcal{L}^*p-\alpha_1q^1\mathbb{1}_{\mathcal{O}_{1,d}}-\alpha_2q^2\mathbb{1}_{\mathcal{O}_{2,d}}\big)\big(\mathcal{L}^*r-\alpha_1w^1\mathbb{1}_{\mathcal{O}_{1,d}}-\alpha_2w^2\mathbb{1}_{\mathcal{O}_{2,d}}\big)\,dx\,dt\\ &\quad+\sum_{i=1}^2\int_0^T\int_0^L e^{-2s\hat\beta_1}\Big(\mathcal{L}q^i+\frac{1}{\mu_i}p\mathbb{1}_{\mathcal{O}_i}\Big)\Big(\mathcal{L}w^i+\frac{1}{\mu_i}r\mathbb{1}_{\mathcal{O}_i}\Big)\,dx\,dt+\int_0^T\int_{\mathcal{O}} e^{-2s\hat\beta_1}\hat\zeta_1^{15}\,p\,r\,dx\,dt\end{aligned}$$

and $\ell:\mathcal{P}_0\to\mathbb{R}$ the linear functional

$$\ell(r,w^1,w^2)=\int_0^T\int_0^L\big(f^0r+f^1w^1+f^2w^2\big)\,dx\,dt+\int_0^L z^0(x)r(x,0)\,dx.$$

Observability inequality (4.9) allows to show that $b(\cdot,\cdot)^{1/2}$ defines a norm in $\mathcal{P}_0$. We denote by $\mathcal{P}$ the closure of $\mathcal{P}_0$ with respect to this norm. Furthermore, $\mathcal{P}$ is a Hilbert space with the inner product coming from b and, again from (4.9), ℓ is bounded in $\mathcal{P}$. Therefore, by Lax–Milgram's Lemma, we deduce that the problem: find $(p,q^1,q^2)\in\mathcal{P}$ such that

$$b\big((p,q^1,q^2),(r,w^1,w^2)\big)=\ell(r,w^1,w^2)\quad\forall(r,w^1,w^2)\in\mathcal{P}\tag{4.11}$$

possesses a unique solution that we call $(\hat p,\hat q^1,\hat q^2)$.

Define

$$\begin{cases}\hat z=e^{-2s\hat\beta_1}\hat\zeta_1^7\big(\mathcal{L}^*\hat p-\alpha_1\hat q^1\mathbb{1}_{\mathcal{O}_{1,d}}-\alpha_2\hat q^2\mathbb{1}_{\mathcal{O}_{2,d}}\big),\\ \hat\phi^i=e^{-2s\hat\beta_1}\Big(\mathcal{L}\hat q^i+\dfrac{1}{\mu_i}\hat p\mathbb{1}_{\mathcal{O}_i}\Big),\quad i=1,2,\\ \hat f=-e^{-2s\hat\beta_1}\hat\zeta_1^{15}\hat p\mathbb{1}_{\mathcal{O}}.\end{cases}\tag{4.12}$$

We will show now that $(\hat z,\hat\phi^1,\phi^2)$ is actually the solution of the linear system (1.20) associated $\hat f$ and that $(\hat z,\hat\phi^1,\hat\phi^2,\hat f)$ belongs to $\mathcal{S}$, therefore solving the null controllability problem.

Since $b\big((\hat{z},\hat{\phi}^1,\hat{\phi}^2)\big) < +\infty$, we have that

$$\int_0^T\int_0^L e^{2s\hat{\beta}_1}\hat{\xi}_1^{-7}|\hat{z}|^2\,dx\,dt + \sum_{i=1}^2\int_0^T\int_0^L e^{2s\hat{\beta}_1}|\hat{\phi}^i|^2\,dx\,dt + \int_0^T\int_{\mathcal{O}} e^{2s\hat{\beta}_1}\hat{\xi}_1^{-15}|\hat{f}|^2\,dx\,dt < +\infty. \tag{4.13}$$

Now, let $(\widetilde{z},\widetilde{\phi}^1,\widetilde{\phi}^2)$ be the weak solution of system (1.20) with $f=\hat{f}$ defined in (4.12). It is clear that $(\widetilde{z},\widetilde{\phi}^1,\widetilde{\phi}^2)$ is also the unique solution by transposition of (1.20) (see Definition 3.3). This means that

$$\int_0^T\int_0^L \big(\widetilde{z}g^0+\widetilde{\phi}^1g^1+\widetilde{\phi}^2g^2\big)\,dx\,dt = \int_0^T\int_0^L \big(f^0r+f^1w^1+f^2w^2\big)\,dx\,dt + \int_0^L z^0(x)r(x,0)\,dx \tag{4.14}$$

for all $(g^0,g^1,g^2)\in L^2((0,T)\times(0,L))^3$, where (r,w^1,w^2) is the solution of the system

$$\begin{cases} \mathcal{L}^*r = g^0+\alpha_1w^1\mathbb{1}_{\mathcal{O}_{1,d}}+\alpha_2w^2\mathbb{1}_{\mathcal{O}_{2,d}} & (x,t)\in(0,L)\times(0,T),\\ \mathcal{L}w^i = g^i-\dfrac{1}{\mu_i}r\mathbb{1}_{\mathcal{O}_i} \quad i=1,2 & (x,t)\in(0,L)\times(0,T),\\ r(0,t)=r(L,t)=w^i(0,t)=w^i(L,t)=0 \quad i=1,2 & t\in(0,T),\\ r_x(0,t)=r_x(L,t)=w^i_x(0,t)=w^i_x(L,t)=0 \quad i=1,2 & t\in(0,T),\\ r(x,T)=0, \quad w^i(x,0)=0 \quad i=1,2 & x\in(0,L). \end{cases}$$

From (4.11) (with $(p,q^1,q^2)=(\hat{p},\hat{q}^1,\hat{q}^2)$) and (4.12), we find that the triplet $(\hat{z},\hat{\phi}^1,\hat{\phi}^2)$ also satisfies (4.14). Thus, $(\hat{z},\hat{\phi}^1,\hat{\phi}^2)$ coincides with $(\widetilde{z},\widetilde{\phi}^1,\widetilde{\phi}^2)$ and must be the weak solution of (1.20) associated to $\hat{f}$.

Finally, let $z_*:=e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2}\hat{z}$ and $\phi^i_*:=e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2}\hat{\phi}^i$, $i=1,2$. Then, (z_*,ϕ^1_*,ϕ^2_*) solves

$$\begin{cases} \mathcal{L}z_* = f^0_*+f_*\mathbb{1}_{\mathcal{O}}-\dfrac{1}{\mu_1}\phi^1_*\mathbb{1}_{\mathcal{O}_1}-\dfrac{1}{\mu_2}\phi^2_*\mathbb{1}_{\mathcal{O}_2}+(e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2})_t\hat{z} & (x,t)\in(0,L)\times(0,T),\\ \mathcal{L}^*\phi^i_* = f^i_*+\alpha_iz_*\mathbb{1}_{\mathcal{O}_{i,d}}-(e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2})_t\hat{\phi}^i \quad i=1,2 & (x,t)\in(0,L)\times(0,T),\\ z_*(0,t)=z_*(L,t)=\phi^i_*(0,t)=\phi^i_*(L,t)=0 \quad i=1,2 & t\in(0,T),\\ z_{*x}(0,t)=z_{*x}(L,t)=\phi^i_{*x}(0,t)=\phi^i_{*x}(L,t)=0 \quad i=1,2 & t\in(0,T),\\ z_*(x,0)=e^{s\beta_1^*(0)}\hat{\xi}_1^{-15/2}(0)z^0(x), \quad \phi^i_*(x,T)=0 & x\in(0,L), \end{cases}$$

where we have denoted as well $(f^0_*,f^i_*,f_*)=e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2}(f^0,f^i,\hat{f})$, $i=1,2$. From the fact that $|(e^{s\hat{\beta}_1}\hat{\xi}_1^{-15/2})_t|\le Ce^{s\hat{\beta}_1}\hat{\xi}_1^{-7/2}$, (4.10), (4.13), the definition of the weight functions and properties (3.4), we see that the terms in the right-hand side of the equations of this system belong to $L^2((0,L)\times(0,T))$.

Therefore, using a regularity result similar to Lemma 4.2 (eventually for μ_i large enough) we deduce that (z_*,ϕ^1_*,ϕ^2_*) belongs to $\Big(L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2(0,L))\Big)^3$. This completes the proof of Proposition 4.5. □

5. Null controllability of the non-linear system

In this section we complete the proof of Theorem 1.2. As mentioned in the introduction, it is equivalent to a local null controllability result for the optimality system (1.18). More precisely, we prove the following result.

Theorem 5.1. *Let the assumptions of Theorem 1.2 be satisfied. Then there exist $\delta > 0$ and positive functions $\hat{\rho}_i = \hat{\rho}_i(t)$ blowing up at $t = T$ such that if z^0 and $z_{i,d}$ satisfy*

$$\|z^0\|^2_{L^2(0,L)} + \sum_{i=1,2} \int_0^T \int_{\mathcal{O}_{i,d}} \hat{\rho}_i^2 |z_{i,d}|^2 \, dx \, dt < \delta,$$

there exist a control $f \in L^2(\mathcal{O} \times (0,T))$ and a associated solution (z, ϕ^1, ϕ^2) of the optimality system (1.18) such that $z(x,T) = 0$ in $(0,L)$.

The proof is based on a local inversion argument, for which we use the following theorem ((see [1])).

Theorem 5.2. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ satisfy $\mathcal{F} \in C^1(\mathcal{B}_1; \mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1$, $\mathcal{F}(b_1) = b_2$ and that $\mathcal{F}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{F}(b) = b', \quad b \in \mathcal{B}_1.$$

Proof. Let us define the spaces

$$\mathcal{B}_1 := \mathcal{S},$$
$$\mathcal{B}_2 := X \times L^2(0,L) \times Y \times Y,$$

where

$$X := \left\{ u : e^{s\beta_1^*} (\zeta_1^*)^{-7/2} u \in L^2((0,L) \times (0,T)) \right\}$$

and

$$Y := \left\{ u : e^{s\beta_1^*} u \in L^2((0,L) \times (0,T)) \right\}.$$

For every $(z, \phi^1, \phi^2, f) \in \mathcal{S}$, let the operator $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be defined by

$$\mathcal{F}(z, \phi^1, \phi^2, f) := \begin{pmatrix} \mathcal{L}z + zz_x - f \mathbb{1}_{\mathcal{O}} + \mu_1^{-1} \phi^1 \mathbb{1}_{\mathcal{O}_1} + \mu_2^{-1} \phi^2 \mathbb{1}_{\mathcal{O}_2} \\ z(x,0) \\ \mathcal{L}^* \phi^1 - z\phi^1 - \alpha_1 z \mathbb{1}_{\mathcal{O}_{1,d}} \\ \mathcal{L}^* \phi^2 - z\phi^2 - \alpha_2 z \mathbb{1}_{\mathcal{O}_{2,d}} \end{pmatrix}.$$

To check the hypothesis of Theorem 5.2, the following lemma will be useful.

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Lemma 5.3. *Let* $Z := \left\{u : e^{s\hat{\beta}}\hat{\zeta}^{-15/2}u \in L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^2(0,L))\right\}$. *The map*

$$(u^1, u^2) \mapsto u^1 u^2_x$$

is continuous from Z to X and from Z to Y.

Proof. It suffices to check that $\|u^1 u^2_x\|_X \leq C\|u^1\|_Z\|u^2\|_Z$ and $\|u^1 u^2_x\|_Y \leq C\|u^1\|_Z\|u^2\|_Z$. We only do the first, since the other is analogous. Notice that the function

$$e^{-2s\hat{\beta}_1 + s\beta^*}\hat{\zeta}_1^{15}(\zeta_1^*)^{-7/2}$$

is bounded (eventually for a large λ). Then for any $u^1, u^2 \in X$ we have

$$\begin{aligned}
\|u^1 u^2_x\|_X &= \|e^{s\beta_1^*}(\zeta_1^*)^{-7/2}u^1 u^2_x\|_{L^2((0,L)\times(0,T))} \\
&\leq C\|e^{2s\hat{\beta}_1}\hat{\zeta}_1^{-15}u^1 u^2_x\|_{L^2((0,L)\times(0,T))} \\
&\leq C\|e^{s\hat{\beta}_1}\hat{\zeta}_1^{-15/2}u^1\|_{L^\infty((0,L);L^2(0,T))}\|e^{s\hat{\beta}_1}\hat{\zeta}_1^{-15/2}u^2_x\|_{L^2((0,L);L^\infty(0,T))} \\
&\leq C\|u^1\|_Z\|u^2\|_Z. \quad \square
\end{aligned}$$

From the definition of $\mathcal{B}_1, \mathcal{B}_2$ and Lemma 5.3, it is fairly simple to check that $\mathcal{F}$ is well defined and of class $C^1(\mathcal{B}_1;\mathcal{B}_2)$. Furthermore,

$$\mathcal{F}'(0,0,0,0)(z,\phi^1,\phi^2,f) = \begin{pmatrix} \mathcal{L}z - f\mathbb{1}_{\mathcal{O}} + \mu_1^{-1}\phi^1\mathbb{1}_{\mathcal{O}_1} + \mu_2^{-1}\phi^2\mathbb{1}_{\mathcal{O}_2} \\ z(x,0) \\ \mathcal{L}^*\phi^1 - \alpha_1 z\mathbb{1}_{\mathcal{O}_{1,d}} \\ \mathcal{L}^*\phi^2 - \alpha_2 z\mathbb{1}_{\mathcal{O}_{2,d}} \end{pmatrix}$$

is surjective from $\mathcal{B}_1$ to $\mathcal{B}_2$ thanks to Proposition 4.5. From Theorem 5.2 with $b_1 = (0,0,0,0)$, $b_2 = (0,0,0,0)$ and $b' = (0, z^0, -\alpha_1 z_{1,d}\mathbb{1}_{\mathcal{O}_{1,d}}, -\alpha_2 z_{2,d}\mathbb{1}_{\mathcal{O}_{2,d}})$ we obtain the existence of a positive number δ such that if

$$\|z^0\|^2_{L^2(0,L)} + \sum_{i=1,2}\int_0^T\int_{\mathcal{O}_{i,d}} e^{2s\beta_1^*}|z_{i,d}|^2\,dx\,dt < \delta,$$

there exists (z,ϕ^1,ϕ^2,f) solution to system (1.18) belonging to $\mathcal{S}$. In particular, $z(x,T) = 0$ in $(0,L)$ and the proof is done. $\square$

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Appendix A. Well-possedness of the optimality system (1.18)

Consider the following linear KS equation

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = f & (x,t) \in (0,L) \times (0,T), \\ y(0,t) = y(L,t) = 0 & t \in (0,T), \\ y_x(0,t) = y_x(L,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L). \end{cases} \tag{A.1}$$

In order to prove the existence of solution for (1.18) we will first study the existence of solution for (A.1). We have the following

Lemma A.1. *Let $A : D(A) \subset L^2(0,L) \to L^2(0,L)$ the operator defined by:*

$$\begin{cases} D(A) = [H^4 \cap H_0^2](0,L), \\ Ay = -y_{xxxx} - \nu y_{xx}. \end{cases}$$

Then A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0} : L^2(0,L) \to L^2(0,L)$. Moreover, if $y_0 \in L^2(0,L)$, $u(t) = T(t)y_0$ is the unique solution of (A.1) in the space

$$u \in C([0,T; L^2(0,L)]) \cap L^2(0,T; H_0^2(0,L)).$$

Proof. The first thing to do is to prove that $(\frac{\nu^2}{4}, \infty) \subset \rho(A)$, more precisely we will show that for each $f \in L^2(0,L)$ and $\sigma \in (\frac{\nu^2}{4}, \infty)$ problem

$$\sigma u - Au = f \tag{A.2}$$

has a unique solution $u \in D(A)$.

Define the bilinear form $B : [H_0^2(0,L)]^2 \to \mathbb{R}$ given by

$$B(u,v) = \sigma(u,v) + (u_{xx}, v_{xx}) + \nu(u_{xx}, v).$$

It is clear that B is continuous in $H_0^2(0,L)$ and it is easy to see that

$$B(u,u) = \sigma|u|^2 + |u_{xx}|^2 + \nu(u_{xx}, u) \geq (1 - \frac{\nu^2}{4\sigma})|u_{xx}|^2.$$

Then, for $\sigma > \frac{\nu^2}{4}$ the bilinear functional B is coercive. We also have that $L : H_0^2(0,L) \to \mathbb{R}$ given by $L(v) = \langle f, v \rangle_{H^{-2}(0,L), H_0^2(0,L)}$ is continuous and

$$|L(v)| \leq \|f\|_{H^{-2}(0,L)} \|v\|_{H_0^2(0,L)}.$$

Using the Lax Milgram theorem, there exists a unique $u^* \in H_0^2(0,L)$ solution of

$$B(u^*, v) = L(v), \quad \forall v \in H_0^2(0,L).$$

We conclude that, if $f \in H^{-2}(0, L)$ and $\sigma > \frac{\nu^2}{4}$, problem (A.2) has a unique solution $u^* \in H_0^2(0, L)$. If $f \in L^2(0, L)$ then, using (A.2), we see that $u^*_{xxxx} = f - \sigma u^* - \nu u^*_{xx}$ and then we obtain that $u^* \in D(A)$. Let us define the resolvent operator $R(\sigma, A) : L^2(0, L) \to L^2(0, L)$ by

$$R(\sigma, A) = (\sigma Id - A)^{-1}, \quad \sigma > \frac{\nu^2}{4}.$$

We observe that

$$\begin{aligned}(R(\sigma, A)f, f) = (u, \sigma u - Au) = (u, \sigma u + u_{xxxx} + \nu u_{xx}) = \sigma |u|^2 + |u_{xx}|^2 + \nu (u_{xx}, u) \\ \geq (\sigma - \frac{\nu^2}{4})|u|^2 = (\sigma - \frac{\nu^2}{4})|R(\sigma, A) f|^2\end{aligned}$$

This proves that

$$|R(\sigma, A)^n f| \leq \frac{1}{(\sigma - \frac{\nu^2}{4})^n} |f|.$$

By [21, Theorem 5.3] we see that A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ and

$$\|T(t)\| \leq e^{\frac{\nu^2}{4} t}. \quad \square$$

As a consequence we have the following:

Corollary A.2. *Let $y^0 \in L^2(0, L)$ and $f \in L^1([0, T]; L^2(0, L))$, then problem (A.1) has a unique* mild solution *$y \in C([0, T]; L^2(0, L))$ given by*

$$y(t) = T(t)y^0 + \int_0^t T(t - s) f(s)\, ds. \tag{A.3}$$

If, moreover, $y_0 \in D(A)$ and $f \in C^1([0, T]; L^2(0, L))$ then the mild solution given by (A.3) is the unique solution for the inhomogeneous boundary value problem (A.1) in the class $C([0, T]; D(A)) \cap C^1([0, T]; L^2(0, L))$.

Proof. The proof comes from the fact that the $T(t)$ is strongly continuous. See Pazy ([21], p. 107). $\square$

Remark A.3. For $y_0 \in L^2(0, L)$ and $f \in L^1(0, T; L^2(0, L))$ the mild solution (A.3) has the regularity $y \in L^2(0, T; H_0^2(0, L))$, moreover the following estimate holds:

$$\|y(t)\|_{L^2(0,L)} + \|y\|_{L^2(0,T;H_0^2(0,L))} \leq C_\nu \left(\|y_0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} \right), \; \forall t \in [0, T]. \tag{A.4}$$

If $a_0, a_{1x} \in L^\infty(Q)$ and $g \in L^1(0, T; L^2(0, L))$, taking $f = g - a_0 y - a_1 y_x$ in (A.1), then we can prove that the equation

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$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} + a_0 y + a_1 y_x = g, \\ y(0,t) = y(L,t) = 0, \\ y_x(0,t) = y_x(L,T) = 0, \\ y(x,0) = y^0(x), \end{cases} \tag{A.5}$$

has a unique solution $y \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^2(0,L))$ and there exists a constant $C_0 := C_0(\nu, \|a_0\|_\infty, \|a_{1x}\|_\infty)$ such that

$$\|y(t)\|_{L^2(0,L)} + \|y\|_{L^2(0,T;H_0^2(0,L))} \le C_0 \left(\|y_0\|_{L^2(0,L)} + \|g\|_{L^1(0,T;L^2(0,L))}\right). \tag{A.6}$$

The next step is the study of the nonlinear problem

$$\begin{cases} y_t + y_{xxxx} + y y_x + \nu y_{xx} + a_0 y + a_1 y_x = f, \\ y(0,t) = y(L,t) = 0, \\ y_x(0,t) = y_x(L,T) = 0, \\ y(x,0) = y^0(x). \end{cases} \tag{A.7}$$

Theorem A.4. *Let $\mathcal{B} = C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^2(0,L))$. There exist $r > 0$ such that for each $y^0 \in L^2(0,L)$ and $f \in L^1(0,T; L^2(0,L))$ satisfying*

$$\|y^0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))} \le r$$

problem (A.7) has a unique solution in $\mathcal{B}$.

Proof. Let $\tilde{r} = \|y^0\|_{L^2(0,L)} + \|f\|_{L^1(0,T;L^2(0,L))}$. By Corollary A.2 and Remark A.3 we can define $\Lambda : L^2(0,T; W^{1,4}(0,L)) \to L^2(0,T; W^{1,4}(0,L))$ such that $\Lambda(\tilde{y}) = y$ where $y \in \mathcal{B}$ is the unique solution of (A.5) with $g = f - \tilde{y}\tilde{y}_x$. Notice that

$$\int_0^T \left(\int_0^L |\tilde{y}\tilde{y}_x|^2 \, dx \right)^{\frac{1}{2}} dt \le \int_0^T \|\tilde{y}\|^2_{W^{1,4}(0,L)} \, dt$$

and then, using (A.6) and the fact that $L^2(0,T; H_0^2(0,L)) \hookrightarrow L^2(0,T; W^{1,4}(0,L))$ we have

$$\|y\|_{L^2(0,T;W^{1,4}(0,L))} \le C_0 \left(\|y^0\|_{L^2(0,L)} + \int_0^T \left(\|f(t)\|_{L^2(0,L)} + \|\tilde{y}(t)\|^2_{W^{1,4}(0,L)} \right) dt \right). \tag{A.8}$$

From the equation (A.5) we have that $y_t \in L^2(0,T; H^{-2}(0,L))$. Also, the immersion $H_0^2(0,L) \hookrightarrow W^{1,4}(0,L)$ is compact. Hence by Aubin–Lions compactness Lemma we conclude that Λ is compact.

Let $\mathbf{B}_K = \{u \in L^2(0,T; W^{1,4}(0,L)); \|u\|_{L^2(0,T;W^{1,4}(0,L))} \le K\}$. Then if $\tilde{y} \in \mathbf{B}_K$ we have by (A.8) that

$$\|y\|_{L^2(0,T;W^{1,4}(0,L))} \le C_0(\tilde{r} + K^2). \tag{A.9}$$

If $K < \frac{1}{C_0}$ and $\tilde{r} \leq \frac{K-C_0K^2}{C_0}$ we have that $\Lambda(\mathbf{B}_K) \subset \mathbf{B}_K$. By Schauder's fixed point theorem, Λ has fixed point and the existence of solution is proved by taking $r = \frac{K-C_0K^2}{C_0}$. It is easy to see that the solutions of (A.7) satisfies the same inequality (A.6) with the constant C_0, this can be used to prove the uniqueness of solution for (A.7). $\square$

We will now study the existence of solution for (1.18).

Theorem A.5. *Let r given by Theorem A.4 and assume that $\|f\|_{L^1(0,T;L^2(0,L))} + \|z_0\|_{L^2(0,L)} \leq \frac{r}{2}$. If μ_i are sufficiently large (depending on α_i, ν, r, $\|\bar{y}_x\|_\infty$, $\|z_{i,d}\|_{L^2(\mathcal{O}_{i,d}\times(0,T))}$), then system (1.18) possesses a unique solution $(z, \phi^1, \phi^2) \in \mathcal{B}^3$.*

Proof. We will use again the Schauder's fixed point theorem. Indeed, let $\tilde{K}$ be a constant to be determined later and fix $(\hat{\phi}^1, \hat{\phi}^2) \in B^1_{\tilde{K}}$ where

$$B^1_{\tilde{K}} = \{(u,v) \in L^1(0,T;L^2(0,L))^2;\ \|(u,v)\|_{L^1(0,T;L^2(0,L))^2} \leq \tilde{K}\}.$$

Notice that

$$\|z_0\|_{L^2(0,L)} + \|f\mathbb{1}_{\mathcal{O}} - \frac{1}{\mu_1}\hat{\phi}^1\mathbb{1}_{\mathcal{O}_1} - \frac{1}{\mu_2}\hat{\phi}^2\mathbb{1}_{\mathcal{O}_2}\|_{L^1(0,T;L^2(0,L))} \leq \frac{r}{2} + \tilde{K}(\frac{1}{\mu_1} + \frac{1}{\mu_2}).$$

If we take $\tilde{K} = \frac{r}{2}(\frac{1}{\mu_1} + \frac{1}{\mu_2})^{-1}$ then, by Theorem A.4, there exists $\hat{z} \in \mathcal{B} \cap \mathbf{B}_K$ the unique solution of

$$\hat{z}_t + \hat{z}_{xxxx} + \nu\hat{z}_{xx} + \hat{z}\hat{z}_x + (\bar{y}\hat{z})_x = f\mathbb{1}_{\mathcal{O}} - \frac{1}{\mu_1}\hat{\phi}^1\mathbb{1}_{\mathcal{O}_1} - \frac{1}{\mu_2}\hat{\phi}^2\mathbb{1}_{\mathcal{O}_2} \quad (x,t) \in (0,L)\times(0,T)$$

with $\hat{z}(0) = z^0$ and satisfying

$$\|\hat{z}(t)\|_{L^2(0,L)} + \|\hat{z}\|_{L^2(0,T;H_0^2(0,L))} \leq C_0 r. \tag{A.10}$$

Consider now $\tilde{\phi}^i \in C([0,T];L^2(0,L)) \cap L^2([0,T];H_0^2(0,L))$ with $\tilde{\phi}^i(\cdot,T) = 0$ solution of

$$-\tilde{\phi}^i_t + \tilde{\phi}^i_{xxxx} + \nu\tilde{\phi}^i_{xx} - (\hat{z}+\bar{y})\tilde{\phi}^i_x = \alpha_i(\hat{z} - z_{i,d})1_{\mathcal{O}_{i,d}} \quad i = 1,2 \quad (x,t) \in (0,L)\times(0,T).$$

Multiplying by $\tilde{\phi}^i$ and integrating on $(0,L)$ we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}^i(t)\|^2_{L^2(0,L)} + \|\tilde{\phi}^i_{xx}(t)\|^2_{L^2(0,L)} &= \alpha_i((\hat{z} - z_{i,d})1_{\mathcal{O}_{i,d}}, \tilde{\phi}^i)_{L^2(0,L)} - \nu(\tilde{\phi}^i_{xx}, \tilde{\phi}^i)_{L^2(0,L)} \\ &\quad + ((\hat{z}+\bar{y})\tilde{\phi}^i_x, \tilde{\phi}^i)_{L^2(0,L)}. \end{aligned} \tag{A.11}$$

Notice that

$$
\begin{aligned}
((\hat{z}+\bar{y})\tilde{\phi}^i_x, \tilde{\phi}^i)_{L^2(0,L)} &= -\frac{1}{2}\int_0^L (\hat{z}_x(t,x)+\bar{y}_x(t,x))|\tilde{\phi}^i(t,x)|^2\,dx \\
&\le (\|\hat{z}_x(t)\|_{L^\infty(0,L)} + \|\bar{y}_x(t)\|_{L^\infty(0,L)})\int_0^L |\tilde{\phi}^i(t,x)|^2\,dx.
\end{aligned}
$$

Replacing the last inequality in (A.11) we obtain for $i = 1, 2$ the inequality

$$
\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|\tilde{\phi}^i(t)\|^2_{L^2(0,L)} + \frac{1}{2}\|\tilde{\phi}^i_{xx}(t)\|^2_{L^2(0,L)} \\
&\le \frac{1}{2}\int_0^L \left(|\hat{z}|^2 + |z_{i,d}|^2\right) dx + \left(\frac{\alpha_i^2}{2} + \frac{\nu^2}{2} + \|\bar{y}_x(t)\|_\infty + \|\hat{z}_x(t)\|_\infty\right)\int_0^L |\tilde{\phi}^i|^2\,dx. \quad \text{(A.12)}
\end{aligned}
$$

By Gronwall's inequality

$$
\|\tilde{\phi}^i(t)\|^2_{L^2(0,L)} + \int_0^T \|\tilde{\phi}^i_{xx}(s)\|^2_{L^2(0,L)}\,ds \le C(\alpha_i, \nu, r, \|\bar{y}_x\|_\infty, \|z_{i,d}\|_{L^2(0,L)}) \quad \text{(A.13)}
$$

where

$$
\begin{aligned}
C(\alpha_i, \nu, r, \|\bar{y}_x\|_\infty, \|z_{i,d}\|_{L^2(0,L)}) = \int_0^T \Big(&\exp\left(T\left(\alpha_i^2 + \nu^2 + 2\|\bar{y}_x\|_\infty + C_0 r\right)\right) \\
&\times \left(C_0^2 r^2 + \|z_{i,d}(s)\|^2_{L^2(0,L)}\right) ds\Big),
\end{aligned}
$$

for $i = 1, 2$. In this way, if μ_1 and μ_2 are sufficiently large, such that

$$
C(\alpha_i, \nu, r, \|\bar{y}_x\|_\infty, \|z_{i,d}\|_{L^2(0,L)}) \le \frac{r}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)^{-1} = \tilde{K}
$$

then $\tilde{\Lambda} : L^1(0,T; L^2(0,L))^2 \to L^1(0,T; L^2(0,L))^2$ given by $\tilde{\Lambda}((\hat{\phi}^1, \hat{\phi}^2)) = (\tilde{\phi}^1, \tilde{\phi}^2)$ is compact and $\tilde{\Lambda}(B^1(\tilde{K})) \subset B^1(\tilde{K})$. By Schauder's fixed point theorem $\tilde{\Lambda}$ has a fixed point $(\hat{\phi}^1, \hat{\phi}^2)$ which is, together with $\hat{z}$, a solution of (1.18). Notice that, thanks to (A.10) and (A.13), the solution $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2)$ satisfies

$$
\|(\hat{z}(t), \hat{\phi}^1(t), \hat{\phi}^2(t))\|_{L^2(0,L)^3} + \|(\hat{z}, \hat{\phi}^1, \hat{\phi}^2)\|_{L^2(0,T;H_0^2(0,L))^3} \le C(\alpha_i, \nu, r, \|\bar{y}_x\|_\infty, \|z_{i,d}\|_{L^2(0,L)}) \quad \text{(A.14)}
$$

provided that μ_1 and μ_2 are large enough.

The uniqueness part can be obtained using (A.14) and standard energy estimates. □

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