

Versal normal form for nonsemisimple singularities

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Abstract

The theory of versal normal form has been playing a role in normal form since the introduction of the concept by V.I. Arnol'd in [1,2]. But there has been no systematic use of it that is in line with the semidirect character of the group of formal transformations on formal vector fields, that is, the linear part should be done completely first, before one computes the nonlinear terms. In this paper, we address this issue by giving a complete description of a first order calculation in the case of the two- and three-dimensional irreducible nilpotent cases, which is then followed up by an explicit almost symplectic calculation to find the transformation to versal normal form in a particular fluid dynamics problem and in the celestial mechanics L_4 problem.

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1. Introduction

In normal form theory for general differential equations or symplectic systems around equilibria, not much attention is usually given to the linear part of the problem. A typical approach in bifurcation theory is to compute the normal form of a general system with respect to a given organizing center and add versal deformation terms (as first considered in [1,2]). One can then analyze all possible bifurcations in a neighborhood of the organizing center. While there is nothing wrong with this approach, it does not answer the question where a given system fits in the

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analysis. In other words, how does one compute where the given system is in this neighborhood of the organizing center?

It is this question that we attempt to answer for a number of examples. Some of these examples will be very concrete, with only one or two parameters to give us a possibility to actually, see the bifurcations, others are completely general systems where one can use the computation by just filling in the parameter values of a given system with the same type of organizing center.

Ideally, before starting the nonlinear computation, the linear system should be brought in versal normal form in a finite number of steps, as is attempted in [16]. In practice what one does is to put the linear part in normal form in the same way as one does the nonlinear part of the equation, but this may involve infinitely many steps. Since the linear terms influence the computation in every step, this is not very desirable (contrary to the nonlinear computations, which cannot influence the linear part unless there is also a constant term to take into account).

In this paper, we address this problem for a very particular system that has been the subject of several papers already from the versal deformation point of view, namely the L_4 -problem as described in [6]. This paper contains a very clear discussion of the arguments involved in the versal deformation computation and we will not repeat these here. The issue we want to address here is to change the infinite series approach into a finite explicit computation. Apart from the L_4 -problem, we have added several examples to illustrate the method and to show that it is indeed a method, not a computation that happens to work in the one example. We treat the 2- and 3-dimensional irreducible nilpotent case in section 3 and 4, respectively. We started this research by computing exponential maps using the generators of the Chevalley normal form of the Lie algebra. In the specific L_4 -problem this leads to quartic equations in the flow parameters and even if one is able to explicitly solve these equations the result is a map full of radical expressions which will be very hard to use if one applies the result to the full nonlinear problem as is our goal. We should mention that in the general linear case this does not occur and one can expect that for simply laced simple Lie algebras this approach will work without problems.

In order to simplify the resulting map that puts the linear system in versal normal form, we then decided to drop the requirement that the symplectic form be preserved. As remarked in [12] there is a strong belief that the symplectic form should be preserved, which is a bit strange if one considers the fact that in order to put the symplectic form in its Darboux normal form, one has to use (by definition) transformations that are not symplectic.

Dropping this requirement, which has anyway no consequence for the further analysis since we work with the symplectic vector fields, not with the Hamiltonians, we then proceed as follows. We first determine a theoretical form of the versal normal form, depending on a finite number of versal deformation parameters. Since we want to reach the versal normal form by conjugation, the characteristic polynomial of the original linear vector field and the versal deformation should be equal. From this equality, we determine the versal deformation parameters (this is in the symplectic case the only nonlinear part of the procedure, in the general linear case this part is completely straightforward).

Once we have, given a linear vector field X_0^ε , where $0 \leq \varepsilon$, which consists of an organizing center X_0^0 plus terms in a neighborhood of the organizing center, in order to compute its versal deformation \bar{X}_0^ε , we need to solve the linear problem $X_0^\varepsilon T^\varepsilon = T^\varepsilon \bar{X}_0^\varepsilon$ in such a way that T^0 reduces to the identity and \bar{X}_0^ε is in versal normal form. We then can obtain reasonable expressions for the transformation, which can then be put to good use in the nonlinear normal form analysis.

2. The algorithm

We start with polynomials $\mathbb{R}[x_1, \dots, x_n]$. We then add to these commuting derivations $\partial_1, \dots, \partial_n$ and consider these as a left $\mathbb{R}[x_1, \dots, x_n]$ module, such that $[\partial_i, x_j] = \delta_j^i$. (One could write ∂_i as $\frac{\partial}{\partial x_i}$.) We write $\frac{\partial P}{\partial x_i} \in \mathbb{R}[x_1, \dots, x_n]$ for $[\partial_i, P]$. We then define a multiplication $P_i \partial_i \star P_j \partial_j = P_i \frac{\partial P_j}{\partial x_i} \partial_j$. This defines a non-associative algebra with an associator $\alpha(x, y, z) = (x \star y) \star z - x \star (y \star z)$ which is symmetric in its first two variables (this ensures that the Jacobi identity holds, [9]) and from it we can define a Lie algebra, the Polynomial Lie algebra by defining the Lie bracket as $[x, y] = x \star y - y \star x$. Apart possibly from the notation, this is the usual way of defining polynomial vector fields. We can put a grading on the polynomial vector field by assigning degree 1 to the x_i 's and degree -1 to the ∂_j 's. We remark that the \star -product is a graded product, that is, the degree of $U \star V$ is the sum of the degree of U and the degree of V , and this makes the Lie algebra into a graded Lie algebra $\mathfrak{g} = \prod_{k=0}^{\infty} \mathfrak{g}_k$. Among the elements in this Lie algebra, as a special position is reserved for those of degree zero. They form a Lie subalgebra $\mathfrak{gl}(n, \mathbb{R})$.

We start with a given linear vector field which we consider as an element in a reductive Lie algebra \mathfrak{g}_0 . In our examples, \mathfrak{g}_0 will be $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{sp}(n, \mathbb{R})$. We chose an organizing center (in all our example this will be characterized by the fact that the real part of all its eigenvalues is zero, since this is where bifurcations happen) and introduce for organizational reasons a deformation parameter $0 \leq \varepsilon$, which at the end of the computation can be set back to 1. In our first two examples, we assume that the organizing center X_0^0 is in real Jordan normal form as is usually done in normal form theory. This is not really necessary and might need the knowledge of the spectrum of X_0^0 , something we try to avoid in this paper, so we stress the fact that the whole construction works well without this choice. Alternatively one might want to put X_0^0 in rational normal form before starting the computation, or not all, as in Section 5. All this is a matter of taste and convenience.

We then split X_0^0 into a semisimple and nilpotent part, $X_0^0 = s_0 + n_0$, with s_0 and n_0 commuting, $s_0, n_0 \in \mathfrak{g}_0$. We remark that this only needs the characteristic polynomial of X_0^0 [10,18]. In $\ker \text{ad}(s_0)$ (where $\text{ad}(X)Y = [X, Y]$, as usual) we construct around n_0 an \mathfrak{sl}_2 -triple $\langle n_0, h_0, m_0 \rangle$ as follows, cf. [11].

Let $z_0 \in \ker \text{ad}(s_0) \cap \mathfrak{g}_0$ be a solution of $2n_0 - \text{ad}^2(n_0)z_0 = 0$. Put $h_0 = [n_0, z_0]$. If

$$[h_0, z_0] = 2z_0,$$

then we let $y_0 = 0$. Otherwise, let $y_0 \in \ker \text{ad}(s_0) \cap \ker \text{ad}(n_0) \cap \mathfrak{g}_0$. Then we solve

$$[h_0, y_0 - z_0] = 2(y_0 - z_0),$$

and put $m_0 = y_0 - z_0$. We now have the following relations:

$$[h_0, n_0] = [[n_0, z_0], n_0] = -\text{ad}^2(n_0)z_0 = -2n_0,$$

$$[h_0, m_0] = 2m_0,$$

$$[m_0, n_0] = [y_0 - z_0, n_0] = [n_0, z_0] = h_0.$$

Now, we have an \mathfrak{sl}_2 -triple $\langle n_0, h_0, m_0 \rangle$, commuting with s_0 .

As in the $\mathfrak{s}_0 + \mathfrak{n}_0$ -decomposition, this is a completely rational procedure [7]. The existence of solutions to the equations is guaranteed by the Jacobson-Morozov theorem, see [11]. We give an explicit example of this computation in Section 5.

Remark 2.1. This construction determines the *style* of the normal form, since we will choose $\ker \operatorname{ad}(\mathfrak{m}_0)$ as the complement to $\operatorname{im} \operatorname{ad}(\mathfrak{n}_0)$ and *costyle* of the normal form transformation, since we will choose $\operatorname{im} \operatorname{ad}(\mathfrak{m}_0)$ as the complement to $\ker \operatorname{ad}(\mathfrak{n}_0)$. The costyle of normal form transformation is the way we choose the free parameters in transformations. As suggested by the terminology, other choices of style are also possible and may in specific problems be preferable.

The versal normal form should be equivalent to the *rational (or Frobenius) normal form* of the matrix of X_0^ε , although for that normal form one usually chooses a different style. The computation of \bar{X}_0^ε from X_0^ε has already been described. The T^ε can be computed by linear elimination. If for some ε_0 , T^ε fails to be invertible, then we should take $\varepsilon < \varepsilon_0$.

Definition 2.2. We say that $X_0^\varepsilon = X_0^0 + \bar{Y}_0^\varepsilon$ is in normal form (in \mathfrak{sl}_2 -style) with respect to X_0^0 if $\bar{Y}_0^\varepsilon \in \ker \mathfrak{s}_0 \cap \ker \mathfrak{m}_0$. We say that $\bar{X}_0^\varepsilon = X_0^0 + \bar{Y}_0^\varepsilon$ is a versal normal form with respect to X_0^0 if \bar{Y}_0^ε is in normal form with respect to X_0^0 and there exists a $T^\varepsilon \in \operatorname{GL}(n, \mathbb{R})$ such that $X_0^\varepsilon T^\varepsilon = T^\varepsilon \bar{X}_0^\varepsilon$ and $T_0^0 = I$.

If the Lie algebra is defined by an invariant bilinear form Ω_0 (for instance, a symplectic form), one has to compute the induced form $\bar{\Omega}_0^\varepsilon = (T^\varepsilon)^t \Omega_0 T^\varepsilon$. In this case we write \mathfrak{g}^Ω and $\mathfrak{g}^{\bar{\Omega}}$. Similar remarks apply to invariant trilinear forms in the less popular (in dynamics) case \mathfrak{g}_2 , the Lie algebra of G_2 , cf. [3], not to be confused with an element of grade two in \mathfrak{g} . This ensures that the versal deformation vector field behaves correctly with respect to $\bar{\Omega}_0^\varepsilon$, that is, $\bar{X}_0^\varepsilon \in \mathfrak{g}_0^{\bar{\Omega}}$. Here we trade symplecticness of the maps involved against computational convenience.

Definition 2.3. Let $T^\varepsilon \in \operatorname{GL}(2n, \mathbb{R})$. Then this induces a new symplectic form $\bar{\Omega}_0^\varepsilon$ and a new vector field \bar{X}_0^ε as follows

$$\bar{\Omega}_0^\varepsilon = (T^\varepsilon)^t \Omega_0 T^\varepsilon, \quad (2.1)$$

$$X_0^\varepsilon T^\varepsilon = T^\varepsilon \bar{X}_0^\varepsilon. \quad (2.2)$$

Lemma 2.4. The vector field \bar{X}_0^ε is $\bar{\Omega}_0^\varepsilon$ -symplectic iff X_0^ε is an Ω_0 -symplectic vector field.

The claim is that \bar{X}_0^ε is a $\bar{\Omega}_0^\varepsilon$ -symplectic vector field, that is, we have to prove that

$$(\bar{X}_0^\varepsilon)^t \bar{\Omega}_0^\varepsilon + \bar{\Omega}_0^\varepsilon \bar{X}_0^\varepsilon = 0. \quad (2.3)$$

Proof. Assume $(X_0^\varepsilon)^t \Omega_0 + \Omega_0 X_0^\varepsilon = 0$. Then

$$\begin{aligned} & (\bar{X}_0^\varepsilon)^t \bar{\Omega}_0^\varepsilon + \bar{\Omega}_0^\varepsilon \bar{X}_0^\varepsilon \\ &= (\bar{X}_0^\varepsilon)^t (T^\varepsilon)^t \Omega_0 T^\varepsilon + (T^\varepsilon)^t \Omega_0 T^\varepsilon \bar{X}_0^\varepsilon \\ &= (T^\varepsilon \bar{X}_0^\varepsilon)^t \Omega_0 T^\varepsilon + (T^\varepsilon)^t \Omega_0 T^\varepsilon \bar{X}_0^\varepsilon \end{aligned}$$

$$\begin{aligned}
&= (X_0^\varepsilon T^\varepsilon)^t \Omega_0 T^\varepsilon + (T^\varepsilon)^t \Omega_0 X_0^\varepsilon T^\varepsilon \\
&= (T^\varepsilon)^t \left((X_0^\varepsilon)^t \Omega_0 + \Omega_0 X_0^\varepsilon \right) T^\varepsilon \\
&= 0,
\end{aligned}$$

proving the statement of the Lemma. \square

The next order step is to compute

$$\exp(\text{ad}(\mathfrak{t}_1^\varepsilon))(\bar{X}_0^\varepsilon + X_1^\varepsilon + \cdots) = \bar{X}_0^\varepsilon + X_1^\varepsilon + [\mathfrak{t}_1^\varepsilon, \bar{X}_0^\varepsilon] + \cdots. \quad (2.4)$$

Then we solve

$$\text{ad}(\mathfrak{s}_0 + \mathfrak{m}_0)(X_1^\varepsilon + [\mathfrak{t}_1^\varepsilon, \bar{X}_0^\varepsilon]) = 0, \quad (2.5)$$

in order to obtain $\bar{X}_0^\varepsilon + \bar{X}_1^\varepsilon + \cdots$ in $\mathfrak{g}^{\bar{\Omega}}$, or, in the general linear case, in \mathfrak{g} , where X_1^ε is the first order nonlinear term and, with $\mathfrak{t}_1^\varepsilon$ a general vector field of order 1 and \bar{X}_1^ε is in normal form with respect to X_0^0 in the \mathfrak{sl}_2 -style.

This procedure can then be repeated until the full system is in normal form up to the fixed degree. The $\text{ad}(\mathfrak{s}_0 + \mathfrak{m}_0)$ ensures that the normal form will automatically have the \mathfrak{sl}_2 -style with respect to X_0^0 .

We should remark here that if we start with a general $\mathfrak{t}_1^\varepsilon$, there may be free parameters in the normal form corresponding to elements in $\ker \text{ad}(\mathfrak{s}_0) \cap \ker \text{ad}(\mathfrak{n}_0)$ in $\mathfrak{t}_1^\varepsilon$. This is analogous to the way unique normal forms are computed [4,17]. The free parameters may be used to simplify the normal form by removing (typically) higher order ε -terms. There is no style known to us that would be preferable to this simple *free-costyle*. In most of our examples the transformation turns out to be in \mathfrak{sl}_2 -costyle.

Remark 2.5. In some problems, when one wants to do the calculations by hand, it pays to view the \mathfrak{g}_k , the polynomial vector fields as representation spaces of \mathfrak{g}_0 , and more specifically of representation spaces of $\langle \mathfrak{s}_0, \mathfrak{n}_0, \mathfrak{h}_0, \mathfrak{m}_0 \rangle$. For instance, in [4] the \mathfrak{g}_k is shown to be a direct sum (as vector spaces, not as Lie algebras) of two irreducible representations of \mathfrak{sl}_2 , \mathfrak{a}_k and \mathfrak{b}_k and this gives rise to a basis that is completely natural with respect to the action of the given \mathfrak{sl}_2 and such that $[\mathfrak{z}_k, \mathfrak{z}_l] \subset \mathfrak{z}_{k+l}$ for $\mathfrak{z} = \mathfrak{a}, \mathfrak{b}$.

As formulated, the algorithm follows what might be called the *rational approach*: no eigenvalues need to be computed, only characteristic polynomials, cf [5]. This makes it suitable not only for Computer Algebra Systems, but also for Symbolic Formula Manipulation Systems like FORM [13] or FERMAT [14], which is nice if the problems get big.

An alternative method, which might also work when the vector fields are not finitely generated at any given order and might be called the *spectral approach*, is to use the spectrum of \mathfrak{s}_0 and \mathfrak{h}_0 , as is done in the averaging method; we refer for this method to [18].

2.1. Nonlinear nilpotent versal normal form

Lemma 2.6. For given $X_k \in \mathfrak{g}_k$, $k > 0$, and parametric vector field $\bar{X}_0^\varepsilon = s_0 + n_0 + \bar{v}_0^\varepsilon$ in which $\bar{v}_0^\varepsilon \in \ker \text{ad}(m_0) \cap \ker \text{ad}(s_0)$ and $\bar{v}_0^0 = 0$, there exists a transformation $t_k^\varepsilon \in \mathfrak{g}_k$ to the following problem

$$\text{ad}(\bar{X}_0^\varepsilon)t_k^\varepsilon = X_k - \bar{X}_k^\varepsilon,$$

where $\bar{X}_k^\varepsilon \in \ker \text{ad}(m_0) \cap \mathfrak{g}_k$. The transformation t_k^ε and the normal form \bar{X}_k^ε can be found explicitly from equations (2.7) and (2.8), respectively.

Proof. It should be noted that this proof follows (but with some minor corrections and clarifications) the proof given in [16, Section 2.3].

Our problem is that to find the admissible transformation t_k^ε and the obstruction term $\bar{X}_0^\varepsilon \in \ker \text{ad}(m_0) \cap \mathfrak{g}_k$ such that the following hold

$$\text{ad}(\bar{X}_0^\varepsilon)t_k^\varepsilon = X_k - \bar{X}_k^\varepsilon.$$

From [18, Chapters 11–12] the procedure is given to solve the following linear problem

$$\text{ad}(n_0)t_k^0 = X_k - \bar{X}_k. \quad (2.6)$$

Denote the transformation t_k^0 in equation (2.6) by $\bar{N}X_k$. Hence from the fact that $V = \ker \text{ad}(m_0) \oplus \text{im ad}(n_0)$ one has

$$\text{ad}(n_0)\bar{N} = \pi_{\text{im ad}(n_0)} = 1 - \pi_{\ker \text{ad}(m_0)}.$$

Note that the notation $\bar{N}X_k$ shows that the operator \bar{N} acts on X_k . Let now $Q = \text{ad}(s_0 + \bar{v}_0^\varepsilon)\bar{N}$ and $\hat{Q} = \bar{N}\text{ad}(s_0 + \bar{v}_0^\varepsilon)$. We will show that Q and \hat{Q} are nilpotent operators, so that $(1 + Q)^{-1}$ and $(1 + \hat{Q})^{-1}$ are both well defined. Observe that $\bar{N}Q = \hat{Q}\bar{N}$.

Lemma 2.7. $\bar{N}(1 + Q)^{-1} = (1 + \hat{Q})^{-1}\bar{N}$.

Proof. We compute

$$\bar{N}(1 + Q)^{-1} = \sum_{i=0}^{\infty} (-1)^i \bar{N}Q^i = \sum_{i=0}^{\infty} (-1)^i \hat{Q}^i \bar{N} = (1 + \hat{Q})^{-1}\bar{N},$$

and the lemma is proved. \square

We claim that t_k^ε is given by

$$t_k^\varepsilon = \bar{N}(1 + Q)^{-1}X_k = (1 + \hat{Q})^{-1}\bar{N}X_k = (1 + \hat{Q})^{-1}t_k^0. \quad (2.7)$$

Therefore we have to first show that Q and \hat{Q} are nilpotent and $X_k - \text{ad}(\bar{X}_0^\varepsilon)t_k^\varepsilon \in \ker \text{ad}(m_0) \cap \mathfrak{g}_k$. Assume that the X_k has $\text{ad}(h_0)$ -eigenvalue λ ; then the $\bar{N}X_k$ has $\text{ad}(h_0)$ -eigenvalue $\lambda + 2$ as follows

from standard finite dimensional representation theory of \mathfrak{sl}_2 since $\text{ad}(\mathfrak{h}_0)\mathfrak{n}_0 = -2\mathfrak{n}_0$ and we can ignore terms in $\ker \text{ad}(\mathfrak{n}_0)$ when we choose a representant of \mathfrak{t}_k^0 .

By assumption, $\bar{\mathfrak{v}}_0^\varepsilon \in \ker \text{ad}(\mathfrak{m}_0) \cap \ker \text{ad}(\mathfrak{s}_0)$; hence $\text{ad}(\mathfrak{m}_0)\bar{\mathfrak{v}}_0^\varepsilon = \text{ad}(\mathfrak{s}_0)\bar{\mathfrak{v}}_0^\varepsilon = 0$. Therefore the $\text{ad}(\mathfrak{h}_0)$ -degree of all terms in $\bar{\mathfrak{v}}_0^\varepsilon$ is ≥ 0 . Since $\text{ad}(\mathfrak{h}_0)\mathfrak{s}_0 = \text{ad}(\mathfrak{m}_0)\mathfrak{s}_0 = 0$ then its $\text{ad}(\mathfrak{h}_0)$ -degree is zero. This implies that the $\text{ad}(\mathfrak{h}_0)$ -degree of $\mathcal{Q} = \text{ad}(\mathfrak{s}_0 + \bar{\mathfrak{v}}_0^\varepsilon)\bar{N} \geq 2$ hence \mathcal{Q} is nilpotent. The proof for $\hat{\mathcal{Q}}$ is the almost the same. It follows that $1 + \mathcal{Q}$ and $1 + \hat{\mathcal{Q}}$ are invertible. What remains to be done is to show $\mathbf{X}_k - \text{ad}(\bar{\mathbf{X}}_0^\varepsilon)\mathfrak{t}_k^\varepsilon \in \ker \text{ad}(\mathfrak{m}_0) \cap \mathfrak{g}_k$:

$$\begin{aligned} \text{ad}(\mathfrak{s}_0 + \mathfrak{n}_0 + \bar{\mathfrak{v}}_0^\varepsilon)\mathfrak{t}_k^\varepsilon &= \text{ad}(\mathfrak{s}_0 + \mathfrak{n}_0 + \bar{\mathfrak{v}}_0^\varepsilon)\bar{N}(1 + \mathcal{Q})^{-1}\mathbf{X}_k \\ &= (\text{ad}(\mathfrak{n}_0)\bar{N} + \mathcal{Q})(1 + \mathcal{Q})^{-1}\mathbf{X}_k \\ &= (1 + \mathcal{Q} - (1 - \text{ad}(\mathfrak{n}_0)\bar{N}))(1 + \mathcal{Q})^{-1}\mathbf{X}_k \\ &= \mathbf{X}_k - (1 - \text{ad}(\mathfrak{n}_0)\bar{N})(1 + \mathcal{Q})^{-1}\mathbf{X}_k \\ &= \mathbf{X}_k - \pi_{\ker \text{ad}(\mathfrak{m}_0)}(1 + \mathcal{Q})^{-1}\mathbf{X}_k. \end{aligned}$$

We rewrite this as

$$\mathbf{X}_k = \pi_{\ker \text{ad}(\mathfrak{m}_0)}(1 + \mathcal{Q})^{-1}\mathbf{X}_k + \text{ad}(\mathfrak{s}_0 + \mathfrak{n}_0 + \bar{\mathfrak{v}}_0^\varepsilon)\bar{N}(1 + \mathcal{Q})^{-1}\mathbf{X}_k,$$

and we define

$$\bar{\mathbf{X}}_k^\varepsilon = \pi_{\ker \text{ad}(\mathfrak{m}_0)}(1 + \mathcal{Q})^{-1}\mathbf{X}_k. \quad (2.8)$$

This concludes the proof of Lemma 2.6. \square

2.2. Nonsemisimple versal normal form

We now extend the versal normal form computation problem from the nilpotent to the non-semisimple case. We follow [16, Section 2.4]. We consider the problem

$$\text{ad}(\mathbf{X}_0^\varepsilon)\mathfrak{t}_k^\varepsilon = \bar{\mathbf{X}}_k^\varepsilon - \bar{\bar{\mathbf{X}}}_k^\varepsilon, \quad \bar{\mathbf{X}}_k^\varepsilon \in \ker \text{ad}(\mathfrak{m}_0), \quad \bar{\bar{\mathbf{X}}}_k^\varepsilon \in \ker \text{ad}(\mathfrak{m}_0) \cap \ker \text{ad}(\mathfrak{s}_0).$$

We observe that the right hand side is by definition in $\ker \text{ad}(\mathfrak{m}_0) \cap \text{im } \text{ad}(\mathfrak{s}_0)$ and $\text{ad}(\mathfrak{s}_0 + \mathfrak{n}_0)$ is invertible on this subspace. We define operators $K_k : \ker \text{ad}(\mathfrak{m}_0)|_{\mathfrak{g}_k} \rightarrow \ker \text{ad}(\mathfrak{m}_0)|_{\mathfrak{g}_k}$ such that $K_k \neq I_{\ker \text{ad}(\mathfrak{m}_0)|_{\mathfrak{g}_k}}$ for $\varepsilon \neq 0$. Let

$$K_k = \text{ad}(\bar{\mathbf{X}}_0^\varepsilon)(1 + \hat{\mathcal{Q}})^{-1}\pi_{\ker \text{ad}(\mathfrak{n}_0)}.$$

The projection on $\ker \text{ad}(\mathfrak{n}_0)$ is necessary, in order not to interfere with the previous normal form calculation in Section 2. We now show that $K_k : \ker \text{ad}(\mathfrak{m}_0)|_{\mathfrak{g}_k} \rightarrow \ker \text{ad}(\mathfrak{m}_0)|_{\mathfrak{g}_k}$:

$$\begin{aligned} K_k &= \text{ad}(\bar{\mathbf{X}}_0^\varepsilon)(1 + \hat{\mathcal{Q}})^{-1}\pi_{\ker \text{ad}(\mathfrak{n}_0)} \\ &= \text{ad}(\bar{\mathbf{X}}_0^\varepsilon)(1 + \hat{\mathcal{Q}})^{-1}(1 - \bar{N}\text{ad}(\mathfrak{n}_0)) \\ &= \text{ad}(\bar{\mathbf{X}}_0^\varepsilon)(1 + \hat{\mathcal{Q}})^{-1}(1 + \hat{\mathcal{Q}} - \bar{N}\text{ad}(\mathbf{X}_0^\varepsilon)) \end{aligned}$$

$$\begin{aligned}
&= \text{ad}(\bar{X}_0^\varepsilon)(1 - (1 + \hat{Q})^{-1}\bar{N}\text{ad}(X_0^\varepsilon)) \\
&= \text{ad}(\bar{X}_0^\varepsilon)(1 - \bar{N}(1 + Q)^{-1}\text{ad}(X_0^\varepsilon)) \\
&= (1 - \text{ad}(\bar{X}_0^\varepsilon)\bar{N}(1 + Q)^{-1})\text{ad}(X_0^\varepsilon) \\
&= (1 - (1 - \pi_{\ker \text{ad}(m_0)})(1 + Q)^{-1} - Q(1 + Q)^{-1})\text{ad}(X_0^\varepsilon) \\
&= (1 - (1 + Q)^{-1} + \pi_{\ker \text{ad}(m_0)}(1 + Q)^{-1} - Q(1 + Q)^{-1})\text{ad}(X_0^\varepsilon) \\
&= \pi_{\ker \text{ad}(m_0)}(1 + Q)^{-1}\text{ad}(X_0^\varepsilon).
\end{aligned}$$

The map $\hat{K}_k = K_k \text{ad}^{-1}(s_0 + n_0)$ is well defined on $\ker \text{ad}(m_0) \cap \text{im ad}(s_0)|_{\mathfrak{g}_k}$ and reduces to $1 - \text{ad}(s_0 + n_0)\bar{N}(1 + Q)^{-1} = 1 - \text{ad}(s_0 + n_0)(1 + \hat{Q})^{-1}\bar{N}$ when the perturbation is zero and this reduces to 1 on $\ker \text{ad}(m_0)$. This in turn implies that \hat{K}_k is invertible in a neighborhood of $\varepsilon = 0$, which means we can find a transformation generator to bring \bar{X}_k^ε into the normal form $\bar{\bar{X}}_k^\varepsilon$. The values of ε for which \hat{K}_k fails to be invertible are called resonances; they play a role in the bifurcation analysis of the L_4 -problem, cf. Section 6.

The method we describe here does prove that it is possible to compute the transformation explicitly and if the dimension of \mathfrak{g}_k is a bit higher, it may help to reduce the dimension of the linear algebra problem, since one can restrict to $\ker \text{ad}(m_0)$.

3. 2D nilpotent – invariant formulation

3.1. The versal normal form of the linear system

In this section, we intend to study the versal normal form of two-dimensional nilpotent singularities. We use this example to illustrate the method in great detail. This leads at times to statements that sound a bit simplistic; these are nevertheless stated explicitly so that it is clear what the flow of the argument is in the later examples, where the complexity of the calculation can obscure what is going on.

Consider the following two-dimensional perturbed singular system.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon \tilde{m}_{1,1} & \varepsilon \tilde{m}_{1,2} \\ \tilde{m}_{2,1} & \varepsilon \tilde{m}_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \tilde{X}_0^\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}, \quad (3.1)$$

where we regard $\tilde{m}_{i,j}$ for all $i, j = 1, 2$ as elements of a commutative ring R of functions of certain parameters taking their values in \mathbb{R} (since we want to work with real differential equations) and $\tilde{m}_{2,1} \in R^*$ where R^* denotes the invertible elements in the ring R . Invertible in this context means that if we use asymptotic estimates, dividing by an *invertible* element does not produce big numbers, which could ruin the asymptotic estimate. As a consequence one is not allowed to divide by the noninvertible elements in the course of the normal form computation.

Since $\tilde{m}_{2,1}$ is invertible, there exists an invertible linear transformation

$$T_{(0)}^\varepsilon = \begin{pmatrix} -\tilde{m}_{2,1}^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

that takes (3.1) (with $\tilde{X}_0^\varepsilon T_{(0)}^\varepsilon = T_{(0)}^\varepsilon X_0^\varepsilon$) to the following

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon m_{1,1} & \varepsilon m_{1,2} \\ -1 & \varepsilon m_{2,2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = X_0^\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}, \quad (3.2)$$

(where $m_{1,1} = \tilde{m}_{1,1}$, $m_{2,2} = \tilde{m}_{2,2}$, and $m_{1,2} = -\tilde{m}_{1,2}\tilde{m}_{2,1}$), so that $-X_0^0$ is in Jordan normal form (the minus sign is there to be consistent with the definitions of the A and B-families to follow shortly).

We now rewrite Equation (3.2) to the operator form

$$X_0^\varepsilon = (\varepsilon m_{1,1}x + \varepsilon m_{1,2}y) \frac{\partial}{\partial x} + (-x + \varepsilon m_{2,2}y) \frac{\partial}{\partial y},$$

and express X_0^ε to the A and B families introduced by [4] (but with A and B interchanged) as

$$X_0^\varepsilon = B_0^1 + \frac{\varepsilon}{2} (m_{1,1} + m_{2,2}) A_0^0 + \varepsilon (m_{1,1} - m_{2,2}) B_0^0 + \varepsilon m_{1,2} B_0^{-1}.$$

We now want (this is the choice of normal form style) $X_0^\varepsilon - B_0^1$ to commute with B_0^{-1} ; a general expression of linear vector fields commuting is $\varepsilon_B B_0^{-1} + \varepsilon_A A_0^0$, corresponding to the differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon_A & \varepsilon_B \\ -1 & \varepsilon_A \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \bar{X}_0^\varepsilon \begin{pmatrix} x \\ y \end{pmatrix}, \quad (3.3)$$

(the fact that the ε_A are on the diagonal and will stay there if we go to higher dimensions prompted the interchange of A and B with respect to the definitions in [4]) and the differential operator

$$\bar{X}_0^\varepsilon = B_0^1 + \varepsilon_A A_0^0 + \varepsilon_B B_0^{-1}. \quad (3.4)$$

We want to find the transformation that is named $T_{(1)}^\varepsilon$ such that $X_0^\varepsilon T_{(1)}^\varepsilon = T_{(1)}^\varepsilon \bar{X}_0^\varepsilon$. The necessary condition under which such transformation exists is that the characteristic polynomials of X_0^ε and \bar{X}_0^ε be the same. In what follows using the characteristic polynomials of X_0^ε and \bar{X}_0^ε we find the $\varepsilon_A, \varepsilon_B$. The characteristic polynomials of X_0^ε and \bar{X}_0^ε are given, respectively by

$$\begin{aligned} \chi(X_0^\varepsilon) &= \lambda^2 - \varepsilon (m_{1,1} + m_{2,2}) \lambda + \varepsilon^2 m_{2,2} m_{1,1} + \varepsilon m_{1,2}, \\ \chi(\bar{X}_0^\varepsilon) &= \lambda^2 - 2\varepsilon_A \lambda + \varepsilon_A^2 + \varepsilon_B. \end{aligned}$$

We define the invariants of X_0^ε as

$$\chi(X_0^\varepsilon) = \lambda^2 - \Delta_1 \lambda + \Delta_2,$$

and we identify Δ_1 as the trace of X_0^ε and Δ_2 as the determinant. Since the equivalent matrices have the same characteristic polynomial then we find that

$$\varepsilon_A = \frac{1}{2} \Delta_1, \quad (3.5)$$

$$\varepsilon_B = \Delta_2 - \frac{1}{4} \Delta_1^2. \quad (3.6)$$

We close this part by the following theorem (this is not much of a theorem in this particular problem, but we formulate it as such because it is a basic step in this paper).

Theorem 3.1. *There exists an invertible transformation $T_{(1)}^\varepsilon$, defined by*

$$T_{(1)}^\varepsilon = \begin{pmatrix} 1 & 0 \\ \frac{\varepsilon}{2}(m_{2,2} - m_{1,1}) & 1 \end{pmatrix}, \quad (3.7)$$

which brings the matrix (3.1) to (3.3).

Proof. The transformation (3.7) is obtained using equation $X_0^\varepsilon T_{(1)}^\varepsilon = T_{(1)}^\varepsilon \bar{X}_0^\varepsilon$. This is a linear equation in $T_{(1)}^\varepsilon$ and the existence of a solution is shown here explicitly. \square

3.2. Some representation theory

Following [4] we describe vector fields of arbitrary order in a bigraded infinite dimensional Lie algebra $\mathfrak{a} \oplus \mathfrak{b}$, where \mathfrak{a} and \mathfrak{b} are bigraded Lie subalgebras and the \oplus denotes the direct sum of modules, not of Lie algebras, as can be seen from the Lie brackets below, and spanned by elements $A_m^n \in \mathfrak{a}_m$, $0 \leq n \leq m$, $B_k^l \in \mathfrak{b}_k$, $-1 \leq l \leq k+1$ (i.e. $\dim \mathfrak{a}_m = m+1$ and $\dim \mathfrak{b}_k = k+3$) where A_m^n and B_k^l are defined as

$$A_m^n := x^n y^{m-n} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad (0 \leq n \leq m), \quad (3.8)$$

$$B_k^l := \frac{x^l y^{k-l}}{k+2} \left((k-l+1)x \frac{\partial}{\partial x} - (l+1)y \frac{\partial}{\partial y} \right), \quad (-1 \leq l \leq k+1), \quad (3.9)$$

with brackets

$$[A_k^l, A_m^n] = (m-k) A_{k+m}^{l+n}, \quad (3.10)$$

$$[B_k^l, A_m^n] = \frac{m(m+1)}{m+k+2} \left(\frac{n}{m} - \frac{l+1}{k+2} \right) A_{k+m}^{l+n} - k B_{k+m}^{l+n}, \quad (3.11)$$

$$[B_k^l, B_m^n] = (k+m+2) \left(\frac{n+1}{m+2} - \frac{l+1}{k+2} \right) B_{k+m}^{l+n}. \quad (3.12)$$

We can now write an arbitrary order s vector field as

$$X_s := \sum_{l=0}^s a_s^l A_s^l + \sum_{l=-1}^{s+1} b_s^l B_s^l.$$

A general element of order s in $\ker \text{ad}(B_0^{-1})$ can be written as

$$\bar{X}_s = \bar{a}_s^0 A_s^0 + \bar{b}_s^{-1} B_s^{-1}. \quad (3.13)$$

3.3. Nonlinear normal form reduction

We now have to solve the equation (in $\mathfrak{t}_s^\varepsilon$)

$$\text{ad}(\mathbf{B}_0^{-1})(\text{ad}(\bar{\mathbf{X}}_0^\varepsilon)\mathfrak{t}_s^\varepsilon - \mathbf{X}_s) = 0, \quad (3.14)$$

where

$$\mathfrak{t}_s^\varepsilon = \sum_{l=0}^s \alpha_s^l \mathbf{A}_s^l + \sum_{l=-1}^{s+1} \beta_s^l \mathbf{B}_s^l. \quad (3.15)$$

Recall that

$$\bar{\mathbf{X}}_s = \bar{a}_s^0 \mathbf{A}_s^0 + \bar{b}_s^{-1} \mathbf{B}_s^{-1}, \quad \mathbf{X}_s = \sum_{l=0}^s a_s^l \mathbf{A}_s^l + \sum_{l=-1}^{s+1} b_s^l \mathbf{B}_s^l.$$

We have to solve

$$\begin{aligned} \bar{a}_s^0 \mathbf{A}_s^0 + \bar{b}_s^{-1} \mathbf{B}_s^{-1} &= \sum_{l=-1}^{s+1} b_s^l \mathbf{B}_s^l + \sum_{l=0}^s a_s^l \mathbf{A}_s^l \\ &\quad + \beta_s^s \mathbf{B}_s^{s+1} - 2s\varepsilon_A \beta_s^{s+1} \mathbf{B}_s^{s+1} + \alpha_s^{s-1} \mathbf{A}_s^s - 2s\varepsilon_A \alpha_s^s \mathbf{A}_s^s \\ &\quad - \sum_{k=0}^s \left(-(s+2-k) \beta_s^{k-1} + 2s\varepsilon_A \beta_s^k - (k+2)\varepsilon_B \beta_s^{k+1} \right) \mathbf{B}_s^k \\ &\quad - \sum_{k=1}^{s-1} \left(-(s+1-k) \alpha_s^{k-1} + 2s\varepsilon_A \alpha_s^k - (k+1)\varepsilon_B \alpha_s^{k+1} \right) \mathbf{A}_s^k \\ &\quad - \left(2s\varepsilon_A \beta_s^{-1} - \varepsilon_B \beta_s^0 \right) \mathbf{B}_s^{-1} - \left(2s\varepsilon_A \alpha_s^0 - \varepsilon_B \alpha_s^1 \right) \mathbf{A}_s^0. \end{aligned}$$

Thus we find, if we look at the \mathbf{B}_s^{s+1} -term, that

$$\beta_s^s = 2s\varepsilon_A \beta_s^{s+1} - b_s^{s+1}, \quad (3.16)$$

where β_s^{s+1} is a free parameter, to be determined later at our convenience.

Similarly, looking at the \mathbf{A}_s^s terms we find

$$\alpha_s^{s-1} = 2s\varepsilon_A \alpha_s^s - a_s^s, \quad (3.17)$$

where α_s^s is the free parameter. For $0 \leq k \leq s$ we find, looking at the \mathbf{B}_s^k ,

$$(s+2-k) \beta_s^{k-1} = 2s\varepsilon_A \beta_s^k - (k+2)\varepsilon_B \beta_s^{k+1} - b_s^k. \quad (3.18)$$

For $1 \leq k \leq s-1$ we find, looking at the \mathbf{A}_s^k ,

$$(s+1-k)\alpha_s^{k-1} = 2s\varepsilon_A\alpha_s^k - (k+1)\varepsilon_B\alpha_s^{k+1} - a_s^k. \quad (3.19)$$

Then

$$\bar{X}_s = \left(b_s^{-1} - 2s\varepsilon_A\beta_s^{-1} + \varepsilon_B\beta_s^0\right)B_s^{-1} + \left(a_s^0 - 2s\varepsilon_A\alpha_s^0 + \varepsilon_B\alpha_s^1\right)A_s^0. \quad (3.20)$$

Let us now specialize to $s = 1$. We find

$$\begin{aligned} \alpha_1^0 &= 2\varepsilon_A\alpha_1^1 - a_1^1, \\ \beta_1^1 &= 2\varepsilon_A\beta_1^2 - b_1^2, \\ \beta_1^0 &= \frac{1}{2}\left(4\varepsilon_A^2 - 3\varepsilon_B\right)\beta_1^2 - \frac{1}{2}b_1^1 - \varepsilon_A b_1^2, \\ \beta_1^{-1} &= \frac{1}{3}\varepsilon_A\left(4\varepsilon_A^2 - 7\varepsilon_B\right)\beta_1^2 + \frac{2}{3}(\varepsilon_B - \varepsilon_A^2)b_1^2 - \frac{1}{3}\left(\varepsilon_A b_1^1 + b_1^0\right), \end{aligned}$$

and

$$\begin{aligned} \bar{X}_1 &= \left(b_1^{-1} - 2\varepsilon_A\beta_1^{-1} + \varepsilon_B\beta_1^0\right)B_1^{-1} + \left(a_1^0 - 2\varepsilon_A\alpha_s^0 + \varepsilon_B\alpha_1^1\right)A_1^0 \\ &= \left(b_1^{-1} + \frac{2}{3}\varepsilon_A b_1^0 - \left(-\frac{4}{3}\varepsilon_A^2 + \varepsilon_B\right)\frac{1}{2}b_1^1 + \frac{7}{3}(\varepsilon_B - \frac{4}{7}\varepsilon_A^2)\varepsilon_A b_1^2\right. \\ &\quad \left.+ \left(\frac{20}{3}\varepsilon_A^2\varepsilon_B - \frac{8}{3}\varepsilon_A^4 - \frac{3}{2}\varepsilon_B^2\right)\beta_1^2\right)B_1^{-1} \\ &\quad + \left(a_1^0 + 2\varepsilon_A a_1^1 + (-4\varepsilon_A^2 + \varepsilon_B)\alpha_1^1\right)A_1^0. \end{aligned}$$

Choosing $\beta_1^2 = 0$ and $\alpha_1^1 = 0$ (in accordance with the \mathfrak{sl}_2 -costyle) we find

$$\begin{aligned} \bar{X}_1 &= \left(b_1^{-1} + \frac{2}{3}\varepsilon_A b_1^0 - \frac{1}{2}\left(-\frac{4}{3}\varepsilon_A^2 + \varepsilon_B\right)b_1^1 + \frac{7}{3}(\varepsilon_B - \frac{4}{7}\varepsilon_A^2)\varepsilon_A b_1^2\right)B_1^{-1} + \left(a_1^0 + 2\varepsilon_A a_1^1\right)A_1^0 \\ &= \left(b_1^{-1} + \frac{1}{3}(\text{Tr } X_0^\varepsilon)b_1^0 + \frac{1}{2}\left(\frac{1}{12}\text{Tr }^2 X_0^\varepsilon + \text{Det } X_0^\varepsilon\right)b_1^1\right. \\ &\quad \left.+ \frac{7}{6}\left(\frac{3}{28}\text{Tr }^2 X_0^\varepsilon - \text{Det } X_0^\varepsilon\right)(\text{Tr } X_0^\varepsilon)b_1^2\right)B_1^{-1} \\ &\quad + \left(a_1^0 + (\text{Tr } X_0^\varepsilon)a_1^1\right)A_1^0. \end{aligned} \quad (3.21)$$

See also [4, Equation 4.2].

4. 3D irreducible nilpotent

4.1. The versal normal form of the linear system

In this section, we discuss versal deformation of three-dimensional nilpotent singularities. Consider the deformed nilpotent system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \varepsilon \tilde{m}_{1,1} & \varepsilon \tilde{m}_{1,2} & \varepsilon \tilde{m}_{1,3} \\ \tilde{m}_{2,1} & \varepsilon \tilde{m}_{2,2} & \varepsilon \tilde{m}_{2,3} \\ \varepsilon \tilde{m}_{3,1} & \tilde{m}_{3,2} & \varepsilon \tilde{m}_{3,3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \tilde{X}_0^\varepsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (4.1)$$

where the elements $\tilde{m}_{2,1}, \tilde{m}_{3,2} \in R^*$. By applying the following invertible transformation

$$T_{(0)}^\varepsilon = \begin{pmatrix} \tilde{m}_{2,1} & 0 & 0 \\ 0 & -\tilde{m}_{2,1}^2 & 0 \\ 0 & -\varepsilon \tilde{m}_{3,1} \tilde{m}_{2,1} & \alpha \end{pmatrix}, \quad (4.2)$$

where

$$\alpha = -\frac{1}{2} \varepsilon^2 \tilde{m}_{3,1} (\varepsilon \tilde{m}_{2,3} \tilde{m}_{3,1} + \tilde{m}_{2,1} \tilde{m}_{2,2}) + \frac{1}{2} \tilde{m}_{2,1} (\varepsilon^2 \tilde{m}_{3,1} \tilde{m}_{3,3} + \tilde{m}_{2,1} \tilde{m}_{3,2}),$$

the system (4.1) transforms to the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \varepsilon m_{1,1} & \varepsilon m_{1,2} & \varepsilon m_{1,3} \\ -1 & \varepsilon m_{2,2} & \varepsilon m_{2,3} \\ 0 & -2 & \varepsilon m_{3,3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = X_0^\varepsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (4.3)$$

in which

$$\begin{aligned} m_{1,1} &= \tilde{m}_{1,1}, \\ m_{1,2} &= -\varepsilon^2 \tilde{m}_{1,3} \tilde{m}_{3,1} - \tilde{m}_{2,1} \varepsilon \tilde{m}_{1,2}, \\ m_{1,3} &= \frac{\tilde{m}_{1,3} \alpha}{\tilde{m}_{2,1}}, \\ m_{2,2} &= \frac{\varepsilon (\varepsilon \tilde{m}_{2,3} \tilde{m}_{3,1} + \tilde{m}_{2,1} m_{2,2})}{\tilde{m}_{2,1}}, \\ m_{2,3} &= -\frac{\tilde{m}_{2,3} \alpha}{\tilde{m}_{2,1}^2}, \\ m_{3,3} &= \frac{\varepsilon (-\varepsilon \tilde{m}_{2,3} \tilde{m}_{3,1} + \tilde{m}_{2,1} m_{3,3})}{\tilde{m}_{2,1}}. \end{aligned}$$

Remark 4.1. Note that due to the assumption $\tilde{m}_{2,1}, \tilde{m}_{3,2} \in R^*$ the transformation given by (4.2) when $\varepsilon = 0$ is invertible.

Now, we writing down (4.3) in terms of vector fields from $\mathcal{A}, \mathcal{B}, \mathcal{C}$ given in [8] to find the following

$$\begin{aligned} X_0^\varepsilon &= B_{0,0}^1 + \varepsilon m_{1,3} C_{0,0}^{-2} + \frac{1}{2} \varepsilon (m_{1,2} - 2m_{2,3}) C_{0,0}^{-1} - \frac{1}{2} \varepsilon m_{2,2} C_{0,0}^0 + \frac{1}{4} \varepsilon (m_{1,2} + 2m_{2,3}) B_{0,0}^{-1} \\ &\quad + \frac{1}{2} \varepsilon (2m_{1,1} + m_{2,2}) B_{0,0}^0 + \varepsilon (m_{1,1} + m_{2,2} + m_{3,3}) A_{0,0}^0. \end{aligned}$$

Due to \mathfrak{sl}_2 -style normal form, in order to find the versal normal form of X_0^ε we seek the vector fields which belong to $\ker \text{ad}(B_{0,0}^{-1})$. Hence the following special structure constants associated to the $B_{0,0}^{-1}$ are given

$$\begin{aligned} [B_{0,0}^{-1}, B_{i,k}^l] &= (l+1)B_{i,k}^{l-1}, \\ [B_{0,0}^{-1}, A_{i,k}^l] &= lA_{i,k}^{l-1}, \\ [B_{0,0}^{-1}, C_{i,k}^l] &= \frac{(l+2)(2i+3-l)}{(2i-l+1)}C_{i,k}^{l-1}, \quad \text{for } l < 2i+1, \\ [B_{0,0}^{-1}, C_{i,k}^l] &= 0, \quad \text{for } l = 2i+1, \\ [B_{0,0}^{-1}, C_{i,k}^l] &= (2i+4)C_{i,k}^{2i+1}, \quad \text{for } l = 2i+2. \end{aligned}$$

Therefore we obtain that

$$\bar{X}_0^\varepsilon = B_{0,0}^1 + \varepsilon_A A_{0,0}^0 + \varepsilon_B B_{0,0}^{-1} + \varepsilon_C C_{0,0}^{-2}, \quad (4.4)$$

and the correspondence differential equation of \bar{X}_0^ε is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \varepsilon_A & 2\varepsilon_B & \varepsilon_C \\ -1 & \varepsilon_A & \varepsilon_B \\ 0 & -2 & \varepsilon_A \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{X}_0^\varepsilon \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (4.5)$$

Now we are ready to find the versal parameters $\varepsilon_A, \varepsilon_B$ and ε_C . As before by computing the characteristic polynomials of \bar{X}_0^ε and \bar{X}_0^ε we get

$$\begin{aligned} \chi(\bar{X}_0^\varepsilon) &= \lambda^3 - 3\varepsilon_A \lambda^2 + (3\varepsilon_A^2 + 4\varepsilon_B)\lambda - \varepsilon_A^3 - 4\varepsilon_A \varepsilon_B - 2\varepsilon_C, \\ \chi(X_0^\varepsilon) &= \lambda^3 - \varepsilon(m_{1,1} + m_{2,2} + m_{3,3})\lambda^2 + \varepsilon(\varepsilon m_{1,1}m_{2,2} + \varepsilon m_{1,1}m_{3,3} + \varepsilon m_{2,2}m_{3,3} \\ &\quad + m_{1,2} + 2m_{2,3})\lambda - \varepsilon(\varepsilon^2 m_{1,1}m_{2,2}m_{3,3} + 2\varepsilon m_{1,1}m_{2,3} + \varepsilon m_{1,2}m_{3,3} + 2m_{1,3}). \end{aligned}$$

These define the invariants $\Delta_i, i = 1, 2, 3$ by

$$\chi(X_0^\varepsilon) = \lambda^3 - \Delta_1 \lambda^2 + \Delta_2 \lambda - \Delta_3.$$

Hence we find

$$\begin{aligned} \varepsilon_A &= \frac{1}{3}\Delta_1, \\ \varepsilon_B &= \frac{1}{4}\Delta_2 - \frac{1}{12}\Delta_1^2, \\ \varepsilon_C &= \frac{1}{2}\Delta_3 - \frac{1}{6}\Delta_1\Delta_2 + \frac{1}{27}\Delta_1^3. \end{aligned}$$

Theorem 4.2. *There exists invertible transformation $T_{(1)}^\varepsilon$ as*

$$T_{(1)}^\varepsilon = \begin{pmatrix} 1 & \varepsilon t_1 & \varepsilon t_2 \\ 0 & 1 & \varepsilon t_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.6)$$

in which

$$\begin{aligned} t_1 &= \frac{1}{3} (m_{3,3} + m_{2,2} - 2m_{1,1}), \\ t_2 &= \frac{5}{36} \varepsilon m_{1,1} (m_{1,1} - m_{2,2} - m_{3,3}) - \frac{1}{36} \varepsilon m_{2,2} (m_{2,2} - 7m_{3,3}) - \frac{1}{36} \varepsilon m_{3,3}^2 \\ &\quad + \frac{1}{2} m_{2,3} - \frac{1}{4} m_{1,2}, \\ t_3 &= \frac{1}{3} \left(m_{3,3} - \frac{1}{2} m_{2,2} - \frac{1}{2} m_{1,1} \right), \end{aligned}$$

which brings the matrix (4.3) to (4.5).

Proof. The transformation $T_{(1)}^\varepsilon$ is obtained using equation $X_0^\varepsilon T_{(1)}^\varepsilon = T_{(1)}^\varepsilon \tilde{X}_0^\varepsilon$. \square

4.2. Quadratic nonlinear versal normal form of triple-zero

In this part we shall compute the nonlinear normal form of the following parametric vector fields with triple zero bifurcation point

$$\tilde{X}_0^\varepsilon + X_1, \quad (4.7)$$

where

$$X_1 = \sum_{j=0}^2 a_j A_{1,0}^j + \sum_{j=-1}^3 b_j B_{1,0}^j + \sum_{j=-2}^0 c_1^j C_{-1,1}^j + \sum_{j=-2}^4 c_0^j C_{1,0}^j, \quad (4.8)$$

or equivalently

$$X_1 = \sum_{i+j+k=2} a_{i,j,k}^{(1)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_1} + \sum_{i+j+k=2} a_{i,j,k}^{(2)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_2} + \sum_{i+j+k=2} a_{i,j,k}^{(3)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_3}. \quad (4.9)$$

The coefficients of (4.8) and (4.9) are related by these relations:

$$\begin{aligned} a_{0,0,2}^{(1)} &= c_1^{-2}, & a_{2,0,0}^{(1)} &= \frac{1}{2} b_2 + c_0^2 + a_2, & a_{0,2,0}^{(1)} &= b_0 - c_1^{-2} + \frac{2}{3} c_0^3, \\ a_{0,2,0}^{(2)} &= c_1^{-1} - c_0^1 + a_1, & a_{0,0,2}^{(2)} &= b_{-1} - \frac{1}{4} c_0^{-1}, & a_{2,0,0}^{(3)} &= 30 c_0^4, \\ a_{0,0,2}^{(3)} &= -\frac{1}{2} b_0 + \frac{1}{6} c_0^0 + a_0, & a_{1,1,0}^{(1)} &= b_1 + c_0^1 + a_1, & a_{1,0,1}^{(1)} &= \frac{1}{3} c_0^0 + \frac{1}{2} b_0 + a_0 + a_1^{-2}, \end{aligned}$$

$$\begin{aligned}
a_{1,1,0}^{(2)} &= -\frac{1}{2}b_2 - 4c_0^2 + a_2, & a_{1,0,1}^{(2)} &= -\frac{1}{2}c_0^1 - c_1^{-1}, & a_{0,1,1}^{(2)} &= -\frac{2}{3}c_0^0 + \frac{1}{2}b_0 + a_0, \\
a_{1,1,0}^{(3)} &= -2b_3 + 20c_0^3, & a_{1,0,1}^{(3)} &= -\frac{1}{2}b_2 + 2c_0^2 + a_2 + 2a_1^0, & c_{0,1,1}^{(1)} &= 2b_{-1} + c_0^{-1}, \\
a_{0,1,1}^{(3)} &= -b_1 + a_1 + c_0^1, & a_{0,2,0}^{(3)} &= -b_2 - 2c_1^0 + 4c_0^2, & c_{2,0,0}^{(2)} &= -b_3 - 5c_0^3.
\end{aligned}$$

Theorem 4.3. *The normal form of (4.7) is given by*

$$\bar{X}_0^\varepsilon = B_{0,0}^1 + \varepsilon_A A_{0,0}^0 + \varepsilon_B B_{0,0}^{-1} + \varepsilon_C C_{0,0}^{-2} + \bar{c}_1^0 C_{1,0}^{-2} + \bar{c}_{-1}^0 C_{-1,1}^{-2} + \bar{b}_1^1 B_{1,0}^{-1} + \bar{a}_0^1 A_{1,0}^0, \quad (4.10)$$

or equivalently

$$\begin{aligned}
\bar{X}_1 &= \left(x_3 \varepsilon_C + 2x_2 \varepsilon_B + x_1 \varepsilon_A + \bar{c}_1^0 x_3^2 + \bar{c}_{-1}^0 (x_1 x_3 - x_2^2) + 2\bar{b}_1^1 x_2 x_3 + \bar{a}_0^1 x_3 x_1 \right) \frac{\partial}{\partial x_1} \\
&\quad + \left(-x_1 + \varepsilon_A x_2 + \varepsilon_B x_3 + \bar{b}_1^1 x_3^2 + \bar{a}_0^1 x_3 x_2 \right) \frac{\partial}{\partial x_2} + \left(-2x_2 + \varepsilon_A x_3 + \bar{a}_0^1 x_3^2 \right) \frac{\partial}{\partial x_3},
\end{aligned}$$

where

$$\begin{aligned}
5\bar{c}_{-1}^0 &:= 3a_{1,0,1}^{(1)} - 2a_{0,0,2}^{(3)} - a_{0,1,1}^{(2)} - 2a_{0,2,0}^{(1)} - \left(\frac{5}{2}a_{1,1,0}^{(3)} + 6a_{2,0,0}^{(2)} \right) \varepsilon_C \\
&\quad - \left(2a_{2,0,0}^{(1)} - 3a_{1,0,1}^{(3)} + 2a_{0,2,0}^{(3)} + a_{1,1,0}^{(2)} \right) \varepsilon_B - \left(\frac{1}{2}a_{1,1,0}^{(1)} - a_{0,2,0}^{(2)} + 4a_{1,0,1}^{(2)} + a_{0,1,1}^{(3)} \right) \varepsilon_A \\
&\quad + \left(4a_{2,0,0}^{(2)} + 3a_{1,1,0}^{(3)} \right) \varepsilon_B \varepsilon_A \\
&\quad - \frac{5}{2}a_{2,0,0}^{(3)} \varepsilon_C \varepsilon_A - \left(\frac{1}{2}a_{1,1,0}^{(2)} - \frac{3}{2}a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} + a_{2,0,0}^{(1)} \right) \varepsilon_A^2 + \left(\frac{3}{4}a_{1,1,0}^{(3)} + a_{2,0,0}^{(2)} \right) \varepsilon_A^3 \\
&\quad + 3a_{2,0,0}^{(3)} \varepsilon_B \varepsilon_A^2 + \frac{3}{4}a_{2,0,0}^{(3)} \varepsilon_A^4, \\
10\bar{a}_0^1 &:= \left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} + 3a_{0,1,1}^{(2)} + 6a_{0,0,2}^{(3)} \right) + \left(-2a_{2,0,0}^{(2)} + \frac{5}{2}a_{1,1,0}^{(3)} \right) \varepsilon_C \\
&\quad + \left(6a_{2,0,0}^{(1)} + 3a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{1,0,1}^{(3)} \right) \varepsilon_B + 2 \left(\frac{3}{4}a_{1,1,0}^{(1)} + a_{0,2,0}^{(2)} + a_{1,0,1}^{(2)} + \frac{3}{4}a_{0,1,1}^{(3)} \right) \varepsilon_A \\
&\quad + \frac{5}{2}a_{2,0,0}^{(3)} \varepsilon_C \varepsilon_A + \left(-12a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_B + \left(-3a_{2,0,0}^{(2)} + \frac{1}{4}a_{1,1,0}^{(3)} \right) \varepsilon_A^3 + a_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_B \\
&\quad + \frac{1}{4}a_{2,0,0}^{(3)} \varepsilon_A^4, \\
6\bar{b}_1^1 &:= a_{0,1,1}^{(1)} + 4a_{0,0,2}^{(2)} + \frac{1}{18} \left(4a_{2,0,0}^{(1)} + a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} - 5a_{1,0,1}^{(3)} \right) \varepsilon_C + \left(a_{1,1,0}^{(1)} - a_{0,1,1}^{(3)} \right) \varepsilon_B \\
&\quad + \frac{1}{2} \left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} + a_{0,1,1}^{(2)} - 2a_{0,0,2}^{(3)} \right) \varepsilon_A - \frac{1}{12} \left(28a_{2,0,0}^{(2)} + 19a_{1,1,0}^{(3)} \right) \varepsilon_C \varepsilon_A \\
&\quad - \frac{10}{3}a_{2,0,0}^{(3)} \varepsilon_B \varepsilon_C - \left(4a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_B^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{5}{6} \left(2a_{2,0,0}^{(1)} - a_{1,1,0}^{(2)} - a_{0,2,0}^{(3)} - a_{1,0,1}^{(3)} \right) \varepsilon_A \varepsilon_B + \frac{1}{4} \left(a_{1,1,0}^{(1)} - a_{0,1,1}^{(3)} \right) \varepsilon_A^2 \\
& - \frac{2}{3} \left(4a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A^2 \varepsilon_B - \frac{4}{9} a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B^2 \\
& + \frac{1}{12} \left(2a_{2,0,0}^{(1)} - b_{1,1,0} - a_{0,2,0}^{(3)} - a_{1,0,1}^{(3)} \right) \varepsilon_A^3 \\
& - \frac{25}{12} a_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_C - \frac{5}{6} a_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_B - \frac{1}{24} \left(4a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A^4 - \frac{1}{24} a_{2,0,0}^{(3)} \varepsilon_A^5, \\
\bar{c}_1^0 := & a_{0,0,2}^{(1)} + \frac{1}{12} \left(a_{1,1,0}^{(1)} + 2a_{0,2,0}^{(2)} - 10a_{1,0,1}^{(2)} - 3a_{0,1,1}^{(3)} \right) \varepsilon_C + \frac{1}{3} \left(\frac{1}{2} a_{0,1,1}^{(1)} - a_{0,0,2}^{(2)} \right) \varepsilon_A \\
& - \frac{1}{3} a_{2,0,0}^{(3)} \varepsilon_C^2 + \frac{1}{5} \left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} - 2a_{0,1,1}^{(2)} + a_{0,0,2}^{(3)} \right) \varepsilon_B \\
& + \left(-\frac{11}{15} a_{2,0,0}^{(2)} + \frac{1}{6} a_{1,1,0}^{(3)} \right) \varepsilon_B \varepsilon_C + \frac{2}{15} \left(a_{1,1,0}^{(1)} - 2a_{0,2,0}^{(2)} - 2a_{1,0,1}^{(2)} + a_{0,1,1}^{(3)} \right) \varepsilon_A \varepsilon_B \\
& + \frac{1}{30} \left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} - 2a_{0,1,1}^{(2)} + a_{0,0,2}^{(3)} \right) \varepsilon_A^2 \\
& + \frac{1}{36} \left(-a_{2,0,0}^{(1)} - 10a_{1,1,0}^{(2)} - 13a_{0,2,0}^{(3)} + 11a_{1,0,1}^{(3)} \right) \varepsilon_A \varepsilon_C + \frac{1}{5} \left(a_{2,0,0}^{(1)} - 2a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} \right. \\
& \left. + a_{1,0,1}^{(3)} \right) \varepsilon_B^2 + \frac{1}{36} a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B \varepsilon_C + a_{2,0,0}^{(3)} \varepsilon_B^3 \\
& + \frac{7}{90} \left(a_{2,0,0}^{(1)} - 2a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{1,0,1}^{(3)} \right) \varepsilon_A^2 \varepsilon_B \\
& + \frac{1}{120} \left(a_{1,1,0}^{(1)} - 2a_{0,2,0}^{(2)} - 2a_{1,0,1}^{(2)} + a_{0,1,1}^{(3)} \right) \varepsilon_A^3 + \left(-\frac{7}{180} a_{2,0,0}^{(2)} + \frac{11}{72} a_{1,1,0}^{(3)} \right) \varepsilon_A^2 \varepsilon_C \\
& + \frac{11}{15} \left(-a_{2,0,0}^{(2)} + \frac{1}{2} a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_B^2 + \frac{1}{9} \left(-a_{2,0,0}^{(2)} + \frac{1}{2} a_{1,1,0}^{(3)} \right) \varepsilon_B \varepsilon_A^3 + \frac{11}{72} a_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_C \\
& + \frac{34}{45} a_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_B^2 + \frac{1}{360} \left(a_{2,0,0}^{(1)} - 2a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{1,0,1}^{(3)} \right) \varepsilon_A^4 \\
& + \frac{1}{720} \left(-2a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A^5 + \frac{5}{72} a_{2,0,0}^{(3)} \varepsilon_B \varepsilon_A^4 + \frac{1}{720} a_{2,0,0}^{(3)} \varepsilon_A^6.
\end{aligned}$$

Proof. In order to find the transformation the following linear system should be solved

$$\text{ad}(\mathbf{B}_{0,0}^{-1})(\text{ad}(\bar{\mathbf{X}}_0^\varepsilon) \mathbf{t}_1^\varepsilon - \mathbf{X}_2) = 0, \quad (4.11)$$

where

$$\mathbf{t}_1^\varepsilon = \sum_{j=0}^2 \alpha_j \mathbf{A}_{1,0}^j + \sum_{j=-1}^3 \beta_j \mathbf{B}_{1,0}^j + \sum_{j=-2}^0 \gamma_j^j \mathbf{C}_{-1,1}^j + \sum_{j=-2}^4 \gamma_0^j \mathbf{C}_{1,0}^j,$$

or in the different basis it equals to

$$\begin{aligned} \mathfrak{t}_1^\varepsilon = & \sum_{i+j+k=2} \alpha_{i,j,k}^{(1)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_1} + \sum_{i+j+k=2} \alpha_{i,j,k}^{(2)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_1} \\ & + \sum_{i+j+k=2} \alpha_{i,j,k}^{(3)} x_1^i x_2^j x_3^k \frac{\partial}{\partial x_3}. \end{aligned} \quad (4.12)$$

By solving Equation (4.11) one can find the coefficients of transformation $\mathfrak{t}_1^\varepsilon$ as are given in Appendix A, see Equation (A.1). On the other hand, by solving the equation below

$$\bar{a}_0^1 A_{1,0}^0 + \bar{b}_1^1 B_{1,0}^{-1} + \bar{c}_1^0 C_{1,0}^{-2} + \bar{c}_{-1}^0 C_{-1,1}^{-2} = \text{ad}(\bar{X}_0^\varepsilon) \mathfrak{t}_B^\varepsilon - X_2,$$

we find the coefficients of normal form which has four free parameters as $\alpha_{2,0,0}^{(3)}$, $\alpha_{2,0,0}^{(2)}$, $\alpha_{0,2,0}^{(3)}$ and $\alpha_{2,0,0}^{(2)}$. In accordance to the \mathfrak{sl}_2 -costyle we can take all of them zero and we get the coefficients as given in the theorem. These coefficients with those free parameters are given in Appendix A, see equations (A.2)–(A.3). \square

5. An example on $\mathfrak{sp}(4, \mathbb{R})$

In [15, Equation (48)] the versal deformation problem is studied using formal power series. We refer to this paper for more references to the literature and a general introduction of the importance of the versal deformation in applied mathematics. We mention that to keep things simple, we use an almost symplectic map to obtain the versal normal form, a trade-off we have discussed in Section 2.

In this section we shall find the near identity transformation T^ε as discussed in the previous section to bring the symplectic matrices given by [15, Equation (48)], describing oscillations of a simply supported elastic pipe conveying fluid, to its versal normal form. Set

$$\rho := \frac{1}{4} \sqrt{(4 + \varepsilon p_1)(3 + 4\varepsilon p_2)},$$

where p_1, p_2 are two real parameters. Define

$$X_0^\varepsilon := \begin{pmatrix} 0 & \rho & 1 & 0 \\ -\rho & 0 & 0 & 1 \\ \varepsilon p_1 - \rho^2 + 3 & 0 & 0 & \rho \\ 0 & 4\varepsilon p_1 - \rho^2 & -\rho & 0 \end{pmatrix}. \quad (5.1)$$

Set $\rho := \frac{\sqrt{3}}{2}$ and define $n_0 = X_0^0$ and apply the Jacobson-Morozov construction as discussed before Remark 2.1 to find \mathfrak{sl}_2 -triple $\langle n_0, h_0, m_0 \rangle$. Let z_0 be an element from $\ker \text{ad}(s_0) \cap \mathfrak{g}_0$ which is as

$$z_0 = \begin{pmatrix} -m_{3,3} & -m_{4,3} & m_{1,3} & m_{2,3} \\ -m_{3,4} & -m_{4,4} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{4,1} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{pmatrix}.$$

z_0 is a solution to $2n_0 - \text{ad}^2(n_0)z_0 = 0$ if

$$m_{1,3} = -4, m_{3,1} = -11, m_{3,4} = -\frac{7}{3}\sqrt{3}, m_{4,3} = \sqrt{3},$$

and we take the remaining parameters to be zero. Then we put $h_0 = [n_0, z_0]$ which is equal to the following

$$h_0 := \begin{pmatrix} 0 & 0 & 0 & -\frac{2}{3}\sqrt{3} \\ 0 & 2 & -\frac{2}{3}\sqrt{3} & 0 \\ 0 & -\sqrt{3} & 0 & 0 \\ -\frac{3}{2}\sqrt{3} & 0 & 0 & -2 \end{pmatrix}.$$

Let y_0 be an general element in $\ker \text{ad}(s_0) \cap \ker \text{ad}(n_0) \cap g_0$, that is,

$$y_0 = \begin{pmatrix} 0 & -\frac{2}{3}\sqrt{3}n_{4,2} & -\frac{4}{3}n_{4,2} & 0 \\ -\frac{4}{3}\sqrt{3}n_{4,2} - \frac{2}{3}\sqrt{3}n_{3,1} & 0 & 0 & -\frac{16}{3}n_{4,2} - \frac{4}{3}n_{3,1} \\ n_{3,1} & 0 & 0 & \frac{4}{3}\sqrt{3}n_{4,2} + \frac{2}{3}\sqrt{3}n_{3,1} \\ 0 & n_{4,2} & \frac{2}{3}\sqrt{3}n_{4,2} & 0 \end{pmatrix}.$$

Then by solving

$$[h_0, y_0 - z_0] = 2(y_0 - z_0),$$

we find

$$n_{3,1} = -\frac{35}{4}, n_{4,2} = \frac{9}{4}.$$

Finally, put

$$m_0 = y_0 - z_0 = \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{3} & 1 & 0 \\ \frac{1}{2}\sqrt{3} & 0 & 0 & -\frac{1}{3} \\ \frac{9}{4} & 0 & 0 & -\frac{1}{2}\sqrt{3} \\ 0 & \frac{9}{4} & \frac{1}{2}\sqrt{3} & 0 \end{pmatrix}.$$

One can see that $m_0^4 = n_0^4 = 0$. The normal form of X_0^ε consists of those elements which are in $\ker \text{ad}(m_0)$; in fact

$$V_1 = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{3} \\ -\frac{9}{4} & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & -\frac{9}{4} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & \frac{9\sqrt{3}}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{81}{4} & -\frac{9\sqrt{3}}{4} & 0 \end{pmatrix}.$$

Hence the normal form is given by

$$\bar{X}_0^\varepsilon = n_0 + \varepsilon_1 V_1 + \varepsilon_2 V_2. \quad (5.2)$$

The characteristic polynomial of X_0^ε given in the (5.1) and \bar{X}_0^ε are as

$$\begin{aligned} \chi(X_0^\varepsilon) &= \lambda^4 + \left(-\frac{17\varepsilon p_1}{4} + \varepsilon^2 p_1 p_2 + 4\varepsilon p_2 \right) \lambda^2 + 4p_1 \varepsilon (\varepsilon p_1 + 3), \\ \chi(\bar{X}_0^\varepsilon) &= \lambda^4 + 10\varepsilon_1 \lambda^2 + 9\varepsilon_1^2 + 54\varepsilon_2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \varepsilon_1 &= \frac{1}{5}\varepsilon \left(2p_2 - \frac{17}{8}p_1 \right) + \frac{1}{10}p_1 p_2 \varepsilon^2, \\ \varepsilon_2 &= \frac{1}{86400}\varepsilon^2 (29p_1 + 48p_2)(131p_1 - 48p_2) \\ &\quad - \frac{1}{3600}\varepsilon p_1 (6\varepsilon^3 p_1 p_2^2 - 51\varepsilon^2 p_1 p_2 + 48\varepsilon^2 p_2^2 - 800). \end{aligned}$$

With $\chi(X_0^\varepsilon) = \lambda^4 + \Delta_2 \lambda^2 + \Delta_4$, we see that

$$\begin{aligned} \Delta_2 &= -\frac{17}{4}\varepsilon p_1 + \varepsilon^2 p_1 p_2 + 4\varepsilon p_2, \\ \Delta_4 &= 4\varepsilon^2 p_1^2 + 12\varepsilon p_1. \end{aligned}$$

To go from p_1, p_2 to Δ_2, Δ_4 is less simple than in the earlier examples.

Theorem 5.1. *There exists invertible transformation T^ε such that brings (5.1) to (5.2),*

$$T^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + \frac{\sqrt{3}}{76800\rho}\varepsilon t_1 & -\frac{1}{115200\rho}\varepsilon t_2 & 0 \\ 0 & \frac{\sqrt{3}}{76800}\varepsilon t_3 & 1 + \frac{1}{115200}\varepsilon t_4 & 0 \\ \frac{1}{25600\rho}\varepsilon t_5 & 0 & 0 & 1 + \frac{\sqrt{3}}{115200\rho}\varepsilon t_6 \end{pmatrix}, \quad (5.3)$$

in which

$$\begin{aligned} t_1 &= 320(29p_1 + 48p_2) + (1079p_1^2 + 11296p_1 p_2 - 2304p_2^2)\varepsilon \\ &\quad + 8p_1 p_2 (233p_1 - 144p_2)\varepsilon^2 - 144\varepsilon^3 p_2^2 p_1^2, \\ t_2 &= 1920(3p_1 + 16p_2) + (1079p_1^2 + 15136p_1 p_2 - 2304p_2^2)\varepsilon \\ &\quad + 8p_1 p_2 (233p_1 - 144p_2)\varepsilon^2 - 144\varepsilon^3 p_2^2 p_1^2, \\ t_3 &= 12800p_1 + (6519p_1^2 + 2336p_1 p_2 - 2304p_2^2)\varepsilon \\ &\quad + 8p_1 p_2 (73p_1 - 144p_2)\varepsilon^2 - 144\varepsilon^3 p_2^2 p_1^2, \end{aligned}$$

$$\begin{aligned}
t_4 &= 960(17p_1 - 16p_2) - (6519p_1^2 + 6176p_1p_2 - 2304p_2^2)\varepsilon \\
&\quad - 8p_1p_2(73p_1 - 144p_2)\varepsilon^2 + 144\varepsilon^3p_2^2p_1^2, \\
t_5 &= 320(17p_1 - 16p_2) - (331p_1^2 + 12704p_1p_2 - 5376p_2^2)\varepsilon \\
&\quad - 168p_1p_2(17p_1 - 16p_2)\varepsilon^2 + 336\varepsilon^3p_2^2p_1^2, \\
t_6 &= 960(3p_1 + 16p_2) + (331p_1^2 + 15264p_1p_2 - 5376p_2^2)\varepsilon \\
&\quad + 168p_1p_2(17p_1 - 16p_2)\varepsilon^2 - 336\varepsilon^3p_2^2p_1^2.
\end{aligned}$$

Proof. The transformation (5.3) is obtained using equation $X_0^\varepsilon T^\varepsilon = T^\varepsilon \bar{X}_0^\varepsilon$. \square

Remark 5.2. In this example, we did not put the X_0^0 into the symplectic normal form.

6. Three body problem

6.1. The versal normal form at L_4

In the theory of the restricted three-body problem, the Langrange equilibria play a very practical role, since they are used to park satellites in orbit, as has been the case for L_1 and L_2 . The Trojan points L_4 and L_5 are considered as positions for space colonies, since they are stable, unlike L_3 which only made it into science fiction sofar.

Consider (cf. [6, Equation 1.8]) the four-dimensional L_4 -singularity

$$X_0^\delta := X_0^0 + (4\sqrt{2} - 3\delta) \begin{pmatrix} 0 & -\frac{\gamma^2}{4} & \frac{\gamma^2}{2} & 0 \\ \frac{1}{8\gamma^2} & 0 & 0 & -\frac{1}{4\gamma^2} \\ \frac{1}{16\gamma^2} & 0 & 0 & -\frac{1}{8\gamma^2} \\ 0 & -\frac{\gamma^2}{8} & \frac{\gamma^2}{4} & 0 \end{pmatrix}, \quad (6.1)$$

where $\gamma = (1 - \frac{\sqrt{2}}{2})^{\frac{1}{2}}$ and

$$X_0^0 := \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 0 & \frac{1}{2}\sqrt{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = s_0 + n_0.$$

The bifurcation point of X_0^δ is $\delta = \delta_0 := \frac{4\sqrt{2}}{3}$. Set $\delta = \delta_0 + \varepsilon$ to find

$$X_0^\varepsilon = X_0^0 + \varepsilon \begin{pmatrix} 0 & \frac{3}{4} - \frac{3}{8}\sqrt{2} & -\frac{3}{2} + \frac{3}{4}\sqrt{2} & 0 \\ -\frac{3}{4} - \frac{3}{8}\sqrt{2} & 0 & 0 & \frac{3}{2} + \frac{3}{4}\sqrt{2} \\ -\frac{3}{8} - \frac{3}{16}\sqrt{2} & 0 & 0 & \frac{3}{4} + \frac{3}{8}\sqrt{2} \\ 0 & \frac{3}{8} - \frac{3}{16}\sqrt{2} & -\frac{3}{4} + \frac{3}{8}\sqrt{2} & 0 \end{pmatrix}. \quad (6.2)$$

The normal form of (6.1) is given by

$$\bar{X}_0^\varepsilon = X_0^0 - \sqrt{2}\varepsilon_S \mathbf{S}_0 + \varepsilon_N \mathbf{n}_0,$$

where $\varepsilon_N, \varepsilon_S$ are obtained as follows. The characteristic polynomial of X_0^ε is

$$\lambda^4 + \lambda^2 - \frac{1}{16} (3\varepsilon + 6 + 4\sqrt{2}) (3\varepsilon - 6 + 4\sqrt{2}). \quad (6.3)$$

Comparing the characteristic polynomial of X_0^ε with characteristic polynomial of \bar{X}_0^ε as

$$\lambda^4 + (2\varepsilon_N + 2\varepsilon_S^2 - 2\sqrt{2}\varepsilon_S + 1)\lambda^2 + \frac{1}{4} (-2\varepsilon_S^2 + 2\sqrt{2}\varepsilon_S + 2\varepsilon_N - 1)^2. \quad (6.4)$$

Then equations (6.3) and (6.4) imply that

$$2\varepsilon_S (\sqrt{2} - \varepsilon_S) = 2\varepsilon_N, \quad (6.5)$$

$$(4\varepsilon_N - 1)^2 = -\frac{1}{4} (3\varepsilon - 6 + 4\sqrt{2}) (3\varepsilon + 6 + 4\sqrt{2}). \quad (6.6)$$

Now, by solving Equation (6.6) respect to ε_N we find two solutions. The negative root is the right solution, since for $\varepsilon = 0$ we have

$$\varepsilon_N = \frac{3\sqrt{2}}{4}\varepsilon + \frac{81}{32}\varepsilon^2 + O(\varepsilon^3),$$

hence,

$$\varepsilon_N = -\frac{\sqrt{-(3\varepsilon - 6 + 4\sqrt{2})(3\varepsilon + 6 + 4\sqrt{2})}}{8} + \frac{1}{4}.$$

Now substitute ε_N into (6.5) and solve for ε_S to get

$$\varepsilon_S := \frac{1}{2} \left(\sqrt{2} - \frac{1}{2} \sqrt{4 + 2\sqrt{-(3\varepsilon - 6 + 4\sqrt{2})(3\varepsilon + 6 + 4\sqrt{2})}} \right).$$

Here also we choose the negative root, since from $\varepsilon = 0$ we obtain

$$\varepsilon_S = \frac{3}{4}\varepsilon + \frac{99\sqrt{2}}{64}\varepsilon^2 + O(\varepsilon^3).$$

Note that to have a real normal form we should make this restriction:

$$-3.885618082 \approx \frac{-6 - 4\sqrt{2}}{3} < \varepsilon < \frac{6 - 4\sqrt{2}}{3} \approx 0.114381918.$$

6.2. Finding the transformation generator $\mathfrak{t}_0^\varepsilon$

Let $\mathfrak{t}_0^\varepsilon \in \text{GL}(4, \mathbb{R})$

$$\mathfrak{t}_0^\varepsilon = \frac{1}{(\alpha_1 + 6)} \begin{pmatrix} t_1 & t_2 & t_3 & -\frac{1}{\varepsilon}t_4 \\ t_5 & t_6 & \frac{1}{\varepsilon}t_7 & t_8 \\ -\frac{1}{\varepsilon}t_9 & \frac{1}{\varepsilon^2}t_{10} & -\frac{1}{\varepsilon^2}t_{11} & \frac{1}{\varepsilon}t_{12} \\ -\frac{1}{\varepsilon^2}t_{13} & \frac{1}{\varepsilon}t_{14} & \frac{1}{\varepsilon}t_{15} & -\frac{1}{\varepsilon^2}t_{16} \end{pmatrix}. \quad (6.7)$$

We solve the following equation

$$(I + \varepsilon \mathfrak{t}_0^\varepsilon) \bar{X}_0^\varepsilon = X_0^\varepsilon (I + \varepsilon \mathfrak{t}_0^\varepsilon), \quad (6.8)$$

for $\{t_i, i = 1 \dots 16\}$. The solutions of above equation respect to four free parameters t_1, t_2, t_5, t_6 are given in Appendix B. Now we should find parameters t_1, t_2, t_5, t_6 . By substituting parameters in $\mathfrak{t}_0^\varepsilon$ and Taylor expansion around $\varepsilon = 0$ we find

$$\mathfrak{t}_0^\varepsilon = \begin{pmatrix} t_1 & t_2 & 0 & 0 \\ t_5 & t_6 & 0 & 0 \\ -\frac{\sqrt{2}(t_2+t_5)}{3\varepsilon} & \frac{\sqrt{2}(4t_1-4t_6-9)}{12\varepsilon} & 0 & 0 \\ -\frac{\sqrt{2}(4t_1-4t_6-9)}{12\varepsilon} & -\frac{\sqrt{2}(t_2+t_5)}{3\varepsilon} & 0 & 0 \end{pmatrix} + O(\varepsilon^0).$$

Due to Equation (6.8) transformation T^ε should be near identity. Hence, it requires $t_2 = -t_5$, $t_1 = t_6 + \frac{9}{4}$. We have two free parameters t_5, t_6 which can be taken as $t_5 = 0, t_6 = -\frac{9}{4}$. Hence $t_1 = t_2 = 0$. Thereby we find

$$\mathfrak{t}_0^\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{9}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + O(\varepsilon^0).$$

Theorem 6.1. The following transformation takes X_0^δ to its versal normal from \bar{X}_0^ε through of equation $(I + \varepsilon \mathfrak{t}_0^\varepsilon) \bar{X}_0^\varepsilon = X_0^\varepsilon (I + \varepsilon \mathfrak{t}_0^\varepsilon)$.

$$\mathfrak{t}_0^\varepsilon = \frac{1}{2(6 + \alpha_1)} \begin{pmatrix} 0 & 0 & 0 & \frac{-1}{8\varepsilon}t_4 \\ 0 & \frac{-9}{4} & \frac{1}{8\varepsilon}t_7 & 0 \\ 0 & \frac{1}{6\varepsilon^2}t_{10} & \frac{1}{48\varepsilon^2}t_{11} & 0 \\ \frac{-1}{6\varepsilon^2}t_{13} & 0 & 0 & \frac{-1}{48\varepsilon^2}t_{16} \end{pmatrix},$$

where

$$\alpha_1 = \sqrt{-(3\varepsilon - 6 + 4\sqrt{2})(3\varepsilon + 6 + 4\sqrt{2})}, \quad \alpha_2 = \sqrt{4 + 2\alpha_1},$$

and

$$\begin{aligned}
 t_4 &= 4 \left(\sqrt{2} - 2 \right) \left(2\sqrt{2}\alpha_2 - 4\sqrt{2} - \alpha_1 + 2\alpha_2 - 6 \right) \\
 &\quad + 3 \left(\sqrt{2} - 2 \right) \left(\sqrt{2}\alpha_2 + 12\sqrt{2} + 3\alpha_1 + 2\alpha_2 + 14 \right) \varepsilon + \left(27\sqrt{2} - 54 \right) \varepsilon^2, \\
 t_7 &= -8 \left(2 + \sqrt{2} \right) \left(2\sqrt{2}\alpha_2 + 4\sqrt{2} - \alpha_1 - 2\alpha_2 - 6 \right) \\
 &\quad - \frac{\left(36\sqrt{2} - 12 \right) \left(-17\alpha_2 + 16 + 14\sqrt{2} \right) \varepsilon}{17} \\
 &\quad + 27\varepsilon^2\alpha_2, \\
 t_{10} &= -4 \left(2 + \sqrt{2} \right) \left(4\sqrt{2}\alpha_1 + 4\sqrt{2}\alpha_2 - \alpha_2\alpha_1 + 8\sqrt{2} - 4\alpha_1 - 6\alpha_2 - 8 \right) \\
 &\quad - \frac{\left(36\sqrt{2} + 12 \right) \left(5\sqrt{2}\alpha_2 - 24\sqrt{2} - 17\alpha_1 + 4\alpha_2 - 26 \right) \varepsilon}{17} + (-27\alpha_1 - 270)\varepsilon^2, \\
 t_{11} &= -16\alpha_2(\alpha_1 - 2) + \left(48\sqrt{2}\alpha_1 + 36\alpha_2\alpha_1 - 96\sqrt{2} - 72\alpha_2 - 384 \right) \varepsilon \\
 &\quad + \left(-144\sqrt{2} - 36\alpha_2 - 288 \right) \varepsilon^2 \\
 &\quad + 81\alpha_2\varepsilon^3, \\
 t_{13} &= 4 \left(\sqrt{2} - 2 \right) \left(4\sqrt{2}\alpha_1 - 4\sqrt{2}\alpha_2 - \alpha_2\alpha_1 + 8\sqrt{2} + 4\alpha_1 - 6\alpha_2 + 8 \right) \\
 &\quad - 3 \left(\sqrt{2} - 2 \right) \left(2\sqrt{2}\alpha_1 - 12\sqrt{2}\alpha_2 - 3\alpha_2\alpha_1 + 20\sqrt{2} + 4\alpha_1 - 14\alpha_2 + 40 \right) \varepsilon \\
 &\quad + 27 \left(\sqrt{2} - 2 \right) \alpha_2 \varepsilon^2, \\
 t_{16} &= -4\alpha_2(\alpha_1 - 2) - 12\sqrt{2} \left(\alpha_1 - 2 + 4\sqrt{2} \right) \varepsilon \\
 &\quad + \frac{9}{2} \left(\sqrt{2} - 2 \right) \left(\sqrt{2}\alpha_2 + 24\sqrt{2} + 6\alpha_1 + 2\alpha_2 + 28 \right) \varepsilon^2 + \left(81\sqrt{2} - 162 \right) \varepsilon^3.
 \end{aligned}$$

Note that t_4, t_7 are in order ε and $t_{10}, t_{11}, t_{13}, t_{16}$ are in order ε^2 .

7. Concluding remarks

We have shown that the correct implementation of versal normal form in normal form computations is possible. It does give, and this was to be expected, an added level of complexity. In any practical computation, this will have to be balanced against the added level of correctness.

It will be interesting to see whether these considerations can also be applied in practice to the theory of unique normal form. This would, after all, be the holy grail of normal form theory: unique versal normal forms!

Appendix A. The coefficients of normal form and transformation of triple-zero

The coefficients of transformation t_1^ε given in the Equation (4.12) respect to four free parameters are as follows:

$$\begin{aligned}
 \alpha_{1,1,0}^{(3)} &= 2\alpha_{2,0,0}^{(2)} + a_{2,0,0}^{(3)} + \alpha_{2,0,0}^{(3)}\varepsilon_A, \\
 \alpha_{2,0,0}^{(1)} &= \alpha_{1,1,0}^{(2)} - a_{2,0,0}^{(2)} - \alpha_{2,0,0}^{(2)}\varepsilon_A + \alpha_{2,0,0}^{(3)}\varepsilon_B, \\
 \alpha_{1,1,0}^{(1)} &= a_{2,0,0}^{(1)} - \alpha_{2,0,0}^{(3)}\varepsilon_C - 2\alpha_{2,0,0}^{(2)}\varepsilon_B + \left(\alpha_{1,1,0}^{(2)} - a_{2,0,0}^{(2)}\right)\varepsilon_A + \alpha_{2,0,0}^{(3)}\varepsilon_B\varepsilon_A - \alpha_{2,0,0}^{(2)}\varepsilon_A^2, \\
 \alpha_{1,0,1}^{(3)} &= \alpha_{1,1,0}^{(2)} + \frac{1}{2}a_{1,1,0}^{(3)} - \alpha_{0,2,0}^{(3)} + 2\alpha_{2,0,0}^{(3)}\varepsilon_B + \left(\alpha_{2,0,0}^{(2)} + \frac{1}{2}a_{2,0,0}^{(3)}\right)\varepsilon_A + \frac{1}{2}\alpha_{2,0,0}^{(3)}\varepsilon_A^2, \\
 3\alpha_{0,2,0}^{(2)} &= a_{1,0,1}^{(3)} + a_{2,0,0}^{(1)} + a_{1,1,0}^{(2)} - \frac{1}{2}a_{0,2,0}^{(3)} + \left(3\alpha_{1,1,0}^{(2)} - a_{2,0,0}^{(2)} + \frac{1}{2}a_{1,1,0}^{(3)} - \frac{1}{2}\alpha_{0,2,0}^{(3)}\right)\varepsilon_A \\
 &\quad + \alpha_{2,0,0}^{(3)}\varepsilon_C - a_{2,0,0}^{(3)}\varepsilon_B + \frac{1}{2}a_{2,0,0}^{(3)}\varepsilon_A^2 + \frac{2}{3}\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_B + \frac{1}{2}\alpha_{2,0,0}^{(3)}\varepsilon_A^3, \\
 3\alpha_{0,1,1}^{(3)} &= a_{2,0,0}^{(1)} + a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{1,0,1}^{(3)} + \alpha_{2,0,0}^{(3)}\varepsilon_C + \left(\frac{1}{2}a_{1,1,0}^{(3)} - a_{2,0,0}^{(2)} + 3\alpha_{1,1,0}^{(2)}\right)\varepsilon_A \\
 &\quad + 2\left(a_{2,0,0}^{(3)} + 3\alpha_{2,0,0}^{(2)}\right)\varepsilon_B + 5\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_B + \frac{1}{2}a_{2,0,0}^{(3)}\varepsilon_A^2 + \frac{1}{2}\alpha_{2,0,0}^{(3)}\varepsilon_A^3, \\
 6\alpha_{1,0,1}^{(2)} &= a_{2,0,0}^{(1)} + a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} - 2a_{1,0,1}^{(3)} - 5\alpha_{2,0,0}^{(3)}\varepsilon_C - a_{2,0,0}^{(3)}\varepsilon_B \\
 &\quad + \left(-a_{2,0,0}^{(2)} - a_{1,1,0}^{(3)} + 3\alpha_{0,2,0}^{(3)}\right)\varepsilon_A \\
 &\quad - 4\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_B + \left(-3\alpha_{2,0,0}^{(2)} - a_{2,0,0}^{(3)}\right)\varepsilon_A^2 - \varepsilon_A^3\alpha_{2,0,0}^{(3)}, \\
 3\alpha_{0,1,1}^{(2)} &= \frac{1}{2}a_{1,1,0}^{(1)} + a_{0,2,0}^{(2)} + a_{1,0,1}^{(2)} + \left(-\frac{1}{2}a_{2,0,0}^{(3)} + \alpha_{2,0,0}^{(2)}\right)\varepsilon_C \\
 &\quad + \left(-\frac{1}{2}a_{1,1,0}^{(3)} + \alpha_{1,1,0}^{(2)} - 2a_{2,0,0}^{(2)}\right)\varepsilon_B \\
 &\quad + \left(a_{2,0,0}^{(1)} + \frac{1}{2}a_{1,1,0}^{(2)}\right)\varepsilon_A - \left(a_{2,0,0}^{(3)} + 4\alpha_{2,0,0}^{(2)}\right)\varepsilon_A\varepsilon_B - \frac{3}{2}\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_C \\
 &\quad + \left(-a_{2,0,0}^{(2)} + \frac{3}{2}\alpha_{1,1,0}^{(2)}\right)\varepsilon_A^2 - \alpha_{2,0,0}^{(2)}\varepsilon_A^3, \\
 3\alpha_{0,2,0}^{(1)} &= a_{1,1,0}^{(1)} + 2a_{1,0,1}^{(2)} - a_{0,2,0}^{(2)} + \left(-a_{2,0,0}^{(3)} + 2\alpha_{2,0,0}^{(2)}\right)\varepsilon_C + \frac{1}{3}\left(\alpha_{0,2,0}^{(3)} - a_{1,1,0}^{(3)} - 4a_{2,0,0}^{(2)}\right)\varepsilon_B \\
 &\quad + \left(-a_{1,0,1}^{(3)} + \frac{1}{2}a_{0,2,0}^{(3)} + a_{2,0,0}^{(1)}\right)\varepsilon_A - \left(8\alpha_{2,0,0}^{(2)} + a_{2,0,0}^{(3)}\right)\varepsilon_A\varepsilon_B - 4\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_C \\
 &\quad + \left(\frac{3}{2}\alpha_{0,2,0}^{(3)} - \frac{1}{2}a_{1,1,0}^{(3)} - a_{2,0,0}^{(2)}\right)\varepsilon_C^2 - 2\alpha_{2,0,0}^{(3)}\varepsilon_A^2\varepsilon_B - \left(\frac{1}{2}a_{2,0,0}^{(3)} + 2\alpha_{2,0,0}^{(2)}\right)\varepsilon_A^3 \\
 &\quad - \frac{1}{2}\varepsilon_A^4\alpha_{2,0,0}^{(3)},
 \end{aligned}
 \tag{A.1}$$

$$\begin{aligned}
12\alpha_{0,0,2}^{(3)} &= a_{1,1,0}^{(1)} + 3a_{0,1,1}^{(3)} + 2a_{1,0,1}^{(2)} + 2a_{0,2,0}^{(2)} + \left(2a_{2,0,0}^{(3)} + 8\alpha_{2,0,0}^{(2)}\right)\varepsilon_C + \alpha_{2,0,0}^{(3)}\varepsilon_B^2 + \alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_C \\
&\quad + \left(12\alpha_{1,1,0}^{(2)} + 2a_{1,1,0}^{(3)} - 4a_{2,0,0}^{(2)}\right)\varepsilon_B + \left(a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} + 2a_{1,1,0}^{(2)} + 3a_{2,0,0}^{(1)}\right)\varepsilon_A \\
&\quad + \left(4\alpha_{2,0,0}^{(2)} + 3a_{2,0,0}^{(3)}\right)\varepsilon_A\varepsilon_B + \left(2a_{1,1,0}^{(3)} - 3a_{2,0,0}^{(2)} + 6\alpha_{1,1,0}^{(2)}\right)\varepsilon_A^2 \\
&\quad + 2\left(a_{2,0,0}^{(3)} - \alpha_{2,0,0}^{(2)}\right)\varepsilon_A^3 + 8\alpha_{2,0,0}^{(3)}\varepsilon_B\varepsilon_A^2 + \alpha_{2,0,0}^{(3)}\varepsilon_A^4, \\
15\alpha_{0,1,1}^{(1)} &= \frac{3}{2}\left(3a_{0,2,0}^{(1)} + 3a_{1,0,1}^{(1)} - 2a_{0,0,2}^{(3)} - a_{0,1,1}^{(2)}\right) - \left(a_{0,2,0}^{(2)} + a_{1,0,1}^{(2)} - \frac{7}{4}a_{1,1,0}^{(1)} + \frac{3}{4}a_{0,1,1}^{(3)}\right)\varepsilon_A \\
&\quad + \frac{1}{2}\left(-a_{0,2,0}^{(3)} + 14a_{2,0,0}^{(1)} - 13a_{1,1,0}^{(2)} - a_{1,0,1}^{(3)}\right)\varepsilon_B \\
&\quad + 3\left(\alpha_{0,2,0}^{(3)} - \frac{5}{4}a_{1,1,0}^{(3)} - 3a_{2,0,0}^{(2)}\right)\varepsilon_C + 15\left(-2\alpha_{2,0,0}^{(2)} + \frac{1}{3}a_{2,0,0}^{(3)}\right)\varepsilon_B^2 \\
&\quad - 15\left(\frac{5a_{2,0,0}^{(3)}}{12} + \frac{47\alpha_{2,0,0}^{(2)}}{30}\right)\varepsilon_A\varepsilon_C - 2\left(7a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)}\right)\varepsilon_A\varepsilon_B \\
&\quad + \frac{1}{4}\left(4a_{2,0,0}^{(1)} - 3a_{1,1,0}^{(2)} - a_{1,0,1}^{(3)} - a_{0,2,0}^{(3)}\right)\varepsilon_A^2 - 14\alpha_{2,0,0}^{(3)}\varepsilon_C\varepsilon_B - \frac{31}{4}\alpha_{2,0,0}^{(3)}\varepsilon_C\varepsilon_A^2 \\
&\quad + 17\alpha_{2,0,0}^{(3)}\varepsilon_B^2\varepsilon_A - \frac{1}{2}\left(a_{2,0,0}^{(3)} + 10\alpha_{2,0,0}^{(2)}\right)\varepsilon_B\varepsilon_A^2 - \left(a_{2,0,0}^{(2)} + \frac{a_{1,1,0}^{(3)}}{8}\right)\varepsilon_A^3 \\
&\quad - \frac{1}{4}\left(5\alpha_{2,0,0}^{(2)} + \frac{1}{2}a_{2,0,0}^{(3)}\right)\varepsilon_A^4 - \frac{\alpha_{2,0,0}^{(3)}}{8}\varepsilon_A^5, \\
6\alpha_{1,0,1}^{(1)} &= a_{1,1,0}^{(1)} - 4a_{1,0,1}^{(2)} + 2a_{0,2,0}^{(2)} + \left(2a_{1,0,1}^{(3)} - a_{0,2,0}^{(3)} + a_{2,0,0}^{(1)}\right)\varepsilon_A - \left(a_{2,0,0}^{(3)} + 10\alpha_{2,0,0}^{(2)}\right)\varepsilon_C \\
&\quad + 6\left(-\alpha_{0,2,0}^{(3)} + \alpha_{1,1,0}^{(2)} + \frac{1}{3}a_{1,1,0}^{(3)} - \frac{2}{3}a_{2,0,0}^{(2)}\right)\varepsilon_B + 12\alpha_{2,0,0}^{(3)}\varepsilon_B^2 + 2\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_C \\
&\quad - 2\left(\alpha_{2,0,0}^{(2)} - a_{2,0,0}^{(3)}\right)\varepsilon_A\varepsilon_B + \left(-3\alpha_{0,2,0}^{(3)}a_{1,1,0}^{(3)} - a_{2,0,0}^{(2)} + 3\alpha_{1,1,0}^{(2)}\right)\varepsilon_A^2 + 7\alpha_{2,0,0}^{(3)}\varepsilon_A^2\varepsilon_B \\
&\quad + \left(a_{2,0,0}^{(3)} + \alpha_{2,0,0}^{(2)}\right)\varepsilon_A^3 + \varepsilon_A^4\alpha_{2,0,0}^{(3)}, \\
60\alpha_{0,0,2}^{(2)} &= 3\left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} + 3a_{0,1,1}^{(2)} - 4a_{0,0,2}^{(3)}\right) + 60\left(-\frac{1}{10}a_{2,0,0}^{(2)} - \frac{1}{8}a_{1,1,0}^{(3)} + \frac{1}{5}\alpha_{0,2,0}^{(3)}\right)\varepsilon_C \\
&\quad + \left(8a_{2,0,0}^{(1)} - a_{1,1,0}^{(2)} - 7a_{0,2,0}^{(3)} - 7a_{1,0,1}^{(3)}\right)\varepsilon_B + \left(2a_{1,1,0}^{(1)} + a_{0,2,0}^{(2)} + a_{1,0,1}^{(2)} - 3a_{0,1,1}^{(3)}\right)\varepsilon_A \\
&\quad - \left(29\alpha_{2,0,0}^{(2)} + \frac{25}{2}a_{2,0,0}^{(3)}\right)\varepsilon_A\varepsilon_C - 46\alpha_{2,0,0}^{(3)}\varepsilon_B\varepsilon_C - 60\left(\alpha_{2,0,0}^{(2)} + \frac{1}{3}a_{2,0,0}^{(3)}\right)\varepsilon_B^2 \\
&\quad - \left(16a_{2,0,0}^{(2)} + 7a_{1,1,0}^{(3)}\right)\varepsilon_A\varepsilon_B + \left(\frac{3}{2}a_{2,0,0}^{(1)} - 2a_{1,1,0}^{(2)} - a_{0,2,0}^{(3)} - a_{1,0,1}^{(3)}\right)\varepsilon_A^2 \\
&\quad - \left(40\alpha_{2,0,0}^{(2)} + \frac{19}{2}a_{2,0,0}^{(3)}\right)\varepsilon_A^2\varepsilon_B - 10\alpha_{2,0,0}^{(3)}\varepsilon_A^3\varepsilon_B - 8\alpha_{2,0,0}^{(3)}\varepsilon_A^2\varepsilon_C - 32\alpha_{2,0,0}^{(3)}\varepsilon_A\varepsilon_B^2
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{3}{2} a_{2,0,0}^{(2)} + 2a_{1,1,0}^{(3)} \right) \varepsilon_A^3 - \left(\frac{5}{2} \alpha_{2,0,0}^{(2)} + 2a_{2,0,0}^{(3)} \right) \varepsilon_A^4 - 2\alpha_{2,0,0}^{(3)} \varepsilon_A^5, \\
\alpha_{0,0,2}^{(1)} = & \frac{1}{3} \left(\frac{1}{2} a_{0,1,1}^{(1)} - a_{0,0,2}^{(2)} \right) + \frac{1}{30} \left(a_{0,0,2}^{(3)} + a_{1,0,1}^{(1)} - 2a_{0,1,1}^{(2)} + a_{0,2,0}^{(1)} \right) \varepsilon_A \\
& + \frac{1}{18} \left(a_{2,0,0}^{(1)} - 2a_{0,2,0}^{(3)} - 2a_{1,1,0}^{(2)} + a_{1,0,1}^{(3)} \right) \varepsilon_C \\
& + \frac{1}{12} \left(a_{0,1,1}^{(3)} + a_{1,1,0}^{(1)} - 2a_{0,2,0}^{(2)} - 2a_{1,0,1}^{(2)} \right) \varepsilon_B \\
& + \left(-\frac{1}{5} \alpha_{0,2,0}^{(3)} - \frac{11}{90} a_{2,0,0}^{(2)} + \frac{1}{36} a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_C \\
& + \frac{1}{60} \left(2a_{1,1,0}^{(1)} + 2a_{0,1,1}^{(3)} - a_{1,0,1}^{(2)} - a_{0,2,0}^{(2)} \right) \varepsilon_A^2 \\
& + \frac{1}{6} \left(-2a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_B^2 + \frac{11}{180} \left(a_{2,0,0}^{(1)} - 2a_{1,1,0}^{(2)} + a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} \right) \varepsilon_A \varepsilon_B \\
& + \frac{1}{18} \alpha_{2,0,0}^{(3)} \varepsilon_C^2 - 18 \left(a_{2,0,0}^{(3)} + \frac{1}{18} \alpha_{2,0,0}^{(2)} \right) \varepsilon_B \varepsilon_C + \left(\frac{1}{36} a_{2,0,0}^{(3)} - \frac{1}{10} \alpha_{2,0,0}^{(2)} \right) \varepsilon_A^2 \varepsilon_C \\
& + \frac{17}{360} \left(a_{1,1,0}^{(3)} - 2a_{2,0,0}^{(2)} \right) \varepsilon_B \varepsilon_A^2 + \frac{1}{360} \left(a_{2,0,0}^{(1)} + a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} - 2a_{1,1,0}^{(2)} \right) \varepsilon_A^3 \\
& + \frac{43}{180} \alpha_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B \varepsilon_C + \alpha_{2,0,0}^{(3)} \varepsilon_B^3 + \frac{17}{36} a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B^2 + \frac{11}{180} a_{2,0,0}^{(3)} \varepsilon_B \varepsilon_A^3 \\
& + \frac{11}{360} \alpha_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_C + \frac{34}{45} \alpha_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_B^2 + \frac{5}{72} \alpha_{2,0,0}^{(3)} \varepsilon_B \varepsilon_A^4 + \frac{1}{720} \left(a_{1,1,0}^{(3)} - 2a_{2,0,0}^{(2)} \right) \varepsilon_A^4 \\
& + \frac{1}{720} a_{2,0,0}^{(3)} \varepsilon_A^5 + \frac{1}{720} \alpha_{2,0,0}^{(3)} \varepsilon_A^6,
\end{aligned}$$

where the coefficients of the normal form are given by

$$\begin{aligned}
10\bar{a}_0^1 = & a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} + 6a_{0,0,2}^{(3)} + 3a_{0,1,1}^{(2)} + \left(-6\alpha_{0,2,0}^{(3)} + \frac{5}{2} a_{1,1,0}^{(3)} - 2a_{2,0,0}^{(2)} + \alpha_{1,1,0}^{(2)} \right) \varepsilon_C \\
& + \left(2a_{0,2,0}^{(2)} + 2a_{1,0,1}^{(2)} + \frac{3}{2} a_{1,1,0}^{(1)} + \frac{3a_{0,1,1}^{(3)}}{2} \right) \varepsilon_A + \left(a_{0,2,0}^{(3)} + 6a_{2,0,0}^{(1)} + 3a_{1,1,0}^{(2)} + a_{1,0,1}^{(3)} \right) \varepsilon_B \\
& + \left(30a_{2,0,0}^{(1)} + 3a_{1,1,0}^{(2)} + a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} \right) \varepsilon_A^2 + 8 \left(\alpha_{2,0,0}^{(3)} \varepsilon_C \varepsilon_B + 2\alpha_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B^2 \right) \\
& + \left(\frac{5}{2} a_{2,0,0}^{(3)} + 7\alpha_{2,0,0}^{(2)} \right) \varepsilon_A \varepsilon_C + \left(-12a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} + 20\alpha_{1,1,0}^{(2)} \right) \varepsilon_A \varepsilon_C^2 \\
& + \left(a_{2,0,0}^{(3)} - 10\alpha_{2,0,0}^{(2)} \right) \varepsilon_A^2 \varepsilon_B + 2\alpha_{2,0,0}^{(3)} \varepsilon_C \varepsilon_A^2 + 5\alpha_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_B \\
& + \left(5\alpha_{1,1,0}^{(2)} - 3a_{2,0,0}^{(2)} + \frac{1}{4} a_{1,1,0}^{(3)} \right) \varepsilon_A^3 + \left(-\frac{5}{2} \alpha_{2,0,0}^{(2)} + \frac{1}{4} a_{2,0,0}^{(3)} \right) \varepsilon_A^4 \\
& + \frac{1}{4} \alpha_{2,0,0}^{(3)} \varepsilon_A^5,
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\bar{c}_{-1}^0 = & \frac{1}{5} \left(-2a_{0,2,0}^{(1)} + 3a_{1,0,1}^{(1)} - 2a_{0,0,2}^{(3)} - a_{0,1,1}^{(2)} \right) + \frac{1}{5} \left(a_{0,2,0}^{(2)} - 4a_{1,0,1}^{(2)} - \frac{1}{2}a_{1,1,0}^{(1)} - a_{0,1,1}^{(3)} \right) \varepsilon_A \\
& + \left(\frac{7}{5}a_{0,2,0}^{(3)} - \frac{1}{2}a_{1,1,0}^{(3)} - \frac{6}{5}a_{2,0,0}^{(2)} \right) \varepsilon_C - \frac{2}{5} \left(a_{0,2,0}^{(3)} + a_{2,0,0}^{(1)} - \frac{1}{2}a_{1,1,0}^{(2)} + \frac{3}{2}a_{1,0,1}^{(3)} \right) \varepsilon_B \\
& - \left(\frac{1}{2}a_{2,0,0}^{(3)} + \frac{19}{5}a_{2,0,0}^{(2)} \right) \varepsilon_C \varepsilon_A + \frac{1}{5} \left(-10a_{0,2,0}^{(3)} + 4a_{2,0,0}^{(2)} + 3a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_B \\
& + \left(\frac{3}{5}a_{2,0,0}^{(3)} + 2a_{2,0,0}^{(2)} \right) \varepsilon_A^2 \varepsilon_B + a_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_B + \frac{4}{5}a_{2,0,0}^{(3)} \varepsilon_C \varepsilon_B + \frac{8}{5}a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B^2 \\
& + \frac{3}{10}a_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_C + \frac{1}{5} \left(-a_{2,0,0}^{(1)} - \frac{1}{2}a_{1,1,0}^{(2)} + \frac{3}{10}a_{1,0,1}^{(3)} - \frac{1}{5}a_{0,2,0}^{(3)} \right) \varepsilon_A^2 \\
& + \left(-\frac{1}{2}a_{0,2,0}^{(3)} + \frac{1}{5}a_{2,0,0}^{(2)} + \frac{3a_{1,1,0}^{(3)}}{20} \right) \varepsilon_A^3 + \frac{1}{20} \left(6a_{2,0,0}^{(1)} + 3a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{1,0,1}^{(3)} \right) \varepsilon_A^2 \\
& + \left(\frac{1}{2}a_{2,0,0}^{(2)} + \frac{3a_{2,0,0}^{(3)}}{20} \right) \varepsilon_A^4 + \frac{3a_{2,0,0}^{(3)}}{20} \varepsilon_A^5, \tag{A.3}
\end{aligned}$$

$$\begin{aligned}
\bar{c}_1^0 = & a_{0,0,2}^{(1)} + \frac{1}{6} \left(\frac{1}{2}a_{1,1,0}^{(1)} - 5, a_{1,0,1}^{(2)} + a_{0,2,0}^{(2)} - \frac{3}{2}a_{0,1,1}^{(3)} \right) \varepsilon_C \\
& + \frac{1}{5} \left(a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} + a_{0,0,2}^{(3)} - 2a_{0,1,1}^{(2)} \right) \varepsilon_B \\
& + \frac{1}{3} \left(\frac{1}{2}a_{0,1,1}^{(1)} - a_{0,0,2}^{(2)} \right) \varepsilon_A + \frac{1}{5} \left(a_{1,0,1}^{(3)} + a_{2,0,0}^{(1)} + a_{0,2,0}^{(3)} - 2a_{1,1,0}^{(2)} \right) \varepsilon_B^2 \\
& + \frac{2}{15} \left(a_{1,1,0}^{(1)} - 2a_{1,0,1}^{(2)} - 2a_{0,2,0}^{(2)} + a_{0,1,1}^{(3)} \right) \varepsilon_B \varepsilon_A \\
& + \left(-\frac{6}{5}a_{0,2,0}^{(3)} + \frac{1}{6}a_{1,1,0}^{(3)} - \frac{11a_{2,0,0}^{(2)}}{15} \right) \varepsilon_C \varepsilon_B \\
& + \frac{1}{36} \left(-13a_{0,2,0}^{(3)} - a_{2,0,0}^{(1)} - 10a_{1,1,0}^{(2)} + 36a_{1,0,1}^{(3)} \right) \varepsilon_C \varepsilon_A - \frac{1}{3} \left(a_{2,0,0}^{(3)} + 7a_{2,0,0}^{(2)} \right) \varepsilon_1^2 \\
& + \frac{1}{30} \left(a_{0,0,2}^{(3)} + a_{1,0,1}^{(1)} - 2a_{0,1,1}^{(2)} + a_{0,2,0}^{(1)} \right) \varepsilon_A^2 \\
& + \left(-\frac{7}{10}a_{0,2,0}^{(3)} + \frac{11}{72}a_{1,1,0}^{(3)} - \frac{7}{180}a_{2,0,0}^{(2)} \right) \varepsilon_A^2 \varepsilon_C \\
& + \frac{1}{120} \left(-2a_{1,0,1}^{(2)} - 2a_{0,2,0}^{(2)} + a_{1,1,0}^{(1)} + a_{0,1,1}^{(3)} \right) \varepsilon_A^3 + \left(\frac{1}{36}a_{2,0,0}^{(3)} - \frac{34}{15}a_{2,0,0}^{(2)} \right) \varepsilon_A \varepsilon_C \varepsilon_B \\
& + a_{2,0,0}^{(3)} \varepsilon_B^3 + \frac{11}{36}a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_C^2 + \frac{8}{5}a_{2,0,0}^{(3)} \varepsilon_C \varepsilon_B^2 + \frac{11}{30} \left(-2a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_B^2 \\
& + \frac{7}{90} \left(-2a_{1,1,0}^{(2)} + a_{0,2,0}^{(3)} + a_{2,0,0}^{(1)} + a_{1,0,1}^{(3)} \right) \varepsilon_A^2 \varepsilon_B + \frac{1}{9} \left(\frac{1}{2}a_{1,1,0}^{(3)} - a_{2,0,0}^{(2)} \right) \varepsilon_A^3 \varepsilon_B \\
& + \frac{16}{5}a_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B^3 + \frac{1}{360} \left(a_{2,0,0}^{(1)} + a_{1,0,1}^{(3)} + a_{0,2,0}^{(3)} - 2a_{1,1,0}^{(2)} \right) \varepsilon_A^4
\end{aligned}$$

$$\begin{aligned}
& + \frac{34}{45} a_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_B^2 + \frac{49}{45} \alpha_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_B^2 + \frac{5}{72} a_{2,0,0}^{(3)} \varepsilon_A^4 \varepsilon_B \\
& + \frac{7}{90} \alpha_{2,0,0}^{(3)} \varepsilon_A^5 \varepsilon_B + \frac{7}{45} \alpha_{2,0,0}^{(3)} \varepsilon_A^4 \varepsilon_C + \left(\frac{7 \alpha_{2,0,0}^{(2)}}{30} + \frac{11}{72} a_{2,0,0}^{(3)} \right) \varepsilon_A^3 \varepsilon_C + \frac{34}{45} \alpha_{2,0,0}^{(3)} \varepsilon_A^2 \varepsilon_C \varepsilon_B \\
& + \frac{1}{360} \left(\frac{a_{1,1,0}^{(3)}}{2} - a_{2,0,0}^{(2)} \right) \varepsilon_A^5 + \frac{1}{720} a_{2,0,0}^{(3)} \varepsilon_A^6 + \frac{1}{720} \alpha_{2,0,0}^{(3)} \varepsilon_A^7, \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
\bar{b}_1^1 = & \frac{1}{6} a_{0,1,1}^{(1)} + \frac{2}{3} a_{0,0,2}^{(2)} + \frac{1}{6} \left(-a_{0,1,1}^{(3)} + a_{1,1,0}^{(1)} \right) \varepsilon_B \\
& + \frac{1}{18} \left(4a_{2,0,0}^{(1)} + a_{0,2,0}^{(3)} + a_{1,1,0}^{(2)} - 5a_{1,0,1}^{(3)} \right) \varepsilon_C \\
& + \frac{1}{12} \left(a_{0,1,1}^{(2)} + a_{0,2,0}^{(1)} + a_{1,0,1}^{(1)} - 2a_{0,0,2}^{(3)} \right) \varepsilon_A - \frac{1}{3} \left(2a_{2,0,0}^{(2)} + \frac{1}{2} a_{1,1,0}^{(3)} \right) \varepsilon_B^2 - \frac{7}{9} \alpha_{2,0,0}^{(3)} \varepsilon_C^2 \\
& - \frac{1}{3} \left(-\frac{5}{3} a_{2,0,0}^{(3)} + 4a_{2,0,0}^{(2)} \right) \varepsilon_B \varepsilon_C + \left(\frac{1}{2} \alpha_{0,2,0}^{(3)} - \frac{7}{18} a_{2,0,0}^{(2)} - \frac{19}{72} a_{1,1,0}^{(3)} \right) \varepsilon_A \varepsilon_C \\
& - \frac{4}{3} \left(2a_{2,0,0}^{(2)} + \frac{1}{3} a_{2,0,0}^{(3)} \right) \varepsilon_A \varepsilon_B^2 - \frac{4}{9} \alpha_{2,0,0}^{(3)} \varepsilon_B^2 \varepsilon_A^2 - \frac{16}{9} \alpha_{2,0,0}^{(3)} \varepsilon_A \varepsilon_B \varepsilon_C \\
& - \frac{1}{12} \left(13 \alpha_{2,0,0}^{(2)} + \frac{25}{6} a_{2,0,0}^{(3)} \right) \varepsilon_A^2 \varepsilon_C \\
& - \frac{29}{72} \alpha_{2,0,0}^{(3)} \varepsilon_A^3 \varepsilon_C + \frac{5}{36} \left(2a_{2,0,0}^{(1)} - a_{1,1,0}^{(2)} - a_{1,0,1}^{(3)} - a_{0,2,0}^{(3)} \right) \varepsilon_A \varepsilon_B + \frac{1}{24} \left(a_{1,1,0}^{(1)} - a_{0,1,1}^{(3)} \right) \varepsilon_A^2 \\
& - \frac{1}{9} \left(a_{1,1,0}^{(3)} + a_{2,0,0}^{(2)} \right) \varepsilon_A^2 \varepsilon_B + \frac{1}{72} \left(-a_{1,0,1}^{(3)} + 2a_{2,0,0}^{(1)} - a_{1,1,0}^{(2)} - a_{0,2,0}^{(3)} \right) \varepsilon_A^3 \\
& - \frac{5}{36} \left(a_{2,0,0}^{(3)} + 6a_{2,0,0}^{(2)} \right) \varepsilon_A^3 \varepsilon_B - \frac{1}{2} \left(4a_{2,0,0}^{(2)} + a_{1,1,0}^{(3)} \right) \varepsilon_A^4 - \frac{1}{144} \left(6a_{2,0,0}^{(2)} + a_{2,0,0}^{(3)} \right) \varepsilon_A^5 \\
& - \frac{5}{36} \alpha_{2,0,0}^{(3)} \varepsilon_A^4 \varepsilon_B - \frac{1}{144} \alpha_{2,0,0}^{(3)} \varepsilon_A^6. \tag{A.5}
\end{aligned}$$

Appendix B. The coefficients of $\mathbf{t}_0^{\mathbf{e}}$ in the L_4 -problem

The coefficients of transformation given in Equation (6.7) with free parameters t_1, t_2, t_5, t_6 are as follows:

$$\begin{aligned}
t_3 = & -2 \left(\sqrt{2} - 2 \right) \left(\alpha_1 t_5 + 6 t_5 + 4 \sqrt{2} t_5 + 2 \sqrt{2} \alpha_2 t_2 + 2 \alpha_2 t_2 \right) \\
& + \left(-6 \sqrt{2} t_5 + 3 \alpha_2 t_2 + 12 t_5 \right) \varepsilon, \\
t_4 = & 2 \left(\sqrt{2} - 2 \right) \left(2 \sqrt{2} \alpha_2 - 4 \sqrt{2} - \alpha_1 + 2 \alpha_2 - 6 \right) \\
& + \frac{1}{2} \left(\sqrt{2} - 2 \right) \left(8 \sqrt{2} \alpha_2 t_1 + 3 \sqrt{2} \alpha_2 - 16 \sqrt{2} t_6 - 4 t_6 \alpha_1 + 8 \alpha_2 t_1 + 6 \alpha_2 - 24 t_6 - 12 \right) \varepsilon \\
& + \left(-6 \sqrt{2} t_6 - 3 \alpha_2 t_1 + 12 t_6 \right) \varepsilon^2,
\end{aligned}$$

$$\begin{aligned}
t_7 &= -2 \left(2 + \sqrt{2} \right) \left(2\sqrt{2}\alpha_2 + 4\sqrt{2} - \alpha_1 - 2\alpha_2 - 6 \right) \\
&\quad - \frac{1}{2} \left(2 + \sqrt{2} \right) \left(8\sqrt{2}\alpha_2 t_6 - 3\sqrt{2}\alpha_2 + 16t_1\sqrt{2} - 4t_1\alpha_1 - 8\alpha_2 t_6 + 6\alpha_2 - 24t_1 + 12 \right) \varepsilon \\
&\quad + \left(-6t_1\sqrt{2} - 3\alpha_2 t_6 - 12t_1 \right) \varepsilon^2, \\
t_8 &= 2 \left(2 + \sqrt{2} \right) \left(\alpha_1 t_2 + 6t_2 - 4\sqrt{2}t_2 + 2\sqrt{2}\alpha_2 t_5 - 2t_5\alpha_2 \right) + \left(-6\sqrt{2}t_2 + 3t_5\alpha_2 - 12t_2 \right) \varepsilon, \\
t_9 &= - \left(2 + \sqrt{2} \right) \left(\alpha_1\alpha_2 t_2 + 6\alpha_2 t_2 - 4\sqrt{2}\alpha_2 t_2 + 4\sqrt{2}t_5\alpha_1 - 4\alpha_1 t_5 + 8\sqrt{2}t_5 - 8t_5 \right) \\
&\quad + \left(3\sqrt{2}\alpha_2 t_2 + 3\alpha_1 t_5 + 6\alpha_2 t_2 + 30t_5 \right) \varepsilon, \\
t_{10} &= - \left(2 + \sqrt{2} \right) \left(4\sqrt{2}\alpha_1 + 4\sqrt{2}\alpha_2 - \alpha_1\alpha_2 + 8\sqrt{2} - 4\alpha_1 - 6\alpha_2 - 8 \right) \\
&\quad - \frac{1}{2} \left(2 + \sqrt{2} \right) \left(8\sqrt{2}\alpha_1 t_6 + 8\sqrt{2}\alpha_2 t_1 - 2\alpha_1\alpha_2 t_1 + 3\sqrt{2}\alpha_1 + 16\sqrt{2}t_6 - 8t_6\alpha_1 - 12\alpha_2 t_1 \right. \\
&\quad \left. + 30\sqrt{2} - 6\alpha_1 + 6\alpha_2 - 16t_6 - 60 \right) \varepsilon + \left(-3\sqrt{2}\alpha_2 t_1 + 3t_6\alpha_1 - 6\alpha_2 t_1 + 30t_6 \right) \varepsilon^2, \\
t_{11} &= -4\alpha_2 (\alpha_1 - 2) + \left(-4\alpha_1\alpha_2 t_6 + 12\sqrt{2}\alpha_1 + 8\alpha_2 t_6 - 24\sqrt{2} - 96 \right) \varepsilon \\
&\quad + \frac{3}{2} \left(2 + \sqrt{2} \right) \left(3\sqrt{2}\alpha_2 - 32t_1\sqrt{2} + 8t_1\alpha_1 - 6\alpha_2 + 48t_1 - 24 \right) \varepsilon^2 \\
&\quad + \left(-36t_1\sqrt{2} - 9\alpha_2 t_6 - 72t_1 \right) \varepsilon^3, \\
t_{12} &= 4\alpha_2 t_5 (\alpha_1 - 2) - 12 \left(2 + \sqrt{2} \right) \left(-\alpha_1 - 6 + 4\sqrt{2} \right) t_2 \varepsilon + \left(-36\sqrt{2}t_2 + 9t_5\alpha_2 - 72t_2 \right) \varepsilon^2, \\
t_{13} &= \left(\sqrt{2} - 2 \right) \left(4\sqrt{2}\alpha_1 - 4\sqrt{2}\alpha_2 - \alpha_1\alpha_2 + 8\sqrt{2} + 4\alpha_1 - 6\alpha_2 + 8 \right) \\
&\quad + \frac{1}{2} \left(\sqrt{2} - 2 \right) \left(8\sqrt{2}\alpha_1 t_1 - 8\sqrt{2}\alpha_2 t_6 - 2\alpha_1\alpha_2 t_6 - 3\sqrt{2}\alpha_1 + 16t_1\sqrt{2} + 8t_1\alpha_1 - 12\alpha_2 t_6 \right. \\
&\quad \left. - 30\sqrt{2} - 6\alpha_1 - 6\alpha_2 + 16t_1 - 60 \right) \varepsilon + \left(-3\sqrt{2}\alpha_2 t_6 + 3t_1\alpha_1 + 6\alpha_2 t_6 + 30t_1 \right) \varepsilon^2, \\
t_{14} &= \left(\sqrt{2} - 2 \right) \left(\alpha_1\alpha_2 t_5 + 6t_5\alpha_2 + 4\sqrt{2}\alpha_2 t_5 + 4\sqrt{2}\alpha_1 t_2 + 4\alpha_1 t_2 + 8\sqrt{2}t_2 + 8t_2 \right) \\
&\quad + \left(3\sqrt{2}\alpha_2 t_5 + 3\alpha_1 t_2 - 6t_5\alpha_2 + 30t_2 \right) \varepsilon, \\
t_{15} &= 4\alpha_2 t_2 (\alpha_1 - 2) - 12 \left(\sqrt{2} - 2 \right) \left(\alpha_1 + 6 + 4\sqrt{2} \right) t_5 \varepsilon + \left(-36\sqrt{2}t_5 + 9\alpha_2 t_2 + 72t_5 \right) \varepsilon^2, \\
t_{16} &= -4\alpha_2 (\alpha_1 - 2) + \left(-4\alpha_1\alpha_2 t_1 - 12\sqrt{2}\alpha_1 + 8\alpha_2 t_1 + 24\sqrt{2} - 96 \right) \varepsilon \\
&\quad + \frac{3}{2} \left(\sqrt{2} - 2 \right) \left(3\sqrt{2}\alpha_2 - 32\sqrt{2}t_6 - 8t_6\alpha_1 + 6\alpha_2 - 48t_6 - 24 \right) \varepsilon^2 \\
&\quad + \left(-36\sqrt{2}t_6 - 9\alpha_2 t_1 + 72t_6 \right) \varepsilon^3.
\end{aligned}$$

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