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# Nonlinear age-structured population models with nonlocal diffusion and nonlocal boundary conditions ☆

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**Abstract**

In this paper, we develop some basic theory for age-structured population models with nonlocal diffusion and nonlocal boundary conditions. We first apply the theory of integrated semigroups and non-densely defined operators to a linear equation, study the spectrum, and analyze the asymptotic behavior via asynchronous exponential growth. Then we consider a semilinear equation with nonlocal diffusion and nonlocal boundary condition, use the method of characteristic lines to find the resolvent of the infinitesimal generator and the variation of constant formula, apply Krasnoselskii's fixed point theorem to obtain the existence of nontrivial steady states, and establish the stability of steady states. Finally we generalize these results to a nonlinear equation with nonlocal diffusion and nonlocal boundary condition.


*MSC:* 35K55; 35K90; 92D25

*Keywords:* Age structure; Nonlocal diffusion; Semigroup theory; Infinitesimal generator; Spectrum theory; Stability

## Contents



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## 1. Introduction

Spatial dispersal of individuals is crucial in determining the nonlinear dynamics of population models. However, individuals need to be mature enough to disperse. So the age structure of the population also plays an important role in modeling population dynamics. Consequently, age-structured population models with diffusion arise very naturally in studying biological, epidemiological and medical problems. Since the pioneer work of Gurtin [19], Gurtin and MacCamy [20], and MacCamy [35], age-structured population models with Laplace diffusion have been extensively studied in the literature, we refer to Chan and Guo [8], Di Blasio [16], Guo and Chan [18], Hastings [21], Huyer [24], Langlais [32], Walker [40,42], Webb [43], and so on. Detailed results and references can be found in a survey by Webb [46].

On the other hand, there is an increasing interest in nonlocal diffusion problems modeled by convolution diffusion operators such as

$$L_0 v := d \int_\Omega J(x-y)\left[v(y) - v(x)\right] dy,$$

where $v$ belongs to a proper Banach space and $\Omega$ is a spatial region, see Bates et al. [3], Bates and Zhao [4], Cao et al. [7], Cortazar et al. [9], Coville [10], Coville et al. [11], García-Melián and Rossi [17], Hutson et al. [23], Kao et al. [30], Rawal and Shen [36], Yang et al. [47], Zhao and Ruan [48], and the references cited therein. We refer to the surveys of Bates [2] and Ruan [37] for applications of nonlocal diffusion equations in material science and epidemiology, respectively, and the monograph of Andreu-Vaillo et al. [1] for fundamental theories and results on nonlocal diffusion problems. As pointed out in Bates et al. [2], $J(x-y)$ is viewed as the probability distribution of jumping from location $y$ to location $x$, namely the convolution $\int_\Omega J(x-y)u(t,y)dy$ is the rate at which individuals are arriving to position $x$ from other places and $\int_\Omega J(y-x)u(t,x)dy$ is the rate at which they are leaving location $x$ to travel to other sites.

Recently, we (Kang et al. [29]) proposed a linear age-structured population model with nonlocal diffusion, studied the semigroup of linear operators associated to the model, and used the spectral properties of its infinitesimal generator to determine the stability of the zero steady state. In this paper we consider the following general nonlinear age-structured population model with nonlocal diffusion and nonlocal boundary condition:

$$\begin{cases} u_t(t,a,x)+u_a(t,a,x)=d(J*u-u)(t,a,x)+G(u(t,\cdot,\cdot))(a,x), \\ \quad t>0, 0<a<a^+, x\in\Omega, \\ u(t,0,x)=F(u(t,\cdot,\cdot))(x), \quad t>0, x\in\Omega, \\ u(t,a,x)=0, \quad t>0, 0<a<a^+, x\notin\Omega, \\ u(0,a,x)=\phi(a,x), \quad 0<a<a^+, x\in\Omega, \end{cases} \tag{1.1}$$

where $u(t,a,x)$ denotes the density of a population at time $t$ with age $a$ at position $x\in\Omega$, where $a^+$ denotes the maximum age which could be finite or infinite, $\Omega\subset\mathbb{R}^N$ is a bounded region and $N\geq 1$ is an integer, $\phi\in E:=L^1((0,a^+),Z)$ is an initial data and $Z$ is an ordered Banach space that represents the distribution of a population with respect to a space structure in $\Omega$. For the sake of deriving estimates and using duality, we assume that $Z=L^2(\Omega)$ in this paper and point out that all results can be established for $Z=L^p(\Omega)(p\geq 1)$ with necessary modifications. Moreover, assume that $G:E\to E$ and $F:E\to Z$ are uniformly bounded and locally Lipschitz continuous (which are reasonable from the point of view of population dynamics). The diffusion kernel $J$ is a $C^0$, compactly supported, nonnegative function with unit integral representing the spatial dispersal; i.e.,

$$\int_{\mathbb{R}^N} J(x)dx=1, \quad J(x)\geq 0, \quad \forall x\in\mathbb{R}^N.$$

Define

$$\mathcal{L}u:=(J*u-u)(t,a,x)=\int_{\mathbb{R}^N} J(x-y)u(t,a,y)dy-u(t,a,x)$$

with the Dirichlet boundary condition. The convolution $\int_{\mathbb{R}^N} J(x-y)u(t,a,y)dy$ represents the rate at which individuals with age $a$ are arriving at position $x$ from other places at time $t$ and $\int_{\mathbb{R}^N} J(y-x)u(t,a,x)dy$ is the rate at which they are leaving location $x$ to travel to other sites. For the sake of simplicity, we assume that the diffusion rate $d=1$.

To the best of our knowledge, the nonlinear age-structured model (1.1) with nonlocal diffusion and nonlocal boundary conditions has not been studied in any literature. One of the technical challenges in analyzing such equations is that the semigroup generated by the associated operator is not compact for any $t\geq 0$ (Andreu-Vaillo et al. [1]). In this paper, we develop some basic theory for the nonlinear age-structured population model (1.1) with nonlocal diffusion and nonlocal boundary conditions. We first apply the theory of integrated semigroups and non-densely defined operators to the linear equations, study the spectrum, and analyze the asymptotic behavior via asynchronous exponential growth. Then we consider a semilinear equation with nonlocal diffusion and nonlocal boundary condition, use the method of characteristic lines to find the resolvent of the infinitesimal generator and the variation of constant formula, apply Krasnoselskii's fixed point theorem to obtain the existence of a steady state, and establish the stability of the steady states. Finally we generalize these results to a nonlinear equation with nonlocal diffusion and nonlocal boundary condition.

We would like to mention that some results we obtain in this paper are parallel to those for age-structured models, including spectrum analysis (with asynchronous exponential growth) for linear models and existence and stability of nontrivial steady states for nonlinear models, see the classical book of Webb [44]. Some results are similar to those for age-structured models with Laplace diffusion, see for instance Guo and Chan [8,18], Delgado et al. [14,15] and Walker [40–42]. Our contribution indeed is to extend those results for either age-structured models or age-structured models with Laplace diffusion to age-structured model (1.1) with nonlocal diffusion. We overcame the lack of compactness of the evolution family generated by the nonlocal diffusion operator by assuming equicontinuity of the integral boundary conditions.

## 2. Preliminaries

In this section we provide some preliminary results, including the integrated semigroup setting of the problem, existence of integral solutions, fixed point theorems, and the principle of linearized stability.

### 2.1. Integrated semigroup setting

First we recall some notations and results on integrated semigroup theory and non-densely defined operators from Thieme [39] and Magal and Ruan [33]. Let $A$ be the sum of a differential operator and the nonlocal dispersal operator $\mathcal{L}$ acting on $E$ defined by

$$A\psi := -\psi_a + \mathcal{L}\psi, \quad D(A) := W^{1,1}\left((0,a^+), Z\right).$$

Then $A$ is densely defined in $E$. Now we introduce an extended state space $X$ by

$$X := Z \times E$$

and its closed subspace $X_0$ by $X_0 := \{0\} \times E$. Define an operator $\mathcal{A}$ acting on $X$ by

$$\mathcal{A}\begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} -\psi(0) \\ -\psi_a + \mathcal{L}\psi \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(\mathcal{A}) := \{0\} \times D(A).$$

Note that $\psi(0)$ is well-defined for any $\psi \in W^{1,1}((0,a^+), Z)$ since the Sobolev embedding $W^{1,1}((0,a^+), Z) \hookrightarrow C([0,a^+), Z)$ holds and we write $\psi(a)(x) = \psi(a,x)$. Throughout the paper, we will hide the variable $x$ by writing $\psi(a)$.

Let $X_{0+} := \{0\} \times E_+$ be the positive cone of $X_0$. Define a bounded operator $\mathcal{B} : X_{0+} \to X$ by

$$\mathcal{B}\begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} F(\psi) \\ G(\psi) \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_{0+}.$$

Using the above notations, we rewrite system (1.1) as an abstract semilinear Cauchy problem with non-densely defined operator on $X$:

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t) + \mathcal{B}U(t), \\ U(0) = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in X_{0+}, \end{cases} \tag{2.1}$$

where

$$U(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}.$$

Following Busenberg et al. [6], we consider the following equivalent system:

$$\begin{cases} \frac{dU(t)}{dt} = \left(\mathcal{A} - \frac{1}{\epsilon}I\right)U(t) + \frac{1}{\epsilon}(I + \epsilon\mathcal{B})U(t) \\ U(0) = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in X_{0+}, \end{cases} \tag{2.2}$$

where $\epsilon$ is chosen so small that the operator $I + \epsilon\mathcal{B}$ maps $X_{0+}$ into the positive cone of $X$, denoted by $X_+$. Since parameter functions $G$ and $F$ are assumed to be uniformly bounded, it can be shown that this choice of $\epsilon$ is possible for our system (2.1). For the sake of simplicity we introduce new notations

$$\mathcal{A}_* = \mathcal{A} - \frac{1}{\epsilon}I, \quad \mathcal{B}_* = \frac{1}{\epsilon}(I + \epsilon\mathcal{B}).$$

Since the operator $\mathcal{A}_*$ is not densely defined, we cannot apply the classical Hille-Yosida theory to solve the ODE (2.2) in the Banach space $X$. However, we can show that the operator $\mathcal{A}_*$ is a Hille-Yosida operator.

**Lemma 2.1.** *$\mathcal{A}_*$ is a closed linear operator with non-dense domain and the following holds: $\overline{D(\mathcal{A}_*)} = X_0$, $\mathcal{A}_*$ satisfies the Hille-Yosida estimate such that for all $\lambda > -\zeta_0 - \frac{1}{\epsilon}$,*

$$\left\| (\lambda I - \mathcal{A}_*)^{-1} \right\|_X \le \frac{2}{\lambda + \zeta_0 + \frac{1}{\epsilon}}, \tag{2.3}$$

*where $\zeta_0$ is the principle eigenvalue of $-\mathcal{L}$ under the Dirichlet boundary condition, and $(\lambda I - \mathcal{A}_*)^{-1}(X_+) \subset X_{0^+}$ for $\lambda > 0$.*

Before proving the above lemma, we recall a result related to the principal eigenvalue of $-\mathcal{L}$ under the Dirichlet boundary condition (see Coville et al. [11], Hutson et al. [23], García-Molián and Rossi [17]).

**Theorem 2.2** *(García-Melián and Rossi [17]). The operator $-\mathcal{L}$ under Dirichlet boundary condition admits an eigenvalue $\zeta_0$ associated with a positive eigenfunction $\varphi_0 \in L^2(\Omega)$. Moreover, it is simple and unique and satisfies $0 < \zeta_0 < 1$. Furthermore, $\zeta_0$ can be variationally characterized as*

$$\zeta_0 = 1 - \left( \sup_{u \in L^2(\Omega), \|u\|_{L^2(\Omega)}=1} \int_\Omega \left( \int_\Omega J(x-y)u(y)dy \right)^2 dx \right)^{1/2}. \tag{2.4}$$

Now we give a proof of Lemma 2.1.

**Proof.** Consider the resolvent of operator $\mathcal{A}_*$, i.e.,

$$(\lambda I - \mathcal{A}_*) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \eta \\ \phi \end{pmatrix} \in X.$$

By the definition of $\mathcal{A}_*$,

$$(\lambda I - \mathcal{A}_*) \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi(0) \\ \frac{\partial \varphi}{\partial a} - \mathcal{L}\varphi + \left(\lambda + \frac{1}{\epsilon}\right)\varphi \end{pmatrix},$$

we have

$$\frac{\partial \varphi}{\partial a} = \mathcal{L}\varphi - (\lambda + 1/\epsilon)\varphi + \phi(a), \tag{2.5}$$

$$\varphi(0) = \eta. \tag{2.6}$$

Solving problem (2.5)-(2.6), we obtain that

$$\varphi(a) = e^{-(\lambda+1/\epsilon)a} e^{\mathcal{L}a}\eta + \int_0^a e^{-(\lambda+1/\epsilon)(a-\sigma)} e^{\mathcal{L}(a-\sigma)}\phi(\sigma)d\sigma, \tag{2.7}$$

where $\{e^{\mathcal{L}a}\}_{a\geq 0}$ is the $C_0$-semigroup generated by $\mathcal{L}$. Thus,

$$(\lambda I - \mathcal{A}_*)^{-1} \begin{pmatrix} \eta \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

and

$$\begin{aligned} \left\| (\lambda I - \mathcal{A}_*)^{-1} \begin{pmatrix} \eta \\ \phi \end{pmatrix} \right\|_X &= \|\varphi\|_E \\ &\leq 2 \int_0^{a^+} e^{-(\lambda+1/\epsilon)a} \left[ \int_\Omega \left( e^{\mathcal{L}a}\eta \right)^2 dx \right]^{\frac{1}{2}} da \end{aligned}$$

$$
\begin{aligned}
&+2\int_0^{a^+}\left[\int_\Omega\left(\int_0^a e^{-(\lambda+1/\epsilon)(a-\sigma)}e^{\mathcal{L}(a-\sigma)}\phi(\sigma)d\sigma\right)^2 dx\right]^{\frac{1}{2}} da\\
\le\, & 2\int_0^{a^+} e^{-(\lambda+1/\epsilon)a}e^{-\zeta_0 a}da\left\|\eta\right\|_Z\\
&+2\int_0^{a^+}\int_0^a e^{-(\lambda+1/\epsilon)(a-\sigma)}e^{-\zeta_0(a-\sigma)}\left(\int_\Omega|\phi(\sigma)|^2dx\right)^{\frac{1}{2}} d\sigma da\\
\le\, & \frac{2}{\lambda+\zeta_0+1/\epsilon}\left\|\eta\right\|_Z+2\int_0^{a^+}\int_0^{a^+} e^{-(\lambda+\zeta_0+1/\epsilon)(a-\sigma)}\left\|\phi(\sigma)\right\|_Z d\sigma da\\
\le\, & \frac{2}{\lambda+\zeta_0+1/\epsilon}\left\|\begin{pmatrix}\eta\\ \phi\end{pmatrix}\right\|_X,
\end{aligned}
\tag{2.8}
$$

in which we used Minkowski inequality and an estimate of the semigroup $\left\|e^{\mathcal{L}a}\right\|_Z \le e^{-\zeta_0 a}, a \ge 0$, where $\zeta_0 \in (0, 1)$ is the principle eigenvalue of $-\mathcal{L}$ with Dirichlet boundary condition. It follows that

$$
\left\|(\lambda I-\mathcal{A}_*)^{-1}\right\|_X \le \frac{2}{\lambda+\zeta_0+1/\epsilon}
$$

for $\lambda > -\zeta_0 - 1/\epsilon$. Hence $\mathcal{A}_*$ is a Hille-Yosida operator with $M = 2$ and $\omega = -\zeta_0 - 1/\epsilon < 0$. Moreover, if $\begin{pmatrix}\eta\\ \phi\end{pmatrix} \in X_+$, we have $\begin{pmatrix}0\\ \varphi\end{pmatrix} \in X_{0+}$, which implies $(\lambda I - \mathcal{A}_*)^{-1}(X_+) \subset X_{0^+}$ for $\lambda > 0$. This completes the proof of Lemma 2.1. □

## 2.2. *Integral solutions*

**Definition 2.3.** A function $u(t) \in C^1(0, T; X) \cup D(\mathcal{A}_*)$ is called a *classical solution* of the Cauchy problem (2.2) if it is satisfied for all $t \in [0, T)$. $u(t) \in C(0, T; X_0)$ is called an *integral solution* of (2.2) (Da Prato and Sinestrari [13] and Benilan et al. [5]) if

$$
\int_0^t u(s)ds \in D(\mathcal{A}_*) \text{ for all } t \in [0, T)
$$

and

$$
u(t) = u(0) + \mathcal{A}_*\int_0^t u(s)ds + \int_0^t \mathcal{B}_*u(s)ds. \tag{2.9}
$$

It can be shown that an integral solution becomes a classical solution if $u(0) \in D(\mathcal{A}_*)$, $\mathcal{A}_* u(0) + \mathcal{B}_* u(0) \in \overline{D(\mathcal{A}_*)}$ (Thieme [39]). Thus, in what follows we focus on integral solutions of (2.2).

Define the *part* $\mathcal{A}_0$ of $\mathcal{A}_*$ in $X_0$ as $\mathcal{A}_0 = \mathcal{A}_*$ on $D(\mathcal{A}_0) = \left\{ \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(\mathcal{A}_*) : \mathcal{A}_* \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in X_0 \right\}$. Then the following result holds (Thieme [39] and Magal and Ruan [33]):

**Lemma 2.4.** *For the part $\mathcal{A}_0$, $\overline{D(\mathcal{A}_0)} = X_0$ holds and $\mathcal{A}_0$ generates a strongly continuous semigroup $\{\mathcal{S}_0(t)\}_{t \geq 0}$ on $X_0$ and $\mathcal{S}_0(X_{0+}) \subset X_{0+}$.*

Using the semigroup $\{\mathcal{S}_0(t)\}_{t \geq 0}$, we can formulate an extended variation of constants formula for (2.2) (see Thieme [39] and Magal and Ruan [33]).

**Proposition 2.5.** *A positive function $u(t) \in C(0, T; X_0)$ is an integral solution for (2.2) if and only if $u(t)$ is the positive continuous solution of the variation of constants formula on $X_0$:*

$$u(t) = \mathcal{S}_0(t)u(0) + \lim_{\lambda \to \infty} \int_0^t \mathcal{S}_0(t-s)\lambda(\lambda I - \mathcal{A}_*)^{-1}\mathcal{B}_* u(s)ds. \tag{2.10}$$

To obtain an integral solution of (2.2), Proposition 2.5 implies that it is sufficient to solve the extended variation of constants formula (2.10). If $\mathcal{B}_*$ is a locally Lipschitz continuous bounded perturbation, modifying the argument of Inaba [26] on the classical variation of constants formula accordingly and applying the contraction mapping principle, we can show the existence of positive local solutions for the extended variation of constants formula (2.10). Since the norm of the local solution grows at most exponentially, a local solution can be extended to a global solution. Hence, we can conclude that the initial boundary value problem (2.2) has a unique global positive integral solution.

### *2.3. Fixed point theorems*

In this subsection, we give the fixed point theorem from Inaba [25] (see also Inaba [27, Proposition 7.7]).

**Theorem 2.6** *(Inaba [25]). Let $E$ be a real Banach space and $E_+$ be its positive cone. Let $\Psi$ be a positive operator from $E_+$ to itself and $T := \Psi'[0]$ be its Fréchet derivative at 0. If*

(i) $\Psi(0) = 0$*;*
(ii) $\Psi$ *is compact and bounded;*
(iii) $T$ *has a positive eigenvector $v_0 \in E_+ \setminus \{0\}$ associated with an eigenvalue $\lambda_0 > 1$;*
(iv) $T$ *has no eigenvector in $E_+$ associated with the eigenvalue 1,*

*then $\Psi$ has at least one nontrivial fixed point in $E_+$.*

In the case where $T$ is a majorant of $\Psi$ (that is, $T$ is a linear operator such that $\Psi(\phi) \leq T\phi$ for any $\phi \in E_+$), the following theorem also holds (see Inaba [27, Proposition 7.8]).

**Theorem 2.7** *(Inaba [27]). Let $E$ be a real Banach space and $E_+$ be its positive cone. Let $\Psi$ be a positive operator from $E_+$ to itself and $T$ be its compact and nonsupporting majorant. Then, $\Psi$ has no trivial fixed point in $E_+$ provided $r(T) \le 1$.*

### 2.4. Principle of linearized stability

Let $\{\mathcal{S}(t)\}_{t\ge 0}$ be a semigroup on $X_0$ induced by setting $\mathcal{S}(t)u(0) = u(t)$, where $u(t)$ is an integral solution of (2.2). Then it follows that $\{\mathcal{S}(t)\}_{t\ge 0}$ is a $C_0$-semigroup generated by the part $\mathcal{A}_* + \mathcal{B}_*$ in $X_0 = \overline{D(\mathcal{A}_*)}$. Let $\omega_0(A)$ and $\omega_1(A)$ denote the growth bound and essential growth bound of the semigroup generated by $A$, respectively. The principle of linearized stability for the evolution system (2.2) with non-densely defined generator can be stated as follows (Thieme [39]).

**Proposition 2.8.** *Let $\mathcal{B}_*$ be continuously Fréchet differentiable in $X_0$, let $u^*$ be a steady state, and let $\mathcal{B}'_*[u^*]$ denote the Fréchet derivative at $u^*$. If $\omega_0(\mathcal{A}_* + \mathcal{B}'_*[u^*]) < 0$, then for any $\omega > \omega_0(\mathcal{A}_* + \mathcal{B}'_*[u^*])$, there exist numbers $M > 0$ and $\delta > 0$ such that*

$$\left\|\mathcal{S}(t)u - u^*\right\| \le M e^{\omega t}\left\|u - u^*\right\|$$

*for all $u \in X_0$ with $\left\|u - u^*\right\| \le \delta, t \ge 0$.*

**Corollary 2.9.** Suppose that $\omega_1(\mathcal{A}_* + \mathcal{B}'_*[u^*]) < 0$. If all eigenvalues of $\mathcal{A}_* + \mathcal{B}'_*[u^*]$ have strictly negative real part, then there exist $\omega < 0, \delta > 0, M > 0$ such that

$$\left\|\mathcal{S}(t)u - u^*\right\| \le M e^{\omega t}\left\|u - u^*\right\|$$

for all $u \in X_0$ with $\left\|u - u^*\right\| \le \delta, t \ge 0$. If at least one eigenvalue of $\mathcal{A}_* + \mathcal{B}'_*[u^*]$ has strictly positive real part, then $u^*$ is an unstable steady state

## 3. Linear equations

In this section, we apply the above techniques to a linear version of problem (1.1) with linear mortality and nonlocal boundary condition; namely,

$$\begin{cases} u_t(t,a,x) + u_a(t,a,x) = d(J * u - u)(t,a,x) - \mu(a,x)u(t,a,x), \\ \quad t>0, 0<a<a^+, x\in\Omega, \\ u(t,0,x) = \int_0^{a^+}\int_\Omega \beta(a,x-y)u(t,a,y)dyda, \quad t>0, x\in\Omega, \\ u(t,a,x) = 0, \quad t>0, 0<a<a^+, x\notin\Omega, \\ u(0,a,x) = \phi(a,x), \quad 0<a<a^+, x\in\Omega, \end{cases} \tag{3.1}$$

where $\mu$ denotes the death rate and $\beta$ represents the birth rate. $\Omega \subset \mathbb{R}^N$ is bounded. In this section, we assume that $a^+ < \infty$ in the proofs of all results. But $a^+$ could be infinity and all proofs throughout this section hold with minor modifications when $a^+ = \infty$, see Remark 3.9.

*3.1. Existence of solutions*

**Assumption 3.1.** Assume that

(i) $\beta \in L^1((0,a^+),Z)$ is bounded a.e. with respect to $a$ and $x$, nonnegative and

$$\lim_{\|h\|\to 0}\int_\Omega |\beta(a,x-y+h)-\beta(a,x-y)|^2 dx = 0$$

uniformly in $a \in (0,a^+)$ and $y \in \Omega$;

(ii) There exists a positive function $\underline{\beta} \in L^1(0,a^+)$ such that

$$\beta(a,x) \ge \underline{\beta}(a) > 0 \quad \text{for almost all} \quad (a,x) \in (0,a^+)\times\Omega;$$

(iii) $\mu$ is bounded a.e. with respect to $a$ and $x$, measurable, positive and

$$\int_0^a \overline{\mu}(\rho)d\rho < \infty \quad \text{for} \quad a < a^+ \quad \text{and} \quad \int_0^{a^+} \underline{\mu}(\rho)d\rho = \infty,$$

in which $\underline{\mu}(a) = \inf_{x\in\Omega}\mu(a,x)$ and $\overline{\mu}(a) = \sup_{x\in\Omega}\mu(a,x)$.

We first give a lemma and its proof is similar to that of Lemma 1 in Guo and Chan [18] (see also Kang et al. [29]). In the following, we will hide the spatial variable $x$ in $\beta$ and $\mu$ for consistence. Note that $\mu(a)(x) = \mu(a,x)$ and $\beta(a)(x) = \beta(a,x)$.

**Lemma 3.2.** *For any $0 \le a_0 < a^+$, there exists a unique mild solution $u(a)$, $0 \le \tau \le a \le a^+ - a_0$, to the evolution equation on $E$ for any initial function $\phi \in Z$:*

$$\begin{cases} \frac{\partial u(a)}{\partial a} = [-\mu(a_0+a)+\mathcal{L}]u(a) \\ u(\tau) = \phi. \end{cases} \tag{3.2}$$

*Define the solution operator of the initial value problem (3.2) by*

$$\mathcal{F}(a_0,\tau,a)\phi = u(a), \quad \forall \phi \in Z. \tag{3.3}$$

*Then $\{\mathcal{F}(a_0,\tau,a)\}_{0\le\tau\le a\le a^+-a_0}$ is a family of uniformly linear bounded positive operators on $E$ and is strongly continuous in $\tau$ and $a$. Furthermore,*

$$e^{-\int_\tau^a \overline{\mu}(a_0+\rho)d\rho} e^{\mathcal{L}(a-\tau)} \le \mathcal{F}(a_0,\tau,a) \le e^{-\int_\tau^a \underline{\mu}(a_0+\rho)d\rho} e^{\mathcal{L}(a-\tau)}, \tag{3.4}$$

*where $\underline{\mu}(a) := \inf_{x\in\Omega}\mu(a,x)$ and $\overline{\mu}(a) := \sup_{x\in\Omega}\mu(a,x)$.*

## 3.2. Spectral analysis

Define

$$\mathcal{X}_1\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}0\\ -\mu\phi\end{pmatrix} \text{ and } \mathcal{X}_2\begin{pmatrix}0\\ \phi\end{pmatrix}=\begin{pmatrix}F(\phi)\\ 0\end{pmatrix}.$$

Then $\mathcal{B}=\mathcal{X}_1+\mathcal{X}_2$. It can be shown that $\mathcal{X}_2 : X \to X$ is a compact operator under the Assumption 3.1-(i) by Kolmogorov compactness theorem (see Lemma 3.3 in the following). Moreover, $\mathcal{A}+\mathcal{X}_1$ generates a nilpotent semigroup $\mathcal{F}_1(t)$ in $X$ which is eventually compact via integrating along characteristics. Then it follows that its perturbed semigroup by the compact perturbation $\mathcal{X}_2$ is still eventually compact. Hence, we have $\omega_1(\mathcal{A}+\mathcal{B})=-\infty$. From Corollary 2.9, we only need to evaluate the eigenvalues of $\mathcal{A}+\mathcal{B}$.

Solving the eigenvalue problem $(\mathcal{A}+\mathcal{B})\begin{pmatrix}0\\ \phi\end{pmatrix}=\lambda\begin{pmatrix}0\\ \phi\end{pmatrix}$, we obtain

$$\begin{cases}\frac{\partial\phi(a)}{\partial a}=-(\lambda+\mu(a))\phi(a)+\mathcal{L}\phi(a),\\ \phi(0)=\int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\phi(a)(y)dyda,\end{cases} \tag{3.5}$$

which has a solution by letting $\mathcal{F}(0,\tau,a)=\mathcal{F}(\tau,a)$:

$$\phi(a)=e^{-\lambda a}\mathcal{F}(0,a)\phi(0).$$

Define an operator $\mathcal{G}_\lambda : Z \to Z$ for $\lambda\in\mathbb{R}$ by

$$\mathcal{G}_\lambda\phi=\int_0^{a^+}\int_\Omega \beta(a,\cdot-y)e^{-\lambda a}\left(\mathcal{F}(0,a)\phi\right)(y)dyda, \quad \forall\phi\in Z. \tag{3.6}$$

Let $B(E)$ be the set of bounded linear operators from $E$ to $E$. $T\in B(E)$ is said to be *positive* if $T(E_+)\subset E_+$. $T\in B(E)$ is said to be *strongly positive* if $\langle f, T\psi\rangle>0$ for every pair $\psi\in E_+\setminus\{0\}$, $f\in E_+^*\setminus\{0\}$. For $T,S\in B(E)$, we say $T\geq S$ if $(T-S)(E_+)\subset E_+$. A positive operator $T\in B(E)$ is said to be *nonsupporting* if for every pair $\psi\in E_+\setminus\{0\}$ and $f\in E_+^*\setminus\{0\}$, there exists a positive integer $p=p(\psi,f)$ such that $\langle f, T^n\psi\rangle>0$ for all $n\geq p$. $r(T)$ denotes the *spectral radius* of $T\in B(E)$, $\sigma(T)$ denotes the *spectrum* of $T$, and $\sigma_P(T)$ denotes the *point spectrum* of $T$.

**Lemma 3.3.** *Under Assumption 3.1, the operator $\mathcal{G}_\lambda$ is compact and nonsupporting.*

**Proof.** We can see by Assumption 3.1-(i) that for $\phi\in K$ which is a uniformly bounded subset in $Z$,

$$\left\|\mathcal{G}_\lambda\phi(\cdot+h)-\mathcal{G}_\lambda(\phi)(\cdot)\right\|_Z$$

$$\leq \left[ \int_{\Omega} \left( \int_0^{a^+} \int_{\Omega} \left| \beta(a, x+h-y) - \beta(x-y) \right| e^{-\lambda a} \left( \mathcal{F}(0,a)\phi \right)(y) dy da \right)^2 dx \right]^{\frac{1}{2}}$$

$$\leq \int_0^{a^+} \int_{\Omega} \left[ \int_{\Omega} \left| \beta(a, x+h-y) - \beta(a, x-y) \right|^2 dx \right]^{\frac{1}{2}} e^{-\lambda a} \left( \mathcal{F}(0,a)\phi \right)(y) dy da$$

$$\to 0 \quad \text{as} \quad h \to 0, \tag{3.7}$$

where we used Minkowski inequality. Thus $\mathcal{G}_\lambda : Z \to Z$ is compact by Kolmogorov compactness theorem. Now we would like to show that $\mathcal{G}_\lambda$ is nonsupporting.

Motivated by Inaba [26], for $\lambda \in \mathbb{R}$, define a positive functional $O_\lambda$ by

$$\langle O_\lambda, \psi \rangle := \int_0^{a^+} \int_{\Omega} \underline{\beta}(a) e^{-\lambda a} \left( \mathcal{F}(0,a)\psi \right)(y) dy da,$$

where $\psi \in L^2_+(\Omega)$. From Assumption 3.1-(ii), $O_\lambda$ is a strictly positive functional and we have

$$\mathcal{G}_\lambda \psi \geq \langle O_\lambda, \psi \rangle e, \qquad \lim_{\lambda \to -\infty} \langle O_\lambda, e \rangle = +\infty, \tag{3.8}$$

where $e \equiv 1$ is a quasi-interior point in $L^2_+(\Omega)$. Moreover, for any integer $n$, we have

$$\mathcal{G}_\lambda^{n+1} \psi \geq \langle O_\lambda, \psi \rangle \langle O_\lambda, e \rangle^n e.$$

Then we obtain $\langle O, \mathcal{G}_\lambda^n \psi \rangle > 0$ for every pair $\psi \in L^2_+ \setminus \{0\}$ and $O \in (L^2_+)^* \setminus \{0\}$; that is, $\mathcal{G}_\lambda$ is a nonsupporting operator. □

**Lemma 3.4.** *Let $r(\mathcal{G}_\lambda)$ denote the spectral radius of $\mathcal{G}_\lambda$. Then the mapping $\lambda \to r(\mathcal{G}_\lambda) : \mathbb{R} \to (\infty, 0)$ is continuous and strictly decreasing from $+\infty$ to 0.*

**Proof.** Since $\mathcal{G}_\lambda$ is compact, $r(\mathcal{G}_\lambda)$ is continuous with respect to $\lambda$. Also, since $\mathcal{G}_\lambda$ is nonsupporting, by the theory of positive operators (Sawashima [38] and Marek [34]), we know that $r(\mathcal{G}_\lambda)$ is a simple eigenvalue of $\mathcal{G}_\lambda$ corresponding to a positive eigenfunction $\Phi_0$ in $Z$ ($\Phi_0$ is in fact a quasi-interior point of $Z$) and a simple pole of the resolvent of $\mathcal{G}_\lambda$. Moreover, $r(\mathcal{G}_\lambda)$ is strictly decreasing with respect to $\lambda$. Furthermore, it is easy to see that $\lim_{\lambda \to \infty} r(\mathcal{G}_\lambda) = 0$. For $\lambda \in \mathbb{R}$, let $f_\lambda$ be a positive eigenfunctional corresponding to the eigenvalue $r(\mathcal{G}_\lambda)$ of the positive operator $\mathcal{G}_\lambda$. Then we have

$$\langle f_\lambda, \mathcal{G}_\lambda e \rangle = r(\mathcal{G}_\lambda) \langle f_\lambda, e \rangle \geq \langle O_\lambda, e \rangle \langle f_\lambda, e \rangle.$$

Since $f_\lambda$ is strictly positive, we obtain $r(\mathcal{G}_\lambda) \geq \langle O_\lambda, e \rangle$. It follows from (3.8) that $\lim_{\lambda \to -\infty} r(\mathcal{G}_\lambda) = +\infty$. □

Next we give a lemma to relate the eigenvalues of $\mathcal{G}_\lambda$ to those of $\mathcal{A}+\mathcal{B}$ inspired by Walker [42, Lemma 3.1].

**Lemma 3.5.** *The following statements hold:*

(i) *Let $\lambda \in \mathbb{C}$ and let $m \in \mathbb{N} \setminus \{0\}$. Then $\lambda \in \sigma_p(\mathcal{A}+\mathcal{B})$ with geometric multiplicity $m$ if and only if $1 \in \sigma_p(\mathcal{G}_\lambda)$ with geometric multiplicity $m$;*
(ii) $1 \in \rho(\mathcal{G}_\lambda) \Rightarrow \lambda \in \rho(\mathcal{A}+\mathcal{B})$.

**Proof.** (i) Let $\lambda \in \mathbb{C}$. Suppose that $\lambda \in \sigma_p(\mathcal{A}+\mathcal{B})$ has geometric multiplicity $m$ so that there are $m$ linearly independent elements

$$\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \phi_m \end{pmatrix} \in D(\mathcal{A}+\mathcal{B}) \text{ with } (\lambda I - \mathcal{A} - \mathcal{B}) \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = 0 \text{ for } j = 1, \dots, m.$$

Then by solving the eigenvalue problem as above, we get

$$\phi_j(a) = e^{-\lambda a}\mathcal{F}(0,a)\phi_j(0) \quad \text{with} \quad \phi_j(0) = \mathcal{G}_\lambda \phi_j(0).$$

Hence, $\phi_1(0), \dots, \phi_m(0)$ are necessarily linearly independent eigenvectors of $\mathcal{G}_\lambda$ corresponding to the eigenvalue 1. Now suppose that $1 \in \sigma_p(\mathcal{G}_\lambda)$ has geometric multiplicity $m$ so that there are linearly independent $\psi_1, \dots, \psi_m \in Z$ with $\mathcal{G}_\lambda \psi_j = \psi_j$ for $j = 1, \dots, m$. Put $\phi_j = e^{-\lambda a}\mathcal{F}(0,a)\psi_j \in E$ and note that for $j = 1, \dots, m$,

$$\partial_a \phi_j + \lambda \phi_j - (\mathcal{L} - \mu)\phi_j = 0, \quad \int_0^{a^+}\int_\Omega \beta(a, \cdot - y)\phi_j(a)(y)dyda = \mathcal{G}_\lambda \psi_j = \psi_j = \phi_j(0),$$

which is equivalent to

$$(\mathcal{A}+\mathcal{B})\begin{pmatrix} 0 \\ \phi_j \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} \in D(\mathcal{A}+\mathcal{B}).$$

Thus $\lambda \in \sigma_p(\mathcal{A}+\mathcal{B})$. If $\alpha_1, \dots, \alpha_m$ are any scalars, the unique solvability of the Cauchy problem

$$\partial_a \phi + \lambda \phi - (\mathcal{L} - \mu)\phi = 0, \quad \phi(0,x) = \sum_{j=1}^m \alpha_j \psi_j$$

ensures that $\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \phi_m \end{pmatrix}$ are linearly independent. Hence, (i) is desired.

(ii) Considering the resolvent equation $(\lambda I - \mathcal{A} - \mathcal{B})\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \eta \\ \psi \end{pmatrix} \in X$, we obtain

$$\begin{cases} \frac{\partial \phi(a)}{\partial a} = -(\lambda + \mu(a))\phi(a) + \mathcal{L}\phi(a) + \psi(a), \\ \phi(0) - \int_0^{a^+} \int_\Omega \beta(a, \cdot - y)\phi(a)(y)dyda = \eta. \end{cases} \tag{3.9}$$

Solving the equation, one has

$$\phi(a) = e^{-\lambda a}\mathcal{F}(0,a)\phi(0) + \int_0^a e^{-\lambda(a-\sigma)}\mathcal{F}(\sigma,a)\psi(\sigma)d\sigma, \tag{3.10}$$

and accordingly

$$\begin{aligned} &\phi(0) - \int_0^{a^+}\int_\Omega \beta(a, \cdot - y)e^{-\lambda a}\left(\mathcal{F}(0,a)\phi(0)\right)(y)dyda \\ &= \int_0^{a^+}\int_\Omega \beta(a, \cdot - y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}(\sigma,a)\psi(\sigma)\right)(y)d\sigma dyda + \eta, \end{aligned}$$

which is equivalent to

$$(I - \mathcal{G}_\lambda)\phi(0) = \int_0^{a^+}\int_\Omega \beta(a, \cdot - y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}(\sigma,a)\psi(\sigma)\right)(y)d\sigma dyda + \eta. \tag{3.11}$$

Thus if $1 \in \rho(\mathcal{G}_\lambda)$, then

$$\phi(0) = (I - \mathcal{G}_\lambda)^{-1}\left[\int_0^{a^+}\int_\Omega \beta(a, \cdot - y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}(\sigma,a)\psi(\sigma)\right)(y)d\sigma dyda + \eta\right], \tag{3.12}$$

which implies that

$$\begin{aligned} \phi(a) = {} & e^{-\lambda a}\mathcal{F}(0,a)(I - \mathcal{G}_\lambda)^{-1}\left[\int_0^{a^+}\int_\Omega \beta(a, \cdot - y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}(\sigma,a)\psi(\sigma)\right)(y)d\sigma dyda + \eta\right] \\ & + \int_0^a e^{-\lambda(a-\sigma)}\mathcal{F}(\sigma,a)\psi(\sigma)d\sigma. \end{aligned} \tag{3.13}$$

It follows that $\lambda \in \rho(\mathcal{A} + \mathcal{B})$ and the result is proved. $\square$

**Lemma 3.6.** *Let $\lambda_0 \in \mathbb{R}$ be such that $r(\mathcal{G}_{\lambda_0}) = 1$. Then $\lambda_0$ is a simple eigenvalue of $\mathcal{A}+\mathcal{B}$.*

**Proof.** We prove the lemma using the idea of Walker [42]. According to Lemma 3.3, there is a quasi-interior eigenvector $\Phi_0 \in Z$ of $\mathcal{G}_{\lambda_0}$ corresponding to the simple eigenvalue $r(\mathcal{G}_{\lambda_0}) = 1$. By Lemma 3.5, $\ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)$ is one-dimensional and spanned by $\phi := \begin{pmatrix} 0 \\ e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0 \end{pmatrix}$. It thus remains to show that $\ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)^2 \subset \ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)$. Since if it is true, $\lambda_0$ will be a simple pole of the resolvent of $\mathcal{A}+\mathcal{B}$ by definition and implies that $\lambda_0$ is a simple eigenvalue of $\mathcal{A}+\mathcal{B}$. Let $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in \ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)^2$ and set

$$\varphi := (\mathcal{A}+\mathcal{B}-\lambda_0 I)\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in \ker(\mathcal{A}+\mathcal{B}-\lambda_0 I).$$

Then $\varphi = \xi\phi$ for some $\xi \in \mathbb{R}$. Suppose $\xi \neq 0$, without loss of generality assume that $\xi > 0$. Let $\tau > 0$ be such that $\tau \begin{pmatrix} 0 \\ \Phi_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \psi(0) \end{pmatrix} \in \{0\} \times Z_+ \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$ and put $q := \tau\phi + \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(\mathcal{A}+\mathcal{B})$. Then $(\mathcal{A}+\mathcal{B}-\lambda_0 I)q = \varphi$ and it follows that

$$\begin{aligned} q(a) &= e^{-\lambda_0 a}\mathcal{F}(0,a)q(0) + \xi\int_0^a e^{-\lambda_0(a-\sigma)}\mathcal{F}(\sigma,a)e^{-\lambda_0\sigma}\mathcal{F}(0,\sigma)\Phi_0 d\sigma \\ &= e^{-\lambda_0 a}\mathcal{F}(0,a)q(0) + a\xi e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0 \end{aligned}$$

and

$$q(0) = \int_0^{a^+}\int_\Omega \beta(a,\cdot - y)q(a)(y)dyda.$$

Plugging the former into the second formula yields

$$(I - \mathcal{G}_{\lambda_0})q(0) = \xi\int_0^{a^+}\int_\Omega \beta(a,\cdot - y)ae^{-\lambda_0 a}\left(\mathcal{F}(0,a)\Phi_0\right)(y)dyda.$$

As $q(0)$ and the right hand side are both positive and nonzero, we derive from Corollary 12.4 of Daners and Koch-Medina [12] a contradiction to $r(\mathcal{G}_{\lambda_0}) = 1$. Consequently, $\xi = 0$ and the claim follows because now $\varphi = 0$. $\square$

### *3.3. Asymptotic behavior*

Now we give the following theorem on the asymptotic behavior of the linear system.

**Theorem 3.7.** *Let $\{\mathcal{S}(t)\}_{t\geq 0}$ denote the $C_0$-semigroup generated by the part of $\mathcal{A}+\mathcal{B}$ in $X_0$ and $L(E)$ be the space of linear operators in $E$. For the linear age-structured model (3.1) with nonlocal diffusion and nonlocal boundary condition, there exists a unique real value $\lambda_0$ (given by the real number $\lambda_0$ such that $r(\mathcal{G}_{\lambda_0})=1$) such that the semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ is exponentially decreasing with decay rate $\lambda_0<0$ or has asynchronous exponential growth with intrinsic growth constant $\lambda_0>0$; that is, $e^{-\lambda_0 t}\mathcal{S}(t)$ converges exponentially to some nonzero rank one projection in $L(E)$ as $t\to\infty$; namely,*

$$e^{-\lambda_0 t}\mathcal{S}(t)\to P_{\lambda_0} \quad \text{in} \quad L(E) \quad \text{as} \quad t\to\infty.$$

**Proof.** First we show that $\lambda_0=s(\mathcal{A}+\mathcal{B}):=\sup\{\mathrm{Re}\lambda:\lambda\in\sigma(\mathcal{A}+\mathcal{B})\}$, where $s(\mathcal{A}+\mathcal{B})$ denotes the spectral bound. Since $\lambda_0\in\mathbb{R}$ satisfies $r(\mathcal{G}_{\lambda_0})=1$, by Lemma 3.6 $\lambda_0$ is a simple eigenvalue of $\mathcal{A}+\mathcal{B}$. Now let $\hat{\lambda}_0:=s(\mathcal{A}+\mathcal{B})$. Then $\hat{\lambda}_0\geq\lambda_0$ and so $\hat{\lambda}_0>\omega_1(\mathcal{A}+\mathcal{B})=-\infty$. Thus $\sigma_0(\mathcal{A}+\mathcal{B})=\{\hat{\lambda}_0\}$ by Webb [45, Proposition 2.5], which states that the peripheral spectrum $\sigma_0$ of the generator of a strongly continuous positive semigroup in a Banach lattice consists exactly of the generator's spectral bound provided that the latter is strictly greater than the essential growth bound. Then $\hat{\lambda}_0\in\sigma(\mathcal{A}+\mathcal{B})$ and thus, by Lemma 3.5 and $\sigma_p(\mathcal{G}_\lambda)\setminus\{0\}=\sigma(\mathcal{G}_\lambda)\setminus\{0\}$ since $\mathcal{G}_\lambda$ is compact, $1\in\sigma_p(\mathcal{G}_{\hat{\lambda}_0})$, which implies that $1\leq r(\mathcal{G}_{\hat{\lambda}_0})$. However, due to $\hat{\lambda}_0\geq\lambda_0$ we have $r(\mathcal{G}_{\hat{\lambda}_0})\leq r(\mathcal{G}_{\lambda_0})=1$, hence $\hat{\lambda}_0=\lambda_0$. Moreover, $\sup\{\mathrm{Re}\lambda:\lambda\in\sigma_e(\mathcal{A}+\mathcal{B})\}\leq\omega_1(\mathcal{A}+\mathcal{B})=-\infty$ implies that $\lambda_0\in\sigma_p(\mathcal{A}+\mathcal{B})\setminus\sigma_e(\mathcal{A}+\mathcal{B})$. Thus $\lambda_0$ is a simple pole of $(\lambda I-\mathcal{A}-\mathcal{B})^{-1}$, where $\sigma_e(A)$ represents the essential spectrum of $\mathcal{A}+\mathcal{B}$. Therefore, by Magal and Ruan [33, Theorem 4.6.2], $\{\mathcal{S}(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant $\lambda_0$. Now if $\mathcal{R}_0:=r(\mathcal{G}_0)<1$, $\lambda_0<0$, it follows that the zero steady state is globally exponentially stable; if $\mathcal{R}_0=1$, $\lambda_0=0$, it follows that the solution $u$ to (3.1) with $\phi\in E$ converges towards an steady state; if $\mathcal{R}_0>1$, $\lambda_0>0$, it follows that the zero steady state is unstable and the solution $u$ to (3.1) with $\phi\in E$ is asymptotic to the stable age distribution $e^{\lambda_0 t}P_{\lambda_0}\phi$ with $\lambda_0>0$. $\square$

Next we derive a formula for the projection $P_{\lambda_0}:X_0\to\ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)$ inspired by Walker [42]. Observe that there is a quasi-interior element $\Phi_0\in Z$ such that

$$\ker(I-\mathcal{G}_{\lambda_0})=\mathrm{span}\{\Phi_0\} \text{ and } \ker(\mathcal{A}+\mathcal{B}-\lambda_0 I)=\mathrm{span}\left\{\begin{pmatrix}0\\ e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0\end{pmatrix}\right\}.$$

Let $\phi\in E$ be fixed and let $c(\phi)\in\mathbb{R}$ be such that

$$P_{\lambda_0}\phi=\begin{pmatrix}0\\ c(\phi)e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0\end{pmatrix}.$$

Note that we only need to find the second component of $P_{\lambda_0}\phi$ since the first one is always zero. Still denote the second component of $P_{\lambda_0}\phi$ by $P_{\lambda_0}\phi$ for convenience and recall that $\lambda_0$ is a simple pole of the resolvent $(\mathcal{A}+\mathcal{B}-\lambda I)^{-1}$. Denote

$$H_\lambda\phi:=\int_0^a e^{-\lambda(a-\sigma)}\mathcal{F}(\sigma,a)\phi(\sigma)d\sigma.$$

Then $H_\lambda \phi$ is holomorphic in $\lambda$ and it follows that from (3.13) and residue theorem that

$$P_{\lambda_0}\phi = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)e^{-\lambda_0 a}\mathcal{F}(0,a)(I-\mathcal{G}_\lambda)^{-1}G_\lambda\phi,$$

where

$$G_\lambda\phi = \int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\int_0^a e^{-\lambda(a-\sigma)}\big(\mathcal{F}(\sigma,a)\phi(\sigma)\big)(y)d\sigma dy da.$$

Let $w' \in Z'$ be a positive eigenfunctional of the dual operator $\mathcal{G}'_{\lambda_0}$ of $\mathcal{G}_{\lambda_0}$ corresponding to the eigenvalue $r(\mathcal{G}_{\lambda_0}) = 1$. Then for $f' \in E'$ defined by

$$\langle f',\psi\rangle := \langle w', \int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\psi(a)(y)dyda\rangle, \quad \psi\in E,$$

we have due to $\mathcal{G}'_{\lambda_0}w' = w'$ that

$$\begin{aligned}
c(\phi)\langle w',\Phi_0\rangle = \langle f', P_{\lambda_0}\phi\rangle &= \lim_{\lambda\to\lambda_0}\langle f',(\lambda-\lambda_0)e^{-\lambda a}\mathcal{F}(0,a)(I-\mathcal{G}_\lambda)^{-1}G_\lambda\phi\rangle\\
&= \lim_{\lambda\to\lambda_0}\langle w',(\lambda-\lambda_0)(I-(I-\mathcal{G}_\lambda))(I-\mathcal{G}_\lambda)^{-1}G_\lambda\phi\rangle\\
&= \lim_{\lambda\to\lambda_0}\langle w',(\lambda-\lambda_0)(I-\mathcal{G}_\lambda)^{-1}G_\lambda\phi\rangle.
\end{aligned}$$

Write

$$G_\lambda\phi = d(G_\lambda\phi)\Phi_0 \oplus (I-\mathcal{G}_{\lambda_0})g(G_\lambda\phi). \tag{3.14}$$

According to the decomposition $Z = \mathbb{R}\cdot\Phi_0 \oplus \mathrm{rg}(I-\mathcal{G}_{\lambda_0})$, it follows that

$$\lim_{\lambda\to\lambda_0}\langle w',(\lambda-\lambda_0)(I-\mathcal{G}_{\lambda_0})^{-1}G_\lambda\phi\rangle = d(G_{\lambda_0}\phi)\lim_{\lambda\to\lambda_0}\langle w',(\lambda-\lambda_0)(I-\mathcal{G}_\lambda)^{-1}\Phi_0\rangle$$

due to the continuity of $\mathcal{G}_\lambda$ in $\lambda$. But it follows from (3.14) that

$$\langle w', G_{\lambda_0}\phi\rangle = d(G_{\lambda_0}\phi)\langle w',\Phi_0\rangle$$

since $\mathcal{G}'_{\lambda_0}w' = w'$, whence $d(G_{\lambda_0}\phi) = \xi\langle w', G_{\lambda_0}\phi\rangle$ with $\xi^{-1} = \langle w',\Phi_0\rangle$. Similarly, decomposing

$$Y_\lambda := (\lambda-\lambda_0)(I-\mathcal{G}_\lambda)^{-1}\Phi_0,$$

we find

$$\lim_{\lambda\to\lambda_0}\langle w', Y_\lambda\rangle = \left(\lim_{\lambda\to\lambda_0} d(Y_\lambda)\right)\langle w',\Phi_0\rangle.$$

Based on these observations, we derive

$$c(\phi)\langle w', \Phi_0\rangle = C_0\langle w', G_{\lambda_0}\phi\rangle\langle w', \Phi_0\rangle$$

for some constant $C_0$. Consequently,

$$P_{\lambda_0}\phi = C_0\langle w', G_{\lambda_0}\phi\rangle e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0.$$

Since $P_{\lambda_0}$ is a projection, i.e. $P_{\lambda_0}^2 = P_{\lambda_0}$, the constant $C_0$ can be easily computed and we obtain the following result.

**Proposition 3.8.** *Under the assumptions of Theorem 3.7, the projection $P_{\lambda_0}$ is given by*

$$P_{\lambda_0}\phi = \frac{\langle w', G_{\lambda_0}\phi\rangle}{\langle w', \int_0^{a^+}\int_\Omega a\beta(a,\cdot-y)e^{-\lambda_0 a}\left(\mathcal{F}(0,a)\Phi_0\right)(y)dyda\rangle} e^{-\lambda_0 a}\mathcal{F}(0,a)\Phi_0 \tag{3.15}$$

*for $\phi \in E$, where*

$$G_{\lambda_0}\phi = \int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\int_0^a e^{-\lambda_0(a-\sigma)}\left(\mathcal{F}(\sigma,a)\phi(\sigma)\right)(y)d\sigma dyda$$

*and $w' \in Z'$ is a positive eigenfunctional of the dual operator $\mathcal{G}'_{\lambda_0}$ of $\mathcal{G}_{\lambda_0}$ corresponding to the eigenvalue $r(\mathcal{G}_{\lambda_0}) = 1$.*

**Remark 3.9.** Note that when $a^+ = \infty$, in order to make $\mathcal{G}_\lambda$ be well-defined, $\lambda$ will take values in the interval $(-\underline{\mu} - \zeta_0, \infty)$. Thus, we need to assume that there exists a $\gamma \in (-\underline{\mu} - \zeta_0, \infty)$ such that $r(\mathcal{G}_\gamma) > 1$ to guarantee the existence of $\lambda_0$ satisfying $r(\mathcal{G}_{\lambda_0}) = 1$. Such an additional assumption is motivated by Walker [42].

## 4. Semilinear equations

In the next two sections we consider semilinear and nonlinear equations derived from the classic nonlinear age-structured population models, where birth and death rates are dependent on the total population, see Webb [44], in which models with nonlinear death rate were referred to as *semilinear* and models with nonlinear death and birth rates were referred to as *nonlinear*. In fact, they are both semilinear in the PDE sense, but we keep using the notations in Webb [44] for consistence. Throughout these two sections, we assume that the maximum age is finite; i.e. $a^+ < \infty$. For $a^+ = \infty$, the results can be proved similarly under an additional assumption, see Remark 3.9. In the following, we use subscripts $S$ and $N$ to represent semilinear equations and nonlinear equations, respectively.

Consider the semilinear equation with nonlocal diffusion and nonlocal boundary condition:

$$\begin{cases} u_t(t,a,x)+u_a(t,a,x)=d(J*u-u)(t,a,x)-\mu(a,x,P)u(t,a,x), \\ \quad t>0, 0<a<a^+, x\in\Omega, \\ u(t,0,x)=\int_0^{a^+}\int_\Omega \beta(a,x-y)u(t,a,y)dyda, \quad t>0, x\in\Omega, \\ u(t,a,x)=0, \quad t>0, 0<a<a^+, x\notin\Omega, \\ u(0,a,x)=\phi(a,x), \quad 0<a<a^+, x\in\Omega, \\ P(t)=\int_0^{a^+}\int_\Omega u(t,a,x)dxda, \quad t>0. \end{cases} \tag{4.1}$$

**Assumption 4.1.**

(i) $\beta(a,x)$ satisfies Assumption 3.1-(i) and (ii);
(ii) $\mu=\mu(a,x,y):[0,a^+]\times\Omega\times\mathbb{R}^+\to\mathbb{R}^+$ is bounded a.e. with respect to $a$, $x$ and $y$, measurable, positive and

$$\int_0^a \overline{\mu}(\rho)d\rho<\infty \quad \text{for} \quad a<a^+ \quad \text{and} \quad \int_0^{a^+}\underline{\mu}(\rho)d\rho=\infty,$$

in which $\underline{\mu}(a)=\inf_{(a,x)\in\Omega\times\mathbb{R}}\mu(a,x,y)$ and $\overline{\mu}(a)=\sup_{(a,x)\in\Omega\times\mathbb{R}}\mu(a,x,y)$. Moreover $\mu$ is differentiable with respect to $y$ in $\mathbb{R}^+$ and denote $\mu_1(\cdot,\cdot,y):=\frac{\partial\mu(\cdot,\cdot,y)}{\partial y}$ and $\mu(\cdot,\cdot,y),\mu_1(\cdot,\cdot,y)$ as functions of $y$ belong to $C(\mathbb{R}^+,L^\infty((0,a^+)\times\Omega))$;
(iii) $\mu(a,x,y)\geq\mu(a,x,0)$ for all $(a,x)\in(0,a^+)\times\Omega$ and $y\in\mathbb{R}^+$.

In order to be consistent with previous notations, we shall hide the spatial variable in the following text to write $\psi(a)(y)=\psi(a,y)$. Suppose that $(0,\hat{\psi})$ is a steady state; i.e.,

$$\mathcal{A}\begin{pmatrix}0\\ \hat{\psi}\end{pmatrix}+\mathcal{B}\begin{pmatrix}0\\ \hat{\psi}\end{pmatrix}=0,$$

where

$$\mathcal{A}\begin{pmatrix}0\\ \psi\end{pmatrix}:=\begin{pmatrix}-\psi(0)\\ -\psi_a+\mathcal{L}\psi\end{pmatrix} \text{ and } \mathcal{B}\begin{pmatrix}0\\ \psi\end{pmatrix}:=\begin{pmatrix}F(\psi)\\ G(\psi)\end{pmatrix},$$

in which

$$F(\psi)=\int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\psi(a)(y)dyda, \tag{4.2}$$

$$G(\psi)(a)=-\mu(a,P\psi)\psi(a), \tag{4.3}$$

where $P\psi:=\int_0^{a^+}\int_\Omega\psi(a,x)dxda$. Now we view $P:E\to\mathbb{R}$ as an operator for the convenience of computation of Fréchet derivatives.

Under the above Assumption 4.1, one can easily obtain the global existence of integral solutions for (4.1). Thus, in what follows we mainly focus on the existence and stability of the nontrivial steady states.

### 4.1. Existence of nontrivial steady states

In this subsection we study the existence of the nontrivial steady state $\hat{\psi} \neq 0$. By the definition, $\hat{\psi}$ satisfies the following equations:

$$\begin{cases} \psi_a - \mathcal{L}\psi + \mu(a, P\psi)\psi = 0, \\ \psi(0) = F(\psi), \\ P\psi = \int_0^{a^+} \int_\Omega \psi(a)(x)dxda. \end{cases} \tag{4.4}$$

Solving the equations, we have

$$\hat{\psi}(a) = \mathcal{F}_{\hat{P}}(0, a)\hat{\psi}(0), \tag{4.5}$$

where $\mathcal{F}_P(0, a)$ is the evolution family in Lemma 3.2, where $\mu(a_0 + a)$ is changed into $\mu(a_0 + a, P)$ and $\hat{P} = P\hat{\psi}$ (the existence of $\mathcal{F}_P(0, a)$ is still guaranteed by Lemma 3.2). Plugging the solution $\hat{\psi}(a)$ into the boundary conditions, we obtain

$$\hat{\psi}(0) = \int_0^{a^+} \int_\Omega \beta(a, \cdot - y) \left( \mathcal{F}_{\hat{P}}(0, a)\hat{\psi}(0) \right)(y)dyda. \tag{4.6}$$

Define $\Gamma_S : \mathbb{R} \times Z \to \mathbb{R} \times Z$ by

$$\Gamma_S \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \begin{pmatrix} \int_0^{a^+} \int_\Omega \left( \mathcal{F}_{\hat{P}}(0, a)\phi \right)(x)dxda \\ \int_0^{a^+} \int_\Omega \beta(a, \cdot - y) \left( \mathcal{F}_{\hat{P}}(0, a)\phi \right)(y)dyda \end{pmatrix}, \quad \forall \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R} \times Z. \tag{4.7}$$

Denote $Y_+ := \mathbb{R}_+ \times Z_+$, i.e. the positive cone of $\mathbb{R} \times Z$. Now the existence of nontrivial steady states is equivalent to the existence of nontrivial fixed points of map $\Gamma_S$. Note that $\Gamma_S$ is a nonlinear operator, so we cannot use the theory for the linear case as above to deal with the problem. However, we still have a fixed point theorem for nonlinear operators which can be regarded as a special case of the Krasnoselskii's theorem (see Krasnoselskii [31, Theorem 4.11]).

The Fréchet derivative of $\Gamma_S$ at $\begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is given by, from Assumption 4.1-(ii),

$$T_S \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \begin{pmatrix} \int_0^{a^+} \int_\Omega (\mathcal{F}_0(0, a)\phi)(x)dxda \\ \int_0^{a^+} \int_\Omega \beta(a, \cdot - y)(\mathcal{F}_0(0, a)\phi)(y)dyda \end{pmatrix}, \quad \forall \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R} \times Z. \tag{4.8}$$

Note that $T_s$ is obtained by linearalizing (4.4), see the derivative of $\mu$ in (4.9). Moreover, it is a positive linear majorant of $\Gamma_S$ (that is, $T_S$ is a linear operator such that $\Gamma_S \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \le T_S \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix}$ for any $\begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R}_+ \times Z_+$) since $\mathcal{F}_{\hat{P}}(0,a) \leq \mathcal{F}_0(0,a)$ by Assumption 4.1-(iii). We shall show that the spectral radius $r(T_S)$ of operator $T_S$ plays the role of a threshold value for the existence of nontrivial fixed points of operator $\Gamma_S$; that is, if $r(T_S) > 1$, then $\Gamma_S$ has a positive nontrivial fixed point.

**Proposition 4.2.** *Let $\Gamma_S$ and $T_S$ defined by* (4.7) *and* (4.8), *respectively.*

(i) *If $r(T_S) \leq 1$, then $\Gamma_S$ only has the trivial fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $Y_+$;*

(ii) *If $r(T_S) > 1$, then $\Gamma_S$ has at least one nontrivial fixed point $\begin{pmatrix} \hat{P} \\ \hat{\psi} \end{pmatrix}$ in $Y_+ \setminus \{0\}$.*

**Proof.** It is obvious that $\Gamma_S$ is bounded and $\Gamma_S \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Moreover,

$$\int_0^{a^+} \int_\Omega \left(\mathcal{F}_{\hat{P}}(0,a)\phi\right)(x)dxda < \infty, \quad \forall \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R} \times Z,$$

which implies that $\Gamma_S^1$ is compact in $\mathbb{R}$, where

$$\Gamma_S^1 \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \int_0^{a^+} \int_\Omega \left(\mathcal{F}_{\hat{P}}(0,a)\phi\right)(x)dxda$$

and

$$\Gamma_S^2 \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \int_0^{a^+} \int_\Omega \beta(a, \cdot - y) \left(\mathcal{F}_{\hat{P}}(0,a)\phi\right)(y)dyda.$$

One can use a similar proof as in Lemma 3.3 to conclude that $\Gamma_S^2$ is also compact. Thus $\Gamma_S$ is compact. For now conditions (i) and (ii) in Theorem 2.6 are satisfied.

By the same argument as above, we can show that $T_S$ is compact and nonsupporting by Lemma 3.3 and $T_S^1(\hat{P},\phi)^T := \int_0^{a^+} \int_\Omega \left(\mathcal{F}_0(0,a)\phi\right)(x)dxda > 0$ for $\phi \in Z_+ \setminus \{0\}$, where $T_S^1$ represents the first component of $T_S$ as above for $\Gamma_S$ and $^T$ represents the transpose.

First we prove (i). Since $\Gamma_S$ is a positive operator from the positive cone $Y_+$ into itself and $T_S$ is a positive linear majorant of $\Gamma_S$, we can apply Theorem 2.7 to conclude that $\Gamma_S$ has no nontrivial fixed point in $Y_+$ provided $r(T_S) \leq 1$.

Let us prove (ii). We apply the theory of nonsupporting operators (see Inaba [27] or Marek [34]) to prove that $r(T_S) > 1$ is an eigenvalue of operator $T_S$ with a corresponding positive nonzero eigenvector and $T_S$ does not have any eigenvector associated with eigenvalue 1. Hence, conditions (iii) and (iv) of Theorem 2.6 follow and consequently, $\Gamma_S$ has at least one nontrivial fixed point in $Y_+$. $\square$

The existence of a nontrivial fixed point of $\Gamma_S$ implies that the existence of the nontrivial steady state solution $\hat{\psi} \in D(A) \setminus \{0\}$ of system (4.1). In conclusion, from Proposition 4.2, the following theorem can be obtained as one the main results of this paper.

**Proposition 4.3.** *Let $T_S$ be defined in* (4.8).

(i) *If $r(T_S) \le 1$, then system* (4.1) *only has the trivial steady state* $0$ *in $D(A)$;*
(ii) *If $r(T_S) > 1$, then system* (4.1) *has at least one nontrivial steady state $\hat{\psi}$ in $D(A) \setminus \{0\}$;*

*4.2. Stability*

Let $\hat{\psi}$ be the positive steady state obtained in the previous subsection. We can see that

$$(G'(\hat{\psi})\psi)(a) = -\mu_1(a, P\hat{\psi})P\psi\hat{\psi}(a) - \mu(a, P\hat{\psi})\psi(a). \tag{4.9}$$

Now define

$$\mathcal{X}_1 \begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} 0 \\ -\mu(a, P\hat{\psi})\psi \end{pmatrix} \text{ and } \mathcal{X}_2 \begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} F(\psi) \\ C(\psi) \end{pmatrix},$$

where $C(\psi) := -\mu_1(\cdot, \hat{P})P\psi\hat{\psi}$. Observe that $C : E \to E$ is a compact operator, thus $\mathcal{X}_2 : X \to X$ is also a compact operator. By the method of characteristic lines, we see that $\mathcal{A} + \mathcal{X}_1$ generates a nilpotent semigroup and its perturbed semigroup by the compact operator $\mathcal{X}_2$ is eventually compact. Hence,

$$\omega_1(\mathcal{A} + \mathcal{B}'[\hat{\psi}]) = \omega_1(\mathcal{A} + \mathcal{X}_1 + \mathcal{X}_2) = \omega_1(\mathcal{A} + \mathcal{X}_1) = -\infty.$$

It follows that the stability of $\hat{\psi}$ is determined by the eigenvalues of $\mathcal{A} + \mathcal{B}'[\hat{\psi}]$. Accordingly, let

$$\lambda \in \mathbb{C} \text{ and } \hat{B}_S \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix} \text{ for } \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(\mathcal{A}) \text{ and } \psi \neq 0,$$

where $\hat{B}_S := \mathcal{A} + \mathcal{B}'[\hat{\psi}]$. From the definition of $\hat{B}_S$, we obtain

$$\begin{cases} \psi_a + \lambda\psi - \mathcal{L}\psi + \mu(a, \hat{P})\psi + \mu_1(a, \hat{P})P\psi\hat{\psi} = 0, \\ \psi(0) = F(\psi), \\ P\psi = \int_0^{a^+} \int_\Omega \psi(a)(x)dxda, \end{cases} \tag{4.10}$$

where $\hat{P} = P\hat{\psi}$. Solving the problem, we get

$$\psi(a) = e^{-\lambda a}\mathcal{F}_{\hat{P}}(0,a)\psi(0) - \int_0^a e^{-\lambda(a-\sigma)}\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})P\psi\hat{\psi}(\sigma)d\sigma. \tag{4.11}$$

First we find $P\psi$ in terms of $\psi(0,x)$. By the definition of $P\psi$, we have

$$\begin{aligned} P\psi = &\int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(x)dxda \\ &- P\psi \int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dxda, \end{aligned}$$

which implies that

$$\begin{aligned} &P\psi\left(1+\int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dxda\right) \\ &= \int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(x)dxda. \end{aligned} \tag{4.12}$$

Now we define a functional $B_\lambda : Z \to \mathbb{R}$ for all $\lambda \in \mathbb{C}$ as follows:

$$B_\lambda(\psi) = \int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi\right)(x)dxda. \tag{4.13}$$

Define

$$A(\lambda) = \int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dxda. \tag{4.14}$$

Note that in order for $A(\lambda)$ to be nonnegative, we require that $\mu_1(a,\hat{P}) \geq 0$ in $Z$ for all $a \in [0,a^+]$. We would like to mention that such an assumption on the sign of $\mu_1(a,\hat{P})$ is not strict, see the Examples 4.11-4.13 in Webb [44] when $\mu = \mu(\hat{P})$. It follows that $P\psi = (1+A(\lambda))^{-1}B_\lambda(\psi(0))$. Now plugging (4.11) into the boundary condition, we obtain

$$\psi(0) = \int_0^{a^+}\int_\Omega \beta(a,\cdot - y)e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(y)dyda$$

$$-P\psi \int_0^{a^+}\int_\Omega \beta(a,\cdot-y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(y)d\sigma dyda. \quad (4.15)$$

Now define $\Gamma_S(\lambda) : Z \to Z$ and $K_S(\lambda) : \Omega \to \mathbb{R}$ for all $\lambda \in \mathbb{C}$ respectively, as follows

$$\Gamma_S(\lambda)\psi = \int_0^{a^+}\int_\Omega \beta(a,\cdot-y)e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi\right)(y)dyda,$$

$$K_S(\lambda)(x) = \int_0^{a^+}\int_\Omega \beta(a,x-y)\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(y)d\sigma dyda. \quad (4.16)$$

Then (4.15) becomes

$$\psi(0) = \Gamma_S(\lambda)\psi(0) - (1 + A(\lambda))^{-1}B_\lambda(\psi(0))K_S(\lambda). \quad (4.17)$$

Next, define $M_S(\lambda) := (1 + A(\lambda))^{-1}K_S(\lambda)$. It follows that

$$\psi(0) = (\Gamma_S(\lambda) - M_S(\lambda)B_\lambda)\psi(0).$$

Denote $\Theta_S(\lambda) := \Gamma_S(\lambda) - M_S(\lambda)B_\lambda$. Since $M_S(\lambda)B_\lambda$ is compact under the Assumption 3.1-(i), $\Theta_S(\lambda) : Z \to Z$ is also compact for all $\lambda \in \mathbb{C}$.

In addition, under the Assumption 3.1-(ii), we can show that $\Gamma_S(\lambda)$ and $M_S(\lambda)B_\lambda$ are nonsupporting, then $\Theta_S(\lambda)$ is also nonsupporting for all $\lambda \in \mathbb{R}$. Thus we have the following results (see also Kang et al. [28]),

**Proposition 4.4.** *Assuming $\mu_1(a, \hat{P}) \geq 0$, where $\hat{P} = \int_0^{a^+}\int_\Omega \hat{\psi}(a)(x)dxda$, we have the following statements*

(i) $\Sigma_S := \{\lambda \in \mathbb{C} : 1 \in \sigma(\Theta_S(\lambda))\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(\Theta_S(\lambda))\}$, *where $\sigma(A)$ and $\sigma_P(A)$ denote the spectrum and point spectrum of the operator $A$, respectively;*

(ii) *There exists a unique real number $\lambda_S \in \Sigma_S$ such that $r(\Theta_S(\lambda_S)) = 1$ and $\lambda_S > 0$ if $r(\Theta_S(0)) > 1$; $\lambda_S = 0$ if $r(\Theta_S(0)) = 1$; and $\lambda_S < 0$ if $r(\Theta_S(0)) < 1$;*

(iii) $\lambda_S > \sup\{\mathrm{Re}\lambda : \lambda \in \Sigma_S \setminus \{\lambda_S\}\}$;

(iv) *$\lambda_S$ is the dominant eigenvalue of $\hat{B}_S$, i.e. $\lambda_S$ is greater than all real parts of the eigenvalues of $\hat{B}_S$. Moreover, it is a simple eigenvalue of $\hat{B}_S$;*

(v) $\{\lambda \in \mathbb{C} : 1 \in \rho(\Theta_S(\lambda))\} \Rightarrow \{\lambda \in \mathbb{C} : \lambda \in \rho(\hat{B}_S)\}$, *where $\rho(A)$ is the resolvent set of $A$;*

(vi) $\lambda_S = s(\hat{B}_S) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\hat{B}_S)\}$.

**Proof.** Note that $\Theta_S(\lambda)$ is strictly decreasing with respect to $\lambda$ in the sense of positive operators, thus (ii) is true by the arguments in Section 2. We only show (iii). And (iii) implies the first part of (iv). The rest are based on the results in Section 2, so we omit them. For any $\lambda \in \Sigma_S$, there is an eigenfunction $\psi_\lambda$ such that $\Theta_S(\lambda)\psi_\lambda = \psi_\lambda$. Then we have $|\psi_\lambda| = |\Theta_S(\lambda)\psi_\lambda| \leq \Theta_S(\mathrm{Re}\lambda)|\psi_\lambda|$. Let $f_{\mathrm{Re}\lambda}$ be the positive eigenfunctional corresponding to the eigenvalue $r(\Theta_S(\mathrm{Re}\lambda))$ of $\Theta_S(\mathrm{Re}\lambda)$. We obtain that

$$\langle f_{\mathrm{Re}\lambda}, \Theta_S(\mathrm{Re}\lambda)|\psi_\lambda|\rangle = r(\Theta_S(\mathrm{Re}\lambda))\langle f_{\mathrm{Re}\lambda}, |\psi_\lambda|\rangle \geq \langle f_{\mathrm{Re}\lambda}, |\psi_\lambda|\rangle.$$

Hence, $r(\Theta_S(\mathrm{Re}\lambda)) \geq 1$ and $\mathrm{Re}\lambda \leq \lambda_S$ since $r(\Theta_S(\lambda))$ is strictly decreasing for $\lambda \in \mathbb{R}$ and $r(\Theta_S(\lambda_S)) = 1$. If $\mathrm{Re}\lambda = \lambda_S$, then $\Theta_S(\lambda_S)|\psi_\lambda| = |\psi_\lambda|$. In fact, if $\Theta_S(\lambda_S)|\psi_\lambda| > |\psi_\lambda|$, taking duality pairing with the eigenfunctional $f_{\lambda_S}$ corresponding to the eigenvalue $r(\Theta_S(\lambda_S)) = 1$ on both sides yields $\langle f_{\lambda_S}, \Theta_S(\lambda_S)|\psi_\lambda|\rangle = \langle f_{\lambda_S}, |\psi_\lambda|\rangle > \langle f_{\lambda_S}, |\psi_\lambda|\rangle$, which is a contradiction. Then we can write that $|\psi_\lambda| = c\psi_S$, where $\psi_S$ is the eigenfunction corresponding to the eigenvalue $r(\Theta_S(\lambda_S)) = 1$. Without loss of generality, we assume that $c = 1$ and write $\psi_\lambda(x) = \psi_S(x)e^{i\alpha(x)}$ for some real function $\alpha(x)$. Substituting this relation into

$$\Theta_S(\lambda_S)\psi_S = \psi_S = |\psi_\lambda| = |\Theta_S(\lambda)\psi_\lambda|,$$

we have

$$\begin{aligned} &\int_0^{a^+}\int_\Omega \left[\beta(a, x-y) - 1\right] e^{-\lambda_S a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi_S\right)(y)dyda \\ &= \left|\int_0^{a^+}\int_\Omega \left[\beta(a, x-y) - 1\right] e^{-(\lambda_S+\mathrm{Im}\lambda)a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi_S\right)(y)e^{i\alpha(y)}dyda\right| \end{aligned}$$

and

$$\frac{K_{\lambda_S}(x)}{1 + A(\lambda_S)} = \frac{|K_{\lambda_S+\mathrm{Im}\lambda}(x)|}{|1 + A(\lambda_S + \mathrm{Im}\lambda)|}.$$

From Heijmans [22, Lemma 6.12], we obtain that $-\mathrm{Im}\lambda a + \alpha(y) = \theta$ for some constant $\theta$. By $\Theta_S(\lambda)\psi_\lambda = \psi_\lambda$, we have $e^{i\theta}\Theta_S(\lambda_S)\psi_S = \psi_S e^{i\alpha}$, so $\theta = \alpha(x)$, which implies that $\mathrm{Im}\lambda = 0$. Then, there is no element $\lambda \in \Gamma$ such that $\mathrm{Re}\lambda = \lambda_S$ and $\lambda \neq \lambda_S$. Therefore, the result (iii) is proved. □

In summary, we have the following theorem.

**Theorem 4.5.** *Assume $\mu_1(a, \hat{P}) \geq 0$, where $\hat{P} = \int_0^{a^+}\int_\Omega \hat{\psi}(a)(x)dxda$, then the steady state $\hat{\psi} \neq 0$ of system (4.1) is locally exponentially asymptotically stable if $r(\Theta_S(0)) < 1$ and unstable if $r(\Theta_S(0)) > 1$.*

**Remark 4.6.** Note that Theorem 4.5 applies to any steady state, not only the steady state $\hat{\psi}$ obtained in the previous subsection. However, when $\hat{\psi}(a) = \mathcal{F}_{\hat{P}}(0,a)\hat{\psi}(0)$ is constructed as in Section 4.1, we can further simplify the operators $A(\lambda)$ and $K_S(\lambda)$. In order to obtain a more explicit operator, let us assume that $\mu(a, x, P) = \mu(P)$ as in [44]. By some computations, we have

$$A(\lambda) = \frac{\mu_1(\hat{P})}{\lambda}\int_0^{a^+}\int_\Omega (1 - e^{-\lambda a})\hat{\psi}(a)(x)dxda$$

and

$$K_S(\lambda)(x) = \frac{\mu_1(\hat{P})}{\lambda} \int_0^{a^+} \int_\Omega \beta(a, x-y)(1-e^{-\lambda a})\hat{\psi}(a)(y)dyda,$$

which not only present explicit expressions of $A(\lambda)$ and $K_S(\lambda)$ but also indicate their dependence on $\mu_1(\hat{P})$.

## 5. Nonlinear equations

Finally, we consider the following nonlinear equation with nonlocal diffusion and nonlocal boundary condition:

$$\begin{cases} u_t(t,a,x) + u_a(t,a,x) = d(J * u - u)(t,a,x) - \mu(a,x,P)u(t,a,x), \\ \quad t > 0, 0 < a < a^+, x \in \Omega, \\ u(t,0,x) = \int_0^{a^+} \int_\Omega \beta(a, x-y, P)u(t,a,y)dyda, \quad t > 0, x \in \Omega, \\ u(t,a,x) = 0, \quad t > 0, 0 < a < a^+, x \notin \Omega, \\ u(0,a,x) = \phi(a,x), \quad 0 < a < a^+, x \in \Omega, \\ P(t) = \int_0^{a^+} \int_\Omega u(t,a,x)dxda, \quad t > 0. \end{cases} \tag{5.1}$$

First we make the following assumptions.

**Assumption 5.1.**

(i) $\mu$ satisfies Assumption 4.1-(ii) and (iii);
(ii) $\beta = \beta(a, x, z) : [0, a^+] \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is bounded a.e. with respect to $a$, $x$ and $z$, measurable, positive and

$$\lim_{\|h\| \to 0} \int_\Omega |\beta(a, x-y+h, z) - \beta(a, x-y, z)|^2 dx = 0$$

uniformly in $a \in (0, a^+)$, $y \in \Omega$ and $z \in \mathbb{R}^+$;
(iii) There exists a positive function $\underline{\beta} \in L^1(0, a^+)$ such that

$$\beta(a, x, z) \geq \underline{\beta}(a) > 0 \quad \text{for almost all} \quad (a, x, z) \in (0, a^+) \times \Omega \times \mathbb{R}^+;$$

(iv) $\beta$ is differentiable with respect to $z$ in $\mathbb{R}^+$ and denote $\beta_1(\cdot, \cdot, z) := \frac{\partial \beta(\cdot,\cdot,z)}{\partial z}$ and $\beta(\cdot, \cdot, z)$, $\beta_1(\cdot, \cdot, z)$ as functions of $z$ belong to $C(\mathbb{R}^+, L^\infty((0, a^+) \times \Omega))$;
(v) $\beta(a, x, z) \leq \beta(a, x, 0)$ for all $(a, x) \in (0, a^+) \times \Omega$ and $z \in \mathbb{R}^+$.

Suppose that $\begin{pmatrix} 0 \\ \hat{\psi} \end{pmatrix}$ is a steady state; i.e.,

$$\mathcal{A}\begin{pmatrix} 0 \\ \hat{\psi} \end{pmatrix} + \mathcal{B}\begin{pmatrix} 0 \\ \hat{\psi} \end{pmatrix} = 0,$$

where

$$\mathcal{A}\begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} -\psi(0) \\ -\psi_a + \mathcal{L}\psi \end{pmatrix} \text{ and } \mathcal{B}\begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} F(\psi) \\ G(\psi) \end{pmatrix},$$

in which

$$F(\psi) = \int_0^{a^+} \int_\Omega \beta(a, \cdot - y, P\psi)\psi(a)(y)dyda, \tag{5.2}$$

$$G(\psi)(a) = -\mu(a, P\psi)\psi(a), \tag{5.3}$$

where $P\psi := \int_0^{a^+} \int_\Omega \psi(a, x)dxda$.

Similarly we can obtain the global existence of integral solutions of (5.1) under Assumption 5.1. In the following we focus on the existence and stability of nontrivial equilibria of (5.1).

### 5.1. Existence of nontrivial steady states

In this subsection we study the existence of a nontrivial steady state $\hat{\psi} \neq 0$. From the definition, $\hat{\psi}$ satisfies the following equations:

$$\begin{cases} \psi_a - \mathcal{L}\psi + \mu(a, P\psi)\psi = 0, \\ \psi(0) = F(\psi), \\ P\psi = \int_0^{a^+} \int_\Omega \psi(a)(x)dxda. \end{cases} \tag{5.4}$$

Solving the problem, we obtain

$$\hat{\psi}(a) = \mathcal{F}_{\hat{P}}(0, a)\hat{\psi}(0), \tag{5.5}$$

where $\mathcal{F}_P(0, a)$ is the evolution family given in Lemma 3.2, where $\mu(a_0 + a)$ is changed into $\mu(a_0 + a, P)$ and $\hat{P} = P\hat{\psi}$. Plugging the solution into the boundary condition, we have

$$\hat{\psi}(0) = \int_0^{a^+} \int_\Omega \beta(a, \cdot - y, \hat{P})\left(\mathcal{F}_{\hat{P}}(0, a)\hat{\psi}(0)\right)(y)dyda. \tag{5.6}$$

Define $\Gamma_N : \mathbb{R} \times Z \to \mathbb{R} \times Z$ by

$$\Gamma_N \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \begin{pmatrix} \int_0^{a^+} \int_\Omega \left( \mathcal{F}_{\hat{P}}(0,a)\phi \right)(y) dy da \\ \int_0^{a^+} \int_\Omega \beta(a, \cdot - y, \hat{P}) \left( \mathcal{F}_{\hat{P}}(0,a)\phi \right)(y) dy da \end{pmatrix}, \quad \forall \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R} \times Z. \quad (5.7)$$

We can compute the Fréchet derivative of $\Gamma_N$ at $\begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as follows

$$T_N \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} := \begin{pmatrix} \int_0^{a^+} \int_\Omega \left( \mathcal{F}_0(0,a)\phi \right)(x) dx da \\ \int_0^{a^+} \int_\Omega \beta(a, \cdot - y, 0) \left( \mathcal{F}_0(0,a)\phi \right)(y) dy da \end{pmatrix}, \quad \forall \begin{pmatrix} \hat{P} \\ \phi \end{pmatrix} \in \mathbb{R} \times Z. \quad (5.8)$$

Once again the existence of nontrivial steady states is equivalent to the existence of nontrivial fixed points of map $\Gamma_N$. Similar as in the semilinear case in the previous section, we have the following results.

**Proposition 5.2.** *Let $\Gamma_N$ and $T_N$ defined by (5.7) and (5.8), respectively.*

(i) *If $r(T_N) \le 1$, then $\Gamma_N$ has only the trivial fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $Y_+$;*

(ii) *If $r(T_N) > 1$, then $\Gamma_N$ has at least one nontrivial fixed point $\begin{pmatrix} \hat{P} \\ \hat{\psi} \end{pmatrix}$ in $Y_+ \setminus \{0\}$.*

**Proposition 5.3.** *Let $T_N$ be defined in (5.8).*

(i) *If $r(T_N) \le 1$, then system (5.1) only has the trivial steady state $0$ in $D(A)$;*
(ii) *If $r(T_N) > 1$, then system (5.1) has at least one nontrivial steady state $\hat{\psi}$ in $D(A) \setminus \{0\}$.*

*5.2. Stability*

To study the stability of the steady state $\hat{\psi}$, we have

$$(G'(\hat{\psi})\psi)(a) = -\mu_1(a, P\hat{\psi}) P\psi \hat{\psi}(a) - \mu(a, P\hat{\psi})\psi(a),$$

$$(F'(\hat{\psi})\psi) = P\psi \int_0^{a^+} \int_\Omega \beta_1(a, \cdot - y, \hat{P}) \hat{\psi}(a)(y) dy da + \int_0^{a^+} \int_\Omega \beta(a, \cdot - y, \hat{P}) \psi(a)(y) dy da.$$

Define

$$\mathcal{X}_1 \begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} 0 \\ -\mu(a, P\hat{\psi})\psi \end{pmatrix} \text{ and } \mathcal{X}_2 \begin{pmatrix} 0 \\ \psi \end{pmatrix} := \begin{pmatrix} F'(\hat{\psi})\psi \\ C(\psi) \end{pmatrix},$$

where $C(\psi) := -\mu_1(\cdot, \cdot, \hat{P}) P\psi \hat{\psi}$. Note that $C : E \to E$ is a compact operator and also $F'(\hat{\psi})\psi : E \to Z$ is a compact operator. Then $\mathcal{X}_2 : X \to X$ is also compact. By the method of characteristic lines, we know that $\mathcal{A} + \mathcal{X}_1$ generates a nilpotent semigroup and its perturbed semigroup by the compact operator $\mathcal{X}_2$ is eventually compact. Thus,

$$\omega_1(\mathcal{A}+\mathcal{B}'[\hat{\psi}])=\omega_1(\mathcal{A}+\mathcal{X}_1+\mathcal{X}_2)=\omega_1(\mathcal{A}+\mathcal{X}_1)=-\infty.$$

So the stability of $\hat{\psi}$ is determined by the eigenvalues of $\mathcal{A}+\mathcal{B}'[\hat{\psi}]$. Let $\lambda \in \mathbb{C}$ and let $\hat{B}_N \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ \psi \end{pmatrix}$ for $\begin{pmatrix} 0 \\ \psi \end{pmatrix} \in D(\mathcal{A})$ and $\psi \neq 0$, where $\hat{B}_N := \mathcal{A}+\mathcal{B}'[\hat{\psi}]$. By the definition of $\hat{B}_N$, we have

$$\begin{cases} \psi_a+\lambda\psi-\mathcal{L}\psi+\mu(a,\hat{P})\psi+\mu_1(a,\hat{P})P\psi\hat{\psi}=0, \\ \psi(0)=F'(\hat{\psi})\psi, \\ P\psi=\int_0^{a^+}\int_\Omega \psi(a)(x)dxda, \end{cases} \tag{5.9}$$

where $\hat{P}=P\hat{\psi}$. Solving the problem yields

$$\psi(a)=e^{-\lambda a}\mathcal{F}_{\hat{P}}(0,a)\psi(0)-P\psi\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma. \tag{5.10}$$

To find $P\psi$ in terms of $\psi(0)$, using the definition of $P\psi$, we obtain

$$\begin{aligned} P\psi &= \int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(x)dxda \\ &\quad - P\psi\int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dxda, \end{aligned}$$

which implies that

$$\begin{aligned} &P\psi\left(1+\int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dxda\right) \\ &=\int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(x)dxda. \end{aligned} \tag{5.11}$$

Again we define a functional $B_\lambda : Z \to \mathbb{R}$ for all $\lambda \in \mathbb{R}$ as follows:

$$B_\lambda\psi=\int_0^{a^+}\int_\Omega e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi\right)(x)dxda. \tag{5.12}$$

Once again, define

$$A(\lambda)=\int_0^{a^+}\int_\Omega\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(x)d\sigma dx da. \tag{5.13}$$

Similarly we require that $\mu_1(a,\hat{P})\geq 0$ as in Section 4.2. We have that $P\psi=(1+A(\lambda))^{-1}\times B_\lambda(\psi(0))$. Now plugging (5.10) into the boundary condition, we obtain

$$\begin{aligned}\psi(0)=&\int_0^{a^+}\int_\Omega \beta(a,\cdot-y,\hat{P})e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi(0)\right)(y)dyda\\ &-P\psi\int_0^{a^+}\int_\Omega \beta(a,\cdot-y,\hat{P})\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(y)d\sigma dyda\\ &+P\psi\int_0^{a^+}\int_\Omega \beta_1(a,\cdot-y,\hat{P})\hat{\psi}(a)(y)dyda. \end{aligned}\tag{5.14}$$

Define $\Gamma_N:Z\to Z$, $K_N:\Omega\to\mathbb{R}$ and $Q:\Omega\to\mathbb{R}$ by

$$\begin{aligned}(\Gamma_N(\lambda)\psi)&=\int_0^{a^+}\int_\Omega \beta(a,\cdot-y,\hat{P})e^{-\lambda a}\left(\mathcal{F}_{\hat{P}}(0,a)\psi\right)(y)dyda,\\ K_N(\lambda)(x)&=\int_0^{a^+}\int_\Omega \beta(a,x-y,\hat{P})\int_0^a e^{-\lambda(a-\sigma)}\left(\mathcal{F}_{\hat{P}}(\sigma,a)\mu_1(\sigma,\hat{P})\hat{\psi}(\sigma)\right)(y)d\sigma dyda,\\ Q(x)&=\int_0^{a^+}\int_\Omega \beta_1(a,x-y,\hat{P})\hat{\psi}(a)(y)dyda.\end{aligned}$$

Then (5.14) becomes

$$\psi(0)=\Gamma_N(\lambda)\psi(0)-(1+A(\lambda))^{-1}B_\lambda(\psi(0))\left(K_N(\lambda)-Q\right). \tag{5.15}$$

Next, let $M_N(\lambda):=(1+A(\lambda))^{-1}(K_N(\lambda)-Q)$. Then it follows that

$$\psi(0)=\left(\Gamma_N(\lambda)-M_N(\lambda)B_\lambda\right)\psi(0).$$

Denote $\Theta_N(\lambda):=\Gamma_N(\lambda)-M_N(\lambda)B_\lambda$. Since $M_N(\lambda)B_\lambda$ is compact under the Assumption 5.1, it follows that $\Theta_N(\lambda):Z\to Z$ is also compact for all $\lambda\in\mathbb{C}$.

Moreover, under the Assumption 5.1, we can see that $\Gamma_N(\lambda)$ and $M_N(\lambda)B_\lambda$ are nonsupporting, then $\Theta_N(\lambda)$ is also nonsupporting for all $\lambda\in\mathbb{R}$. Thus, we have the following conclusions based on the results in Section 2.

**Proposition 5.4.** *Assuming that $\mu_1(a, \hat{P}) \geq 0$, where $\hat{P} = \int_0^{a^+} \int_\Omega \hat{\psi}(a)(x)dxda$, we have the following statements*

(i) $\Sigma_N := \{\lambda \in \mathbb{C} : 1 \in \sigma(\Theta_N(\lambda))\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(\Theta_N(\lambda))\}$, *where $\sigma(A)$ and $\sigma_P(A)$ denote the spectrum and point spectrum of the operator $A$, respectively;*
(ii) *There exists a unique real number $\lambda_N \in \Sigma_N$ such that $r(\Theta_N(\lambda_N)) = 1$ and $\lambda_N > 0$ if $r(\Theta_N(0)) > 1$; $\lambda_N = 0$ if $r(\Theta_N(0)) = 1$; and $\lambda_N < 0$ if $r(\Theta_N(0)) < 1$;*
(iii) $\lambda_N > \sup\{\mathrm{Re}\lambda : \lambda \in \Sigma_N \setminus \{\lambda_N\}\}$;
(iv) *$\lambda_N$ is the dominant eigenvalue of $\hat{B}_N$; i.e., $\lambda_N$ is greater than all real parts of the eigenvalues of $\hat{B}_N$. Moreover, it is a simple eigenvalue of $\hat{B}_N$;*
(v) $\{\lambda \in \mathbb{C} : 1 \in \rho(\Theta_N(\lambda))\} \Rightarrow \{\lambda \in \mathbb{C} : \lambda \in \rho(\hat{B}_N)\}$, *where $\rho(A)$ denotes the resolvent set of $A$;*
(vi) $\lambda_N = s(\hat{B}_N) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\hat{B}_N)\}$.

**Theorem 5.5.** *Assume that $\mu_1(a, \hat{P}) \geq 0$, where $\hat{P} = \int_0^{a^+} \int_\Omega \hat{\psi}(a)(x)dxda$, then the steady state $\hat{\psi} \neq 0$ of problem (5.1) is locally exponentially asymptotically stable if $r(\Theta_N(0)) < 1$ and unstable if $r(\Theta_N(0)) > 1$.*

**Remark 5.6.** Again the conclusions in Theorem 5.5 apply to any steady state, not only the one $\hat{\psi}$ constructed in the previous subsection. However, when $\hat{\psi}(a) = \mathcal{F}_{\hat{P}}(0, a)\hat{\psi}(0)$ is constructed as in Section 5.1, we can further simplify the operators $A(\lambda)$ and $K_N(\lambda)$ as Remark 4.6.

## 6. Discussion

In this paper, we studied the linear, semilinear, and nonlinear age-structured population models with nonlocal diffusion and nonlocal boundary conditions via the integrated semigroup theory and non-densely defined operators. For the linear case, we analyzed the spectrum of the infinitesimal generator associated with the integrated semigroup and studied the asymptotic behavior by asynchronous exponential growth. We then considered the semilinear and nonlinear cases and established the existence and stability of nontrivial steady states.

Observe that we considered a nonlocal boundary condition in this paper, which looks complicated compared with the regular boundary condition of the form

$$u(t, 0, x) = \int_0^{a^+} \beta(a, x)u(t, a, x)da.$$

However, the nonlocal boundary condition problem has a good property, i.e., the compactness, under an appropriate condition. Since the semigroup or evolution family itself generated by nonlocal diffusion does not have regularity, the constructed operator $\mathcal{G}_\lambda$ does not have compactness under the regular boundary condition. So we do not have much tools to carry out the spectrum analysis, see Kang et al. [29]. On the contrary, one will obtain the compactness and thus spectrum and asymptotic behavior if the nonlocal diffusion is changed to Laplace diffusion, see [8,18,42,41,14,15]. Recently we (Kang et al. [28]) coupled a second physiological structure to the classical age-structured models and analyzed the spectrum and asymptotic behavior under some appropriate boundary conditions. Combining our previous results in [28] with that in this paper, we can see that no matter what extra structure is coupled into the classical age-structured models (age or size corresponding to the first order operator and spatial diffusion corresponding to the Laplace or nonlocal operator), one can always obtain some reasonable results as long as the boundary conditions satisfy certain suitable assumptions (for example equicontinuity).

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