

Stochastic Versions of the LaSalle Theorem*

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The main aim of this paper is to establish stochastic versions of the well-known LaSalle stability theorem. From these stochastic versions follow many classical results on stochastic stability. This shows clearly the power of our new results.

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1. INTRODUCTION

In 1892, Lyapunov introduced the concept of stability of a dynamic system and created a very powerful tool known as Lyapunov's second method in the study of stability. The Lyapunov method has been developed and applied by many authors during the past century. One of the important developments in this direction is the LaSalle theorem for locating limit sets of nonautonomous systems (cf. Hale and Lunel [3] or LaSalle [10]), from which follow many of the classical Lyapunov results on stability. On the other hand, since Itô introduced his stochastic calculus about 50 years ago, the theory of stochastic differential equations has been developed very quickly. In particular, the Lyapunov method has been developed to deal with stochastic stability by many authors, and here we only mention Arnold [1], Friedman [2], Has'minskii [4], Kushner [7], Kolmanovskii and Myshkis [8], Ladde and Lakshmikantham [9], Mohammed [15], and Mao [12–14]. The classical results by Has'minskii, Kushner, and others typically state that solutions of a stochastic differential equation converge in probability to some invariant set if the initial condition approaches this invariant set. However, so far there seems to be no stochastic version of the LaSalle theorem (i.e., Theorem 1 of LaSalle [10]) that locates limit sets of a system, and the main aim of this paper is to

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extend it from ordinary differential equations to stochastic differential equations. We shall show that many classical results on stochastic stability follow from our stochastic versions of the LaSalle theorem. This shows clearly the power of our new results.

2. NONAUTONOMOUS STOCHASTIC SYSTEMS

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and \mathcal{F}_0 contains all P -null sets). Let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in R^n . If A is a vector or matrix, its transpose is denoted by A^T . If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$.

In this section we consider a nonautonomous n -dimensional stochastic differential equation

$$dx(t) = f(x(t), t) dt + g(x(t), t) dB(t) \quad (2.1)$$

on $t \geq 0$ with initial value $x(0) = x_0 \in R^n$. As a standing condition, we shall impose a hypothesis:

(H1) Both $f: R^n \times R_+ \rightarrow R^n$ and $g: R^n \times R_+ \rightarrow R^{n \times m}$ are measurable functions. They satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \dots$, there is a $c_k > 0$ such that

$$|f(x, t) - f(y, t)| \vee |g(x, t) - g(y, t)| \leq c_k |x - y|$$

for all $t \geq 0$ and those $x, y \in R^n$ with $|x| \vee |y| \leq k$, and there is moreover a $c > 0$ such that

$$|f(x, t)| \vee |g(x, t)| \leq c(1 + |x|)$$

for all $(x, t) \in R^n \times R_+$.

It is known (cf. Arnold [1] or Friedman [2]) that under hypothesis (H1), Eq. (2.1) has a unique continuous solution on $t \geq 0$, which is denoted by $x(t; x_0)$ in this paper. Moreover, for every $p > 0$,

$$E\left[\sup_{0 \leq s \leq t} |x(s; x_0)|^p\right] < \infty \quad \text{on } t \geq 0.$$

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(x, t)$ on $R^n \times R_+$ which are continuously twice differentiable in x and

once differentiable in t . Define an operator L acting on $C^{2,1}(R^n \times R_+; R_+)$ functions by

$$LV(x, t) = V_t(x, t) + V_x(x, t) f(x, t) + \frac{1}{2} \text{trace}[g^T(x, t) V_{xx}(x, t) g(x, t)],$$

where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t}, \quad V_x(x, t) = \left(\frac{\partial V(x, t)}{\partial x_1}, \dots, \frac{\partial V(x, t)}{\partial x_n} \right),$$

$$V_{xx}(x, t) = \left(\frac{\partial^2 V(x, t)}{\partial x_i \partial x_j} \right)_{n \times n}.$$

Moreover, let \mathcal{K} denote the class of continuous (strictly) increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let \mathcal{K}_∞ denote the class of functions μ in \mathcal{K} with $\mu(r) \rightarrow \infty$ as $r \rightarrow \infty$. Functions in \mathcal{K} and \mathcal{K}_∞ are called class \mathcal{K} and \mathcal{K}_∞ functions, respectively. If $\mu \in \mathcal{K}$, its inverse function is denoted by μ^{-1} . We also denote by $L^1(R_+; R_+)$ the family of all functions $\gamma: R_+ \rightarrow R_+$ such that $\int_0^\infty \gamma(t) dt < \infty$.

We can now formulate our first result, which is a stochastic version of the well-known LaSalle theorem (i.e., Theorem 1 of LaSalle [10]) for locating limit sets of a system.

THEOREM 2.1. *Let (H1) hold. Assume that there is a function $V \in C^{2,1}(R^n \times R_+; R_+)$, a function $\gamma \in L^1(R_+; R_+)$ and a continuous function $w: R^n \rightarrow R_+$ such that*

$$\lim_{|x| \rightarrow \infty} \inf_{0 \leq t < \infty} V(x, t) = \infty \quad (2.2)$$

and

$$LV(x, t) \leq \gamma(t) - w(x), \quad (x, t) \in R^n \times R_+. \quad (2.3)$$

Moreover, for each initial value $x_0 \in R^n$ there is a $p > 2$ such that

$$\sup_{0 \leq t < \infty} E |x(t; x_0)|^p < \infty. \quad (2.4)$$

Then, for every $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} V(x(t; x_0), t) \text{ exists and is finite almost surely}$$

and, moreover,

$$\lim_{t \rightarrow \infty} w(x(t; x_0)) = 0 \quad a.s. \quad (2.5)$$

A function satisfying (2.2) is known as radially unbounded in the literature (cf. Arnold [1]). To interpret (2.5), let us define $D_w = \{x \in R^n: w(x) = 0\}$. We claim that D_w is nonempty. In fact, for any fixed $x_0 \in R^n$ we can, by Theorem 2.1, choose an $\omega \in \Omega$ such that

$$\lim_{t \rightarrow \infty} V(x(t, \omega; x_0), t) < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} w(x(t, \omega; x_0)) = 0.$$

By (2.2), $\{x(t, \omega; x_0): t \geq 0\}$ must be bounded so there is a subsequence $\{x(t_k, \omega; x_0): k \geq 1\}$ which converges to some $\bar{x} \in R^n$. Since w is continuous, we have $w(\bar{x}) = 0$, i.e., $\bar{x} \in D_w$ whence $D_w \neq \emptyset$. Let $d(x, D_w)$ denote the distance between x and set D_w , that is $d(x, D_w) = \min_{y \in D_w} |x - y|$. Then (2.5) means

$$\lim_{t \rightarrow \infty} d(x(t; x_0), D_w) = 0 \quad \text{a.s.} \quad (2.5)'$$

In other words, all the solutions of Eq. (2.1) will asymptotically approach D_w with probability one. To prove the theorem let us present three useful lemmas.

LEMMA 2.2. *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative \mathcal{F}_0 -measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \quad \text{for } t \geq 0.$$

If $X(t)$ is nonnegative, then

$$\left\{ \lim_{t \rightarrow \infty} A(t) < \infty \right\} \subset \left\{ \lim_{t \rightarrow \infty} X(t) \text{ exists and is finite} \right\} \cap \left\{ \lim_{t \rightarrow \infty} U(t) < \infty \right\} \quad \text{a.s.}$$

where $B \subset D$ a.s. means $P(B \cap D^c) = 0$. In particular, if $\lim_{t \rightarrow \infty} A(t) < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t \rightarrow \infty} X(t, \omega) \text{ exists and is finite, and } \lim_{t \rightarrow \infty} U(t, \omega) < \infty.$$

This lemma is established by Liptser and Shirayev [11, Theorem 7, p. 139]. The following lemma is the well-known Kolmogorov-Čentsov theorem on the continuity of a stochastic process derived from the moment property.

LEMMA 2.3. *Suppose that an n -dimensional stochastic process $X(t)$ on $t \geq 0$ satisfies the condition*

$$E |X(t) - X(s)|^\alpha \leq C |t - s|^{1+\beta}, \quad 0 \leq s, \quad t < \infty,$$

for some positive constants α , β , and C . Then there exists a continuous modification $\tilde{X}(t)$ of $X(t)$, which has the property that for every $\gamma \in (0, \beta/\alpha)$, there is a positive random variable $h(\omega)$ such that

$$P \left\{ \omega: \sup_{\substack{0 < t-s < h(\omega) \\ 0 \leq s, t < \infty}} \frac{|\tilde{X}(t, \omega) - \tilde{X}(s, \omega)|}{|t-s|^\gamma} \leq \frac{2}{1-2^{-\gamma}} \right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder-continuous with exponent γ .

The proof of this result can be found in Karatzas and Shreve [6] in the case when the stochastic process $X(t)$ is on the finite interval $[0, T]$ but a little bit of modification of the proof works for the case when $X(t)$ is on the entire R_+ .

LEMMA 2.4. *Let (H1) and (2.4) hold. Set*

$$y(t) := \int_0^t g(x(s), s) dB(s) \quad \text{on } t \geq 0,$$

where we write $x(t; x_0) = x(t)$ simply. Then almost every sample path of $y(t)$ is uniformly continuous on $t \geq 0$.

Proof. By the moment inequality for stochastic integrals (cf. Friedman [2] or Mao [14]) we have that for $0 \leq s < t < \infty$ and $p > 2$,

$$E |y(t) - y(s)|^p \leq \left[\frac{p(p-1)}{2} \right]^{p/2} (t-s)^{(p-2)/2} \int_s^t E |g(x(r), r)|^p dr.$$

But by (H1) and (2.4) we can derive that

$$E |g(x(r), r)|^p \leq E [c(1 + |x(r)|)]^p \leq (2c)^p (1 + E |x(r)|^p) \leq (2c)^p (1 + K),$$

where $K := \sup_{0 \leq t < \infty} E |x(t)|^p < \infty$. Therefore

$$E |y(t) - y(s)|^p \leq \left[\frac{p(p-1)}{2} \right]^{p/2} (2c)^p (1 + K) (t-s)^{1+(p-2)/2}.$$

Bearing in mind that $y(t)$ is continuous, we see from Lemma 2.3 that almost every sample path of $y(t)$ is locally but uniformly Hölder-continuous with exponent γ for every $\gamma \in (0, (p-2)/2p)$ and therefore almost every sample path of $y(t)$ must be uniformly continuous. The proof is complete.

We can now begin to prove Theorem 2.1.

Proof of Theorem 2.1. Fix any initial value x_0 and for simplicity write $x(t; x_0) = x(t)$. By Itô's formula and condition (2.3),

$$\begin{aligned} V(x(t), t) &= V(x_0, 0) + \int_0^t \gamma(s) ds - \int_0^t [\gamma(s) - LV(x(s), s)] ds \\ &\quad + \int_0^t V_x(x(s), s) g(x(s), s) dB(s) \\ &\leq V(x_0, 0) + \int_0^t \gamma(s) ds - \int_0^t w(x(s)) ds \\ &\quad + \int_0^t V_x(x(s), s) g(x(s), s) dB(s). \end{aligned}$$

Since $\int_0^\infty \gamma(s) ds < \infty$ and $\gamma(s) - LV(x(s), s) \geq 0$, we obtain, by Lemma 2.2, that for almost every $\omega \in \Omega$,

$$\int_0^\infty w(x(t, \omega)) dt < \infty \quad (2.6)$$

and

$$\lim_{t \rightarrow \infty} V(x(t, \omega), t) \text{ exists and is finite.} \quad (2.7)$$

We first claim that almost every sample path of $x(t)$ is uniformly continuous on $t \geq 0$. Write $x(t) = x_0 + z(t) + y(t)$, where

$$z(t) = \int_0^t f(x(s), s) ds \quad \text{and} \quad y(t) = \int_0^t g(x(s), s) dB(s).$$

By Lemma 2.4 and facts (2.6)–(2.7) shown above, we see that there is an $\bar{\Omega} \subset \Omega$ with $P(\bar{\Omega}) = 1$ such that for every $\omega \in \bar{\Omega}$ (2.6) and (2.7) hold, moreover, $y(t, \omega)$ is uniformly continuous on $t \geq 0$. Now, fix any $\omega \in \bar{\Omega}$. By (2.7),

$$\sup_{0 \leq t < \infty} V(x(t, \omega), t) < \infty.$$

Hence, by (2.2), there is a positive number $h(\omega)$ such that

$$|x(t, \omega)| \leq h(\omega) \quad \text{for all } t \geq 0.$$

From this and hypothesis (H1) we compute that for $0 \leq s < t < \infty$,

$$\begin{aligned} |z(t, \omega) - z(s, \omega)| &\leq \int_s^t |f(x(r, \omega), r)| dr \\ &\leq c \int_s^t (1 + |x(r, \omega)|) dr \leq c(1 + h(\omega))(t - s), \end{aligned}$$

which implies that $z(t, \omega)$ is uniformly continuous on $t \geq 0$. Since $\omega \in \bar{\Omega}$ is arbitrary, we have showed that for every $\omega \in \bar{\Omega}$, $x(t, \omega)$ is uniformly continuous on $t \geq 0$.

We next claim that

$$\lim_{t \rightarrow \infty} w(x(t, \omega)) = 0 \quad \text{for all } \omega \in \bar{\Omega}. \quad (2.8)$$

If this is not true, then for some $\hat{\omega} \in \bar{\Omega}$

$$\limsup_{t \rightarrow \infty} w(x(t, \hat{\omega})) > 0.$$

So there is some $\varepsilon > 0$ and a sequence $\{t_k\}_{k \geq 1}$ of positive numbers with $t_k + 1 < t_{k+1}$ such that

$$w(x(t_k, \hat{\omega})) > \varepsilon \quad \text{for all } k \geq 1. \quad (2.9)$$

Set $\bar{S}_h = \{x \in R^n : |x| \leq h\}$, where $h = h(\hat{\omega})$ has been defined above in the way that $\{x(t, \hat{\omega}) : t \geq 0\} \subset \bar{S}_h$. Since it is continuous, $w(\cdot)$ must be uniformly continuous in \bar{S}_h and there is a $\delta_1 > 0$ such that

$$|w(x) - w(y)| < \frac{\varepsilon}{2} \quad \text{if } x, y \in \bar{S}_h, \quad |x - y| < \delta_1. \quad (2.10)$$

On the other hand, recalling that $x(t, \hat{\omega})$ is uniformly continuous on $t \geq 0$, we can find a $\delta_2 \in (0, 1)$ such that

$$|x(t, \hat{\omega}) - x(s, \hat{\omega})| < \delta_1 \quad \text{if } 0 \leq t, s < \infty, \quad |t - s| \leq \delta_2. \quad (2.11)$$

Combining (2.10) and (2.11) we see that for every $n \geq 1$,

$$|w(x(t_k, \hat{\omega})) - w(x(t, \hat{\omega}))| < \frac{\varepsilon}{2} \quad \text{if } t_k \leq t \leq t_k + \delta_2.$$

This, together with (2.9), yields

$$w(x(t, \hat{\omega})) \geq w(x(t_k, \hat{\omega})) - |w(x(t_k, \hat{\omega})) - w(x(t, \hat{\omega}))| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Therefore

$$\int_0^\infty w(x(t, \hat{\omega})) dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k + \delta_2} w(x(t, \hat{\omega})) dt \geq \sum_{k=1}^\infty \frac{\varepsilon \delta_2}{2} = \infty,$$

which contradicts (2.6) since we have already shown that (2.6) holds for all $w \in \bar{Q}$ and of course for $\hat{\omega}$. Hence, (2.8) must be true and the theorem has been proved.

From the proof above, we see clearly that condition (2.4) is only used to show the uniform continuity of almost every sample path of $\int_0^t g(x(s; x_0), s) dB(s)$ on $t \geq 0$. This condition can be replaced by others and the following theorem gives some alternatives.

THEOREM 2.5. *The conclusions of Theorem 2.1 still hold if (2.4) is replaced by one of the following:*

- (i) *g is bounded*
- (ii) *for every x_0 there are three positive constants α , β , and C such that*

$$E \left| \int_s^t g(x(r; x_0), r) dB(r) \right|^\alpha \leq C(t-s)^{1+\beta}, \quad 0 \leq s < t < \infty;$$

- (iii) *there are functions $U \in C^{2,1}(R^n \times R_+; R_+)$ and $\bar{\gamma} \in L^1(R_+; R_+)$, and moreover a convex function $\mu \in \mathcal{K}_\infty$ and a number $p > 2$ such that*

$$\mu(|x|^p) \leq U(x, t) \quad \text{and} \quad LU(x, t) \leq \bar{\gamma}(t), \quad (x, t) \in R^n \times R_+;$$

- (iv) *there is a convex function $\mu \in \mathcal{K}_\infty$ and $p > 2$ such that*

$$\mu(|x|^p) \leq V(x, t), \quad (x, t) \in R^n \times R_+.$$

(Note this implies condition (2.2) so in this case (2.2) is satisfied automatically.)

Proof. The result is obvious in the cases of (i) and (ii). In the case of (iii), by Itô's formula and Jensen's inequality we have

$$\mu(E|x(t; x_0)|^p) \leq E\mu(|x(t; x_0)|^p) \leq U(x_0, 0) + \int_0^\infty \bar{\gamma}(s) ds := C < \infty,$$

which implies

$$E |x(t; x_0)|^p \leq \mu^{-1}(C) \quad \text{for all } t \geq 0.$$

In other words, (2.4) is satisfied so the conclusion follows. Finally, case (iv) follows from (iii). The proof is complete.

Sometimes it is difficult to verify the uniform continuity of almost every sample path of $\int_0^t g(x(s; x_0), s) dB(s)$ on $t \geq 0$, although Theorems 2.1 and 2.5 give several sufficient conditions. The following theorem may be useful in this case.

THEOREM 2.6. *Let (H1) hold. Assume that there is a function $V \in C^{2,1}(R^n \times R_+; R_+)$, a function $\gamma \in L^1(R_+; R_+)$ and a continuous function $\eta: R_+ \rightarrow R_+$ such that*

$$LV(x, t) \leq \gamma(t) - \eta(V(x, t)) \quad (2.12)$$

for all $(x, t) \in R^n \times R_+$. Then, for every $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} V(x(t; x_0), t) \text{ exists and is finite almost surely}$$

and, moreover,

$$\lim_{t \rightarrow \infty} \eta(V(x(t; x_0), t)) = 0 \quad \text{a.s.} \quad (2.13)$$

Proof. Fix any initial value x_0 and write $x(t; x_0) = x(t)$ again. By Itô's formula and condition (2.12),

$$\begin{aligned} V(x(t), t) &= V(x_0, 0) + \int_0^t \gamma(s) ds - \int_0^t [\gamma(s) - LV(x(s), s)] ds \\ &\quad + \int_0^t V_x(x(s), s) g(x(s), s) dB(s) \\ &\leq V(x_0, 0) + \int_0^t \gamma(s) ds - \int_0^t \eta(V(x(s), s)) ds \\ &\quad + \int_0^t V_x(x(s), s) g(x(s), s) dB(s). \end{aligned}$$

By Lemma 2.2, we see that there is an $\bar{\Omega} \subset \Omega$ with $P(\bar{\Omega}) = 1$ such that for every $\omega \in \bar{\Omega}$

$$\int_0^\infty \eta(V(x(t, \omega), t)) dt < \infty \quad (2.14)$$

and

$$\lim_{t \rightarrow \infty} V(x(t, \omega), t) \text{ exists and is finite.} \quad (2.15)$$

We claim that for every $\omega \in \bar{\Omega}$

$$\lim_{t \rightarrow \infty} \eta(V(x(t, \omega), t)) = 0. \quad (2.16)$$

If this is not true, then for some $\hat{\omega} \in \bar{\Omega}$

$$\limsup_{t \rightarrow \infty} \eta(V(x(t, \hat{\omega}), t)) > 0.$$

So there is some $\varepsilon > 0$ and a sequence $\{t_k\}_{k \geq 1}$ of positive numbers with $t_k + 1 < t_{k+1}$ such that

$$\eta(V(x(t_k, \hat{\omega}), t_k)) > \varepsilon \quad \text{for all } k \geq 1. \quad (2.17)$$

On the other hand, we see from (2.15) that $V(x(t, \hat{\omega}), t)$ is bounded and uniformly continuous on $t \geq 0$. Let h be its bound and δ_1 be a positive number such that

$$|\eta(u) - \eta(v)| \leq \frac{\varepsilon}{2} \quad \text{if } 0 \leq u, \quad v \leq h, \quad |u - v| < \delta_1. \quad (2.18)$$

Moreover, there is a $\delta_2 \in (0, 1)$ such that

$$|V(x(t, \hat{\omega}), t) - V(x(s, \hat{\omega}), s)| < \delta_1 \quad \text{if } 0 \leq t, \quad s < \infty, \quad |t - s| \leq \delta_2. \quad (2.19)$$

Combining (2.18) and (2.19) we see that for every $n \geq 1$,

$$|\eta(V(x(t_k, \hat{\omega}), t_k)) - \eta(V(x(t, \hat{\omega}), t))| < \frac{\varepsilon}{2} \quad \text{if } t_k \leq t \leq t_k + \delta_2.$$

This, together with (2.17), yields

$$\begin{aligned} \eta(V(x(t, \hat{\omega}), t)) &\geq \eta(V(x(t_k, \hat{\omega}), t_k)) \\ &\quad - |\eta(V(x(t_k, \hat{\omega}), t_k)) - \eta(V(x(t, \hat{\omega}), t))| > \frac{\varepsilon}{2}. \end{aligned}$$

Therefore

$$\int_0^\infty \eta(V(x(t, \hat{w}), t)) dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k + \delta_2} \eta(V(x(t, \hat{w}), t)) dt \geq \sum_{k=1}^\infty \frac{\varepsilon \delta_2}{2} = \infty,$$

which contradicts (2.14) so (2.16) must be true. The proof is complete.

As we interpreted (2.5) as (2.5)', let us define $D_\eta = \{u \geq 0 : \eta(u) = 0\}$. We can show $D_\eta \neq \emptyset$ in the same way as the proof of $D_w \neq \emptyset$ before. Then (2.13) means

$$\lim_{t \rightarrow \infty} d(V(x(t; x_0), t), D_\eta) = 0 \quad a.s. \quad (2.13)'$$

that is, $V(x(t; x_0), t)$ will asymptotically approach D_η with probability one. Comparing (2.13)' with (2.5)' we observe that the result of Theorem 2.6 seems not so precise as the result of Theorem 2.1. The conditions of Theorem 2.6 seem simpler than Theorem 2.1 since there is no need of (2.2) and (2.4), but it is in fact more difficult for condition (2.12) to be satisfied than condition (2.3). Nevertheless, Theorem 2.6 is very useful. In particular, we can easily apply it to obtain the following classical result on the globally stochastically asymptotic stability (cf. Arnold [1], Has'minskii [4], or Kushner [7]).

COROLLARY 2.7. *Let (H1) hold. Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$, $\mu_1, \mu_2 \in \mathcal{K}_\infty$, and $\mu_3 \in \mathcal{K}$ such that*

$$\mu_1(|x|) \leq V(x, t) \leq \mu_2(|x|) \quad (2.20)$$

and

$$LV(x, t) \leq -\mu_3(|x|) \quad (2.21)$$

for all $(x, t) \in R^n \times R_+$. Then, for every $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} |x(t; x_0)| = 0 \quad a.s. \quad (2.22)$$

Proof. It follows from (2.20) that

$$\mu_2^{-1}(V(x, t)) \leq |x|.$$

Substituting this into (2.21) gives

$$LV(x, t) \leq -\mu_3(\mu_2^{-1}(V(x, t))).$$

By Theorem 2.6, for any $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} \mu_3(\mu_2^{-1}(V(x(t; x_0), t))) = 0 \quad \text{a.s.}$$

Note that $\mu_3(\mu_2^{-1}(u)) = 0$ if and only if $u = 0$. The above implies

$$\lim_{t \rightarrow \infty} V(x(t; x_0), t) = 0 \quad \text{a.s.}$$

This, together with (2.20), yields

$$\lim_{t \rightarrow \infty} \mu_1(|x(t; x_0)|) = 0 \quad \text{a.s.}$$

But $\mu_1(u) = 0$ if and only if $u = 0$. We must therefore have

$$\lim_{t \rightarrow \infty} |x(t; x_0)| = 0 \quad \text{a.s.}$$

as required. The proof is complete.

Remark 2.8. It is easy to see that the conclusion of Corollary 2.7 still holds if condition (2.21) is replaced by the following weaker one

$$LV(x, t) \leq \gamma(t) - \mu_3(|x|), \quad (x, t) \in R^n \times R_+ \quad (2.23)$$

for some $\gamma \in L^1(R_+; R_+)$. The reason why we would rather use (2.21) instead of (2.23) is just to keep it the same as the classical result.

We can not only apply Theorem 2.6 to show the convergence of the solutions but also to obtain the rate of convergence. The following corollary on almost sure exponential stability is a simple application of Theorem 2.6.

COROLLARY 2.9. *Let (H1) hold. Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$ and $\gamma \in L^1(R_+; R_+)$, and moreover a pair of two positive constants λ and p , such that*

$$e^{\lambda t} |x|^p \leq V(x, t) \quad \text{and} \quad LV(x, t) \leq \gamma(t) \quad (2.24)$$

for all $(x, t) \in R^n \times R_+$. Then, for every $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |x(t; x_0)| \leq -\frac{\lambda}{p} \quad \text{a.s.} \quad (2.25)$$

Proof. By Theorem 2.6 with $\eta \equiv 0$ we see that, for any $x_0 \in R^n$,

$$\lim_{t \rightarrow \infty} V(x(t; x_0), t) \text{ exists and is finite almost surely.}$$

Hence there is a finite random variable ξ such that

$$e^{\lambda t} |x(t; x_0)|^p \leq V(x(t; x_0), t) \leq \xi \quad \text{for all } t \geq 0.$$

This implies the required assertion (2.25) immediately. The proof is complete.

Theorem 2.6 can also be applied to obtain other types of convergence, e.g., polynomial convergence as shown in Example 5.4 below.

3. FURTHER RESULTS

The theory in the previous section can be generalized to cope with the convergence of the solutions in an invariant set. Let us first recall the definition of an invariant set with respect to the solutions of equation (2.1) (cf. Friedman [2]).

DEFINITION 3.1. An open subset G of R^n is said to be invariant with respect to the solutions of Eq. (2.1) if

$$P\{x(t; x_0) \in G \text{ for all } t \geq 0\} = 1 \quad \text{for every } x_0 \in G,$$

that is, the solutions starting in G will remain in G .

A particular but important case of the invariant set is $R^n - \{0\}$ when hypothesis (H1) holds and the coefficients f and g of Eq. (2.1) satisfy

$$f(0, t) \equiv 0, \quad g(0, t) \equiv 0. \quad (3.1)$$

In this case, almost every sample path of any solution starting from a non-zero state will never reach the origin (cf. Mao [14]). Condition (3.1) is always imposed when one discusses the stability of the trivial solution and the related Lyapunov function is often only defined for $x \in R^n - \{0\}$ instead of for all $x \in R^n$. As another example, consider the one-dimensional equation

$$dx(t) = -\sin(x(t)) dt + \sin(x(t)) dB(t)$$

where $B(t)$ is a scalar Brownian motion. The open interval $(0, \pi)$ is an invariant set with respect to the solutions of this equation. For other examples of invariant sets please see Friedman [2].

To deal with the general invariant case, let us denote by $C^{2,1}(G \times R_+; R_+)$ the family of all nonnegative functions $V(x, t)$ on $G \times R_+$ which are continuously twice differentiable in $x \in G$ and once differentiable in $t \geq 0$. We also let \bar{G} denote the closure of G .

THEOREM 3.1. *Let (H1) hold and G be an invariant set with respect to the solutions of equation (2.1). Assume that there are functions $V \in C^{2,1}(G \times R_+; R_+)$ and $\gamma \in L^1(R_+; R_+)$, and a continuous function $w: \bar{G} \rightarrow R_+$, such that*

$$LV(x, t) \leq \gamma(t) - w(x), \quad (x, t) \in G \times R_+. \quad (3.2)$$

If G is bounded; or otherwise if

$$\lim_{\substack{x \in G \\ |x| \rightarrow \infty}} \inf_{0 \leq t < \infty} V(x, t) = \infty \quad (3.3)$$

and, moreover, for each initial value $x_0 \in G$ there is a $p > 2$ such that

$$\sup_{0 \leq t < \infty} E |x(t; x_0)|^p < \infty, \quad (3.4)$$

then for every $x_0 \in G$, $\lim_{t \rightarrow \infty} V(x(t; x_0), t)$ exists and is finite almost surely and, furthermore,

$$\lim_{t \rightarrow \infty} w(x(t; x_0)) = 0 \quad a.s. \quad (3.5)$$

This theorem can be proved in the same way as Theorem 2.1. Also condition (3.4) can be replaced by similar conditions described in Theorem 2.5. Moreover, (3.5) means that all the solutions of equation (2.1) starting in G will asymptotically approach $D_w^G := \{x \in \bar{G} : w(x) = 0\}$, which is nonempty, with probability one.

THEOREM 3.2. *Let (H1) hold and G be an invariant set with respect to the solutions of Eq. (2.1). Assume that there are functions $V \in C^{2,1}(G \times R_+; R_+)$ and $\gamma \in L^1(R_+; R_+)$ and, moreover, a continuous function $\eta: R_+ \rightarrow R_+$ such that*

$$LV(x, t) \leq \gamma(t) - \eta(V(x, t)), \quad (x, t) \in G \times R_+. \quad (3.6)$$

Then, for every $x_0 \in G$, $\lim_{t \rightarrow \infty} V(x(t; x_0), t)$ exists and is finite almost surely and, more precisely,

$$\lim_{t \rightarrow \infty} \eta(V(x(t; x_0), t)) = 0 \quad a.s. \quad (3.7)$$

This theorem can be proved in the same way as Theorem 2.6. A straightforward application of this theorem gives the following result on the asymptotic stability of a set, which is a generalization of Theorem 12.2.4 of Friedman [2].

COROLLARY 3.3. *Let (H1) hold and G be an invariant set with respect to the solutions of Eq. (2.1). Assume that G can be decomposed as $G = G_1 - G_0$ with G_1 an open set and G_0 a closed subset of G_1 . Assume that there are functions $V \in C^{2,1}(G \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, $\mu_1, \mu_2 \in \mathcal{K}_\infty$ and $\mu_3 \in \mathcal{K}$ such that*

$$\mu_1(d(x, G_0)) \leq V(x, t) \leq \mu_2(d(x, G_0)) \quad (3.8)$$

and

$$LV(x, t) \leq \gamma(t) - \mu_3(d(x, G_0)) \quad (3.9)$$

for all $(x, t) \in G \times R_+$. Then, for every $x_0 \in G$,

$$\lim_{t \rightarrow \infty} d(x(t; x_0), G_0) = 0 \quad a.s. \quad (3.10)$$

In other words, all the solutions of Eq. (2.1) starting within G will approach G_0 with probability one.

4. AUTONOMOUS STOCHASTIC SYSTEMS

If the coefficients of Eq. (2.1) are independent of time t , it becomes an autonomous stochastic differential equation

$$dx(t) = f(x(t)) dt + g(x(t)) dB(t). \quad (4.1)$$

Accordingly, hypothesis (H1) becomes

(H2) For each $k = 1, 2, \dots$, there is a $c_k > 0$ such that

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq c_k |x - y|$$

for those $x, y \in R^n$ with $|x| \vee |y| \leq k$, and there is moreover a $c > 0$ such that

$$|f(x)| \vee |g(x)| \leq c(1 + |x|)$$

for all $x \in R^n$.

Under (H2), there is a unique solution, still denoted by $x(t; x_0)$, of Eq. (4.1) for every initial value $x_0 \in R^n$. Moreover, let $C^2(G; R_+)$ denote the family of all C^2 -functions from G to R_+ . The corresponding operator L from $C^2(G; R_+)$ to R becomes

$$LV(x) = V_x(x) f(x) + \frac{1}{2} \text{trace}[g^T(x) V_{xx}(x) g(x)].$$

Applying the results obtained in the previous section we immediately obtain the following useful results.

COROLLARY 4.1. *Let (H2) hold and G be an invariant set with respect to the solutions of Eq. (4.1). Assume that there is a function $V \in C^2(G; R_+)$ such that*

$$LV(x) \leq 0, \quad x \in G. \quad (4.2)$$

If G is bounded; or otherwise if

$$\lim_{\substack{x \in G \\ |x| \rightarrow \infty}} V(x) = \infty \quad (4.3)$$

and, moreover, for each initial value $x_0 \in G$ there is a $p > 2$ such that

$$\sup_{0 \leq t < \infty} E|x(t; x_0)|^p < \infty; \quad (4.4)$$

then for every $x_0 \in G$, $\lim_{t \rightarrow \infty} V(x(t; x_0))$ exists and is finite almost surely and, more precisely,

$$\lim_{t \rightarrow \infty} LV(x(t; x_0)) = 0 \quad a.s. \quad (4.5)$$

This corollary follows from Theorem 3.1. In particular, (4.5) follows from (3.5) by letting $w(x) = -LV(x)$.

COROLLARY 4.2. *Let (H2) hold and G be an invariant set with respect to the solutions of Eq. (4.1). Assume that there is a function $V \in C^2(G; R_+)$ and a continuous function $\eta: R_+ \rightarrow R_+$ such that*

$$LV(x) \leq -\eta(V(x)), \quad x \in G. \quad (4.6)$$

Then, for every $x_0 \in G$, $\lim_{t \rightarrow \infty} V(x(t; x_0))$ exists and is finite almost surely and, more precisely,

$$\lim_{t \rightarrow \infty} \eta(V(x(t; x_0))) = 0 \quad a.s. \quad (4.7)$$

COROLLARY 4.3. *Let (H2) hold and G be an invariant set with respect to the solutions of Eq. (4.1). Assume that G can be decomposed as $G = G_1 - G_0$ with G_1 an open set and G_0 a closed subset of G_1 . Assume that there are functions $V \in C^2(G; R_+)$, $\mu_1, \mu_2 \in \mathcal{K}_\infty$ and $\mu_3 \in \mathcal{K}$ such that*

$$\mu_1(d(x, G_0)) \leq V(x) \leq \mu_2(d(x, G_0)) \quad (4.8)$$

and

$$LV(x) \leq -\mu_3(d(x, G_0)) \quad (4.9)$$

for all $x \in G$. Then, for every $x_0 \in G$,

$$\lim_{t \rightarrow \infty} d(x(t; x_0), G_0) = 0 \quad a.s. \quad (4.10)$$

5. EXAMPLES

In this section we shall discuss a number of examples to illustrate our theory. In the following examples we will let $B(t)$ be a scalar Brownian motion.

EXAMPLE 5.1. Let α and β be bounded real-valued Borel measurable functions defined on R_+ . Consider a one-dimensional stochastic differential equation

$$dx(t) = \alpha(t) x(t) dt + \beta(t) x(t) dB(t), \quad t \geq 0. \quad (5.1)$$

It is known that $G = R - 0$ is an invariant set with respect to the solutions. Assume that there is a $\delta \in (0, 1)$ such that

$$\inf_{0 \leq t < \infty} \left(\frac{1-\delta}{2} \beta^2(t) - \alpha(t) \right) > 0. \quad (5.2)$$

Let $V(x, t) = |x|^\delta$ for $x \neq 0$ and $t \geq 0$. Then

$$LV(x, t) = -\delta \left(\frac{1-\delta}{2} \beta^2(t) - \alpha(t) \right) |x|^\delta \leq -\delta \varepsilon |x|^\delta,$$

where ε denotes the left hand side of (5.2) which is positive. By Corollary 3.3 we can conclude that all the solutions of Eq. (5.1) will asymptotically tend to zero with probability one. To verify this conclusion, let us solve Eq. (5.1) explicitly

$$x(t) = x_0 \exp \left[- \int_0^t \left(\frac{1}{2} \beta^2(s) - \alpha(s) \right) ds + \int_0^t \beta(s) dB(s) \right]$$

given initial value $x(0) = x_0 \neq 0$. By condition (5.2) we have

$$\bar{\varepsilon} := \inf_{0 \leq t < \infty} \left(\frac{1}{2} \beta^2(t) - \alpha(t) \right) > 0.$$

Hence

$$\frac{1}{t} \log(|x(t)|) \leq -\bar{\varepsilon} + \frac{1}{t} \left[\log(|x_0|) + \int_0^t \beta(s) dB(s) \right].$$

Since $\beta(\cdot)$ is bounded, it is well known that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \beta(s) dB(s) = 0 \quad \text{a.s.}$$

Therefore

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|x(t)|) \leq -\bar{\varepsilon} \quad \text{a.s.}$$

This confirms that all the solutions of Eq. (5.1) will asymptotically tend to zero with probability one.

EXAMPLE 5.2. Consider a one-dimensional stochastic differential equation

$$dx(t) = -\sin(x(t)) dt + \sin(x(t)) dB(t), \quad t \geq 0. \quad (5.3)$$

It is known that $G = (0, \pi)$ is an invariant set with respect to the solutions. Let $V(x) = |x|^2$ for $x \in (0, \pi)$. Then

$$LV(x) = -\sin(x)[2x - \sin(x)] \leq 0, \quad x \in (0, \pi).$$

By Corollary 4.1 we conclude that for any initial value $x_0 \in (0, \pi)$, $\lim_{t \rightarrow \infty} |x(t; x_0)|^2$ exists and is finite almost surely and, more precisely,

$$\lim_{t \rightarrow \infty} (-\sin(x(t; x_0))[2x(t; x_0) - \sin(x(t; x_0))]) = 0 \quad \text{a.s.}$$

This means that almost every sample path of $x(t; x_0)$ tends to either 0 or π .

EXAMPLE 5.3. Let α be a function on R_+ such that $0 < \delta_1 \leq \alpha(t) \leq \delta_2 < \infty$. Consider a stochastic oscillator

$$\ddot{y}(t) + [\alpha(t) + \sqrt{\alpha(t)} \dot{B}(t)] \dot{y}(t) + y(t) = 0, \quad t \geq 0. \quad (5.4)$$

Introducing a new variable $x = (x_1, x_2)^T = (y, \dot{y})^T$, this oscillator can be written as an Itô equation

$$dx(t) = \begin{bmatrix} x_2(t) \\ -x_1(t) - \alpha(t) x_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ -\sqrt{\alpha(t)} x_2(t) \end{bmatrix} dB(t). \quad (5.5)$$

Let $2 < p < 3$ and define $V(x, t) = |x|^p$ for $(x, t) \in R^2 \times R_+$. Then

$$\begin{aligned} LV(x, t) &\leq p |x|^{p-2} x_1 x_2 + p |x|^{p-2} x_2 (-x_1 - \alpha(t) x_2) \\ &\quad + \frac{p(p-1)}{2} \alpha(t) |x|^{p-2} x_2^2 \\ &= -\frac{p(3-p)}{2} \alpha(t) |x|^{p-2} x_2^2 \leq -\frac{p(3-p)}{2} \delta_1 |x|^{p-2} x_2^2. \end{aligned} \quad (5.6)$$

By Theorem 2.5 we see that $\lim_{t \rightarrow \infty} |x(t; x_0)|^p$ exists and is finite almost surely and

$$\lim_{t \rightarrow \infty} |x(t; x_0)|^{p-2} x_2^2(t; x_0) = 0 \quad \text{a.s.}$$

We can therefore conclude that for almost every sample path of each solution of the stochastic oscillator not only $\lim_{t \rightarrow \infty} \dot{y}(t; x_0) \rightarrow 0$ but also $\lim_{t \rightarrow \infty} y(t; x_0)$ exists and is finite.

EXAMPLE 5.4. Consider a scalar linear stochastic differential equation

$$dx(t) = -\frac{p}{1+t} x(t) dt + (1+t)^{-p} dB(t), \quad t \geq 0, \quad (5.7)$$

where $p > \frac{1}{2}$. Let $0 < \varepsilon < p - \frac{1}{2}$ be arbitrary and define

$$V(x, t) = (1+t)^{2p-1-2\varepsilon} x^2 \quad \text{for } (x, t) \in R \times R_+.$$

Clearly

$$\lim_{|x| \rightarrow \infty} \inf_{0 \leq t < \infty} V(x, t) = \lim_{|x| \rightarrow \infty} x^2 = \infty.$$

Compute

$$\begin{aligned} LV(x, t) &= (2p-1-2\varepsilon)(1+t)^{2p-2-2\varepsilon} x^2 \\ &\quad - 2p(1+t)^{2p-2-2\varepsilon} x^2 + (1+t)^{-(1+2\varepsilon)} \\ &\leq (1+t)^{-(1+2\varepsilon)}. \end{aligned}$$

Note that

$$\int_0^{\infty} (1+t)^{-(1+2\varepsilon)} dt = \frac{1}{2\varepsilon} < \infty.$$

By Theorem 2.6, we see that for any given initial value $x_0 \in R$ the solution of equation (5.7) has the property that

$$\lim_{t \rightarrow \infty} V(x(t; x_0), t) \text{ exists and is finite almost surely.}$$

Hence there is a finite random variable ξ such that

$$(1+t)^{2p-1-2\varepsilon} x^2(t; x_0) \leq \xi,$$

i.e.,

$$x^2(t; x_0) \leq \xi(1+t)^{-(2p-1-2\varepsilon)} \quad \text{for all } t \geq 0.$$

This yields

$$\limsup_{t \rightarrow \infty} \frac{\log |x(t; x_0)|}{\log t} \leq -\frac{2p-1-2\varepsilon}{2} \quad \text{a.s.}$$

Letting $\varepsilon \rightarrow 0$ we obtain that

$$\limsup_{t \rightarrow \infty} \frac{\log |x(t; x_0)|}{\log t} \leq -\frac{2p-1}{2} \quad \text{a.s.}$$

In other words almost every sample path of each solution will tend to zero polynomially.

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