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Analytic smoothing of geometric maps with applications to KAM theory

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Abstract

We show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving, or contact can be approximated with analytic diffeomorphisms that are, respectively, symplectic, volume-preserving or contact. We prove that the approximating functions are uniformly bounded on some complex domains and that the rate of convergence, in C^r-norms, of the approximation can be estimated in terms of the size of such complex domains and the order of differentiability of the approximated function. As an application to this result, we give a proof of the existence, the local uniqueness and the bootstrap of regularity of KAM tori for finitely differentiable symplectic maps. The symplectic maps considered here are not assumed either to be written in action-angle variables or to be perturbations of integrable systems. Our main assumption is the existence of a finitely differentiable parameterization of a maximal dimensional torus that satisfies a non-degeneracy condition and that is approximately invariant. The symplectic, volume-preserving and contact forms are assumed to be analytic.

Keywords: Approximation; Smoothing; Symplectic maps; Volume-preserving maps; Contact maps; KAM tori; Uniqueness; Bootstrap of regularity

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1. Introduction

It is known that finitely differentiable functions can be approximated by C^∞ or analytic ones, in such a way that the quantitative properties of the approximation are related to the order of differentiability of the approximated function [1–4]. In view of applications to KAM theory, it is natural to ask whether it is possible to approximate finitely differentiable diffeomorphisms preserving a symplectic or volume form with C^∞ or analytic diffeomorphisms preserving the same form. Here we show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving, or contact can be approximated with analytic diffeomorphisms that are, respectively, symplectic, volume-preserving or contact. We prove that the approximating functions are uniformly bounded on some complex domains and give quantitative relations between: the rate of convergence in C^r-norms, the degree of regularity of the approximated function, and the size of the complex domains where the approximating functions are uniformly bounded. As an application we give a proof of the existence, the local uniqueness and the bootstrap of regularity of KAM tori for finitely differentiable symplectic diffeomorphisms. The symplectic diffeomorphisms considered here are not assumed to be either written in action-angle variables or perturbations of integrable systems. Our main assumption is the existence of a finitely differentiable parameterization of a maximal dimensional torus that is approximately invariant and that satisfies a non-degeneracy condition. We emphasize that the approximately invariant torus is not assumed to be equal to $(\theta, 0)$. Besides the mentioned results, in this work we also obtain several results which may be of independent interest. We present a detailed study of the relation between an analytic linear *smoothing operator* (cf. [4], see also Definition 5) and the nonlinear operators: composition and pull-back.

The case of approximating finitely differentiable symplectic or volume-preserving diffeomorphisms on a compact manifold with symplectic, respectively volume-preserving, C^∞-diffeomorphisms has been considered in [5], where it was proved that: (i) *symplectic C^k-diffeomorphisms, with $k \geqslant 1$, can be approximated in the C^k-norms with symplectic C^∞-diffeomorphisms*, and (ii) *volume-preserving $C^{k+\alpha}$-diffeomorphisms, with $k \geqslant 1$ an integer and $0 < \alpha < 1$, can be approximated by volume-preserving C^∞-diffeomorphisms in the C^k-norms.* The method used in the present work differs from that in [5], because we avoid the use of *generating functions*. As it is well known, generating functions may fail to be globally defined for some maps. One advantage of not using generating functions is that the result given here can be applied directly to *non-twist* maps. We are currently working on such an application [6].

The KAM results we present here are of 'polishing' type. More precisely, given a Diophantine frequency vector ω, a finitely differentiable symplectic map f, and a finitely differentiable parameterization of a maximal dimensional torus K, satisfying a non-degeneracy condition, we give an explicit condition on the size of the error $f \circ K - K \circ R_\omega$, in finitely differentiable norms, that guarantees the existence of a true invariant torus near the approximately invariant one, here R_ω represents the translation by the vector ω. One of the motivations for obtaining such KAM result is the *validation* of numerical computation where the only input is a parameterization of a maximal dimensional torus, which is often only approximately invariant and different from $(\theta, 0)$. Having a condition on the size of the error in finitely differentiable norms is useful because for some numerical methods it is easier to estimate the finitely differentiable norms than the analytic ones, for example when using splines. The fact that we do not assume that the system is close to integrable or written in action-angle variables makes the KAM results presented here more applicable, because we do not have to compute local coordinates before the verification of the size of the error. Another application of this KAM result is when studying invariant tori restricted to normally hyperbolic manifolds – which are only finitely differentiable. This analysis occurs in some mechanisms for the study of instability. In particular, in [7,8] it is shown that secondary tori close to resonances play an important role. The present KAM result is particularly useful for this study since for these tori, the action-angle coordinates are singular and their construction and their estimates require extra work and extra assumptions, see [8, 8.5.4]. The present work allows to simplify the proof of some of the results in [8] and lowers the regularity assumptions of the main result of [8]. This improvements are crucial in the higher dimensional extensions of the model.

This paper is divided into three parts. To make the reading easier, we have included, in Section 2, an outline of the methods used here, emphasizing the main ideas. In Section 3 we show how to approximate finitely differentiable functions that preserve a geometric structure (exact symplectic, volume or contact) with analytic functions preserving the same geometric structure. In Section 4 we give an application of the symplectic smoothing result to KAM theory, proving of the existence, the local uniqueness and the bootstrap of regularity of Diophantine invariant tori for finitely differentiable symplectic maps. In Section 4 we also prove the bootstrap of regularity of KAM tori for analytic exact symplectic maps. That is, we prove that given an analytic exact symplectic map and an invariant torus with Diophantine frequency vector, if the invariant torus is sufficiently differentiable, then it is analytic.

2. Brief description of the methodology

Even though the proofs of our results involve many technicalities, the main ideas are rather simple. In what follows we give a brief description of the methodology used in this work emphasizing the main ideas. First, we define an analytic linear smoothing operator S_t, taking differentiable functions into analytic ones. The definition of S_t depends on the domain of definition of the functions we wish to smooth. We consider three situations: (i) *the* d*-dimensional torus* $\mathbb{T}^d \stackrel{\text{def}}{=} \mathbb{R}^d/\mathbb{Z}^d$; (ii) $U \subset \mathbb{R}^d$ *satisfying certain conditions, specified in Section* 3.1*, that guarantee the existence of a bounded linear extension operator* [1] (*see Definition* 9) *and the validity of the Mean Value Theorem*; and (iii) $\mathbb{T}^n \times U$ *with* $U \subset \mathbb{R}^{d-n}$ *as in* (ii). Following [4] we smooth functions defined on $\mathbb{R}^d$ by an operator S_t defined by the convolution operator with an analytic kernel (see Section 3.1). By defining S_t in this way, we obtain a linear operator which takes periodic functions into periodic functions. Hence, by considering *lifts* to $\mathbb{R}^d$, the universal covering of $\mathbb{T}^n$, S_t can be applied to differentiable functions defined on the torus $\mathbb{T}^d$: this is important in applications to KAM theory. It is known [1,3] that if $U \subset \mathbb{R}^d$ has smooth boundary then there exists a bounded linear extension operator taking differentiable functions defined on U into differentiable functions defined on $\mathbb{R}^d$. Hence, for functions defined on $U \subset \mathbb{R}^d$ with smooth boundary, we define an analytic linear smoothing operator by taking extensions and then applying the operator S_t described above. It is easy to check that if $U \subset \mathbb{R}^{d-n}$ has smooth boundary then $\mathbb{R}^n \times U \subset \mathbb{R}^d$ also has smooth boundary. Hence functions defined on $\mathbb{T}^n \times U$ are smoothed by considering the universal covering $\mathbb{R}^n \times U$ and using a linear extension operator (see Section 3.1).

Given a finite differentiable diffeomorphism f that preserves a form Ω, it is not necessarily true that $S_t[f]$ preserves Ω. More generally, the form $S_t[f]^*\Omega$ is not necessarily equal to $f^*\Omega$. So we use Moser's *deformation method* [9] to prove that, for t sufficiently large, there is a diffeomorphism φ_t such that $\varphi_t^*(S_t[f]^*\Omega) = f^*\Omega$. Hence, given a finitely differentiable diffeomorphism f which is either symplectic, volume-preserving, or contact, for t sufficiently large, $T_t[f] = S_t[f] \circ \varphi_t$ gives a symplectic, respectively, volume-preserving, or contact diffeomorphism approximating f. Furthermore, using the calculus of deformations [10], we prove that if f is exact symplectic, then it is possible to construct analytic approximating functions $T_t[f]$ which are also exact. The method used in the present work produces quantitative properties of the nonlinear operators T_t in terms of the degree of differentiability of f. More precisely, for t sufficiently large, $T_t[f]$ is bounded uniformly, with respect to t, on some complex domains and the rate of convergence of $T_t[f]$ to f is given in terms of t and the degree of differentiability of f. Obtaining such quantitative properties involves estimates on complex domains of the difference between: (i) *smoothing a composition of two functions and composing their smoothings*, and (ii) *smoothing the pulled-back form $f^*\alpha$ and pulling-back the form α with the smoothed function $S_t[f]$, for a k-form α*. Estimating these differences on complex domains requires many technicalities but, once this is done, proving the quantitative properties of T_t is rather easy as we show in Section 3.4. An estimate, on complex domains, of the difference between smoothing a composition of two functions and composing their smoothings was previously obtained in [11].

We emphasize that the geometric form Ω is assumed to be analytic. This is important because in this case, if f is symplectic, respectively, volume-preserving or contact, we have that both $f^*\Omega$ and $S_t[f]^*\Omega$ are analytic so that Moser's deformation method produces, for t sufficiently large, an analytic diffeomorphism φ_t such that: $\varphi_t^* S_t[f]^*\Omega = f^*\Omega$. The analyticity assumption on Ω is of particular importance in the volume case because the existence of a diffeomorphism φ such that $\varphi^*\alpha = \beta$ for two arbitrary volume forms depends on the regularity of the forms and on their domain of definition. The existence of such a diffeomorphism for volume forms has been studied under different hypotheses in [5,9,12–14]. Nevertheless, to the best knowledge of the authors the question proposed in [5] whether C^1-volume diffeomorphisms can be approximated in C^1-norm by C^∞-volume diffeomorphisms on d-dimensional manifolds, with $d \geqslant 3$, is still open.

The existence of invariant tori for finitely differentiable symplectic maps, formulated in Theorem 47, is a finitely differentiable version of Theorem 1 in [15] (the latter is reported as Theorem 46 in the present work). Roughly, Theorem 46 establishes the existence of a maximal dimensional invariant torus K^* with Diophantine rotation vector ω for a given analytic exact symplectic map f. The main hypotheses of Theorem 46 are the existence of an analytic parameterization of an n-dimensional torus K such that (i) *certain non-degeneracy conditions are satisfied*, and (ii) *K is approximately invariant, in the sense that the sup norm of the error function $f \circ K - K \circ R_\omega$ on a complex set $\{x \in C^n\colon |\mathrm{Im}(x)| < \rho\}$, for some $\rho > 0$, is 'sufficiently small,' where R_ω represents the translation by ω*. Theorem 46 also gives an estimate of the distance between the initial, approximately invariant torus K and the invariant torus K^* in terms of the size of the initial error. Theorem 47 is a finitely differentiable version of Theorem 46: the analyticity hypotheses for f and K are replaced by 'sufficiently large' differentiability of both f and K and by asking the norm, in suitable spaces of differentiable functions, of $f \circ K - K \circ R_\omega$ to be 'sufficiently small.' In Theorem 49 we prove that finitely differentiable invariant tori for finitely differentiable symplectic diffeomorphisms are locally unique. Theorem 49 is a finitely differentiable version of Theorem 2 in [15].

Moser's smoothing technique [2,4,16] provides a method to obtain finitely differentiable versions of *Generalized Implicit Function* theorems from the corresponding analytic ones. Briefly, Moser's method goes as follows: Let F be defined on Banach spaces of analytic functions and assume that a Generalized Implicit Function Theorem holds in these Banach spaces. Assume that the functional equation $F(f, K) = 0$ has an analytic solution (f_0, K_0), and that there exists an analytic smoothing operator. Then one finds, using the analytic smoothing operator, a solution $(f, \Phi(f))$ for f in a small neighbourhood of f_0 in a space of finitely differentiable functions. One important hypothesis of Moser's technique is the existence of an approximate right inverse of the linear operator $D_2F(f, K)$. The approximate right-invertibility yields a loss of differentiability: in KAM theory this is related to the so-called *'small denominators.'* At this point it becomes crucial to have quantitative properties of the smoothing in terms of the degree of differentiability of the smoothed functions. For a more detailed explanation of Moser's method see for example [2,4,16]. Some variations of Moser's technique have been used previously to give proofs of the existence of KAM tori for finitely differentiable Hamiltonian vector fields. In [17] the authors assume the existence of a finitely differentiable invariant torus, i.e. they assume the existence of a solution of the functional equation in finitely differentiable spaces. In [18,19] the authors assume that $(\theta, 0)$ parameterizes an approximately invariant torus, i.e. they assume that the given finitely differentiable Hamiltonian vector field and the identity form an approximate solution of the functional equation.

To prove the existence of finitely differentiable solutions of the equation $f \circ K = K \circ R_\omega$ we use the following 'modified' smoothing technique: Rather than assuming the existence of an analytic initial solution of the functional equation we just assume the existence of a finitely differentiable approximate solution and find conditions under which there is an analytic solution nearby. The analytic Generalized Implicit Function Theorem for the functional $f \circ K - K \circ R_\omega$ is provided by Theorem 46, which only holds for exact symplectic maps. Hence, to apply the smoothing technique we use the nonlinear operator T_t, described above, to smooth the exact symplectic map f. Parameterizations of approximately invariant tori are smoothed using the operator S_t described at the beginning of this introduction. Then, given a finitely differentiable approximate solution (f, K) of $f \circ K = K \circ R_\omega$, the existence of an analytic solution close to (f, K) is guaranteed by: (i) *a non-degeneracy condition on* K, and (ii) *a 'smallness' condition on the* sup *norm on complex domains of the difference* $T_t[f] \circ S_t[K] - S_t[f \circ K]$ *in terms of the size of the initial error* $f \circ K - K \circ R_\omega$ *in a finite differentiable norm.*

The modified smoothing technique described above has been previously used, although in a more abstract setting, in [20]. In this reference, given a general functional F satisfying certain hypotheses and assuming the existence of a smoothing operator, the author proves the existence of an analytic solution near a given finitely differentiable approximate solution for F. Here we use the same approach of [20] for the operator given by $f \circ K - K \circ R_\omega$. As a necessary step, a smoothing operator is here explicitly constructed for this concrete operator. As a matter of fact, two distinct smoothing operators are constructed: one for the parameterization K and a second one for the symplectic map f, the last one preserving the symplectic character.

As a consequence of the fact that, under certain general conditions, near a finitely differentiable solution (f, K) of the equation $f \circ K = K \circ R_\omega$ there is an analytic solution, we obtain the bootstrap of regularity of invariant tori with Diophantine rotation vector for exact symplectic maps that are either finitely differentiable or analytic. The bootstrap of regularity is stated in Theorem 50. To prove Theorem 50, first in Theorem 49 we prove a finitely differentiable version of the local uniqueness of invariant tori for symplectic maps. Theorems 49 and 50 are similar to Theorems 4 and 5 in [17]. However, while the results in [17] are stated and proved for Hamiltonian vector fields written in the Lagrangian formalism, Theorems 49 and 50 in the present work are stated and proved for exact symplectic maps that are not necessarily written either in action-angle variables or as perturbation of integrable ones. Moreover, rather that assuming the existence of a finitely differentiable invariant torus, we assume the existence of a finitely differentiable torus which is approximately invariant.

3. Smoothing geometric diffeomorphisms

In this section we show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving or contact can be approximated by analytic diffeomorphisms having the same geometric property. We give quantitative properties of the approximation in terms of the degree of differentiability of the approximated functions.

Since obtaining such geometric approximating functions involves many technicalities, we have divided the present section as follows. In Section 3.1 we define the norms used and set the conditions on the domain of definition of the diffeomorphism to be smoothed. In Section 3.2 the geometric smoothing results are stated. The technical part of the proofs is given in Section 3.3 and the proofs are concluded in Sections 3.4 and 3.5.

3.1. Setting

Informally, the method we use to smooth symplectic, volume-preserving or contact diffeomorphism with analytic diffeomorphism having the same geometric property is the following. First, for $t \geqslant 1$, we define a linear operator S_t that takes finitely differentiable functions into analytic ones and such that $S_t[f]$ tends to f when t goes to infinity. Then, if f is a finitely differentiable symplectic, volume-preserving, or contact diffeomorphism we find, for t sufficiently large, a diffeomorphism φ_t such that $\varphi_t^*(S_t[f]^*\Omega) = f^*\Omega$. The analytic approximating functions satisfying the same geometric property of f are then defined by $S_t[f] \circ \varphi_t$. In view of the applications we are interested in symplectic, volume-preserving or contact diffeomorphisms defined on either $\mathbb{T}^d$, $\mathbb{U} \subset \mathbb{R}^d$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$. First, by using the convolution operator with an analytic kernel, we define S_t for continuous and bounded functions defined on $\mathbb{R}^d$. It turns out that, if f is a $\mathbb{Z}^d$-periodic (or partially periodic) continuous and bounded function defined on $\mathbb{R}^d$ then $S_t[f]$ is also $\mathbb{Z}^d$-periodic (respectively, partially periodic). Hence to extend the definition of S_t to torus maps we use lifts of torus maps to $\mathbb{R}^d$ (the universal covering of $\mathbb{T}^d$). To define S_t on functions with domain $U \subset \mathbb{R}^d$ we use a linear bounded extension operator. Then, by taking lifts, the definition of S_t is extended to functions defined on the annulus $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$. Before making these definitions explicit, let us introduce the Banach spaces of functions we work with.

Definition 1. Let $\mathbb{Z}_+$ denote the set of positive integers. Given $U \subset \mathbb{C}^d$ an open set, $C^0(U)$ denotes the space of continuous functions $f : U \to \mathbb{R}$, such that

$$|f|_{C^0(U)} \stackrel{\text{def}}{=} \sup_{x \in U} \bigl|f(x)\bigr| < \infty.$$

For $\ell \in \mathbb{N}$, $C^\ell(U)$ denotes the space of functions $f : U \to \mathbb{R}$ with continuous derivatives up to order ℓ such that

$$|f|_{C^\ell(U)} \stackrel{\text{def}}{=} \sup_{\substack{x \in U \\ |k| \leqslant \ell}} \bigl\{\bigl|D^k f(x)\bigr|\bigr\} < \infty.$$

Let $\ell = p + \alpha$, with $p \in \mathbb{Z}_+$ and $0 < \alpha < 1$. Define the Hölder space $C^\ell(U)$ to be the set of all functions $f : U \to \mathbb{R}$ with continuous derivatives up to order p for which

$$|f|_{C^\ell(U)} \stackrel{\text{def}}{=} |f|_{C^p} + \sup_{\substack{x, y \in U,\ x \neq y \\ |k| = p}} \left\{ \frac{|D^k f(x) - D^k f(y)|}{|x - y|^\alpha} \right\} < \infty.$$

For $\rho > 0$ and $U \subseteq \mathbb{R}^d$ let $U + \rho$ denote the complex strip:

$$U + \rho = \{x + iy \in \mathbb{C}^d\colon\ x \in U,\ |y| < \rho\}.$$

Definition 2. Let $\ell \geqslant 0$. Given $U \subseteq \mathbb{R}^d$ open, define the Banach space $\mathcal{A}(U + \rho, C^\ell)$ to be the set of all holomorphic functions $f : U + \rho \to \mathbb{C}$ which are real-valued on U (i.e. $\overline{f(x)} = f(\bar{x})$ for all $x \in U$) and such that $|f|_{C^\ell(U+\rho)} < \infty$.

For a matrix or vector-valued function G with components $G_{i,j}$ in either $C^\ell(U)$ or in $\mathcal{A}(U + \rho, C^\ell)$ we use the norm, respectively,

$$|G|_{C^\ell(U)} \stackrel{\text{def}}{=} \max_{i,j} |G_{i,j}|_{C^\ell(U)} \quad \text{or} \quad |G|_{C^\ell(U+\rho)} \stackrel{\text{def}}{=} \max_{i,j} |G_{i,j}|_{C^\ell(U+\rho)}.$$

The space of all functions $g = (g_1, \ldots, g_d) : V \subseteq \mathbb{C}^n \to U \subseteq \mathbb{C}^d$ such that $g_i \in C^\ell(U)$, for $i = 1, \ldots, d$, is denoted by $C^\ell(U, V)$. Since it will not lead to confusion, $\mathcal{A}(U + \rho, C^\ell)$ will also denote the set of functions $g = (g_1, \ldots, g_d)$ with components in $\mathcal{A}(U + \rho, C^\ell)$.

Definition 3. Let $U \subset \mathbb{R}^m$. A *lift* of a continuous map f, defined on the annulus $\mathbb{T}^n \times U$, to $\mathbb{R}^n \times U$ (the universal cover of $\mathbb{T}^n \times U$) is a continuous map $\hat{f}$ defined on $\mathbb{R}^n \times U$ such that:

(i) $\hat{f}(x, y) = f(x \bmod \mathbb{Z}^n, y)$, if f takes values in $\mathbb{R}$.
(ii) $\hat{f}(x, y) \bmod \mathbb{Z} = f(x \bmod \mathbb{Z}^n, y)$ for $(x, y) \in \mathbb{R}^n \times U$, if f takes values in $\mathbb{T}$.

It is well known that given a continuous map f defined on $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^m$, any lift $\hat{f}$: $\mathbb{R}^n \times U$ has the following form

$$\hat{f}(x, y) = Px + u(x, y), \quad (x, y) \in \mathbb{R}^n \times U, \tag{1}$$

where $u \in C^0(\mathbb{R}^n \times U, \mathbb{R}^s)$ is $\mathbb{Z}$-periodic in the first n-variables and P is an $(n \times 1)$-matrix with components in $\mathbb{Z}$. Furthermore, if f takes values in $\mathbb{R}$ then $P = 0$. Moreover, if f has additional regularity, the corresponding function u has the same regularity. Even though lifts of continuous annulus maps are not unique, they differ by a constant vector in $\mathbb{Z}$. This, together with the fact that any map of the form (1) defines an annulus map, enable us to work with lifts of torus and annulus maps (considering torus maps as particular cases of annulus maps). For notational reasons we use the same symbol to denote the annulus (torus) map and a lift of it.

Definition 4. For $\ell \geqslant 0$, denote by $C^\ell(\mathbb{T}^n \times U, V)$, and $\mathcal{A}(\mathbb{T}^n \times U + \rho, C^\ell)$ the set of annulus maps with lift of the form (1) with $u \in C^\ell(\mathbb{R}^n \times U, V)$ (in $\mathcal{A}(\mathbb{R}^n \times U + \rho, C^\ell)$, respectively) $\mathbb{Z}$-periodic in the first n-variables. The corresponding norms are defined as follows:

$$|f|_{C^\ell(\mathbb{T}^d \times U)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d \times U)}$$

and

$$|f|_{C^\ell(\mathbb{T}^d \times U+\rho)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d \times U+\rho)}.$$

In the case of torus maps, denote by $C^\ell(\mathbb{T}^d, V)$, and $\mathcal{A}(\mathbb{T}^d + \rho, C^\ell)$ the set of torus maps with lift of the following form:

$$f(x) = Px + u(x), \tag{2}$$

where P is a matrix with components in $\mathbb{Z}$ and $u \in C^\ell(\mathbb{R}^d, V)$ ($u \in \mathcal{A}(\mathbb{R}^d + \rho, C^\ell)$, respectively) is $\mathbb{Z}^d$-periodic. The corresponding norms are defined as follows:

$$|f|_{C^\ell(\mathbb{T}^d)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d)}$$

and

$$|f|_{C^\ell(\mathbb{T}^d+\rho)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d+\rho)}.$$

Moreover, for $r \geqslant 0$ denote by $\mathrm{Diff}^r(\mathbb{U})$ the set of C^r-diffeomorphisms of $\mathbb{U}$, where $\mathbb{U}$ is either $\mathbb{U} \subseteq \mathbb{R}^d$ open, $\mathbb{T}^d$, or $\mathbb{T}^n \times U$.

For $U \subseteq \mathbb{R}^d$ open, denote by $\Lambda^k(U)$ the space of real analytic k-forms in U. Let $\Omega \in \Lambda^k(U)$ have the following form:

$$\Omega(x) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant d} \Omega_{\mathbf{i}}(x)\, dx_{\mathbf{i}},$$

where $\mathbf{i}$ represents the multi-index $(i_1, \ldots, i_k)$ and $dx_{\mathbf{i}} \stackrel{\text{def}}{=} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. If $|\Omega_{\mathbf{i}}|_{C^\ell(U)} < \infty$ for all $1 \leqslant i_1 < \cdots < i_k \leqslant d$, define

$$|\Omega|_{C^\ell(U)} \stackrel{\text{def}}{=} \max_{1 \leqslant i_1 < \cdots < i_k \leqslant d} |\Omega_{\mathbf{i}}|_{C^\ell(U)}.$$

Definition 5. Let $\mathbb{U}$ be either $\mathbb{U} \subseteq \mathbb{R}^d$ open, $\mathbb{T}^d$, $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ an open set. We say that the linear operator $S_t : C^\ell(\mathbb{U}) \to \mathcal{A}(\mathbb{U} + t^{-1}, C^0)$ is an *analytic smoothing* operator if the following properties hold for any $f \in C^\ell(U)$:

(i) $|S_t[f]|_{C^0(\mathbb{U}+t^{-1})} \leqslant c|f|_{C^\ell(\mathbb{U})}$ for all $t \geqslant 1$;
(ii) $\lim_{t\to\infty} |(S_t - \mathrm{Id})[f]|_{C^0(\mathbb{U})} = 0$;
(iii) $|S_t - S_\tau[f]|_{C^0(\mathbb{U}+\tau^{-1})} \leqslant c|f|_{C^\ell(\mathbb{U})} t^{-\ell}$, for $\tau \geqslant t \geqslant 1$,

for some constant c depending on ℓ and $\mathbb{U}$, but independent of t.

Now we define the smoothing operator S_t we work with. First we define $S_t[f]$ for $f \in C^0(\mathbb{R}^d)$.

Definition 6. Let $u : \mathbb{R}^d \to \mathbb{R}$ be C^∞, even, identically equal to 1 in a neighbourhood of the origin, and with support contained in the ball with center in the origin and radius 1. Let $\hat{u} : \mathbb{R}^d \to \mathbb{R}$

be the Fourier transform of u and denote by s the holomorphic continuation of $\hat{u}$. Define the linear operator S_t as

$$S_t[f](z) \stackrel{\text{def}}{=} t^d \int_{\mathbb{R}^d} s\big(t(y-z)\big) f(y)\,dy, \quad \text{for } f \in C^0\big(\mathbb{R}^d\big). \tag{3}$$

Applying obvious modifications, Definition 6 can be extended to functions in $C^0(\mathbb{R}^n, \mathbb{R}^d)$. In the sequel these latter operators are denoted by the same symbol S_t. We now summarize some elementary properties of S_t that follow from Definition 6.

Remark 7.

(i) S_t transforms functions in $C^0(\mathbb{R}^d)$ into entire functions on $\mathbb{C}^d$.
(ii) Using the change of variables $\xi = t\,\mathrm{Re}(y-z) = ty - t\,\mathrm{Re}(z)$, one has for $f \in C^0(\mathbb{R}^d)$

$$S_t[f](z) = \int_{\mathbb{R}^d} s\big(\xi - it\,\mathrm{Im}(z)\big) f\big(\mathrm{Re}(z) + \xi/t\big)\,d\xi. \tag{4}$$

(iii) S_t commutes with constant coefficient differential operators.
(iv) S_t acts as the identity on polynomials.
(v) From (4) one has that S_t takes (partially) periodic functions into (partially) periodic functions.
(vi) From (4) we have that $S_t[f](x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d$.

Remark 8. In the applications of Moser's smoothing method to KAM theory it is of particular importance to know how to define S_t for functions defined on the d-dimensional torus $\mathbb{T}^d$ as well as functions defined on $\mathbb{T}^n \times \mathbb{R}^m$. Notice that since S_t in Definition 6 acts as the identity on polynomials and it takes partially periodic functions into partially periodic functions, we have that for any annulus map f, with lift of the form (1) $S_t[\hat{f}]$ is also of the form (1):

$$S_t[\hat{f}](x, y) = Px + S_t[u](x, y).$$

Hence to extend the definition of S_t to torus maps as well as to maps defined on $\mathbb{T}^n \times \mathbb{R}^m$, we apply S_t in Definition 6 to any lift of it. This is well defined because two lifts of the same torus map (respectively annulus map) differ by a constant vector with components in $\mathbb{Z}$.

Definition 9. Let $\ell > 0$ be not an integer. A *bounded linear extension operator* is a linear operator $\mathscr{E}_U : C^\ell(U) \to C^\ell(\mathbb{R}^d)$ such that $\mathscr{E}_U(f)|_U = f$ for all $f \in C^\ell(U)$ and $|\mathscr{E}_U(f)|_{C^\ell(\mathbb{R}^d)} \leqslant c_U |f|_{C^\ell(U)}$.

In order to extend the definition of the linear operator S_t to functions defined on $U \subset \mathbb{R}^d$ and to the annulus $\mathbb{T}^n \times U$, it suffices to have a linear bounded linear extension operator from $C^\ell(U)$ to $C^\ell(\mathbb{R}^d)$. The sufficient condition we adopt here to have such extension operator is that given in Theorem 14.9 in [1]. It amounts to the regularity of the boundary of U.

Definition 10. Let $\varrho : \mathbb{R}^d \to \mathbb{R}$ be a function with continuous derivatives up to order m, for some $m \in \mathbb{N}$, and assume that $\operatorname{grad} \varrho(x) \neq 0$ for all $x \in \{x\colon \varrho(x) = 0\}$. The set $U = \{x \in \mathbb{R}^d\colon \varrho(x) \leqslant 0\}$ is called a closed domain with C^m-boundary. An open domain is defined by $\{x \in \mathbb{R}^d\colon \varrho(x) < 0\}$.

The following result guarantees the existence of a bounded extension operator for functions with domain of definition $U \subset \mathbb{R}^d$ provided that U has smooth boundary. For a proof we refer the reader to [1,3].

Theorem 11. *If $0 < \ell < m \in \mathbb{N}$ with $\ell \notin \mathbb{N}$, and $U \subset \mathbb{R}^d$ has C^m-boundary, then there is a linear extension operator $\mathscr{E}_U^\ell : C^\ell(U) \to C^\ell(\mathbb{R}^d)$ such that*

$$\left|\mathscr{E}_U^\ell(f)\right|_{C^\ell(\mathbb{R}^d)} \leqslant c_U |f|_{C^\ell(U)}, \tag{5}$$

for some constant c_U, depending on U.

Hence, for functions that are defined on a subset of $\mathbb{R}^d$ with regular boundary we have the following

Definition 12. Let $0 < \ell < m$, with $m \in \mathbb{N}$ and $\ell \notin \mathbb{N}$. Let $U \subset \mathbb{R}^d$ be an open domain with C^m-boundary and $\mathscr{E}_U^\ell$ a linear extension operator as in Theorem 11. For $f \in C^\ell(U)$ and for any $x \in \mathbb{C}^d$ we define

$$\hat{S}_t[f](x) \stackrel{\text{def}}{=} S_t\left[\mathscr{E}_U^\ell(f)\right](x), \tag{6}$$

where S_t is as in (3).

The following remark is related to Remark 7.

Remark 13. Notice that the operator $\hat{S}_t$, defined in Definition 12 for functions in $C^\ell(U)$, satisfies the following properties:

(i) $\hat{S}_t$ is linear.
(ii) $\hat{S}_t$ transforms functions in $C^\ell(U)$ into entire functions on $\mathbb{C}^d$.

Remark 14. Notice that if $U \subset \mathbb{R}^{d-n}$ is an open domain with C^m-boundary then $\mathbb{U} = \mathbb{R}^n \times U \subset \mathbb{R}^d$ also is an open domain with C^m-boundary. Moreover, it follows from Remark 8 and Definition 12 that, if $u(x, y)$ is defined for $(x, y) \in \mathbb{U} = \mathbb{R}^n \times U$ and $\mathbb{Z}^n$-periodic on the x-variable and $\hat{S}_t$ is as in Definition 12, then $\hat{S}_t[u]$ is also $\mathbb{Z}^n$-periodic on the x-variable. Indeed, since $\mathscr{E}_{\mathbb{U}}^\ell(u)(x, y) = u(x, y)$ for all $(x, y) \in \mathbb{R}^n \times U$, we have that $\mathscr{E}_{\mathbb{U}}^\ell(u)$ is $\mathbb{Z}^n$-periodic in the x-variable, where $\mathscr{E}_{\mathbb{U}}^\ell$ is as in Theorem 11. Therefore, any map defined on the annulus $\mathbb{T}^n \times U$ with U as in Theorem 11 and lift given by (1), is smoothed by

$$\hat{S}_t[f](x, y) = Px + S_t\left[\mathscr{E}_{\mathbb{U}}^\ell(u)\right](x, y),$$

where S_t is as in Definition 6 and $\mathscr{E}_{\mathbb{U}}^\ell$ is as in Theorem 11.

Since it will not lead to confusion, the operator $\hat{S}_t$ defined in (6) will be denoted (dropping the hat) as the operator S_t in (3).

Remark 15. Summarizing, a function f is smoothed, depending on its domain of definition as follows:

(i) If $f \in C^0(\mathbb{R}^d)$, $S_t[f]$ is given by (3).
(ii) If $f \in C^\ell(\mathbb{U})$, with $\mathbb{U} \subset \mathbb{R}^d$ an open domain with C^m-boundary, we define

$$S_t[f] = S_t\big[\mathscr{E}^\ell_{\mathbb{U}}(f)\big],$$

where $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 11 and S_t on the right-hand side is defined by (3).
(iii) If $f \in C^0(\mathbb{T}^d)$, with lift as in (2), where $u \in C^0(\mathbb{R}^d)$ is $\mathbb{Z}^d$-periodic, and P a $(d \times 1)$-matrix with components in $\mathbb{Z}$, then

$$S_t[f](x) \stackrel{\text{def}}{=} Px + S_t[u](x),$$

where S_t on the right-hand side is defined by (3).
(iv) For $U \subset \mathbb{R}^{d-n}$ an open domain with C^m-boundary and $f \in C^\ell(\mathbb{T}^n \times U)$, with lift given by (1) where $u \in C^\ell(\mathbb{R}^n \times U)$ $\mathbb{Z}$-periodic on the first n-variables, and P an $(n \times 1)$-matrix with components in $\mathbb{Z}$, we define

$$S_t[f](x, y) \stackrel{\text{def}}{=} Px + S_t\big[\mathscr{E}^\ell_{\mathbb{U}}(u)\big](x, y),$$

where $\mathbb{U} = \mathbb{R}^n \times U$, $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 11, and S_t on the right-hand side is defined by (3).

To define an analytic smoothing operator such that it takes finitely differentiable diffeomorphisms preserving either an exact symplectic, volume or contact form into analytic diffeomorphisms preserving the same structure, we need to estimate the C^ℓ-norm of the composition of two functions in terms of the C^ℓ-norms of the composed functions. We use an estimate given in [21], which is guaranteed to hold for functions defined on domains satisfying a geometric condition that is established in the following definition.

Definition 16. Given $U \subseteq \mathbb{C}^n$, for $x, y \in U$ denote by $d_U(x, y)$ the minimum length of arcs inside U joining x and y. We say that U is *compensated* if there exists a constant c_U such that $d_U(x, y) \leqslant c_U|x - y|$, for all $x, y \in U$.

We finish this section recalling some geometric definitions.

Definition 17.

(i) Given a k-form Ω on a d-dimensional manifold, denote by $\mathscr{I}_\Omega$ the application $X \to i_X\Omega$, sending the vector field X into the inner product $i_X\Omega \stackrel{\text{def}}{=} \Omega(x)(X(x), \cdot)$. A k-form Ω is *non-degenerate* if $\mathscr{I}_\Omega$ is an isomorphism.
(ii) A *volume* element on a d-dimensional manifold is a d-form which is non-degenerate.
(iii) A *symplectic* form on a $2n$-dimensional manifold is a non-degenerate closed 2-form.

(iv) A *contact* form on a $(2n+1)$-dimensional manifold is a 1-form Ω, such that $\Omega \wedge (d\Omega)^n$ is a volume element.
(v) A diffeomorphism f of a contact manifold (M, Ω) is a contact diffeomorphism if there exists a nowhere zero function $\lambda : M \to \mathbb{R}$ such that $f^*\Omega = \lambda\Omega$.
(vi) Let $\Omega = d\alpha$ be an exact symplectic form on a symplectic manifold. The diffeomorphism f is exact symplectic if $f^*\alpha - \alpha$ is an exact 1-form.

3.2. Statement of results

In this section we formulate the results guaranteeing the existence of an analytic smoothing operator that preserves the prescribed geometric structure. In Theorem 18 the symplectic and volume cases are considered; the contact case is considered in Theorem 20.

Theorem 18. *Let $2 < \ell < m$, with $m \in \mathbb{N}$ and $\ell \notin \mathbb{N}$, and let $C, \beta > 0$ and $1 < \mu < \ell - 1$ be given. Assume that the following hypotheses hold*:

H1. $\mathbb{U}$ *is either*: (i) $\mathbb{T}^d$, (ii) *a compensated bounded open domain in $\mathbb{R}^d$ with C^m-boundary* (*see Definitions* 10 *and* 16), *or* (iii) $\mathbb{T}^n \times U$, *with $U \subset \mathbb{R}^{d-n}$ a compensated bounded open domain with C^m-boundary and $n < d$.*
H2. *$\mathbb{V}$ is C^m-diffeomorphic to $\mathbb{U}$ and such that $\mathbb{U} \subseteq \mathbb{V}$. $\Omega = d\alpha$ is either a real analytic exact symplectic form* (*with $d = 2n$*) *or volume element on $\mathbb{V}$ such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)} < \infty$ for some $\rho > 0$.*
H3. *Let $\mathscr{I}_\Omega$ be as in Definition* 17 *and let $\mathscr{I}_\Omega^{-1}$ denote the inverse of $\mathscr{I}_\Omega$. Let $k = 2$ if Ω is a symplectic form, and $k = d$ if Ω is a volume form and assume that for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho')} < \infty$, with $\rho' \geqslant 0$, the following holds*

$$\left|\mathscr{I}_\Omega^{-1}\theta\right|_{C^0(\mathbb{U}+\rho')} \leqslant M_\Omega |\theta|_{C^0(\mathbb{U}+\rho')}.$$

Then, there exist two constants $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$ and $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and a family of nonlinear operators $\{T_t\}_{t \geqslant t^*}$, taking functions belonging to

$$\left\{f \in \mathrm{Diff}^\ell(\mathbb{U}) \colon |f|_{C^\ell(\mathbb{U})} \leqslant \beta,\ f^*\Omega = \Omega,\ \textit{closure of } f(\mathbb{U}) \subseteq \mathbb{V}\right\}$$

into real analytic functions. Moreover, if $\mathbb{U}_t$ is defined as follows:

$$\mathbb{U}_t \stackrel{\text{def}}{=} \begin{cases} \mathbb{T}^d, & \mathbb{U} = \mathbb{T}^d, \\ \{x \in \mathbb{U} \colon \bar{B}(x, t^{-1}) \subset \mathbb{U}\}, & \mathbb{U} \subset \mathbb{R}^d, \\ \mathbb{T}^n \times \{x \in U \colon \bar{B}(x, t^{-1}) \subset U\}, & \mathbb{U} = \mathbb{T}^n \times U, \end{cases} \tag{7}$$

where $\bar{B}(x, t^{-1})$ represents the closed ball with center at x and radius t^{-1}, then the following properties hold:

T0. *$T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$.*
T1. *$T_t[f]^*\Omega = f^*\Omega$.*
T2. *$|T_t[f]|_{C^1(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa M_f$.*
T3. *$|T_t[f] - S_t[f]|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa M_f t^{-\mu+1}$.*
T4. *If $2 < \mu < \ell - 1$, then $|T_t[f]|_{C^2(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa M_f$.*

T5. $|(T_t - \mathrm{Id})[f]|_{C^r(\mathbb{U}_t)} \leqslant \kappa M_f t^{-(\mu-r-1)}$, *for all* $0 \leqslant r \leqslant \mu - 1$.
T6. $|(T_\tau - T_t)[f]|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \kappa M_f t^{-\mu+1}$, *for all* $\tau \geqslant t \geqslant t^*$.
T7. *If f is exact symplectic so is $T_t[f]$.*

Here M_f depends on ℓ, k, $|\Omega|_{C^\ell(\mathbb{U}+\rho)}$, and β, but it is independent of t.

Remark 19. We remark that, in hypothesis H2 of Theorem 18, if $\mathbb{U} = \mathbb{T}^d$ then $\mathbb{V}$ can be chosen to be also $\mathbb{T}^d$. Actually, we asked Ω to be defined on a neighbourhood of $\mathbb{U}$ that contains the closure of $f(\mathbb{U})$, to guarantee that $S_t[f](\mathbb{U})$ is contained in the domain of definition of Ω, for t sufficiently large. And so $S_t[f]^*\Omega$ is defined on $\mathbb{U}$. If Ω is defined only on $\mathbb{U}$ and it cannot be extended to a neighbourhood of $\mathbb{U}$, then $S_t[f]^*\Omega$ is defined on $S_t[f]^{-1}(\mathbb{U})$. By modifying the definition of $\mathbb{U}_t$ in (7), the proof of Theorem 18 given in Section 3.4 also works in this latter case. However this just yields a more complicated notation and does not change the proof of Theorem 18. To avoid this notational complication we assume that Ω is defined on a neighbourhood of $\mathbb{U}$ as in H2 in Theorem 18.

Theorem 20. *Let m, ℓ, $\mathbb{V}$, and $\mathbb{U}$ be as in Theorem* 18. *Let Ω be a contact form on $\mathbb{V}$ such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)}, |d\Omega|_{C^\ell(\mathbb{V}+\rho)} < \infty$, for some $\rho > 0$. Assume that for any $\theta \in \mathscr{I}_{d\Omega}(\mathrm{Ker}(\Omega))$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho')} < \infty$, the following holds*:

$$\left|(\mathscr{I}_{d\Omega}|_{\mathrm{Ker}(\Omega)})^{-1}\theta\right|_{C^0(\mathbb{U}+\rho')} \leqslant M_\Omega |\theta|_{C^0(\mathbb{U}+\rho')}.$$

Then, given $N, C, \beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa' = \kappa'(d, \ell, C, \beta, \mu, M_\Omega)$ and $t^{**} = t^{**}(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)}, |d\Omega|_{C^\ell(\mathbb{U}+\rho)})$, and a family of – nonlinear – operators $\{T_t\}_{t\geqslant t^{**}}$, taking contact diffeomorphisms belonging to the set of diffeomorphisms $f \in \mathrm{Diff}^\ell(\mathbb{U})$ such that: (i) $|f|_{C^\ell(\mathbb{U})} \leqslant \beta$ *and* (ii) $\mathbb{V}$ *contains the closure of $f(\mathbb{U})$ into real analytic functions such that properties* T0–T6 *in Theorem* 18 *hold.*

3.3. Analytic smoothing

This section contains the technical part of our proof of Theorem 18 and Theorem 20. We begin by collecting the properties of the operator S_t defined in Section 3.1 (see Remark 15). First we prove that S_t is a linear smoothing operator (see Definition 5) and then, using the fact that S_t is a linear smoothing operator, we show that given a k-form Ω, the C^0-norm of the k-form given by

$$S_t[f]^*\Omega - f^*\Omega \tag{8}$$

goes to zero as t goes to infinity. However, to prove Theorem 18 we need more accurate estimates. Actually, as we will see in Section 3.4, we need an estimate for the C^0-norm of (8) on complex strips, which is given in Proposition 29. To obtain such an estimate we extend the definition of S_t to k-forms and prove several analytic estimates which are given in Section 3.3.1. Estimates of particular importance are those given in Proposition 31, and in Proposition 37. Proposition 31 contains an estimate of the norm of $S_t[f]^*\Omega - S_t[f^*\Omega]$ on complex strips of width Ct^{-1} for arbitrary $C \geqslant 0$. In Proposition 37 we give an estimate of the difference between smoothing a composition of two functions and composing their smoothings.

To describe the behaviour of S_t we find it very useful to write $S_t[f]$ in terms of the Taylor expansion of f, for $f \in C^\ell(\mathbb{R}^d)$. This is done in the following:

Lemma 21. *For any $f \in C^\ell(\mathbb{R}^d)$, with ℓ not an integer, we have*

$$S_t[f](z) = P_{f,\ell}\big(\operatorname{Re}(z), i\operatorname{Im}(z)\big) + \hat{R}_{f,\ell}(z,t), \tag{9}$$

where

$$P_{f,\ell}(x,y) \stackrel{\text{def}}{=} \sum_{|k|_1<\ell} \frac{1}{k!} D^k f(x) y^k$$

and

$$\big|\hat{R}_{f,\ell}(z,t)\big| \leqslant \tilde{c}|f|_{C^\ell(\mathbb{R}^d)} t^{-\ell} e^{t|\operatorname{Im}(z)|}, \tag{10}$$

where $\tilde{c} = \tilde{c}(\ell, d)$.

Proof. Following [4,19], we apply Taylor's Theorem to f:

$$f(x+y) = P_{f,\ell}(x,y) + R_{f,\ell}(x;y),$$

where $R_{f,\ell}$ is the remainder. Then, using (4) and since S_t acts as the identity on polynomials, we have (9) with

$$\hat{R}_{f,\ell}(z,t) \stackrel{\text{def}}{=} \int s\big(\xi - it\operatorname{Im}(z)\big) R_{f,\ell}\big(\operatorname{Re}(z); \xi/t\big)\, d\xi.$$

We note that from the properties of s in Definition 6, for any $r, N > 0$ there exists a constant $c = c(r,N) > 0$ such that for all $k \in \mathbb{N}^d$ with $|k|_1 \leqslant r$ then

$$\big|D^k s(z)\big| \leqslant c\big(1 + \big|\operatorname{Re}(z)\big|\big)^{-N} e^{|\operatorname{Im}(z)|}.$$

Then, from Taylor's Theorem we have

$$\begin{aligned}
\big|\hat{R}_{f,\ell}(z,t)\big| &\leqslant ct^{-\ell}|f|_{C^\ell(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d} \big|s\big(\xi - it\operatorname{Im}(z)\big)\big| |\xi|^\ell\, d\xi \\
&\leqslant t^{-\ell}|f|_{C^\ell(\mathbb{R}^d)} c e^{|t\operatorname{Im}(z)|} \int\limits_{\mathbb{R}^d} \frac{|\xi|^\ell}{(1+|\xi|)^N}\, d\xi \\
&\leqslant \tilde{c}|f|_{C^\ell(\mathbb{R}^d)} t^{-\ell} e^{|t\operatorname{Im}(z)|},
\end{aligned}$$

where we have fixed $N > \ell + d$. $\square$

Remark 22. The constants appearing in our estimates depend on certain quantities. In particular if $f \in C^\ell(\mathbb{U})$, with $\mathbb{U}$ an open domain with smooth boundary, these constants also depend on $\mathbb{U}$. In what follows we do not write explicitly this dependence and represent a generic constant by κ.

The following result ensures that S_t is an analytic linear smoothing operator in the sense of [4]. The case in which S_t is applied to functions in $C^0(\mathbb{R}^d)$ is proved in [4].

Proposition 23. *Let $1 < \ell < m$ with $\ell \notin \mathbb{N}$, $m \in \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in* H1 *in Theorem* 18. *Assume that S_t is as in Remark* 15. *Then, for any $C \geqslant 0$, there exists a constant $\kappa = \kappa(d, \ell, C)$ such that for all $t \geqslant 1$ and $f \in C^\ell(\mathbb{U})$ the following holds*:

(i) $|(S_t - \mathrm{Id})[f]|_{C^r(\mathbb{U})} \leqslant \kappa |f|_{C^\ell(\mathbb{U})} t^{-\ell+r}$, $0 \leqslant r < \ell$.
(ii) $|S_t[f]|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa |f|_{C^0(\mathbb{U})}$.
(iii) $|(S_\tau - S_t)[f]|_{C^0(\mathbb{U}+\tau^{-1})} \leqslant \kappa |f|_{C^\ell(\mathbb{U})} t^{-\ell}$, *for all $\tau \geqslant t$.*
(iv) $|\mathrm{Im}(S_t[f])|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa C t^{-1} |f|_{C^\ell(\mathbb{U})}$.

Proof. We first prove Proposition 23 for functions in $C^\ell(\mathbb{U})$, with $\mathbb{U} \subset \mathbb{R}^d$ a compensated open domain with C^m-boundary. In this case S_t is defined by Eq. (6). The linearity of S_t follows from the linearity of the extension operator $\mathscr{E}^\ell_U$ in Theorem 11 and from the linearity of the convolution operator. To prove part (i), first notice that if $f \in C^\ell(\mathbb{U})$ and $x \in \mathbb{U}$ then $\mathscr{E}^\ell_{\mathbb{U}}(f)(x) = f(x)$, then using the fact that part (i) holds for functions in $C^0(\mathbb{R}^d)$ and estimate (5) we have

$$\begin{aligned}\left|(S_t - \mathrm{Id})[f]\right|_{C^r(\mathbb{U})} &= \left|(S_t - \mathrm{Id})\left[\mathscr{E}^\ell_{\mathbb{U}}(f)\right]\right|_{C^r(\mathbb{U})} \\ &\leqslant \kappa' \left|\mathscr{E}^\ell_{\mathbb{U}}(f)\right|_{C^\ell(\mathbb{R}^d)} t^{-\ell+r} \\ &\leqslant \kappa |f|_{C^\ell(\mathbb{U})} t^{-\ell+r}.\end{aligned}$$

To prove part (ii) we use (4) and Theorem 11 to obtain

$$\left|S_t[f]\right|_{C^0(\mathbb{R}^d+Ct^{-1})} \leqslant \left(\sup_{0\leqslant\eta<C} \int_{\mathbb{R}^d} \left|s(\xi - i\eta)\right| d\xi\right) \left|\mathscr{E}^\ell_U(f)\right|_{C^0(\mathbb{R}^d)} \leqslant \kappa |f|_{C^0(U)}.$$

Part (iii) is a consequence of Lemma 21 and Theorem 11. To prove part (iv) we use the Mean Value Theorem and the fact that the convolution commutes with the derivative to obtain

$$\begin{aligned}\left|\mathrm{Im}\big(S_t[f]\big)\right|_{C^0(\mathbb{R}^d+Ct^{-1})} &\leqslant Ct^{-1}\left|DS_t[f]\right|_{C^0(\mathbb{R}^d+Ct^{-1})} \\ &\leqslant Ct^{-1}\left|D\mathscr{E}^\ell_{\mathbb{U}}(f)\right|_{C^0(\mathbb{R}^d)} \\ &\leqslant Ct^{-1}\left|\mathscr{E}^\ell_{\mathbb{U}}(f)\right|_{C^\ell(\mathbb{R}^d)}.\end{aligned}$$

To prove Proposition 23 for $f \in C^\ell(\mathbb{T}^d)$ we use a lift of f. Let $f \in C^\ell(\mathbb{T}^d)$ with lift (see Definition 3) given by $Px + u(x)$, where P is a $(d \times 1)$-matrix with components in $\mathbb{Z}$ and $u \in C^\ell(\mathbb{R}^d)$ is a $\mathbb{Z}^d$-periodic function. Then (see part (iii) in Remark 15):

$$(S_t - \mathrm{Id})[f](x) = Px + S_t[u](x) - \big(Px + u(x)\big) = (S_t - \mathrm{Id})[u]. \tag{11}$$

Moreover from Definition 4 one has

$$\begin{gathered}\left|S_t[f]\right|_{C^0(\mathbb{T}^d+Ct^{-1})} \leqslant |P| + \left|S_t[u]\right|_{C^0(\mathbb{R}^d+Ct^{-1})}, \\ (S_t - S_\tau)[f] = (S_t - S_\tau)[u], \\ \left|\mathrm{Im}\big(S_t[f]\big)\right|_{C^0(\mathbb{T}^d+Ct^{-1})} \leqslant |P|Ct^{-1} + \left|\mathrm{Im}\big(S_t[u]\big)\right|_{C^0(\mathbb{R}^d+Ct^{-1})},\end{gathered}$$

where S_t on the right-hand side is given by (3). Hence properties (i)–(iv) of Proposition 23 follow from the same properties for $u \in C^\ell(\mathbb{R}^d)$ and the fact that $S_t[u]$ is $\mathbb{Z}^d$-periodic. The annulus case $\mathbb{U} = \mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ a bounded compensated open domain with C^m-boundary, is proved in a similar way. □

Remark 24. From (11) we have that in the case that $\mathbb{U} = \mathbb{T}^d$, a better estimate holds than that given in part (i) of Proposition 23:

$$\left|(S_t - \mathrm{Id})[f]\right|_{C^0(\mathbb{U})} \leqslant \kappa |u|_{C^0(\mathbb{R}^d)} t^{-\ell},$$

where u is the periodic part of a lift of f. A similar result holds for $f \in C^\ell(\mathbb{T}^n \times U)$, with U a compensated open domain in $\mathbb{R}^{d-n}$ with C^m-boundary for some $\ell < m \in \mathbb{N}$.

Remark 25. From the proof of Proposition 23 one notices that, if $\mathbb{U} \subset \mathbb{R}^d$ is an open domain with C^m-boundary, then the estimates in parts (ii)–(iv) in Proposition 23 also hold if one replaces $\mathbb{U} + Ct^{-1}$ with $\mathbb{R}^d + Ct^{-1}$.

Remark 26. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 18 and let $\mathbb{V} \subseteq \mathbb{R}^p$ be open, and assume that $f \in C^\ell(\mathbb{U}, \mathbb{V})$. Then for any $\Omega \in \Lambda^k(\mathbb{V})$ one has $f^*\Omega \in \Lambda^k(\mathbb{U})$. Notice that since the domain of definition of $\Omega \circ S_t[f]$ is $S_t[f]^{-1}(\mathbb{V})$, and since we know an estimate of $|S_t[f](x) - f(x)|$ only when $x \in \mathbb{U}$, to estimate the norm of the difference between $S_t[f]^*\Omega(x)$ and $f^*\Omega(x)$ we have to restrict x to be in $S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U} \subseteq \mathbb{U}$. It is not difficult to see that

$$\mathbb{U} = \bigcup_{t \geqslant 1} \left(S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U}\right).$$

Furthermore,

$$S_t[f](\mathbb{U}) \subseteq \mathbb{V} \quad \Longleftrightarrow \quad S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U} = \mathbb{U}.$$

We consider functions $f : \mathbb{U} \to \mathbb{V}$, with $\mathbb{U}$ either $\mathbb{R}^d$ or as in H1 in Theorem 18, and $\mathbb{V}$ either $\mathbb{R}^p$ or H2 in Theorem 18. In Lemma 27 we prove that if moreover $\mathbb{V}$ contains the closure of $f(\mathbb{U})$ then, for t sufficiently large, $S_t[f]^*\Omega \in \Lambda^k(\mathbb{U})$.

Lemma 27. *Let $1 < \ell < m$ with $\ell \notin \mathbb{N}$ and $m \in \mathbb{N}$. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in Theorem 18, let $\mathbb{V}$ be either $\mathbb{V} \subseteq \mathbb{R}^p$ an open subset, or $\mathbb{T}^p$, or $\mathbb{T}^s \times V$, with $V \subset \mathbb{R}^{p-s}$ an open subset. Let $f \in C^\ell(\mathbb{U}, \mathbb{V})$, and assume that $\mathbb{V}$ contains the closure of $f(\mathbb{U})$. Then there exists $\bar{t} = \bar{t}(\ell, d, |f|_{C^\ell(\mathbb{U})}, \mathbb{V})$ such that for all $t \geqslant \bar{t}$ the following holds*:

(i) $S_t[f](\mathbb{U}) \subseteq \mathbb{V}$.
(ii) $S_t[f](\mathbb{U} + Ct^{-1}) \subseteq \mathbb{V} + (C\beta\kappa)t^{-1}$.

Proof. To prove part (i) of Lemma 27 first, notice that from Remark 15 and part (vi) of Remark 7, one has that $S_t[f](x)$ is real if x is real. Hence, if $\mathbb{V} = \mathbb{R}^p$, or $\mathbb{V} = \mathbb{T}^p$, we have $S_t[f](\mathbb{U}) \subseteq \mathbb{V}$.

Now, assume that $\mathbb{V} \subsetneq \mathbb{R}^d$ is open. Then from part (i) in Proposition 23 we have that for $t \geqslant 1$, the following holds:

$$S_t[f](\mathbb{U}) \subset \left\{ y \in \mathbb{R}^d \colon \sup_{x\in\mathbb{U}} \bigl| y - f(x) \bigr| \leqslant \kappa |f|_{C^\ell(U)} t^{-\ell} \right\}. \tag{12}$$

Hence, if $\mathbb{V} \subsetneq \mathbb{R}^d$ is open and the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, one has from (12) that for t sufficiently large $S_t[f](\mathbb{U}) \subset \mathbb{V}$. By taking coordinate functions, the case $\mathbb{V} = \mathbb{T}^s \times V$, with $V \subset \mathbb{R}^{p-s}$ an open subset, follows from the previous two cases.

Part (ii) of Lemma 27 follows from part (i) of Lemma 27 and part (iv) of Proposition 23. □

A consequence of Proposition 23 is the following.

Proposition 28. *Let $1 < \ell \notin \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in* H1 *in Theorem* 18. *Let $\mathbb{V}$ be as in Lemma* 27, *and let $\Omega \in \Lambda^k(\mathbb{V})$ be such that $|\Omega|_{C^1(\mathbb{V}+\rho)} < \infty$, for some $\rho > 0$. Let $f \in C^\ell(\mathbb{U}, \mathbb{V})$, and assume that $\mathbb{V}$ contains the closure of $f(\mathbb{U})$. Then there exist two positive constants $\kappa = \kappa(d, \ell, k, |\Omega|_{C^1(\mathbb{V}+\rho)})$ and $\bar{\bar{t}} = \bar{\bar{t}}(d, \ell, \rho, |f|_{C^\ell(\mathbb{U})}, \mathbb{V})$ such that for all $t \geqslant \bar{\bar{t}}$ the following holds*:

(i) $|S_t[f]^*\Omega - f^*\Omega|_{C^0(\mathbb{U})} \leqslant \kappa (t^{-k(\ell-1)} |f|^k_{C^\ell(\mathbb{U})} + t^{-\ell} |f|_{C^\ell(\mathbb{U})})$.
(ii) $|S_t[f]^*\Omega|_{C^0(\mathbb{U}+t^{-1})} \leqslant \kappa |f|^k_{C^\ell(\mathbb{U})}$.

Proof. Assume that Ω has the following form:

$$\Omega(x) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant p} \Omega_{\mathbf{i}}(x)\, dx_{\mathbf{i}}.$$

Since $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, we have that part (i) of Lemma 27 implies that, for index $\mathbf{i} = (i_1, \ldots, i_k)$ with $1 \leqslant i_1 < \cdots < i_k \leqslant p$, $\Omega_{\mathbf{i}} \circ S_t[f]$ is defined on $\mathbb{U}$ for all $t \geqslant \bar{t}$, where $\bar{t}$ is as in Lemma 27. Hence for $t \geqslant \bar{t}$ the following holds:

$$\bigl(S_t[f]^*\Omega\bigr)(x) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant p} \Omega_{\mathbf{i}}\bigl(S_t[f](x)\bigr) S_t[f]^*\, dx_{\mathbf{i}}, \quad \forall x \in \mathbb{U}. \tag{13}$$

Then part (i) follows from Proposition 23 and the following equality

$$\begin{aligned}\bigl(S_t[f]^*\Omega - f^*\Omega\bigr)(x) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant p} & \bigl[\Omega_{\mathbf{i}}\bigl(f(x)\bigr)\bigl\{S_t[f] - f\bigr\}^*\, dx_{\mathbf{i}} \\ & + \bigl\{\Omega_{\mathbf{i}} \circ S_t[f] - \Omega_{\mathbf{i}} \circ f\bigr\}(x) S_t[f]^*\, dx_{\mathbf{i}}\bigr],\end{aligned}$$

for $x \in \mathbb{U}$, where we have used the equality $\{f^* - g^*\}\, dx_{\mathbf{i}} = \{f - g\}^*\, dx_{\mathbf{i}}$, which is true because the k-form $dx_{\mathbf{i}}$ does not depend on the base point.

Let us prove part (ii) of Proposition 28. From part (iv) of Proposition 23 and Lemma 27 we have that

$$S_t[f]\bigl(\mathbb{U} + t^{-1}\bigr) \subseteq S_t[f](\mathbb{U}) + \kappa t^{-1} |f|_{C^\ell(\mathbb{U})} \subseteq \mathbb{V} + \kappa t^{-1} |f|_{C^\ell(\mathbb{U})}.$$

Hence, if $t \geqslant \bar{t}$ is sufficiently large so that $t^{-1}\kappa|f|_{C^\ell(\mathbb{U})} < \rho$, then for any multi-index $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \leqslant i_1 < \dots < i_k \leqslant p$, the following holds:

$$\left|\Omega_{\mathbf{i}} \circ S_t[f]\right|_{C^0(\mathbb{U}+t^{-1})} \leqslant |\Omega_{\mathbf{i}}|_{C^0(S_t[f](\mathbb{U}+t^{-1}))} \leqslant |\Omega_{\mathbf{i}}|_{C^0(\mathbb{V}+\rho)}. \tag{14}$$

Hence, part (ii) of Proposition 28 follows from (13) and (14). □

To prove Theorems 18 and 20 we need more accurate estimates than those given in Proposition 28. Actually (see Sections 3.4 and 3.5) we need an estimate for

$$\left|S_t[f]^*\Omega - f^*\Omega\right|_{C^0(\mathbb{U}+Ct^{-1})},$$

with $C \geqslant 0$, in the case that both Ω and $f^*\Omega$ are real analytic k-forms. This is given in the following.

Proposition 29. *Let $1 < \ell < m$, with $\ell \notin \mathbb{N}$ and $m \in \mathbb{N}$. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in* H1 *in Theorem* 18 *and let $\mathbb{V}$ be either $\mathbb{R}^p$, $\mathbb{T}^p$, or $\mathbb{T}^s \times V$, $V \subset \mathbb{R}^{p-s}$ a compensated open domain with C^m-boundary, or $\mathbb{V} \subset \mathbb{R}^p$ a compensated open domain with C^m-boundary.*

Assume that $\Omega \in \Lambda^k(\mathbb{V})$ and $\tilde{\Omega} \in \Lambda^k(\mathbb{U})$ are two real analytic k-forms such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)}$, $|\tilde{\Omega}|_{C^\ell(\mathbb{U}+\rho)} < \infty$, for some $\rho > 0$. Then, for each $C \geqslant 0$, $\beta > 0$, and $0 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, p, \ell, C, \beta, \mu, k)$ and $\hat{t} = \hat{t}(d, p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for all $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying: (i) *the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$,* (ii) *$|f|_{C^\ell(\mathbb{U})} \leqslant \beta$, and* (iii) *$f^*\Omega = \tilde{\Omega}$, the following holds*:

$$\left|S_t[f]^*\Omega - \tilde{\Omega}\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu}, \quad \forall t \geqslant \hat{t},$$

where $\hat{M}_f$ depends on k, $|\Omega|_{C^\ell(\mathbb{U}+\rho)}$, $|\tilde{\Omega}|_{C^\ell(\mathbb{U}+\rho)}$, and β, but is independent of t.

To prove Proposition 29 we extend the definition of the analytic smoothing operator S_t to k-forms in the following way. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 18. Let $\tilde{\Omega} \in \Lambda^k(\mathbb{U})$ be of the form

$$\tilde{\Omega}(x) = \sum_{1 \leqslant i_1 < \dots < i_k \leqslant d} \tilde{\Omega}_{\mathbf{i}}(x)\, dx_{\mathbf{i}},$$

with $\tilde{\Omega}_{\mathbf{i}} \in C^\ell(\mathbb{U})$ for all $1 \leqslant i_1 < \dots < i_k \leqslant d$, define the k-form $S_t[\tilde{\Omega}] \in \Lambda^k(\mathbb{U})$ by

$$S_t[\tilde{\Omega}] \stackrel{\text{def}}{=} \sum_{1 \leqslant i_1 < \dots < i_k \leqslant d} S_t[\Omega_{\mathbf{i}}]\, dx_{\mathbf{i}}. \tag{15}$$

Notice that

$$S_t[f]^*\Omega - \tilde{\Omega} = \left\{S_t[f]^*\Omega - S_t[f^*\Omega]\right\} + (S_t - \mathrm{Id})[\tilde{\Omega}], \tag{16}$$

so to prove Proposition 29 it suffices to estimate the norm of the differences on the right-hand side of (16). An estimate of the norm of the second difference on the right-hand side of (16) follows from the following lemma and (15).

Lemma 30. *Let $1 < \ell \notin \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in* H1 *in Theorem* 18. *Then, there exists a constant $\kappa = \kappa(d, \ell, C)$ such that if $g \in \mathcal{A}(\mathbb{U} + Ct^{-1}, C^\ell)$ then the following holds*:

$$\left|(S_t - \mathrm{Id})[g]\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa |g|_{C^\ell(\mathbb{U}+Ct^{-1})} t^{-\ell}. \tag{17}$$

Proof. First one proves Lemma 30 in the case that $\mathbb{U} \subseteq \mathbb{R}^d$ is either $\mathbb{R}^d$ or a compensated open domain with C^m-boundary. Then the cases $g \in \mathcal{A}(\mathbb{T}^d + Ct^{-1}, C^\ell)$ and $g \in \mathcal{A}(\mathbb{T}^n \times U + Ct^{-1}, C^\ell)$, with U a compensated open domain in $\mathbb{R}^{d-n}$ with C^m-boundary, follow by taking a lift of g and using that (17) holds for the periodic (respectively partially periodic) part of the lift of g (see Remarks 8, 14, and 15).

We prove Lemma 30 in the case that $\mathbb{U} \subset \mathbb{R}^d$ is a compensated open domain with C^m-boundary. The case $\mathbb{U} = \mathbb{R}^d$ is proved in the same way. From Lemma 21 and Theorem 11 we have that if $z \in \mathbb{U} + Ct^{-1}$ then

$$\left|S_t[g](z) - P_{g,\ell}\big(\mathrm{Re}(z), i\,\mathrm{Im}(z)\big)\right| \leqslant e^C \tilde{c} c_U |g|_{C^\ell(\mathbb{U})} t^{-\ell}. \tag{18}$$

Moreover, from the Taylor Theorem we have for all $z \in \mathbb{U} + Ct^{-1}$

$$\left|g(z) - P_{g,\ell}\big(\mathrm{Re}(z), i\,\mathrm{Im}(z)\big)\right| \leqslant \hat{c} |g|_{C^\ell(\mathbb{U}+Ct^{-1})} \left|\mathrm{Im}(z)\right|^\ell, \tag{19}$$

for some constant $\hat{c}$. Therefore Lemma 30 follows from (18) and (19). □

Giving an estimate for the norm of the first difference on the right-hand side of (16) is more intricate. In Section 3.3.1 we give several results from which the following proposition follows easily (see Section 3.3.2).

Proposition 31. *Assume that ℓ, m, $\mathbb{U}$, $\mathbb{V}$ and $\Omega \in \Lambda^k(\mathbb{V})$ are as in Proposition* 29. *Then, for each $C \geqslant 0$, $\beta > 0$, and $0 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, p, \ell, C, \beta, \mu, k)$ and $\tilde{t} = \tilde{t}(d, p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for all $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying* (i) *and* (ii) *in Proposition* 29, *and for all $t \geqslant \tilde{t}$, the following holds*:

$$\left|S_t[f^*\Omega] - S_t[f]^*\Omega\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa \tilde{M}_f t^{-\mu},$$

where $\tilde{M}_f$ depends on $|\Omega|_{C^\ell(\mathbb{V}+\rho)}$ and $|f|_{C^\ell(\mathbb{U})}$, but is independent of t.

3.3.1. Analytic estimates

In this section we give analytic estimates of certain quantities that enable us to estimate the norm on complex strips of the difference between $S_t[f]^*\Omega$ and $S_t[f^*\Omega]$: Since the pull-back involves the composition and the multiplication of functions, the quantities to be estimated depend on the norm of the difference between:

(i) smoothing a multiplication of two functions and multiplying their smoothings (see Lemma 33),
(ii) smoothing a composition of two functions and composing their smoothings (see Proposition 37).

Let us start by estimating the C^ℓ-norms on complex strips of the product and composition of functions in terms of the C^ℓ-norms of the original functions.

Lemma 32.

(i) *Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in Theorem* 18. *Assume that $g_1, \dots, g_k$ belong to $C^r(\mathbb{U}+\rho)$, for some $\rho \geqslant 0$. Then, the following holds*:

$$|g_1 g_2|_{C^r(\mathbb{U}+\rho)} \leqslant \kappa \big(|g_1|_{C^r(\mathbb{U}+\rho)} |g_2|_{C^0(\mathbb{U}+\rho)} + |g_1|_{C^0(\mathbb{U}+\rho)} |g_2|_{C^r(\mathbb{U}+\rho)}\big) \tag{20}$$

and

$$|g_1 g_2 \cdots g_k|_{C^r(\mathbb{U}+\rho)} \leqslant \kappa \sum_{i=1}^{k} \Bigg(|g_i|_{C^r(\mathbb{U}+\rho)} \prod_{\substack{j\in\{1,\dots,k\} \\ j\neq i}} |g_j|_{C^0(\mathbb{U}+\rho)} \Bigg). \tag{21}$$

(ii) *Let $W \subset \mathbb{C}^n$ and $Z \subset \mathbb{C}^p$ be compensated domains* (*Definition* 16), *$s, \sigma \geqslant 0$, and $h \in \mathcal{A}(Z, C^s)$. Assume that $f \in \mathcal{A}(W, C^\sigma)$ is such that $f(W) \subset Z$, then*:

(a) *If* $\max(s,\sigma) < 1$, *then $h \circ f \in \mathcal{A}(W, C^{s\sigma})$ and*

$$|h \circ f|_{C^{s\sigma}(W)} \leqslant |h|_{C^s(Z)} |f|^s_{C^\sigma(W)} + |h|_{C^0(Z)}.$$

(b) *If* $\max(s,\sigma) \geqslant 1$, *then $h \circ f \in \mathcal{A}(W, C^\ell)$, with $\ell = \min(s,\sigma)$. Moreover*:

(i) *If* $0 \leqslant s < 1 \leqslant \sigma$, *then*

$$|h \circ f|_{C^s(W)} \leqslant \kappa |h|_{C^s(Z)} |f|^s_{C^1(W)} + |h|_{C^0(Z)}.$$

(ii) *If* $0 \leqslant \sigma \leqslant 1 \leqslant s$, *then*

$$|h \circ f|_{C^\sigma(W)} \leqslant \kappa |h|_{C^1(Z)} |f|_{C^\sigma(W)} + |h|_{C^0(Z)}.$$

(iii) *If* $\ell = \min(s,\sigma) \geqslant 1$, *then*

$$|h \circ f|_{C^\ell(W)} \leqslant \kappa |h|_{C^\ell(Z)} \big(1 + |f|^\ell_{C^\ell(W)}\big).$$

Proof. To prove estimate (20) use the Leibniz's rule to write the derivative of the product function $h = g_1 g_2$ in terms of the derivatives of g_1 and g_2 and use the interpolation estimates [4,21]. Estimate (21) follows from (20). Part (ii) follows from Theorem 4.3 in [21]. □

In the following lemma we give an estimate for the norm of the difference between smoothing a multiplication of two functions and multiplying their smoothings.

Lemma 33. *Let $1 < \ell < m$, with $m \in \mathbb{N}$ and $\ell \notin \mathbb{N}$, and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in* H1 *in Theorem* 18. *Then for each $C \geqslant 0$, $0 \leqslant \mu < \ell$ and $r \in (0,1)$, with $0 < r + \mu < \ell$, there exists a constant $\kappa = \kappa(d, \ell, C, \mu, r)$, such that for all $t \geqslant e^{1/r}$ satisfying*

$$t^{-1}\big(C + r\log(t)\big) \leqslant 1, \tag{22}$$

the following holds:

(i) $|S_t[g]|_{C^\mu(\mathbb{U}+Ct^{-1})} \leqslant \kappa |g|_{C^\ell(\mathbb{U})}$, *for* $g \in C^\ell(\mathbb{U})$.

(ii) *For* $g_1, g_2 \in C^\ell(\mathbb{U})$

$$\big|S_t[g_1]S_t[g_2]\big|_{C^\mu(\mathbb{U}+Ct^{-1})} \leqslant \kappa\big(|g_1|_{C^0(\mathbb{U})}|g_2|_{C^\ell(\mathbb{U})} + |g_1|_{C^\ell(\mathbb{U})}|g_2|_{C^0(\mathbb{U})}\big).$$

(iii) *For* $g_1, g_2 \in C^\ell(\mathbb{U})$

$$\big|S_t[g_1 g_2] - S_t[g_1]S_t[g_2]\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa\big(|g_1|_{C^0(\mathbb{U})}|g_2|_{C^\ell(\mathbb{U})} + |g_1|_{C^\ell(\mathbb{U})}|g_2|_{C^0(\mathbb{U})}\big)t^{-\mu}.$$

Proof. To prove part (i) of Lemma 33, one first proves that it holds for $g \in C^\ell(\mathbb{U})$, when $\mathbb{U}$ is either $\mathbb{R}^d$ or an open domain with C^m-boundary. That part (i) of Lemma 33 holds for $g \in C^\ell(\mathbb{U})$, when $\mathbb{U}$ is either $\mathbb{T}^d$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ an open domain with C^m-boundary, follows by taking a lift of g, applying part (i) of Lemma 33 to the periodic (respectively, partially periodic) part of the lift of g, and using the norms introduced in Definition 4 (see Remarks 8, 14, and 15).

We only prove part (i) of Lemma 33 in the case that $\mathbb{U}$ is an open domain with C^m-boundary. The case $\mathbb{U} = \mathbb{R}^d$ is proved in the same way. For $t \geqslant 1$, define $\rho(t) = t^{-1}(C + r\log(t))$ and let k be such that $2^k \leqslant t < 2^{k+1}$. Using Lemma 21 and Theorem 11 one proves that if $g \in C^\ell(\mathbb{U})$ then the following estimates hold:

$$\big|(S_{2t} - S_t)[g]\big|_{C^0(\mathbb{R}^d+\rho(2t))} \leqslant \kappa |g|_{C^\ell(\mathbb{U})} t^{-\ell+r},$$
$$\big|(S_t - S_{2^k})[g]\big|_{C^0(\mathbb{R}^d+\rho(t))} \leqslant \kappa |g|_{C^\ell(\mathbb{U})} 2^{-k\ell} t^r.$$

Then part (i) of Lemma 33 in the case that $\mathbb{U}$ is an open subset of $\mathbb{R}^d$ with C^m-boundary, follows using Cauchy's estimates and the following inequality:

$$\begin{aligned}\big|S_t[g]\big|_{C^\mu(\mathbb{R}^d+t^{-1}C)} \leqslant{}& \big|(S_t - S_{2^k})[g]\big|_{C^\mu(\mathbb{R}^d+t^{-1}C)} + \sum_{j=1}^{k}\big|(S_{2^j} - S_{2^{j-1}})[g]\big|_{C^\mu(\mathbb{R}^d+t^{-1}C)}\\ &+ \big|S_1[g]\big|_{C^\mu(\mathbb{R}^d+t^{-1}C)}.\end{aligned}$$

Part (ii) of Lemma 33 follows from estimate (20), and part (i) of Lemma 33. To prove part (iii) of Lemma 33 write

$$\begin{aligned}S_t[g_1 g_2] - S_t[g_1]S_t[g_2] ={}& S_t\big[(\mathrm{Id} - S_t)[g_1]g_2\big] + S_t\big[S_t[g_1](\mathrm{Id} - S_t)[g_2]\big]\\ &+ (S_t - 1)\big[S_t[g_1]S_t[g_2]\big].\end{aligned} \tag{23}$$

Parts (ii) and (i) of Proposition 23 imply

$$\begin{aligned}\big|S_t\big[(1 - S_t)[g_1]g_2\big]\big|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant \kappa\big|(1 - S_t)[g_1]g_2\big|_{C^0(\mathbb{U})} \leqslant \kappa |g_2|_{C^0(\mathbb{U})}\big|(1 - S_t)[g_1]\big|_{C^0(\mathbb{U})}\\ &\leqslant \kappa |g_2|_{C^0(\mathbb{U})}|g_1|_{C^\ell(\mathbb{U})}t^{-\ell}\end{aligned} \tag{24}$$

and

$$\begin{aligned}\big|S_t\big[S_t[g_1](1-S_t)[g_2]\big]\big|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant \kappa\big|S_t[g_1](1-S_t)[g_2]\big|_{C^0(\mathbb{U})}\\ &\leqslant \kappa\big|S_t[g_1]\big|_{C^0(\mathbb{U})}\big|(1-S_t)[g_2]\big|_{C^0(\mathbb{U})}\\ &\leqslant \kappa|g_1|_{C^0(\mathbb{U})}|g_2|_{C^\ell(\mathbb{U})}t^{-\ell}.\end{aligned}\tag{25}$$

Moreover, because of part (ii) of Lemma 33 we have that $S_t[g_1]S_t[g_1]$ belongs to $\mathcal{A}(\mathbb{R}^d+Ct^{-1}, C^\mu)$, then Lemma 30 and part (ii) of Lemma 33 imply

$$\begin{aligned}\big|(S_t-1)\big[S_t[g_1]S_t[g_2]\big]\big|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant \kappa\big|S_t[g_1]S_t[g_2]\big|_{C^\mu(\mathbb{R}^d+Ct^{-1})}t^{-\mu}\\ &\leqslant \kappa\big(|g_1|_{C^0(\mathbb{U})}|g_2|_{C^\ell(\mathbb{U})}+|g_1|_{C^\ell(\mathbb{U})}|g_2|_{C^0(\mathbb{U})}\big)t^{-\mu}.\end{aligned}\tag{26}$$

Hence part (iii) of Lemma 33 follows from equality (23) and estimates (24), (25), and (26). □

We emphasize that the proof of Lemma 33 is based on the linearity of S_t. As a consequence of Lemma 33 we have the following.

Lemma 34. *Let ℓ and $\mathbb{U}$ be as in Lemma* 33*. Let k, n be two non-negative integers such that $0 \leqslant n+k \leqslant d$. For each $0 \leqslant \mu < \ell$, $C \geqslant 0$ and $r \in (0,1)$, with $0 < r+\mu < \ell$, there exists a constant $\kappa = \kappa(d, \ell, C, \mu, r, k, n)$, such that for all $\vartheta \in \Lambda^n(\mathbb{U})$ and $\alpha \in \Lambda^k(\mathbb{U})$ with*

$$|\vartheta|_{C^\ell(\mathbb{U})} < \infty \quad \textit{and} \quad |\alpha|_{C^\ell(\mathbb{U})} < \infty,$$

and for all $t \geqslant e^{1/r}$ satisfying (22) *the following holds*:

$$\big|S_t[\vartheta]\wedge S_t[\alpha] - S_t[\vartheta\wedge\alpha]\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa\big(|\vartheta|_{C^0(\mathbb{U})}|\alpha|_{C^\ell(\mathbb{U})}+|\vartheta|_{C^\ell(\mathbb{U})}|\alpha|_{C^0(\mathbb{U})}\big)t^{-\mu}.$$

Proof. Let $\vartheta \in \Lambda^n(\mathbb{U})$ and $\alpha \in \Lambda^k(\mathbb{U})$ be given by

$$\vartheta(x) = \sum_{1\leqslant i_1<\cdots<i_n\leqslant d} \vartheta_{\mathbf{i}}(x)\,dx_{\mathbf{i}}, \qquad \alpha(x) = \sum_{1\leqslant j_1<\cdots<j_k\leqslant d} \alpha_{\mathbf{j}}(x)\,dx_{\mathbf{j}},$$

with $\vartheta_{\mathbf{i}}, \alpha_{\mathbf{j}} \in C^\ell(\mathbb{U})$ for all $\mathbf{i} = (i_1, \dots, i_n)$, with $1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant d$, and $\mathbf{j} = (j_1, \dots, j_k)$, with $1 \leqslant j_1 < \cdots < j_k \leqslant d$. Then, performing some simple computations one obtains

$$\begin{aligned}&S_t[\vartheta]\wedge S_t[\alpha] - S_t[\vartheta\wedge\alpha](x)(\xi_1,\dots,\xi_{m+k})\\ &\quad = \sum_{\sigma\in\mathcal{S}(n,k)} \sum_{\substack{1\leqslant i_1<\cdots<i_n\leqslant d\\ 1\leqslant j_1<\cdots<j_k\leqslant d}} c_{\mathbf{i},\mathbf{j}}(x)\,dx_{\mathbf{i}}(\xi_{\sigma(1)},\dots,\xi_{\sigma(n)})\,dx_{\mathbf{j}}(\xi_{\sigma(n+1)},\dots,\xi_{\sigma(n+k)}),\end{aligned}$$

where $\mathcal{S}(n,k)$ represents the set of all permutations σ of $\{1, 2, \dots, n+k\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+k)$, and

$$c_{\mathbf{i},\mathbf{j}}(x) \stackrel{\text{def}}{=} \big(S_t[\vartheta_{\mathbf{i}}]S_t[\alpha_{\mathbf{j}}] - S_t[\vartheta_{\mathbf{i}}\alpha_{\mathbf{j}}]\big)(x).$$

Hence the proof is finished applying part (iii) of Lemma 33. □

The following lemma is the same result as Proposition 31, applied to the components of $\Lambda^k(\mathbb{V})$ with respect to the basis: $dx_{\mathbf{i}} = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, with $1 \leqslant i_1 < \cdots < i_k \leqslant p$.

Lemma 35. *Let ℓ, $\mathbb{U}$ and $\mathbb{V}$ be as in Lemma 27. For each natural number $1 \leqslant k \leqslant p$, and all real numbers $0 \leqslant \mu < \ell - 1$, $C \geqslant 0$, and $r \in (0,1)$, satisfying $0 < r + \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \mu, r, k)$ and $t_0 = t_0(d, \ell, \beta, \mathbb{V}, r)$ such that for any $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying* (i) *the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$,* (ii) *$|f|_{C^\ell(\mathbb{U})} \leqslant \beta$, and any multi-index $\mathbf{i} = (i_1, \ldots, i_k)$, with $1 \leqslant i_1 < \cdots < i_k \leqslant p$, the following holds for all $t \geqslant t_0$ satisfying* (22)

$$\bigl|S_t[f^* dx_{\mathbf{i}}] - S_t[f]^* dx_{\mathbf{i}}\bigr|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa |f|^k_{C^\ell(\mathbb{U})} t^{-\mu}.$$

Proof. Let $\mathbf{i} = (i_1, \ldots, i_k)$ be a multi-index with $1 \leqslant i_1 < \cdots < i_k \leqslant p$, and let $f \in C^\ell(\mathbb{U}, \mathbb{V})$ be such that the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and $|f|_{C^\ell(\mathbb{U})} \leqslant \beta$. Performing some computations and using the linearity of S_t one obtains

$$S_t[f^* dx_{\mathbf{i}}] - S_t[f]^* dx_{\mathbf{i}} = \sum_{n=1}^{k-1} \alpha_n \wedge \varphi_n, \tag{27}$$

where, for $n \in \{1, \ldots, k-1\}$,

$$\varphi_n \stackrel{\text{def}}{=} \begin{cases} S_t[f^* dx_{i_{n+2}}] \wedge \cdots \wedge S_t[f^* dx_{i_k}], & n \in \{1, \ldots, k-2\}, \\ 1, & n = k-1, \end{cases}$$

and

$$\alpha_n \stackrel{\text{def}}{=} S_t[\vartheta_n \wedge f^* dx_{i_{n+1}}] - S_t[\vartheta_n] \wedge S_t[f^* dx_{i_{n+1}}], \tag{28}$$

where

$$\vartheta_n \stackrel{\text{def}}{=} f^*(dx_{i_1} \wedge \cdots \wedge dx_{i_n}). \tag{29}$$

Notice that, because of part (ii) of Proposition 23, for all $n \in \{1, \ldots, k-2\}$ the following estimate holds:

$$\begin{aligned} |\varphi_n|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant \kappa \bigl(\bigl|S_t[Df]\bigr|_{C^0(\mathbb{U}+Ct^{-1})}\bigr)^{k-(n+1)} \\ &\leqslant \kappa \bigl(|Df|_{C^0(\mathbb{U})}\bigr)^{k-(n+1)}. \end{aligned}$$

Hence using (27) one has

$$\begin{aligned} \bigl|S_t[f^* dx_{\mathbf{i}}] - S_t[f]^* dx_{\mathbf{i}}\bigr|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant |\alpha_{k-1}|_{C^0(\mathbb{U}+Ct^{-1})} \\ &\quad + \kappa \sum_{n=1}^{k-2} |\alpha_n|_{C^0(\mathbb{U}+Ct^{-1})} |Df|^{k-(n+1)}_{C^0(\mathbb{U})}, \end{aligned} \tag{30}$$

where we assumed, without loss of generality, that $\kappa \geqslant 1$. Moreover, from (21) and (29) we have for all $n \in \{1, \dots, k-1\}$

$$|\vartheta_n|_{C^{\ell-1}(\mathbb{U})} \leqslant \kappa n |Df|^{n-1}_{C^0(\mathbb{U})} |Df|_{C^{\ell-1}(\mathbb{U})} < \infty$$

and

$$|f^* \, dx_{i_{n+1}}|_{C^{\ell-1}(\mathbb{U})} \leqslant \kappa |Df|_{C^{\ell-1}(\mathbb{U})} < \infty,$$

for some constant κ. Hence, using (28) and Lemma 34, we have that given $0 \leqslant \mu < \ell - 1$, $C \geqslant 0$, and $r \in (0, 1)$, with $0 < r < \ell - 1 - \mu$, there exists a constant $\kappa = \kappa(d, \ell, \mu, r, C)$, such that

$$\begin{aligned} |\alpha_n|_{C^0(\mathbb{U}+Ct^{-1})} &\leqslant \kappa |\vartheta_n|_{C^{\ell-1}(\mathbb{U})} \bigl(|f^* \, dx_{i_{n+1}}|_{C^{\ell-1}(\mathbb{U})} + |f^* \, dx_{i_{n+1}}|_{C^0(\mathbb{U})}\bigr) t^{-\mu} \\ &\leqslant \kappa (n+1) |Df|^n_{C^0(\mathbb{U})} |Df|_{C^{\ell-1}(\mathbb{U})} t^{-\mu}. \end{aligned} \tag{31}$$

Hence Lemma 35 follows from (30) and (31). □

In order to prove Proposition 31 for an arbitrary k-form we need an estimate for the norm of the difference between the composition of the smoothing and the smoothing of the composition. This was considered in [11] for functions in $C^\ell(\mathbb{R}^d)$. We use the following.

Lemma 36. *Let ℓ, m, $\mathbb{U}$ and $\mathbb{V}$ be as in Proposition* 29. *Given $0 < \mu < \ell$ and $C \geqslant 0$, $\beta > 0$ there exist two constants $\kappa = \kappa(d, p, \ell, C, \mu, \beta)$ and $t_1 = t_1(p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for each $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$, satisfying* (i) *the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and* (ii) *$|f|_{C^\ell(\mathbb{U})} \leqslant \beta$, the following holds for all $t \geqslant t_1$:*

$$\bigl|S_t[h] \circ S_t[f]\bigr|_{C^\mu(\mathbb{U}+Ct^{-1})} \leqslant \kappa |h|_{C^\ell(\mathbb{V})} \bigl(1 + |f|^\tau_{C^\ell(\mathbb{U})}\bigr), \tag{32}$$

where

$$\begin{aligned} &\tau \text{ is any number in } (\mu, 1), && \text{if } 0 < \mu < 1 < \ell, \\ &\tau = \mu, && \text{if } 1 \leqslant \mu < \ell. \end{aligned}$$

Proof. Let $0 \leqslant s, \sigma < \ell$, fix $r_1, r_2 \in (0, 1)$ in such a way that $0 \leqslant s + r_1 < \ell$, and $0 \leqslant \sigma + r_2 < \ell$ (e.g. $r_1 = \min(1/2, (\ell - s)/2)$, $r_2 = \min(1/2, (\ell - \sigma)/2)$). Let κ be as in Proposition 23 and assume that $t \geqslant \max(e^{1/r_1}, e^{1/r_2})$ is sufficiently large such that

$$t^{-1} \max\bigl(C + r_2 \log(t), C\kappa\beta + r_1 \log(t)\bigr) \leqslant 1.$$

Then Lemma 33 implies for $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$,

$$\begin{aligned} \bigl|S_t[f]\bigr|_{C^\sigma(\mathbb{U}+Ct^{-1})} &\leqslant \kappa(d, \ell, C, \sigma, r_2) |f|_{C^\ell(\mathbb{U})}, \quad 0 \leqslant \sigma < \ell, \\ \bigl|S_t[h]\bigr|_{C^s(\mathbb{V}+(C\beta\kappa)t^{-1})} &\leqslant \kappa(p, \ell, C, s, r_1) |h|_{C^\ell(\mathbb{V})}, \quad 0 \leqslant s < \ell. \end{aligned} \tag{33}$$

Hence for all $0 \leqslant s, \sigma < \ell$ there exists $\tilde{t}_1 = \tilde{t}_1(p, \ell, C, \beta, s, \sigma)$ such that, for all $t \geqslant \tilde{t}_1$,

$$S_t[h] \in \mathcal{A}\big(\mathbb{V} + (C\beta\kappa)t^{-1}, C^s\big),$$
$$S_t[f] \in \mathcal{A}\big(\mathbb{U} + Ct^{-1}, C^\sigma\big). \tag{34}$$

Inclusions in (34) and part (ii) of Lemma 27 enable us to apply part (ii) of Lemma 32 to the composition $S_t[h] \circ S_t[f]$ as follows. If $1 \leqslant \mu < \ell$, estimate (32) follows from estimates (33), and part (b)(iii) of Lemma 32. Finally, if $0 < \mu < 1 < \ell$, write $\mu = \sigma s$ with $s \in (\mu, 1) \subset [0, \ell)$ and $\sigma = \mu/s \in (0, 1) \subset [0, \ell)$, then estimate (32) follows from estimates (33), and part (a) of Lemma 32 with

$$t_1(p, \ell, C, \beta, \mu) \stackrel{\text{def}}{=} \tilde{t}_1\big(p, \ell, C, \beta, s(\mu), \sigma(\mu)\big). \qquad \square$$

Proposition 37. *Let ℓ, m, $\mathbb{U}$ and $\mathbb{V}$ be as in Proposition* 29. *Given the real numbers $C \geqslant 0$, $\beta > 0$, and $0 < \mu < \ell$, there exist two positive constants $\kappa = \kappa(p, d, \ell, C, \mu, \beta)$ and $t_2 = t_2(p, \ell, \mathbb{V}, C, \mu, \beta)$ such that for every $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$, satisfying* (i) *the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and* (ii) $|f|_{C^\ell(\mathbb{U})} < \beta$, *the following holds for all $t \geqslant t_2$:*

$$\big|S_t[h] \circ S_t[f] - S_t[h \circ f]\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa M_1 t^{-\mu}, \tag{35}$$

where

$$M_1 \stackrel{\text{def}}{=} |h|_{C^\ell(\mathbb{V})}\big(1 + |f|^\tau_{C^\ell(\mathbb{U})}\big) + |h|_{C^\ell(\mathbb{V})}|f|_{C^\ell(\mathbb{U})},$$

and

$$\begin{array}{ll} \tau \text{ is any number in } (\mu, 1), & \text{if } 0 < \mu < 1 < \ell, \\ \tau = \mu, & \text{if } 1 \leqslant \mu < \ell. \end{array}$$

Proof. That the composition $h \circ f$ belongs to $C^\ell(\mathbb{U})$ follows from part (ii) of Lemma 32 (the torus and annulus cases this is obtained by using lifts). To prove estimate (35), first write

$$\begin{aligned} S_t[h] \circ S_t[f] - S_t[h \circ f] = {} & (1 - S_t)\big[S_t[h] \circ S_t[f]\big] \\ & + S_t\big[S_t[h] \circ S_t[f]\big] - S_t\big[S_t[h] \circ f\big] \\ & + S_t\big[S_t[h] \circ f - h \circ f\big]. \end{aligned} \tag{36}$$

Let us estimate the first term on the right-hand side of (36). Let $C \geqslant 0$, $\beta > 0$, and $0 < \mu < \ell$ be given and let κ and t_1 be as in Lemma 36. Then from Lemmas 36 and 30 one obtains for all $t \geqslant t_1$

$$\begin{aligned} \big|(\mathrm{Id} - S_t)\big[S_t[h] \circ S_t[f]\big]\big|_{C^0(\mathbb{U}+Ct^{-1})} & \leqslant \kappa \big|S_t[h] \circ S_t[f]\big|_{C^\mu(\mathbb{U}+Ct^{-1})} t^{-\mu} \\ & \leqslant \kappa |h|_{C^\ell(\mathbb{V})}\big(1 + |f|^\tau_{C^\ell(\mathbb{U})}\big) t^{-\mu}, \end{aligned} \tag{37}$$

where τ is as in Lemma 36. Now we consider the third term on the right-hand side of (36). Using again part (ii) of Proposition 23 we have

$$\begin{aligned}\bigl|S_t\bigl[S_t[h]\circ f-h\circ f\bigr]\bigr|_{C^0(\mathbb{U}+Ct^{-1})}&\leqslant\kappa\bigl|S_t[h]\circ f-h\circ f\bigr|_{C^0(\mathbb{U})}\\&\leqslant\kappa\bigl|(S_t-1)[h]\bigr|_{C^0(\mathbb{V})}\\&\leqslant\kappa|h|_{C^\ell(\mathbb{V})}t^{-\ell},\end{aligned}\tag{38}$$

where in the last inequality we have used part (i) of Proposition 23. To estimate the second term on the right-hand side of (36), we first consider the case $\mathbb{U}\subset\mathbb{R}^d$ is a compensated open domain with C^m-boundary. Notice that from Remark 15 one has

$$S_t\bigl[S_t[h]\circ S_t[f]\bigr]-S_t\bigl[S_t[h]\circ f\bigr]=S_t\bigl[S_t[h]\circ S_t[f]-\mathscr{E}^\ell_{\mathbb{U}}\bigl(S_t[h]\circ f\bigr)\bigr].\tag{39}$$

Moreover, if $x\in\mathbb{U}$, then

$$\mathscr{E}^\ell_{\mathbb{U}}\bigl(S_t[h]\circ f\bigr)(x)=\bigl(S_t[h]\circ f\bigr)(x).\tag{40}$$

Then, from Proposition 23, and equalities (39) and (40) we have

$$\bigl|S_t\bigl[S_t[h]\circ S_t[f]\bigr]-S_t\bigl[S_t[h]\circ f\bigr]\bigr|_{C^0(\mathbb{U}+Ct^{-1})}\leqslant\kappa|h|_{C^\ell(\mathbb{U})}|f|_{C^\ell(\mathbb{U})}.\tag{41}$$

In the same way, one proves that estimate (41) also holds in the case $\mathbb{U}=\mathbb{R}^d$. Indeed, if $\mathbb{U}=\mathbb{R}^d$ then (compare with (39))

$$S_t\bigl[S_t[h]\circ S_t[f]\bigr]-S_t\bigl[S_t[h]\circ f\bigr]=S_t\bigl[S_t[h]\circ S_t[f]-S_t[h]\circ f\bigr].$$

Furthermore, taking lifts, using the norms introduced in Definition 4, and using that (41) holds when $\mathbb{U}$ is either $\mathbb{R}^d$ or a compensated open domain in $\mathbb{R}^d$ with C^m-boundary, one proves that estimate (41) also holds in the following cases: (i) $\mathbb{U}=\mathbb{T}^d$, $\mathbb{V}=\mathbb{T}^p$, (ii) $\mathbb{U}=\mathbb{T}^d$ and $\mathbb{V}\subset\mathbb{R}^p$ is a compensated open domain with C^m-boundary, (iii) $\mathbb{U}=\mathbb{T}^n\times U$, with $U\subset\mathbb{R}^d$ a compensated open domain with C^m-boundary, and $\mathbb{V}=\mathbb{T}^p$, (iv) $\mathbb{U}=\mathbb{T}^n\times U$, with $U\subset\mathbb{R}^d$ a compensated open domain with C^m-boundary, and $\mathbb{V}$ is a compensated open domain with C^m-boundary. Hence, estimate (41) holds for $\mathbb{U}$ and $\mathbb{V}$ as in the hypotheses of Proposition 37.

Proposition 37 follows from equality (36) taking t_2 sufficiently large such that estimates (37), (38), and (41), holds for all $t\geqslant t_2$. □

3.3.2. Smoothing and pull-back (Proof of Proposition 31)

We now have all the ingredients to prove Proposition 31. Let $\mathbb{U}$, $\mathbb{V}$, and $\Omega\in\Lambda^k(\mathbb{V})$ be as in Proposition 29. Throughout this section we assume that $C\geqslant 0$, $\beta>0$, and $0<\mu<\ell-1$ are given. Fix $r\in(0,1)$ in terms of ℓ and μ in such a way that $0<\mu+r<\ell-1$ (e.g. $r=\min(1/2,(\ell-1-\mu)/2)$) so that the constants depending on r will actually depend on μ and ℓ. Let $f\in C^\ell(\mathbb{U},\mathbb{V})$ be such that the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, then Lemma 27 implies $S_t[f]^*\Omega\in\Lambda^k(\mathbb{U})$ for $t\geqslant\bar t$. Hence, to have $S_t[f]^*\Omega$ defined on $\mathbb{U}$ we assume from now on that $t\geqslant\bar t$. To prove Proposition 31 we first write

$$\begin{aligned}S_t[f]^*\Omega-S_t[f^*\Omega]&=\bigl\{S_t[f]^*\Omega-S_t[f]^*\bigl(S_t[\Omega]\bigr)\bigr\}\\&\quad+\bigl\{S_t[f]^*\bigl(S_t[\Omega]\bigr)-S_t[f^*\Omega]\bigr\}.\end{aligned}\tag{42}$$

Let us estimate the first term in brackets on the right-hand side:

$$\big(S_t[f]^*\Omega - S_t[f]^*\big(S_t[\Omega]\big)\big)(x) = \sum_{1\leqslant i_i<\cdots<i_k\leqslant d} (1-S_t)[\Omega_{\mathbf{i}}]\big(S_t[f](x)\big)S_t[f]^*\,dx_{\mathbf{i}}. \tag{43}$$

From Lemma 27 we have

$$S_t[f]\big(\mathbb{U}+Ct^{-1}\big) \subseteq \mathbb{V} + (C\beta\kappa)t^{-1} \subseteq \mathbb{V}+\rho, \quad \forall t \geqslant \max\big(\rho^{-1}\beta C\kappa\bar{t}\big).$$

Assume that $\Omega \in \mathcal{A}(\mathbb{V}+\rho, C^\ell)$, then for all $\mathbf{i} = (i_1,\dots,i_k)$, with $1 \leqslant i_i < \cdots < i_k \leqslant p$ and $t \geqslant \max(\rho^{-1}\beta C\kappa, \bar{t})$, Lemma 30 implies

$$\big|(\mathrm{Id}-S_t)[\Omega_{\mathbf{i}}]\circ S_t[f]\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \big|(\mathrm{Id}-S_t)[\Omega_{\mathbf{i}}]\big|_{C^0(\mathbb{V}+\rho)} \leqslant \big(\kappa|\Omega_{\mathbf{i}}|_{C^\sigma(\mathbb{V}+\rho)}\big)t^{-\sigma}, \tag{44}$$

for all $0 \leqslant \sigma \leqslant \ell$. Hence, part (ii) of Proposition 23, estimate (44), and equality (43), yield for all $t \geqslant \max(\rho^{-1}\beta C\kappa, \bar{t})$,

$$\big|S_t[f]^*\Omega - S_t[f]^*\big(S_t[\Omega]\big)\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa|\Omega|_{C^\mu(\mathbb{V}+\rho)}|f|^k_{C^\ell(\mathbb{U})}t^{-\mu}, \tag{45}$$

where $\kappa = \kappa(p,d,\ell,C,\mu,\beta,k)$.

Now write the second term on the right-hand side of (42) in the following way:

$$\begin{aligned} S_t[f]^*\big(S_t[\Omega]\big) - S_t[f^*\Omega] = \sum_{1\leqslant i_i<\cdots<i_k\leqslant d} &\big\{\big(S_t[\Omega_{\mathbf{i}}]\circ S_t[f] - S_t[\Omega_{\mathbf{i}}\circ f]\big)S_t[f]^*\,dx_{\mathbf{i}} \\ &+ S_t[\Omega_{\mathbf{i}}\circ f]\big(S_t[f]^*\,dx_{\mathbf{i}} - S_t[f^*\,dx_{\mathbf{i}}]\big) \\ &+ S_t[\Omega_{\mathbf{i}}\circ f]S_t[f^*\,dx_{\mathbf{i}}] - S_t\big[(\Omega_{\mathbf{i}}\circ f)f^*\,dx_{\mathbf{i}}\big]\big\}. \end{aligned} \tag{46}$$

In what follows we give estimates for the three terms on the right-hand side of (46). The first term is estimated as follows: Let t_2 be as in Proposition 37, then Proposition 37 and part (ii) of Proposition 23 yield for all $t \geqslant t_2$:

$$\begin{aligned} &\big|\big(S_t[\Omega_{\mathbf{i}}]\circ S_t[f] - S_t[\Omega_{\mathbf{i}}\circ f]\big)S_t[f]^*\,dx_{\mathbf{i}}\big|_{C^0(\mathbb{R}^d+Ct^{-1})} \\ &\quad \leqslant \kappa|f|^k_{C^\ell(\mathbb{U})}\big\{|\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{U})}\big(1+|f|^\tau_{C^\ell(\mathbb{U})}\big) + |\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{U})}|f|_{C^\ell(\mathbb{U})}\big\}t^{-\mu}, \end{aligned} \tag{47}$$

where $\kappa = \kappa(p,d,\ell,C,\mu,\beta,k)$ and τ is as in Proposition 37.

An estimate for the second term on the right-hand side of (46) follows from part (ii) of Proposition 23 and Lemma 35:

$$\big|S_t[\Omega_{\mathbf{i}}\circ f]\big(S_t[f]^*\,dx_{\mathbf{i}} - S_t(f^*\,dx_{\mathbf{i}})\big)\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa|\Omega_{\mathbf{i}}|_{C^0(\mathbb{V})}|f|^k_{C^\ell(\mathbb{U})}t^{-\mu}, \tag{48}$$

where $t \geqslant t_0$, with t_0 as in Lemma 35, and $\kappa = \kappa(d,\ell,C,\mu,k)$.

Finally, applying Lemma 34 to the 0-form $\Omega_{\mathbf{i}}\circ f$ and the k-form $f^*\,dx_{\mathbf{i}}$ and using Lemma 32, one has that there exists a constant $\kappa = \kappa(d,\ell,C,\mu,k)$ such that, for all $t \geqslant e^{1/r}$ satisfying (22), the following holds:

$$
\begin{aligned}
&\big|S_t[\Omega_{\mathbf{i}}\circ f]S_t[f^* dx_{\mathbf{i}}] - S_t\big[(\Omega_{\mathbf{i}}\circ f)f^* dx_{\mathbf{i}}\big]\big|_{C^0(\mathbb{U}+Ct^{-1})}\\
&\quad\leqslant \kappa\big(|\Omega_{\mathbf{i}}\circ f|_{C^0(\mathbb{U})}|f^* dx_{\mathbf{i}}|_{C^{\ell-1}(\mathbb{U})} + |\Omega_{\mathbf{i}}\circ f|_{C^{\ell-1}(\mathbb{U})}|f^* dx_{\mathbf{i}}|_{C^0(\mathbb{U})}\big)t^{-\mu}\\
&\quad\leqslant \kappa\big\{|\Omega_{\mathbf{i}}|_{C^0(\mathbb{V})}|Df|^{k-1}_{C^0(\mathbb{U})}|f|_{C^\ell(\mathbb{U})} + |\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{V})}\big(1+|f|^{\ell}_{C^\ell(\mathbb{U})}\big)|Df|^{k}_{C^0(\mathbb{U})}\big\}t^{-\mu},
\end{aligned}
\tag{49}
$$

where we have used the inequality $|\Omega_{\mathbf{i}}\circ f|_{C^{\ell-1}(\mathbb{U})} \leqslant |\Omega_{\mathbf{i}}\circ f|_{C^\ell(\mathbb{U})}$ and part (b)(iii) of Lemma 32.

Define

$$\tilde{t} \stackrel{\text{def}}{=} \max\big(C\beta\kappa\rho^{-1}, e^{1/r}, t_2, \bar{t}\big),$$

where $\kappa = \kappa(d, C)$ is as in Proposition 23, $\bar{t}$ is as in Lemma 27, and t_2 is as in Proposition 37. Let $t \geqslant \tilde{t}$ satisfy (22), then equality (46) and estimates (47), (48) and (49) imply

$$\big|S_t[f]^*\big(S_t[\Omega]\big) - S_t[f^*\Omega]\big|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa M_2 t^{-\mu}, \tag{50}$$

where κ is a constant depending on d, ℓ, k, r, μ, and C, and M_2 is defined by

$$M_2 \stackrel{\text{def}}{=} |f|^k_{C^\ell(\mathbb{U})}|\Omega|_{C^\ell(\mathbb{U})}\big\{1 + |f|^\tau_{C^\ell(\mathbb{U})} + |f|^\ell_{C^\ell(\mathbb{U})} + |f|_{C^\ell(\mathbb{U})}\big\}.$$

Hence Proposition 31 follows from estimates (45) and (50). □

3.4. The symplectic and volume cases (Proof of Theorem 18)

Let ℓ and $\mathbb{U}$ be as in Theorem 18 and let $f \in \mathrm{Diff}^\ell(\mathbb{U})$. We prove Theorem 18 in several lemmas. First in Lemma 39 we prove that if Ω is a non-degenerate form, then for sufficiently large t, the form defined by

$$\Omega_t^\varepsilon \stackrel{\text{def}}{=} \Omega + \varepsilon\big(S_t[f]^*\Omega - \Omega\big), \tag{51}$$

is also non-degenerate for all $\varepsilon \in [0, 1]$. We also give explicit estimates for the norm of $\mathscr{I}^{-1}_{\Omega_t^\varepsilon}\theta$ on complex strips in terms of the corresponding norm of θ. In Lemma 41 we use the deformation method [9] to prove that, for t sufficiently large, there exists a diffeomorphism such that $(\phi_t^\varepsilon)^*\Omega_t^\varepsilon = \Omega$. Moreover, in Lemma 41 we also give quantitative properties of ϕ_t^ε. More precisely, using Lemma 39 we prove that the diffeomorphism ϕ_t^ε is real analytic, close to the identity and with first and second derivatives bounded on the complex strips $\mathbb{U}_t + Ct^{-1}$, with $\mathbb{U}_t$ defined in (7). In Lemma 42 we prove that if $\varphi_t \stackrel{\text{def}}{=} \phi_t^1$, then $T_t[f] \stackrel{\text{def}}{=} S_t[f]\circ\varphi_t$ satisfies properties T1–T6 of Theorem 18. Property T7 is proved in Section 3.4.1.

Remark 38. Notice that if $f \in \mathrm{Diff}^\ell(\mathbb{U})$ then from part (i) of Proposition 23 we have that, for t sufficiently large, $S_t[f]$ is a diffeomorphism on $\mathbb{U}$.

Lemma 39. *Let ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω and $\mathscr{I}_\Omega$ satisfy the hypotheses of Theorem* 18. *Then, given $C \geqslant 0$ and $\beta > 0$ there exists a constant t_3, depending on d, ℓ, $\mathbb{V}$, C, β, M_Ω, and $|\Omega|_{C^\ell(\mathbb{U}+\rho)}$, such that for all $t \geqslant t_3$ and for all $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying* (i) $|f|_{C^\ell(\mathbb{U})} < \beta$, (ii) $\mathbb{V}$ *contains the closure of $f(\mathbb{U})$, and* (iii) $f^*\Omega = \Omega$, *the k-form defined by* (51) *is non-degenerate for all $\varepsilon \in [0, 1]$. Furthermore, for any real analytic $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$, and any $t \geqslant t_3$, the* *application taking* (ε, x) *into* $\mathscr{I}^{-1}_{\Omega^\varepsilon_t}(\theta)(x)$ *is continuous on* $(\varepsilon, x) \in [0,1] \times \mathbb{U} + Ct^{-1}$ *and real analytic with respect to* x. *Moreover*

$$\left|\mathscr{I}^{-1}_{\Omega^\varepsilon_t}(\theta)\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant 2M_\Omega |\theta|_{C^0(\mathbb{U}+Ct^{-1})}, \quad \forall \varepsilon \in [0,1].$$

Proof. This follows from Proposition 29. Indeed, first notice that for all $t \geqslant 1$ and $\varepsilon \in [0,1]$ the following equality holds:

$$\mathscr{I}_{\Omega^\varepsilon_t} = \mathscr{I}_\Omega + \varepsilon \mathscr{I}_{(S_t[f]^*\Omega - \Omega)} = \left(\mathrm{Id} + \varepsilon \mathscr{I}_{(S_t[f]^*\Omega - \Omega)} \circ \mathscr{I}^{-1}_\Omega\right) \circ \mathscr{I}_\Omega, \tag{52}$$

where Id represents the identity map on $\Lambda^{k-1}(\mathbb{U})$. Let $\kappa = \kappa(d, d, \ell, C, \beta, (\ell-1)/2, k)$, $\hat{t} = \hat{t}(d, d, \ell, \mathbb{V}, C, \beta, (\ell-1)/2)$, and $\hat{M}_f$ be as in Proposition 29, then for all $t \geqslant \max(\hat{t}, C)$ and for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$, the following estimate holds:

$$\left|\mathscr{I}_{(S_t[f]^*\Omega - \Omega)}\left(\mathscr{I}^{-1}_\Omega \theta\right)\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-(\ell-1)/2} M_\Omega |\theta|_{C^0(\mathbb{U}+Ct^{-1})}. \tag{53}$$

Assume that t_3 is sufficiently large so that for all $t \geqslant t_3$ estimate (53) holds and moreover

$$t^{-(\ell-1)/2} \kappa \hat{M}_f M_\Omega \leqslant 1/2.$$

Then for all $t \geqslant t_3$, $\varepsilon \in [0,1]$, the application $(\mathrm{Id} - \varepsilon \mathscr{I}_{S_t[f]^*\Omega - \Omega} \circ \mathscr{I}^{-1}_\Omega)$ is an isomorphism on $\Lambda^{k-1}(\mathbb{U})$, and moreover the following holds for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$:

$$\left|\left(\mathrm{Id} - \varepsilon \mathscr{I}_{S_t[f]^*\Omega - \Omega} \circ \mathscr{I}^{-1}_\Omega\right)^{-1} \theta\right|_{C^0(\mathbb{U}+Ct^{-1})} \leqslant 2|\theta|_{C^0(\mathbb{U}+Ct^{-1})}.$$

Hence from (52) we have that for all $t \geqslant t_3$, and $\varepsilon \in [0,1]$, the application $\mathscr{I}_{\Omega^\varepsilon_t}$ is invertible with inverse given by

$$\mathscr{I}^{-1}_{\Omega^\varepsilon_t} = \mathscr{I}^{-1}_\Omega \circ \left(\mathrm{Id} + \varepsilon \mathscr{I}_{S_t[f]^*\Omega - \Omega} \circ \mathscr{I}^{-1}_\Omega\right)^{-1},$$

from which Lemma 39 follows. □

Lemma 40. *Let* ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω *and* $\mathscr{I}_\Omega$ *satisfy the hypotheses of Theorem* 18. *Then, given* $C \geqslant 0$, $\beta > 0$ *and* $1 < \mu < \ell - 1$, *there exist two constants* $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ *and* $t_4 = t_4(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, *such that for all* $t \geqslant t_4$ *and for all* $f \in \mathrm{Diff}^\ell(\mathbb{U})$ *satisfying* (i) $|f|_{C^\ell(\mathbb{U})} < \beta$, (ii) $\mathbb{V}$ *contains the closure of* $f(\mathbb{U})$, *and* (iii) $f^*\Omega = \Omega$, *there exists a vector field* u^ε_t *satisfying*

$$d i_{u^\varepsilon_t}\left(\Omega^\varepsilon_t\right) = -\left(S_t[f]^*\Omega - f^*\Omega\right), \tag{54}$$

where d *represents the exterior derivative and* Ω^ε_t *is defined in* (51). *Furthermore, the vector field* u^ε_t *is continuous on* $(\varepsilon, x) \in [0,1] \times \mathbb{U} + 2Ct^{-1}$, *real analytic with respect to* x *on* $\mathbb{U} + 2Ct^{-1}$, *and it satisfies the following estimates*:

$$\left|u^\varepsilon_t\right|_{C^0(\mathbb{U}+2Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu}, \quad \forall \varepsilon \in [0,1], \tag{55}$$

where $\hat{M}_f$ *is as in Proposition* 29.

Proof. First notice that since $\Omega = d\alpha$ is exact and analytic, then the right-hand side of (54) is also exact and analytic. Then, the Poincaré's formula implies the existence of an analytic 1-form γ_t such that: $d\gamma_t = S_t[f]^*\Omega - \Omega$ and

$$|\gamma_t|_{\mathbb{U}+2Ct^{-1}} \leqslant \hat{\kappa}\left|S_t[f]^*\Omega - \Omega\right|_{\mathbb{U}+Ct^{-1}} \leqslant \kappa \hat{M}_f t^{-\mu},$$

where we have used Proposition 29 and the fact that $\mathbb{U}$ is bounded. Lemma 40 follows from Lemma 39 by solving the following equation:

$$i_{u_t^\varepsilon}\Omega_t^\varepsilon = -\gamma_t. \qquad \square$$

Lemma 41. *Let ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω and $\mathscr{I}_\Omega$ satisfy the hypotheses of Theorem* 18. *Then, given $C \geqslant 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and $t_5 = t_5(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for any $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying* (i) $|f|_{C^\ell(\mathbb{U})} < \beta$, (ii) *$\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and* (iii) *$f^*\Omega = \Omega$, any $t \geqslant t_5$, and any $\varepsilon \in [0,1]$, there exists an analytic diffeomorphism ϕ_t^ε on $\mathbb{U}_t$, with $\mathbb{U}_t$ defined in* (7), *such that the following hold*:

(i) $(\phi_t^\varepsilon)^*\Omega_t^\varepsilon = \Omega$.
(ii) $\phi_t^0 = \mathrm{id}$.
(iii) $|\phi_t^1 - \mathrm{id}|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu}$, *where* id *represents the identity map.*
(iv) $|D\phi_t^1|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \exp(C^{-1}\kappa \hat{M}_f t^{-\mu+1})$.
(v) $|D^2\phi_t^1|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant C^{-2}\kappa \hat{M}_f t^{-\mu+2}\exp(3C^{-1}\kappa \hat{M}_f t^{-\mu+1})$.

Proof. Following the proof of Theorem 2 in [9], we determine ϕ_t^ε by solving the differential equation

$$\frac{d}{d\varepsilon}\phi_t^\varepsilon = u_t^\varepsilon \circ \phi_t^\varepsilon, \quad 0 \leqslant \varepsilon \leqslant 1, \tag{56}$$

with ϕ_t^0 the identity mapping, where the vector field u_t^ε is as in Lemma 40. Notice that, in the case $\mathbb{U} = \mathbb{T}^d$, the properties of u_t^ε given in Lemma 40 imply the existence of a unique solution ϕ_t^ε of (56) for all $\varepsilon \in [0,1]$ and all x in the closure of $\mathbb{T}^d + Ct^{-1}$. To guarantee a solution of (56) for all $\varepsilon \in [0,1]$ in the non-compact cases: (i) $\mathbb{U} \subset \mathbb{R}^d$ a compensated bounded open domain with C^m-boundary, and (ii) $\mathbb{U} = \mathbb{T}^n \times U$ with $U \subset \mathbb{R}^{d-n}$ a compensated bounded open domain with C^m-boundary, we solve (56) for initial conditions in the closure of $\mathbb{U}_t + Ct^{-1}$, with $\mathbb{U}_t \subset \mathbb{U}$ defined in (7). Notice that if t_4 as in Lemma 40 and $t \geqslant t_4$ is sufficiently large so that

$$\kappa \hat{M}_f t^{-\mu+1} < 1, \tag{57}$$

which is possible because $1 < \mu < \ell - 1$, then (55) and (57) imply the existence of a unique solution ϕ_t^ε of (56) for all $\varepsilon \in [0,1]$ and all x in the closure of $\mathbb{U}_t + Ct^{-1}$, with $\mathbb{U}_t \subset \mathbb{U}$ defined in (7).

Hence, if $\mathbb{U}$ and $\mathbb{U}_t$ are as in Theorem 18 and $t \geqslant t_4$ satisfies (57), then Eq. (56) has a unique solution $\phi_t^\varepsilon(x)$, defined for $\varepsilon \in [0,1]$, and x in the closure of $\mathbb{U}_t + Ct^{-1}$. Moreover, the following holds:

$$\left|\phi_t^1 - \mathrm{id}\right|_{C^0(\mathbb{U}_t + Ct^{-1})} = \sup_{x \in \mathbb{U}_t + Ct^{-1}} \left| \int_0^1 u_t^s\left(\phi_t^s(x)\right) ds \right| \leqslant \sup_{s \in [0,1]} \left|u_t^s\right|_{C^0(\mathbb{U} + Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu},$$

from which part (iii) of Lemma 41 follows.

Part (i) follows from (54), (56) and the E. Cartan's formula for the Lie derivatives (cf. [22]):

$$\frac{d}{d\varepsilon}\left(\left(\phi_t^\varepsilon\right)^* \Omega_t^\varepsilon\right) = \left(\phi_t^\varepsilon\right)^* \left\{ d\left(i_{u_t^\varepsilon} \Omega_t^\varepsilon\right) + i_{u_t^\varepsilon}\left(d\Omega_t^\varepsilon\right) + \frac{d}{d\varepsilon} \Omega_t^\varepsilon \right\} = 0.$$

Parts (iv) and (v) follow from the Gronwall's and Cauchy's estimates, and (55) as follows: From (56) we have for $t \geqslant t_4$ satisfying (57) and x in the closure of $\mathbb{U}_t + Ct^{-1}$

$$\left|D\phi_t^\varepsilon(x)\right| \leqslant 1 + \int_0^\varepsilon \left|Du_t^s\right|_{C^0(\mathbb{U} + Ct^{-1})} \left|D\phi_t^s(x)\right| ds, \quad \varepsilon \in [0,1],$$

then the Gronwall's and Cauchy's estimates and (55) imply

$$\left|D\phi_t^\varepsilon\right|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \exp\Bigl(\sup_{s \in [0,1]} \left|Du_t^s\right|_{C^0(\mathbb{U} + Ct^{-1})} \Bigr) \leqslant \exp\bigl(C^{-1} \kappa \hat{M}_f t^{-\mu+1}\bigr).$$

Similarly

$$\begin{aligned}
\left|D^2\phi_t^1\right|_{C^0(\mathbb{U}_t + Ct^{-1})} &\leqslant \int_0^1 \left|D^2 u_t^s\right|_{C^0(\mathbb{U} + Ct^{-1})} \left|D\phi_t^s\right|^2_{C^0(\mathbb{U}_t + Ct^{-1})} ds \\
&\quad + \int_0^1 \left|Du_t^s\right|_{C^0(\mathbb{U} + Ct^{-1})} \left|D^2\phi_t^s\right|_{C^0(\mathbb{U}_t + Ct^{-1})} ds \\
&\leqslant \sup_{\varepsilon \in [0,1]} \left|u_t^\varepsilon\right|_{C^2(\mathbb{U} + Ct^{-1})} \exp\bigl(2C^{-1} \kappa \hat{M}_f t^{-\mu+1}\bigr) \\
&\quad + \left|u_t^\varepsilon\right|_{C^1(\mathbb{U} + Ct^{-1})} \int_0^1 \left|D^2\phi_t^s\right|_{C^0(\mathbb{U}_t + Ct^{-1})} ds \\
&\leqslant \kappa C^{-2} \hat{M}_f t^{-\mu+2} \exp\bigl(2C^{-1} \kappa \hat{M}_f t^{-\mu+1}\bigr) \\
&\quad + \kappa C^{-1} \hat{M}_f t^{-\mu+1} \int_0^1 \left|D^2\phi_t^s\right|_{C^0(\mathbb{U}_t + Ct^{-1})} ds,
\end{aligned}$$

from which part (v) of Lemma 41 follows. □

Lemma 42. *Assume that the hypotheses of Theorem* 18 *hold. Let* t_5 *and* ϕ_t^1 *be as in Lemma* 41*, define for* $t \geqslant t_5$

$$\varphi_t \stackrel{\text{def}}{=} \phi_t^1.$$

Then, given $C \geqslant 0$*,* $\beta > 0$ *and* $1 < \mu < \ell - 1$*, there exist two constants* $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ *and* $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$*, such that if the elements of the family of – nonlinear – operators* $\{T_t\}_{t \geqslant t^*}$ *are defined for* $f \in \mathrm{Diff}^\ell(\mathbb{U})$ *satisfying* (i) $|f|_{C^\ell(\mathbb{U})} < \beta$*,* (ii) $\mathbb{V}$ *contains the closure of* $f(\mathbb{U})$*, and* (iii) $f^*\Omega = \Omega$*, by*

$$T_t[f](x) \stackrel{\text{def}}{=} S_t[f]\big(\varphi_t(x)\big), \quad x \in \mathbb{U}_t,$$

where $\mathbb{U}_t$ *is as in Theorem* 18*, then* $T_t[f]$ *satisfies* T0, T1, T2, T4 *of Theorem* 18 *and the following properties*:

T3′. $|T_t[f] - S_t[f]|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa M_f t^{-\mu}$.
T5′. $|(T_t - \mathrm{Id})[f]|_{C^r(\mathbb{U}_t)} \leqslant \kappa M_f t^{-(\mu - r)}$*, for all* $0 \leqslant r \leqslant \mu$.
T6′. $|(T_\tau - T_t)[f]|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa M_f t^{-\mu}$*, for all* $\tau \geqslant t \geqslant t^*$.

Proof. That $T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$ follows from Remark 38 and Lemma 41. Notice that property T1 of Theorem 18 follows from part (i) of Lemma 41. Now, assume that $t^* \geqslant t_5$ is sufficiently large so that for all $t \geqslant t^*$ the following holds:

$$\kappa \hat{M}_f t^{-(\mu-1)} < C \log(2), \tag{58}$$

which is possible because $1 < \mu < \ell - 1$. Then using (58) and parts (iii) and (iv) of Lemma 41 one has for all $t \geqslant t_5$

$$|\varphi_t - \mathrm{id}|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu} < Ct^{-1} \tag{59}$$

and

$$|D\varphi_t|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant \exp\big(C^{-1} \kappa \hat{M}_f t^{-(\mu-1)}\big) < 2. \tag{60}$$

Notice that if $2 < \mu < \ell - 1$ then it is possible to choose t^* sufficiently large such that (compare with (58))

$$\kappa \hat{M}_f t^{-(\mu-2)} < \min\big(C^2, C\log(2)\big), \quad \forall t \geqslant t^*.$$

Then part (v) of Lemma 41 implies for such t that

$$\big|D^2 \varphi_t\big|_{C^0(\mathbb{U}_t + Ct^{-1})} < 2^3. \tag{61}$$

A consequence of (60) is that we can control the domain of the composition $S_t[f] \circ \varphi_t$ on complex strips because of the following estimate:

$$\big|\mathrm{Im}(\varphi_t)\big|_{C^0(\mathbb{U}_t + Ct^{-1})} \leqslant Ct^{-1} |D\varphi_t|_{C^0(\mathbb{U}_t + Ct^{-1})} < 2Ct^{-1},$$

from which we have

$$\varphi_t\big(\mathbb{U}_t + Ct^{-1}\big) \subset \mathbb{U} + 2Ct^{-1}. \tag{62}$$

Now property T2 of Theorem 18 follows easily. First, using (62) and part (ii) of Proposition 23 one has

$$\big|T_t[f]\big|_{C^0(\mathbb{U}_t+Ct^{-1})} = \big|S_t[f] \circ \varphi_t\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \big|S_t[f]\big|_{C^0(\mathbb{U}+(2Ct^{-1}))} \leqslant \kappa |f|_{C^0(\mathbb{U})}.$$

Now, using (62), part (i) of Lemma 33, and estimate (60) one has

$$\big|DT_t[f]\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \big|S_t[f]\big|_{C^1(\mathbb{U}+2Ct^{-1})}\big|D\varphi_t(x)\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \kappa |f|_{C^\ell(\mathbb{U})}.$$

To prove T3′ use (62), (59) and part (i) of Lemma 33 to obtain

$$\big|(T_t - S_t)[f]\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \big|S_t[f]\big|_{C^1(\mathbb{U}+2Ct^{-1})}|\varphi_t - \mathrm{id}|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant \kappa \hat{M}_f |f|_{C^\ell(\mathbb{U})} t^{-\mu}.$$

Furthermore, if $2 < \mu < \ell - 1$, then the chain rule, (62), part (i) of Lemma 33, and estimates (60) and (61) imply

$$\big|D^2 T_t[f]\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \leqslant 2^2\big|S_t[f]\big|_{C^2(\mathbb{U}+2Ct^{-1})} + 2^3\big|S_t[f]\big|_{C^1(\mathbb{U}+2Ct^{-1})} \leqslant \kappa |f|_{C^\ell(\mathbb{U})}.$$

This proves property T4 of Theorem 18.

Finally, properties T5′ and T6′ of Lemma 42 follow from Proposition 23, property T3′ of Lemma 42, Cauchy's estimates, and the following inequalities

$$\big|(T_t - 1)[f]\big|_{C^r(\mathbb{U}_t)} \leqslant \big|(T_t - S_t)[f]\big|_{C^r(\mathbb{U}_t)} + \big|(S_t - 1)[f]\big|_{C^r(\mathbb{U})},$$

and for $\tau \geqslant t$

$$\begin{aligned}\big|(T_\tau - T_t)[f]\big|_{C^0(\mathbb{U}_t+C\tau^{-1})} &\leqslant \big|(T_\tau - S_\tau)[f]\big|_{C^0(\mathbb{U}_t+C\tau^{-1})} + \big|(S_t - T_t)[f]\big|_{C^0(\mathbb{U}_t+Ct^{-1})} \\ &\quad + \big|(S_\tau - S_t)[f]\big|_{C^0(\mathbb{U}+C\tau^{-1})}. \qquad \square\end{aligned}$$

3.4.1. Exactness considerations

In this section we show that in the case that the diffeomorphism f is exact symplectic, then it is possible to construct analytic approximating functions $T_t[f]$ which are also exact symplectic, as claimed in part T7 of Theorem 18. Here use the calculus of deformations, similar constructions are obtained in [10].

Let $\mathbb{U}$ be as in Theorem 18. Of course, exactness is a problem only in the case that $\mathbb{U} = \mathbb{T}^n \times U$. In the other cases, Poincaré's Lemma shows that all symplectic maps are exact. Hence, throughout this section we assume that $\mathbb{U} = \mathbb{T}^n \times U$.

Let $T_t[f]$ be as in Lemma 42, we show that for t sufficiently large, there exists a diffeomorphism h_t such that $h_t \circ T_t[f]$ is exact and satisfies properties T1–T6 of Theorem 18. Notice that since $T_t[f]^*\Omega = \Omega$ we have that the form $(T_t[f]^*\alpha - \alpha)$ is closed. Recall that if $\Omega = d\alpha$, then $h_t \circ T_t[f]$ is exact if and only if the form $(h_t \circ T_t[f])^*\alpha - \alpha$ is exact. Equivalently

$$\big[T_t[f]^*\big(h_t^*\alpha - \alpha\big)\big] = -\big[T_t[f]^*\alpha - \alpha\big], \tag{63}$$

where $[\beta]$ represents the de Rham cohomology class of the closed form β. The existence of a diffeomorphism h_t satisfying (63) is proved in the following lemma where, moreover, we estimate the distance between h_t and the identity.

Lemma 43. *Let $\Omega = d\alpha$ be an exact symplectic form and $2 < \ell \notin \mathbb{Z}$. Assume that the hypotheses of Theorem* 18 *hold. Let $\mathbb{U}_t$ be defined by* (7)*. Then, given $C \geqslant 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, M_\Omega)$ and $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for any $t \geqslant t^*$ and any $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying* (i) $|f|_{C^\ell(\mathbb{U})} < \beta$, (ii) $\mathbb{V}$ *contains the closure of $f(\mathbb{U})$, and* (iii) f *is exact, there exists a diffeomorphism h_t satisfying equality* (63) *and such that the following holds*:

$$|h_t - \mathrm{id}|_{C^0(\mathbb{U}+\rho)} \leqslant \kappa \hat{M}_f t^{-\mu+1}, \tag{64}$$

$$|h_t|_{C^1(\mathbb{U}+\rho)} \leqslant \kappa. \tag{65}$$

Proof. Let $H^1(M, \mathbb{R})$ denote *the first de Rham cohomology group* of the manifold M. Let $\mathbb{V}$ be as in Theorem 18 and let $\mathbb{U}_t$ be as in (7). We note that if $\mathbb{U} = \mathbb{T}^n \times U$, and if $\mathbb{V}$ diffeomorphic to $\mathbb{U}$, then $H^1(\mathbb{U}_t, \mathbb{R}) = \mathbb{R}^n$ and $H^1(\mathbb{V}, \mathbb{R}) = \mathbb{R}^n$. For t sufficiently large, let $T_t[f]$ be as in Lemma 42. Consider

$$\begin{aligned} T_t[f]_\# : H^1(\mathbb{V}, \mathbb{R}) &\to H^1(\mathbb{U}_t, \mathbb{R}), \\ [\gamma] &\to \big[T_t[f]^*\gamma\big]. \end{aligned}$$

Notice that since $T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$ and since the pull-back commutes with the exterior derivative one has: (a) $T_t[f]_\#$ *is well defined*, (b) $T_t[f]_\#$ *at zero is equal to zero*, and (c) $T_t[f]_\#$ *is differentiable with invertible derivative at zero*. If moreover, f is exact we have

$$\big|\big[T_t[f]^*\alpha - \alpha\big]\big| = \big|\big[T_t[f]^*\alpha - f^*\alpha\big]\big| \leqslant \hat{\kappa}\big|(T_t - \mathrm{Id})[f]\big|_{C^1(\mathbb{U}_t)} \leqslant \kappa M_f t^{-\mu+1},$$

where we have used the fact that $T_t[f]$ satisfies property T5′ of Lemma 42. Hence a finite dimensional version of the Implicit Function Theorem implies that, for t sufficiently large, there exists $[\gamma_t] \in H^1(\mathbb{V}, \mathbb{R})$ such that

$$T_t[f]_\#\big([\gamma_t]\big) = -\big[T_t[f]^*\alpha - \alpha\big] \tag{66}$$

and

$$\big|[\gamma_t]\big| \leqslant \kappa M_f t^{-\mu+1}. \tag{67}$$

Let $\gamma_1, \ldots, \gamma_n$ be closed forms, analytic on $\mathbb{V} + \rho$, and such that $\{[\gamma_j]\}_{j=1}^n$ is a basis of $H^1(\mathbb{V}, \mathbb{R})$. For t sufficiently large, let $\eta_t = (\eta_t^1, \ldots, \eta_t^n) \in \mathbb{R}^n$ be such that

$$[\gamma_t] = \sum_{j=1}^n \eta_t^j [\gamma_i].$$

Then estimate (67) implies

$$|\eta_t| \leqslant \tilde{\kappa} M_f t^{-\mu+1}. \tag{68}$$

Following [23], we construct, for t sufficiently large, a diffeomorphism h_t satisfying

$$\left[h_t^* \alpha - \alpha\right] = \sum_{j=1}^{n} \eta_t^j [\gamma_i]. \tag{69}$$

The non-degeneracy of Ω implies the existence of a vector field X_t such that

$$i_{X_t}(\Omega) = \sum_{j=1}^{n} \eta_t^j \gamma_j. \tag{70}$$

From (68) and (70) we have

$$|X_t|_{C^0(\mathbb{V}+\rho)} \leqslant \kappa' |\eta| \leqslant \kappa M_f t^{-\mu+1}. \tag{71}$$

Let h_t^ε be the flow generated by X_t:

$$\frac{d}{d\varepsilon} h_t^\varepsilon = X_t \circ h_t^\varepsilon, \qquad h_t^0 = \mathrm{id}. \tag{72}$$

The existence of h_t^ε for all $\varepsilon \in [0,1]$ is obtained by assuming that t is sufficiently large and using (70). Using Proposition I.1.3. in [23] we have

$$\left(h_t^\varepsilon\right)^* \alpha - \alpha = \int_0^\varepsilon \frac{d}{ds} \left(h_t^s\right)^* \alpha \, ds = \varepsilon \sum_{j=1}^{n} \eta_j \gamma_j + d\beta_t^\varepsilon, \tag{73}$$

with

$$\beta_t^\varepsilon = \int_0^\varepsilon \left(\int_0^s \left(h_t^r\right)^* i_{X_t} \left(\sum_{j=1}^{n} \eta_j \gamma_j \right) dr \right) ds + \int_0^\varepsilon \left(h_t^s\right)^* i_{X_t}(\alpha) \, ds,$$

where we have used the Cartan's formula and the fact that the right-hand side of (70) is closed. From (73) one has that, for all for $\varepsilon \in [0,1]$, h_t^ε preserves the exact symplectic form $\Omega = d\alpha$. Define $h_t \stackrel{\mathrm{def}}{=} h_t^1$, then considering the first cohomology class in (73) we have that h_t satisfies (69). Finally notice that (69) and (66) imply (63).

Estimate (64) follows from (71) and (72). Now taking t sufficiently large, using (71), (72), and Gronwall's inequality we obtain, for t sufficiently large, the following estimate:

$$|Dh_t|_{C^0(\mathbb{U}+\rho)} < 2,$$

from which and (64) estimate (65) follows, for t sufficiently large. $\square$

It is clear from Lemma 43 that the composition $\tilde{T}_t[f] \stackrel{\text{def}}{=} h_t \circ T_t[f]$ is exact symplectic on $\mathbb{U}_t$. The verification of properties T1–T6 of Theorem 18 for the diffeomorphism $\tilde{T}_t[f]$ is performed by using Lemmas 43, 42, and the following estimates

$$\begin{aligned}
\left|h_t \circ T_t[f]\right|_{C^1(\mathbb{U}_t+Ct^{-1})} &\leqslant \kappa |h_t|_{C^1(\mathbb{U}+\rho)}\left(1+\left|T_t[f]\right|_{C^1(\mathbb{U}_t+Ct^{-1})}\right),\\
\left|h_t \circ T_t[f]\right|_{C^2(\mathbb{U}_t+Ct^{-1})} &\leqslant \kappa |h_t|_{C^2(\mathbb{U}+\rho)}\left(1+\left|T_t[f]\right|_{C^2(\mathbb{U}_t+Ct^{-1})}\right),\\
\left|h_t \circ T_t[f]-f\right|_{C^0(\mathbb{U}_t)} &\leqslant \kappa |h_t-\mathrm{id}|_{C^0(\mathbb{U})}+\kappa\left|T_t[f]-f\right|_{C^0(\mathbb{U}_t)},\\
\left|h_t \circ T_t[f]-S_t[f]\right|_{C^0(\mathbb{U}_t+Ct^{-1})} &\leqslant \kappa |h_t|_{C^1(\mathbb{U}+\rho)}\left|T_t[f]-S_t[f]\right|_{C^0(\mathbb{U}_t+Ct^{-1})}\\
&\quad+\kappa |h_t-\mathrm{id}|_{C^0(\mathbb{U}+\rho)}.
\end{aligned}$$

3.5. The contact case (Proof of Theorem 20)

Theorem 20 is proved following the same steps of the proof of Theorem 18 given in Section 3.4. Here we just mention the necessary modifications. Let ℓ, $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{U}_t$ be as in Theorem 18. Let Ω be a contact form on $\mathbb{V}$, $f \in \mathrm{Diff}^{\ell}(\mathbb{U})$ a contact diffeomorphism, and let λ be a nowhere zero function such that $f^*\Omega = \lambda\Omega$. Define for $t \geqslant 1$ and $\varepsilon \in [0,1]$

$$\Omega_t^{\varepsilon} \stackrel{\text{def}}{=} \lambda\Omega + \varepsilon\left(S_t[f]^*\Omega - \lambda\Omega\right).$$

Notice that, since the 2-form $d\Omega$ is a symplectic form on the fibres of the $2n$-dimensional subbundle $\mathrm{Ker}(\Omega) \subset T(\mathbb{U})$, with the obvious modifications, Lemma 39 holds for the isomorphism $\mathscr{I}_{d\Omega}|_{\mathrm{Ker}(\Omega)}$. Roughly speaking, this means that, for t sufficiently large, $\mathscr{I}_{d\Omega_t^{\varepsilon}}|_{\mathrm{Ker}(\Omega_t^{\varepsilon})}$ is also an isomorphism. Hence, for t sufficiently large and $\varepsilon \in [0,1]$, there exists a vector field u_t^{ε} satisfying

$$u_t^{\varepsilon} = -\left(\mathscr{I}_{d\Omega_t^{\varepsilon}}|_{\mathrm{Ker}(\Omega_t^{\varepsilon})}\right)^{-1}\left(\frac{\partial}{\partial\varepsilon}\Omega_t^{\varepsilon}\right), \quad u_t^{\varepsilon} \in \mathrm{Ker}\left(\Omega_t^{\varepsilon}\right),$$

equivalently,

$$\begin{aligned}
i_{u_t^{\varepsilon}}\left(d\Omega_t^{\varepsilon}\right) &= -\left(S_t[f]^*\Omega - \lambda\Omega\right),\\
i_{u_t^{\varepsilon}}\left(\Omega_t^{\varepsilon}\right) &= 0.
\end{aligned} \tag{74}$$

Applying Proposition 29 to the 1-forms Ω and $\lambda\Omega$ we obtain, for t sufficiently large,

$$\left|S_t[f]^*\Omega - \lambda\Omega\right|_{C^0(\mathbb{U}+(2Ct^{-1}))} \leqslant \kappa \hat{M}_f t^{-\mu}.$$

Then, following the same steps of the proof of Lemma 39 we obtain that the solution $u_t^{\varepsilon}(x)$ of (74), is continuous on $(\varepsilon, x) \in [0,1] \times \mathbb{U}+2Ct^{-1}$, real analytic with respect to $x \in \mathbb{U}+2Ct^{-1}$, and moreover

$$\left|u_t^{\varepsilon}\right|_{C^0(\mathbb{U}+2Ct^{-1})} \leqslant 2M_{\Omega}\left|S_t[f]^*\Omega - \lambda\Omega\right|_{C^0(\mathbb{U}+2Ct^{-1})} \leqslant \kappa \hat{M}_f t^{-\mu}. \tag{75}$$

Now, for t sufficiently large, let ϕ_t^{ε} be the solution of the following differential equation

$$\frac{d}{d\varepsilon}\phi_t^\varepsilon = u_t^\varepsilon, \qquad \phi_t^0 = \mathrm{id},$$

then (74) and the Cartan's formula for the Lie derivative imply

$$\frac{d}{d\varepsilon}\left((\phi_t^\varepsilon)^* \Omega_t^\varepsilon\right) = (\phi_t^\varepsilon)^* \left[d\left(i_{u_t^\varepsilon}\Omega_t^\varepsilon\right) + i_{u_t^\varepsilon}\left(d\Omega_t^\varepsilon\right) + \frac{d}{d\varepsilon}\Omega_t^\varepsilon \right] = 0.$$

Hence

$$(\phi_t^\varepsilon)^* \Omega_t^\varepsilon = \lambda\Omega, \quad \forall \varepsilon \in [0, 1].$$

Moreover, following the proof of Lemma 41 and using (75) one obtains that ϕ_t^1 satisfies estimates (iii)–(v) of Lemma 41. The proof of Theorem 20 is finished by following the same steps in the proof of Lemma 42. □

4. An application: KAM theory without action-angle variables for finitely differentiable symplectic maps

Let $(\mathbb{U}, \Omega = d\alpha)$ be a $2n$-dimensional analytic exact symplectic manifold and let $f \in \mathrm{Diff}^\ell(\mathbb{U})$ be an exact symplectic map. The study of the existence of n-dimensional invariant tori with quasi-periodic motion is based on the study of the equation

$$F(f, K) = 0, \tag{76}$$

where

$$F(f, K)(\theta) \stackrel{\mathrm{def}}{=} (f \circ K)(\theta) - K(\theta + \omega), \tag{77}$$

$K : \mathbb{T}^n \to \mathbb{U}$ is the function to be determined, and $\omega \in \mathbb{T}^n$ satisfies a Diophantine condition. In [15] it is proved that if f is a real analytic diffeomorphism and if there exists a real analytic parameterization of an n-dimensional torus K, satisfying a non-degeneracy condition, such that (f, K) is an approximate solution of (76) in the sense that $|F(f, K)|_{C^0(\mathbb{T}^n + \rho)}$ is 'sufficiently small,' with respect to the Diophantine and non-degeneracy conditions, then there exists a true real analytic solution of (76), which moreover is close to the approximate solution. In Theorem 46 we give the rigorous formulation of this result. We emphasize that in Theorem 46 we do not assume the symplectic map is written either in action-angle variables or as perturbation of an integrable one. Moreover, the proof of Theorem 46 produces an algorithm to compute invariant tori for exact symplectic maps.

In this section we show that a finitely differentiable version of Theorem 46 also holds, see Theorem 47 for the formulation. The proof of Theorem 47 we present here is a slightly modified Moser's analytic smoothing method. We remark that, since Theorem 46 holds for exact symplectic maps, then the symplectic map f is smoothed using the operator T_t of Theorem 18. Moreover, rather than assuming the existence of an analytic approximate solution of (76) we assume the existence of a finitely differentiable initial approximate solution of (76) and give conditions guaranteeing the existence of an analytic solution, which is close to the approximately invariant one in finitely differentiable norms. This is achieved by using the estimates given in Theorem 18 and Proposition 37.

In this section we also prove the bootstrap of regularity of invariant tori with Diophantine rotation vector for exact symplectic diffeomorphisms. To obtain the bootstrap of regularity first, we prove the local uniqueness of finitely differentiable invariant tori for finitely differentiable symplectic maps. We remark that the uniqueness result stated in Theorem 49 is the finitely differentiable version of Theorem 2 in [15].

The local uniqueness and the bootstrap of regularity are stated in Theorems 49 and 50, respectively. Theorems 49 and 50 are similar to Theorems 4 and 5 in [17], respectively. However, while the latter are stated and proved for Hamiltonian vector fields written in a Lagrangian formalism, Theorems 49 and 50 are stated for symplectic maps that are not assumed either to be written in action-angle variables or to be perturbations of integrable systems, and proved using the symplectic formalism rather than the Lagrangian one.

The existence of the operator T_t in Theorem 18, in the exact symplectic case, enables us to obtain analytic approximate solutions of Eq. (76) close to a given finitely differentiable one. This together with the uniqueness argument yield the bootstrap of regularity for solutions of (76).

Let $\mathbb{U}$ be either an open subset of $\mathbb{R}^{2n}$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$. In addition to the notation introduced in Section 3.1 we use the following notation. For each $x \in \mathbb{U}$ let $J(x) : T_x\mathbb{U} \to T_x\mathbb{U}$ be linear isomorphism satisfying

$$\Omega(x)(\xi, \eta) = \xi^\top \cdot J(x)\eta, \tag{78}$$

where $\cdot$ is the Euclidean scalar product on $\mathbb{R}^{2n}$. The average of a mapping $K \in C^0(\mathbb{T}^n, \mathbb{U})$ is defined by

$$\operatorname{avg}\{K\}_\theta \stackrel{\text{def}}{=} \int_{\mathbb{T}^n} K(\theta)\, d\theta.$$

Definition 44. Given $\gamma > 0$ and $\sigma \geqslant n$, we define $D(\gamma, \sigma)$ as the set of frequency vectors $\omega \in \mathbb{T}^n$ satisfying the Diophantine condition:

$$|\ell \cdot \omega - m| \geqslant \gamma |\ell|_1^{-\sigma}, \quad \forall \ell \in \mathbb{Z}^n \setminus \{0\},\ m \in \mathbb{Z},$$

where $|\ell|_1 = |\ell_1| + \cdots + |\ell_n|$.

Definition 45. Given a symplectic diffeomorphism $f \in \operatorname{Diff}^1(\mathbb{U})$ and $\omega \in D(\gamma, \sigma)$, let $\mathcal{N}$ denote the set of functions in $K \in C^1(\mathbb{T}^n, \mathbb{U})$ satisfying the following conditions:

N1. There exists an $(n \times n)$-matrix-valued function $N(\theta)$ such that

$$N(\theta)\big(DK(\theta)^\top DK(\theta)\big) = I_n,$$

where I_n is the n-dimensional identity matrix.

N2. The average of the matrix-valued function

$$A(\theta) \stackrel{\text{def}}{=} P(\theta+\omega)^\top \big[Df\big(K(\theta)\big) J\big(K(\theta)\big)^{-1} P(\theta) - J\big(K(\theta+\omega)\big)^{-1} P(\theta+\omega)\big], \tag{79}$$

with J defined in (78) and

$$P(\theta) \stackrel{\text{def}}{=} DK(\theta)N(\theta),$$

is non-singular.

By the Rank Theorem, condition N1 guarantees that $\dim K(\mathbb{T}^n) = n$. For the KAM theorems 46 and 47, the main non-degeneracy condition is N2, which is a twist condition. Note that N1 only depends on K whereas N2 depends on both K and f.

From now on we assume that $\Omega = d\alpha$ is analytic exact symplectic form as in Theorem 18. Let J be the isomorphism defined by (78), and let J^{-1} denote its inverse. The following Theorem 46 is the main theorem from [15].

Theorem 46. *Let $\mathbb{U}$ be either a compensated open domain in $\mathbb{R}^{2n}$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$ a compensated open domain. Let f be an exact symplectic diffeomorphism on $\mathbb{U}$ and $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$. Assume that the following hypotheses hold*:

A1. $K \in \mathcal{N} \cap \mathcal{A}(\mathbb{T}^n + \rho, C^1)$ (*see Definitions* 45 *and* 4).

A2. *The map f is real analytic and it can be holomorphically extended to $\mathcal{B}$, a complex neighbourhood of $K(\mathbb{T}^n + \rho)$, such that* $\operatorname{dist}(K(\mathbb{T}^n + \rho), \partial\mathcal{B}) > \eta > 0$. *Furthermore,* $|f|_{C^2(\mathcal{B})} < \infty$.

A3. $|J|_{C^1(\mathcal{B})}, |J^{-1}|_{C^1(\mathcal{B})}, |\alpha|_{C^2(\mathcal{B})} < \infty$.

Then, there exists a constant $c > 0$ depending on σ, n, $|f|_{C^2(\mathcal{B})}$, $|\alpha|_{C^2(\mathcal{B})}$, $|J|_{C^1(\mathcal{B})}$, $|J^{-1}|_{C^1(\mathcal{B})}$, $|DK|_{C^0(\mathbb{T}^n+\rho)}$, $|N|_{C^0(\mathbb{T}^n+\rho)}$, $|(\operatorname{avg}\{A\}_\theta)^{-1}|$ (where N and A are as in Definition 45) *such that, if*

$$c\gamma^{-4}\rho^{-4\sigma}\left|F(f,K)\right|_{C^0(\mathbb{T}^n+\rho)} < \min(1,\eta),$$

then there exists $K^* \in \mathcal{N} \cap \mathcal{A}(\mathbb{T}^n + \rho/2, C^1)$ such that $F(f, K^*) = 0$. Moreover,

$$|K^* - K|_{C^0(\mathbb{T}^n+\rho/2)} \leqslant c\gamma^{-2}\rho^{-2\sigma}\left|F(f,K)\right|_{C^0(\mathbb{T}^n+\rho)} \tag{80}$$

and

$$|DK^* - DK|_{C^0(\mathbb{T}^n+\rho/2)} \leqslant c\gamma^{-2}\rho^{-(2\sigma+1)}\left|F(f,K)\right|_{C^0(\mathbb{T}^n+\rho)}.$$

The finitely differentiable version of Theorem 46 we present here is the following. For additional smoothness results see Theorem 50.

Theorem 47. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$. Let $m \in \mathbb{N}$, $\ell \notin \mathbb{N}$ be such that $4\sigma + 3 < \ell < m$. Let $\mathbb{U}$ be either a compensated open domain in $\mathbb{R}^{2n}$ with C^m-boundary, or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$ a compensated open domain with C^m-boundary. Let $f \in \operatorname{Diff}^\ell(\mathbb{U})$ be an exact symplectic diffeomorphism and let $K \in C^\ell(\mathbb{T}^n, \mathbb{U})$ be a parameterization of an n-dimensional torus. Assume that the following hypotheses hold*:

S1. $|DK|_{C^0(\mathbb{T}^n)} < \beta$ *and* $K(\mathbb{T}^n) \subset \mathbb{U}$, *with* $\eta \stackrel{\text{def}}{=} 2^{-1}\operatorname{dist}(K(\mathbb{T}^n), \partial\mathbb{U}) > 0$.

S2. $K \in \mathcal{N}$ (*see Definition* 45).

S3. *$\Omega = d\alpha$ is real analytic on $\mathbb{U}+\rho$, and $|J|_{C^\ell(\mathbb{U}+\rho)}, |\alpha|_{C^\ell(\mathbb{U}+\rho)} < \zeta$, and $|J^{-1}|_{C^1(\mathbb{U}+\rho)} < M_\Omega$, for some $\rho, \zeta > 0$.*

Then, given $4\sigma + 2 < \mu < \ell - 1$ there exist two positive constants c and $\rho^* < 1$, depending on μ, n, ℓ, σ, ζ, β, M_Ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $|(\operatorname{avg}\{A\}_\theta)^{-1}|$, such that: given $0 < \rho_1 \leqslant \rho^*$, if $\mu - 2\sigma \notin \mathbb{Z}$ and

$$c\gamma^{-4}\rho_1^{-(4\sigma+1)}\bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)} \leqslant \min(1, \kappa\beta, \eta), \tag{81}$$

where $\kappa = \kappa(n, \ell, 1)$ is as in Proposition 23, then there exists a parameterization of an n-dimensional torus, $K^* \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n, \mathbb{U})$ such that $F(f, K^*) = 0$ and

$$|K - K^*|_{C^\nu(\mathbb{T}^n)} \leqslant \tilde{c}\gamma^{-2}\rho_1^{-(2\sigma+\nu)}\bigl(\rho_1^{\mu-1} + \bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)}\bigr),$$

for all $0 \leqslant \nu < \mu - (2\sigma + 1)$, where F is as in (77) and $\tilde{c}$ is a constant depending on the same quantities as c.

Remark 48. Let f and K be as in Theorem 47. Notice that since $|f|_{C^\ell(U)}$ and $|K|_{C^\ell(\mathbb{T}^n)}$ are bounded we have that $|F(f,K)|_{C^\ell(\mathbb{T}^n)}$ is also bounded. If moreover assumption (81) holds then there is a constant κ such that

$$\bigl|F(f,K)\bigr|^{\ell}_{C^\ell(\mathbb{T}^n)} \leqslant \kappa \quad \text{and} \quad \bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)} \leqslant \kappa\rho_1^{4\sigma+1}.$$

Thus, by using the interpolation estimates [4,21], we have for any $0 \leqslant s \leqslant \ell$

$$\bigl|F(f,K)\bigr|_{C^s(\mathbb{T}^n)} \leqslant \kappa\bigl|F(f,K)\bigr|^{s}_{C^\ell(\mathbb{T}^n)}\bigl|F(f,K)\bigr|^{1-s/\ell}_{C^0(\mathbb{T}^n)} \leqslant \hat{\kappa}\rho_1^{(4\sigma+1)(1-s/\ell)}.$$

Hence assumption (81) implies that all the intermediate norms $|F(f,K)|_{C^s(\mathbb{T}^n)}$ with $0 \leqslant s < \ell$ are also small. We therefore have that hypothesis (81) is equivalent to assuming that the C^s-norms of the error function are small, for $0 \leqslant s < \ell$.

The local uniqueness is stated in the following

Theorem 49. *Let $\omega \in D(\gamma, \sigma)$ for some $\gamma > 0$ and $\sigma > n$. Let $\ell > 2\sigma$ be such that $\ell, \ell - 2\sigma \notin \mathbb{Z}$. Let $f \in \mathrm{Diff}^{\ell+2}(\mathbb{U})$ be a symplectic diffeomorphism. Assume that (f, K_1) and (f, K_2) satisfy* (76)*, with $K_1, K_2 \in C^{\ell+1}(\mathbb{T}^n, \mathbb{U})$ satisfying* N1 *and* N2 *in Definition* 45*. Then, there exists a constant κ, depending on n, $|J^{-1}|_{C^0(\mathbb{U})}$, $|f|_{C^{\ell+2}(\mathbb{U})}$, $|K_2|_{C^{\ell+1}(\mathbb{T}^n)}$, $|K_1|_{C^{\ell+1}(\mathbb{T}^n)}$, and $|N_2|_{C^0}$, with N_2 defined as in* N1 *in Definition* 45 *by replacing K with K_2, such that if*

$$\kappa\gamma^{-2}|K_1 - K_2|_{C^\ell(\mathbb{T}^n)} < 1,$$

then $K_1 \circ R_{\hat{\theta}} = K_2$ on $\mathbb{T}^n$, for some constant $\hat{\theta} \in \mathbb{R}^n$.

The bootstrap of regularity is stated in the following theorem.

Theorem 50. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$ and let $\varrho > 0$. Let $4\sigma + 3 < \ell_1 < m$, with $m \in \mathbb{N}$ and $\ell_1 \notin \mathbb{N}$. Let $\mathbb{U}$ be as in Theorem* 47. *Let (K, f) be a solution of* (76)*, with $f \in \mathrm{Diff}^{\ell_1}(\mathbb{U})$ an exact symplectic diffeomorphism and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ satisfying* N1 *and* N2 *in Definition* 45. *Let $\ell \in [\ell_1, m)$ be not an integer and assume that $f \in \mathrm{Diff}^{\ell}(\mathbb{U})$, and that hypotheses* S1–S3 *in Theorem* 47 *hold* (*replacing ρ with ϱ in* S3). *Then, for any $4\sigma + 2 < \mu < \ell - 1$ satisfying $\mu - (2\sigma + 1) \notin \mathbb{Z}$ we have that $K \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n)$. Moreover if $m = \infty$ and $f \in \mathrm{Diff}^{\infty}(\mathbb{U})$ then $K \in C^{\infty}(\mathbb{T}^n, \mathbb{U})$. Furthermore, if $f \in \mathcal{A}(\mathbb{U} + \varrho, C^{\ell_1})$, then $K \in \mathcal{A}(\mathbb{T}^n, C^1)$.*

4.1. Existence (Proof of Theorem 47)

Throughout this section we assume that the hypotheses of Theorem 47 hold. As we already mentioned, the proof of Theorem 47 given here is based on Moser's technique of analytic smoothing [2,4]. What we do is the following:

Step 1: Obtain an analytic approximate solution (f_1, K_1) of (76), with f_1 exact symplectic map and K_1 satisfying properties N1–N2 of Definition 45.

Step 2: Apply Moser's smoothing technique and Theorem 46 to construct a sequence of analytic solutions of (76) converging to a finitely differentiable solution (f, K^*). More concretely, starting with (f_1, K_1), we assume that we have computed (f_m, K_m) an analytic solution of (76) and verify that, if T_t is as in Theorem 18 then, for a suitable t_m, $(T_{t_m}[f], K_m)$ is the approximate solution of (76) that satisfies the hypotheses of Theorem 46, so that one obtains a new analytic solution (f_{m+1}, K_{m+1}) of (76), with $f_{m+1} = T_{t_m}[f]$. The convergence of the method is obtain by using (80) in Theorem 46.

In order to perform Step 1 we use pairs of functions of the form $(T_t[f], S_t[K])$, where T_t is as in Theorem 18, and S_t is as in Definition 6. Notice that since S_t takes periodic functions into periodic functions then $S_t[K] \in \mathcal{A}(\mathbb{T}^n + t^{-1}, C^1)$ is an analytic parameterization of an n-dimensional torus that is close to K (see Remark 8).

To prove that $S_t[K]$ satisfies properties N1–N2 of Definition 45, since condition N2 in Definition 45 depends on both the parameterization and the map, it is necessary to fix the constants appearing in Theorem 18 and verify that $S_t[K](\mathbb{T}^n + t^{-1})$ belongs to the domain of $T_t[f]$. This is done in the following

Lemma 51. *Let $K \in C^{\ell}(\mathbb{T}^n, \mathbb{U})$ satisfy hypothesis* S1 *of Theorem* 47. *Let κ be as in Proposition* 23*, define $r \stackrel{\text{def}}{=} \kappa\beta$, then*

$$\left|DS_t[K]\right|_{C^0(\mathbb{T}^n + t^{-1})} < r, \quad \text{for all } t \geqslant 1. \tag{82}$$

Moreover, if $\mathbb{U}_t$ is defined by (7)*, then there exists $t_6 \geqslant 1$, depending on n, ℓ, $|K|_\ell$, and η such that for all $t > t_6$, $S_t[K](\mathbb{T}^n) \subset \mathbb{U}_t$, and*

$$\left|S_t[K] - K\right|_{C^0(\mathbb{T}^n)} < \frac{1}{2}\eta. \tag{83}$$

Furthermore, if $2 < \mu < \ell - 1$ is given, let $t^* = t^*(d, \ell, \mathbb{V}, 2r, \mu, |f|_{C^{\ell}(\mathbb{U})}, M_\Omega, \zeta)$ be as in Theorem 18*, then for all $t \geqslant \max(t^*, t_6)$, the components of the symplectic map $T_t[f]$ belong to $\mathcal{A}(\mathbb{U}_t + 2rt^{-1}, C^2)$ and properties* T0–T7 *of Theorem* 18 *hold.*

Proof. Part (ii) of Proposition 23 implies (82), from which we have

$$\left|\mathrm{Im}\left(S_t[K]\right)\right|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant t^{-1}\left|DS_t[K]\right|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant rt^{-1}.$$

Let $t_6 > 1$ be sufficiently large so that for any $t \geqslant t_6$ the following holds:

$$\max\left(\kappa|K|_{C^\ell(\mathbb{T}^n)}t^{-\ell}, t^{-1}\right) < 2^{-1}\eta,$$

then part (i) of Proposition 23 implies (83). So we have $S_t[K](\mathbb{T}^n + t^{-1}) \subset \mathbb{U}_t + rt^{-1}$. Now apply Theorem 18 to the constants $C = 2r$, and $\beta = |f|_{C^\ell(\mathbb{U})}$. □

Now we prove that, for t sufficiently large, $S_t[K]$ satisfies N1–N2 of Definition 45.

Lemma 52. *Let r and t_6 be as in Lemma* 51, *and let $2 < \mu < \ell - 1$ be fixed. Assume that $K \in C^\ell(\mathbb{T}^n, \mathbb{U})$ satisfies the hypothesis of Theorem* 47 *and let N and A be as in Definition* 45. *Then, there exists $t_7 \geqslant t_6$, depending on n, ℓ, $2r$, μ, η, M_Ω, $|K|_{C^\ell(\mathbb{T}^d)}$, $|N|_{C^0(\mathbb{T}^d)}$, $|\mathrm{avg}\{A\}_\theta^{-1}|$, $|f|_{C^\ell(\mathcal{B})}$ and M_f, with M_f as in Theorem* 18, *such that $S_t[K] \in \mathcal{A}(\mathbb{T}^n + t^{-1}, C^1) \cap \mathcal{N}$, for all $t \geqslant t_7$. Moreover, if*

$$N_t(\theta) \stackrel{\text{def}}{=} \left(DS_t[K](\theta)^\top DS_t[K](\theta)\right)^{-1}$$

and

$$\begin{aligned} A_t(\theta) \stackrel{\text{def}}{=}\ & P_t(\theta+\omega)^\top DT_t[f]\left(S_t[K](\theta)\right)J\left(S_t[K](\theta)\right)^{-1}P_t(\theta) \\ & - P_t(\theta+\omega)^\top J\left(S_t[K](\theta+\omega)\right)^{-1}P_t(\theta+\omega), \end{aligned}$$

where $P_t(\theta) \stackrel{\text{def}}{=} S_t[DK](\theta)N_t(\theta)$, then the following hold

$$|N_t|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant 2|N|_{C^0(\mathbb{T}^n)}\left(1 + \kappa\hat{M}_{K,f}|N|_{C^0(\mathbb{T}^n)}t^{-1}\right) \tag{84}$$

and

$$\left|\mathrm{avg}\{A_t\}_\theta^{-1}\right| \leqslant \left|\mathrm{avg}\{A\}_\theta^{-1}\right|\left(1 + \kappa\hat{M}_{K,f}\left|\mathrm{avg}\{A\}_\theta^{-1}\right|t^{-\mu+2}\right),$$

where κ is a constant depending on n and ℓ and $\hat{M}_{K,f}$ depends on $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathcal{B})}$ and M_f.

Proof. Notice that the conditions N1–N2 in Definition 45 deal with invertibility of matrices, hence Lemma 52 is a consequence of the openness of the invertibility of matrices. In what follows we obtain explicit estimates for the size of t. Performing some simple computations and using Proposition 23 one has

$$\left|DS_t[K](\theta)^\top DS_t[K](\theta) - N(\theta)^{-1}\right|_{C^0(\mathbb{T}^n)} \leqslant \kappa|K|^2_{C^\ell(\mathbb{T}^n)}t^{-\ell+1}.$$

Hence if t is sufficiently large such that

$$\kappa |K|^2_{C^\ell(\mathbb{T}^n)} |N|_{C^0(\mathbb{T}^n)} t^{-\ell+1} \leqslant 1/2,$$

the Neuman's series theorem implies that, $DS_t[K](\theta)^\top DS_t[K](\theta)$ is invertible, for all $\theta \in \mathbb{R}^d$, and its inverse, denoted by N_t, satisfies

$$|N_t - N|_{C^0(\mathbb{T}^n)} \leqslant 2\kappa |K|^2_{C^\ell(\mathbb{T}^n)} |N|^2_{C^0(\mathbb{T}^n)} t^{-\ell+1} \leqslant |N|_{C^0(\mathbb{T}^n)}. \tag{85}$$

Now, let $\theta \in \mathbb{R}^n + t^{-1}$, then part (i) of Lemma 33 implies, for t sufficiently large,

$$\left| DS_t[K](\theta) - DS_t[K]\big(\mathrm{Re}(\theta)\big)\right| \leqslant \left| D^2 S_t[K]\right|_{C^0(\mathbb{T}^n + t^{-1})} \left|\mathrm{Im}(\theta)\right| \leqslant \kappa |K|_{C^\ell(\mathbb{T}^n)} t^{-1},$$

κ is a constant depending on n, and ℓ. So one obtains

$$\left| DS_t[K]^\top(\theta) DS_t[K](\theta) - N_t^{-1}\big(\mathrm{Re}(\theta)\big)\right| \leqslant \kappa |K|^2_{C^\ell(\mathbb{T}^n)} t^{-1}.$$

Then, if t is sufficiently large so that

$$2\kappa |K|^2_{C^\ell(\mathbb{T}^n)} |N|_{C^0(\mathbb{T}^n)} t^{-1} \leqslant 1/2,$$

we have from (85)

$$\kappa |K|^2_{C^\ell(\mathbb{T}^n)} |N_t|_{C^0(\mathbb{T}^n)} t^{-1} \leqslant 1/2.$$

Hence, Neuman's series theorem implies that $DS_t[K](\theta)^\top DS_t[K](\theta)$ is invertible for all $\theta \in \mathbb{R}^n + t^{-1}$ and

$$|N_t|_{C^0(\mathbb{T}^n + t^{-1})} \leqslant |N_t|_{C^0(\mathbb{T}^n)} + 2|N_t|^2_{C^0(\mathbb{T}^n)} \kappa |K|^2_{C^\ell(\mathbb{T}^n)} t^{-1},$$

from which and (85) estimate (84) follows.

It is clear that, for t sufficiently large, A_t is a perturbation of A defined in (79). In what follows we give an estimation of the size of $|A_t - A|_{C^0(\mathbb{T}^n)}$. Let $P(\theta) = DK(\theta)N(\theta)$, then using (85) and Proposition 23 we have

$$|P_t - P|_{C^0(\mathbb{T}^n)} \leqslant \kappa M_K t^{-\ell+1}, \tag{86}$$

where κ depends on n and ℓ, and M_K is a constant depending on $|K|_{C^\ell(\mathbb{T}^n)}$ and $|N|_{C^0(\mathbb{T}^n)}$. Moreover, performing some simple computations and using Theorem 18, Proposition 23, and the Cauchy's estimates we have

$$\left| DT_t[f]\big(S_t[K](\theta)\big) - Df\big(K(\theta)\big)\right|_{C^0(\mathbb{T}^n)} \leqslant \kappa t^{-\mu+2}, \tag{87}$$

where κ is a generic constant independent of t. Finally, using again Proposition 23 we have

$$\left| J \circ S_t[K] - J \circ K\right|_{C^0(\mathbb{T}^n)} \leqslant |J|_{C^1(\mathbb{U})} \kappa |K|_{C^\ell(\mathbb{T}^n)} t^{-\ell+2}. \tag{88}$$

Performing some computations and using (86), (87), and (88) one gets

$$|A_t - A|_{C^0(\mathbb{T}^n)} \leqslant \kappa M_{f,K} t^{-\mu+2},$$

where κ is a constant depending on n, ℓ, C^{-1}, κ, κ, $|J|_{C^1(\mathbb{U})}$, and $M_{f,K}$ is a constant depending on $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathcal{B})}$ and M_f. Hence the proof of Lemma 52 is finished by applying Neuman's series theorem and taking t is sufficiently large so that

$$\left|\operatorname{avg}\{A\}_\theta^{-1}\right| \kappa M_{f,K} t^{-\mu+2} \leqslant 1/2. \qquad \square$$

From Lemmas 51 and 52 we have that, for t sufficiently large, $(T_t[f], S_t[K])$ is a candidate for an analytic approximate solution of Eq. (76). In the following lemma we summarize the results of Lemmas 51 and 52 and give an estimate of $|F(T_t[f], S_t[K])|_{C^0(\mathbb{T}^n+t^{-1})}$.

Lemma 53. *Let t_7 be as in Lemma 52, and let $2 < \mu < \ell - 1$. Assume that $K \in C^\ell(\mathbb{T}^n)$, $f \in \mathrm{Diff}^\ell(\mathbb{U})$ and that hypotheses of Theorem 47 hold. Then there exists $t_8 \geqslant t_7$, depending on n, ℓ, β, μ, ζ, M_Ω, η, $|K|_{C^\ell(\mathbb{T}^d)}$, $|\operatorname{avg}\{A\}_\theta^{-1}|$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathbb{U})}$, and M_f, with M_f as in Theorem 18, such that for all $t \geqslant t_8$ the following hold:*

(i) *$S_t[K] \in \mathcal{A}(\mathbb{T}^n + t^{-1}, C^1)$, and $|DS_t[K]|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant r$, $r = r(n, \beta)$ as in Lemma 51.*
(ii) *$S_t[K](\mathbb{T}^n) \subset \mathbb{U}_t$, with $|S_t[K] - K|_{C^0(\mathbb{T}^n)} < \eta/2$.*
(iii) *$T_t[f] \in \mathcal{A}(\mathbb{U}_t + 2rt^{-1}, C^2)$ with $|T_t[f]|_{C^2(\mathbb{U}_t+(2rt^{-1}))} \leqslant \kappa M_f$.*
(iv) *$S_t[K] \in \mathcal{N}$ and if N_t and A_t are as in Lemma 52, then*

$$|N_t|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant 3|N|_{C^0}, \qquad \left|\operatorname{avg}\{A_t\}_\theta^{-1}\right| \leqslant \frac{3}{2}\left|\operatorname{avg}\{A\}_\theta^{-1}\right|.$$

(v) *There is a constant κ, depending on n, ℓ, C, μ, M_Ω, and $|Df|_{C^0(\mathbb{U})}$ such that*

$$\left|F\big(T_t[f], S_t[K]\big)\right|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant \kappa \hat{M}_{K,f} t^{-\mu+1} + \kappa \left|F(f, K)\right|_{C^0(\mathbb{T}^n)},$$

where

$$\hat{M}_{K,f} = \max\big(M_f, |f|_{C^\ell(\mathbb{U})}\big(1 + |K|^\mu_{C^\ell(\mathbb{T}^n)} + |K|_{C^\ell(\mathbb{T}^n)}\big)\big),$$

with M_f as in Theorem 18.

Proof. Parts (i)–(iii) are stated in Lemma 51. Part (iv) is consequence of Lemma 52. Part (v) is a consequence of property T3 in Theorem 18, Proposition 37, and part (ii) in Proposition 23. Indeed, let t_2 be as in Proposition 37, and let t^* and $\{T_t\}_{t \geqslant t^*}$ be as in Lemma 51. Then part (v) follows by taking $t_8 = \max(t_2, t^*, t_7)$ and using the following equality

$$\begin{aligned} F\big(T_t[f], S_t[K]\big) &= \big\{\big(T_t[f] - S_t[f]\big) \circ S_t[K]\big\} + \big\{S_t[f] \circ S_t[K] - S_t[f \circ K]\big\} \\ &\quad + S_t\big[F(f, K)\big]. \qquad \square \end{aligned}$$

Now we give some sufficient conditions that ensure that we can construct a sequence of analytic solutions (f_j, K_j^*) of Eq. (76):

Lemma 54. *Let $r = r(n,\beta)$ be as in Lemma 51. Let $2 < \mu < \ell - 1$ be given and assume that for fixed $m \geqslant 1$ there is a $\tau_m \geqslant 1$ such that $(f_m, K_m) = (T_{\tau_m}[f], S_{\tau_m}[K])$ satisfies the following conditions*:

A1(m) *$K_m \in \mathcal{A}(\mathbb{T}^n + \rho_m, C^1)$, and $|DK_m|_{C^0(\mathbb{T}^n+\rho_m)} \leqslant r_m$, with $\rho_m \stackrel{\text{def}}{=} \tau_m^{-1}$ and $r_m = r\sum_{j=0}^{m-1} 2^{-j}$.*

A2(m) *$K_m(\mathbb{T}^n) \subset \mathbb{U}_{\tau_m}$, with $|K_m - K|_{C^0(\mathbb{T}^n)} < \eta_m$ where $\eta_m \stackrel{\text{def}}{=} \eta \sum_{j=1}^{m} 2^{-j}$.*

A3(m) *$f_m \in A(\mathbb{U}_{\tau_m} + (2r_m\rho_m), C^2)$ with $|f_m|_{C^2(\mathbb{U}_{\tau_m}+2r_m\rho_m)} \leqslant \kappa M_f$.*

A4(m) *If N_m and A_m are defined as in Lemma 52, by replacing $S_t[K]$ with K_m, then*

$$|N_m|_{C^0(\mathbb{T}^n+\rho_m)} \leqslant 2|N|_{C^0(\mathbb{T}^n)} \prod_{j=1}^{m}\bigl(1+2^{-j}\bigr),$$

$$\bigl|\operatorname{avg}\{A_m\}_\theta^{-1}\bigr| \leqslant \bigl|\operatorname{avg}\{A\}_\theta^{-1}\bigr| \prod_{j=1}^{m}\bigl(1+2^{-j}\bigr),$$

where N and A are as in Definition 45.

Then there exist two constants $\tilde{\lambda}$ and λ, depending on σ, n, η, M_Ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $|(\operatorname{avg}\{A\}_\theta)^{-1}|$, such that, if

$$\gamma^{-4}\tilde{\lambda}\rho_m^{-(4\sigma+1)}\bigl|F(f_m,K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1,r,\eta), \tag{89}$$

then there exists a parameterization $K_{m+1} \in \mathcal{A}(\mathbb{T}^n + \rho_{m+1}, C^1) \cap \mathcal{N}$, with $\rho_{m+1} = \rho_m/2$, such that $F(f_m, K_{m+1}) = 0$,

$$|K_{m+1} - K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \gamma^{-2}\tilde{\lambda}\rho_m^{-2\sigma}\bigl|F(f_m,K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)}, \tag{90}$$

and

$$|DK_{m+1} - DK_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \tilde{\lambda}\gamma^{-2}\rho_m^{-(2\sigma+1)}\bigl|F(f_m,K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)}. \tag{91}$$

Furthermore, if $f_{m+1} \stackrel{\text{def}}{=} T_{2\tau_m}[f]$ and

$$2^{m+1}\lambda\bigl(\rho_m^{\mu-1} + \gamma^{-2}\rho_m^{-(2\sigma+1)}\bigl|F(f_m,K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)}\bigr) < \min(1,r,\eta), \tag{92}$$

then (f_{m+1}, K_{m+1}) satisfies properties A1($m+1$)–A4($m+1$) *and the following estimate holds*:

$$\bigl|F(f_{m+1},K_{m+1})\bigr|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \kappa M_f\rho_m^{\mu-1}, \tag{93}$$

where κ and M_f are as in Theorem 18.

Proof. From properties A1(m)–A4(m) and Theorem 46 there exists a constant λ_m depending on $\sigma, n, \gamma^{-4}, \zeta, M_\Omega, |f_m|_{C^2(\mathbb{U}_{\tau_m}+(2r_m\rho_m))}, |DK_m|_{C^0(\mathbb{T}^n+\rho_m)}, |N_m|_{C^0(\mathbb{T}^n+\rho_m)}, |(\mathrm{avg}\{A_m\}_\theta)^{-1}|$ such that, if

$$\gamma^{-4}\lambda_m\rho_m^{-(4\sigma+1)}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1, r_m), \tag{94}$$

then there exists $K_{m+1} \in \mathcal{A}(\rho_m/2, C^1) \cap \mathcal{N}$ such that $F(f_m, K_{m+1}) = 0$,

$$|K_{m+1} - K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \lambda_m\gamma^{-2}\rho_m^{-2\sigma}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)},$$

and

$$|DK_{m+1} - DK_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \lambda_m\gamma^{-2}\rho_m^{-(2\sigma+1)}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)}.$$

It turns out that λ_m depends in a polynomial way of the following quantities (see Remark 15 in [15]):

$$|f_m|_{C^2(\mathbb{U}_{\tau_m}+(2r_m\rho_m))}, \quad |DK_m|_{C^0(\mathbb{T}^n+\rho_m)}, \quad |N_m|_{C^0(\mathbb{T}^n+\rho_m)}, \quad \left|\left(\mathrm{avg}\{A_m\}_\theta\right)^{-1}\right|. \tag{95}$$

Let $\tilde{\lambda}$ be the constant obtained by replacing in the definition of $\tilde{\lambda}_m$ the quantities in (95), respectively, by

$$\kappa M_f, \quad 2r, \quad 2e|N|_{C^0(\mathbb{T}^n)}, \quad e\left|\left(\mathrm{avg}\{A\}_\theta\right)^{-1}\right|.$$

Assume that

$$\tilde{\lambda}\gamma^{-4}\rho_m^{-(4\sigma+1)}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1, r, \eta),$$

then using the estimates in A2(m) and A3(m) and $r < r_m < 2r$, we have that (94) holds. In particular estimates (90) and (91) hold. Now we prove properties A($m+1$). First from (91) we have that if

$$2^m\tilde{\lambda}\gamma^{-2}\rho_m^{-(2\sigma+1)}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)} < r,$$

then

$$|DK_{m+1}|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant r_m + \tilde{\lambda}\gamma^{-2}\rho_m^{-2\sigma+1}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)} \leqslant r_m + 2^{-m}r.$$

Hence A1($m+1$) holds. Property A2($m+1$) follows from (90) by assuming the following estimate

$$2^{m+1}\gamma^{-2}\tilde{\lambda}\rho_m^{-2\sigma}\left|F(f_m, K_m)\right|_{C^0(\mathbb{T}^n+\rho_m)} < \eta.$$

Notice that A1($m+1$) and A2($m+1$) imply

$$K_{m+1}\left(\mathbb{T}^n + \rho_{m+1}\right) \subset \mathbb{U}_{\tau_{m+1}} + 2r\rho_m,$$

so the composition $f_{m+1} \circ K_{m+1}$ is well defined on $\mathbb{T}^n + \rho_{m+1}$.

Property A3$(m+1)$ follows from Theorem 18. Now we prove A4$(m+1)$. Using $|DK_m| \leqslant r_m < 2r$ and (91) we have

$$\begin{aligned}&\bigl|DK_{m+1}(\theta)^\top DK_{m+1}(\theta) - N_n^{-1}\bigr|_{C^0(\mathbb{T}^n+\rho_{m+1})}\\&\quad\leqslant 2(2r+1)\tilde{\lambda}\gamma^{-2}\rho_m^{-(2\sigma+1)}\bigl|F(f_m, K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)},\end{aligned}$$

then if $\hat{\lambda} \stackrel{\text{def}}{=} \tilde{\lambda}2^3 e(2r+1)|N|_{C^0(\mathbb{T}^n)}$ and

$$2^{m+1}\gamma^{-2}\hat{\lambda}\rho_m^{-(2\sigma+1)}\bigl|F(f_m, K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)} \leqslant 1, \tag{96}$$

then we have that N_{m+1} exists and

$$|N_{m+1} - N_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant |N_m|_{C^0(\mathbb{T}^n+\rho_m)}\hat{\lambda}\gamma^{-2}\rho_m^{-(2\sigma+1)}\bigl|F(f_m, K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)}, \tag{97}$$

where we have used $|N_m|_{C^0(\mathbb{T}^n+\rho_m)} < 2e|N|_{C^0(\mathbb{T}^n)}$, which follows from A4$(m)$. Then using (96) we have

$$\begin{aligned}|N_{m+1}|_{C^0(\mathbb{T}^n+\rho_{m+1})} &\leqslant |N_m|_{C^0(\mathbb{T}^n+\rho_m)} + |N_{m+1} - N_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\\&\leqslant |N_m|_{C^0(\mathbb{T}^n+\rho_m)}\bigl(1 + 2^{-(m+1)}\bigr),\end{aligned}$$

from which the first estimate in A4$(m+1)$ holds. Let us now prove the second one. Define

$$P_{m+1} \stackrel{\text{def}}{=} DK_{m+1}N_{m+1},$$

then estimates (91) and (97) imply

$$|P_{m+1} - P_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \hat{\lambda}'\gamma^{-2}\rho_m^{-2(\sigma+1)}\bigl|F(f_m, K_m)\bigr| \tag{98}$$

and

$$\bigl|J(K_{m+1})^{-1} - J(K_m)^{-1}\bigr|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \hat{\lambda}'\gamma^{-2}\rho_m^{-2\sigma}\bigl|F(f_m, K_m)\bigr|, \tag{99}$$

where $\hat{\lambda}'$ depends on r, $|N|_{C^0(\mathbb{T}^n)}$, $|J^{-1}|_{C^1(\mathbb{U})}$, $\tilde{\lambda}$ and $\hat{\lambda}$. Moreover, using property T6 of Theorem 18 we have

$$|f_{m+1} - f_m|_{C^0(\mathbb{U}+2r\rho_{m+1})} \leqslant \kappa M_f \rho_m^{\mu-1}. \tag{100}$$

From estimates (90) and (100) and property T4 of Theorem 18 we have that

$$\Delta_m \stackrel{\text{def}}{=} Df_{m+1}\bigl(K_{m+1}(\theta)\bigr) - Df_m\bigl(K_m(\theta)\bigr)$$

satisfies

$$|\Delta_m|_{C^0(\mathbb{T}^n)} \leqslant \tilde{\tilde{\lambda}}\bigl(\rho_m^{\mu-2} + \gamma^{-2}\rho_m^{-(2\sigma+1)}\bigl|F(f_m, K_m)\bigr|_{C^0(\mathbb{T}^n+\rho_m)}\bigr), \tag{101}$$

where we have used the Cauchy's estimates. Performing some computations and using (98), (99), and (101) we have that there exists a constant $\bar{\lambda}$, depending on σ, n, γ^{-4}, η, M_Ω, μ, $|f|_{C^\ell(\mathbb{U})}$, β, $|N|_{C^0(\mathbb{T}^n)}$, $|\mathrm{avg}\{A\}_\theta^{-1}|$, and M_f, such that

$$|A_{m+1} - A_m|_{C^0(\mathbb{T}^n)} \leqslant \bar{\lambda}\big(\rho_m^{\mu-2} + \gamma^{-2}\rho_m^{-(2\sigma+1)}\big|F(f_m, K_m)\big|_{C^0(\mathbb{T}^n+\rho_m)}\big),$$

from which we have that if $\lambda \stackrel{\text{def}}{=} \bar{\lambda}|\mathrm{avg}\{A\}_\theta^{-1}|e$ and

$$2^{m+1}\lambda\big(\rho_m^{\mu-2} + \gamma^{-2}\rho_m^{-(2\sigma+1)}\big|F(f_m, K_m)\big|_{C^0(\mathbb{T}^n+\rho_m)}\big) \leqslant 1,$$

then, since $|\mathrm{avg}\{A_m\}_\theta^{-1}| \leqslant |\mathrm{avg}\{A\}_\theta^{-1}|e$ (which follows from A4(m)), we have that $\mathrm{avg}\{A_{m+1}\}_\theta$ is invertible and

$$\big|\mathrm{avg}\{A_{m+1}\}_\theta^{-1}\big| \leqslant \mathrm{avg}\{A_{m+1}\}_\theta\big(1 + 2^{-(m+1)}\big),$$

this proves A4($m+1$). Finally using the equality $F(f_m, K_{m+1}) = 0$ and (100) we have

$$\begin{aligned}\big|F(f_{m+1}, K_{m+1})\big|_{C^0(\mathbb{T}^n+\rho_{m+1})} &= |f_{m+1} \circ K_{m+1} - f_m \circ K_{m+1}|_{C^0(\mathbb{T}^n+\rho_{m+1})} \\ &\leqslant \kappa M_f \rho_m^{\mu-1}. \qquad \square\end{aligned}$$

Summarizing, from Lemma 53 we have that $(T_t[f], S_t[K])$ is an analytic approximate solution of the functional equation (76), for t sufficiently large. Lemma 54 provides the iterative scheme to construct a sequence of analytic solutions (f_j, K_{j+1}) of Eq. (76). Hence we have all the ingredients to apply the Moser's smoothing technique to prove Theorem 47.

Lemma 55. *Assume that the hypotheses of Theorem* 47 *hold. Let* $4\sigma + 2 < \mu < \ell - 1$, *with* $\ell \notin \mathbb{N}$, *then there exist two positive constants* c *and* $\rho^* < 1$, *depending* μ, n, ℓ, σ, ζ, β, M_ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, *and* $|(\mathrm{avg}\{A\}_\theta)^{-1}|$, *such that*: *given* $0 < \rho_1 \leqslant \rho^*$, *if*

$$c\gamma^{-4}\rho_1^{-(4\sigma+1)}\big|F(f, K)\big|_{C^0(\mathbb{T}^n)} \leqslant \min(1, r, \eta), \tag{102}$$

then there exist two sequences of functions $\{f_m\}_{m\geqslant 1} \subset \mathcal{A}(\mathbb{U} + 2r\rho_m, C^2)$ *and* $\{K_m\}_{m\geqslant 1} \subset \mathcal{A}(\mathbb{T}^n + \rho_m, C^1)$, *with* $\rho_m \stackrel{\text{def}}{=} 2^{-(m-1)}\rho_1$, *satisfying properties* A(m) *of Lemma* 54, *and such that* $f_m = T_{\tau_m}[f]$, *with* $\tau_m = \rho_m^{-1}$, *and for* $m \geqslant 2$

$$|K_{m+1} - K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \tilde{c}\gamma^{-2}\rho_m^{\mu-(2\sigma+1)}, \tag{103}$$

where $\tilde{c}$ *is a constant depending on the same variables as* c. *Furthermore if* $\mu - (2\sigma + 1) \notin \mathbb{N}$, *then the sequence* $\{K_m\}_{m\geqslant 1}$ *converges to a function* $K^* \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n, \mathbb{U})$ *such that*

$$F(f, K^*) = 0$$

and

$$|K - K^*|_{C^\nu(\mathbb{T}^n)} \leqslant M\gamma^{-2}\rho_1^{-(2\sigma+\nu)}\bigl(\rho_1^{\mu-1} + \bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)}\bigr),$$

for all $0 \leqslant \nu < \mu - (2\sigma + 1)$*, where* M *is a constant depending on the same variables as* c*.*

Proof. Let λ, $\tilde{\lambda}$ be as in Lemma 54, let M_f and κ be as in Theorem 18, and let κ and $\hat{M}_{K,f}$ be as in part (v) of Lemma 53, define

$$c \stackrel{\text{def}}{=} 2^\mu \kappa \max(4\lambda, \tilde{\lambda}) \max(1, \hat{M}_{K,f}, \kappa M_f). \tag{104}$$

Let t_8 be as in Lemma 53 and let $0 < \rho^* < 1$ be sufficiently small such that $\rho^* \leqslant t_8^{-1}$ and such that following inequality holds:

$$c\gamma^{-4}(\rho^*)^{\mu-(4\sigma+2)} < \min(1, r, \eta). \tag{105}$$

Let $0 < \rho_1 < \rho^*$ and define $\tau_1 \stackrel{\text{def}}{=} \rho_1^{-1}$ and $f_1 \stackrel{\text{def}}{=} T_{\tau_1}[f]$, and $K_1 \stackrel{\text{def}}{=} S_{\tau_1}[K]$, then, because of Lemma 53, (f_1, K_1) satisfies properties A1(1)–A4(1) of Lemma 54. Moreover if (102) holds, then part (v) of Lemma 53, Eq. (104), and estimate (105) imply conditions (89) and (92) in Lemma 54 for $m = 1$. Therefore, if $f_2 = T_{2\tau_1}[f]$, Lemma 54 implies the existence of $K_2 \in \mathcal{A}(\rho_2, C^1)$, with $\rho_2 = \rho_1/2$, such that (f_2, K_2) satisfies properties A1$(m+1)$–A4$(m+1)$ and estimate (93) in Lemma 54 for $m = 1$. Moreover, estimate (90) and part (v) of Lemma 53 imply

$$|K_2 - K_1|_{C^0(\mathbb{T}^n+\rho_2)} < c\gamma^{-2}\rho_1^{-2\sigma}\bigl(\rho_1^{\mu-1} + \bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)}\bigr).$$

Now assume that, for $m \geqslant 2$ we have (f_m, K_m) satisfying properties A1(m)–A4(m) and estimate (93) in Lemma 54 for $(m-1)$. Performing some simple computations and using the definition of c in (104), and estimates (102), (105) one obtains that estimates (89) and (92) hold for m. Hence Lemma 54 can be iterated to obtain an analytic invariant torus K_m for f_m. Moreover, using estimates (90) and (93) one obtains (103).

The convergence of the sequence $\{K_m\}_{m\geqslant 1}$ follows from the *Inverse Approximation Lemma* (see, for example, Lemma 2.2 in [4] or Lemma 6.14 in [24]). Indeed, define $u_m \stackrel{\text{def}}{=} K_m - K_1$, then the following properties hold:

(i) $u_m \in \mathcal{A}(\rho_m, C^1)$, for all $m \geqslant 1$ and $u_1 = 0$.
(ii) $\sup_{m\geqslant 2} \rho_m^{-\mu+(2\sigma+1)} |u_{m+1} - u_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leqslant \tilde{c}$.
(iii) If $0 \leqslant \nu < \mu - (2\sigma + 1)$, then

$$\begin{aligned}|u_m|_{C^\nu(\mathbb{T}^n)} &\leqslant \sum_{j=1}^{m-1} \rho_{j+1}^\nu |K_{j+1} - K_j|_{C^0(\mathbb{T}^n+\rho_{j+1})} \\ &\leqslant c\gamma^{-2}2^\nu\rho_1^{-(\nu+2\sigma)}\bigl(\rho_1^{\mu-1} + \bigl|F(f,K)\bigr|_{C^0(\mathbb{T}^n)}\bigr),\end{aligned}$$

where $\hat{c} \stackrel{\text{def}}{=} \tilde{c}2^{\mu-2\sigma}\sum_{j=2}^\infty 2^{-(\mu-(2\sigma+1)-\alpha)}$.

The Inverse Approximation Lemma implies the existence of a function u^* in $C^{\mu-(2\sigma+1)}(\mathbb{T}^n, \mathbb{U})$ such that

$$\lim_{m\to\infty} |u^* - u_m|_{C^\nu(\mathbb{T}^n)} = 0,$$

for any $\nu < \mu - (2\sigma + 1)$. The proof of Lemma 55 is finished by defining $K^* \stackrel{\text{def}}{=} u^* + u_1$. $\square$

4.2. Local uniqueness (Proof of Theorem 49)

Throughout this section we assume that the hypotheses of Theorem 49 hold. The proof of Theorem 49 we give here is rather standard, as it is proved in [4] it suffices to show that the operator $D_2F(f, K)$, with F defined in (77), has an *approximate left inverse* for each f fixed. In our context the existence of the approximate left inverse amounts to the uniqueness of the solutions of the following linear equation

$$D_2F(f, K)\Delta = Df\big(K(\theta)\big)\Delta - \Delta \circ R_\omega = g(\theta). \tag{106}$$

The uniqueness of Eq. (106) depends on the arithmetic properties of ω because the so-called small divisors are involved. The following result is well known in KAM theory, for completeness we state it here, for a proof see [25–27].

Lemma 56. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$ and let $r > \sigma$ be not an integer. Let $h \in C^r(\mathbb{T}^n)$ be such that* $\operatorname{avg}\{h\}_\theta = 0$, *and assume that $r - \sigma \notin \mathbb{Z}$, then the linear difference equation*

$$u - u \circ R_\omega = h$$

has a unique zero average solution $u \in C^{r-\sigma}(\mathbb{T}^n)$. Moreover, the following holds:

$$|u|_{C^{r-\sigma}(\mathbb{T}^n)} \leqslant \kappa\gamma^{-1}|h|_{C^\sigma(\mathbb{T}^n)},$$

where κ is a constant depending on n, σ, and r.

Now we prove the uniqueness of the solution of (106).

Lemma 57. *Let $\omega \in D(\gamma, \sigma)$ for some $\gamma > 0$ and $\sigma > n$. Let $\ell > 2\sigma$ be such that $\ell, \ell - 2\sigma \notin \mathbb{Z}$. Let $f \in \operatorname{Diff}^{\ell+1}(\mathbb{U})$ be symplectic. Assume that (f, K) is a solution of* (76), *with $K \in \mathcal{N} \cap C^{\ell+1}(\mathbb{T}^n, \mathbb{U})$ (see Definition* 45*). Then, for any $g \in C^\ell(\mathbb{T}^n, \mathbb{U})$ satisfying*

$$\operatorname{avg}\big\{DK(\theta)^\top J\big(K(\theta)\big)g(\theta - \omega)\big\}_\theta = 0, \tag{107}$$

the linear equation (106) *has a unique solution $\Delta \in C^{\ell-2\sigma}(\mathbb{T}^n)$, satisfying*

$$\operatorname{avg}\big\{T(\theta)\Delta(\theta)\big\}_\theta = 0,$$

where

$$T(\theta) \stackrel{\text{def}}{=} N(\theta)^\top DK(\theta)^\top \{ I_n - J(K(\theta))^{-1} DK(\theta) N(\theta) DK(\theta)^\top J(K(\theta)) \}, \tag{108}$$

with N defined as in N1 *in Definition* 45. *Furthermore, the following estimate holds*:

$$|\Delta|_{C^{\ell-2\sigma}(\mathbb{T}^n)} \leqslant \kappa\gamma^{-2}|g|_{C^\ell(\mathbb{T}^n)},$$

where κ is a constant depending on n, σ, ℓ, $|N|_{C^\ell(\mathbb{T}^n)}$, $|K|_{C^{\ell+1}(\mathbb{T}^n)}$, and $|\operatorname{avg}\{A\}_\theta^{-1}|$, with A defined by (79).

Proof. Let $M(\theta)$ be the $(2n \times 2n)$-matrix-valued function defined by

$$M(\theta) = \big(DK(\theta) \,\big|\, J(K(\theta))^{-1} DK(\theta) N(\theta) \big).$$

It is clear that the components of M belong to $C^\ell(\mathbb{T}^n)$. In Section 4.2 of [15] it is proved that if K is a parameterization of an invariant torus for the symplectic map f, then:

(i) M is invertible with inverse given by

$$M(\theta)^{-1} = \begin{pmatrix} T(\theta) \\ DK(\theta)^\top J(K(\theta)) \end{pmatrix}.$$

(ii) If $\Delta = M\xi$, then in the variable ξ the linear equation (106) becomes

$$\begin{aligned} \xi_1 - \xi_1 \circ R_\omega &= T(\theta+\omega) g(\theta) - A(\theta)\xi_2, \\ \xi_2 - \xi_2 \circ R_\omega &= DK(\theta+\omega)^\top J\big(K(\theta+\omega)\big) g(\theta). \end{aligned} \tag{109}$$

Notice that, by Lemma 56 and the assumption (107), there exists a unique zero average function $\tilde{\xi}_2$ satisfying

$$\tilde{\xi}_2 - \tilde{\xi}_2 \circ R_\omega = DK(\theta+\omega)^\top J\big(K(\theta+\omega)\big) g(\theta).$$

The proof of Lemma 57 is finished by using Lemma 56 to find a unique solution of the triangular system (109) satisfying:

$$\begin{aligned} &\operatorname{avg}\{\xi_1\}_\theta = 0, \\ &\operatorname{avg}\{\xi_2\}_\theta = \operatorname{avg}\{A\}_\theta^{-1} \operatorname{avg}\big\{ T(\theta+\omega) g(\theta) - A(\theta)\tilde{\xi}_2(\theta) \big\}_\theta. \qquad \square \end{aligned}$$

Lemma 58. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$. Let $f \in \operatorname{Diff}^{\ell+1}(\mathbb{U})$ be symplectic. Assume that (f, K_1) and (f, K_2) satisfy* (76), *with $K_1, K_2 \in \mathcal{N} \cap C^{\ell+1}(\mathbb{T}^n, \mathbb{U})$ (see Definition* 45). *Then, there exists a constant κ, depending on n, ℓ, $|J^{-1}|_{C^0(\mathbb{U})}$, $|K_1|_{C^2(\mathbb{T}^n)}$, $|K_2|_{C^1(\mathbb{T}^n)}$, $|N_2|_{C^0}$, with N_2 defined as in* N1 *in Definition* 45 *by replacing K with K_2, such that if*

$$\kappa |K_1 - K_2|_{C^1(\mathbb{T}^n)} < 1, \tag{110}$$

then there exists $\theta_0 \in \mathbb{R}^n$ *such that*

$$\operatorname{avg}\{T_2(\theta)(K_1 \circ R_{\theta_0} - K_2)\}_\theta = 0, \tag{111}$$

where T_2 *is defined by replacing* K *with* K_2 *in* (108). *Moreover, the following estimate holds*:

$$|K_1 \circ R_{\theta_0} - K_2|_{C^\ell(\mathbb{T}^n)} \leqslant \tilde{\kappa}|K_1 - K_2|^{1-\alpha}_{C^0(\mathbb{T}^n)} + \tilde{\kappa}|K_1 - K_2|_{C^\ell(\mathbb{T}^n)}, \tag{112}$$

where $0 \leqslant \alpha < 1$ *is such that* $\ell - \alpha \in \mathbb{N}$ *and* $\tilde{\kappa}$ *is a constant depending on the same variables as* κ *and on* $|K_1|_{C^{\ell+1}(\mathbb{T}^n)}$.

Proof. Lemma 58 is consequence of the Implicit Function Theorem. Indeed, let $\mathcal{M}_{n\times n}(\mathbb{R})$ represent the space of $n \times n$ matrices with components in $\mathbb{R}$. Define

$$\begin{gathered} \Phi : \mathbb{R}^n \times C^1(\mathbb{T}^n) \to \mathcal{M}_{n\times n}(\mathbb{R}), \\ (x, K) \to \operatorname{avg}\{T_2(K \circ R_x - K_2)\}_\theta, \end{gathered}$$

where T_2 is defined by (108) by replacing K with K_2. Notice that

$$\begin{gathered} \Phi(0, K_2) = 0, \\ D_1\Phi(x, K)\Delta x = \operatorname{avg}\{T_2(\theta)DK(\theta + x)\}_\theta \Delta x. \end{gathered}$$

Moreover, since $K_2(\mathbb{T}^n)$ is Lagrangian [15], from the definition of T_2 one easily verifies that $T_2(\theta)DK_2(\theta) = I_n$, this implies

$$D_1\Phi(x, K)|_{(x,K)=(0,K_2)} = I_n.$$

Hence the Implicit Function Theorem guarantees the existence of a constant κ as in Lemma 58 such that if (110) holds, then there is a $\theta_0 \in \mathbb{R}^n$ satisfying (111) and such that

$$|\theta_0| \leqslant \kappa\big|\Phi(0, K_1)\big| \leqslant \kappa|T_2|_{C^0(\mathbb{T}^n)}|K_1 - K_2|_{C^0(\mathbb{T}^n)}. \tag{113}$$

It is not difficult to prove the following estimate (see [21])

$$|K_1 \circ R_{\theta_0} - K_1|_{C^\ell(\mathbb{T}^n)} \leqslant \tilde{\kappa}|K_1|_{C^{\ell+1}(\mathbb{T}^n)}|\theta_0|^{1-\alpha}, \tag{114}$$

where $0 < \alpha < 1$ is such that $\ell - \alpha \in \mathbb{N}$. Finally, estimate (112) follows from (113) and (114). □

The proof of Theorem 49 is concluded using Lemmas 57, 58 and Taylor's Theorem as follows. Assume that $|K_1 - K_2|_{C^\ell(\mathbb{T}^n)}$ is sufficiently small such that Lemma 58 holds, let θ_0 be as in Lemma 58. Define

$$\Delta(\theta) \stackrel{\text{def}}{=} K_1 \circ R_{\theta_0} - K_2.$$

Using that $(f, K_1 \circ R_{\theta_0})$ and (f, K_2) satisfy (76) and Lemma 58 we have

$$D_2F(f,K_2)\Delta = \mathcal{R}(K_1 \circ R_{\theta_0}, K_2),$$
$$\operatorname{avg}\{T_2(\theta)\Delta(\theta)\}_\theta = 0,$$

where F is as in (77), T_2 is as in Lemma 58, and

$$\mathcal{R}(K_1 \circ R_{\theta_0}, K_2)(\theta) = f \circ K_1 \circ R_{\theta_0} - f \circ K_2(\theta) - Df\big(K_2(\theta)\big)\Delta(\theta).$$

Then, from the Taylor's Theorem and Lemma 57 we have the following estimate:

$$|\Delta|_{C^{\ell-2\sigma}} \leqslant \hat{\kappa}\gamma^{-2}|\mathcal{R}|_{C^\ell(\mathbb{T}^n)} \leqslant \kappa\gamma^{-2}|\Delta|_{C^\ell(\mathbb{T}^n)}|\Delta|_{C^{\ell-2\sigma}(\mathbb{T}^n)},$$

from which and (112) we have that if $|K_1 - K_2|_{C^\ell(\mathbb{T}^n)}$ is sufficiently small such that

$$\kappa\gamma^{-2}|\Delta|_{C^\ell(\mathbb{T}^n)} < 1,$$

then $\Delta = 0$.

4.3. Bootstrap of regularity (Proof of Theorem 50)

Theorem 50 is a consequence of Theorems 46, 49, and the fact that, near to a finitely differentiable approximate solution (f,K) of (76) it is possible to obtain an analytic approximate solution of the same equation by means of the operators S_t and T_t of Theorem 18. More precisely, using Theorems 18 and 46 we prove that, under certain conditions, if (f,K) belongs to either $\operatorname{Diff}^\ell(\mathbb{U}) \times C^{\ell_1}(\mathbb{T}^n,\mathbb{U})$ or $\mathcal{A}(\mathbb{U}+\varrho, C^{\ell_1}) \times C^{\ell_1}(\mathbb{T}^n,\mathbb{U})$, with ℓ and ℓ_1 as in Theorem 50, then there exists a finitely differentiable parameterization of an n-dimensional torus K^* such that: (a) *(f,K) is a solution of* (76), (b) *K^* is close to K in certain norms*, and (c) *K^* has the wished regularity*. Then Theorem 50 follows from the local uniqueness result Theorem 49.

Lemma 59. *Let γ, σ, ω, m, ℓ_1, and $\mathbb{U}$ be as in Theorem* 50. *Let (K,f) be a solution of* (76) *with $f \in \operatorname{Diff}^\ell(\mathbb{U})$ an exact symplectic diffeomorphism, and $K \in C^{\ell_1}(\mathbb{T}^n,\mathbb{U})$. Let $\ell \in [\ell_1, m)$ and let $f \in C^\ell(\mathbb{U})$. Assume that hypotheses* S1–S3 *(replacing ρ with ϱ in* S3*) in Theorem* 47 *hold. Then, for any $4\sigma+2 < \mu < \ell - 1$, satisfying $\mu - (2\sigma+1) \notin \mathbb{N}$, there is positive constant c, depending on μ, ℓ, ℓ_1, σ, ζ, M_Ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^{\ell_1}(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $|(\operatorname{avg}\{A\}_\theta)^{-1}|$, such that for any $0 < \rho < 1$ satisfying*

$$c\gamma^{-4}\rho^{\mu-(4\sigma+2)} < \min(1,\beta,\eta), \tag{115}$$

there exists $K^* \in C^{\mu-2\sigma}(\mathbb{T}^n,\mathbb{U})$ satisfying N1 *and* N2 *in Definition* 45 *and such that (f,K^*) is a solution of* (76). *Moreover, for any $0 \leqslant \nu < \mu - (2\sigma+1)$ the following estimate holds*:

$$|K^* - K|_{C^\nu(\mathbb{T}^n)} \leqslant \kappa\gamma^{-2}\rho^{\mu-(2\sigma+1+\nu)},$$

for some positive constant κ.

Proof. The proof of Lemma 59 follows the same steps as the proof of Theorem 47 the only thing one has to be careful is that $f \in C^{\ell}(\mathbb{U})$ and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ (and not in $C^{\ell}(\mathbb{T}^n, \mathbb{U})$), hence we replace ℓ with ℓ_1 in the estimates of the norms involving the term $S_t[K]$. Moreover, the assumption $F(f, K) = 0$, with F as in (77), simplifies many estimates. □

Lemma 60. *Let γ, σ, ω, m, ℓ_1, and $\mathbb{U}$ be as in Theorem* 50. *Let (K, f) be a solution of* (76) *with $f \in \mathcal{A}(\mathbb{U} + \varrho, C^{\ell_1})$, an exact symplectic diffeomorphism, and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$. Assume that hypotheses* S1–S3 *in Theorem* 47 *hold. Then, for any $4\sigma < \mu < \ell_1 - 1$, there is positive constant c, depending on n, μ, ℓ_1, σ, ζ, ϱ, β, M_Ω, $|f|_{C^{\ell_1}(\mathbb{U}+\varrho)}$, $|K|_{C^{\ell_1}(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $|(\mathrm{avg}\{A\}_\theta)^{-1}|$, such that for any $0 < \rho < 1$ satisfying*

$$c\gamma^{-4}\rho^{\mu-4\sigma} < \min(1, \varrho, \eta), \tag{116}$$

there exists $K^* \in \mathcal{A}(\mathbb{T}^n + \rho/2, C^1)$ satisfying N1 *and* N2 *in Definition* 45 *and such that (f, K^*) is a solution of* (76). *Moreover, for any $0 \leqslant \nu < \mu - 2\sigma$, the following estimate holds*:

$$|K^* - K|_{C^\nu(\mathbb{T}^n)} \leqslant \kappa\big(\gamma^{-2}\rho^{\mu-(2\sigma+\nu)} + \rho^{\ell_1-\nu}\big), \tag{117}$$

for some positive constant κ.

Proof. We prove Lemma 60 applying again the smoothing technique. Since f is already analytic we only smooth the parameterization $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ by using the smoothing operator S_t, defined in Section 3.1. Let $\kappa = \kappa(n, \ell_1, 1)$ be as in Proposition 23, and assume that t is sufficiently large so that

$$\kappa\beta t^{-\ell_1}|K|_{C^{\ell_1}(\mathbb{T}^n)} < \min(\varrho/2, \eta/2), \tag{118}$$

then Proposition 23 implies $S_t[K](\mathbb{T}^n + t^{-1}) \subset \mathbb{U} + \varrho$, so that the composition $f \circ S_t[K]$ is well defined on $\mathbb{T}^n + t^{-1}$. Now, write

$$f \circ S_t[K] - S_t[K] \circ R_\omega = S_t[f] \circ S_t[K] - S_t[f \circ K],$$

where we have used that (f, K) satisfies Eq. (76). Then, using Proposition 37 and Lemma 30 one has that for any $4\sigma < \mu < \ell_1$, there exists a constant $\tilde{c}$, depending on n, ℓ_1, β, μ, $|f|_{C^{\ell_1}(\mathbb{U}+\varrho)}$, and $|K|_{C^{\ell_1}(\mathbb{T}^n)}$ such that

$$\big|f \circ S_t[K] - S_t[K] \circ R_\omega\big|_{C^0(\mathbb{T}^n+t^{-1})} \leqslant \tilde{c}t^{-\mu}.$$

Hence for t satisfying (118) $(f, S_t[K])$ is an approximate solution of Eq. (76), with error bounded in (116). Moreover, it can be proved, as we did in Lemma 52, that for t sufficiently large, $S_t[K]$ satisfies N1 and N2 in Definition 45 and the estimates given in part (iv) of Lemma 53. Hence, applying Theorem 46 to the analytic approximate solution $(f, S_t[K])$ one has that there is a positive constant c, depending on σ, n, β, μ, $|f|_{C^2(\mathbb{U}+\varrho)}$, ζ, M_ω, $|K|_{C^{\ell_1}(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n+t^{-1})}$, and $|(\mathrm{avg}\{A\}_\theta)^{-1}|$ such that, if $\rho = t^{-1}$, with t is sufficiently large so that (116) and (118), then there exists $K^* \in \mathcal{A}(\mathbb{T}^n + \rho/2, C^1)$ satisfying

$$|K^* - K|_{C^0(\mathbb{T}^n)} \leqslant \hat{c}\big(\gamma^{-2}\rho^{\mu-2\sigma} + \rho^{\ell_1}\big).$$

Estimate (117) follows from the Cauchy's estimates. □

Now Theorem 50 follows easily from the local uniqueness formulated in Theorem 49, Lemmas 59 and 60. Indeed, in the case that $f \in \mathrm{Diff}^{\ell}(\mathbb{U})$ with $\ell \in [\ell_1, m) - \mathbb{Z}$, let K^* be as in Lemma 59. Fix $\nu \in (2\sigma, \mu - (2\sigma + 2)) \cap (2\sigma, \ell_1 - 1)$ such that $\nu, \nu - 2\sigma \notin \mathbb{Z}$, then $f \in \mathrm{Diff}^{\nu+2}$, and $K, K^* \in C^{\nu+1}(\mathbb{T}^n, \mathbb{U}) \cap \mathcal{N}$. Assume that in (115) ρ is sufficiently small such that Theorem 49 holds, then $K = K^* \circ T_{\theta^*}$, for some $\theta^* \in \mathbb{R}^n$ and hence $K \in C^{\ell-2\sigma}(\mathbb{T}^n, \mathbb{U})$. The case $f \in \mathcal{A}(\mathbb{U} + \varrho, C^{\ell})$ is proved similarly using Lemma 60 instead of Lemma 59 and fixing $\nu \in (2\sigma, \mu - 2\sigma - 1)$ such that $\nu, \nu - 2\sigma \notin \mathbb{Z}$ and applying Lemma 60 and Theorem 49.

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