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Large time behavior of the semigroup on L^p spaces associated with the linearized compressible Navier–Stokes equation in a cylindrical domain

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A B S T R A C T

Large time behavior of solutions to the linearized compressible Navier–Stokes equation around the motionless state in a cylindrical domain is investigated. The L^p decay estimates of the associated semigroup are established for all $1 < p < \infty$. It is also shown that the time-asymptotic leading part of the semigroup is given by a one-dimensional heat semigroup.

1. Introduction

This paper studies large time behavior of solutions to the following system of equations

$$\partial_t u + Lu = 0, \tag{1.1}$$

where $u = {}^T(\phi, v)$, $\phi = \phi(x,t) \in \mathbf{R}$, $v = {}^T(v^1(x,t), v^2(x,t), v^3(x,t)) \in \mathbf{R}^3$, and L is an operator defined by

$$L = \begin{pmatrix} 0 & \gamma \operatorname{div} \\ \gamma \nabla & -\nu\Delta - \tilde{\nu}\nabla \operatorname{div} \end{pmatrix}$$

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with positive constants ν, $\tilde{\nu}$ and γ. Here $t \geqslant 0$ is the time variable and $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ is the space variable and ${}^T\cdot$ stands for the transposition.

In this paper we consider (1.1) in a cylindrical domain

$$\Omega = D \times \mathbf{R} = \{x = (x', x_3);\ x' = (x_1, x_2) \in D,\ x_3 \in \mathbf{R}\}$$

under the boundary condition

$$v|_{\partial\Omega} = 0. \tag{1.2}$$

Here D is a bounded domain in $\mathbf{R}^2$ with smooth boundary ∂D.

The system (1.1) arises from the linearization of the compressible Navier–Stokes equation

$$\begin{aligned} &\partial_t \rho + \operatorname{div} m = 0, \\ &\partial_t m + \operatorname{div}\left(\frac{m \otimes m}{\rho}\right) + \nabla P(\rho) = \mu \Delta\left(\frac{m}{\rho}\right) + (\mu + \mu')\nabla \operatorname{div}\left(\frac{m}{\rho}\right) \end{aligned} \tag{1.3}$$

around the constant motionless state $(\rho, m) = (\rho_*, 0)$, where $\rho = \rho(x,t)$ is the density; $m = {}^T(m^1(x,t), m^2(x,t), m^3(x,t))$ is the momentum; and ρ_* is a given positive number.

Large time behavior of solutions of (1.3) in unbounded domains has been widely studied, which presents interesting aspects. Concerning the Cauchy problem for (1.3) on the whole space $\mathbf{R}^3$, it was shown in [13,17,18] that if the initial perturbation $(\rho(0) - \rho_*, m(0))$ is sufficiently small in H^3, then there exists a unique global solution to (1.3) and the leading part of the perturbation $u(t) = (\rho(t) - \rho_*, m(t))$ in large time is given by the solution of the linearized problem, which exhibits a hyperbolic–parabolic aspect of system (1.3). (See [12] for the case of a general class of quasilinear hyperbolic–parabolic systems.) The solution of the linearized problem is approximated in large time by the sum of two terms; one is given by the convolution of the heat kernel and the fundamental solution of the wave equation, the so-called diffusion wave; and the other is the solution of the heat equation. It was found in [3,4] that hyperbolic and parabolic aspects of the diffusion wave exhibits an interesting interaction phenomena in the decay properties of L^p norms with $1 \leqslant p \leqslant \infty$. (See also [16].) Such an interaction phenomena also appears in the exterior domain problem [14,15] and the half space problem [8,9]. Furthermore, in the case of the half space problem, some different aspect appears in the decay property of spatial derivatives due to the presence of unbounded boundary.

On the other hand, solutions on the infinite layer $\mathbf{R}^{n-1} \times (0,1)$ behave in a different manner from the ones on the domains mentioned above. The leading part of the solution on the infinite layer is given by a solution of an $(n-1)$-dimensional heat equation [7]. This is due to the fact that the infinite layer has an infinite extent in $n-1$ unbounded directions and the remaining one direction has a finite thickness. An analogous result was obtained in [10] for the cylindrical domain Ω that has one unbounded direction x_3 and two-dimensional bounded cross section D. In this case, under suitable assumptions on the initial value, the perturbation $u(t) = (\rho(t) - \rho_*, m(t))$ satisfies

$$\|u(t)\|_{L^2} = O\left(t^{-1/4}\right), \qquad \|u(t) - u^{(0)}(t)\|_{L^2} = O\left(t^{-3/4}\log t\right)$$

as $t \to \infty$. Here $u^{(0)} = (\phi^{(0)}(x_3, t), 0)$ with $\phi^{(0)}(x_3, t)$ satisfying

$$\partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}|_{t=0} = \frac{1}{|D|} \int_D \left(\rho_0(x', x_3) - \rho_*\right) dx', \tag{1.4}$$

where κ is a positive constant and $|D|$ denotes the Lebesgue measure of D. In [10] large time behavior was investigated only in the L^2 space, while in the case of the infinite layer [5–7] it was investigated in general L^p spaces. The analysis in L^p spaces in the case of the infinite layer relies on a solution formula [5] whose analogous version seems to be unavailable in the case of cylindrical domains since D is a general bounded domain of $\mathbf{R}^2$.

In this paper we will extend the analysis in the L^2 space in [10] to general L^p spaces. We here treat only the linearized problem (1.1)–(1.2), since the nonlinear problem (1.3)–(1.2) can be treated as in [7] based on the energy method by Matsumura and Nishida [19] and the analysis of the linearized problem (1.1)–(1.2).

The main result of this paper is summarized as follows. Let $1 < p < \infty$ and let $u(t)$ be a solution of (1.1)–(1.2) with $u|_{t=0} = {}^T(\phi_0, v_0) \in [W^{1,p} \times L^p] \cap L^1$. Then

$$\begin{aligned} \|u(t)\|_{L^p} &= O\big(t^{-\frac{1}{2}(1-\frac{1}{p})}\big), \\ \|u(t) - u^{(0)}(t)\|_{L^p} &= O\big(t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}\big) \end{aligned} \tag{1.5}$$

as $t \to \infty$. Here $u^{(0)} = {}^T(\phi^{(0)}(x_3, t), 0)$ with $\phi^{(0)}(x_3, t)$ satisfying the equation in (1.4) and $\phi^{(0)}|_{t=0} = \frac{1}{|D|}\int_D \phi_0(x', x_3)\,dx'$.

To prove (1.5) we will consider the Fourier transform of problem (1.1)–(1.2) with respect to x_3 variable which is written in the form

$$\begin{aligned} &\partial_t \hat{u} + \hat{L}_\xi \hat{u} = 0, \\ &\hat{v}|_{\partial D} = 0, \qquad \hat{u}|_{t=0} = \hat{u}_0. \end{aligned} \tag{1.6}$$

Here $\hat{u} = \hat{u}(x', \xi, t) = {}^T(\hat{\phi}(x', \xi, t), \hat{v}(x', \xi, t))$ ($x' \in D$, $\xi \in \mathbf{R}$, $t \geqslant 0$) denotes the Fourier transform of $u(x', x_3, t) = {}^T(\phi(x', x_3, t), v(x', x_3, t))$ with respect to x_3 variable. We investigate problem (1.6) according to the following three cases:

$$\text{(i) } |\xi| \ll 1, \qquad \text{(ii) } |\xi| \gg 1, \qquad \text{(iii) } r \leqslant |\xi| \leqslant M$$

with suitable constants $0 < r < M < \infty$. The case (i) is treated similarly as in [6,10]. We regard problem (1.6) as a perturbation from the one with $\xi = 0$ and analyze the spectral properties of $\hat{L}_\xi$ by applying the analytic perturbation theory [11]. As for the case (ii), we treat it as a perturbation from the problem on the half space and derive necessary estimates for the corresponding part of the resolvent in L^p spaces by using the Fourier Multiplier Theorem. As for the case (iii), we derive estimates for the derivatives of $(\lambda + \hat{L}_\xi)^{-1}$ with respect to ξ and then obtain necessary estimates for the resolvent by employing the Riemann–Lebesgue lemma. To investigate the cases (ii) and (iii), we will use the solution formula for the half space problem [8,9].

This paper is organized as follows. In Section 2 we state our main result of this paper. The analysis for the cases (ii) and (iii) are done in Sections 3 and 4. Section 5 is devoted to the analysis for the case (i). Based on the analysis in Sections 3–5, we prove our main result in Section 6.

2. Main result

We first introduce some notation.

For $1 \leqslant p \leqslant \infty$ we denote by $L^p(\Omega)$ the usual Lebesgue space on Ω and its norm is denoted by $\|\cdot\|_p$. Let ℓ be a nonnegative integer. The symbol $W^{\ell,p}(\Omega)$ denotes the ℓ-th order L^p Sobolev space on Ω with norm $\|\cdot\|_{W^{\ell,p}}$. When $p = 2$, the space $W^{\ell,2}(\Omega)$ is denoted by $H^\ell(\Omega)$ and its norm is denoted by $\|\cdot\|_{H^\ell}$. $C_0^{\ell,p}(\Omega)$ stands for the set of all C^ℓ functions which have compact support in Ω. We denote by $W_0^{\ell,p}(\Omega)$ the completion of $C_0^{\ell,p}(\Omega)$ in $W^{\ell,p}(\Omega)$. In particular, $W_0^{\ell,2}(\Omega)$ is denoted by $H_0^\ell(\Omega)$.

We simply denote by $L^p(\Omega)$ (resp., $W^{\ell,p}(\Omega)$, $H^\ell(\Omega)$) the set of all vector fields $v = {}^T(v^1, v^2, v^3)$ on Ω with $v^j \in L^p(\Omega)$ (resp., $W^{\ell,p}(\Omega)$, $H^\ell(\Omega)$), $j = 1, 2, 3$, and its norm is also denoted by $\|\cdot\|_{L^p}$

(resp., $\|\cdot\|_{W^{\ell,p}}$, $\|\cdot\|_{H^\ell}$). For $u = {}^T(\phi, v)$ with $\phi \in W^{\ell,p}(\Omega)$ and $v = {}^T(v^1, v^2, v^3) \in W^{j,p}(\Omega)$, we define $\|u\|_{W^{\ell,p}\times W^{j,p}} = \|\phi\|_{W^{\ell,p}} + \|v\|_{W^{j,p}}$. When $\ell = j$, we simply write $\|u\|_{W^{\ell,p}}$ for $\|u\|_{W^{\ell,p}\times W^{\ell,p}}$.

Similarly we introduce the function spaces $L^p(D)$, $W^{\ell,p}(D)$, $H^\ell(D)$ and $H^\ell_0(D)$. Their norms are denoted by

$$|\cdot|_p, \quad |\cdot|_{W^{\ell,p}}, \quad |\cdot|_{H^\ell}.$$

For $u = {}^t(\phi, v)$ with $\phi \in W^{\ell,p}(D)$ and $v = {}^T(v^1, v^2, v^3) \in W^{j,p}(D)$, we denote $|u|_{W^{\ell,p}\times W^{j,p}} \equiv |\phi|_{W^{\ell,p}} + |v|_{W^{j,p}}$.

The inner product of $L^2(D)$ is denoted by

$$(f, g) \equiv \int_D f(x')\overline{g(x')}\,dx'$$

for $f, g \in L^2(D)$. We also denote the inner product of $L^2(\Omega)$ by the same symbol if no confusion occurs. We define $\langle\cdot,\cdot\rangle$ by

$$\langle f, g\rangle \equiv \frac{1}{|D|}(f, g)$$

for $f, g \in L^2(D)$. In particular, when $g = 1$, $\langle f, 1\rangle$ is denoted by $\langle f\rangle$, i.e.,

$$\langle f\rangle \equiv \frac{1}{|D|}\int_D f(x')\,dx'.$$

For a Banach space X, we denote by $\mathcal{S}(\mathbf{R}; X)$ the set of all rapidly decreasing functions on $\mathbf{R}$ with values in X.

We next introduce some notations about integral operators. We denote the Fourier transform of $f = f(z)$ $(z \in \mathbf{R}^k)$ by

$$[\mathcal{F}_{z\to\zeta} f](\zeta) = \int_{\mathbf{R}^k} f(z)e^{-i\zeta\cdot z}\,dz,$$

and the inverse Fourier transform is denoted by

$$\left[\mathcal{F}^{-1}_{\zeta\to z} f\right](z) = (2\pi)^{-k}\int_{\mathbf{R}^k} f(\zeta)e^{i\zeta\cdot z}\,d\zeta.$$

In particular, the Fourier transform of $f = f(x_3)$ $(x_3 \in \mathbf{R})$ is denoted by $\hat{f}$ or $\mathcal{F}f$, i.e.,

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbf{R}} f(x_3)e^{-i\xi\cdot x_3}\,dx_3,$$

and the inverse Fourier transform is denoted by $\mathcal{F}^{-1}f$, i.e.,

$$\mathcal{F}^{-1}f(x_3) = (2\pi)^{-1}\int_{\mathbf{R}} f(\xi)e^{i\xi\cdot x_3}\,d\xi.$$

For a function $K(y,z)$ on $(0,\infty)\times(0,\infty)$ we will denote by Kf the integral operator $\int_0^\infty K(y,z)f(z)\,dz$.

We denote the resolvent set of a closed operator A by $\rho(A)$ and the spectrum by $\sigma(A)$. For $c\in\mathbf{R}$ and $\theta\in(\frac{\pi}{2},\pi)$, we will denote

$$\Sigma(c,\theta)=\bigl\{\lambda\in\mathbf{C};\ \bigl|\arg(\lambda-c)\bigr|\leqslant\theta\bigr\}.$$

We denote by Q_0, $\tilde{Q}$ and Q' the 4×4 diagonal matrices

$$Q_0=\operatorname{diag}(1,0,0,0),\qquad \tilde{Q}=\operatorname{diag}(0,1,1,1),\qquad Q'=\operatorname{diag}(0,1,1,0),$$

respectively. We then have for $u={}^T(\phi,v)$, $v={}^T(v^1,v^2,v^3)$,

$$Q_0u=\begin{pmatrix}\phi\\0\end{pmatrix},\qquad \tilde{Q}u=\begin{pmatrix}0\\v\end{pmatrix},\qquad Q'u=\begin{pmatrix}0\\v'\\0\end{pmatrix}\quad\bigl(v'={}^T\bigl(v^1,v^2\bigr)\bigr).$$

We now state our main result. Let $1<p<\infty$. We define an operator L on $W^{1,p}(\Omega)\times L^p(\Omega)$ with domain of definition $D(L)$ by

$$L=\begin{pmatrix}0 & \gamma\operatorname{div}\\ \gamma\nabla & -\nu\Delta-\tilde{\nu}\nabla\operatorname{div}\end{pmatrix},$$
$$D(L)=W^{1,p}(\Omega)\times\bigl[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\bigr]$$

with positive constants ν, $\tilde{\nu}$ and γ.

Theorem 2.1. *Let $1<p<\infty$. Then $-L$ generates an analytic semigroup e^{-tL} on $W^{1,p}(\Omega)\times L^p(\Omega)$ and e^{-tL} has the following properties.*

(i) *There hold the estimates*

$$\bigl\|e^{-tL}u_0\bigr\|_{W^{1,p}\times L^p}\leqslant C\|u_0\|_{W^{1,p}\times L^p}$$

and

$$\bigl\|\partial_x^\ell\tilde{Q}e^{-tL}u_0\bigr\|_p\leqslant Ct^{-\frac{\ell}{2}}\|u_0\|_{W^{1,p}\times L^p}\quad(\ell=1,2)$$

for $0<t<1$.

(ii) *If $u_0={}^T(\phi_0,v_0)\in[W^{1,p}(\Omega)\times L^p(\Omega)]\cap L^1(\Omega)$, then $e^{-tL}u_0$ is decomposed as*

$$e^{-tL}u_0=\mathcal{U}_0(t)u_0+\mathcal{U}_\infty(t)u_0.$$

Here $\mathcal{U}_0(t)u_0$ and $\mathcal{U}_\infty(t)u_0$ satisfy the following (ii-a) *and* (ii-b).

(ii-a) *$\mathcal{U}_0(t)u_0$ is written as*

$$\mathcal{U}_0(t)u_0=\mathcal{W}_0(t)u_0+\mathcal{R}_0(t)u_0.$$

Here $\mathcal{W}_0(t)u_0$ takes the form

$$\mathcal{W}_0(t)u_0 = \begin{pmatrix} \phi^{(0)}(t) \\ 0 \end{pmatrix},$$

and $\phi^{(0)}(x_3, t)$ satisfies the following heat equation on $\mathbf{R}$:

$$\partial_t \phi^{(0)} - \kappa \partial_{x_3}^2 \phi^{(0)} = 0, \quad \phi^{(0)}\big|_{t=0} = \frac{1}{|D|} \int\limits_D \phi_0(x', x_3)\, dx'$$

with some positive constant κ.
For $1 \leqslant r \leqslant \infty$ and $\ell = 0, 1$, the function $\mathcal{R}_0(t)u_0$ satisfies the estimate

$$\left\| \partial_x^\ell \mathcal{R}_0(t)u_0 \right\|_r \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{r})-\frac{1}{2}} \|u_0\|_1$$

uniformly for $t \geqslant 1$. Furthermore,

$$\mathcal{U}_0(t)\tilde{Q} = \mathcal{R}_0(t)\tilde{Q}$$

and the estimates hold for $t \geqslant 1$:

$$\|\partial_x \mathcal{U}_0(t)\tilde{Q}u_0\|_r \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{r})-1} \|\tilde{Q}u_0\|_1,$$
$$\left\| \mathcal{U}_0(t)[\partial_x \tilde{Q}u_0] \right\|_r \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{r})-\frac{1}{2}} \|\tilde{Q}u_0\|_1.$$

(ii-b) *For $\ell = 0, 1$, the function $\mathcal{U}_\infty(t)u_0$ satisfies the estimate*

$$\left\| \partial_x^\ell \mathcal{U}_\infty(t)u_0 \right\|_p \leqslant Ce^{-ct} \|u_0\|_{W^{1,p} \times L^p}$$

uniformly for $t \geqslant 1$.

Remark 2.2. Since $\phi^{(0)}$ is a solution of a one-dimensional heat equation, we have

$$\left\| \mathcal{W}_0(t)u_0 \right\|_r \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{r})} \|u_0\|_1.$$

This implies that

$$\left\| \mathcal{U}_0(t)u_0 \right\|_r \leqslant Ct^{-\frac{1}{2}(1-\frac{1}{r})} \|u_0\|_1.$$

The proof of Theorem 2.1 is based on the resolvent problem associated with (1.1)–(1.2):

$$(\lambda + L)u = f, \quad v|_{\partial\Omega} = 0. \tag{2.1}$$

Here $u = {}^T(\phi, v)$.
Hereafter we will often write

$$x = {}^t(x', x_3), \qquad x' = {}^t(x_1, x_2) \in D, \qquad \nabla' = {}^t(\partial_{x_1}, \partial_{x_2}), \qquad \Delta' = \partial_{x_1}^2 + \partial_{x_2}^2.$$

We take the Fourier transform of (2.1) with respect to x_3 to obtain

$$\begin{cases} \lambda\hat{\phi} + \gamma\nabla'\cdot\hat{v}' + i\gamma\xi\hat{v}^3 = \hat{f}^0, \\ \lambda\hat{v}' - \nu\Delta'\hat{v}' + \nu\xi^2\hat{v}' - \tilde{\nu}\nabla'\big(\nabla'\cdot\hat{v}' + i\xi\hat{v}^3\big) + \gamma\nabla'\hat{\phi} = \hat{g}', \\ \lambda\hat{v}^3 - \nu\Delta'\hat{v}^3 + \nu\xi^2\hat{v}^3 - i\tilde{\nu}\xi\big(\nabla'\cdot\hat{v}' + i\xi\hat{v}^3\big) + i\gamma\xi\hat{\phi} = \hat{g}^3, \\ \hat{v}|_{\partial D} = 0, \end{cases} \tag{2.2}$$

which is written in the form

$$(\lambda + \hat{L}_\xi)\hat{u} = \hat{f}, \quad \hat{v}|_{\partial D} = 0. \tag{2.3}$$

Here $\xi \in \mathbf{R}$ denotes the dual variable; the unknown $\hat{u} = {}^T(\hat{\phi}, \hat{v}', \hat{v}^3)$ is a function on D with values in $\mathbf{C}$; and

$$\hat{L}_\xi = \begin{pmatrix} 0 & \gamma^T\nabla' & i\gamma\xi \\ \gamma\nabla' & -\nu\Delta' + \nu\xi^2 - \tilde{\nu}\nabla'^T\nabla' & -i\tilde{\nu}\xi\nabla' \\ i\gamma\xi & -i\tilde{\nu}\xi^T\nabla' & -\nu\Delta' + (\nu+\tilde{\nu})\xi^2 \end{pmatrix}.$$

Problem (2.3) will be investigated according to the cases

$$\text{(i) } |\xi| \ll 1, \qquad \text{(ii) } |\xi| \gg 1, \qquad \text{(iii) } r \leqslant |\xi| \leqslant M$$

with constants $0 < r < M < \infty$. We will study the cases (ii) and (iii) in Sections 3 and 4, respectively. The case (i) will be studied in Section 5. Based on the analysis of problem (2.3), we will prove Theorem 2.1 in Section 6.

3. Resolvent problem for high frequency part

In this section we establish estimates on $(\lambda + \hat{L}_\xi)^{-1}$ for $|\xi| \gg 1$.

Let $M > 0$ and set $\hat{u}_M = \kappa_M(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}$. Here κ_M is a C^∞ function on $\mathbf{R}$ satisfying

$$0 \leqslant \kappa_M \leqslant 1, \quad \kappa_M(\xi) = \begin{cases} 1 & (|\xi| > M), \\ 0 & (|\xi| < \frac{M}{2}). \end{cases}$$

We will show the following estimate.

Theorem 3.1. *Let $1 < p < \infty$ and let κ_M be a function defined as above. Then there exist $M_0 > 0$, $c_\infty > 0$ and $\theta_\infty \in (\frac{\pi}{2}, \pi)$ such that if $M \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$, then $\mathcal{F}^{-1}[\kappa_M(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}]$ satisfies the estimate*

$$\big\|\mathcal{F}^{-1}\big[\kappa_M(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}\big]\big\|_{W^{1,p}\times W^{2,p}} \leqslant C\|f\|_{W^{1,p}\times L^p}$$

uniformly in $M \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$.

Theorem 3.1 is proved by establishing interior and boundary estimates. We here give a proof of the estimate near the boundary only, since the interior estimate can be proved similarly.

We see from (2.2) that $\hat{u}_M = \kappa_M(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}$ satisfies

$$\begin{cases} \lambda\hat{\phi}_M + \gamma\nabla'\cdot\hat{v}'_M + i\gamma\xi\hat{v}^3_M = \kappa_M\hat{f}^0, \\ \lambda\hat{v}'_M - \nu\Delta'\hat{v}'_M + \nu\xi^2\hat{v}'_M - \tilde{\nu}\nabla'\big(\nabla'\cdot\hat{v}'_M + i\xi\hat{v}^3_M\big) + \gamma\nabla'\hat{\phi}_M = \kappa_M\hat{g}', \\ \lambda\hat{v}^3_M - \nu\Delta'\hat{v}^3_M + \nu\xi^2\hat{v}^3_M - i\tilde{\nu}\xi\big(\nabla'\cdot\hat{v}'_M + i\xi\hat{v}^3_M\big) + i\gamma\xi\hat{\phi}_M = \kappa_M\hat{g}^3, \\ \hat{v}_M|_{\partial D} = 0. \end{cases} \tag{3.1}$$

We take a point $\bar{x}'=(\bar{x}_1,\bar{x}_2)\in\partial D$ and an open neighborhood $\mathcal{O}\subset\mathbf{R}^2$ of $\bar{x}'$. Let $\chi\in C_0^\infty(\mathcal{O})$. Then, by (3.1), we have

$$\begin{cases} \lambda(\chi\hat{\phi}_M)+\gamma\nabla'\cdot\big(\chi\hat{v}'_M\big)+i\gamma\xi\big(\chi\hat{v}^3_M\big)=F_M,\\ \lambda\big(\chi\hat{v}'_M\big)+\nu\big(\xi^2-\Delta'\big)\big(\chi\hat{v}'_M\big)-\tilde{\nu}\nabla'\big\{\nabla'\cdot\big(\chi\hat{v}'_M\big)+i\xi\big(\chi\hat{v}^3_M\big)\big\}+\gamma\nabla'(\chi\hat{\phi})=G'_M,\\ \lambda\big(\chi\hat{v}^3_M\big)+\nu\big(\xi^2-\Delta'\big)\big(\chi\hat{v}^3_M\big)-i\tilde{\nu}\xi\big\{\nabla'\cdot\big(\chi\hat{v}'_M\big)+i\xi\big(\chi\hat{v}^3_M\big)\big\}+i\gamma\xi(\chi\hat{\phi}_M)=G^3_M,\\ (\chi\hat{v}_M)|_{\partial D\cap\mathcal{O}}=0. \end{cases} \tag{3.2}$$

Here

$$\begin{cases} F_M=\chi\kappa_M\hat{f}^0+(\nabla'\chi)\cdot\hat{v}'_M,\\ G'_M=\chi\kappa_M\hat{g}'-\nu(\Delta'\chi)\hat{v}'_M-2\nu\nabla'\chi\cdot\nabla'\hat{v}'_M-\tilde{\nu}\nabla'\big(\nabla'\chi\cdot\hat{v}'_M\big)\\ \qquad -\tilde{\nu}\nabla'\chi\big(\nabla'\cdot\hat{v}'_M\big)-i\xi\tilde{\nu}(\nabla'\chi)\hat{v}^3_M+\gamma(\nabla'\chi)\hat{\phi}_M,\\ G^3_M=\chi\kappa_M\hat{g}^3_M-\nu(\Delta'\chi)\hat{v}^3_M-2\nu\nabla'\chi\cdot\nabla'\hat{v}^3_M-i\tilde{\nu}\xi(\nabla'\chi)\cdot\hat{v}'_M. \end{cases}$$

For any $\eta>0$, if the diameter of $\mathcal{O}$ is sufficiently small, then one can find a function h with the following properties (i)–(iii).

(i) $h\in C^\infty(\mathbf{R})$, $\bar{x}_1=h(\bar{x}_2)$, $h'(\bar{x}_2)=0$.
(ii) $D\cap\mathcal{O}\subset\{x'=(x_1,x_2);\ x_1>h(x_2)\}$, $\partial D\cap\mathcal{O}\subset\{x'=(x_1,x_2);\ x_1=h(x_2)\}$.
(iii) There are an open neighborhood $\tilde{\mathcal{O}}$ of the origin of $\mathbf{R}^2$ and a diffeomorphism $\omega={}^T(\omega_1,\omega_2)$ from $\mathcal{O}$ to $\tilde{\mathcal{O}}$ such that

$$\begin{cases} y'=\omega(x')=\begin{pmatrix} x_1-h(x_2)\\ x_2-\bar{x}_2\end{pmatrix}=\begin{pmatrix}\omega_1(x')\\ \omega_2(x')\end{pmatrix}, \quad \omega(\bar{x}')=0,\\ \omega(D\cap\mathcal{O})\subset\big\{y=(y_1,y_2);\ y_1>0\big\},\\ \omega(\partial D\cap\mathcal{O})\subset\big\{y=(y_1,y_2);\ y_1=0\big\},\\ x'=\omega^{-1}(y')=\begin{pmatrix} y_1+h(y_2+\bar{x}_2)\\ y_2+\bar{x}_2\end{pmatrix},\\ \sup\limits_{y_2}\big|h'(y_2+\bar{x}_2)\big|<\eta. \end{cases}$$

Using the map ω, we define $V_M(y',\xi)$, $\Phi_M(y',\xi)$, $\tilde{F}_M(y',\xi)$, $\tilde{G}'_M(y',\xi)$ and $\tilde{G}^3_M(y',\xi)$ by

$$V_M(y',\xi)=\chi\hat{v}_M\big(\omega^{-1}(y'),\xi\big),\qquad \Phi_M(y',\xi)=\chi\hat{\phi}_M\big(\omega^{-1}(y'),\xi\big),$$
$$\tilde{F}_M(y',\xi)=F_M\big(\omega^{-1}(y'),\xi\big),\qquad \tilde{G}'_M(y',\xi)=G'_M\big(\omega^{-1}(y'),\xi\big),$$
$$\tilde{G}^3_M(y',\xi)=G^3_M\big(\omega^{-1}(y'),\xi\big).$$

Problem (3.2) is then transformed into the following one on the half space $\mathbf{R}^2_+=\{y=(y_1,y_2)\in\mathbf{R}^2;\ y_1>0,\ y_2\in\mathbf{R}\}$:

$$\begin{cases} \lambda\Phi_M+\gamma\nabla'\cdot V'_M+i\gamma\xi V^3_M=\tilde{R}^0_M,\\ \lambda V'_M+\nu\big(\xi^2-\Delta'\big)V'_M-\tilde{\nu}\nabla'\big(\nabla'\cdot V'_M+i\xi V^3_M\big)+\gamma\nabla'\Phi_M=\tilde{R}'_M,\\ \lambda V^3_M+\nu\big(\xi^2-\Delta'\big)V^3_M-i\tilde{\nu}\xi\big(\nabla'\cdot V'_M+i\xi V^3_M\big)+i\gamma\xi\Phi_M=\tilde{R}^3_M,\\ V_M|_{y_1=0}=0. \end{cases} \tag{3.3}$$

Here $V_M = {}^T(V_M', V_M^3)$, $V_M' = {}^T(V_M^1, V_M^2)$,

$$\begin{aligned}
\tilde{R}_M^0 &= \tilde{F}_M + \gamma h' \partial_{y_1} V_M^2,\\
\tilde{R}_M' &= {}^T\big(\tilde{R}_M^1, \tilde{R}_M^2\big),\\
\tilde{R}_M^1 &= \tilde{G}_M^1 - \nu\big\{h''\partial_{y_1} V_M^1 + 2h'\partial_{y_1 y_2} V_M^1 - (h')^2\partial_{y_1}^2 V_M^1\big\} - \tilde{\nu} h' \partial_{y_1}^2 V_M^2,\\
\tilde{R}_M^2 &= \tilde{G}_M^2 - \nu\big\{h''\partial_{y_1} V_M^2 + 2h'\partial_{y_1 y_2} V_M^2 - (h')^2\partial_{y_1}^2 V_M^2\big\}\\
&\quad - \tilde{\nu}\big\{h'\partial_{y_1}^2 V_M^1 + h''\partial_{y_1} V_M^2 + 2h'\partial_{y_1 y_2} V_M^2 - (h')^2\partial_{y_1}^2 V_M^2 + i\xi h'\partial_{y_1} V_M^3\big\}\\
&\quad + \gamma h' \partial_{y_1} \Phi_M,\\
\tilde{R}_M^3 &= \tilde{G}_M^3 - \nu\big\{h''\partial_{y_1} V_M^3 + 2h'\partial_{y_1 y_2} V_M^3 - (h')^2\partial_{y_1}^2 V_M^3\big\} - i\tilde{\nu}\xi h'\partial_{y_1} V_M^2.
\end{aligned}$$

As for problem (3.3), we have the following estimates. In what follows we will write y for $(y_1, y_2, x_3) \in \mathbf{R}_+^3$.

Proposition 3.2. *Let $1 < p < \infty$ and let $M_0 > 0$ be given. Then there exists a number $\delta > 0$ such that if* $\operatorname{diam}(\mathcal{O}) \leqslant \delta$, *$M \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$, then the solution $U_M = {}^T(\Phi_M, V_M', V_M^3)$ of* (3.3) *satisfies the following estimates with $C = C(\chi, \mathcal{O}) > 0$ uniformly for $M \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$:*

$$\begin{aligned}
&\text{(i)}\ \ \big\|\mathcal{F}_{\xi\to x_3}^{-1} V_M\big\|_p \leqslant \frac{C}{M}\big\{\big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\kappa_M f]\big\|_{W^{1,p}\times L^p} + \big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi \hat{u}_M]\big\|_{L^p\times W^{1,p}}\big\},\\
&\text{(ii)}\ \ \big\|\mathcal{F}_{\xi\to x_3}^{-1}\Phi_M\big\|_p + \big\|\partial_y \mathcal{F}_{\xi\to x_3}^{-1} V_M\big\|_p\\
&\qquad \leqslant C\bigg\{\big\|\mathcal{F}_{\xi\to x_3}^{-1}\big[\chi\kappa_M \hat{f}^0\big]\big\|_{W^{1,p}} + \big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\hat{v}_M]\big\|_p\\
&\qquad\quad + \frac{1}{M}\big(\big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\kappa_M\hat{g}]\big\|_p + \big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\hat{u}_M]\big\|_{L^p\times W^{1,p}}\big)\bigg\},\\
&\text{(iii)}\ \ \big\|\partial_y\mathcal{F}_{\xi\to x_3}^{-1}\Phi_M\big\|_p + \big\|\partial_y^2\mathcal{F}_{\xi\to x_3}^{-1} V_M\big\|_p\\
&\qquad \leqslant C\big\{\big\|\partial_y\mathcal{F}_{\xi\to x_3}^{-1}\big[\chi\kappa_M\hat{f}^0\big]\big\|_p + \big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\kappa_M\hat{g}]\big\|_p + \big\|\mathcal{F}_{\xi\to x_3}^{-1}[\chi\hat{u}_M]\big\|_{L^p\times W^{1,p}}\big\}.
\end{aligned}$$

To prove Proposition 3.2, we consider the Fourier transform of (3.3) in y_2 variable. In what follows we will write $\tilde{\zeta}$ for ${}^T(\zeta_2, \xi) \in \mathbf{R}^2$. Then the Fourier transform of (3.3) in y_2 gives

$$\begin{cases} (\lambda + \hat{A}_{\tilde{\zeta}})\mathcal{F}_{y_2\to\zeta_2} U_M = \mathcal{F}_{y_2\to\zeta_2}\tilde{R}_M \quad (y_1 > 0),\\ \mathcal{F}_{y_2\to\zeta_2} V_M|_{y_1=0} = 0.\end{cases} \tag{3.4}$$

Here $U_M = {}^T(\Phi_M(y_1, y_2, \xi), V_M(y_1, y_2, \xi))$ with $V_M' = {}^T(V_M'(y_1, y_2, \xi), V_M^3(y_1, y_2, \xi))$ is the solution of (3.3); $\tilde{R}_M = {}^T(\tilde{R}_M^0(y_1, y_2, \xi), \tilde{R}_M'(y_1, y_2, \xi), \tilde{R}_M^3(y_1, y_2, \xi))$; and

$$\hat{A}_{\tilde{\zeta}} = \begin{pmatrix} 0 & \gamma\partial_{y_1} & i\gamma\, {}^T\tilde{\zeta}\\ \gamma\partial_{y_1} & \nu(|\tilde{\zeta}|^2 - \partial_{y_1}^2) - \tilde{\nu}\partial_{y_1}^2 & -i\tilde{\nu}\, {}^T\tilde{\zeta}\partial_{y_1}\\ i\gamma\tilde{\zeta} & -i\tilde{\nu}\tilde{\zeta}\partial_{y_1} & \nu(|\tilde{\zeta}|^2 - \partial_{y_1}^2)I_2 + \tilde{\nu}\tilde{\zeta}\, {}^T\tilde{\zeta}\end{pmatrix},$$

where I_2 is the 2×2 identity matrix.

As for problem (3.4) we make use of some results by [9]. For a given $f \in C_0^\infty(\overline{\mathbf{R}_+^3}) \times C_0^\infty(\mathbf{R}_+^3)$ let us consider the problem

$$\begin{cases} (\lambda + \hat{A}_{\tilde{\zeta}})u = \mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f \quad (y_1 > 0), \\ v|_{y_1=0} = 0 \end{cases} \tag{3.5}$$

for the unknown $u = {}^T(\phi, v)$. To investigate problem (3.5) we introduce some quantities. We set

$$\lambda_1 = -\nu|\tilde{\zeta}|^2,$$
$$\lambda_\pm = -\frac{\nu_1}{2}|\tilde{\zeta}|^2 \pm \frac{1}{2}\sqrt{\nu_1^2|\tilde{\zeta}|^4 - 4\gamma^2|\tilde{\zeta}|^2}$$

and

$$\tilde{\lambda}_\pm = -\frac{\tilde{\nu}_1}{2}|\tilde{\zeta}|^2 \pm \frac{1}{2}\sqrt{\tilde{\nu}_1^2|\tilde{\zeta}|^4 - 4\gamma^2|\tilde{\zeta}|^2},$$

where $\nu_1 = \nu + \tilde{\nu}$ and $\tilde{\nu}_1 = 2\nu + \tilde{\nu}$. It was shown in [8,9] that if $\tilde{\zeta} \neq 0$ and $\lambda \notin \{\lambda_1, \lambda_\pm, \tilde{\lambda}_\pm, -\gamma^2/\nu_1\}$, then (3.5) has a unique solution u. We denote the solution operator for (3.5) by $\hat{S}(\lambda, \tilde{\zeta})$. Then for the solution $U_M = {}^T(\Phi_M, V_M)$ of (3.3) we have

$$\mathcal{F}_{y_2\to\zeta_2} U_M = \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{y_2\to\zeta_2}\tilde{R}_M,$$

and, therefore,

$$\begin{aligned} \mathcal{F}^{-1}_{\xi\to x_3} U_M &= \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}} \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{y_2\to\zeta_2}\tilde{R}_M \\ &= \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}} \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}[\mathcal{F}^{-1}_{\xi\to x_3}\tilde{R}_M]. \end{aligned}$$

As for $\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}} \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}$, we have the following estimates.

Lemma 3.3. *For any $M_0 > 0$ there exist $c_\infty > 0$ and $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that if $\mathrm{supp}(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f) \subset \{|\xi| \geqslant M/2\}$ with $M \geqslant 2M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$, then there hold the following estimates uniformly for $M \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty, \theta_\infty)$:*

$$\big\|\partial_y^\alpha Q_0 \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[\hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f\big]\big\|_p \leqslant C\left\{\big\|\partial_y^\alpha f^0\big\|_p + \frac{1}{M^{1-|\alpha|}}\|g\|_p\right\} \quad \big(|\alpha| = 0, 1\big), \tag{3.6}$$

$$\begin{aligned} &\big\|\partial_y^\alpha \tilde{Q}\, \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[\hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f\big]\big\|_p \\ &\quad \leqslant C\left\{\frac{1}{M^{(1-|\alpha|)_+}}\big\|\partial_y^{(|\alpha|-1)_+} f^0\big\|_p + \frac{1}{M^{2-|\alpha|}}\|g\|_p\right\} \quad \big(|\alpha| = 0, 1, 2\big). \end{aligned} \tag{3.7}$$

Lemma 3.3 will be proved in a similar argument to that given in [5, Sections 4 and 5], but we here need to pay attention to the dependence on M. The spectral bound, $\sup \mathrm{Re}\, \sigma(\hat{A}_{\tilde{\zeta}})$, of $\hat{A}_{\tilde{\zeta}}$ satisfies $\sup \mathrm{Re}\, \sigma(\hat{A}_{\tilde{\zeta}}) = O(1) < 0$ as $|\tilde{\zeta}| \to \infty$, and so we in general have $\|\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}} \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}\kappa_M\| = O(1)$ as $M \to \infty$, but we can gain a factor M^{-1} as in (3.6) and (3.7) which work well to obtain the desired estimate of Theorem 3.1.

To prove Lemma 3.3, we will make use of an integral representation of the solution $u = \hat{S}(\lambda, \tilde{\zeta})\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f$ of (3.5) given by [9].

We introduce the characteristic roots of the ordinary differential system $(\lambda + \hat{A}_{\tilde{\zeta}})u = 0$, which are given by $\pm\mu_j(\lambda, \tilde{\zeta})$, $j = 1, 2$, where

$$\mu_1 = \mu_1\big(\lambda, |\tilde\xi|^2\big) = \sqrt{\frac{\lambda + \nu|\tilde\xi|^2}{\nu}},$$

$$\mu_2 = \mu_2\big(\lambda, |\tilde\xi|^2\big) = \sqrt{\frac{\lambda^2 + \nu_1|\tilde\xi|^2\lambda + \gamma^2|\tilde\xi|^2}{\nu_1\lambda + \gamma^2}}, \quad \nu_1 = \nu + \tilde\nu.$$

We next introduce the Green functions $g^{(+)}_{\mu_j}(y_1, z_1)$ and $g^{(-)}_{\mu_j}(y_1, z_1)$ of the equation $\mu_j^2 w - \partial_{y_1}^2 w = f$ under the Neumann boundary condition and the Dirichlet one at $y_1 = 0$ respectively. We define $g^{(\pm)}_{\mu_j}(y_1, z_1)$ by

$$g^{(\pm)}_{\mu_j}(y_1, z_1) = \frac{1}{2\mu_j}\big(e^{-\mu_j|y_1 - z_1|} \pm e^{-\mu_j(y_1+z_1)}\big) \quad (j = 1, 2).$$

We set

$$g^{(\pm)}_{\mu_1,\mu_2}(y_1, z_1) = g^{(\pm)}_{\mu_1}(y_1, z_1) - g^{(\pm)}_{\mu_2}(y_1, z_1).$$

We also define functions $h_{\mu_j}(y_1)$, $h_{\mu_1,\mu_2}(y_1)$, $\beta_0(z_1)$, $\beta(z_1)$ and $\mathbf{b}(z_1)$ by

$$h_{\mu_j}(y_1) = \frac{1}{\mu_j}e^{-\mu_j y_1} \quad (j = 1, 2), \qquad h_{\mu_1,\mu_2}(y_1) = h_{\mu_1}(y_1) - h_{\mu_2}(y_1),$$

$$\beta_0(z_1) = \frac{\gamma\lambda}{d(\lambda)}e^{-\mu_2 z_1}, \qquad \beta(z_1) = |\tilde\xi|^2\big(e^{-\mu_1 z_1} - e^{-\mu_2 z_1}\big),$$

$$\mathbf{b}(z_1) = i\tilde\xi\mu_2\big(e^{-\mu_1 z_1} - e^{-\mu_2 z_1}\big).$$

Using the functions defined above, we have an integral representation of the solution $u = \hat S(\lambda, \tilde\xi)f$ of (3.5).

Lemma 3.4. *If $\tilde\xi \neq 0$ and $\lambda \notin \{\lambda_1, \lambda_\pm, \tilde\lambda_\pm, -\gamma^2/\nu_1\}$, then the solution $u = \hat S(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f$ of* (3.5) *is represented as*

$$\hat S(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f = \hat G(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f + \hat H(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f,$$

where $\hat G(\lambda, \tilde\xi)$ and $\hat H(\lambda, \tilde\xi)$ are the integral operators given by

$$\big(\hat G(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f\big)(y_1) = \int_0^\infty \hat G(\lambda, \tilde\xi, y_1, z_1)\mathcal{F}_{\tilde y\to\tilde\xi} f(z_1)\,dz_1$$

and

$$\big(\hat H(\lambda, \tilde\xi)\mathcal{F}_{\tilde y\to\tilde\xi} f\big)(y_1) = \int_0^\infty \hat H(\lambda, \tilde\xi, y_1, z_1)\mathcal{F}_{\tilde y\to\tilde\xi} f(z_1)\,dz_1.$$

Here $\hat G(\lambda, \tilde\xi, y_1, z_1)$ is a 4×4 matrix of the form

$$\hat{G}(\lambda, \tilde{\zeta}, y_1, z_1) = \frac{\nu_1}{d(\lambda)}\delta(y_1 - z_1)Q_0 + \hat{G}_1(\lambda, \tilde{\zeta}, y_1, z_1) + \hat{G}_2(\lambda, \tilde{\zeta}, y_1, z_1),$$

where $\delta(y_1)$ denotes the Dirac delta function;

$$\hat{G}_1 = \frac{1}{d(\lambda)}\begin{pmatrix} \frac{\gamma\lambda}{d(\lambda)} g^{(+)}_{\mu_2}(y_1, z_1) & -\partial_{y_1} g^{(-)}_{\mu_2}(y_1, z_1) & -i\,{}^T\tilde{\zeta} g^{(-)}_{\mu_2}(y_1, z_1) \\ -\partial_{y_1} g^{(+)}_{\mu_2}(y_1, z_1) & 0 & 0 \\ -i\tilde{\zeta} g^{(+)}_{\mu_2}(y_1, z_1) & 0 & 0 \end{pmatrix};$$

$$\hat{G}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\nu} g^{(-)}_{\mu_1}(y_1, z_1) - \frac{1}{\lambda}\partial^2_{y_1} g^{(-)}_{\mu_1,\mu_2}(y_1, z_1) & -\frac{i\,{}^T\tilde{\zeta}}{\lambda}\partial_{y_1} g^{(-)}_{\mu_1,\mu_2}(y_1, z_1) \\ 0 & -\frac{i\tilde{\zeta}}{\lambda}\partial_{y_1} g^{(-)}_{\mu_1,\mu_2}(y_1, z_1) & \frac{1}{\nu} g^{(-)}_{\mu_1}(y_1, z_1) I_2 + \frac{\tilde{\zeta}\,{}^T\tilde{\zeta}}{\lambda} g^{(-)}_{\mu_1,\mu_2}(y_1, z_1) \end{pmatrix}$$

with $d(\lambda) = \nu_1\lambda + \gamma^2$; and $\hat{H}(\lambda, \tilde{\zeta}, y_1, z_1)$ is a 4×4 matrix of the form

$$\begin{aligned}\hat{H}(y_1, z_1) = {}& \begin{pmatrix} 0 & 0 & \frac{i\gamma\,{}^T\tilde{\zeta}}{d(\lambda)} h_{\mu_2}(y_1) e^{-\mu_1 z_1} \\ 0 & 0 & \frac{i\,{}^T\tilde{\zeta}}{\lambda}\partial_{y_1} h_{\mu_1,\mu_2}(y_1) e^{-\mu_1 z_1} \\ 0 & 0 & -\frac{\tilde{\zeta}\,{}^T\tilde{\zeta}}{\lambda} h_{\mu_1,\mu_2}(y_1) e^{-\mu_1 z_1} \end{pmatrix} \\ & + \frac{1}{\mu_1\mu_2 - |\tilde{\zeta}|^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{i\tilde{\zeta}}{\nu} h_{\mu_1}(y_1)\beta_0(z_1) & \frac{1}{\nu} h_{\mu_1}(y_1)\mathbf{b}(z_1) & -\frac{1}{\nu} h_{\mu_1}(y_1)\frac{\tilde{\zeta}\,{}^T\tilde{\zeta}}{|\tilde{\zeta}|^2}\beta(z_1) \end{pmatrix} \\ & + \frac{1}{\mu_1\mu_2 - |\tilde{\zeta}|^2} \\ & \times \begin{pmatrix} \frac{\gamma|\tilde{\zeta}|^2}{d(\lambda)} h_{\mu_2}(y_1)\beta_0(z_1) & -\frac{i\gamma\,{}^T\tilde{\zeta}}{d(\lambda)} h_{\mu_2}(y_1)\mathbf{b}(z_1) & \frac{i\gamma\,{}^T\tilde{\zeta}}{d(\lambda)} h_{\mu_2}(y_1)\beta(z_1) \\ \frac{|\tilde{\zeta}|^2}{\lambda}\partial_{y_1} h_{\mu_1,\mu_2}(y_1)\beta_0(z_1) & -\frac{i\,{}^T\tilde{\zeta}}{\lambda}\partial_{y_1} h_{\mu_1,\mu_2}(y_1)\mathbf{b}(z_1) & \frac{i\,{}^T\tilde{\zeta}}{\lambda}\partial_{y_1} h_{\mu_1,\mu_2}(y_1)\beta(z_1) \\ \frac{i\tilde{\zeta}|\tilde{\zeta}|^2}{\lambda} h_{\mu_1,\mu_2}(y_1)\beta_0(z_1) & \frac{|\tilde{\zeta}|^2}{\lambda} h_{\mu_1,\mu_2}(y_1)\mathbf{b}(z_1) & -\frac{\tilde{\zeta}\,{}^T\tilde{\zeta}}{\lambda} h_{\mu_1,\mu_2}(y_1)\beta(z_1) \end{pmatrix}.\end{aligned}$$

The solution formula above is given in [9, Section 3]. (See also [8, Section 3 and Appendix] and [5, Theorem 3.8].)

Remark 3.5.

(i) For $g^{(\pm)}_{\mu_j}$ $(j = 1, 2)$, we have

$$\partial^2_{y_1}\bigl(g^{(\pm)}_{\mu_j} f\bigr) = \mu_j^2 g^{(\pm)}_{\mu_j} f - f \quad (j = 1, 2), \qquad \partial^2_{y_1}\bigl(g^{(\pm)}_{\mu_1,\mu_2} f\bigr) = \mu_1^2 g^{(\pm)}_{\mu_1} f - \mu_2^2 g^{(\pm)}_{\mu_2} f.$$

(ii) As for μ_j $(j = 1, 2)$, an elementary observation shows that $\mu_1 = \sqrt{\frac{\lambda - \lambda_1}{\nu}}$; $\mu_2 = \sqrt{\frac{(\lambda - \lambda_+)(\lambda - \lambda_-)}{(\nu_1\lambda + \gamma^2)}}$; and if $|\tilde{\zeta}| < 2\gamma/\nu_1$, then $\lambda_- = \overline{\lambda_+}$ and $\operatorname{Im}\lambda_+ = \gamma|\tilde{\zeta}|\sqrt{1 - \frac{\nu_1^2}{4\gamma^2}|\tilde{\zeta}|^2}$, while if $|\tilde{\zeta}| > 2\gamma/\nu_1$, then $\lambda_\pm \in \mathbf{R}$. We also have

$$\frac{1}{\mu_1\mu_2 - |\tilde{\zeta}|^2} = \frac{\nu(\nu_1\lambda + \gamma^2)(\mu_1\mu_2 + |\tilde{\zeta}|^2)}{\lambda(\lambda - \tilde{\lambda}_+)(\lambda - \tilde{\lambda}_-)}.$$

Furthermore,

$$\lambda_\pm = -\frac{\nu_1}{2}|\tilde\zeta|^2 \pm i\gamma|\tilde\zeta| + O\big(|\tilde\zeta|^3\big) \quad \text{as } |\tilde\zeta| \to 0,$$

$$\lambda_+ = -\frac{\gamma^2}{\nu_1} + O\big(|\tilde\zeta|^{-2}\big), \qquad \lambda_- = -\nu_1|\tilde\zeta|^2 + O(1) \quad \text{as } |\tilde\zeta| \to \infty,$$

and similar asymptotics also hold for $\tilde\lambda_\pm$.

To estimate $\mathcal{F}^{-1}_{\tilde\zeta\to\tilde y}\hat G(\lambda,\tilde\zeta)\mathcal{F}_{\tilde y\to\tilde\zeta}$ and $\mathcal{F}^{-1}_{\tilde\zeta\to\tilde y}\hat H(\lambda,\tilde\zeta)\mathcal{F}_{\tilde y\to\tilde\zeta}$, we prepare several lemmas. We proceed as in [5, Sections 4 and 5].

Lemma 3.6 *(Fourier Multiplier Theorem). Let $1 < p < \infty$ and let s be an integer satisfying $s \geqslant [k/2]+1$. Suppose that $\Psi(\omega) \in C^s(\mathbf{R}^k - \{0\}) \cap L^\infty(\mathbf{R}^k)$ and that there exists a constant $C_0 > 0$ such that*

$$|\omega|^{|\alpha|}\big|\partial_\omega^\alpha \Psi(\omega)\big| \leqslant C_0$$

for all $\omega \in \mathbf{R}^k - \{0\}$ and $|\alpha| \leqslant s$. Then the operator $\mathcal{F}^{-1}_{\omega\to w}[\Psi(\omega)(\mathcal{F}_{w\to\omega} f)(\omega)]$ is extended to a bounded linear operator on $L^p(\mathbf{R}^k)$ and there holds the estimate

$$\big\|\mathcal{F}^{-1}_{\omega\to w}\big[\Psi(\omega)(\mathcal{F}_{w\to\omega} f)(\omega)\big]\big\|_{L^p(\mathbf{R}^k)} \leqslant CC_0\|f\|_{L^p(\mathbf{R}^k)}.$$

See, e.g., [2] for the proof of Lemma 3.6.

An elementary observation yields the following lemma.

Lemma 3.7. *Let $g^{(1)}_{\mu_j}(y_1,z_1) = \frac{1}{2\mu_j}e^{-\mu_j|y_1-z_1|}$ with $\mu_j = \mu_j(\lambda,|\tilde\zeta|^2)$ $(j=1,2)$. Then*

$$\mathcal{F}^{-1}_{\tilde\zeta\to\tilde y}\Bigg[\int_0^\infty g^{(1)}_{\mu_j}(y_1,z_1,\tilde\zeta)(\mathcal{F}_{\tilde y\to\tilde\zeta} f)(z_1,\tilde\zeta)\,dz_1\Bigg] = \mathcal{F}^{-1}_{\zeta\to y}\Bigg[\frac{1}{\mu_j^2+\zeta_1^2}\mathcal{F}_{y\to\zeta}(Ef)\Bigg].$$

Here $y = (y_1,\tilde y)$, $\zeta = (\zeta_1,\tilde\zeta)$ with $\tilde\zeta = (\zeta_2,\xi)$ and

$$(Ef)(y) = \begin{cases} f(y) & (y_1 \geqslant 0), \\ 0 & (y_1 < 0). \end{cases}$$

The following lemma follows from the boundedness of the Hilbert transform. (See [1, Lemma 2.6].)

Lemma 3.8. *Let $1 < p < \infty$ and set*

$$Tf(y_1) = \int_0^\infty \frac{1}{y_1+z_1} f(z_1)\,dz_1, \quad y_1 \in (0,\infty),\ f \in L^p(0,\infty).$$

Then there exists a positive constant $C = C(p) > 0$ such that

$$\|Tf\|_{L^p(0,\infty)} \leqslant C\|f\|_{L^p(0,\infty)}.$$

By Remark 3.5(ii) one can obtain the following estimates. (Cf. [5, Lemma 4.5].) In what follows we will denote

$$\sigma(\lambda,\tilde{\zeta}) = |\lambda| + |\tilde{\zeta}|^2.$$

Lemma 3.9. *Let $M_0 > 0$. Then there exist $c_\infty > 0$ and $\theta_\infty \in (\frac{\pi}{2},\pi)$ such that if $|\tilde{\zeta}| \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty,\theta_\infty)$, then for any multi-index $\tilde{\alpha}$ the following estimates hold with some positive constants $c = c(\tilde{\alpha})$ and $C = C(\tilde{\alpha})$ uniformly for $|\tilde{\zeta}| \geqslant M_0$ and $\lambda \in \Sigma(-c_\infty,\theta_\infty)$:*

$$\begin{aligned}
&\text{(i)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \mu_j\bigr| \leqslant C\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{\frac{1}{2}-\frac{|\tilde{\alpha}|}{2}} \quad (j=1,2),\\
&\text{(ii)} \quad \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \frac{1}{\mu_j}\right| \leqslant C\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{1}{2}-\frac{|\tilde{\alpha}|}{2}},\\
&\text{(iii)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} (\mu_1-\mu_2)\bigr| \leqslant C|\lambda|\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{1}{2}-\frac{|\tilde{\alpha}|}{2}},\\
&\text{(iv)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \bigl(\mu_1\mu_2 - |\tilde{\zeta}|^2\bigr)\bigr| \leqslant C|\lambda|\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{|\tilde{\alpha}|}{2}},\\
&\text{(v)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \bigl(\mu_1\mu_2 - |\tilde{\zeta}|^2\bigr)^{-1}\bigr| \leqslant C\frac{1}{|\lambda|}\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{|\tilde{\alpha}|}{2}},\\
&\text{(vi)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} e^{-\mu_j y_1}\bigr| \leqslant C\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} y_1} \quad (j=1,2),\\
&\text{(vii)} \quad \bigl|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \bigl(e^{-\mu_1 y_1} - e^{-\mu_2 y_1}\bigr)\bigr| \leqslant C|\lambda|\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-1-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} y_1}.
\end{aligned}$$

We are now in a position to prove Lemma 3.3.

Proof of Lemma 3.3. Let $M_0 > 0$ and let $M \geqslant 2M_0$. Suppose that $\operatorname{supp}(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f) \subset \{(y_1,\tilde{\zeta}), \tilde{\zeta} = (\zeta_2,\xi); |\xi| \geqslant M/2\}$.

We first estimate the $\hat{G}(\lambda,\tilde{\zeta})$ part of $\hat{S}(\lambda,\tilde{\zeta})$. We begin with the terms concerning $g_{\mu_1}^{(-)}$. We write

$$g_{\mu_1}^{(-)}\bigl[\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f^0\bigr](y_1,\tilde{\zeta}) = I_1 - I_2,$$

where

$$I_1 = \frac{1}{2\mu_1}\int_0^\infty e^{-\mu_1|y_1-z_1|}\bigl(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f^0\bigr)(z_1)\,dz_1,$$

$$I_2 = \frac{1}{2\mu_1}\int_0^\infty e^{-\mu_1(y_1+z_1)}\bigl(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f^0\bigr)(z_1)\,dz_1.$$

As for I_1, by Lemma 3.7,

$$\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}[I_1] = \mathcal{F}_{\zeta\to y}^{-1}\left[\frac{1}{\mu_1^2+\zeta_1^2}\mathcal{F}_{y\to\zeta}\bigl(Ef^0\bigr)\right].$$

If $|\xi| \geqslant M/2$, then

$$\left|\partial_\zeta^\alpha\left[\frac{1}{\mu_1^2+\zeta_1^2}\right]\right| \leqslant C_\alpha\frac{1}{|\zeta|^2}|\zeta|^{-|\alpha|} \leqslant \frac{C_\alpha}{M^2}|\zeta|^{-|\alpha|}$$

for any α ($|\alpha| \geqslant 0$). It then follows from Lemma 3.6 that

$$\bigl\|\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}[I_1]\bigr\|_p \leqslant \frac{C}{M^2}\bigl\|f^0\bigr\|_p.$$

Similarly one can obtain

$$\bigl\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}[I_1]\bigr\|_p \leqslant \frac{C}{M^{2-|\beta|}}\bigl\|f^0\bigr\|_p \quad \bigl(|\beta| = 1, 2\bigr).$$

We next consider I_2. Let $|\tilde{\beta}| + \ell = 0, 1, 2$. By Lemma 3.9, we have

$$\begin{aligned}
\left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}\left[\tilde{\zeta}^{\tilde{\beta}}\partial_{y_1}^\ell \frac{1}{2\mu_1}e^{-\mu_1(y_1+z_1)}\right]\right| &\leqslant C_{\tilde{\alpha}}\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{1}{2}+\frac{\ell}{2}+\frac{\tilde{\beta}}{2}-\frac{\tilde{\alpha}}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}}(y_1+z_1)} \\
&\leqslant C_{\tilde{\alpha}}\bigl(|\lambda| + |\tilde{\zeta}|^2\bigr)^{-\frac{1}{2}+\frac{\ell+|\tilde{\beta}|}{2}-\frac{\tilde{\alpha}}{2}} \frac{1}{|\tilde{\zeta}|(y_1+z_1)} \\
&\leqslant \frac{C_\alpha}{M^{2-\ell-|\tilde{\beta}|}}\cdot\frac{|\tilde{\zeta}|^{-|\tilde{\alpha}|}}{y_1+z_1}
\end{aligned}$$

for $|\xi| \geqslant M/2$. It then follows from Lemma 3.6 that

$$\left\|\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\left[\tilde{\zeta}^{\tilde{\beta}}\partial_{y_1}^\ell \frac{1}{2\mu_1}e^{-\mu_1(y_1+z_1)}\bigl(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f^0\bigr)(z_1)\right]\right\|_{L^p_{\tilde{y}}(\mathbf{R}^2)} \leqslant \frac{C}{M^{2-\ell-|\tilde{\beta}|}}\cdot\frac{\|f^0(z_1,\tilde{y})\|_{L^p_{\tilde{y}}}}{y_1+z_1},$$

and, therefore, by Minkowski's inequality for integrals, we have

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\left[\tilde{\zeta}^{\tilde{\beta}}\partial_{y_1}^\ell \frac{1}{2\mu_1}\int_0^\infty e^{-\mu_1(y_1+z_1)}\bigl(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f^0\bigr)(z_1)\,dz_1\right]\right\|_p \\
&\quad\leqslant \frac{C}{M^{2-\ell-|\tilde{\beta}|}}\left(\int_0^\infty\left(\int_0^\infty \frac{\|f^0\|_{L^p_{\tilde{y}}}}{y_1+z_1}\,dz_1\right)^p dy_1\right)^{\frac{1}{p}}.
\end{aligned}$$

Using Lemma 3.8, we see that for $|\beta| = |\tilde{\beta}| + \ell = 0, 1, 2$,

$$\bigl\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}[I_2]\bigr\|_p = \bigl\|\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\bigl[\tilde{\zeta}^{\tilde{\beta}}\partial_{y_1}^\ell I_2\bigr]\bigr\|_p \leqslant \frac{C}{M^{2-|\beta|}}\bigl\|f^0\bigr\|_p.$$

From the estimates for I_1 and I_2 obtained above, we conclude

$$\bigl\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\bigl[g^{(-)}_{\mu_1}\bigl(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f^0\bigr)(y_1)\bigr]\bigr\|_p \leqslant \frac{C}{M^{2-|\beta|}}\bigl\|f^0\bigr\|_p \quad \bigl(|\beta| = 0, 1, 2\bigr).$$

Also, since

$$\begin{aligned}
\partial_{y_1}\bigl[g^{(+)}_{\mu_j}\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f\bigr] &= g^{(-)}_{\mu_j}\bigl[\partial_{y_1}[\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f]\bigr] \quad (j = 1, 2), \\
\tilde{\zeta}\bigl[g^{(+)}_{\mu_j}\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f\bigr] &= g^{(+)}_{\mu_j}\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}[\partial_{\tilde{y}}f] \quad (j = 1, 2),
\end{aligned}$$

one can similarly conclude for $\partial_{y_1} g_{\mu_2}^{(+)}$ and $i\tilde{\zeta} g_{\mu_2}^{(+)}$ that

$$\big\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[\partial_{y_1} g_{\mu_2}^{(+)}\big(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f^0\big)(y_1)\big]\big\|_p \leqslant \frac{C}{M^{(1-|\beta|)_+}}\big\|\partial_y^{(|\beta|-1)_+} f^0\big\|_p \quad \big(|\beta|=0,1,2\big),$$

$$\big\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[i\tilde{\zeta} g_{\mu_2}^{(+)}\big(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f^0\big)(y_1)\big]\big\|_p \leqslant \frac{C}{M^{(1-|\beta|)_+}}\big\|\partial_y^{(|\beta|-1)_+} f^0\big\|_p \quad \big(|\beta|=0,1,2\big).$$

It remains to estimate the terms concerning $g_{\mu_1,\mu_2}^{(\pm)}$. This will be complete if we show the estimates

$$\left\|\partial_y^\beta \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\left[\frac{\tilde{\zeta}^{\tilde{a}}}{\lambda}\partial_{y_1}^b g_{\mu_1,\mu_2}^{(\pm)}(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f)(y_1)\right]\right\|_p \leqslant \frac{C}{M^{2-|\beta|}}\|f\|_p \tag{3.8}$$

for any $\tilde{a}$ and b with $|\tilde{a}|+b=2$ and $|\beta|=0,1,2$.

Let us prove (3.8). We write

$$\frac{\tilde{\zeta}^{\tilde{a}}}{\lambda}\partial_{y_1}^b g_{\mu_1,\mu_2}^{(\pm)}(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f)(y_1) = J_1 \pm J_2,$$

where

$$J_1 = \int_0^\infty \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda}\partial_{y_1}^b\left(\frac{1}{2\mu_1}e^{-\mu_1|y_1-z_1|} - \frac{1}{2\mu_2}e^{-\mu_2|y_1-z_1|}\right)(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f)(z_1)\,dz_1,$$

$$J_2 = \int_0^\infty \frac{\tilde{\zeta}^{\tilde{a}}}{\lambda}\partial_{y_1}^b\left(\frac{1}{2\mu_1}e^{-\mu_1(y_1+z_1)} - \frac{1}{2\mu_2}e^{-\mu_2(y_1+z_1)}\right)(\mathcal{F}_{\tilde{y}\to\tilde{\zeta}} f)(z_1)\,dz_1.$$

As for J_1, by Lemma 3.7,

$$\mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}[J_1] = \mathcal{F}^{-1}_{\zeta\to y}\left[\frac{\tilde{\zeta}^{\tilde{a}}}{\lambda}\partial_{y_1}^b N(\lambda,\zeta)\mathcal{F}_{y\to\zeta}(Ef)\right].$$

Here

$$N(\lambda,\zeta) = \frac{1}{\mu_1^2+\zeta_1^2} - \frac{1}{\mu_2^2+\zeta_1^2}.$$

An elementary computation gives

$$N(\lambda,\zeta) = -\frac{\lambda(\tilde{\nu}\lambda+\gamma^2)}{(\lambda+\nu|\zeta|^2)(\lambda^2+\nu_1|\zeta|^2\lambda+\gamma^2|\zeta|^2)}.$$

In view of Remark 3.5 (ii), one can see that

$$\left|\partial_\zeta^\alpha\left(\frac{\zeta^j N(\lambda,\zeta)}{\lambda}\right)\right| \leqslant \frac{C_\alpha}{M^2}|\zeta|^{-|\alpha|} \quad \big(|j|=|\tilde{a}|+b\big)$$

for $|\xi| \geqslant M/2$. Lemma 3.6 then implies that

$$\left\| \mathcal{F}^{-1}_{\tilde{\xi} \to \tilde{y}}[J_1] \right\|_p \leqslant \frac{C}{M^2} \|f\|_p.$$

Similarly we have

$$\left\| \partial_y^{\beta} \mathcal{F}^{-1}_{\tilde{\xi} \to \tilde{y}}[J_1] \right\|_p \leqslant \frac{C}{M^{2-|\beta|}} \|f\|_p \quad \bigl(|\beta| = 1, 2\bigr).$$

We next consider J_2. We set

$$g^{(2)}_{\mu_1,\mu_2}(y_1, z_1) = \frac{1}{2\mu_1} e^{-\mu_1 (y_1+z_1)} - \frac{1}{2\mu_2} e^{-\mu_2 (y_1+z_1)}.$$

By Lemma 3.9, we have, for $|\xi| \geqslant M/2$,

$$\begin{aligned}
&\left| \partial_{\tilde{\xi}}^{\tilde{\alpha}} \left[\frac{\tilde{\xi}^{\tilde{\beta}+\tilde{\alpha}}}{\lambda} \partial_{y_1}^{\ell+b} \frac{1}{2\mu_1} \bigl(e^{-\mu_1 (y_1+z_1)} - e^{-\mu_2 (y_1+z_1)} \bigr) \right] \right| \\
&\quad \leqslant C_{\tilde{\alpha}} \bigl(|\lambda| + |\tilde{\xi}|^2 \bigr)^{-\frac{1}{2} + \frac{\ell+b}{2} + \frac{|\tilde{\beta}|+|\tilde{\alpha}|}{2} - \frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\xi})^{\frac{1}{2}} (y_1+z_1)} \\
&\quad \leqslant C_{\tilde{\alpha}} \bigl(|\lambda| + |\tilde{\xi}|^2 \bigr)^{-\frac{1}{2} + \frac{\ell+b+|\tilde{\beta}|+|\tilde{\alpha}|}{2} - \frac{|\tilde{\alpha}|}{2}} \frac{1}{|\tilde{\xi}|(y_1+z_1)} \\
&\quad \leqslant \frac{C_\alpha}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{\alpha}|}} \cdot \frac{|\tilde{\xi}|^{-|\tilde{\alpha}|}}{y_1+z_1}
\end{aligned} \tag{3.9}$$

and

$$\left| \partial_{\tilde{\xi}}^{\tilde{\alpha}} \left[\frac{\tilde{\xi}^{\tilde{\beta}+\tilde{\alpha}}}{\lambda} \partial_{y_1}^{\ell+b} \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) e^{-\mu_2 (y_1+z_1)} \right] \right| \leqslant \frac{C_\alpha}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{\alpha}|}} \cdot \frac{|\tilde{\xi}|^{-|\tilde{\alpha}|}}{y_1+z_1}. \tag{3.10}$$

Since

$$g^{(2)}_{\mu_1,\mu_2}(y_1, z_1) = \frac{1}{2\mu_1} \bigl(e^{-\mu_1 (y_1+z_1)} - e^{-\mu_2 (y_1+z_1)} \bigr) + \frac{1}{2} \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) e^{-\mu_2 (y_1+z_1)},$$

we see from Lemma 3.6, (3.9) and (3.10) that

$$\left\| \mathcal{F}^{-1}_{\tilde{\xi} \to \tilde{y}} \left[\frac{\tilde{\xi}^{\tilde{\beta}+\tilde{\alpha}}}{\lambda} \partial_{y_1}^{\ell+b} g^{(2)}_{\mu_1,\mu_2}(y_1, z_1) (\mathcal{F}_{\tilde{y} \to \tilde{\xi}} f)(z_1) \right] \right\|_{L^p_{\tilde{y}}(\mathbf{R}^2)} \leqslant \frac{C}{M^{2-\ell-b-|\tilde{\beta}|-|\tilde{\alpha}|}} \cdot \frac{\|f\|_{L^p_{\tilde{y}}}}{y_1+z_1}$$

for $|\xi| \geqslant M/2$. Therefore, by Lemma 3.8 and (3.9), we have, for $|\beta| = |\tilde{\beta}| + \ell = 0, 1, 2$,

$$\left\| \partial_y^{\beta} \mathcal{F}^{-1}_{\tilde{\xi} \to \tilde{y}}[J_2] \right\|_p = \left\| \mathcal{F}^{-1}_{\tilde{\xi} \to \tilde{y}} \bigl[\tilde{\xi}^{\tilde{\beta}} \partial_{y_1}^{\ell} J_2 \bigr] \right\|_p \leqslant \frac{C}{M^{2-|\beta|}} \|f\|_p.$$

Combining the estimates for J_1 and J_2 we obtain (3.8); and the desired estimates for the $\hat{G}$-part are obtained.

We next consider the $\hat{H}(\lambda,\tilde{\zeta})$ part of $\hat{S}(\lambda,\tilde{\zeta})$. By Lemma 3.9, we have

$$\begin{cases} \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} h_{\mu_j}(y_1)\right| \leqslant C\big(|\lambda|+|\tilde{\zeta}|^2\big)^{-\frac{1}{2}-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} y_1}, \\ \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} h_{\mu_1,\mu_2}(y_1)\right| \leqslant C|\lambda|\big(|\lambda|+|\tilde{\zeta}|^2\big)^{-\frac{3}{2}-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} y_1}, \\ \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \beta_0(z_1)\right| \leqslant C|\lambda|\big(|\lambda|+|\tilde{\zeta}|^2\big)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} z_1}, \\ \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \beta(z_1)\right| \leqslant C|\lambda|\big(|\lambda|+|\tilde{\zeta}|^2\big)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} z_1}, \\ \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}} \mathbf{b}(z_1)\right| \leqslant C|\lambda|\big(|\lambda|+|\tilde{\zeta}|^2\big)^{-\frac{|\tilde{\alpha}|}{2}} e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}} z_1}. \end{cases} \tag{3.11}$$

These inequalities yield the desired estimates for the $\hat{H}$-part. For example, let us consider the term $\frac{1}{\mu_1\mu_2-|\tilde{\zeta}|^2}\cdot\frac{\tilde{\zeta}}{\nu}h_{\mu_1}(y_1)\beta_0(z_1)$. By (3.11) and Lemma 3.9(v), we have

$$\begin{aligned} \left|\partial_{\tilde{\zeta}}^{\tilde{\alpha}}\left[\tilde{\zeta}^{\tilde{\beta}}\partial_{y_1}^{\ell}\frac{1}{\mu_1\mu_2-|\tilde{\zeta}|^2}\cdot\frac{\tilde{\zeta}}{\nu}h_{\mu_1}(y_1)\beta_0(z_1)\right]\right| &\leqslant C|\tilde{\zeta}|^{|\tilde{\beta}|+\ell-|\alpha|}e^{-c\sigma(\lambda,\tilde{\zeta})^{\frac{1}{2}}(y_1+z_1)} \\ &\leqslant \frac{C}{M^{1-|\tilde{\beta}|-\ell}}\cdot\frac{1}{y_1+z_1}|\tilde{\zeta}|^{-|\tilde{\alpha}|} \end{aligned}$$

for $|\xi|\geqslant M/2$. As in the estimates for I_2 and J_2 above, we see from Lemmas 3.6 and 3.8 that

$$\left\|\partial_y^{\beta}\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}\left[\frac{1}{\mu_1\mu_2-|\tilde{\zeta}|^2}\cdot\frac{\tilde{\zeta}}{\nu}h_{\mu_1}(y_1)\beta_0(z_1)\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f^0\right]\right\|_p \leqslant \frac{C}{M^{(1-|\beta|)_+}}\left\|\partial_y^{(|\beta|-1)_+}f^0\right\|_p$$

for $|\beta|=0,1,2$. Similarly one can obtain

$$\begin{gathered} \left\|\partial_y^{\beta}\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}[Q_0\hat{H}Q_0\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f]\right\|_p \leqslant C\left\|\partial_y^{\beta}f^0\right\|_p \quad \big(|\beta|=0,1\big), \\ \left\|\partial_y^{\beta}\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}[\tilde{Q}\hat{H}Q_0\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f]\right\|_p \leqslant \frac{C}{M^{(1-|\beta|)_+}}\left\|\partial_y^{(|\beta|-1)_+}f^0\right\|_p \quad \big(|\beta|=0,1,2\big), \\ \left\|\partial_y^{\beta}\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}[Q_0\hat{H}\tilde{Q}\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f]\right\|_p \leqslant \frac{C}{M^{1-|\beta|}}\|g\|_p \quad \big(|\beta|=0,1\big), \\ \left\|\partial_y^{\beta}\mathcal{F}_{\tilde{\zeta}\to\tilde{y}}^{-1}[\tilde{Q}\hat{H}\tilde{Q}\mathcal{F}_{\tilde{y}\to\tilde{\zeta}}f]\right\|_p \leqslant \frac{C}{M^{2-|\beta|}}\|g\|_p \quad \big(|\beta|=0,1,2\big). \end{gathered}$$

This completes the proof. $\square$

We now prove Proposition 3.2.

Proof of Proposition 3.2. By Lemma 3.3, if $\sup_{y_2}|h'(y_2)|\leqslant\eta$, then

$$\begin{aligned} &\left\|\partial_y\mathcal{F}_{\xi\to x_3}^{-1}\Phi_M\right\|_p+\left\|\partial_y^2\mathcal{F}_{\xi\to x_3}^{-1}V_M\right\|_p \\ &\quad\leqslant C\left\{\left\|\mathcal{F}_{\xi\to x_3}^{-1}[\partial_y Q_0\tilde{R}_M]\right\|_p+\left\|\mathcal{F}_{\xi\to x_3}^{-1}[\tilde{Q}\tilde{R}_M]\right\|_p\right\} \\ &\quad\leqslant C\left\{\left\|\partial_y\mathcal{F}_{\xi\to x_3}^{-1}[\tilde{F}_M]\right\|_p+\left\|\mathcal{F}_{\xi\to x_3}^{-1}[\partial_{y'}V_M]\right\|_p+\eta\left\|\mathcal{F}_{\xi\to x_3}^{-1}\big[\partial_{y'}^2V_M\big]\right\|_p\right. \\ &\qquad\left.+\left\|\mathcal{F}_{\xi\to x_3}^{-1}[\tilde{G}_M]\right\|_p+\eta\left\|\mathcal{F}_{\xi\to x_3}^{-1}\big[\partial_{y'}^2V_M\big]\right\|_p+\eta\left\|\mathcal{F}_{\xi\to x_3}^{-1}[\partial_{y'}\Phi_M]\right\|_p\right\}. \end{aligned} \tag{3.12}$$

We now take $\eta > 0$ in such a way that $C\eta \leqslant \frac{1}{2}$ and then choose $\delta > 0$ so small that $\sup_{y_2} |h'(y_2)| \leqslant \eta$ whenever $\operatorname{diam}(\mathcal{O}) \leqslant \delta$. It then follows from (3.12) that

$$\begin{aligned}
&\left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} \Phi_M\right\|_p + \left\|\partial_y^2 \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p \\
&\quad \leqslant C\left\{\left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3}[\tilde{F}_M]\right\|_p + \left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{G}_M]\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p\right\}. \qquad (3.13)
\end{aligned}$$

Similarly, by Lemma 3.3,

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}_{\xi\to x_3} \Phi_M\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p \\
&\quad \leqslant C\left\{\left\|\mathcal{F}^{-1}_{\xi\to x_3}[Q_0 \tilde{R}_M]\right\|_p + \frac{1}{M}\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{Q}\tilde{R}_M]\right\|_p\right\} \\
&\quad \leqslant C\left\{\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{F}_M]\right\|_p + \eta\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\partial_{y'} V_M]\right\|_p\right. \\
&\qquad + \frac{1}{M}\left(\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{G}_M]\right\|_p + \left\|\mathcal{F}^{-1}_{\xi\to x_3}\left[\partial_{y'}^2 V_M\right]\right\|_p\right. \\
&\qquad \left.\left. + \eta\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\partial_{y'} V_M]\right\|_p + \eta\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\partial_{y'} \Phi_M]\right\|_p\right)\right\}. \qquad (3.14)
\end{aligned}$$

We see from (3.13) and (3.14) that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}_{\xi\to x_3} \Phi_M\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p \\
&\quad \leqslant C\left\{\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{F}_M]\right\|_p + \frac{1}{M}\left(\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{G}_M]\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3}[\tilde{F}_M]\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p\right)\right\} \quad (3.15)
\end{aligned}$$

by taking η and δ smaller if necessary. It then follows from Lemma 3.3, (3.13) and (3.15) that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p &\leqslant \frac{C}{M}\left\{\left\|\mathcal{F}^{-1}_{\xi\to x_3}[Q_0 \tilde{R}_M]\right\|_p + \left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{Q}\tilde{R}_M]\right\|_p\right\} \\
&\leqslant \frac{C}{M}\left\{\left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{F}_M]\right\|_{W^{1,p}} + \left\|\mathcal{F}^{-1}_{\xi\to x_3}[\tilde{G}_M]\right\|_p + \left\|\partial_y \mathcal{F}^{-1}_{\xi\to x_3} V_M\right\|_p\right\}. \qquad (3.16)
\end{aligned}$$

Proposition 3.2 now follows from (3.13), (3.15) and (3.16). This completes the proof. □

We finally prove Theorem 3.1.

Proof of Theorem 3.1. For each $\bar{x}' \in \partial D$ we take $\mathcal{O}_{\bar{x}'}$ so that the estimates in Proposition 3.2 hold with $\mathcal{O}$ replaced by $\mathcal{O}_{\bar{x}'}$. Since D is bounded, one can find an open covering $\{\mathcal{O}_j\}_{j=0}^k$ of D and $\{\chi_j\}_{j=0}^k \subset C_0^\infty$ such that $U_M^{(j)} = \chi_j \hat{u}_M$ $(j = 1, \dots, k)$ satisfy the estimates in Proposition 3.2 with $\mathcal{O}$ replaced by $\mathcal{O}_j$. Here $\{\mathcal{O}_j\}_{j=0}^k$ satisfies $\overline{\mathcal{O}}_0 \subset D$, $\mathcal{O}_j = \mathcal{O}_{\bar{x}_j}$, for some $\bar{x}_j \in \partial D$ $(j = 1, 2, \dots, k)$, and $D \subset \bigcup_{j=0}^k \mathcal{O}_j$; and $\{\chi_j\}_{j=0}^k \subset C^\infty$ is a partition of unity subordinate to $\{\mathcal{O}_{j=0}^k\}$, i.e., there hold $\chi_j \in C_0^\infty(\mathcal{O}_j)$ and $\sum_{j=0}^k \chi_j \equiv 1$ on D. One can see that $U_M^{(0)} = \chi_0 \hat{u}_M$ satisfies similar estimates to those in Proposition 3.2. Furthermore, the constants C appearing in the estimates for $U_M^{(j)}$ $(j = 0, \dots, k)$ can be taken uniformly in $j = 0, \dots, k$.

Proposition 3.2(i) then yields

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\hat{v}_M\right\|_p &\leqslant \sum_{j=0}^{k}\left\|\mathcal{F}^{-1}V_M^{(j)}\right\|_p \\
&\leqslant \frac{C}{M}\sum_{j=0}^{k}\left\{\left\|\mathcal{F}^{-1}[\chi_j\kappa_M\hat{f}]\right\|_{W^{1,p}\times L^p}+\left\|\mathcal{F}^{-1}[\chi_j\hat{u}_M]\right\|_{L^p\times W^{1,p}}\right\} \\
&\leqslant \frac{C}{M}\left\{\left\|\mathcal{F}^{-1}[\kappa_M\hat{f}]\right\|_{W^{1,p}\times L^p}+\left\|\mathcal{F}^{-1}\hat{u}_M\right\|_{L^p\times W^{1,p}}\right\}.
\end{aligned}\tag{3.17}$$

By (3.17) and Proposition 3.2(ii), we have

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\hat{u}_M\right\|_{L^p\times W^{1,p}} &\leqslant \sum_{j=0}^{k}\left\|\mathcal{F}^{-1}U_M^{(j)}\right\|_{L^p\times W^{1,p}} \\
&\leqslant C\left\{\left\|\mathcal{F}^{-1}[\kappa_M\hat{f}]\right\|_{W^{1,p}\times L^p}+\frac{1}{M}\left\|\mathcal{F}^{-1}\hat{u}_M\right\|_{L^p\times W^{1,p}}\right\}.
\end{aligned}$$

Therefore, if $M>0$ is taken so large, we obtain

$$\left\|\mathcal{F}^{-1}\hat{u}_M\right\|_{L^p\times W^{1,p}} \leqslant C\left\|\mathcal{F}^{-1}[\kappa_M\hat{f}]\right\|_{W^{1,p}}.\tag{3.18}$$

It then follows from (3.18) and Proposition 3.2(iii) that

$$\begin{aligned}
\left\|\mathcal{F}^{-1}\hat{u}_M\right\|_{W^{1,p}\times W^{2,p}} &\leqslant \sum_{j=0}^{k}\left\|\mathcal{F}^{-1}U_M^{(j)}\right\|_{W^{1,p}\times W^{2,p}} \\
&\leqslant Ck\left\|\mathcal{F}^{-1}[\kappa_M\hat{f}]\right\|_{W^{1,p}\times L^p}.
\end{aligned}$$

This completes the proof. □

4. Resolvent problem for the middle frequency part

Let $M>0$ and $r>0$. In this section we establish estimates on $(\lambda+\hat{L}_\xi)^{-1}$ for $r/2\leqslant|\xi|\leqslant M$.

We begin with estimating $(\lambda+\hat{L}_\xi)^{-1}$ for λ in compact sets. We first estimate $\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}f$ for $f\in W^{1,p}(D)\times L^p(D)$. Here $\kappa_{r,M}$ is a function in $C_0^\infty(\mathbf{R})$ satisfying

$$0\leqslant\kappa_{r,M}\leqslant 1,\quad \kappa_{r,M}(\xi)=\begin{cases}1 & (r\leqslant|\xi|\leqslant\frac{M}{2}),\\ 0 & (|\xi|<\frac{r}{2},\ |\xi|>M).\end{cases}$$

Note that here f is a function of $x'\in D$ and does not depend on ξ.

Proposition 4.1. *Let r and M be numbers satisfying $0<r<\frac{M}{2}$ and let $\Lambda_1>0$. Then there exist constants $c_1=c_1(r,M)>0$ and $\theta_1=\theta_1(r,M)\in(\frac{\pi}{2},\pi)$ such that if $\lambda\in\Sigma(-c_1,\theta_1)\cap\{|\lambda|\leqslant\Lambda_1\}$, then for any integer $k\geqslant 0$ the function $\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}f$ satisfies the following estimate*

$$\left|\partial_\xi^k\left[\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}f\right]\right|_{W^{1,p}\times W^{2,p}}\leqslant C_k|f|_{W^{1,p}\times L^p}$$

with some constant C_k uniformly for ξ and $\lambda\in\Sigma(-c_1,\theta_1)\cap\{|\lambda|\leqslant\Lambda_1\}$.

Proof. Let $\tilde{\zeta} = (\zeta_2, \xi) \in \mathbf{R}^2$ and let $\hat{S}(\lambda, \tilde{\zeta})$ be the solution operator for problem (3.5) introduced in Section 3. We consider the following problem on $\{y_1 > 0\}$

$$\begin{cases} (\lambda + \hat{A}_{\tilde{\zeta}})w = F \quad (y_1 > 0), \\ \tilde{Q}\, w|_{y_1=0} = 0 \end{cases} \tag{4.1}$$

for the unknown $w = w(y_1, \tilde{\zeta})$ and a given $F = F(y_1, \zeta_2)$ with $\tilde{\zeta} = (\zeta_2, \xi)$ regarded as a parameter. Note that F does not depend on ξ. In view of Lemma 3.4 and Remark 3.5(ii), similarly to the proof of Lemma 3.3, we see that there exist $c_1 = c_1(r, M) > 0$ and $\theta_1 = \theta_1(r, M) \in (\frac{\pi}{2}, \pi)$ such that if $\frac{r}{2} \leqslant |\xi| \leqslant M$ and $\lambda \in \Sigma(-c_1, \theta_1)$, then (4.1) has a unique solution $w(y_1, \tilde{\zeta}) = \hat{S}(\lambda, \tilde{\zeta})F(y_1)$. Furthermore, $\mathcal{F}^{-1}_{\zeta_2 \to y_2} w = \mathcal{F}^{-1}_{\zeta_2 \to y_2} \hat{S}(\lambda, \tilde{\zeta})F$, which is a function of $y' = (y_1, y_2) \in \mathbf{R}^2_+$ with parameter ξ, satisfies the estimates

$$\begin{aligned} &\bigl\| \partial^{\alpha'}_{y'} Q_0 \mathcal{F}^{-1}_{\zeta_2 \to y_2}\bigl[\hat{S}(\lambda, \tilde{\zeta})F\bigr](y', \xi)\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)} \\ &\quad \leqslant C\bigl\{\bigl\|\partial^{\alpha'}_{y'} \mathcal{F}^{-1}_{\zeta_2 \to y_2}[Q_0 F]\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)} + \bigl\|\mathcal{F}^{-1}_{\zeta_2 \to y_2} \tilde{Q} F\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)}\bigr\} \end{aligned} \tag{4.2}$$

for $|\alpha'| = 0, 1$, and

$$\begin{aligned} &\bigl\| \partial^{\alpha'}_{y'} \tilde{Q} \mathcal{F}^{-1}_{\zeta_2 \to y_2}\bigl[\hat{S}(\lambda, \tilde{\zeta})F\bigr](y', \xi)\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)} \\ &\quad \leqslant C\bigl\{\bigl\|\partial^{(|\alpha|-1)}_{y'} \mathcal{F}^{-1}_{\zeta_2 \to y_2}[Q_0 F]\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)} + \bigl\|\mathcal{F}^{-1}_{\zeta_2 \to y_2} \tilde{Q} F\bigr\|_{L^p_{y'}(\mathbf{R}^2_+)}\bigr\} \end{aligned} \tag{4.3}$$

for $|\alpha'| = 0, 1, 2$.

We write f and $\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1} f$ as

$$f = {}^T\bigl(f^0, g\bigr), \qquad \kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1} f = {}^T(\hat{\phi}_{r,M}, \hat{v}_{r,M}).$$

Based on (4.2) and (4.3), by using the localization argument as in the proof of Theorem 3.1, one can obtain the estimate

$$|\hat{\phi}_{r,M}|_{W^{1,p}} + |\hat{v}_{r,M}|_{W^{2,p}} \leqslant C\bigl\{\bigl|f^0\bigr|_{W^{1,p}} + |g|_p + |\hat{\phi}_{r,M}|_p + |\hat{v}_{r,M}|_{W^{1,p}}\bigr\}. \tag{4.4}$$

Let us prove

$$|\hat{\phi}_{r,M}|_p + |\hat{v}_{r,M}|_{W^{1,p}} \leqslant C\bigl\{\bigl|f^0\bigr|_{W^{1,p}} + |g|_p\bigr\}. \tag{4.5}$$

We will prove (4.5) by a contradiction. Assume that (4.5) does not hold. Then for any $n \in \mathbf{N}$, there are $f_n = {}^T(f^0_n, g'_n, g^3_n) \in W^{1,p}(D) \times L^p(D)^3$, $\xi_n \in \mathbf{R}$, $\lambda_n \in \mathbf{C}$ and $\hat{u}_n = {}^T(\hat{\phi}_n, \hat{v}_n) \in W^{1,p}(D) \times [W^{2,p}(D) \cap W^{1,p}_0(D)]$ satisfying the following (4.6)–(4.8):

$$\frac{r}{2} \leqslant |\xi_n| \leqslant M, \quad \lambda_n \in \Sigma(-c_1, \theta_1), \quad |\lambda_n| \leqslant \Lambda_1, \tag{4.6}$$

$$(\lambda + \hat{L}_{\xi_n})\hat{u}_n = \kappa_{r,M}(\xi_n) f_n, \tag{4.7}$$

$$|\hat{\phi}_n|_p + |\hat{v}_n|_{W^{1,p}} \geqslant n\bigl\{\bigl|f^0_n\bigr|_{W^{1,p}} + |g_n|_p\bigr\}. \tag{4.8}$$

We may assume that

$$|\hat{\phi}_n|_p + |\hat{v}_n|_{W^{1,p}} = 1. \tag{4.9}$$

By (4.4), we have

$$|\hat{\phi}_n|_{W^{1,p}} + |\hat{v}_n|_{W^{2,p}} \leqslant C\left(\frac{1}{n}+1\right) \leqslant 2C.$$

Therefore, we can find a subsequence of $\{f_n, \xi_n, \lambda_n, \hat{u}_n\}$, which we again denote by $\{f_n, \xi_n, \lambda_n, \hat{u}_n\}$, such that, as $n \to \infty$,

$$\begin{gathered}
f_n^0 \to 0 \quad \text{in } W^{1,p}(D), \qquad g_n \to 0 \quad \text{in } L^p(D), \\
\xi_n \to \xi \quad \left(\frac{r}{2} \leqslant |\xi| \leqslant M\right), \qquad \lambda_n \to \lambda \in \Sigma(-c_1, \theta_1) \cap \big\{|\lambda| \leqslant \Lambda_1\big\}, \\
\phi_n \rightharpoonup \phi \quad \text{in } W^{1,p}(D), \qquad \phi_n \to \phi \quad \text{in } L^p(D), \\
v_n \rightharpoonup v \quad \text{in } W^{2,p}(D), \qquad v_n \to v \quad \text{in } W_0^{1,p}(D).
\end{gathered}$$

Letting $n \to \infty$ in (4.7) and (4.9), we have

$$(\lambda + \hat{L}_\xi)\hat{u} = 0, \quad \hat{u} \in W^{1,p}(D) \times \big[W^{2,p}(D) \cap W_0^{1,p}(D)\big], \quad |\hat{u}|_{L^p \times W^{1,p}} = 1.$$

But, by Lemma 4.2 below, if $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leqslant \Lambda_1\}$, then $\hat{u} = 0$, which contradicts $|\hat{u}|_{L^p \times W^{1,p}} = 1$. Therefore, we have (4.5).

It now follows from (4.4) and (4.5) that

$$\big|\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1} f\big|_{W^{1,p} \times W^{2,p}} \leqslant C|f|_{W^{1,p} \times L^p}. \tag{4.10}$$

We next estimate $\partial_\xi^k[\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1} f]$. We set

$$\hat{u}_{(k)} = \partial_\xi^k\big[\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1} f\big] = {}^T\big(\hat{\phi}_{(k)}, \hat{v}'_{(k)}, \hat{v}^3_{(k)}\big).$$

Then $\hat{u}_{(k)}$ is a solution of the problem

$$\begin{cases}
\lambda\hat{\phi}_{(k)} + \gamma\nabla' \cdot \hat{v}'_{(k)} + i\gamma\xi\hat{v}^3_{(k)} = \big(\partial_\xi^k\kappa_1\big)f^0 + \displaystyle\sum_{j=0}^{k-1}\binom{k-j}{j}\partial_\xi^{k-1}(i\gamma\xi)\hat{v}^3_{(j)}, \\
\lambda\hat{v}'_{(k)} - \nu\Delta'\hat{v}'_{(k)} + \nu\xi^2\hat{v}'_{(k)} - \tilde{\nu}\nabla'\big(\nabla' \cdot \hat{v}'_{(k)} + i\xi\hat{v}^3_{(k)}\big) + \gamma\nabla'\hat{\phi}_{(k)} \\
\quad = \big(\partial_\xi^k\kappa_1\big)g' + \displaystyle\sum_{j=0}^{k-1}\binom{k-j}{j}\big\{\partial_\xi^{k-1}\big(\nu\xi^2\big)\hat{v}'_{(j)} - \partial_\xi^{k-1}(\tilde{\nu}i\xi)\nabla'\hat{v}^3_{(j)}\big\}, \\
\lambda\hat{v}^3_{(k)} - \nu\Delta'\hat{v}^3_{(k)} + \nu\xi^2\hat{v}^3_{(k)} - i\tilde{\nu}\xi\big(\nabla' \cdot \hat{v}'_{(k)} + i\xi\hat{v}^3_{(k)}\big) + i\gamma\xi\hat{\phi}_{(k)} \\
\quad = \big(\partial_\xi^k\kappa_1\big)g^3 + \displaystyle\sum_{j=0}^{k-1}\binom{k-j}{j}\big\{\partial_\xi^{k-1}\big(\nu\xi^2\big)\hat{v}^3_{(j)} - \partial_\xi^{k-1}(\tilde{\nu}i\xi)\nabla' \cdot \hat{v}'_{(j)} \\
\qquad + \partial_\xi^{k-1}\big(\tilde{\nu}\xi^2\big)\hat{v}^3_{(j)} + \partial_\xi^{k-1}(i\gamma\xi)\hat{\phi}_{(j)}\big\}, \\
\hat{v}_{(k)}|_{\partial D} = 0.
\end{cases}$$

By (4.10), we have

$$|\hat{\phi}_{(k)}|_{W^{1,p}} + |\hat{v}_{(k)}|_{W^{2,p}} \leqslant C_k\big\{\big|f^0\big|_{W^{1,p}} + |g|_p\big\} + \sum_{j=0}^{k-1}\big\{|\hat{\phi}_{(j)}|_p + |\hat{v}_{(j)}|_{W^{1,p}}\big\}.$$

The desired estimate now follows by an induction argument. This completes the proof. □

Lemma 4.2. *Let $1 < p < \infty$. If $u \in W^{1,p}(D) \cap [W^{2,p}(D) \times W_0^{1,p}(D)]$, $(\lambda + \hat{L}_\xi)u = 0$, $\frac{r}{2} \leqslant |\xi| \leqslant M$ and $\lambda \in \Sigma(-c_1, \theta_1)$, then $u = 0$.*

To prove Lemma 4.2, we prepare some propositions.

Proposition 4.3. *Let $k \in \mathbf{N}$. If $\frac{r}{2} \leqslant |\xi| \leqslant M$ and $\lambda \in \Sigma(-c_1, \theta_1)$, then for any $f \in H^k(D) \times H^{k-1}(D)$ there exists a unique solution $u \in H^k(D) \times [H^{k+1}(D) \cap H_0^1(D)]$ of $(\lambda + \hat{L}_\xi)u = f$ and u satisfies the estimate*

$$|u|_{H^k \times H^{k+1}} \leqslant C|f|_{H^k \times H^{k-1}}.$$

Proposition 4.3 for $k = 1$ was proved in [10]. (See [10, Proposition 3.14].) The proof for $k \geqslant 2$ is done in a similar line to that of [10, Proposition 3.14] by using the Matsumura–Nishida energy method [19]. We here omit the details.

Remark 4.4. Proposition 4.3 remains true for the adjoint problem $(\lambda + \hat{L}_\xi^*)u = f$, where

$$\hat{L}_\xi^* = \begin{pmatrix} 0 & -\gamma^T \nabla' & -i\gamma\xi \\ -\gamma\nabla' & -\nu\Delta' + \nu\xi^2 - \tilde{\nu}\nabla'^T\nabla' & -i\tilde{\nu}\xi\nabla' \\ -i\gamma\xi & -i\tilde{\nu}\xi^T\nabla' & -\nu\Delta' + (\nu + \tilde{\nu})\xi^2 \end{pmatrix}.$$

Proposition 4.5. *Let $2 \leqslant q < \infty$. If $\frac{r}{2} \leqslant |\xi| \leqslant M$ and $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leqslant \Lambda_1\}$, then for any $f \in W^{1,q}(D) \times L^q(D)$ there exists a unique solution $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ of $(\lambda + \hat{L}_\xi^*)u^* = f$.*

Proof. Let $f \in C^\infty(\bar{D}) \times C_0^\infty(D)$. Then, by Remark 4.4, there exists a unique solution u^* of $(\lambda + \hat{L}_\xi^*)u^* = f$, which belongs to $H^k(D) \times [H^{k+1}(D) \cap H_0^1(D)]$ for any $k \in \mathbf{N}$. By the Sobolev embedding theorem, we have $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$. Similarly to the proof of Proposition 4.1, we can obtain the estimate

$$|u^*|_{W^{1,q} \times W^{2,p}} \leqslant C|f|_{W^{1,q} \times L^q}, \tag{4.11}$$

if we show that $(\lambda + \hat{L}_\xi^*)u = 0$ and $u \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ implies that $u = 0$. But, since $q \geqslant 2$, we have $W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)] \subset H^1(D) \times [H^2(D) \cap H_0^1(D)]$; and, hence, by Remark 4.4, $u = 0$. We thus obtain (4.11).

We next assume that $f \in W^{1,q}(D) \times L^q(D)$. Then there exists $\{f^{(n)}\}_{n=1}^\infty \subset C^\infty(\bar{D}) \times C_0^\infty(D)$ such that

$$f^{(n)} \to f \quad \text{in } W^{1,q}(D) \times L^q(D).$$

By the preceding argument, for each n, there exists $u^{*(n)} \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ such that

$$\big(\lambda + \hat{L}_\xi^*\big)u^{*(n)} = f^{(n)}$$

and

$$\big|u^{*(n)} - u^{*(m)}\big|_{W^{1,q} \times W^{2,q}} \leqslant C\big|f^{(n)} - f^{(m)}\big|_{W^{1,q} \times L^q}.$$

Therefore, $\{u^{*(n)}\}$ is a Cauchy sequence in $W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$, and we can find a function $u^* \in W^{1,q}(D) \times [W^{2,q}(D) \cap W_0^{1,q}(D)]$ such that

$$\big|u^{*(n)} - u^*\big|_{W^{1,q} \times W^{2,q}} \to 0 \quad (n \to \infty).$$

Letting $n \to \infty$ in $(\lambda + \hat{L}_\xi^*)u^{*(n)} = f^{(n)}$, we obtain

$$\big(\lambda + \hat{L}_\xi^*\big)u^* = f.$$

The uniqueness of u^* follows from Remark 4.4 since $q \geqslant 2$. This completes the proof. $\square$

We now prove Lemma 4.2.

Proof of Lemma 4.2. It suffices to prove Lemma 4.2 for $1 < p < 2$. Let $q \in (2, \infty)$ be the Hölder conjugate to p. Assume that $u \in W^{1,p}(D) \cap [W^{2,p}(D) \times W_0^{1,p}(D)]$ satisfies $(\lambda + \hat{L}_\xi)u = 0$. By Proposition 4.5, for any $f \in C^\infty(\bar{D}) \times C_0^\infty(D)$, there exists a unique solution $u^* \in W^{1,q}(D) \cap (W^{2,q}(D) \times W_0^{1,q}(D))$ of $(\bar{\lambda} + \hat{L}_\xi^*)u^* = f$. By integration by parts,

$$(u, f) = \big(u, \big(\bar{\lambda} + \hat{L}_\xi^*\big)u^*\big) = \big((\lambda + \hat{L}_\xi)u, u^*\big) = 0,$$

which implies $u = 0$. This completes the proof. $\square$

We now establish the estimate on $\mathcal{F}^{-1}[\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}]$ for $f \in W^{1,p}(\Omega) \times L^p(\Omega)$.

Theorem 4.6. *Let $f \in W^{1,p}(\Omega) \times L^p(\Omega)$. If $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leqslant \Lambda_1\}$, there holds the estimate*

$$\big\|\mathcal{F}^{-1}\big[\kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{f}\big]\big\|_{W^{1,p} \times W^{2,p}} \leqslant C\|f\|_{W^{1,p} \times L^p}$$

uniformly for $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leqslant \Lambda_1\}$.

It suffices to prove Theorem 4.6 for $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$. In fact, since $f \in W^{1,p}(\Omega) \times L^p(\Omega)$ can be approximated by elements in $\mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$, Theorem 4.6 immediately follows from the following proposition.

Proposition 4.7. *Let $\lambda \in \Sigma(-c_1, \theta_1) \cap \{|\lambda| \leqslant \Lambda_1\}$ and set $\hat{K}(\lambda, \xi) = \kappa_{r,M}(\xi)(\lambda + \hat{L}_\xi)^{-1}$. Define $K(\lambda, x_3)$ by*

$$K(\lambda, x_3)F = \mathcal{F}^{-1}\big[\hat{K}(\lambda, \xi)F\big] = \frac{1}{2\pi}\int_{\mathbf{R}} e^{ix_3\xi}\hat{K}(\lambda, \xi)F\,d\xi$$

for $F \in W^{1,p}(D) \times L^p(D)$. Then for $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$, the function $u = \mathcal{F}^{-1}[\hat{K}(\lambda, \xi)\hat{f}(\xi)]$ satisfies $u = K(\lambda, \cdot) * f$ and the estimate

$$\|u\|_{W^{1,p} \times W^{2,p}} \leqslant C\|f\|_{W^{1,p} \times L^p}.$$

Here $*$ means the convolution in x_3.

Proof. We first show

$$\hat{K}(\lambda, \cdot)\hat{f}(\xi) = \mathcal{F}\big[K(\lambda, \cdot) * f\big]$$

for any $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$. Since

$$e^{ix_3\xi} = \frac{1}{(ix_3)^k}\partial_\xi^k e^{ix_3\xi},$$

we see from Proposition 4.1 that for $F \in W^{1,p}(D) \times L^p(D)$

$$\begin{aligned}
&\bigl|\partial_{x_3}^{\ell} K(\lambda, x_3) F\bigr|_{W^{1,p}\times W^{2,p}} \\
&\quad \leqslant \left| \frac{1}{2\pi} \int\limits_{\mathbf{R}} (i\xi)^{\ell} e^{ix_3\xi} \hat{K}(\lambda, \xi) F \, d\xi \right|_{W^{1,p}\times W^{2,p}} \\
&\quad \leqslant \left| \frac{1}{2\pi} \frac{(-1)^k}{(ix_3)^k} \int\limits_{\mathbf{R}} e^{ix_3\xi} \partial_{\xi}^{k} \bigl[(i\xi)^{\ell} \hat{K}(\lambda, \xi)\bigr] F \, d\xi \right|_{W^{1,p}\times W^{2,p}} \\
&\quad \leqslant C(1+M)^{\ell} |x_3|^{-k} \sum_{j=0}^{k} \int\limits_{\{r/2 \leqslant |\xi| \leqslant M\}} \bigl|\partial_{\xi}^{j} \hat{K}(\lambda, \xi) F\bigr|_{W^{1,p}\times W^{2,p}} \, d\xi \\
&\quad \leqslant C(1+M)^{\ell} |x_3|^{-k} \int\limits_{\{r/2 \leqslant |\xi| \leqslant M\}} |F|_{W^{1,p}\times L^p} \, d\xi \\
&\quad \leqslant C_{r,M} |x_3|^{-k} |F|_{W^{1,p}\times L^p} \quad (\ell = 0, 1, 2).
\end{aligned}$$

It then follows that

$$\bigl|\partial_{x_3}^{\ell} K(\lambda, x_3)\bigr|_{\mathcal{L}(W^{1,p}(D)\times L^p(D), W^{1,p}(D)\times W^{2,p}(D))} \leqslant \frac{C}{1+|x_3|^2}. \tag{4.12}$$

Here $|T|_{\mathcal{L}(X,Y)}$ denotes the operator norm of a bounded operator $T : X \to Y$. By (4.12), for any $f \in \mathcal{S}(\mathbf{R}; W^{1,p}(D) \times L^p(D))$, there hold the estimates

$$\bigl|K(\lambda, x_3) f(y_3)\bigr|_{W^{1,p}\times W^{2,p}} \leqslant \frac{C}{1+|x_3|^2} \bigl|f(y_3)\bigr|_{W^{1,p}\times L^p} \in L^1(\mathbf{R}_{x_3} \times \mathbf{R}_{y_3})$$

and

$$\bigl|K(\lambda, z_3 - y_3) f(y_3)\bigr|_{W^{1,p}\times W^{2,p}} \leqslant \frac{C}{1+|z_3 - y_3|^2} \bigl|f(y_3)\bigr|_{W^{1,p}\times L^p} \in L^1(\mathbf{R}_{z_3} \times \mathbf{R}_{y_3}).$$

Therefore, by Fubini's theorem, we have $\hat{K}(\lambda, \xi)\hat{f}(\xi) = \mathcal{F}[K(\lambda, \cdot) * f]$, which implies $u = K(\lambda, \cdot) * f$.

Furthermore, we see from (4.12) that

$$\begin{aligned}
&\bigl\|Q_0 K(\lambda, \cdot) * f\bigr\|_{W^{1,p}}^{p} \\
&\quad = \sum_{\ell+k \leqslant 1} \bigl\|\partial_{x_3}^{\ell} Q_0 K(\lambda, \cdot) * f\bigr\|_{L^p(\mathbf{R}; W^{k,p}(D))}^{p} \\
&\quad = \sum_{\ell+k \leqslant 1} \int\limits_{\mathbf{R}} \left| \partial_{x_3}^{\ell} \int\limits_{\mathbf{R}} Q_0 K(\lambda, x_3 - y_3) f(\cdot, y_3) \, dy_3 \right|_{W^{k,p}}^{p} dx_3 \\
&\quad \leqslant C \sum_{\ell+k \leqslant 1} \int\limits_{\mathbf{R}} \left(\int\limits_{\mathbf{R}} \frac{1}{1+|x_3 - y_3|^2} \bigl|f(\cdot, y_3)\bigr|_{W^{1,p}\times L^p} \, dy_3 \right)^{p} dx_3 \\
&\quad \leqslant C \|f\|_{W^{1,p}\times L^p}^{p}.
\end{aligned}$$

Similarly one can estimate $\|\tilde{Q}K(\lambda,\cdot)*f\|_{W^{2,p}}$ and the desired estimate is obtained. This completes the proof. $\square$

We next consider estimates on $\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}\hat{f}$ for large $|\lambda|$, which can be obtained by a similar argument as in Section 3.

Let $f \in C_0^\infty(\overline{\Omega}) \times C_0^\infty(\Omega)$. Then $\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}\hat{f} = {}^T(\hat{\phi}_{r,M}, \hat{v}_{r,M})$ is a solution of (3.1) with κ_M replaced by $\kappa_{r,M}$.

Similarly to the proof of Lemma 3.3, one can prove the following estimate (cf. [5, Sections 4 and 5]).

Lemma 4.8. *There are $\tilde{\Lambda} > 0$ and $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that if $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta})$, then there hold the estimates*

$$\big\|\partial_y^\alpha \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[\hat{S}(\lambda,\tilde{\zeta})\mathcal{F}^{-1}_{\tilde{y}\to\tilde{\zeta}} f\big]\big\|_p \leqslant C\left\{\frac{\|\partial_y^\alpha f^0\|_p}{|\lambda|} + \frac{\|g\|_p}{|\lambda|^{1-\frac{|\alpha|}{2}}}\right\} \quad \big(|\alpha| = 0, 1\big)$$

and

$$\big\|\partial_y^2 \mathcal{F}^{-1}_{\tilde{\zeta}\to\tilde{y}}\big[\tilde{Q}\,\hat{S}(\lambda,\tilde{\zeta})\mathcal{F}^{-1}_{\tilde{y}\to\tilde{\zeta}} f\big]\big\|_p \leqslant C\big\{\big\|\partial_y f^0\big\|_p + \|g\|_p\big\}.$$

Based on Lemma 4.8 and the localization argument as in the proof of Theorem 3.1, we have the following estimate (by taking $\tilde{\Lambda}$ larger if necessary).

Theorem 4.9. *There are $\tilde{\Lambda} > 0$ and $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that if $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta})$, then there holds the estimate*

$$\big\|\mathcal{F}^{-1}\big[\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}\hat{f}\big]\big\|_{W^{1,p}\times W^{2,p}} \leqslant C\|f\|_{W^{1,p}\times L^p}.$$

Combining Theorems 4.6 and 4.9, we obtain the following estimate for $\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}\hat{f}$.

Theorem 4.10. *Let r and M be numbers satisfying $0 < r < \frac{M}{2}$. Then there are constants $\tilde{\Lambda} > 0$, $c_1 > 0$ and $\tilde{\theta} \in (\frac{\pi}{2}, \pi)$ such that if $\lambda \in \Sigma(\tilde{\Lambda}, \tilde{\theta}) \cup \{\mathrm{Re}\,\lambda \geqslant -c_1\}$, then*

$$\big\|\mathcal{F}^{-1}\big[\kappa_{r,M}(\xi)(\lambda+\hat{L}_\xi)^{-1}\hat{f}\big]\big\|_{W^{1,p}\times W^{2,p}} \leqslant C\|f\|_{W^{1,p}\times L^p}.$$

5. Spectral properties of low frequency part

In this section we investigate spectral properties of $-\hat{L}_\xi$ for $|\xi| \ll 1$. This case is treated as a perturbation from the case $\xi = 0$.

We begin with some spectral properties of $-\hat{L}_0$. We set $\xi = 0$ in (2.2) to obtain

$$(\lambda + \hat{L}_0)\hat{u} = \hat{f}, \quad \hat{v}|_{\partial D} = 0, \tag{5.1}$$

where $\hat{u} = {}^T(\hat{\phi}, \hat{v}', \hat{v}^3)$, $\hat{f} = {}^T(\hat{f}^0, \hat{g}', \hat{g}^3)$ and

$$\hat{L}_0 = \begin{pmatrix} 0 & \gamma\,{}^T\nabla' & 0 \\ \gamma\nabla' & -\nu\Delta' - \tilde{\nu}\nabla'\,{}^T\nabla' & 0 \\ 0 & 0 & -\nu\Delta' \end{pmatrix}.$$

We decompose $\hat{\phi}$ and $\hat{f}^0$ into

$$\hat{\phi} = \hat{\phi}_0 + \hat{\phi}_1, \quad \hat{\phi}_0 \equiv \frac{1}{|D|}\int_D \hat{\phi}(x')\,dx',$$

$$\hat{f}^0 = \hat{f}^0_0 + \hat{f}^0_1, \quad \hat{f}^0_0 \equiv \frac{1}{|D|}\int_D \hat{f}(x')\,dx',$$

respectively. This gives an orthogonal decomposition in $L^2(D)$, and we have

$$|\hat{\phi}|_2^2 = |\hat{\phi}_0|_2^2 + |\hat{\phi}_1|_2^2.$$

Furthermore, since $\hat{\phi}_1$-component has vanishing mean value, by the Poincaré inequality, there holds the estimate

$$|\hat{\phi}_1|_p \leqslant C|\partial_{x'}\hat{\phi}_1|_p = C|\partial_{x'}\hat{\phi}|_p.$$

In terms of this decomposition, problem (5.1) is reduced to the following problem (5.2)–(5.5):

$$\lambda\hat{\phi}_0 = \hat{f}^0_0, \tag{5.2}$$

$$\lambda\hat{\phi}_1 + \gamma\nabla'\cdot\hat{v}' = \hat{f}^0_1, \tag{5.3}$$

$$\lambda\hat{v}' - \nu\Delta'\hat{v}' - \tilde{\nu}\nabla'(\nabla'\cdot\hat{v}') + \gamma\nabla'\hat{\phi}_1 = \hat{g}', \quad \hat{v}'|_{\partial D} = 0, \tag{5.4}$$

$$\lambda\hat{v}^3 - \nu\Delta'\hat{v}^3 = \hat{g}^3, \quad \hat{v}^3|_{\partial D} = 0. \tag{5.5}$$

As for the solvability of (5.2)–(5.5) we have the following facts.

It is clear that (5.2) is uniquely solvable if and only if $\lambda \neq 0$, and in this case the solution is given by $\hat{\phi}_0 = \frac{1}{\lambda}\hat{f}^0_0$. It is also easy to see that $\lambda = 0$ is a simple eigenvalue with eigenfunction $\hat{\phi}_0 = 1$.

As for (5.5), it is well known that there are $\{\lambda_j\}_{j=1}^{\infty}$ ($\lambda_j < 0$, $|\lambda_1| < |\lambda_2| \leqslant |\lambda_3| \leqslant \cdots \to \infty$) such that each λ_j is a semi-simple eigenvalue and, for $\lambda \notin \{\lambda_j\}_{j=1}^{\infty}$, (5.5) has a unique solution $\hat{v}^3 \in W^{2,p}(D) \cap W^{1,p}_0(D)$. Furthermore, if $|\arg(\lambda - \frac{1}{2}\lambda_1)| \leqslant \pi - \varepsilon$ ($\varepsilon > 0$), then the solution $\hat{v}^3$ satisfies the estimate

$$|\lambda|\big|\hat{v}^3\big|_p + |\lambda|^{\frac{1}{2}}\big|\partial_{x'}\hat{v}^3\big|_p + \big|\partial_{x'}^2\hat{v}^3\big|_p \leqslant C_\varepsilon\big|\hat{g}^3\big|_p.$$

As for the solvability of (5.3)–(5.4), we have the following result.

Proposition 5.1. *Let $1 < p < \infty$. Then there exist constants $c_0 > 0$, $\Lambda > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that if $\lambda \in \Sigma(\Lambda_0/2, \theta_0) \cup \{\mathrm{Re}\,\lambda \geqslant -2c_0\}$, then for any ${}^T(\hat{f}^0_1, \hat{g}') \in W^{1,p}(D) \times L^p(D)$ with $\int_D \hat{f}^0_1\,dx' = 0$, there exists a unique solution ${}^T(\hat{\phi}_1, \hat{v}') \in W^{1,p}(D) \times [W^{2,p}(D) \cap W^{1,p}_0(D)]$ with $\int_D \hat{\phi}_1\,dx' = 0$ of (5.3)–(5.4), which satisfies the estimate*

$$|\lambda|\big\{|\hat{\phi}_1|_{W^{1,p}} + |\hat{v}'|_p\big\} + |\lambda|^{\frac{1}{2}}|\partial_{x'}\hat{v}'|_p + \big|\partial_{x'}^2\hat{v}'\big|_p \leqslant C\big\{\big|\hat{f}^0_1\big|_{W^{1,p}} + |\hat{g}'|_p\big\}.$$

Proposition 5.1 was proved by [21]. (See also [20].) We summarize the spectral properties of $-\hat{L}_0$ obtained above.

Proposition 5.2. *There are constants $c_0 > 0$, $\Lambda_0 > 0$ and $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that*

$$\big(\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -2c_0\}\big) \cap \big\{|\lambda| \geqslant c_0\big\} \subset \rho(-\hat{L}_0)$$

and

$$\sigma(-\hat{L}_0) \cap \big\{|\lambda| < c_0\big\} = \{0\}.$$

If $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -2c_0\}) \cap \{|\lambda| \geqslant c_0\}$, then

$$\big|(\lambda + \hat{L}_0)^{-1} f\big|_{W^{1,p} \times L^p} \leqslant \frac{C}{|\lambda| + 1} |f|_{W^{1,p} \times L^p},$$

$$\big|\partial_{x'}^{\ell} \tilde{Q}(\lambda + \hat{L}_0)^{-1} f\big|_p \leqslant \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} |f|_{W^{1,p} \times L^p} \quad (\ell = 1, 2).$$

Furthermore, 0 is a simple eigenvalue and the associated eigenprojection $\hat{P}^{(0)}$ is given by

$$\hat{P}^{(0)} u = {}^{T}\big(\langle \phi \rangle, 0\big) \quad \text{for } u = {}^{T}(\phi, v).$$

Based on Proposition 5.2, one can obtain the following result by a perturbation argument as in the proof of [10, Propositions 4.3 and 4.4].

Theorem 5.3. *There exists a positive constant $r_1 > 0$ such that the following assertions hold.*

(i) *If $|\xi| \leqslant r_1$, then*

$$\big(\Sigma(\Lambda_0, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -c_0\}\big) \cap \left\{|\lambda| \geqslant \frac{c_0}{2}\right\} \subset \rho(-\hat{L}_\xi).$$

(ii) *If $\lambda \in (\Sigma(\Lambda_0, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -c_0\}) \cap \{|\lambda| \geqslant \frac{c_0}{2}\}$, then*

$$\big|(\lambda + \hat{L}_\xi)^{-1} f\big|_{W^{1,p} \times L^p} \leqslant \frac{C}{|\lambda| + 1} |f|_{W^{1,p} \times L^p},$$

$$\big|\partial_{x'}^{\ell} \tilde{Q}(\lambda + \hat{L}_\xi)^{-1} f\big|_p \leqslant \frac{C}{(|\lambda| + 1)^{1 - \frac{\ell}{2}}} |f|_{W^{1,p} \times L^p} \quad (\ell = 1, 2).$$

(iii) *If $|\xi| \leqslant r_1$, then*

$$\sigma(-\hat{L}_\xi) \cap \left\{|\lambda| < \frac{c_0}{2}\right\} = \big\{\lambda_0(\xi)\big\}.$$

Here $\lambda_0(\xi)$ is a simple eigenvalue of $-\hat{L}_\xi$, which satisfies

$$\lambda_0(\xi) = -\frac{a_1 \gamma}{\nu} |\xi|^2 + O\big(|\xi|^4\big) \quad \big(|\xi| \to 0\big)$$

for some constant $a_1 > 0$.

We next give an estimate for the eigenprojection $\hat{P}(\xi)$ associated with the eigenvalue $\lambda_0(\xi)$. For this purpose we write $\hat{L}_\xi$ as

$$\hat{L}_\xi = \hat{L}_0 + \xi \hat{L}^{(1)} + \xi^2 \hat{L}^{(2)}.$$

Here

$$\hat{L}^{(1)} = \begin{pmatrix} 0 & 0 & i\gamma \\ 0 & 0 & -i\tilde{\nu}\nabla' \\ i\gamma & -i\tilde{\nu}^T\nabla' & 0 \end{pmatrix}, \qquad \hat{L}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nu I & 0 \\ 0 & 0 & \nu + \tilde{\nu} \end{pmatrix}.$$

We begin with the following

Proposition 5.4. *Let $R > 0$. Then the following estimate holds for $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -2c_0\}) \cap \{c_0 \leqslant |\lambda| \leqslant R\}$:*

$$\left|(\lambda + \hat{L}_0)^{-1}\hat{f}\right|_{H^3 \times H^4} \leqslant C_R |\hat{f}|_{H^3 \times H^2}.$$

Proof. We here give an outline of the proof. As it was shown in [10], an application of the Matsumura–Nishida energy method [19] to (5.3)–(5.4) gives

$$|\hat{v}'|_{H^2} + |\hat{\phi}_1|_{H^1} \leqslant C_R\left(\left|\hat{f}_1^0\right|_{H^1} + |\hat{g}'|_2\right)$$

for some constant $C = C(R) > 0$. Then higher order derivatives can also be estimated by the Matsumura–Nishida energy method to obtain

$$|\hat{v}'|_{H^4} + |\hat{\phi}_1|_{H^3} \leqslant C_R\left(\left|\hat{f}_1^0\right|_{H^3} + |\hat{g}'|_{H^2}\right). \tag{5.6}$$

Applying the elliptic regularity estimate to (5.5), we have

$$\left|\hat{v}^3\right|_{H^4} \leqslant C_R\left|\hat{g}^3\right|_{H^2}. \tag{5.7}$$

Proposition 5.4 now follows from (5.2), (5.6) and (5.7). This completes the proof. □

Lemma 5.5. *There hold the following estimates*

$$\left|L^{(j)}(\lambda + \hat{L}_0)^{-1}\hat{f}\right|_{H^3} \leqslant C|\hat{f}|_{H^3} \quad (j = 1, 2)$$

for $\lambda \in (\Sigma(\Lambda_0/2, \theta_0) \cup \{\operatorname{Re}\lambda \geqslant -2c_0\}) \cap \{c_0 \leqslant |\lambda| \leqslant R\}$.

Proof. Let ${}^T(\hat{\phi}, \hat{v}) = (\lambda + \hat{L}_0)^{-1}\hat{f}$. It follows from Proposition 5.4 that

$$\begin{aligned}\left|L^{(1)}(\lambda + \hat{L}_0)^{-1}\hat{f}\right|_{H^3} = \left|L^{(1)}\hat{u}\right|_{H^3} &\leqslant C\left\{\left|\hat{v}^3\right|_{H^3} + \left|\nabla'\hat{v}^3\right|_{H^3} + |\hat{\phi}|_{H^3} + |\nabla' \cdot \hat{v}'|_{H^3}\right\} \\ &\leqslant C|\hat{f}|_{H^3 \times H^2}\end{aligned}$$

and

$$\left|L^{(2)}(\lambda + \hat{L}_0)^{-1}\hat{f}\right|_{H^3} = \left|L^{(2)}\hat{u}\right|_{H^3} \leqslant C\left\{|\hat{v}'|_{H^3} + \left|\hat{v}^3\right|_{H^3}\right\} \leqslant C|\hat{f}|_{H^3 \times H^2}.$$

This completes the proof. □

We now estimate the integral kernel of the eigenprojection $\hat{P}(\xi)$ associated with the eigenvalue $\lambda_0(\xi)$ for $|\xi| \ll 1$.

Theorem 5.6. *There exists $r_2 > 0$ such that if $|\xi| \leqslant r_2$, then the following assertions hold.*

(i) *The eigenprojection $\hat{P}(\xi)$ associated with the eigenvalue $\lambda_0(\xi)$ is written in the form*

$$\hat{P}(\xi) = \hat{P}^{(0)} + \xi \hat{P}^{(1)} + \hat{P}^{(2)}(\xi),$$
$$\hat{P}^{(j)} u = \int_D \hat{P}^{(j)}(x', y') u(y')\, dy', \quad j = 0, 1,$$
$$\hat{P}^{(2)} u = \int_D \hat{P}^{(2)}(\xi, x', y') u(y')\, dy'.$$

Here $\hat{P}^{(0)} = \frac{1}{|D|} Q_0$; and $\hat{P}^{(1)}(x', y')$ and $\hat{P}^{(2)}(\xi, x', y')$ satisfy

$$\partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \hat{P}^{(1)}(x', y'),\ \partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \hat{P}^{(2)}(\xi, x', y') \in L^\infty(D \times D)$$

for $|\alpha'| \leqslant 1$ and $|\beta'| \leqslant 1$. Furthermore, for any $\alpha \geqslant 0$, $\hat{P}^{(2)}(\xi, x', y')$ satisfies the estimate

$$\left| \partial_\xi^\alpha \partial_{x'}^{\alpha'} \partial_{y'}^{\beta'} \hat{P}^{(2)}(\xi, \cdot, \cdot) \right|_{L^\infty(D \times D)} \leqslant C_\alpha |\xi|^{2-\alpha}.$$

(ii) *$\lambda_0(\xi)$ is a simple eigenvalue of the adjoint operator $-\hat{L}_\xi^*$ and the associated eigenprojection $\hat{P}^*(\xi)$ is written as*

$$\hat{P}^*(\xi) = \hat{P}^{(0)*} + \xi \hat{P}^{(1)*} + \hat{P}^{(2)*}(\xi),$$
$$\hat{P}^{(j)*} u = \int_D \hat{P}^{(j)*}(x', y') u^*(y')\, dy', \quad j = 0, 1,$$
$$\hat{P}^{(2)*}(\xi) u = \int_D \hat{P}^{(2)*}(\xi, x', y') u^*(y')\, dy'.$$

Here $\hat{P}^{(j)*}(x', y')$ $(j = 0, 1)$ and $P^{(2)*}(\xi, x', y')$ satisfy

$$\hat{P}^{(0)*} = \hat{P}^{(0)}, \qquad \hat{P}^{(1)*}(x', y') = \overline{\hat{P}^{(1)}(y', x')},$$
$$\hat{P}^{(2)*}(\xi, x', y') = \overline{\hat{P}^{(2)}(\xi, y', x')}.$$

(iii) *There hold the following relations*

$$\left(\hat{P}(\xi)[\partial_{x'} \tilde{Q} u], u^*\right) = -\left(u, \partial_{x'} \tilde{Q} \hat{P}^*(\xi) u^*\right),$$
$$\left(\hat{P}^{(1)}[\partial_{x'} \tilde{Q} u], u^*\right) = -\left(u, \partial_{x'} \tilde{Q} \hat{P}^{(1)} u^*\right),$$
$$\left(\hat{P}^{(2)}[\partial_{x'} \tilde{Q} u], u^*\right) = -\left(u, \partial_{x'} \tilde{Q} \hat{P}^{(2)} u^*\right).$$

Proof. As for (i), we here give an outline of the proof, since it is similar to that of [6, Theorem 3.3].

Let $\psi_0 = {}^T(1,0)$, which is an eigenfunction for the eigenvalue 0 of $-\hat{L}_0$ and $-\hat{L}_0^*$. Clearly, $|\psi_0|_{H^k} = |D|^{\frac{1}{2}}$ for all $k \geqslant 0$.

We define $\psi(\xi)$ and $\psi^*(\xi)$ by

$$\psi(\xi) = \frac{1}{2\pi i}\int_\Gamma (\lambda + \hat{L}_\xi)^{-1}\psi_0\, d\lambda$$

and

$$\psi^*(\xi) = \frac{1}{\langle \psi(\xi), \tilde{\psi}^*(\xi)\rangle}\tilde{\psi}^*(\xi)$$

with

$$\tilde{\psi}^*(\xi) = \frac{1}{2\pi i}\int_\Gamma \left(\lambda + \hat{L}_\xi^*\right)^{-1}\psi_0\, d\lambda,$$

where $\Gamma = \{|\lambda| = \frac{c_0}{2}\}$. Then as in the proof of [6, Theorem 3.3], one can see the following estimates on $\psi(\xi)$ and $\psi^*(\xi)$. By Lemma 5.5 and the Neumann series expansion of $(\lambda + \hat{L}_\xi)^{-1}$, $\psi(\xi)$ is expanded as

$$\psi(x', \xi) = \psi_0 + \xi\psi^{(1)}(x') + \psi^{(2)}(x', \xi),$$

and, with the aid of the Sobolev embedding $H^3 \hookrightarrow W^{1,\infty}$,

$$\left|\psi^{(1)}\right|_{W^{1,\infty}} \leqslant C\left|\psi^{(1)}\right|_{H^3} \leqslant C,$$
$$\left|\partial_\xi^\alpha \psi^{(2)}(\xi)\right|_{W^{1,\infty}} \leqslant C\left|\partial_\xi^\alpha \psi^{(2)}(\xi)\right|_{H^3} \leqslant C|\xi|^{2-\alpha}.$$

The same expansion also holds for $\psi^*(\xi)$:

$$\psi^*(x', \xi) = \psi_0 + \xi\psi^{(1)*}(x') + \psi^{(2)*}(x', \xi),$$

where $\psi^{(j)*}$ $(j = 1, 2)$ satisfy the estimates

$$\left|\psi^{(1)*}\right|_{W^{1,\infty}} \leqslant C \quad \text{and} \quad \left|\partial_\xi^\alpha \psi^{(2)*}(\xi)\right|_{W^{1,\infty}} \leqslant C|\xi|^{2-\alpha}.$$

In terms of $\psi(x', \xi)$ and $\psi^*(x', \xi)$, $\hat{P}(\xi)$ is given in the form

$$\hat{P}(\xi)u = \left\langle u, \psi^*(\xi)\right\rangle \psi(\xi) = \int_D \hat{P}(\xi, x', y')u(y')\, dy'$$

with

$$\begin{aligned}\hat{P}(\xi, x', y') &= \frac{1}{|D|}\psi(x', \xi)^T \psi^*(y', \xi) \\ &= \frac{1}{|D|}Q_0 + \xi \hat{P}^{(1)}(x', y') + \hat{P}^{(2)}(\xi, x', y').\end{aligned}$$

Here

$$\hat{P}^{(1)}(x', y') = \frac{1}{|D|}\left\{\psi^{(1)}(x')^T\psi_0 + {\psi_0}^T\psi^{(1)*}(y')\right\},$$
$$\begin{aligned}\hat{P}^{(2)}(\xi, x', y') = \frac{1}{|D|}\big\{&{\psi_0}^T\psi^{(2)*}(y', \xi) + \psi^{(2)}(x', \xi)^T\psi_0 \\ &+ \psi^{(2)}(x', \xi)^T\psi^{(2)*}(y', \xi) + \xi\psi^{(1)}(x')^T\psi^{(2)*}(y', \xi) \\ &+ \xi\psi^{(2)}(x', \xi)^T\psi^{(1)*}(y') + \xi^2\psi^{(1)}(x')^T\psi^{(1)*}(y')\big\}.\end{aligned}$$

It follows from the estimates for $\psi(\xi)$ and $\psi^*(\xi)$ obtained above that

$$\left|\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\hat{P}^{(1)}(x', y')\right|_{L^\infty(D\times D)} \leqslant C$$

and

$$\left|\partial_\xi^\alpha\partial_{x'}^{\alpha'}\partial_{y'}^{\beta'}\hat{P}^{(2)}(\xi, x', y')\right|_{L^\infty(D\times D)} \leqslant C|\xi|^{2-\alpha}.$$

For the details, see the proof of [6, Theorem 3.3].

Assertion (ii) easily follows from the relation

$$\big((\lambda + \hat{L}_\xi)^{-1}u, u^*\big) = \big(u, \big(\bar{\lambda} + \hat{L}_\xi^*\big)^{-1}u^*\big)$$

for $u, u^* \in W^{1,p}(D) \times L^p(D)$.

As for (iii), since $\tilde{Q}(\bar{\lambda} + \hat{L}_\xi^*)^{-1}u^*|_{\partial D} = 0$, by integration by parts, we have

$$\big((\lambda + \hat{L}_\xi)^{-1}[\partial_{x'}\tilde{Q}u], u^*\big) = -\big(u, \partial_{x'}\tilde{Q}(\lambda + \hat{L}_\xi)^{-1}u^*\big),$$

which yields the desired results. This completes the proof. □

6. Proof of Theorem 2.1

In this section we give an outline of the proof of Theorem 2.1.

The following proposition implies that $-L$ generates an analytic semigroup.

Proposition 6.1. *There are $\Lambda > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\Lambda, \theta) \subset \rho(-L)$ and there hold the following estimates uniformly for $\lambda \in \Sigma(\Lambda, \theta)$:*

$$\text{(i)}\ \left\|(\lambda + L)^{-1}f\right\|_{W^{1,p}\times L^p} \leqslant \frac{C}{|\lambda|}\|f\|_{W^{1,p}\times L^p},$$
$$\text{(ii)}\ \left\|\partial_x^\ell\tilde{Q}(\lambda + L)^{-1}f\right\|_p \leqslant \frac{C}{|\lambda|^{1-\frac{\ell}{2}}}\|f\|_{W^{1,p}\times L^p} \quad (\ell = 1, 2).$$

Proof. We here give an outline of the proof. Let $\lambda \neq 0$. By (2.1),

$$\phi = \frac{1}{\lambda}\big(f^0 - \gamma\,\mathrm{div}\,v\big). \tag{6.1}$$

Substituting this into the second equation of (2.1), we have

$$\lambda v - \nu\Delta v - \tilde{\nu}\nabla\,\mathrm{div}\,v = F, \quad v|_{\partial\Omega} = 0. \tag{6.2}$$

Here

$$F \equiv g - \frac{\gamma}{\lambda}\nabla\big(f^0 - \gamma \operatorname{div} v\big).$$

Since $Bv = -\nu\Delta v - \tilde{\nu}\nabla \operatorname{div} v$ is strongly elliptic, it holds that there are $\Lambda' > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ such that for $\lambda \in \Sigma(\Lambda', \theta)$

$$\begin{aligned}|\lambda|\|v\|_p + |\lambda|^{\frac{1}{2}}\|\partial_x v\|_p + \big\|\partial_x^2 v\big\|_p &\leqslant C\|F\|_p \\ &\leqslant C\left\{\|f\|_{W^{1,p}\times L^p} + \frac{1}{|\lambda|}\big\|\partial_x^2 v\big\|_p\right\}.\end{aligned}$$

We take $\Lambda > 0$ large enough so that $\frac{Cp}{|\lambda|} \leqslant \frac{1}{2}$ for $\lambda \in \Sigma(\Lambda, \theta)$. Then

$$|\lambda|\|v\|_p + |\lambda|^{\frac{1}{2}}\|\partial_x v\|_p + \big\|\partial_x^2 v\big\|_p \leqslant 2C\|f\|_{W^{1,p}\times L^p}.$$

This, together with (6.1), yields

$$\|\phi\|_{W^{1,p}} \leqslant \frac{C}{|\lambda|}\big(\big\|f^0\big\|_{W^{1,p}} + \|\operatorname{div} v\|_{W^{1,p}}\big) \leqslant \frac{C}{|\lambda|}\|f\|_{W^{1,p}\times L^p}.$$

This completes the proof. □

By Proposition 6.1, $-L$ generates an analytic semigroup e^{-tL} on $W^{1,p}(\Omega) \times L^p(\Omega)$; and e^{-tL} is represented as

$$e^{-tL} = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda + L)^{-1}\, d\lambda.$$

Here $\Gamma = \{\lambda = \Lambda + se^{\pm\theta};\ s \geqslant 0\}$.

Using the estimates (i) and (ii) in Proposition 6.1, one can show Theorem 2.1(i) by a standard argument.

We now give a proof of asymptotic behavior of e^{-tL} given in Theorem 2.1(ii).

Proof of Theorem 2.1(ii). The proof is done by a similar argument to that in [6, Section 4]. We here give an outline of the proof.

We decompose e^{-tL} as

$$e^{-tL} = \mathcal{V}_0(t) + \mathcal{V}_\infty(t).$$

Here

$$\mathcal{V}_0(t) = \mathcal{F}^{-1}\big[\kappa_0(\xi)e^{-t\hat{L}_\xi}\big], \qquad \mathcal{V}_\infty(t) = \mathcal{F}^{-1}\big[\big(1-\kappa_0(\xi)\big)e^{-t\hat{L}_\xi}\big],$$

where κ_0 is a function satisfying

$$\kappa_0(\xi) \in C_0^\infty(\mathbf{R}), \quad \kappa_0(\xi) = \begin{cases} 1 & (|\xi| \leqslant \frac{r}{2}), \\ 0 & (|\xi| \geqslant r) \end{cases}$$

and

$$e^{-t\hat{L}_\xi} = \frac{1}{2\pi i}\int_{\Gamma} e^{\lambda t}(\lambda + \hat{L}_\xi)^{-1}\,d\lambda$$

with $\Gamma = \{\lambda = \Lambda + se^{\pm\theta};\ s \geqslant 0\}$. We here take $r > 0$ in such a way that $0 < r \leqslant \min\{r_1, r_2\}$ with r_1 and r_2 given in Theorems 5.3 and 5.6 respectively.

To prove Theorem 2.1(ii), we will deform the contour Γ in a suitable way.

We first consider $\mathcal{V}_0(t)$. By Theorem 5.3, we can deform Γ into $\Gamma_0 \cup \tilde{\Gamma}_0$ and a suitable circle around 0, where

$$\Gamma_0 = \{\lambda = -c_0 + is;\ |s| \leqslant s_0\}, \qquad \tilde{\Gamma}_0 = \{\lambda = \Lambda_0 + se^{\pm i\theta_0}\colon s \geqslant \tilde{s}_0\}.$$

Here Λ_0 and θ_0 are the numbers given in Theorem 5.3; and we choose s_0 and $\tilde{s}_0$ in such a way that Γ_0 connects with $\tilde{\Gamma}_0$ at the end points of Γ_0. It then follows from Theorems 5.3, 5.6 and the residue theorem that $\mathcal{V}_0(t)$ is written as

$$\mathcal{V}_0(t)u_0 = W^{(0)}(t)u_0 + W^{(1)}(t)u_0,$$

where

$$\begin{aligned}
&W^{(j)}(t)u_0 = \mathcal{F}^{-1}\big[\hat{W}^{(j)}(t)\hat{u}_0\big] \quad (j = 0, 1),\\
&\hat{W}^{(0)}(t)\hat{u}_0 = \kappa_0(\xi)e^{\lambda_0(\xi)t}\hat{P}(\xi)\hat{u}_0,\\
&\hat{W}^{(1)}(t)\hat{u}_0 = \frac{1}{2\pi i}\int_{\Gamma_0\cup\tilde{\Gamma}_0} e^{\lambda t}\kappa_0(\xi)(\lambda + \hat{L}_\xi)^{-1}\hat{u}_0\,d\lambda.
\end{aligned}$$

By using Theorems 5.3 and 5.6, one can show that $W^{(0)}(t)u_0$ is written in the form

$$W^{(0)}(t)u_0 = \mathcal{W}^{(0)}(t)u_0 + \mathcal{R}^{(0)}(t)u_0,$$

where $\mathcal{W}^{(0)}(t)u_0$ and $\mathcal{R}^{(0)}(t)u_0$ have the properties in Theorem 2.1(ii-a). We here omit the details since it can be shown in the same way as in [6, Section 4]. Also, by using Theorem 5.3, one can show that $W^{(1)}(t)$ satisfies the estimate

$$\big\|W^{(1)}(t)u_0\big\|_{W^{1,p}\times W^{2,p}} \leqslant Ce^{-c_0 t}\|u_0\|_{W^{1,p}\times L^p}.$$

As for $\mathcal{V}_\infty(t)$, by Theorems 3.1 and 4.10, one can deform the contour Γ into $\Gamma = \Gamma_\infty \cup \tilde{\Gamma}_\infty$, where

$$\Gamma_\infty = \{\lambda;\ \lambda = -c_\infty + is\ \big(|s| \leqslant s_\infty\big)\}, \qquad \tilde{\Gamma}_\infty = \{\lambda;\ \lambda = \Lambda_0 + se^{\pm i\theta_0},\ s \geqslant \tilde{s}_\infty\}$$

for some $c_\infty > 0$. We here take s_∞ and $\tilde{s}_\infty$ so that Γ_∞ connects with $\tilde{\Gamma}_\infty$ at the end points of Γ_∞. It then follows from Theorems 3.1 and 4.10 that

$$\big\|\mathcal{V}_\infty(t)u_0\big\|_{W^{1,p}\times W^{2,p}} \leqslant Ce^{-c_\infty t}\|u_0\|_{W^{1,p}\times L^p}.$$

Setting $\mathcal{U}_\infty(t) = W^{(1)}(t) + \mathcal{V}_\infty(t)$, we see that $\mathcal{U}_\infty(t)$ satisfies the estimate in Theorem 2.1(ii-b). This completes the proof. □

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