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Global classical large solutions to 1D compressible Navier–Stokes equations with density-dependent viscosity and vacuum

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ABSTRACT

In this paper, we investigate an initial boundary value problem for 1D compressible isentropic Navier–Stokes equations with large initial data, density-dependent viscosity, external force, and vacuum. Making full use of the local estimates of the solutions in Cho and Kim (2006) [3] and the one-dimensional properties of the equations and the Sobolev inequalities, we get a unique global classical solution (ρ, u) where $\rho \in C^1([0,T]; H^1([0,1]))$ and $u \in H^1([0,T]; H^2([0,1]))$ for any $T > 0$. As it is pointed out in Xin (1998) [31] that the smooth solution $(\rho, u) \in C^1([0,T]; H^3(R^1))$ (T is large enough) of the Cauchy problem must blow up in finite time when the initial density is of nontrivial compact support. It seems that the regularities of the solutions we obtained can be improved, which motivates us to obtain some new estimates with the help of a new test function $\rho^2 u_{tt}$, such as Lemmas 3.2–3.6. This leads to further regularities of (ρ, u) where $\rho \in C^1([0,T]; H^3([0,1]))$, $u \in H^1([0,T]; H^3([0,1]))$. It is still open whether the regularity of u could be improved to $C^1([0,T]; H^3([0,1]))$ with the appearance of vacuum, since it is not obvious that the solutions in $C^1([0,T]; H^3([0,1]))$ to the initial boundary value problem must blow up in finite time.

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Contents

1. Introduction

In this paper, we consider the initial boundary value problem of compressible isentropic Navier–Stokes equations in one dimension:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \quad \rho \geqslant 0, \\ (\rho u)_t + \left(\rho u^2\right)_x + \left[P(\rho)\right]_x = \left[\mu(\rho) u_x\right]_x + \rho f, \end{cases} \tag{1.1}$$

for $(x,t) \in (0,1) \times (0,+\infty)$, with the initial condition:

$$(\rho, u)|_{t=0} = \left(\rho_0(x), u_0(x)\right) \quad \text{for } x \in [0,1], \tag{1.2}$$

and the boundary condition:

$$u|_{x=0,1} = 0, \quad t \geqslant 0, \tag{1.3}$$

where ρ and u, denoting density and velocity of the fluids respectively, are unknown functions; $P(\rho) = K\rho^\gamma$, for some constants $\gamma > 1$ and $K > 0$, is the pressure function; f is a given external force.

Our main concern here is to show the existence, uniqueness and further regularity of global classical large solutions to (1.1)–(1.3). It worths mentioning that the initial density may vanish (i.e., the initial vacuum may exist), and the viscosity μ depends on the density ρ. The restrictions on μ in the following will imply that the case for constant viscosity is included.

We begin with a rough review in this direction. When the viscosity μ is constant, the local classical solutions to the Navier–Stokes equations with heat-conducting fluid in Hölder spaces was obtained respectively by Itaya [11] for the Cauchy problem and by Tani [28] for IBVP with $\inf \rho_0 > 0$, where the spatial dimension $N = 3$. Using delicate energy methods in Sobolev spaces, Matsumura and Nishida showed in [21,22] that the global classical solutions exist provided that the initial data is small in some sense and away from vacuum and the spatial dimension $N = 3$. For large initial data, Kazhikhov, Shelukhi in [17] (for polytropic perfect gas with constant viscosity) and Kawohl in [15] (for real gas with $\mu = \mu(\rho)$) got global classical solutions in dimension $N = 1$ with $\inf \rho_0 > 0$, respectively. The viscosity μ in [15] satisfies

$$0 < \mu_0 \leqslant \mu(\rho) \leqslant \mu_1, \quad \text{for } \rho \geqslant 0, \tag{1.4}$$

where μ_0 and μ_1 are constants. In fact, such the condition includes the case $\mu(\rho) \equiv \text{const}$. There are also some results about the existence of global strong (classical) solutions to the Navier–Stokes equations for isentropic fluid with $\inf \rho_0 > 0$, refer for instance to [1,16,29].

In the presence of vacuum, the existence of global weak solutions with large initial data in $\mathbb{R}^N$ was first obtained by Lions in [18], where $\gamma \geqslant \frac{3N}{N+2}$ for $N = 2$ or 3. Feireisl et al. in [9] extended the work in [18] to the case $\gamma > \frac{3}{2}$ for $N = 3$. For solutions with spherical symmetry, Jiang and Zhang in [14] relaxed the restriction on γ in [18] to the case $\gamma > 1$, and got the global existence of the weak solutions for $N = 2$ or 3. On the existence and regularity of weak solutions with density connecting to vacuum continuously in 1D, please refer to [20]. During the pass two decades, Salvi, Choe, Kim and Jiang et al made great progress towards the local or global existence of strong (classical) solutions with vacuum, see [27,2,4,5,7,3]. Particularly, Choe and Kim in [5] showed that the radially symmetric strong solutions exist globally in time for $\gamma \geqslant 2$ and $N \geqslant 2$. This result had been generalized to the case $\gamma \geqslant 1$ by Fan, Jiang and Ni in [7]. Note that [27,2,4,5,7] all considered the existence and uniqueness of strong solutions. On the classical solutions, Cho and Kim in [3] in 2006 got the local existence and uniqueness result for $N = 3$ by using successive approximations, based on some *a priori* estimates for the solutions to the corresponding linearized problems. The existence of global classical solutions with vacuum is still open.

It is well known that the physically important case related to vacuum is the case when μ is not constant. It can be seen from the derivation of the Navier–Stokes equations from the Boltzmann equation through Chapman–Enskog expansion to the second order, cf. [10], where the viscosity coefficient depends on the temperature. For isentropic flow, this dependence is translated into the dependence on the density by the laws of Boyle and Gay–Lussac for ideal gas as discussed in [19]. For the case $\mu(\rho) = \alpha\rho^\theta$ with α and θ being positive constants, when the initial density was assumed to be connected to vacuum discontinuously, the global existence of weak solutions for isentropic flow in one dimension was obtained by Okada, Matušů-Nečasová and Makino in [24] for $0 < \theta < \frac{1}{3}$ and by Yang, Yao and Zhu in [32] for $0 < \theta < \frac{1}{2}$ and by Jiang, Xin and Zhang in [13] for $0 < \theta < 1$. Qin and Yao in [26] relaxed the restriction on θ in [13] to the case $0 < \theta \leqslant 1$ with more regular initial data. On the global existence of classical solutions to the Navier–Stokes equations with heat-conducting fluid, $\inf \rho_0 > 0$, and viscosity $\mu(\rho) = \alpha\rho^\theta$, please refer to [12,25] where Jiang in [12] and Qin, Yao in [25] considered the case $\theta \in (0, \frac{1}{4})$ and $\theta \in (0, \frac{1}{2})$ respectively. For the Cauchy problem of (1.1)–(1.3), Mellet and Vasseur in [23] showed the existence of global strong solutions when μ satisfies

$$\begin{cases} \mu(\rho) \geqslant \alpha\rho^\theta, & \text{for any } \rho \leqslant 1 \text{ and } \theta \in [0, \frac{1}{2}), \\ \mu(\rho) \geqslant \alpha, & \text{for } \rho \geqslant 1, \\ \mu(\rho) \leqslant c + c\rho^\gamma, & \text{for any } \rho \geqslant 0, \end{cases} \tag{1.5}$$

where α and c are some positive constants. Moreover, if $\mu(\rho) \geqslant \alpha > 0$ for all $\rho \geqslant 0$, $\mu(\rho)$ is uniformly Lipschitz and $\gamma \geqslant 2$, then this global strong solution is unique in the large class of weak solutions satisfying the usual entropy inequality.

When the density function connects to vacuum continuously, please refer to [8,30,33] for the isentropic case, where the global existence of weak solutions was obtained for $\mu(\rho) = \alpha\rho^\theta$.

To sum up, the existence of global classical solutions of the Navier–Stokes equations with vacuum and large initial data is still open whether the viscosity μ is constant or depends on the density. This is our motivation in this paper.

There are two main theorems in the paper. In Theorem 1.1, we get the existence and uniqueness of global classical solutions with relatively weak initial data. In Theorem 1.2, we get further regularities of the solutions with more regular initial data. The difficulties in Theorem 1.1 compared with [3] are that we need some *a priori* estimates globally in time, and that we have to handle the density-dependent viscosity. To handle these, we make full use of one-dimensional properties of the equations and the Sobolev inequalities. Since the initial density may vanish, similarly to [3], we need the compatibility condition (1.6) to handle $\int_I \rho u_t^2|_{t=0}$ and $\int_I u_{xt}^2|_{t=0}$ in Lemma 2.6 and Lemma 2.10 respectively. The main difficulties in Theorem 1.2 lie in the higher order estimates of u. We get them by some new useful estimates with the help of a new test function $\rho^2 u_{tt}$, such as Lemmas 3.2–3.5. Among them, Lemma 3.3 is the most important one, where we take $\rho^2 u_{tt}$ instead of u_{tt} as the test function in order to handle the vanishing effect of the density.

We have to emphasize that the smooth solution $(\rho, u) \in C^1([0, T]; H^3(R^1))$($T$ is large enough) to the Cauchy problem of compressible isentropic Navier–Stokes equations in 1D must blow up in finite time when the initial density is of nontrivial compact support (please refer to [31]). In Theorem 1.2, we get $\rho \in C^1([0, T]; H^3([0, 1]))$, $u \in H^1([0, T]; H^3([0, 1]))$. It is open whether regularity of u could be improved to $C^1([0,T];H^3([0,1]))$ as vacuum appears, since it is not obvious that the solutions in $C^1([0,T];H^3([0,1]))$ to the initial boundary value problem must blow up in finite time.

As in [3], we assume that the initial data satisfies the compatibility condition:

$$\left[\mu(\rho_0)u_{0x}\right]_x(x)-\left[p(\rho_0)\right]_x(x)=\rho_0(x)\left[-f(x,0)+g(x)\right],\quad x\in[0,1], \tag{1.6}$$

for a given function $g\in H_0^1$, and the viscosity μ satisfies

$$\mu\in C^2[0,\infty),\quad 0<M_1\leqslant\mu(\rho)\leqslant M_2\left(1+\rho^\gamma\right),\quad \text{for any } \rho\geqslant 0. \tag{1.7}$$

We give a remark on (1.6) and (1.7).

Remark 1.1. (i) The function g in (1.6) plays a crucial role in handling $\int_0^1\rho u_t^2(0)$, $\int_0^1 u_{xt}^2(0)$ and $\int_0^1\rho^3u_{tt}^2(0)$ when we use the Gronwall inequality in Lemma 2.6, Lemma 2.10 and Lemma 3.3, respectively.

(ii) The restrictions on μ in (1.7) cover the condition (1.4) in [15]. The upper bound of μ in (1.7) is the same as $(1.5)_3$ in [23]. Unfortunately, the case $\mu(0)=0$ is excluded by (1.7). The lower bound in (1.7) is imposed here because of the requirement of analysis. For the case $\mu(0)=0$, we will consider in the next future.

Notations. (1) $I=[0,1]$, $\partial I=\{0,1\}$, $Q_T=I\times[0,T]$ for $T>0$.

(2) For $p\geqslant 1$, $L^p=L^p(I)$ denotes the L^p space with the norm $\|\cdot\|_{L^p}$. For $k\geqslant 1$ and $p\geqslant 1$, $W^{k,p}=W^{k,p}(I)$ denotes the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, $H^k=W^{k,2}(I)$.

(3) For an integer $k\geqslant 0$ and $0<\alpha<1$, let $C^{k+\alpha}(I)$ denote the Schauder space of functions on I, whose kth order derivative is Hölder continuous with exponents α, with the norm $\|\cdot\|_{C^{k+\alpha}}$.

Our main results are stated as follows.

Theorem 1.1. *Assume that $\rho_0\geqslant 0$, $\rho_0\in H^2$, $\rho_0^\gamma\in H^2$, $u_0\in H^3\cap H_0^1$, $f\in C([0,\infty);H^1)$, $f_t\in L^2_{loc}([0,\infty);L^2)$, and that the initial data and μ satisfy* (1.6) *and* (1.7) *respectively. Then for any $T>0$ there exists a unique global classical solution (ρ,u) to* (1.1)–(1.3) *satisfying*

$$\begin{gathered}\left(\rho,\rho^\gamma\right)\in C\left([0,T];H^2\right),\quad \left(\rho_t,\left(\rho^\gamma\right)_t\right)\in C\left([0,T];H^1\right),\quad \rho_{tt}\in C\left([0,T];L^2\right),\\ \rho\geqslant 0,\quad \left(\rho^\gamma\right)_{tt}\in L^\infty\left([0,T];L^2\right),\quad (\rho u)_t\in C\left([0,T];H^1\right),\\ u\in C\left([0,T];H^3\cap H_0^1\right),\quad u_t\in L^\infty\left([0,T];H_0^1\right)\cap L^2\left([0,T];H^2\right).\end{gathered}$$

Remark 1.2. (i) The classical solution here means that it satisfies the equations and the corresponding initial and boundary conditions everywhere.

(ii) If $\gamma\geqslant 2$, the assumption $\rho_0^\gamma\in H^2$ is not necessary, since $\rho_0\in H^2$ implies $\rho_0^\gamma\in H^2$. For $\gamma\in(1,2)$, such the conditions of ρ_0, ρ_0^γ and u_0 in Theorem 1.1 are not empty. For example, $\rho_0=x^\beta$ ($\beta>\frac{3}{2}$), and u_0 is the solution to the elliptic equation (1.6) with zero boundary condition. By the standard elliptic theory, we get $u_0\in H^3\cap H_0^1$.

(iii) By the Sobolev embedding theorems, we have

$$H^k(I)\hookrightarrow C^{k-\frac{1}{2}}(I),\quad \text{for } k=1,2,3.$$

This together with regularities of (ρ,u) gives

$$\left(\rho,\rho^\gamma\right)\in C\left([0,T];C^{1+\frac{1}{2}}(I)\right),\quad \left(\rho_t,(\rho u)_t\right)\in C\left([0,T];C^{\frac{1}{2}}(I)\right),\quad u\in C\left([0,T];C^{2+\frac{1}{2}}(I)\right),$$

which means (ρ, u) is the classical solution to (1.1)–(1.3). Note that we get the continuity of $(\rho u)_t$ with respect to space–time variables instead of getting that of u_t. Anyway, ρ and u satisfy (1.1)–(1.3) everywhere.

The next result shows further regularities of the solutions.

Theorem 1.2. *Consider the same assumptions as in Theorem* 1.1, *and in addition assume that* $\rho_0 \in H^4$, $\rho_0^\gamma \in H^4$, $(\sqrt{\rho_0})_x \in L^\infty$, $(\sqrt{\rho_0} g_x)_x \in L^2$, $u_0 \in H^4$ *and* $(f_{xt}, f_{tt}) \in L^2_{loc}([0,\infty); L^2)$, $f \in L^2_{loc}([0,\infty); H^3)$, $\mu \in C^4[0,\infty)$. *Then the regularities of the solutions obtained in Theorem* 1.1 *can be improved as follows*:

$$\begin{gathered}
(\rho, \rho^\gamma) \in C\big([0,T]; H^4\big), \quad \big((\sqrt{\rho})_x, (\sqrt{\rho})_t\big) \in L^\infty(Q_T), \quad \big(\rho_t, (\rho^\gamma)_t\big) \in C\big([0,T]; H^3\big), \\
\big(\rho_{tt}, (\rho^\gamma)_{tt}\big) \in C\big([0,T]; H^1\big) \cap L^2\big([0,T]; H^2\big), \quad \big(\rho_{ttt}, (\rho^\gamma)_{ttt}\big) \in L^2(Q_T), \\
u \in C\big([0,T]; H^4\big) \cap L^2\big([0,T]; H^5\big), \quad u_t \in L^\infty\big([0,T]; H_0^1\big) \cap L^2\big([0,T]; H^3\big), \\
\sqrt{\rho} u_{xxt} \in L^\infty\big([0,T]; L^2\big).
\end{gathered}$$

Remark 1.3. (i) The conditions of ρ_0, ρ_0^γ and u_0 in Theorem 1.2 are not empty. For example, $\rho_0 = x^\beta (\beta > \frac{7}{2})$, and u_0 is the solution to the elliptic equation (1.6) with zero boundary condition. By the standard elliptic theory, we get $u_0 \in H^4 \cap H_0^1$.

(ii) From Theorem 1.2, we get

$$(\rho, u) \in C\big([0,T]; H^4(I)\big), \quad \rho \in C^1\big([0,T]; H^3(I)\big), \; u \in H^1\big([0,T]; H^3(I)\big).$$

Whether could regularity of (ρ, u) here be improved further? This will be considered in our forthcoming paper.

The constant K in the pressure function $P(\rho)$ doesn't play any role in the analysis, we assume henceforth that $K = 1$.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1 by giving the initial density a lower bound $\delta > 0$, getting a sequence of approximate solutions to (1.1)–(1.3) and taking $\delta \to 0$ after making some estimates uniformly for δ. The regularities guarantee the uniqueness of the solutions. In Section 3, based on some new useful estimates such as Lemmas 3.2–3.6, we prove Theorem 1.2 by the similar arguments as in Section 2.

2. Proof of Theorem 1.1

In this section, we get a unique classical solution of (1.1)–(1.3) with initial density bounded below away from zero by using some *a priori* estimates of the solutions based on the local existence.

In this section, we denote "$c(T)$" to be a generic constant depending on $\|\rho_0\|_{H^2}$, $\|\rho_0^\gamma\|_{H^2}$, $\|u_0\|_{H^3}$, T and some other known constants but independent of δ, for any $\delta \in (0,1)$.

Before proving Theorem 1.1, we need the following auxiliary theorem.

Theorem 2.1. *Consider the same assumptions as in Theorem* 1.1, *and in addition assume that* $\rho_0 \geqslant \delta > 0$. *Then for any* $T > 0$ *there exists a unique global classical solution* (ρ, u) *to* (1.1)–(1.3) *satisfying*

$$\begin{gathered}
\rho \in C\big([0,T]; H^2\big), \quad \rho_t \in C\big([0,T]; H^1\big), \quad \rho_{tt} \in C\big([0,T]; L^2\big), \quad \rho \geqslant \frac{\delta}{c(T)}, \\
u_{tt} \in L^2\big([0,T]; L^2\big), \quad u \in C\big([0,T]; H^3 \cap H_0^1\big), \quad u_t \in C\big([0,T]; H_0^1\big) \cap L^2\big([0,T]; H^2\big).
\end{gathered}$$

Proof. The local solutions in Theorem 2.1 can be obtained by the successive approximations like in [3]. We omit it here for simplicity. The regularities guarantee the uniqueness (refer for instance to [4]). Based on the local existence of the solutions, Theorem 2.1 can be proved by some *a priori* estimates globally in time.

For any given $T \in (0, \infty)$, let (ρ, u) be the classical solutions to (1.1)–(1.3) as in Theorem 2.1. Then we have the following basic energy estimate.

Lemma 2.1. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I \left(\rho u^2 + \rho^\gamma\right)(t) + \int_0^t \int_I u_x^2 \leqslant c(T). \tag{2.1}$$

Proof. Multiplying $(1.1)_2$ by u, integrating the resulting equation over I, and using integration by parts, $(1.1)_1$, (1.7), the Cauchy inequality and the Gronwall inequality, we get (2.1).

This completes the proof of Lemma 2.1. $\square$

Lemma 2.2. *For any* $(y, s) \in Q_T$, *it holds*

$$\frac{\delta}{c(T)} \leqslant \rho(y, s) \leqslant c(T). \tag{2.2}$$

Proof. *Step* 1: we claim $\rho(y, s) \leqslant c(T)$, for $(y, s) \in Q_T$.

Denote

$$w(x, t) = \int_0^t \left[\mu(\rho) u_x - \rho u^2 - \rho^\gamma\right] + \int_0^x \rho_0 u_0 + \int_0^t \int_0^x \rho f. \tag{2.3}$$

Differentiating (2.3) with respect to x, and using $(1.1)_2$, we have

$$w_x = \rho u.$$

This together with Lemma 2.1 and the Cauchy inequality gives

$$\int_I |w_x| \leqslant c(T).$$

It follows from (2.3), (1.7), and Lemma 2.1 that

$$\int_I |w| \leqslant c(T).$$

Since $W^{1,1} \hookrightarrow L^\infty$, we get

$$\|w\|_{L^\infty(Q_T)} \leqslant c(T). \tag{2.4}$$

For any $(y, s) \in Q_T$, let $x(t, y)$ satisfy

$$\begin{cases} \dfrac{dx(t,y)}{dt} = u\big(x(t,y),t\big), \quad 0 \leqslant t < s, \\ x(s,y) = y. \end{cases} \tag{2.5}$$

Denote

$$F(x,t) = \exp\left\{ \int_1^{\rho(x,t)} \frac{\mu(\xi)}{\xi}\, d\xi + w(x,t) \right\}.$$

It is easy to verify

$$\begin{aligned} \frac{dF(x(t,y),t)}{dt} &= \partial_t F + u \partial_x F \\ &= F\left(\frac{\mu(\rho)}{\rho}\rho_t + w_t + \frac{\mu(\rho)}{\rho}\rho_x u + \rho u^2 \right) \\ &= \left(-\rho^\gamma + \int_0^{x(t,y)} \rho f \right) F \\ &\leqslant F \int_0^{x(t,y)} \rho f, \end{aligned}$$

which implies

$$\frac{d}{dt}\left\{ F \exp\left(-\int_0^t \int_0^{x(\tau,y)} \rho f(\xi,\tau)\, d\xi\, d\tau \right) \right\} \leqslant 0. \tag{2.6}$$

Integrating (2.6) over $(0,s)$, we have

$$F(y,s)\exp\left(-\int_0^s \int_0^{x(\tau,y)} \rho f(\xi,\tau)\, d\xi\, d\tau \right) \leqslant F\big(x(0,y),0\big),$$

which implies

$$F(y,s) \leqslant \exp\left(\int_0^s \int_0^{x(\tau,y)} \rho f(\xi,\tau)\, d\xi\, d\tau \right) F\big(x(0,y),0\big) \leqslant c(T),$$

where we have used (1.7). This gives

$$\exp\left(\int_1^{\rho(y,s)} \frac{\mu(\xi)}{\xi}\, d\xi \right) \leqslant c(T)\exp\big\{-w(y,s)\big\} \leqslant c(T),$$

where we have used (2.4). This together with (1.7) completes Step 1.

Step 2: we claim $\rho(y,s) \geqslant \frac{\delta}{c(T)}$, for $(y,s) \in Q_T$.

Since $\frac{\partial w}{\partial t} = \mu(\rho)u_x - \rho u^2 - \rho^\gamma + \int_0^x \rho f$, $\frac{\partial w}{\partial x} = \rho u$, we have

$$\begin{aligned} \frac{dw(x(t,y),t)}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} \\ &= \mu(\rho)u_x\big(x(t,y),t\big) - \rho^\gamma\big(x(t,y),t\big) + \int_0^{x(t,y)} \rho f. \end{aligned} \tag{2.7}$$

By $(1.1)_1$, we deduce

$$\begin{aligned} \mu(\rho)u_x\big(x(t,y),t\big) &= \mu(\rho)\left(-\frac{\rho_x u}{\rho} - \frac{\rho_t}{\rho}\right) \\ &= -\frac{d}{dt}\left(\int_0^{\ln\rho(x(t,y),t)} \mu\big(\exp\{\xi\}\big)\,d\xi\right). \end{aligned}$$

This together with (2.7) gives

$$\begin{aligned} &\frac{d}{dt}\left(w\big(x(t,y),t\big) + \int_0^{\ln\rho(x(t,y),t)} \mu\big(\exp\{\xi\}\big)\,d\xi\right) \\ &\quad = -\rho^\gamma\big(x(t,y),t\big) + \int_0^{x(t,y)} \rho f. \end{aligned} \tag{2.8}$$

Integrating (2.8) over $(0,s)$ with respect to t, we get

$$\begin{aligned} &\int_0^{\ln\rho(y,s)} \mu\big(\exp\{\xi\}\big)\,d\xi \\ &= w\big(x(0,y),0\big) - w(y,s) + \int_0^{\ln\rho_0(x(0,y))} \mu\big(\exp\{\xi\}\big)\,d\xi - \int_0^s \rho^\gamma + \int_0^s\int_0^{x(t,y)} \rho f \\ &\geqslant -2\|w\|_{L^\infty(Q_T)} + \int_0^{\ln\delta} \mu\big(\exp\{\xi\}\big)\,d\xi - c(T) \\ &\geqslant M_1 \ln\delta - c(T), \end{aligned} \tag{2.9}$$

where we have used Step 1, (1.7) and (2.4).

If $\rho(y,s) < 1$, we have

$$\int_0^{\ln\rho(y,s)} \mu\big(\exp\{\xi\}\big)\,d\xi \leqslant M_1 \ln\rho(y,s). \tag{2.10}$$

(2.9) and (2.10) give

$$\rho(y,s) \geqslant \frac{\delta}{c(T)}. \tag{2.11}$$

If $\rho(y,s) \geqslant 1$, (2.11) is obviously true. Step 2 is obtained.

By Steps 1 and 2, the proof of Lemma 2.2 is complete. $\square$

Lemma 2.3. *For any $0 \leqslant t \leqslant T$, it holds*

$$\int_I u_x^2 + \int_{Q_T} \rho u_t^2 \leqslant c(T).$$

Proof. Using $(1.1)_1$, we rewrite $(1.1)_2$ as

$$\rho u_t + \rho u u_x + \left(\rho^\gamma\right)_x = \left[\mu(\rho)u_x\right]_x + \rho f. \tag{2.12}$$

Multiplying (2.12) by u_t, integrating it over I, and using integration by parts and the Cauchy inequality, we have

$$\begin{aligned}
\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2 &= \frac{d}{dt}\int_I \rho^\gamma u_x + \frac{1}{2}\int_I \mu'(\rho)\rho_t u_x^2 - \int_I \rho u u_x u_t \\
&\quad - \gamma\int_I \rho^{\gamma-1}\rho_t u_x + \int_I \rho f u_t \\
&\leqslant \frac{d}{dt}\int_I \rho^\gamma u_x - \frac{1}{2}\int_I \mu'(\rho)(\rho u)_x u_x^2 + \frac{1}{8}\int_I \rho u_t^2 + c(T)\int_I \rho u^2 u_x^2 \\
&\quad + \gamma\int_I \rho^{\gamma-1}(\rho u)_x u_x + \frac{1}{8}\int_I \rho u_t^2 + c(T)\int_I \rho f^2.
\end{aligned}$$

This deduces

$$\begin{aligned}
\frac{3}{4}\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2 &\leqslant \frac{d}{dt}\int_I \rho^\gamma u_x - \frac{1}{2}\int_I \mu'(\rho)(\rho u)_x u_x^2 + c(T)\int_I \rho u^2 u_x^2 \\
&\quad + \gamma\int_I \rho^{\gamma-1}(\rho u)_x u_x + c(T)\int_I \rho f^2.
\end{aligned}$$

By (1.7), Lemma 2.2, the Sobolev inequality and integration by parts, we have

$$\begin{aligned}
&\frac{3}{4}\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2 \\
&\quad \leqslant \frac{d}{dt}\int_I \rho^\gamma u_x - \frac{1}{2}\int_I \mu'(\rho)\rho u_x^3 - \frac{1}{2}\int_I \mu'(\rho)\rho_x u u_x^2 + c(T)\left(\int_I \mu(\rho)u_x^2\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+\gamma\int_I \rho^{\gamma-1}\rho_x u u_x+\gamma\int_I \rho^{\gamma}u_x^2+c(T)\\
&\leqslant \frac{d}{dt}\int_I \rho^{\gamma}u_x+c(T)\biggl(\int_I \mu(\rho)u_x^2\biggr)^2+c(T)\bigl\|\mu(\rho)u_x\bigr\|_{L^\infty}\int_I \mu(\rho)u_x^2\\
&-\frac{1}{2}\int_I \mu'(\rho)\rho_x u u_x^2+\gamma\int_I \rho^{\gamma-1}\rho_x u u_x+c(T).
\end{aligned}
$$

We will estimate the fourth and the fifth terms of the right side above.

$$
\begin{aligned}
&-\frac{1}{2}\int_I \mu'(\rho)\rho_x u u_x^2+\gamma\int_I \rho^{\gamma-1}\rho_x u u_x\\
&=-\frac{1}{2}\int_I \frac{\mu'(\rho)}{\mu^2(\rho)}\rho_x u\bigl[\mu(\rho)u_x\bigr]^2+\gamma\int_I \frac{\rho^{\gamma-1}\rho_x}{\mu(\rho)}u\mu(\rho)u_x\\
&=-\frac{1}{2}\int_I \frac{\mu'(\rho)}{\mu^2(\rho)}\rho_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]^2+\frac{1}{2}\int_I \frac{\mu'(\rho)}{\mu^2(\rho)}\rho_x u\bigl[-2\mu(\rho)u_x\rho^{\gamma}+\rho^{2\gamma}\bigr]\\
&\quad+\gamma\int_I \frac{\rho^{\gamma-1}\rho_x}{\mu(\rho)}u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]+\gamma\int_I \frac{\rho^{2\gamma-1}\rho_x}{\mu(\rho)}u\\
&=\frac{1}{2}\int_I \biggl(\frac{1}{\mu(\rho)}\biggr)_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]^2-\int_I \frac{\mu'(\rho)}{\mu^2(\rho)}\rho^{\gamma}\rho_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]-\frac{1}{2}\int_I \frac{\mu'(\rho)}{\mu^2(\rho)}\rho^{2\gamma}\rho_x u\\
&\quad+\gamma\int_I \biggl(\int_0^{\rho}\frac{s^{\gamma-1}}{\mu(s)}\,ds\biggr)_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]+\gamma\int_I \biggl(\int_0^{\rho}\frac{s^{2\gamma-1}}{\mu(s)}\,ds\biggr)_x u\\
&=\frac{1}{2}\int_I \biggl(\frac{1}{\mu(\rho)}\biggr)_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]^2-\int_I \biggl(\int_0^{\rho}\frac{\mu'(s)}{\mu^2(s)}s^{\gamma}\,ds\biggr)_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]\\
&\quad-\frac{1}{2}\int_I \biggl(\int_0^{\rho}\frac{\mu'(s)}{\mu^2(s)}s^{2\gamma}\,ds\biggr)_x u+\gamma\int_I \biggl(\int_0^{\rho}\frac{s^{\gamma-1}}{\mu(s)}\,ds\biggr)_x u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]+\gamma\int_I \biggl(\int_0^{\rho}\frac{s^{2\gamma-1}}{\mu(s)}\,ds\biggr)_x u.
\end{aligned}
$$

Using integration by parts, we have

$$
\begin{aligned}
&-\frac{1}{2}\int_I \mu'(\rho)\rho_x u u_x^2+\gamma\int_I \rho^{\gamma-1}\rho_x u u_x\\
&\quad=-\frac{1}{2}\int_I \frac{1}{\mu(\rho)}u_x\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]^2-\int_I \frac{1}{\mu(\rho)}u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]_x\\
&\qquad+\int_I \biggl(\int_0^{\rho}\frac{\mu'(s)}{\mu^2(s)}s^{\gamma}ds\biggr)u_x\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]+\int_I \biggl(\int_0^{\rho}\frac{\mu'(s)}{\mu^2(s)}s^{\gamma}\,ds\biggr)u\bigl[\mu(\rho)u_x-\rho^{\gamma}\bigr]_x
\end{aligned}
$$

$$+\frac{1}{2}\int_I\left(\int_0^\rho \frac{\mu'(s)}{\mu^2(s)}s^{2\gamma}\,ds\right)u_x-\gamma\int_I\left(\int_0^\rho \frac{s^{\gamma-1}}{\mu(s)}\,ds\right)u_x\big[\mu(\rho)u_x-\rho^\gamma\big]$$

$$-\gamma\int_I\left(\int_0^\rho \frac{s^{\gamma-1}}{\mu(s)}\,ds\right)u\big[\mu(\rho)u_x-\rho^\gamma\big]_x-\gamma\int_I\left(\int_0^\rho \frac{s^{2\gamma-1}}{\mu(s)}\,ds\right)u_x.$$

This together with Lemma 2.2, (2.12), (1.7), the Cauchy inequality, and the Sobolev inequality gives

$$\begin{aligned}
&-\frac{1}{2}\int_I \mu'(\rho)\rho_x u u_x^2+\gamma\int_I \rho^{\gamma-1}\rho_x u u_x\\
&\leqslant c(T)\big\|\mu(\rho)u_x\big\|_{L^\infty}\int_I \mu(\rho)u_x^2+c(T)\int_I \mu(\rho)u_x^2+c(T)\\
&\quad+\int_I\left\{-\frac{(\mu(\rho)u_x-\rho^\gamma)u}{\mu(\rho)}+\int_0^\rho \frac{\mu'(s)s^\gamma}{\mu^2(s)}\,ds\,u-\gamma\int_0^\rho \frac{s^{\gamma-1}}{\mu(s)}\,ds\,u\right\}(\rho u_t+\rho u u_x-\rho f)\\
&\leqslant c(T)\big\|\mu(\rho)u_x\big\|_{L^\infty}\int_I \mu(\rho)u_x^2+\frac{1}{8}\int_I \rho u_t^2+c(T)\left(\int_I \mu(\rho)u_x^2\right)^2+c(T).
\end{aligned}$$

From above, we get

$$\begin{aligned}
\frac{5}{8}\int_I \rho u_t^2+\frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2 &\leqslant \frac{d}{dt}\int_I \rho^\gamma u_x+c(T)\left(\int_I \mu(\rho)u_x^2\right)^2\\
&\quad+c(T)\big\|\mu(\rho)u_x\big\|_{L^\infty}\int_I \mu(\rho)u_x^2+c(T). \qquad (2.13)
\end{aligned}$$

By Lemma 2.2, (2.12), $W^{1,1}\hookrightarrow L^\infty$, and the Cauchy inequality, we get

$$\begin{aligned}
\big\|\mu(\rho)u_x\big\|_{L^\infty} &\leqslant \big\|\mu(\rho)u_x-\rho^\gamma\big\|_{L^\infty}+c(T)\\
&\leqslant c(T)\int_I\big[\big|\mu(\rho)u_x-\rho^\gamma\big|+\big|\big(\mu(\rho)u_x-\rho^\gamma\big)_x\big|\big]+c(T)\\
&\leqslant c(T)\int_I \mu(\rho)|u_x|+c(T)\int_I|\rho u_t+\rho u u_x-\rho f|+c(T)\\
&\leqslant c(T)\int_I \mu(\rho)u_x^2+c(T)\|\sqrt{\rho}u_t\|_{L^2(I)}+c(T). \qquad (2.14)
\end{aligned}$$

By (2.13)–(2.14) and the Cauchy inequality, we get

$$\frac{5}{8}\int_I \rho u_t^2+\frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2\leqslant \frac{d}{dt}\int_I \rho^\gamma u_x+c(T)\left(\int_I \mu(\rho)u_x^2\right)^2+\frac{1}{8}\int_I \rho u_t^2+c(T).$$

Thus,

$$\frac{1}{2}\int_I \rho u_t^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_x^2 \leqslant \frac{d}{dt}\int_I \rho^\gamma u_x + c(T)\left(\int_I \mu(\rho)u_x^2\right)^2 + c(T). \tag{2.15}$$

Integrating (2.15) over $(0,t)$ and using the Cauchy inequality, we have

$$\begin{aligned}\frac{1}{2}\int_0^t\int_I \rho u_t^2 + \frac{1}{2}\int_I \mu(\rho)u_x^2 &\leqslant \int_I \rho^\gamma u_x + c(T)\int_0^t\left(\int_I \mu(\rho)u_x^2\right)^2 + c(T)\\ &\leqslant \frac{1}{4}\int_I \mu(\rho)u_x^2 + c(T)\int_0^t\left(\int_I \mu(\rho)u_x^2\right)^2 + c(T),\end{aligned}$$

which implies

$$\frac{1}{2}\int_0^t\int_I \rho u_t^2 + \frac{1}{4}\int_I \mu(\rho)u_x^2 \leqslant c(T)\int_0^t\left(\int_I \mu(\rho)u_x^2\right)^2 + c(T).$$

Using the Gronwall inequality and Lemma 2.1, we complete the proof of Lemma 2.3. □

Lemma 2.4. *For any $0 \leqslant t \leqslant T$, it holds*

$$\int_I \left(\rho_x^2 + \rho_t^2\right) \leqslant c(T).$$

Proof. Differentiating $(1.1)_1$ with respect to x, multiplying the resulting equation by ρ_x, integrating over I, and using integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_x^2 = -\frac{3}{2}\int_I \rho_x^2 u_x - \int_I \frac{\rho\rho_x[\mu(\rho)u_x]_x}{\mu(\rho)} + \int_I \frac{\rho\rho_x^2\mu'(\rho)u_x}{\mu(\rho)}.$$

By (2.12), (2.14), Lemmas 2.2–2.3 and the Cauchy inequality, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_I \rho_x^2 &\leqslant c(T)\left\|\mu(\rho)u_x\right\|_{L^\infty}\int_I \rho_x^2 - \int_I \frac{\rho\rho_x}{\mu(\rho)}\left[\rho u_t + \rho u u_x + \left(\rho^\gamma\right)_x - \rho f\right]\\ &\leqslant c(T)\int_I \rho u_t^2\int_I \rho_x^2 + c(T)\int_I \rho_x^2 + c(T)\int_I \rho u_t^2 + c(T).\end{aligned}$$

By the Gronwall inequality, and Lemma 2.3, we get

$$\int_I \rho_x^2 \leqslant c(T).$$

This together with $(1.1)_1$ and Lemma 2.3 gives

$$\int_I \rho_t^2 \leqslant c(T).$$

The proof of Lemma 2.4 is complete. $\square$

Lemma 2.5. *For any* $0 \leqslant t \leqslant T$, *we have*

$$\int_{Q_T} u_{xx}^2 \leqslant c(T).$$

Proof. It follows from (2.12), (2.14), (1.7), Lemmas 2.2–2.4 that

$$\begin{aligned}\int_{Q_T} u_{xx}^2 \leqslant{}& c(T)\int_{Q_T} \rho u_t^2 + c(T)\int_0^T \left(\int_I u_x^2\right)^2 + c(T)\int_{Q_T} \rho_x^2 \\ &+ c(T)\int_0^T \|u_x\|_{L^\infty}^2 \int_I \rho_x^2 + c(T)\int_I f^2 \\ \leqslant{}& c(T).\end{aligned}$$

This proves Lemma 2.5. $\square$

Lemma 2.6. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I \rho u_t^2 + \int_{Q_T} u_{xt}^2 \leqslant c(T).$$

Proof. Differentiating (2.12) with respect to t, we have

$$\rho u_{tt} + \rho_t u_t + \rho_t u u_x + \rho u_t u_x + \rho u u_{xt} + \left(\rho^\gamma\right)_{xt} = \left[\mu(\rho)u_x\right]_{xt} + \rho_t f + \rho f_t. \tag{2.16}$$

Multiplying (2.16) by u_t, integrating over I, and using $(1.1)_1$ and integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_I \rho u_t^2 + \int_I \mu(\rho)u_{xt}^2 \\ &\quad= -2\int_I \rho u u_t u_{xt} - \int_I \rho_t u u_x u_t - \int_I \rho u_t^2 u_x + \int_I \gamma\rho^{\gamma-1}\rho_t u_{tx} \\ &\qquad - \int_I \mu'(\rho)\rho_t u_x u_{xt} + \int_I \rho_t f u_t + \int_I \rho f_t u_t \\ &\quad\leqslant 2\|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^\infty}\|u_{xt}\|_{L^2} + \|u_t\|_{L^\infty}\|u\|_{L^\infty}\|\rho_t\|_{L^2}\|u_x\|_{L^2}\end{aligned}$$

$$+\|u_x\|_{L^\infty}\int_I \rho u_t^2+\gamma\|\rho\|_{L^\infty}^{\gamma-1}\|\rho_t\|_{L^2}\|u_{xt}\|_{L^2}+c(T)\|\rho_t\|_{L^2}\|u_x\|_{L^\infty}\|u_{xt}\|_{L^2}$$
$$+\|u_t\|_{L^\infty}\|\rho_t\|_{L^2}\|f\|_{L^2}+\|u_t\|_{L^\infty}\|\rho\|_{L^2}\|f_t\|_{L^2}.$$

This, combining Lemmas 2.2–2.4 and the Cauchy inequality, gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho u_t^2+\int_I \mu(\rho)u_{xt}^2\\
&\leqslant c(T)\big(\|\sqrt{\rho}u_t\|_{L^2}+\|u_{xx}\|_{L^2}+\|f_t\|_{L^2}+1\big)\big\|\sqrt{\mu(\rho)}u_{xt}\big\|_{L^2}+c(T)\big(\|u_{xx}\|_{L^2}+1\big)\int_I \rho u_t^2\\
&\leqslant \frac{1}{2}\int_I \mu(\rho)u_{xt}^2+c(T)\big(\|u_{xx}\|_{L^2}+1\big)\int_I \rho u_t^2+c(T)\int_I u_{xx}^2+c(T)\int_I f_t^2+c(T),
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{d}{dt}\int_I \rho u_t^2+\int_I \mu(\rho)u_{xt}^2\leqslant c(T)\big(\|u_{xx}\|_{L^2}+1\big)\int_I \rho u_t^2&\\
+c(T)\int_I u_{xx}^2+c(T)\int_I f_t^2+c(T).&
\end{aligned}\tag{2.17}$$

By (1.6) and (2.12), we have

$$\begin{aligned}
u_t(x,0)&=\frac{1}{\rho_0(x)}\big[\mu(\rho_0)u_{0x}-\rho_0^\gamma\big]_x(x)-(u_0u_{0x})(x)+f(x,0)\\
&=g(x)-(u_0u_{0x})(x).
\end{aligned}\tag{2.18}$$

Integrating (2.17) over $(0,t)$, and using Lemma 2.5 and (2.18), we have

$$\int_I \rho u_t^2+\int_0^t\int_I \mu(\rho)u_{xt}^2\leqslant c(T)\int_0^t\big(\|u_{xx}\|_{L^2}+1\big)\int_I \rho u_t^2+c(T).$$

Using the Gronwall inequality together with Lemma 2.5 and (1.7), we complete the proof of Lemma 2.6. □

Lemma 2.7. *For any $0\leqslant t\leqslant T$, it holds*

$$\int_I u_{xx}^2\leqslant c(T).$$

Proof. First, we claim

$$\|u_x\|_{L^\infty(Q_T)}\leqslant c(T).\tag{2.19}$$

In fact, this can be obtained directly by (2.14), Lemmas 2.3 and 2.6.

It follows from (2.12), (2.19), (1.7), Lemmas 2.2–2.4, and Lemma 2.6 that

$$\int_I u_{xx}^2 \leqslant c(T).$$

This completes the proof of Lemma 2.7. □

Lemma 2.8. *For any $0 \leqslant t \leqslant T$, it holds*

$$\int_I \left[\rho_{xx}^2 + \rho_{xt}^2 + \left|(\rho^\gamma)_{xx}\right|^2 + \left|(\rho^\gamma)_{xt}\right|^2\right] + \int_{Q_T} \left[\rho_{tt}^2 + \left|(\rho^\gamma)_{tt}\right|^2\right] \leqslant c(T).$$

Proof. From $(1.1)_1$, we have

$$\rho_{xxt} = -\rho_{xxx}u - 3\rho_{xx}u_x - 3\rho_x u_{xx} - \rho u_{xxx}. \tag{2.20}$$

Multiplying (2.20) by ρ_{xx}, integrating over I, and using integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_I \rho_{xx}^2 &= -\frac{5}{2}\int_I \rho_{xx}^2 u_x - 3\int_I \rho_x\rho_{xx}u_{xx} - \int_I \rho\rho_{xx}u_{xxx} \\
&\leqslant c(T)\|u_x\|_{L^\infty}\int_I \rho_{xx}^2 + 3\|\rho_x\|_{L^\infty}\|\rho_{xx}\|_{L^2}\|u_{xx}\|_{L^2} + \|\rho\|_{L^\infty}\|\rho_{xx}\|_{L^2}\|u_{xxx}\|_{L^2}.
\end{aligned}$$

By the Sobolev inequality, the Cauchy inequality, Lemmas 2.2–2.4, Lemma 2.7 and (2.19), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_I \rho_{xx}^2 &\leqslant c(T)\int_I \rho_{xx}^2 + c(T)\big(1 + \|\rho_{xx}\|_{L^2}\big)\|\rho_{xx}\|_{L^2} + c(T)\|\rho_{xx}\|_{L^2}\|u_{xxx}\|_{L^2} \\
&\leqslant c(T)\int_I \rho_{xx}^2 + c(T)\int_I u_{xxx}^2 + c(T).
\end{aligned} \tag{2.21}$$

Multiplying $(1.1)_1$ by $\gamma\rho^{\gamma-1}$, we have

$$(\rho^\gamma)_t + (\rho^\gamma)_x u + \gamma\rho^\gamma u_x = 0, \tag{2.22}$$

which implies

$$(\rho^\gamma)_{xxt} + (\rho^\gamma)_{xxx}u + (\gamma+2)(\rho^\gamma)_{xx}u_x + (2\gamma+1)(\rho^\gamma)_x u_{xx} + \gamma\rho^\gamma u_{xxx} = 0.$$

Similarly to (2.21), we get

$$\frac{1}{2}\frac{d}{dt}\int_I \left|(\rho^\gamma)_{xx}\right|^2 \leqslant c(T)\int_I \left|(\rho^\gamma)_{xx}\right|^2 + c(T)\int_I u_{xxx}^2 + c(T). \tag{2.23}$$

(2.21) and (2.23) imply

$$\frac{d}{dt}\int_I \left(\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right) \leqslant c(T)\int_I \left(\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right)+c(T)\int_I u_{xxx}^2+c(T). \tag{2.24}$$

The next step is to estimate the term $\int_I u_{xxx}^2$.

Differentiating (2.12) with respect to x, we have

$$\begin{aligned}\mu(\rho)u_{xxx} = {}& -2\mu'(\rho)\rho_x u_{xx}+\rho_x u_t+\rho u_{xt}+\rho_x u u_x+\rho u_x^2+\rho u u_{xx}+\left(\rho^\gamma\right)_{xx}\\ &-\mu''(\rho)\rho_x^2 u_x-\mu'(\rho)\rho_{xx}u_x-\rho_x f-\rho f_x.\end{aligned} \tag{2.25}$$

(2.25), combining (1.7), Lemmas 2.2–2.4, Lemma 2.7 and the Sobolev inequality, gives

$$\int_I u_{xxx}^2 \leqslant c(T)\int_I \rho_{xx}^2+c(T)+c(T)\int_I u_{xt}^2+c(T)\int_I \left|(\rho^\gamma)_{xx}\right|^2+c(T)\|f\|_{H^1}^2.$$

Since $f\in C([0,T];H^1)$, we have

$$\int_I u_{xxx}^2 \leqslant c(T)\int_I \left[\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right]+c(T)\int_I u_{xt}^2+c(T). \tag{2.26}$$

By (2.24) and (2.26), we get

$$\frac{d}{dt}\int_I \left(\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right) \leqslant c(T)\int_I \left(\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right)+c(T)\int_I u_{xt}^2+c(T).$$

By Lemma 2.6 and the Gronwall inequality, we get

$$\int_I \left(\rho_{xx}^2+\left|(\rho^\gamma)_{xx}\right|^2\right) \leqslant c(T).$$

This, combining $(1.1)_1$, (2.22), Lemmas 2.2–2.4, Lemmas 2.6–2.7, immediately gives

$$\int_I \left[\rho_{xt}^2+\left|(\rho^\gamma)_{xt}\right|^2\right]+\int_{Q_T} \left[\rho_{tt}^2+\left|(\rho^\gamma)_{tt}\right|^2\right] \leqslant c(T).$$

The proof of Lemma 2.8 is complete. □

From (2.26), Lemmas 2.6 and 2.8, we immediately get the next result.

Lemma 2.9. *For any $0\leqslant t\leqslant T$, it holds*

$$\int_{Q_T} u_{xxx}^2 \leqslant c(T).$$

Lemma 2.10. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I u_{xt}^2 + \int_{Q_T} \rho u_{tt}^2 \leqslant c(T).$$

Proof. Multiplying (2.16) by u_{tt}, integrating over I, and using integration by parts, (1.7), Lemma 2.2 and the Cauchy inequality, we have

$$\begin{aligned}
&\int_I \rho u_{tt}^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_{xt}^2 \\
&\quad = \frac{1}{2}\int_I \mu'(\rho)\rho_t u_{xt}^2 - \int_I \mu'(\rho)\rho_t u_x u_{xtt} + \int_I (\rho_t f + \rho f_t)u_{tt} \\
&\qquad - \int_I \big[\rho_t u_t u_{tt} + \rho_t u u_x u_{tt} + \rho u_t u_x u_{tt} + \rho u u_{xt} u_{tt} - \big(\rho^\gamma\big)_t u_{xtt}\big] \\
&\quad \leqslant c(T)\|\rho_t\|_{H^1}\int_I \mu(\rho)u_{xt}^2 - \int_I \mu'(\rho)\rho_t u_x u_{xtt} - \int_I (\rho_t u_t u_{tt} + \rho_t u u_x u_{tt}) + \frac{1}{4}\int_I \rho u_{tt}^2 \\
&\qquad + c(T)\|u_x\|_{L^\infty}^2 \int_I \rho u_t^2 + c(T)\|u\|_{L^\infty}^2 \int_I \mu(\rho)u_{xt}^2 + \int_I \big(\rho^\gamma\big)_t u_{xtt} + \int_I \rho_t f u_{tt} + c(T)\int_I f_t^2.
\end{aligned}$$

By Lemmas 2.3–2.4, Lemmas 2.6–2.8, (1.7), the Cauchy inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\frac{3}{4}\int_I \rho u_{tt}^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_{xt}^2 \\
&\quad \leqslant c(T)\int_I \mu(\rho)u_{xt}^2 - \int_I \mu'(\rho)\rho_t u_x u_{xtt} - \int_I \rho_t u_t u_{tt} - \int_I \rho_t u u_x u_{tt} + \int_I \big(\rho^\gamma\big)_t u_{xtt} \\
&\qquad + \int_I \rho_t f u_{tt} + c(T)\int_I f_t^2 + c(T) \\
&\quad \leqslant c(T)\int_I \mu(\rho)u_{xt}^2 - \frac{d}{dt}\int_I \mu'(\rho)\rho_t u_x u_{xt} + \int_I \big[\mu''(\rho)\rho_t^2 u_x u_{xt} + \mu'(\rho)\rho_{tt} u_x u_{xt} + \mu'(\rho)\rho_t u_{xt}^2\big] \\
&\qquad - \frac{1}{2}\frac{d}{dt}\int_I \rho_t u_t^2 + \frac{1}{2}\int_I \rho_{tt} u_t^2 - \frac{d}{dt}\int_I \rho_t u u_x u_t + \int_I \big(\rho_{tt} u u_x u_t + \rho_t u_t^2 u_x + \rho_t u u_{xt} u_t\big) \\
&\qquad + \frac{d}{dt}\int_I \big(\rho^\gamma\big)_t u_{xt} - \int_I \big(\rho^\gamma\big)_{tt} u_{xt} + \frac{d}{dt}\int_I \rho_t f u_t - \int_I (\rho_{tt} f + \rho_t f_t)u_t + c(T)\int_I f_t^2 + c(T) \\
&\quad \leqslant c(T)\int_I \mu(\rho)u_{xt}^2 - \frac{d}{dt}\int_I \Big\{\mu'(\rho)\rho_t u_x u_{xt} + \frac{1}{2}\rho_t u_t^2 + \rho_t u u_x u_t - \big(\rho^\gamma\big)_t u_{xt} - \rho_t f u_t\Big\} \\
&\qquad + c(T)\int_I \rho_{tt}^2 + c(T)\|u_t\|_{L^\infty}^4 + c(T)\int_I \big|\big(\rho^\gamma\big)_{tt}\big|^2 + c(T)\int_I f_t^2 + c(T).
\end{aligned}$$

Since $\|u_t\|_{L^\infty}^4 \leqslant (\int_I u_{xt}^2)^2 \leqslant c(T)(\int_I \mu(\rho)u_{xt}^2)^2$, we have

$$\begin{aligned}
&\frac{3}{4}\int_I \rho u_{tt}^2 + \frac{1}{2}\frac{d}{dt}\int_I \mu(\rho)u_{xt}^2 \\
&\quad\leqslant -\frac{d}{dt}\int_I \left\{\mu'(\rho)\rho_t u_x u_{xt} + \frac{1}{2}\rho_t u_t^2 + \rho_t u u_x u_t - \left(\rho^\gamma\right)_t u_{xt} - \rho_t f u_t\right\} \\
&\qquad + c(T)\left(\int_I \mu(\rho)u_{xt}^2\right)^2 + c(T)\int_I \left[\rho_{tt}^2 + \left|\left(\rho^\gamma\right)_{tt}\right|^2\right] + c(T)\int_I f_t^2 + c(T).
\end{aligned} \tag{2.27}$$

Integrating (2.27) over $(0,t)$, and using Lemma 2.8, we have

$$\begin{aligned}
&\frac{3}{4}\int_0^t\int_I \rho u_{tt}^2 + \frac{1}{2}\int_I \mu(\rho)u_{xt}^2 \\
&\quad\leqslant \frac{1}{2}\int_I \mu(\rho_0)u_{xt}^2(x,0) - \int_I \left\{\mu'(\rho)\rho_t u_x u_{xt} + \frac{1}{2}\rho_t u_t^2 + \rho_t u u_x u_t - \left(\rho^\gamma\right)_t u_{xt} - \rho_t f u_t\right\} \\
&\qquad + \int_I \left\{\mu'(\rho)\rho_t u_x u_{xt} + \frac{1}{2}\rho_t u_t^2 + \rho_t u u_x u_t - \left(\rho^\gamma\right)_t u_{xt} - \rho_t f u_t\right\}(x,0) \\
&\qquad + c(T)\int_0^t \left(\int_I \mu(\rho)u_{xt}^2\right)^2 + c(T).
\end{aligned} \tag{2.28}$$

By $(1.1)_1$, we get

$$\rho_t|_{t=0} = -(\rho_0 u_0)_x \in H^1. \tag{2.29}$$

(2.18), (2.28) and (2.29) show

$$\begin{aligned}
&\frac{3}{4}\int_0^t\int_I \rho u_{tt}^2 + \frac{1}{2}\int_I \mu(\rho)u_{xt}^2 \\
&\quad\leqslant -\int_I \left\{\mu'(\rho)\rho_t u_x u_{xt} + \frac{1}{2}\rho_t u_t^2 + \rho_t u u_x u_t - \left(\rho^\gamma\right)_t u_{xt} - \rho_t f u_t\right\} \\
&\qquad + c(T)\int_0^t \left(\int_I \mu(\rho)u_{xt}^2\right)^2 + c(T) \\
&\quad\leqslant c(T)\|\rho_t\|_{L^2}\|u_x\|_{L^\infty}\|u_{xt}\|_{L^2} + \frac{1}{2}\int_I (\rho u)_x u_t^2 + \|\rho_t\|_{L^2}\|u\|_{L^\infty}\|u_x\|_{L^2}\|u_t\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\left(\rho^\gamma\right)_t\right\|_{L^2}\|u_{xt}\|_{L^2}+\|u_t\|_{L^\infty}\|f\|_{L^2}\|\rho_t\|_{L^2}+c(T)\int_0^t\left(\int_I \mu(\rho)u_{xt}^2\right)^2+c(T)\\
&\leqslant c(T)\left\|\sqrt{\mu(\rho)}u_{xt}\right\|_{L^2}-\int_I \rho u u_t u_{xt}+c(T)\|u_t\|_{L^\infty}+c(T)\int_0^t\left(\int_I \mu(\rho)u_{xt}^2\right)^2+c(T)\\
&\leqslant \frac{1}{4}\int_I \mu(\rho)u_{xt}^2+c(T)\int_0^t\left(\int_I \mu(\rho)u_{xt}^2\right)^2+c(T),
\end{aligned}
$$

which implies

$$
\int_I \mu(\rho)u_{xt}^2+\int_0^t\int_I \rho u_{tt}^2\leqslant c(T)\int_0^t\left(\int_I \mu(\rho)u_{xt}^2\right)^2+c(T). \tag{2.30}
$$

Here we have used Lemmas 2.2–2.4, Lemmas 2.6–2.8, (1.7), the Hölder inequality, the Sobolev inequality, and the Cauchy inequality. By Lemma 2.6 and the Gronwall inequality, we can complete the proof of Lemma 2.10. □

Lemma 2.11. *For any* $0\leqslant t\leqslant T$, *it holds*

$$
\int_I u_{xxx}^2\leqslant c(T), \tag{2.31}
$$

$$
\int_{Q_T} u_{xxt}^2\leqslant c(T). \tag{2.32}
$$

$$
\int_I \left[\rho_{tt}^2+\left|\left(\rho^\gamma\right)_{tt}\right|^2\right]\leqslant c(T). \tag{2.33}
$$

Proof. By (2.26), Lemmas 2.8 and 2.10, we get (2.31). (2.32) can be obtained by (2.16), Lemmas 2.2–2.4, Lemmas 2.7–2.8, Lemma 2.10. From $(1.1)_1$, Lemmas 2.2–2.4, Lemmas 2.8 and 2.10, we have (2.33). □

From above, we get the following estimates

$$
\frac{\delta}{c(T)}\leqslant \rho(x,t)\leqslant c(T), \tag{2.34}
$$

and

$$
\begin{aligned}
&\left\|\left(\rho,\rho^\gamma\right)\right\|_{H^2}+\left\|\left(\rho_t,\left(\rho^\gamma\right)_t\right)\right\|_{H^1}+\left\|\left(\rho_{tt},\left(\rho^\gamma\right)_{tt}\right)\right\|_{L^2}+\|u\|_{H^3}+\|u_t\|_{H^1}\\
&\quad+\int_{Q_T}\left(\rho u_{tt}^2+u_{xxt}^2\right)\leqslant c(T),
\end{aligned} \tag{2.35}
$$

where $c(T)$ is a positive constant, independent of δ and $\|(\cdot,\cdot)\|_X=\|\cdot\|_X+\|\cdot\|_X$, for some Banach space X.

The proof of Theorem 2.1 is complete. □

Proof of Theorem 1.1. To prove Theorem 1.1, we construct a sequence of approximate solutions to (1.1)–(1.3) by giving the initial density a positive lower bound δ and using Theorem 2.1. Then, we get some estimates uniformly on δ, and take limit $\delta \to 0$ (take a subsequence if necessary). After that, we get existence of classical solutions to (1.1)–(1.3).

More precisely, we denote $\rho_0^\delta = \rho_0 + \delta$. u_0^δ is the unique solution of the following elliptic problems for each $\delta > 0$:

$$\left[\mu\left(\rho_0^\delta\right)u_{0x}^\delta\right]_x(x) - \left[p\left(\rho_0^\delta\right)\right]_x(x) = \rho_0^\delta(x)\left[-f(x,0) + g(x)\right], \quad x \in (0,1), \tag{2.36}$$

and

$$u_0^\delta\big|_{x=0,1} = 0. \tag{2.37}$$

Since $\rho_0,\ \rho_0^\gamma \in H^2$, we have as $\delta \to 0$

$$\rho_0^\delta \to \rho_0, \quad \text{in } H^2, \tag{2.38}$$

$$\left(\rho_0^\delta\right)^\gamma \to \rho_0^\gamma, \quad \text{in } H^2. \tag{2.39}$$

From (2.36)–(2.39) and the elliptic theory (please refer to [6])

$$u_0^\delta \to u_0, \quad \text{in } H^3, \tag{2.40}$$

as $\delta \to 0$.

Consider (1.1), (2.36) with initial boundary data

$$\left(\rho^\delta, u^\delta\right)\big|_{t=0} = \left(\rho_0^\delta, u_0^\delta\right), \quad \text{in } I,$$

and

$$u^\delta\big|_{\partial I} = 0, \quad \text{for } t \geqslant 0.$$

We can get a unique solution (ρ^δ, u^δ) for each $\delta > 0$ by Theorem 2.1, with the following estimates:

$$\frac{\delta}{c(T)} \leqslant \rho^\delta(x,t) \leqslant c(T), \tag{2.41}$$

and

$$\begin{aligned} &\left\|\left(\rho^\delta, P\left(\rho^\delta\right)\right)\right\|_{H^2} + \left\|\left(\rho_t^\delta, P\left(\rho^\delta\right)_t\right)\right\|_{H^1} + \left\|\left(\rho_{tt}^\delta, P\left(\rho^\delta\right)_{tt}\right)\right\|_{L^2} + \left\|u^\delta\right\|_{H^3} \\ &\quad + \left\|u_t^\delta\right\|_{H^1} + \int_{Q_T} \left(\rho^\delta\left|u_{tt}^\delta\right|^2 + \left|u_{xxt}^\delta\right|^2\right) \leqslant c(T), \end{aligned} \tag{2.42}$$

where $c(T)$ is a positive constant, independent of δ.

Based on the estimates (2.41) and (2.42), we get a solution (ρ, u) to (1.1)–(1.3) after taking limit $\delta \to 0$ (take the subsequence if necessary):

$$\rho \in L^\infty\big([0,T];H^2\big), \quad \rho_t \in L^\infty\big([0,T];H^1\big), \quad \rho_{tt} \in L^\infty\big([0,T];L^2\big), \quad \rho \geqslant 0,$$
$$\big(\rho^\gamma\big)_{xx} \in L^\infty\big([0,T];L^2\big), \quad \big(\rho^\gamma\big)_{xt} \in L^\infty\big([0,T];L^2\big), \quad \big(\rho^\gamma\big)_{tt} \in L^\infty\big([0,T];L^2\big),$$
$$u \in L^\infty\big([0,T];H^3 \cap H_0^1\big), \quad u_t \in L^\infty\big([0,T];H_0^1\big) \cap L^2\big([0,T];H^2\big).$$

Since $u \in L^\infty([0,T];H^3)$ and $u_t \in L^\infty([0,T];H_0^1)$, we have $u \in C([0,T];H^2)$ (please refer to [6]). This together with $(1.1)_1$, (2.22) and [3] implies

$$\rho \in C\big([0,T];H^2\big), \tag{2.43}$$

and

$$\rho^\gamma \in C\big([0,T];H^2\big). \tag{2.44}$$

Denote $G = [\mu(\rho)u_x - \rho^\gamma]_x + \rho f$. By (2.12), (1.7) and the regularities of (ρ, u), we have

$$G = \rho u_t + \rho u u_x \in L^2\big([0,T];H^2\big),$$
$$G_t = \big[\mu(\rho)u_x - \rho^\gamma\big]_{xt} + (\rho f)_t \in L^2\big([0,T];L^2\big).$$

From the embedding theorem ([6]), we have $G \in C([0,T];H^1)$. Since $\rho f \in C([0,T];H^1)$, we get

$$\big[\mu(\rho)u_x - \rho^\gamma\big]_x \in C\big([0,T];H^1\big).$$

This means

$$\mu(\rho)u_x - \rho^\gamma \in C\big([0,T];H^2\big). \tag{2.45}$$

By (2.44) and (2.45), we get

$$\mu(\rho)u_x \in C\big([0,T];H^2\big).$$

This together with (1.7) and (2.43) implies

$$u \in C\big([0,T];H^3\big). \tag{2.46}$$

By $(1.1)_1$, (2.43) and (2.46), we obtain

$$\rho_t \in C\big([0,T];H^1\big). \tag{2.47}$$

It follows from $(1.1)_2$, (2.43)–(2.44), (2.46) and (1.7) that

$$(\rho u)_t \in C\big([0,T];H^1\big). \tag{2.48}$$

Differentiating $(1.1)_1$ with respect to t, we have

$$\rho_{tt} = -(\rho u)_{tx}.$$

This combining (2.48) gives

$$\rho_{tt} \in C\big([0,T];L^2\big). \tag{2.49}$$

By (2.22), (2.44) and (2.46), we get

$$\big(\rho^\gamma\big)_t \in C\big([0,T];H^1\big).$$

The existence of global classical solutions as in Theorem 1.1 is obtained. The uniqueness of the solutions can be proved by the standard method like in [4], we omit it for simplicity. The proof of Theorem 1.1 is complete. □

3. Proof of Theorem 1.2

Before proving Theorem 1.2, we need the following auxiliary theorem.

Theorem 3.1. *Consider the same assumptions as in Theorem* 1.2*, and in addition assume that* $\rho_0 \geqslant \delta > 0$. *Then for any* $T > 0$ *there exists a unique global classical solution* (ρ, u) *to* (1.1)–(1.3) *satisfying*

$$\begin{gathered}
\rho \in C\big([0,T];H^4\big), \quad \rho_t \in C\big([0,T];H^3\big), \quad \rho_{tt} \in C\big([0,T];H^1\big) \cap L^2\big([0,T];H^2\big), \\
\rho_{ttt} \in L^2(Q_T), \quad \rho \geqslant \frac{\delta}{c(T)}, \quad u \in C\big([0,T];H^4 \cap H_0^1\big) \cap L^2\big([0,T];H^5\big), \\
u_t \in C\big([0,T];H^2\big) \cap L^2\big([0,T];H^3\big), \quad u_{tt} \in C\big([0,T];L^2\big) \cap L^2\big([0,T];H_0^1\big).
\end{gathered}$$

Proof. Similarly to the proof of Theorem 2.1, Theorem 3.1 can be proved by some *a priori* estimates globally in time. Since (2.34) and (2.35) are also valid here, we need other *a priori* estimates about higher order derivatives of (ρ, u). The generic positive constant $c(T)$ may depend on the initial data presented in Theorem 1.2 and some other known constants but independent of δ.

Lemma 3.1. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I \big[\rho_{xxx}^2 + \rho_{xxt}^2 + \big|(\rho^\gamma)_{xxx}\big|^2 + \big|(\rho^\gamma)_{xxt}\big|^2\big] + \int_{Q_T} \big[\rho_{xtt}^2 + \big|(\rho^\gamma)_{xtt}\big|^2 + u_{xxxx}^2\big] \leqslant c(T).$$

Proof. Differentiating (2.20) with respect to x, we have

$$\rho_{xxxt} = -\rho_{xxxx}u - 4\rho_{xxx}u_x - 6\rho_{xx}u_{xx} - 4\rho_x u_{xxx} - \rho u_{xxxx}. \tag{3.1}$$

Multiplying (3.1) by ρ_{xxx}, integrating the resulting equation over I, and using integration by parts, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_I \rho_{xxx}^2 &= -\frac{7}{2}\int_I \rho_{xxx}^2 u_x - 6\int_I \rho_{xx}\rho_{xxx}u_{xx} - 4\int_I \rho_x\rho_{xxx}u_{xxx} - \int_I \rho\rho_{xxx}u_{xxxx} \\
&\leqslant \frac{7}{2}\|u_x\|_{L^\infty}\int_I \rho_{xxx}^2 + 6\|u_{xx}\|_{L^\infty}\|\rho_{xx}\|_{L^2}\|\rho_{xxx}\|_{L^2} \\
&\quad + 4\|\rho_x\|_{L^\infty}\|\rho_{xxx}\|_{L^2}\|u_{xxx}\|_{L^2} + \|\rho\|_{L^\infty}\|\rho_{xxx}\|_{L^2}\|u_{xxxx}\|_{L^2}.
\end{aligned}$$

By the Sobolev inequality, (2.34)–(2.35) and the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_I \rho_{xxx}^2 \leqslant c(T)\int_I \rho_{xxx}^2 + c(T)\int_I u_{xxxx}^2 + c(T). \tag{3.2}$$

Similarly to (3.2), we get from (2.22)

$$\frac{1}{2}\frac{d}{dt}\int_I \left|\left(\rho^\gamma\right)_{xxx}\right|^2 \leqslant c(T)\int_I \left|\left(\rho^\gamma\right)_{xxx}\right|^2 + c(T)\int_I u_{xxxx}^2 + c(T). \tag{3.3}$$

By (3.2) and (3.3), we have

$$\frac{d}{dt}\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) \leqslant c(T)\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) + c(T)\int_I u_{xxxx}^2 + c(T). \tag{3.4}$$

Differentiating (2.25) with respect to x, we have

$$\begin{aligned}\mu(\rho)u_{xxxx} = {} & -3\mu'(\rho)\rho_x u_{xxx} - 3\mu'(\rho)\rho_{xx}u_{xx} - 3\mu''(\rho)\rho_x^2 u_{xx} + \rho_{xx}u_t + 2\rho_x u_{xt} \\ & + \rho u_{xxt} + (\rho_x u u_x)_x + \left(\rho u_x^2\right)_x + (\rho u u_{xx})_x + \left(\rho^\gamma\right)_{xxx} - \mu'''(\rho)\rho_x^3 u_x \\ & - 3\mu''(\rho)\rho_x\rho_{xx}u_x - \mu'(\rho)\rho_{xxx}u_x - \rho_{xx}f - 2\rho_x f_x - \rho f_{xx}.\end{aligned} \tag{3.5}$$

By (3.5) and (2.34)–(2.35), we have

$$\int_I u_{xxxx}^2 \leqslant c(T)\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) + c(T)\int_I u_{xxt}^2 + c(T)\int_I f_{xx}^2 + c(T). \tag{3.6}$$

Combining (3.4) and (3.6), we get

$$\frac{d}{dt}\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) \leqslant c(T)\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) + c(T)\int_I u_{xxt}^2 + c(T)\int_I f_{xx}^2 + c(T).$$

By the Gronwall inequality and (2.35), we obtain

$$\int_I \left(\rho_{xxx}^2 + \left|\left(\rho^\gamma\right)_{xxx}\right|^2\right) \leqslant c(T). \tag{3.7}$$

It follows from (2.20), (2.22), $(1.1)_1$, (2.34), (2.35), (3.6) and (3.7) that

$$\int_I \left[\rho_{xxt}^2 + + \left|\left(\rho^\gamma\right)_{xxt}\right|^2\right] + \int_{Q_T} \left[\rho_{xtt}^2 + \left|\left(\rho^\gamma\right)_{xtt}\right|^2 + u_{xxxx}^2\right] \leqslant c(T).$$

The proof of Lemma 3.1 is complete. □

Lemma 3.2. *For any $T > 0$, we have*

$$\left\|(\sqrt{\rho})_x\right\|_{L^\infty(Q_T)} + \left\|(\sqrt{\rho})_t\right\|_{L^\infty(Q_T)} \leqslant c(T).$$

Proof. Multiplying $(1.1)_1$ by $\frac{1}{2\sqrt{\rho}}$, we have

$$(\sqrt{\rho})_t + (\sqrt{\rho})_x u + \frac{1}{2}\sqrt{\rho}u_x = 0. \tag{3.8}$$

Differentiating (3.8) with respect to x, we get

$$(\sqrt{\rho})_{xt} + (\sqrt{\rho})_{xx} u + \frac{3}{2}(\sqrt{\rho})_x u_x + \frac{1}{2}\sqrt{\rho}u_{xx} = 0.$$

Denote $h = (\sqrt{\rho})_x$, we have

$$h_t + h_x u + \frac{3}{2}hu_x + \frac{1}{2}\sqrt{\rho}u_{xx} = 0,$$

which implies

$$\frac{d}{dt}\left\{h\exp\left(\frac{3}{2}\int_0^t u_x\big(x(\tau,y),\tau\big)\,d\tau\right)\right\} = -\frac{1}{2}\sqrt{\rho}u_{xx}\exp\left(\frac{3}{2}\int_0^t u_x\big(x(\tau,y),\tau\big)\,d\tau\right), \tag{3.9}$$

where $x(t,y)$ is the solution to (2.5).

Integrating (3.9) over $(0,s)$, we get

$$\begin{aligned} h(y,s) = {} & \exp\left(-\frac{3}{2}\int_0^s u_x\big(x(\tau,y),\tau\big)\,d\tau\right)h\big(x(0,y),0\big) \\ & - \frac{1}{2}\exp\left(-\frac{3}{2}\int_0^s u_x\big(x(\tau,y),\tau\big)\,d\tau\right)\int_0^s \sqrt{\rho}u_{xx}\exp\left(\frac{3}{2}\int_0^t u_x\big(x(\tau,y),\tau\big)\,d\tau\right)dt. \end{aligned}$$

This together with (2.35) implies

$$\left\|(\sqrt{\rho})_x\right\|_{L^\infty(Q_T)} \leqslant c(T). \tag{3.10}$$

From (2.34)–(2.35), (3.8) and (3.10), we get

$$\left\|(\sqrt{\rho})_t\right\|_{L^\infty(Q_T)} \leqslant c(T).$$

The proof of Lemma 3.2 is complete. $\square$

The next lemma plays the most important role in this section.

Lemma 3.3. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I (\rho^3 u_{tt}^2 + \rho u_{xxt}^2) + \int_{Q_T} (\rho^2 u_{xtt}^2 + u_{xxxt}^2) \leqslant c(T).$$

Proof. Differentiating (2.16) with respect to t, multiplying the resulting equation by $\rho^2 u_{tt}$, integrating over I, and using integration by parts and the Cauchy inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho^3 u_{tt}^2 + \int_I \mu(\rho)\rho^2 u_{xtt}^2 \\
&= -\frac{1}{2}\int_I \rho^2 \rho_t u_{tt}^2 - \int_I \big[\rho_{tt} u_t + \rho_{tt} u u_x + 2\rho_t u_t u_x + 2\rho_t u u_{xt} + 2\rho u_t u_{xt} + \big(\rho^\gamma\big)_{xtt}\big]\big(\rho^2 u_{tt}\big) \\
&\quad - \int_I \rho^3 u_{tt}^2 u_x - \int_I \rho^3 u u_{tt} u_{xtt} - \int_I \big[\mu''(\rho)\rho_t^2 u_x + \mu'(\rho)\rho_{tt} u_x + 2\mu'(\rho)\rho_t u_{xt}\big]\big(\rho^2 u_{xtt}\big) \\
&\quad - 2\int_I \big[\mu''(\rho)\rho_t^2 u_x + \mu'(\rho)\rho_{tt} u_x + 2\mu'(\rho)\rho_t u_{xt} + \mu(\rho) u_{xtt}\big](\rho\rho_x u_{tt}) \\
&\quad + \int_I (\rho_{tt} f + 2\rho_t f_t + \rho f_{tt})\big(\rho^2 u_{tt}\big) \\
&\leqslant c(T)\int_I \rho u_{tt}^2 + c(T)\big\|\big(\rho^\gamma\big)_{xtt}\big\|_{L^2}^2 + \frac{1}{3}\int_I \mu(\rho)\rho^2 u_{xtt}^2 \\
&\quad - 4\int_I \mu(\rho)\rho u_{xtt}\sqrt{\rho}\, u_{tt}(\sqrt{\rho})_x + c(T)\big(\|f_t\|_{L^2}^2 + \|f_{tt}\|_{L^2}^2\big) + c(T) \\
&\leqslant \frac{2}{3}\int_I \mu(\rho)\rho^2 u_{xtt}^2 + c(T)\int_I \rho u_{tt}^2 + c(T)\big\|\big(\rho^\gamma\big)_{xtt}\big\|_{L^2}^2 \\
&\quad + c(T)\int_I \rho u_{tt}^2 \big|(\sqrt{\rho})_x\big|^2 + c(T)\int_I f_t^2 + c(T)\int_I f_{tt}^2 + c(T).
\end{aligned}$$

This together with Lemma 3.2 implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_I \rho^3 u_{tt}^2 + \frac{1}{3}\int_I \mu(\rho)\rho^2 u_{xtt}^2 \\
&\quad \leqslant c(T)\int_I \rho u_{tt}^2 + c(T)\int_I \big|\big(\rho^\gamma\big)_{xtt}\big|^2 + c(T)\int_I \big(f_t^2 + f_{tt}^2\big) + c(T).
\end{aligned} \tag{3.11}$$

Integrating (3.11) over $(0, t)$, and using (2.35) and Lemma 3.1, we get

$$\frac{1}{2}\int_I \rho^3 u_{tt}^2 + \frac{1}{3}\int_0^t \int_I \mu(\rho)\rho^2 u_{xtt}^2 \leqslant \frac{1}{2}\int_I \rho^3 u_{tt}^2(x, 0) + c(T). \tag{3.12}$$

By (2.16), (2.18), (2.29), (3.12), (1.7) and $(\sqrt{\rho_0}g_x)_x \in L^2$, we have

$$\int_I \rho^3 u_{tt}^2 + \int_{Q_T} \rho^2 u_{xtt}^2 \leqslant c(T). \tag{3.13}$$

Differentiating (2.16) with respect to x, we get

$$\begin{aligned} \mu(\rho)u_{xxxt} = {} & -2\mu'(\rho)\rho_x u_{txx} - 2\mu''(\rho)\rho_x\rho_t u_{xx} - 2\mu'(\rho)\rho_{xt}u_{xx} - 2\mu''(\rho)\rho_{xt}\rho_x u_x \\ & - \mu'(\rho)\rho_t u_{xxx} - \mu'''(\rho)\rho_x^2\rho_t u_x - \mu''(\rho)\rho_t\rho_{xx}u_x - \mu'(\rho)\rho_{xxt}u_x \\ & - \mu''(\rho)\rho_x^2 u_{xt} - \mu'(\rho)\rho_{xx}u_{xt} + 2(\sqrt{\rho})_x\sqrt{\rho}u_{tt} + \rho u_{xtt} + \rho_{xt}u_t + \rho_t u_{xt} \\ & + \rho_{xt}uu_x + \rho_t u_x^2 + \rho_t uu_{xx} + \rho_x u_t u_x + 2\rho u_{xt}u_x + \rho u_t u_{xx} + \rho_x uu_{xt} \\ & + \rho uu_{xxt} + \left(\rho^\gamma\right)_{xxt} - \rho_{xt}f - \rho_t f_x - \rho_x f_t - \rho f_{xt}. \end{aligned}$$

This together with (1.7), (2.35), (3.13) and Lemmas 3.1–3.2 implies

$$\int_{Q_T} u_{xxxt}^2 \leqslant c(T).$$

By (2.16), (2.34)–(2.35), and (3.13), we have

$$\int_I \rho u_{xxt}^2 \leqslant c(T).$$

The proof of Lemma 3.3 is complete. $\square$

Lemma 3.4. *For any $0 \leqslant t \leqslant T$, it holds*

$$\int_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) + \int_{Q_T} u_{xxxxx}^2 \leqslant c(T).$$

Proof. Differentiating (3.1) with respect to x, multiplying the resulting equation by ρ_{xxxx}, integrating over I, and using integration by parts, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_I \rho_{xxxx}^2 = {} & -\frac{9}{2}\int_I \rho_{xxxx}^2 u_x - 10\int_I \rho_{xxx}\rho_{xxxx}u_{xx} - 10\int_I \rho_{xx}\rho_{xxxx}u_{xxx} \\ & - 5\int_I \rho_x\rho_{xxxx}u_{xxxx} - \int_I \rho\rho_{xxxx}u_{xxxxx} \\ \leqslant {} & c(T)\int_I \rho_{xxxx}^2 + c(T)\int_I u_{xxxx}^2 + c(T)\int_I u_{xxxxx}^2 + c(T). \end{aligned} \tag{3.14}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_I \left|\left(\rho^\gamma\right)_{xxxx}\right|^2 \leqslant c(T)\int_I \left|\left(\rho^\gamma\right)_{xxxx}\right|^2 + c(T)\int_I u_{xxxx}^2 + c(T)\int_I u_{xxxxx}^2 + c(T). \tag{3.15}$$

By (3.14) and (3.15), we obtain

$$\frac{d}{dt}\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) \leqslant c(T)\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) + c(T)\int\limits_I u_{xxxx}^2 + c(T)\int\limits_I u_{xxxxx}^2 + c(T). \tag{3.16}$$

Now we estimate the third term of the right-hand side of (3.16).

Differentiating (3.5) with respect to x, we have

$$\begin{aligned}
\mu(\rho)u_{xxxxx} = {} & -4\mu'(\rho)\rho_x u_{xxxx} - 6\mu''(\rho)\rho_x^2 u_{xxx} - 6\mu'(\rho)\rho_{xx}u_{xxx} \\
& - 12\mu''(\rho)\rho_x\rho_{xx}u_{xx} - 4\mu'(\rho)\rho_{xxx}u_{xx} - 4\mu'''(\rho)\rho_x^3 u_{xx} + \rho_{xxx}u_t \\
& + 3\rho_{xx}u_{xt} + 3\rho_x u_{xxt} + \rho u_{xxxt} + (\rho_x u u_x)_{xx} + \left(\rho u_x^2\right)_{xx} \\
& + (\rho u u_{xx})_{xx} + \left(\rho^\gamma\right)_{xxxx} - \mu''''(\rho)\rho_x^4 u_x - 6\mu'''(\rho)\rho_x^2\rho_{xx}u_x \\
& - 3\mu''(\rho)\rho_{xx}^2 u_x - 4\mu''(\rho)\rho_x\rho_{xxx}u_x - \mu'(\rho)\rho_{xxxx}u_x \\
& - \rho_{xxx}f - 3\rho_{xx}f_x - 3\rho_x f_{xx} - \rho f_{xxx}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int\limits_I u_{xxxxx}^2 \leqslant {} & c(T)\int\limits_I u_{xxxx}^2 + c(T)\int\limits_I u_{xxt}^2 + c(T)\int\limits_I u_{xxxt}^2 \\
& + c(T)\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) + c(T)\int\limits_I \left(f_{xx}^2 + f_{xxx}^2\right) + c(T),
\end{aligned} \tag{3.17}$$

where we have used (1.7), (2.34)–(2.35), Lemma 3.1.

By (3.16) and (3.17), we obtain

$$\begin{aligned}
\frac{d}{dt}\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) \leqslant {} & c(T)\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) + c(T)\int\limits_I u_{xxxx}^2 \\
& + c(T)\int\limits_I u_{xxt}^2 + c(T)\int\limits_I u_{xxxt}^2 + c(T)\int\limits_I \left(f_{xx}^2 + f_{xxx}^2\right) + c(T).
\end{aligned}$$

Using the Gronwall inequality, (2.35), Lemmas 3.1 and 3.3, we get

$$\int\limits_I \left(\rho_{xxxx}^2 + \left|\left(\rho^\gamma\right)_{xxxx}\right|^2\right) \leqslant c(T). \tag{3.18}$$

It follows from (3.17), (3.18), (2.35), Lemmas 3.1 and 3.3 that

$$\int\limits_{Q_T} u_{xxxxx}^2 \leqslant c(T). \tag{3.19}$$

This proves Lemma 3.4. □

Lemma 3.5. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I u_{xxxx}^2 \leqslant c(T).$$

Proof. From (2.35), Lemmas 3.1, 3.3, and 3.4, we get

$$\|u\|_{L^2(0,T;H^5)} \leqslant c(T), \tag{3.20}$$

and

$$\|u_t\|_{L^2(0,T;H^3)} \leqslant c(T). \tag{3.21}$$

By the embedding theorem (cf. [6]), (3.20), (3.21) show

$$\|u\|_{C([0,T];H^4)} \leqslant c(T).$$

This proves Lemma 3.5. □

From $(1.1)_1$, (2.22), (2.34)–(2.35), Lemma 3.1, Lemmas 3.3–3.5, we immediately get the following lemma.

Lemma 3.6. *For any* $0 \leqslant t \leqslant T$, *it holds*

$$\int_I \left[\rho_{xtt}^2 + \left|(\rho^\gamma)_{xtt}\right|^2 + \rho_{xxxt}^2 + \left|(\rho^\gamma)_{xxxt}\right|^2\right] + \int_{Q_T} \left[\rho_{ttt}^2 + \left|(\rho^\gamma)_{ttt}\right|^2 + \rho_{xxtt}^2 + \left|(\rho^\gamma)_{xxtt}\right|^2\right] \leqslant c(T).$$

Here we have used the following inequality

$$\rho_x^2 u_{tt}^2 = 2\left\{(\sqrt{\rho})_x \sqrt{\rho}\right\}^2 u_{tt}^2 \leqslant c(T)\rho u_{tt}^2.$$

From above estimates, we get

$$\frac{\delta}{c(T)} \leqslant \rho(x,t) \leqslant c(T), \tag{3.22}$$

and

$$\begin{aligned}
&\left\|(\sqrt{\rho})_x\right\|_{L^\infty} + \left\|(\sqrt{\rho})_t\right\|_{L^\infty} + \left\|\left(\rho, \rho^\gamma\right)\right\|_{H^4} + \left\|\left(\rho_t, (\rho^\gamma)_t\right)\right\|_{H^3} \\
&\quad + \left\|\left(\rho_{tt}, (\rho^\gamma)_{tt}\right)\right\|_{H^1} + \|u\|_{H^4} + \|u_t\|_{H^1} + \left\|\rho^{\frac{3}{2}} u_{tt}\right\|_{L^2} + \|\sqrt{\rho} u_{xxt}\|_{L^2} \\
&\quad + \int_{Q_T} \left\{\rho^2 u_{xtt}^2 + \rho u_{tt}^2 + u_{xxt}^2 + u_{xxxt}^2 + u_{xxxxx}^2 + \rho_{ttt}^2 + \left|(\rho^\gamma)_{ttt}\right|^2 + \rho_{xxtt}^2 + \left|(\rho^\gamma)_{xxtt}\right|^2\right\} \\
&\leqslant c(T).
\end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we get

$$\begin{aligned}&\|\rho\|_{H^4}+\|\rho_t\|_{H^3}+\|\rho_{tt}\|_{H^1}+\|u\|_{H^4}+\|u_t\|_{H^2}+\|u_{tt}\|_{L^2}\\&\quad+\int\limits_{Q_T}\big(u_{xtt}^2+u_{xxxt}^2+u_{xxxxx}^2+\rho_{ttt}^2+\rho_{xxtt}^2\big)\leqslant c(T,\delta),\end{aligned}\tag{3.24}$$

where $c(T,\delta)$ is a positive constant, and may depend on δ.

The proof of Theorem 3.1 is complete. □

Proof of Theorem 1.2. To prove Theorem 1.2, we follow the similar arguments as in Section 2. After taking $\delta\to 0$ (take subsequence if necessary), we get a solution (ρ,u) to (1.1)–(1.3) satisfying

$$\begin{gathered}\big(\rho,\rho^\gamma\big)\in L^\infty\big([0,T];H^4\big),\quad \big((\sqrt{\rho})_x,(\sqrt{\rho})_t\big)\in L^\infty(Q_T),\quad \big(\rho_t,\big(\rho^\gamma\big)_t\big)\in L^\infty\big([0,T];H^3\big),\\ \big(\rho_{tt},\big(\rho^\gamma\big)_{tt}\big)\in L^\infty\big([0,T];H^1\big)\cap L^2\big([0,T];H^2\big),\quad \big(\rho_{ttt},\big(\rho^\gamma\big)_{ttt}\big)\in L^2(Q_T),\\ u\in L^\infty\big([0,T];H^4\big)\cap L^2\big([0,T];H^5\big),\quad u_t\in L^\infty\big([0,T];H_0^1\big)\cap L^2\big([0,T];H^3\big),\\ \sqrt{\rho}u_{xxt}\in L^\infty\big([0,T];L^2\big).\end{gathered}$$

Since

$$\begin{gathered}u\in L^2\big([0,T];H^5\big),\qquad u_t\in L^2\big([0,T];H^3\big),\\ \big(\rho_{tt},\big(\rho^\gamma\big)_{tt}\big)\in L^2\big([0,T];H^2\big),\qquad \big(\rho_{ttt},\big(\rho^\gamma\big)_{ttt}\big)\in L^2\big([0,T];L^2\big),\end{gathered}$$

we apply the embedding theorem in [6] to get

$$u\in C\big([0,T];H^4\big),\qquad \big(\rho_{tt},\big(\rho^\gamma\big)_{tt}\big)\in C\big([0,T];H^1\big).\tag{3.25}$$

By (3.25), $u\in L^2([0,T];H^5)$, $(1.1)_1$, (2.22) and [3], we have

$$\rho\in C\big([0,T];H^4\big),\qquad \rho^\gamma\in C\big([0,T];H^4\big),\qquad \big(\rho_t,\big(\rho^\gamma\big)_t\big)\in C\big([0,T];H^3\big).$$

The proof of Theorem 1.2 is complete. □

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