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Continuity properties of the solution map for the generalized reduced Ostrovsky equation

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ABSTRACT

It is shown that the data-to-solution map for the generalized reduced Ostrovsky (gRO) equation is not uniformly continuous on bounded sets in Sobolev spaces on the circle with exponent $s > 3/2$. Considering that for this range of exponents the gRO equation is well posed with continuous dependence on initial data, this result makes the continuity of the solution map an optimal property. However, if a weaker H^r-topology is used then it is shown that the solution map becomes Hölder continuous in H^s.

1. Introduction and results

We consider the initial value problem for the generalized reduced Ostrovsky equation (gRO)

$$\partial_t u + \frac{1}{k+1}\partial_x\left(u^{k+1}\right) - \gamma\,\partial_x^{-1} u = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{1.2}$$

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where k is a positive integer, γ is a constant, $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ is the torus, and $H^s(\mathbb{T})$ is the Sobolev space on the torus with exponent s. For $s>3/2$ we prove that the Cauchy problem (1.1)–(1.2) is locally well posed in $H^s(\mathbb{T})$ and that the data-to-solution map is continuous but not uniformly continuous. Furthermore, we show that the solution map is Hölder continuous in $H^s(\mathbb{T})$ if it is equipped with an $H^r(\mathbb{T})$-norm, $0\leqslant r<s$.

The gRO equation appears in the literature more often in the following local form

$$\left(u_t+u^k u_x\right)_x=\gamma u. \tag{1.3}$$

When $k=1$ this equation is obtained from the Ostrovsky equation found in Ostrovsky [20]

$$(u_t+c_0u_x+\alpha uu_x+\beta u_{xxx})_x=\gamma u. \tag{1.4}$$

By choosing $\beta=0$, the equation is called the reduced Ostrovsky equation in Cai, Xie, and Yang [2], in Parkes [21] and in Stepanyants [27]. With $c_0=0=\beta$ and $\alpha=1=\gamma$, it is called the Ostrovsky–Hunter equation by Boyd [1] and the Vakhnenko equation by Morrison, Parkes and Vakhnenko [19] and Vakhnenko and Parkes [32]. This equation was first presented by Vakhnenko [31] to describe high-frequency waves in a relaxing medium. Furthermore, it has been shown to be integrable in the sense that an inverse scattering problem can be formulated [32]. Hunter [13] noted that writing the equation as (1.4) meant that there is no long wave dispersion. The equation would now model waves with a wave frequency that is significantly larger than the Coriolis frequency. A numerical solution was then found with an initial guess of a sine wave for the lowest amplitude wave. Hunter showed that the equation is well posed if the initial condition and the corresponding solution $u(x,t)$ are mean zero. An exact N loop soliton solution was found for an arbitrary integer $N\geqslant 2$ in [19]. A transformation of the independent variables enabled the use of Hirota's method. The solution was then found in its implicit form. In Boyd [1], it is noted that the dilational symmetry of the Ostrovsky–Hunter equation allows a restriction to the torus without loss of generality. Since the equation has traveling wave and limiting wave properties that are found in other wave equations, they are used in finding a solution. The method of matched asymptotic expansions is used to find an approximation and the derived approximations are then tested for accuracy.

For $k=2$ this equation becomes the short pulse equation (SPE). It is used in nonlinear optics as a model for very short pulse propagation in nonlinear media by Schäfer and Wayne [25]. It has been shown that the SPE provides a better approximation to the solution of Maxwell's equation than the nonlinear Schrödinger equation (NLSE). Sakovich and Sakovich discovered that the SPE is integrable as well by finding a Lax pair and then transforming the equation into the sine-Gordon equation [23]. This allowed the authors then to show it has explicit analytical solutions of loop and breather form [24]. Global well-posedness with small initial data was shown in Pelinovsky and Sakovich [22]. Studying the SPE under its guise as the Ostrovsky–Hunter equation, Liu, Pelinovsky and Sakovich found sufficient conditions for wave breaking on an infinite line and in a periodic domain [18]. Local well-posedness for $s\geqslant 2$ was shown and conditions for wave breaking are found that are sharper than previous results. From an applications standpoint, the accuracy of the SPE increases as the pulse width shortens. This has led to the derivation of a regularized SPE which has smooth traveling wave solutions, under appropriate conditions by Costanzino, Manukian and Jones [4].

Well-posedness for the Ostrovsky equation ($\gamma\neq 0$) has been presented in many sources. We consider a few here and refer the reader to the references of the papers for further study. Local-in-time well-posedness for $H^s(\mathbb{R})$, $s>3/2$, was proved using parabolic regularization while continuous dependence came from Bona–Smith approximations in Varlamov and Liu [33]. This result was extended to $s>3/4$ in Linares and Milanés [17]. The result was further extended for $s\geqslant -1/8$ locally and globally to $L^2(\mathbb{R})$ by Huo and Jia [14]. Using a variation of Bourgain's called the "I-method", global well-posedness in $H^s(\mathbb{R})$ for $s>-3/10$ is shown in Isaza and Mejía [15]. Other well-posedness results are presented in [7,30,34].

The first result of our work states that the periodic initial value problem for the gRO equation is well posed in the sense of Hadamard in Sobolev spaces with exponent greater than 3/2. More precisely, we have the following result.

Theorem 1.1. *If $s > 3/2$ and $u_0 \in H^s(\mathbb{T})$ then there exist $T > 0$ and a unique solution $u \in C([0,T]; H^s(\mathbb{T}))$ of the initial value problem* (1.1)–(1.2)*, which depends continuously on the initial data u_0. Furthermore, we have the estimate*

$$\left\|u(t)\right\|_{H^s(\mathbb{T})} \leqslant \sqrt[k]{2} \cdot e^{c_s T} \cdot \|u_0\|_{H^s(\mathbb{T})}, \quad \textit{for } 0 \leqslant t \leqslant T \leqslant \frac{1}{2kc_s} \ln\left(1 + \frac{1}{\|u_0\|^k_{H^s(\mathbb{T})}}\right), \tag{1.5}$$

where $c_s > 0$ is a constant depending on s.

Well-posedness of the gRO equation on the line and for $s > 3/2$ has been proved by Stefanov, Shen and Kevrekidis [26]. Also, in [26] they proved some global well-posedness results. The proof of Theorem 1.1 is based on a Galerkin-type approximation method, which for quasi-linear symmetric hyperbolic systems can be found in Taylor [28,29].

The second result of this paper, which is motivated by the works of Himonas and Holliman [8], Himonas and Kenig [9], Himonas, Kenig and Misiołek [10], Himonas, Misiołek and Ponce [11], and Holliman [12], demonstrates that continuous dependence of the solution on initial data is sharp.

Theorem 1.2. *If $s > 3/2$ then the data-to-solution map for the generalized reduced Ostrovsky equation defined by the Cauchy problem* (1.1)–(1.2) *is not uniformly continuous from any bounded subset of $H^s(\mathbb{T})$ into $C([0,T]; H^s(\mathbb{T}))$.*

The proof of the analogous result for the Camassa–Holm equation relies on the conservation of the H^1-norm [9,10], while the proof of the corresponding result for the Degasperis–Procesi equation relies on the conservation of a twisted L^2-norm [8]. However, the method of the proof for Theorem 1.2 does not invoke any conserved quantities as in Grayshan [6]. To demonstrate the sharpness of continuity, sequences of approximate solutions that contain terms of both high and low frequency are constructed. The actual solutions are then found by solving the Cauchy problem with initial data given by the approximate solutions at time $t = 0$. Then the error produced by solving the Cauchy problem and by using approximate solutions is shown to be inconsequential. The proof is also based on the well-posedness result Theorem 1.1 and in particular on estimate for the size of the solution and its lifespan.

We note that our method of proof of well-posedness does not rely upon a fixed point theorem for contraction mappings. Since there is not uniform continuity on the space where the solutions live, the local well-posedness of gRO in $H^s(\mathbb{T})$ cannot be shown by use of this method.

Although the data-to-solution map is not uniformly continuous on $H^s(\mathbb{T})$, in our third result we show that if a properly weakened topology is chosen then it becomes Hölder continuous. More precisely, we prove the following result.

Theorem 1.3. *Let $s > 3/2$ and $0 \leqslant r < s$. Then the data-to-solution map of the Cauchy problem for the generalized reduced Ostrovsky equation* (1.1)–(1.2) *is Hölder continuous with exponent*

$$\alpha = \begin{cases} 1, & \textit{if } 0 \leqslant r \leqslant s-1, \\ s-r, & \textit{if } s-1 < r < s \end{cases} \tag{1.6}$$

as a map from $B(0,\rho) \subset H^s(\mathbb{T})$, with $H^r(\mathbb{T})$-norm, into $C([0,T]; H^r(\mathbb{T}))$. More precisely, we have

$$\left\|u(t) - w(t)\right\|_{C([0,T]; H^r(\mathbb{T}))} \leqslant c\left\|u(0) - w(0)\right\|^{\alpha}_{H^r(\mathbb{T})} \tag{1.7}$$

for all $u(0), w(0) \in B(0,\rho) = \{u \in H^s(\mathbb{T}): \|u\|_{H^s(\mathbb{T})} < \rho\}$ *and* $u(t)$, $w(t)$ *the solutions corresponding to the initial data* $u(0)$, $w(0)$*, respectively. The lifespan* $T > 0$ *and the constant* $c > 0$ *depend on* s*,* r *and* ρ*.*

The proof of Theorem 1.3 follows the work of Chen, Liu, and Zhang [3] and Grayshan [6]. The paper is structured as follows. In Section 2, we define the inverse partial and approximate solutions, estimate the $H^\sigma(\mathbb{T})$ norm of the error, and construct sequences of initial data that demonstrate the non-uniformity of the data-to-solution map. In Section 3, we provide an overview of the local well-posedness result for $s > 3/2$ with an accompanying solution size estimate. Finally, in Section 4, we prove Theorem 1.3.

2. Proof of Theorem 1.2

Before giving the proof of the non-uniform dependence for the gRO equation we shall state the definition of the inverse ∂_x^{-1} and its basic properties. A more detailed discussion of ∂_x^{-1} can be found in Fokas and Himonas [5] and Holliman [12]. For $f \in H^s(\mathbb{T})$ the inverse of ∂_x is defined by the formula

$$\partial_x^{-1} f(x) \doteq \tilde{\partial}_x^{-1} f(x) - \frac{1}{2\pi} \int_0^{2\pi} \tilde{\partial}_y^{-1} f(y)\, dy, \tag{2.1}$$

where $\tilde{\partial}_x^{-1}$ is given by

$$\tilde{\partial}_x^{-1} f(x) \doteq \int_0^x f(y)\, dy - \frac{x}{2\pi} \int_0^{2\pi} f(y)\, dy.$$

It defines a continuous linear map $\partial_x^{-1} : H^s(\mathbb{T}) \to \dot{H}^{s+1}(\mathbb{T})$ satisfying the estimate

$$\left\| \partial_x^{-1} f \right\|_{\dot{H}^{s+1}(\mathbb{T})} \leqslant \|f\|_{\dot{H}^s(\mathbb{T})} \leqslant \|f\|_{H^s(\mathbb{T})}. \tag{2.2}$$

Also, it is both a left and right inverse of the operator $\partial_x : \dot{H}^{s+1}(\mathbb{T}) \to \dot{H}^s(\mathbb{T})$. Moreover, it satisfies the relations

$$\begin{aligned} \partial_x^{-1} \partial_x f(x) &= f(x) - \widehat{f}(0), \quad f \in H^s(\mathbb{T}), \\ \partial_x \partial_x^{-1} f(x) &= f(x) - \frac{1}{2\pi} \widehat{f}(0), \quad f \in H^s(\mathbb{T}). \end{aligned}$$

Since for any $r \geqslant 0$ and $g \in \dot{H}^r(\mathbb{T})$ we have $\|g\|_{H^r(\mathbb{T})} \leqslant 2^{r/2} \|f\|_{\dot{H}^r(\mathbb{T})}$, from (2.2) we also obtain the following inequality

$$\left\| \partial_x^{-1} f \right\|_{H^r(\mathbb{T})} \leqslant 2^{r/2} \left\| \partial_x^{-1} f \right\|_{\dot{H}^r(\mathbb{T})} \leqslant 2^{r/2} \|f\|_{\dot{H}^{r-1}(\mathbb{T})}. \tag{2.3}$$

Recall that for any real number s the Sobolev space $H^s(\mathbb{T})$ is defined by the norm

$$\|f\|_{H^s(\mathbb{T})}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \left(1 + k^2\right)^s \left|\widehat{f}(k)\right|^2.$$

The homogeneous Sobolev space $\dot{H}^s(\mathbb{T})$ is the subspace of $H^s(\mathbb{T})$ defined by the condition $\widehat{f}(0) = 0$ and its norm is defined by

$$\|f\|^2_{\dot{H}^s(\mathbb{T})} = \sum_{k\in\dot{\mathbb{Z}}} |k|^{2s} \big|\widehat{f}(k)\big|^2,$$

where $\dot{\mathbb{Z}} = \mathbb{Z} - \{0\}$. Here, the Fourier transform is defined by

$$\widehat{f}(k) = \int_{\mathbb{T}} e^{-ixk} f(x)\,dx.$$

Next, we will prove Theorem 1.2 for Sobolev exponents $s > 3/2$. The basis of our proof rests upon finding two sequences of solutions, $\{u_n\}$, $\{v_n\}$ in $C([0,T]; H^s(\mathbb{T}))$, to the gRO i.v.p. (1.1)–(1.2) that share a common lifespan and satisfy

$$\big\|u_n(t)\big\|_{H^s(\mathbb{T})} + \big\|v_n(t)\big\|_{H^s(\mathbb{T})} \lesssim 1,$$
$$\lim_{n\to\infty}\big\|u_n(0) - v_n(0)\big\|_{H^s(\mathbb{T})} = 0,$$

and

$$\liminf_n \big\|u_{1,n}(t) - u_{-1,n}(t)\big\|_{H^s(\mathbb{T})} \gtrsim \big|\sin(t)\big| \quad \text{for } t\in(0,T] \text{ for } k\text{-odd},$$
$$\liminf_n \big\|u_{1,n}(t) - u_{0,n}(t)\big\|_{H^s(\mathbb{T})} \gtrsim \big|\sin(t/2)\big| \quad \text{for } t\in(0,T] \text{ for } k\text{-even}.$$

2.1. Approximate gRO solutions

For any n, a positive integer, we define the approximate solution $u^{\omega,n} = u^{\omega,n}(x,t)$ as

$$u^{\omega,n}(x,t) \doteq \omega n^{-1/k} + n^{-s}\cos(nx - \omega t), \tag{2.4}$$

where

$$\omega = -1, 1 \quad \text{if } k \text{ is odd} \quad \text{and} \quad \omega = 0, 1 \quad \text{if } k \text{ is even}. \tag{2.5}$$

Then substituting the above approximate solutions into gRO (1.1) gives the error

$$\begin{aligned} E(t) = E = {} & \omega n^{-s}\sin(nx - \omega t) \\ & + \big[\omega n^{-1/k} + n^{-s}\cos(nx - \omega t)\big]^k \cdot \big[-n^{-s+1}\sin(nx-\omega t)\big] \\ & + \gamma\,\partial_x^{-1}\big[u^{\omega,n}\big]. \end{aligned} \tag{2.6}$$

Taking into account that for ω satisfying condition (2.5) we have $\omega^k = \omega$; thus, the first term in the error, which has H^s norm of 1, cancels. Therefore, we have

$$E = -\sum_{j=1}^{k} \binom{k}{j} \omega^{k-j} n^{-\frac{1}{k}(k-j)-js-s+1} \cos^j(nx-\omega t)\sin(nx-\omega t) + \gamma\,\partial_x^{-1}\big[u^{\omega,n}\big].$$

Next, using the identity $\sin(2\alpha) = 2\sin\alpha\cos\alpha$, we write E in the following form

$$E = -\frac{1}{2}\sum_{j=1}^{k}\binom{k}{j}\omega^{k-j}n^{-\frac{1}{k}(k-j)-js-s+1}\cos^{j-1}(nx-\omega t)\sin\big[2(nx-\omega t)\big] + \gamma\,\partial_x^{-1}\big[u^{\omega,n}\big].$$

Finally, using the formulas

$$\big\|\cos(nx-\alpha)\big\|_{H^\sigma(\mathbb{T})} \approx n^\sigma \quad \text{and} \quad \big\|\sin(nx-\alpha)\big\|_{H^\sigma(\mathbb{T})} \approx n^\sigma \tag{2.7}$$

and the Algebra Property, we estimate the H^σ-norm of E as follows

$$\|E\|_{H^\sigma(\mathbb{T})} \leqslant \frac{1}{2}\sum_{j=1}^{k}\binom{k}{j}\omega^{k-j}n^{-\frac{1}{k}(k-j)-js-s+1}n^{(j-1)\sigma}\cdot n^\sigma + |\gamma|\big\|\partial_x^{-1}u^{\omega,n}\big\|_{H^\sigma(\mathbb{T})}. \tag{2.8}$$

Also, using inequality (2.3) we have

$$\big\|\partial_x^{-1}u^{\omega,n}\big\|_{H^\sigma(\mathbb{T})} \leqslant 2^{\sigma/2}\big\|u^{\omega,n}\big\|_{H^{\sigma-1}(\mathbb{T})} \lesssim n^{-s+(\sigma-1)}. \tag{2.9}$$

Since $|\omega| \leqslant 1$ from (2.8) and (2.9) we get

$$\|E\|_{H^\sigma(\mathbb{T})} \lesssim \sum_{j=1}^{k} n^{(j-k)\frac{1}{k}-(j+1)s+1+\sigma(j-1+1)} + n^{-s+(\sigma-1)} = \sum_{j=1}^{k} n^{j(\frac{1}{k}-s+\sigma)-s} + n^{-s-1+\sigma}. \tag{2.10}$$

Recall that $\sigma < s-1 < s-\frac{1}{k}$. Thus, $\sigma - s + \frac{1}{k} \leqslant s - \frac{1}{k} - s + \frac{1}{k} = 0$. Along with $k \geqslant 1$, we now have

$$\sum_{j=1}^{k} n^{j(\frac{1}{k}-s+\sigma)-s} \lesssim n^{\frac{1}{k}-s+\sigma-s} \lesssim n^{1-s+\sigma-s} = n^{1-2s+\sigma}. \tag{2.11}$$

Therefore, from (2.10) and (2.11) we see that for $s > 3/2$ the error of the approximate solutions satisfies the estimate

$$\|E\|_{H^\sigma(\mathbb{T})} \lesssim n^{-r_s}, \tag{2.12}$$

where

$$r_s = \begin{cases} 2s-\sigma-1, & \text{if } s \leqslant 2, \\ s+1-\sigma, & \text{if } s \geqslant 2. \end{cases} \tag{2.13}$$

2.2. Actual gRO solutions

Let $u_{\omega,n}(x,t)$ be the actual solutions to the gRO i.v.p. (1.1)–(1.2) with initial data $u_{\omega,n}(x,0) = u^{\omega,n}(x,0)$. More precisely, $u_{\omega,n}(x,t)$ solves

$$\partial_t u_{\omega,n} = -u_{\omega,n}^k \partial_x u_{\omega,n} + \gamma \partial_x^{-1} u_{\omega,n}, \tag{2.14}$$

$$u_{\omega,n}(x,0) = \omega n^{-1/k} + n^{-s}\cos(nx), \quad x \in \mathbb{T},\ t \in \mathbb{R}. \tag{2.15}$$

Notice that (2.7) implies that the initial data $u_{\omega,n}(x,0) \in H^s(\mathbb{T})$ for all $s \geqslant 0$, since

$$\|u_{\omega,n}(\cdot,0)\|_{H^s(\mathbb{T})} \leqslant \omega n^{-1/k} + n^{-s}\|\cos(nx)\|_{H^s(\mathbb{T})} \approx 1,$$

for n sufficiently large. Hence, by Theorem 1.1, there is a $T > 0$ such that for $n > 1$, the Cauchy problem (2.14)–(2.15) has a unique solution in $C([0,T]; H^s(\mathbb{T}))$ with lifespan $T > 0$ such that $u^{\omega,n}$ satisfies (1.5) for $t \in [0,T]$.

To estimate the difference between the actual solutions and the approximate solutions, we define $v = u^{\omega,n} - u_{\omega,n}$ which satisfies the following i.v.p.

$$\partial_t v = E - \left[\frac{1}{k+1}\partial_x\left(u^{\omega,n}\right)^{k+1} - \frac{1}{k+1}\partial_x(u_{\omega,n})^{k+1}\right] + \gamma \partial_x^{-1} v,$$

where

$$v(x,0) = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}.$$

For ease of notation, we rewrite the i.v.p. as

$$\partial_t v = E - \frac{1}{k+1}\partial_x[w \cdot v] + \gamma \partial_x^{-1} v, \tag{2.16}$$

where

$$w = \left(u^{\omega,n}\right)^k + \left(u^{\omega,n}\right)^{k-1} u_{\omega,n} + \cdots + u^{\omega,n}(u_{\omega,n})^{k-1} + (u_{\omega,n})^k$$

and E satisfies the estimate in (2.12).

We will now show that the H^s norm of the difference v decays to zero as n goes to infinity. Apply the operator D^σ to both sides of (2.16), multiply by $D^\sigma v$, and integrate over the torus to get

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^\sigma(\mathbb{T})}^2 = \int_{\mathbb{T}} D^\sigma E D^\sigma v\, dx \tag{2.17}$$

$$-\frac{1}{k+1}\int_{\mathbb{T}} D^\sigma \partial_x(w \cdot v) D^\sigma v\, dx \tag{2.18}$$

$$+\gamma \int_{\mathbb{T}} D^\sigma \partial_x^{-1} v D^\sigma v\, dx. \tag{2.19}$$

Applying the Cauchy–Schwarz inequality to the first integral, we get (2.17)

$$\left|(2.17)\right| \leqslant \|E\|_{H^\sigma(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}. \tag{2.20}$$

To estimate the second term, we commute $D^\sigma \partial_x$ and w to find

$$(2.18) \approx \int_{\mathbb{T}} \left[D^\sigma \partial_x, w\right] v \cdot D^\sigma v \, dx \tag{2.21}$$

$$+ \int_{\mathbb{T}} w D^\sigma \partial_x v \cdot D^\sigma v \, dx. \tag{2.22}$$

Starting with (2.21), we apply Cauchy–Schwarz to find

$$\left|(2.21)\right| \leqslant \left\|\left[D^\sigma \partial_x, w\right] v\right\|_{L^2(\mathbb{T})} \left\|D^\sigma v\right\|_{L^2(\mathbb{T})}. \tag{2.23}$$

The commutator can be handled by the following Calderon–Coifman–Meyer type commutator estimate that can be found in Himonas, Kenig and Misiołek [10].

Lemma 2.1. *If $\sigma + 1 \geqslant 0$, $\rho > 3/2$ and $\sigma + 1 \leqslant \rho$, then*

$$\left\|\left[D^\sigma \partial_x, w\right] v\right\|_{L^2} \leqslant C\|w\|_{H^\rho}\|v\|_{H^\sigma}. \tag{2.24}$$

Applying Lemma 2.1 with $\sigma + 1 \geqslant 0$, $\rho = s > 3/2$ and $\sigma + 1 \leqslant s$ tells us

$$\left|(2.21)\right| \lesssim \|w\|_{H^s(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})} = \|w\|_{H^s(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{2.25}$$

For the second term, we apply integration-by-parts to find

$$\left|(2.22)\right| \approx \left|\int_{\mathbb{T}} \partial_x w \left(D^\sigma v\right)^2 dx\right|. \tag{2.26}$$

Applying Hölder's inequality and the Sobolev Embedding Theorem with $s > 3/2$ results in

$$\left|(2.22)\right| \lesssim \|\partial_x w\|_{L^\infty(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})} \lesssim \|w\|_{H^s(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})}. \tag{2.27}$$

We now consider the non-local term, where we apply the Cauchy–Schwarz inequality and use the continuity of ∂_x^{-1} to find

$$\left|(2.19)\right| \lesssim \|v\|^2_{H^\sigma(\mathbb{T})}. \tag{2.28}$$

Finally, combining (2.20), (2.25), (2.27) and (2.28) we obtain an energy estimate for v

$$\frac{1}{2}\frac{d}{dt}\left\|v(t)\right\|^2_{H^\sigma(\mathbb{T})} \lesssim \|E\|_{H^\sigma(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})} + \|w\|_{H^s(\mathbb{T})}\|v\|^2_{H^\sigma(\mathbb{T})} + \|v\|^2_{H^\sigma(\mathbb{T})}. \tag{2.29}$$

We shall show that $\|w\|_{H^s(\mathbb{T})} \lesssim 1$. Substituting in the definition of w gives us

$$\|w\|_{H^s(\mathbb{T})} = \left\|\left(u^{\omega,n}\right)^k + \left(u^{\omega,n}\right)^{k-1} u_{\omega,n} + \cdots + (u_{\omega,n})^k\right\|_{H^s(\mathbb{T})}.$$

The triangle inequality, the Algebra Property and (2.7) result in

$$\|w\|_{H^s(\mathbb{T})} \lesssim n^{-1/k} + 1.$$

The actual solutions are bounded by the solution size estimate in Theorem 1.1

$$\|u_{\omega,n}(t)\|_{H^s(\mathbb{T})} \lesssim \|u_{\omega,n}(0)\|_{H^s(\mathbb{T})} \lesssim n^{-1/k} + 1.$$

We consider a general term $\|u^{\omega,n}\|_{H^s(\mathbb{T})}^{j-k}\|u_{\omega,n}\|_{H^s(\mathbb{T})}^{j}$ and use the above two inequalities to find

$$\begin{aligned}\|w\|_{H^s(\mathbb{T})} &\leqslant \|u^{\omega,n}\|_{H^s(\mathbb{T})}^{k} + \|u^{\omega,n}\|_{H^s(\mathbb{T})}^{k-1}\|u_{\omega,n}\|_{H^s(\mathbb{T})} + \cdots + \|u_{\omega,n}\|_{H^s(\mathbb{T})}^{k} \\ &\lesssim \left(n^{-1/k}+1\right)^k + \left(n^{-1/k}+1\right)^{k-1}\left(n^{-1/k}+1\right) + \cdots + \left(n^{-1/k}+1\right)^k \lesssim 1^k = 1.\end{aligned}$$

Thus, we have

$$\|w\|_{H^s(\mathbb{T})} \lesssim 1. \tag{2.30}$$

We can refine (2.29) by first using (2.30) to write

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^\sigma(\mathbb{T})}^2 \lesssim \|E\|_{H^\sigma(\mathbb{T})}\|v\|_{H^\sigma(\mathbb{T})} + \|v\|_{H^\sigma(\mathbb{T})}^2. \tag{2.31}$$

Solving (2.31) and using the error estimate (2.12) gives

$$\|v(t)\|_{H^\sigma} \lesssim n^{-r_s}, \tag{2.32}$$

which shows that the H^σ-norm of $v(t)$ decays for all $t \in [0,T]$. Also from the well-posedness estimates for gRO we have that

$$\|v(t)\|_{H^{s+1}} \lesssim \|u^{\omega,n}(0)\|_{H^{s+1}} \lesssim n, \tag{2.33}$$

which shows that the H^{s+1}-norm of $v(t)$ may grow for $t \in [0,T]$. Next, using interpolation between σ and $s+1$ we show that the H^s-norm of $v(t)$ decays. In fact, using (2.32) and (2.33) we have

$$\|v(t)\|_{H^s} \leqslant \|v(t)\|_{H^\sigma}^{1/(s+1-\sigma)}\|v(t)\|_{H^{s+1}}^{(s-\sigma)/(s+1-\sigma)} \lesssim n^{\frac{-1}{s+1-\sigma}(r_s-s+\sigma)}.$$

Finally, from this inequality and the definition of r_s (2.35) we see that for given $s > 3/2$ and $1/2 < \sigma < s-1$ the difference $v(t)$ between approximate solutions and solutions with the same initial data satisfies the estimate

$$\|v(t)\|_{H^s(\mathbb{T})} \lesssim n^{-\rho_s}, \quad t \in [0,T], \tag{2.34}$$

where

$$\rho_s = \begin{cases} (s-1)/(s+1-\sigma), & \text{if } s \leqslant 2, \\ 1/(s+1-\sigma), & \text{if } s \geqslant 2. \end{cases} \tag{2.35}$$

Proof of non-uniform dependence. Here we will prove Theorem 1.2 for Sobolev exponents $s > 3/2$. The basis of our proof rests upon finding two sequences of solutions to the gRO i.v.p. (1.1)–(1.2) that share a common lifespan and satisfy three conditions. For k-odd, we take the sequence of solutions with $\omega = \pm 1$ and for k-even the sequence of solutions with $\omega = 0, 1$. The three conditions they satisfy are as follows

(1) $\|u_{\omega,n}(t)\|_{H^s(\mathbb{T})} \lesssim 1$ for $t \in [0, T]$,
(2) $\|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s(\mathbb{T})} \to 0$ as $n \to \infty$ for k-odd,
$\|u_{1,n}(0) - u_{0,n}(0)\|_{H^s(\mathbb{T})} \to 0$ as $n \to \infty$ for k-even, and
(3) $\liminf_n \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s(\mathbb{T})} \gtrsim |\sin(t)|$ for $t \in (0, T]$ for k-odd,
$\liminf_n \|u_{1,n}(t) - u_{0,n}(t)\|_{H^s(\mathbb{T})} \gtrsim |\sin(t/2)|$ for $t \in (0, T]$ for k-even.

Property (1) for k-even or odd follows from the solution size estimate in Theorem 1.1. We have

$$\left\|u_{\omega,n}(t)\right\|_{H^s(\mathbb{T})} \lesssim \left\|u^{\omega,n}(0)\right\|_{H^s(\mathbb{T})} \lesssim 1.$$

Property (2), for k-odd, follows from the definition of our approximate solutions (2.4). We have

$$\begin{aligned}\left\|u_{1,n}(0) - u_{-1,n}(0)\right\|_{H^s(\mathbb{T})} &= \left\|u^{1,n}(0) - u^{-1,n}(0)\right\|_{H^s(\mathbb{T})} \\ &= \left\|n^{-1/k} + n^{-s}\cos(nx) + n^{-1/k} - n^{-s}\cos(nx)\right\|_{H^s(\mathbb{T})} \\ &= 4\pi n^{-1/k} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Similarly, for k-even we have

$$\begin{aligned}\left\|u_{1,n}(0) - u_{0,n}(0)\right\|_{H^s(\mathbb{T})} &= \left\|u^{1,n}(0) - u^{0,n}(0)\right\|_{H^s(\mathbb{T})} \\ &= \left\|n^{-1/k} + n^{-s}\cos(nx) - n^{-s}\cos(nx)\right\|_{H^s(\mathbb{T})} \\ &= 2\pi n^{-1/k} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

For property (3), we consider k-odd first. Using the reverse triangle inequality we get

$$\begin{aligned}\left\|u_{1,n}(t) - u_{-1,n}(t)\right\|_{H^s} &\geqslant \left\|u^{1,n}(t) - u^{-1,n}(t)\right\|_{H^s} - \left\|u^{1,n}(t) - u_{1,n}(t)\right\|_{H^s} - \left\|u^{-1,n}(t) - u_{-1,n}(t)\right\|_{H^s} \\ &\gtrsim \left\|u^{1,n}(t) - u^{-1,n}(t)\right\|_{H^s} - n^{-\rho_s},\end{aligned}$$

from which we obtain that

$$\liminf_{n\to\infty}\left\|u_{1,n}(t) - u_{-1,n}(t)\right\|_{H^s} \gtrsim \liminf_{n\to\infty}\left\|u^{1,n}(t) - u^{-1,n}(t)\right\|_{H^s}. \tag{2.36}$$

Since, by the trigonometric identity $\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta$, we have

$$u^{1,n}(t) - u^{-1,n}(t) = 2n^{1/k} + 2n^{-s}\sin(nx)\sin(t),$$

inequality (2.36) gives

$$\liminf_{n\to\infty}\left\|u_{1,n}(t) - u_{-1,n}(t)\right\|_{H^s(\mathbb{T})} \gtrsim \liminf_{n\to\infty}\left(\left|\sin(t)\right| - n^{1/k}\right) \gtrsim \left|\sin(t)\right|, \tag{2.37}$$

which completes the proof of property (3) in the case that k is odd.

We now consider k-even. Similarly, using the reverse triangle inequality, the definition of approximate solutions, and the fact that the difference between solutions and approximate solutions decays, we get

$$\liminf_{n\to\infty}\left\|u_{1,n}(t) - u_{0,n}(t)\right\|_{H^s} \gtrsim \liminf_{n\to\infty}\left\|u^{1,n}(t) - u^{0,n}(t)\right\|_{H^s}, \tag{2.38}$$

where now

$$u^{1,n}(t) - u^{0,n}(t) = n^{1/k} + 2n^{-s}\sin(nx - t/2)\sin(t/2).$$

Therefore (2.38) gives

$$\liminf_{n\to\infty}\big\|u_{1,n}(t) - u_{0,n}(t)\big\|_{H^s(\mathbb{T})} \gtrsim \liminf_{n\to\infty}\big(\big|\sin(t/2)\big| - n^{1/k}\big) \gtrsim \big|\sin(t/2)\big|, \tag{2.39}$$

which completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.1

The generalized reduced Ostrovsky equation (1.1) (as is) cannot be thought of as an o.d.e. on the space $H^s(\mathbb{T})$. Specifically, we note that for $u \in H^s(\mathbb{T})$ the product $u^k\partial_x u \in H^{s-1}(\mathbb{T})$. This difficulty is overcome by showing the existence of solutions to a mollified smooth version of the gRO i.v.p., which we later show gives rise to solutions of (1.1)–(1.2). Consider the mollified smooth version of the gRO i.v.p.

$$\partial_t u = -J_\varepsilon\big[(J_\varepsilon u)^k \cdot \partial_x J_\varepsilon u\big] + \gamma\partial_x^{-1}u := F_\varepsilon, \tag{3.1}$$

$$u(x,0) = u_0(x), \quad u_0 \in H^s(\mathbb{T}),\ x \in \mathbb{T},\ t \in \mathbb{R}, \tag{3.2}$$

where for each $\varepsilon \in (0,1]$, the operator J_ε is the Friedrichs mollifier. We fix a Schwartz function $j \in \mathcal{S}(\mathbb{R})$ that satisfies $0 \leqslant \widehat{j}(\xi) \leqslant 1$ for every $\xi \in \mathbb{R}$ and $\widehat{j}(\xi) = 1$ for $\xi \in [-1,1]$. This allows us to define the periodic functions j_ε as

$$j_\varepsilon(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\widehat{j}(\varepsilon n)e^{inx}.$$

Then J_ε is given by $J_\varepsilon f(x) = j_\varepsilon * f(x)$. This construction of j_ε results in a lemma that will prove repeatedly useful throughout the paper.

Lemma 3.1. *If $r \leqslant s$ and $I - J_\varepsilon : H^s \to H^r$, then*

$$\|f - J_\varepsilon f\|_{H^r} = o\big(\varepsilon^{s-r}\big). \tag{3.3}$$

Proof. Let $f \in H^s(\mathbb{T})$, then applying $H^r(\mathbb{T})$ norm to $(I - J_\varepsilon)f$ gives us

$$\lim_{\varepsilon\to 0}\frac{\|f - J_\varepsilon f\|^2_{H^r(\mathbb{T})}}{\varepsilon^{2(s-r)}} = \lim_{\varepsilon\to 0}\sum_{k\in\mathbb{Z}}\frac{(1+k^2)^r|\widehat{f}(k)|^2|1-\widehat{j}(\varepsilon k)|^2}{\varepsilon^{2(s-r)}}$$

$$\leqslant \lim_{\varepsilon\to 0}\sum_{\substack{k\in\mathbb{Z}\\ |\varepsilon k|>1}}\big|1-\widehat{j}(\varepsilon k)\big|^2\big|\widehat{f}(k)\big|^2\big(1+k^2\big)^s = 0. \qquad \square$$

Using the Algebra Property, we see that the mollified i.v.p. is an o.d.e. on $H^s(\mathbb{T})$. Moreover, for each $\varepsilon \in (0,1]$, the map $F_\varepsilon : H^s(\mathbb{T}) \to H^s(\mathbb{T})$ in (3.1) is continuously differentiable with the derivative at $u_0 \in H^s(\mathbb{T})$ given by

$$F'_\varepsilon(u_0)u = -J_\varepsilon\big(k(J_\varepsilon u_0)^{k-1}J_\varepsilon u \cdot \partial_x J_\varepsilon u_0 + (J_\varepsilon u_0)^k \cdot \partial_x J_\varepsilon u\big) + \gamma\partial_x^{-1}u.$$

Hence, for each $\varepsilon \in (0,1]$, (3.1)–(3.2) has a unique solution u_ε with lifespan $T_\varepsilon > 0$.

3.1. Energy estimate

For each ε, there is a solution u_ε to the mollified gRO (3.1)–(3.2). The lifespan of each of these solutions has a lower bound T_ε. In this subsection, we shall demonstrate that there is actually a lower bound $T > 0$ that does not depend upon ε. This estimate is crucial in our proofs. To show the existence of T, we shall derive an energy estimate for the u_ε. Applying the operator D^s to both sides of (3.1), multiplying by $D^s u_\varepsilon$, and integrating over the torus yields the $H^s(\mathbb{T})$-energy of u_ε

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^s(\mathbb{T})} = -\int_{\mathbb{T}} D^s\big[J_\varepsilon\big((J_\varepsilon u_\varepsilon)^k \cdot \partial_x J_\varepsilon u_\varepsilon\big)\big] \cdot D^s u_\varepsilon \, dx \tag{3.4}$$

$$+\gamma \int_{\mathbb{T}} D^s \partial_x^{-1} u_\varepsilon \cdot D^s u_\varepsilon \, dx. \tag{3.5}$$

To bound the energy, we will need the following Kato–Ponce [16] commutator estimate.

Lemma 3.2 *(Kato–Ponce). If $s > 0$, then there is a $c_s > 0$ such that*

$$\big\|\big[D^s, f\big]g\big\|_{L^2} \leqslant c_s\big(\big\|D^s f\big\|_{L^2}\|g\|_{L^\infty} + \|\partial_x f\|_{L^\infty}\big\|D^{s-1}g\big\|_{L^2}\big). \tag{3.6}$$

To estimate the Burgers term (3.4), we rewrite it using the self-adjointness of J_ε in L^2, commute D^s and v^k, and apply Lemma 3.2 and the Sobolev Embedding Theorem with $s > 3/2$. For the non-local term (3.5), the Cauchy–Schwarz inequality and the continuity of ∂_x^{-1} provide the upper bound. Given that $s > 3/2$, we have the resulting estimate

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(t)\|^2_{H^s(\mathbb{T})} \leqslant c_s\big[\|u_\varepsilon(t)\|^{k+2}_{H^s(\mathbb{T})} + \|u_\varepsilon(t)\|^2_{H^s(\mathbb{T})}\big], \quad 0 \leqslant t \leqslant T_\varepsilon,$$

where c_s is a constant and T_ε is the lifespan of u_ε. Solving this differential inequality we see that the solution u_ε exists at least until time T_0, where

$$T_0 = \frac{1}{kc_s}\ln\left(1 + \frac{1}{\|u_0\|^k_{H^s(\mathbb{T})}}\right).$$

For simplicity, we define $T = T_0/2$ and find

$$\|u_\varepsilon(t)\|^k_{H^s(\mathbb{T})} \leqslant 2\|u_0\|^k_{H^s(\mathbb{T})} e^{kc_s T}, \quad \text{for } 0 \leqslant t \leqslant T,$$

which can be written as (1.5). We have now obtained a lower bound for T_ε and an upper bound for $\|u_\varepsilon\|_{H^s(\mathbb{T})}$ that are both independent of ε.

3.2. Existence of solutions

From the sequence of solutions to the mollified i.v.p. (3.1)–(3.2), we extract a sufficiently refined subsequence which converges to a solution of the gRO i.v.p. (1.1)–(1.2). The main tools utilized in this argument are Alaoglu's Theorem, Ascoli's Theorem, the compactness of the torus, and Rellich's Theorem. After refining the subsequence adequately, we obtain a subsequence which converges to a solution of the gRO i.v.p. (1.1)–(1.2) in $C([0, T]; H^s(\mathbb{T}))$. For more details, we refer the reader to Himonas and Holliman [8].

3.3. Uniqueness

Here we will show the solution to the gRO i.v.p. (1.1)–(1.2) is unique. Our strategy is to develop an energy estimate for the difference of two solutions. Fix the initial data $u_0 \in H^s(\mathbb{T})$ and let u and w be two solutions to the gRO i.v.p. (1.1)–(1.2) with $u(x,0) = u_0(x) = w(x,0) \in H^s(\mathbb{T})$. Then the difference $v = u - w$ satisfies the following Cauchy problem

$$\partial_t v = -\frac{1}{k+1}\partial_x\left((u-w)\cdot\sum_{j=0}^{k} u^j w^{k-j}\right) + \gamma\partial_x^{-1} v,$$

with initial data $v(0) = 0$. For convenience, let $\tilde{w} = \sum_{j=0}^{k} u^j w^{k-j}$. Then we have

$$\partial_t v = -\frac{1}{k+1}\partial_x(v\tilde{w}) + \gamma\partial_x^{-1} v. \tag{3.7}$$

Let $\sigma \in [0, s-1]$. The $H^\sigma(\mathbb{T})$-energy estimate is then given by

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^\sigma(\mathbb{T})} = -\frac{1}{k+1}\int_{\mathbb{T}} D^\sigma\partial_x(v\tilde{w})\cdot D^\sigma v\,dx + \gamma\int_{\mathbb{T}} D^\sigma\partial_x^{-1} v\cdot D^\sigma v\,dx. \tag{3.8}$$

To bound (3.8), we commute $D^\sigma\partial_x$ and $\tilde{w}$, which results in two integrals. The commutator integral is estimated by applying the Cauchy–Schwarz inequality followed by Lemma 2.1 and the solution size estimate (1.5). The second integral is bounded using integration by parts, the Sobolev Embedding Theorem and the solution size bound (1.5). The non-local term is estimated by the Cauchy–Schwarz inequality and the continuity of ∂_x^{-1}. The resulting energy estimate is

$$\frac{1}{2}\frac{d}{dt}\left\|v(t)\right\|^2_{H^\sigma(\mathbb{T})} \lesssim \left\|v(t)\right\|^2_{H^\sigma(\mathbb{T})}, \tag{3.9}$$

which we solve to find the inequality

$$\left\|v(t)\right\|^2_{H^\sigma(\mathbb{T})} \leqslant \left\|v(0)\right\|^2_{H^\sigma(\mathbb{T})} e^{2c_s T}.$$

We recall that $v = u - w$ where u and w are both solutions to the gRO i.v.p. (1.1)–(1.2). This means we have

$$\begin{aligned}\left\|v(t)\right\|_{H^\sigma(\mathbb{T})} &\leqslant \left\|v(0)\right\|_{H^\sigma(\mathbb{T})} e^{c_s T}\\ &\leqslant \|u_0 - u_0\|_{H^\sigma(\mathbb{T})} e^{c_s T} = 0.\end{aligned} \tag{3.10}$$

Thus, we have uniqueness. □

3.4. Continuity of the data-to-solution map

Proposition 3.3. *The data-to-solution map for the gRO i.v.p.* (1.1)–(1.2) *from* $H^s(\mathbb{T})$ *to* $C([0,T]; H^s(\mathbb{T}))$ *given by* $u_0 \mapsto u$ *is continuous.*

Proof. Fix $u_0 \in H^s(\mathbb{T})$ and let $\{u_{0,n}\} \subset H^s(\mathbb{T})$ be a sequence such that

$$\lim_{n\to\infty} u_{0,n} = u_0 \quad \text{in } H^r(\mathbb{T}).$$

Then if u is the solution to the gRO i.v.p. (1.1)–(1.2) with initial data u_0 and if u_n is the solution to the gRO i.v.p. with initial data $u_{0,n}$, we will prove that

$$\lim_{n\to\infty} u_n = u, \quad \text{in } C\big([0,T]; H^s\big).$$

Our approach is to use energy estimates. To avoid some of the difficulties of estimating the term involving $u^k \partial_x u$, we will use the J_ε convolution operator to smooth out the initial data. Let $\varepsilon \in (0,1]$. We take u^ε to be the solution to the gRO i.v.p. with smoothed initial data $J_\varepsilon u_0 = j_\varepsilon * u_0$. Similarly, let u_n^ε be the solution with initial data $J_\varepsilon u_{0,n}$. Applying the triangle inequality, we arrive at

$$\|u - u_n\|_{C([0,T];H^s)} \leqslant \big\|u - u^\varepsilon\big\|_{C([0,T];H^s)} + \big\|u^\varepsilon - u_n^\varepsilon\big\|_{C([0,T];H^s)} + \big\|u_n^\varepsilon - u_n\big\|_{C([0,T];H^s)}. \tag{3.11}$$

We will prove that for any $\eta > 0$, there exists an N such that for all $n > N$, each of these terms can be bounded by $\eta/3$ for suitable choices of ε and N. We note that the choice of a sufficiently small ε will be independent of N and will only depend on η; whereas, the choice of N will depend on both η and ε. However, after ε has been chosen, N can be chosen so as to force each of the three terms to be small.

3.4.1. Estimation of $\|u^\varepsilon - u_n^\varepsilon\|_{C([0,T];H^s(\mathbb{T}))}$

We can bound this term directly using an $H^s(\mathbb{T})$-energy estimate. Let $v = v(n,\varepsilon) = u^\varepsilon - u_n^\varepsilon$. Then v satisfies the following Cauchy problem

$$\partial_t v = -\frac{1}{k+1}\partial_x\left[\big(u^\varepsilon - u_n^\varepsilon\big)\sum_{j=0}^{k}\big(u^\varepsilon\big)^j\big(u_n^\varepsilon\big)^{k-j}\right] + \gamma\,\partial_x^{-1}(v),$$

where $v(0) = u^\varepsilon(0) - u_n^\varepsilon(0) = J_\varepsilon u_0 - J_\varepsilon u_{0,n}$. For ease of notation, define

$$\tilde{w} = \sum_{j=0}^{k}\big(u^\varepsilon\big)^j\big(u_n^\varepsilon\big)^{k-j} \tag{3.12}$$

and we can write

$$\partial_t v = -\frac{1}{k+1}\partial_x(v\tilde{w}) + \gamma\,\partial_x^{-1} v. \tag{3.13}$$

Apply the operator D^s to both sides of (3.13), multiply by $D^s v$ and integrate over the torus to obtain the $H^s(\mathbb{T})$-energy

$$\frac{1}{2}\frac{d}{dt}\big\|v(t)\big\|_{H^s(\mathbb{T})}^2 = -\frac{1}{k+1}\int_{\mathbb{T}} D^s\partial_x(v\tilde{w})\cdot D^s v\,dx \tag{3.14}$$

$$+\gamma\int_{\mathbb{T}} D^s\partial_x^{-1} v\cdot D^s v\,dx. \tag{3.15}$$

To estimate (3.14), we commute $D^s\partial_x$ and $\tilde{w}$ and apply Lemma 2.1 and the Sobolev Embedding theorem to get

$$\left|(3.14)\right| \lesssim \|\tilde{w}\|_{H^{s+1}(\mathbb{T})}\|v\|^2_{H^s(\mathbb{T})}. \tag{3.16}$$

We shall consider $\|\tilde{w}\|_{H^{s+1}(\mathbb{T})}$. From our construction of J_ε, we can see that our initial data $J_\varepsilon u_0$, $J_\varepsilon u_{0,n} \in C^\infty$. Therefore, we may apply our solution size estimate (1.5), in conjunction with the Algebra Property for $s+1 > 1/2$, to the definition of $\|\tilde{w}\|_{H^{s+1}(\mathbb{T})}$ to find

$$\|\tilde{w}\|_{H^{s+1}(\mathbb{T})} \lesssim \sum_{j=0}^{k} \|J_\varepsilon u_0\|^j_{H^{s+1}(\mathbb{T})}\|J_\varepsilon u_{0,n}\|^{k-j}_{H^{s+1}(\mathbb{T})}.$$

In examining $\|J_\varepsilon u_0\|_{H^{s+1}(\mathbb{T})}$ we shall use that if $f \in H^{s+1}(\mathbb{T})$, then

$$\|f\|_{H^{s+1}(\mathbb{T})} \leqslant \|f\|_{H^s(\mathbb{T})} + \|\partial_x f\|_{H^s(\mathbb{T})}.$$

Using this inequality and $\|\partial_x J_\varepsilon f\|_{H^s(\mathbb{T})} \leqslant \frac{\alpha}{\varepsilon}\|f\|_{H^s(\mathbb{T})}$, we can write

$$\|J_\varepsilon u_0\|_{H^{s+1}(\mathbb{T})} \leqslant \|J_\varepsilon u_0\|_{H^s(\mathbb{T})} + \|\partial_x J_\varepsilon u_0\|_{H^s(\mathbb{T})} \lesssim \frac{1}{\varepsilon}.$$

Similarly, we have

$$\|J_\varepsilon u_{0,n}\|_{H^{s+1}(\mathbb{T})} \lesssim \frac{1}{\varepsilon}\|u_0\|_{H^s(\mathbb{T})} \lesssim \frac{1}{\varepsilon}.$$

Therefore, we have

$$(3.16) \lesssim \frac{1}{\varepsilon}\|v\|^2_{H^s(\mathbb{T})}. \tag{3.17}$$

Now we consider the non-local term (3.15). Using the Cauchy–Schwarz inequality and the continuity of ∂_x^{-1} results in

$$\left|(3.15)\right| \lesssim \|v\|^2_{H^s(\mathbb{T})}. \tag{3.18}$$

Combining (3.17) and (3.18), we have the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\left\|v(t)\right\|^2_{H^s(\mathbb{T})} \lesssim \frac{1}{\varepsilon}\|v\|^2_{H^s(\mathbb{T})}, \tag{3.19}$$

which implies

$$\left\|v(t)\right\|^2_{H^s(\mathbb{T})} \leqslant \left\|v(0)\right\|^2_{H^s(\mathbb{T})} e^{2\frac{c_s}{\varepsilon}t} = \left\|u_0^\varepsilon - u_{0,n}^\varepsilon\right\|^2_{H^s(\mathbb{T})} e^{2\frac{c_s}{\varepsilon}t}.$$

Recalling that the solutions are mollified, we write

$$\left\|u_0^\varepsilon - u_{0,n}^\varepsilon\right\|_{H^s(\mathbb{T})} e^{\frac{c_s}{\varepsilon}T} = \left\|J_\varepsilon\left(u_0^\varepsilon - u_{0,n}^\varepsilon\right)\right\|_{H^s(\mathbb{T})} e^{\frac{c_s}{\varepsilon}T} \leqslant \|u_0 - u_{0,n}\|_{H^s(\mathbb{T})} e^{\frac{c_s}{\varepsilon}T}.$$

When we bound the first and third terms of (3.11), we will force ε to be small. After ε (independent of N) is chosen, we can bound $\|u^\varepsilon - u_n^\varepsilon\|_{C([0,T];H^s(\mathbb{T}))}$ by taking N large enough that

$$\|u_0 - u_{0,n}\|_{H^s(\mathbb{T})} < \frac{\eta}{3} e^{-\frac{c_\varepsilon T}{\varepsilon}}.$$

Then we have

$$\left\|u^\varepsilon - u_n^\varepsilon\right\|_{C([0,T];H^s(\mathbb{T}))} \leqslant \frac{\eta}{3}.$$

3.4.2. Estimation of $\|u^\varepsilon - u\|_{C([0,T];H^s(\mathbb{T}))}$ *and* $\|u_n^\varepsilon - u_n\|_{C([0,T];H^s(\mathbb{T}))}$

Since the arguments will be largely the same for both terms, we will omit the subscript n until such a time as differences in their handlings emerge. For convenience, let $v = u^\varepsilon - u$. We use Newton's Binomial and the product rule to write

$$\partial_t v = \frac{1}{k+1}\left[\sum_{j=1}^{k+1}\binom{k+1}{j}(-1)^j j\big(u^\varepsilon\big)^{k+1-j} v^{j-1}\partial_x v \right. \tag{3.20}$$

$$\left. + \sum_{j=1}^{k+1}\binom{k+1}{j}(-1)^j (k+1-j)\big(u^\varepsilon\big)^{k-j} v^j \partial_x u^\varepsilon\right] \tag{3.21}$$

$$+ \gamma \partial_x^{-1} v, \tag{3.22}$$

with initial condition $v_0 = J_\varepsilon u_0 - u_0$.

As in previous arguments, our strategy is to prove an $H^s(\mathbb{T})$-energy estimate for v. To bound the terms that arise from the generalized Burgers equation, we consider an arbitrary term within the sum. We rewrite each term as a commutator and then apply the Cauchy–Schwarz inequality before using Lemma 3.2, the Sobolev Embedding Theorem, and the Algebra Property. For the non-local terms, we employ the Cauchy–Schwarz inequality and the continuity of ∂_x^{-1}. The resulting energy estimate is

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_{H^s(\mathbb{T})}^2 \lesssim \sum_{j=1}^{k+1}\Big(&\left\|u^\varepsilon\right\|_{H^s(\mathbb{T})}^{k-j+1}\|v\|_{H^s(\mathbb{T})}^{j+1} \\ &+ \left\|u^\varepsilon\right\|_{H^s(\mathbb{T})}^{k-j}\left\|u^\varepsilon\right\|_{H^{s+1}(\mathbb{T})}\|v\|_{H^{s-1}(\mathbb{T})}^{j}\|v\|_{H^s(\mathbb{T})} + \|v\|_{H^s(\mathbb{T})}^2\Big). \end{aligned} \tag{3.23}$$

By interpolating between 0 and s, we have

$$\|v\|_{H^{s-1}(\mathbb{T})} \leqslant \|v\|_{H^0(\mathbb{T})}^{\frac{1}{s}}\|v\|_{H^s(\mathbb{T})}^{1-\frac{1}{s}} \lesssim \|v\|_{L^2(\mathbb{T})}^{\frac{1}{s}}.$$

Note that the solution size estimate (1.5) implies that $\|v(t)\|_{H^s(\mathbb{T})} \lesssim 1$. By utilizing an L^2-energy estimate, it can be shown that $\|v(t)\|_{L^2(\mathbb{T})} = o(\varepsilon^s)$. This is used to reduce the energy estimate to the differential inequality

$$\frac{dy}{dt} \lesssim y + \delta,$$

where $y = y(t) = \|v(t)\|^2_{H^s(\mathbb{T})}$ and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Integrating from 0 to t, with $t \in [0, T]$, we have

$$y(t) \lesssim y(0)e^T + \delta\big(e^T - 1\big).$$

3.4.3. Case of $y = \|v(t)\|_{H^s(\mathbb{T})}$

We first note that

$$\big\|v(t)\big\|_{H^s(\mathbb{T})} = \|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} + \delta(\varepsilon).$$

We can make δ arbitrary small by our choice of ε. Since

$$\|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} = o\big(\varepsilon^{s-s}\big) = o(1),$$

we have that $\|v(t)\|_{H^s(\mathbb{T})} \to 0$ as $\varepsilon \to 0$. Therefore, if we choose ε sufficiently small, we can bound the first term of (3.11) by $\frac{\eta}{3}$.

3.4.4. Case of $y = \|v_n(t)\|_{H^s(\mathbb{T})}$

We first note that

$$\big\|v_n(t)\big\|_{H^s(\mathbb{T})} = \|J_\varepsilon u_{0,n} - u_{0,n}\|_{H^s(\mathbb{T})} + \delta(\varepsilon).$$

Thus, we have

$$\begin{aligned}\|J_\varepsilon u_{0,n} - u_{0,n}\|_{H^s(\mathbb{T})} + \delta &\leqslant \big\|J_\varepsilon(u_{0,n} - u_0)\big\|_{H^s(\mathbb{T})} + \|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} + \|u_{0,n} - u_0\|_{H^s(\mathbb{T})} + \delta \\ &\lesssim \|u_{0,n} - u_0\|_{H^s(\mathbb{T})} + \|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} + \delta.\end{aligned}$$

We pick ε small so that $\|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} + \delta(\varepsilon)$ is bounded. Then we pick n (dependent on ε) large so that

$$\big\|u_n^\varepsilon - u_n\big\|_{H^s(\mathbb{T})} \leqslant \|u_{0,n} - u_0\|_{H^s(\mathbb{T})} + \|J_\varepsilon u_0 - u_0\|_{H^s(\mathbb{T})} + \delta(\varepsilon) < \frac{\eta}{3}$$

for n sufficiently large and ε sufficiently small.

This completes our proof of the well-posedness Theorem 1.1. □

4. Proof of Theorem 1.3

In this section, we will prove that the solution map for the gRO i.v.p. (1.1)–(1.2) is Hölder continuous. We begin with the case $0 \leqslant r \leqslant s - 1$, where $s > 3/2$. Let $u_0, w_0 \in B(0, \rho) \subset H^s(\mathbb{T})$ and $u, w \in C([0, T]; H^s(\mathbb{T}))$ denote the corresponding solutions to the gRO i.v.p. (1.1)–(1.2) with initial data u_0, w_0, respectively. Notice that we can find a common lifespan $T > 0$ for all initial data $u_0 \in B(0, \rho)$. This follows from the fact that the lifespan T_u of any solution $u \in B(0, \rho)$ to gRO (1.1)–(1.2) satisfies the following inequality

$$T_u \geqslant \frac{1}{2kc_s} \ln\left(1 + \frac{1}{\|u_0\|_{H^s(\mathbb{T})}}\right) > \frac{1}{2kc_s} \ln\left(1 + \frac{1}{\rho}\right) = T,$$

where T does not depend upon u.

We need to show that there exist constant c and Hölder exponent α such that

$$\left\| u(t) - w(t) \right\|_{C([0,T];H^r(\mathbb{T}))} \leqslant c\|u_0 - w_0\|_{H^r(\mathbb{T})}^{\alpha}.$$

If we let $v(t) = u(t) - w(t)$, then v satisfies the following energy estimate

$$\frac{1}{2}\frac{d}{dt}\left\| v(t) \right\|_{H^r(\mathbb{T})}^2 \lesssim \left\| v(t) \right\|_{H^r(\mathbb{T})}^2,$$

as shown in the proof of uniqueness, see (3.9), provided that $r \in [0, s-1]$.

Solving the differential inequality (3.10) gives

$$\left\| u(t) - w(t) \right\|_{H^r(\mathbb{T})} \leqslant e^{c_s T}\|u_0 - w_0\|_{H^r(\mathbb{T})}, \quad 0 \leqslant t \leqslant T, \tag{4.1}$$

where $u_0, w_0 \in B(0,\rho)$. Thus, we have shown that the data-to-solution map is Lipschitz continuous as a map

$$\underset{\text{with } H^r\text{-norm}}{H^s(\mathbb{T}) \supset B(0,\rho)} \to C\big([0,T]; H^r(\mathbb{T})\big)$$

for $0 \leqslant r \leqslant s-1$.

Let $s-1 < r < s$, where $s > 3/2$. Interpolating between $s-1$ and s we have

$$\left\| v(t) \right\|_{H^r(\mathbb{T})} \leqslant \left\| v(t) \right\|_{H^{s-1}(\mathbb{T})}^{s-r} \left\| v(t) \right\|_{H^s(\mathbb{T})}^{r+1-s}. \tag{4.2}$$

Since for $s-1$ we can apply inequality (4.1), we have

$$\left\| v(t) \right\|_{H^{s-1}(\mathbb{T})} \leqslant e^{c_s T}\|u_0 - w_0\|_{H^{s-1}(\mathbb{T})}. \tag{4.3}$$

Also, using the solution size estimate (1.5), we have

$$\left\| v(t) \right\|_{H^s(\mathbb{T})} \leqslant \sqrt[k]{2} \cdot e^{c_s T}\big(\|u_0\|_{H^s(\mathbb{T})} + \|w_0\|_{H^s(\mathbb{T})}\big) \leqslant \sqrt[k]{2} \cdot e^{c_s T} \cdot 2\rho. \tag{4.4}$$

Combining (4.2), (4.3), and (4.4) gives

$$\left\| v(t) \right\|_{H^r(\mathbb{T})} \leqslant e^{c_s T\cdot(s-r)}\big(\sqrt[k]{2} \cdot e^{c_s T} \cdot 2\rho\big)^{r+1-s}\|u_0 - w_0\|_{H^{s-1}(\mathbb{T})}^{s-r}.$$

Finally, taking into consideration the fact that $s-1 < r$ from the last inequality we obtain

$$\left\| u(t) - w(t) \right\|_{H^r(\mathbb{T})} \leqslant c_{r,T,\rho}\|u_0 - w_0\|_{H^r(\mathbb{T})}^{s-r}, \quad 0 \leqslant t \leqslant T,$$

which shows that the solution map is Hölder continuous with exponent $s-r$. □

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