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Weak almost periodic motions, minimality and stability in impulsive semidynamical systems

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Abstract

In this paper, we study topological properties of semidynamical systems whose continuous dynamics are interrupted by abrupt changes of state. First, we establish results which relate various concepts as stability of Lyapunov, weakly almost periodic motions, recurrence and minimality. In the sequel, we study the stability of Zhukovskij for impulsive systems and we obtain some results about uniform attractors.

Keywords: Impulsive systems; Semidynamical systems; Stability; Recurrence; Minimality; Attractor

1. Introduction

The theory of impulsive systems has been intensively investigated since this theory presents important applications. The reader may consult [1,9,13,14,16,23,24], for instance.

The present paper concerns with the topological study of semidynamical systems with impulses. We deal with results which encompass periodicity, recurrence, minimality, attractors and stability.

In [2], many recursive concepts (minimality, recurrence and almost periodic motions) are presented for continuous dynamical systems. Most of these results were generalized for impulsive systems, see [8,18,22] for example. However, many other properties of minimality, recurrence, weakly almost periodic motions, stability and attractors still need to be studied for the impulsive

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case. In this way, we present in this paper a study of recursive motions via stability theory (Lyapunov stability and Zhukovskij quasi stability). In the next lines we describe the organization of the paper and the main results.

We start by presenting a summary of the basis of the theory of semidynamical systems with impulse effects. In Section 2, we give some basic definitions and notations about impulsive semidynamical systems. In Section 3, we present some additional definitions and various results that will be very useful in the proof of the new results. Section 4 and Section 5 concern the main results of this paper. In the sequel, we mention some of these results.

In Section 4, we start by presenting the concept of weak almost $\tilde{\pi}$-periodic motions which was introduced by Saroop Kaul in [18]. We give sufficient conditions for a weakly almost $\tilde{\pi}$-periodic point to be $\tilde{\pi}$-recurrent, see Theorem 4.1. In Theorem 4.2, we show that a given point belongs to its limit set provided this point is weakly almost $\tilde{\pi}$-periodic and it does not belong to the impulsive set. The converse of Theorem 4.2 does not hold in general and we show a counterexample in Example 4.1. Following the results proved in the continuous case, we show that a $\tilde{\pi}$-recurrent point is weakly almost $\tilde{\pi}$-periodic provided it is Lyapunov $\tilde{\pi}$-stable, see Theorem 4.3.

In Theorem 4.4, we present sufficient conditions for the limit set of a given point to be minimal. Moreover, it is shown that if the points from the limit set which are outside of the impulsive set are Lyapunov $\tilde{\pi}$-stable, then these points are weakly almost $\tilde{\pi}$-periodic. Theorem 4.5 shows that the limit set of a point is minimal if and only if the trajectory of this point approaches uniformly to its limit set.

Section 5 deals with the quasi stability of Zhukovskij. In impulsive semidynamical systems, this kind of stability was introduced by Changming Ding in [15]. In Theorem 5.1, we show that the limit set of a given point is minimal provided the points of this limit set which are not in the impulsive set are Zhukovskij quasi $\tilde{\pi}$-stable. As consequence of this result, we give sufficient conditions for the points of a limit set which are not in the impulsive set to be $\tilde{\pi}$-recurrent, see Corollary 5.1.

Theorem 5.2 presents some conditions for a point to be periodic. In the last result, namely Theorem 5.3, we establish sufficient conditions for a limit set to be a uniform $\tilde{\pi}$-attractor.

2. Preliminaries

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and $t, s \in \mathbb{R}_+$. We denote such system simply by (X, π). For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t)\colon\ t \in \mathbb{R}_+\}$. Given $A \subset X$ and $t \geqslant 0$, we define

$$\pi^+(A) = \bigcup_{x \in A} \pi^+(x) \quad \text{and} \quad \pi(A, t) = \bigcup_{x \in A} \pi(x, t).$$

For $t \geqslant 0$ and $x \in X$, we define $F(x, t) = \{y \in X\colon\ \pi(y, t) = x\}$ and, for $\Delta \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \Delta) = \bigcup \big\{F(x, t)\colon\ x \in D \text{ and } t \in \Delta\big\}.$$

Then a point $x \in X$ is called an *initial point* if $F(x, t) = \emptyset$ for all $t > 0$.

An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a nonempty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F\big(x, (0, \varepsilon_x)\big) \cap M = \emptyset \quad \text{and} \quad \pi\big(x, (0, \varepsilon_x)\big) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ whose action we explain below in the description of the impulsive trajectory of an impulsive semidynamical system. The set M is called the *impulsive set* and the function I is called *impulse function*. We also define

$$M^+(x) = \left(\bigcup_{t>0} \pi(x, t)\right) \cap M.$$

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We define the function $\phi : X \to (0, +\infty]$ by

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \text{ and } \pi(x, t) \notin M \text{ for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an X-valued function $\tilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such trajectory follows inductively as described in the following lines.

If $M^+(x) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x, t)$ for all $t \in \mathbb{R}_+$ and $\phi(x) = +\infty$. However if $M^+(x) \neq \emptyset$, there is the smallest positive number s_0 such that $\pi(x, s_0) = x_1 \in M$ and $\pi(x, t) \notin M$, for $0 < t < s_0$. Then we define $\tilde{\pi}_x$ on $[0, s_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leqslant t < s_0, \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$ and $\phi(x) = s_0$. Let us denote x by x_0^+.

Since $s_0 < +\infty$, the process now continues from x_1^+ onwards. If $M^+(x_1^+) = \emptyset$, then we define $\tilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$ for $s_0 \leqslant t < +\infty$ and $\phi(x_1^+) = +\infty$. When $M^+(x_1^+) \neq \emptyset$, there is the smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in M$ and $\pi(x_1^+, t - s_0) \notin M$, for $s_0 < t < s_0 + s_1$. Then we define $\tilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leqslant t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = I(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Notice that $\tilde{\pi}_x$ is defined on each interval $[t_n, t_{n+1}]$, where $t_{n+1} = \sum_{i=0}^{n} s_i$. Hence $\tilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$, $n = 0, 1, 2, \ldots$, and in this case the function $\tilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Given $x \in X$, the *impulsive positive orbit* of x is defined by the set

$$\tilde{\pi}^+(x) = \{\tilde{\pi}(x, t): t \in [0, T(x))\}.$$

Given $A \subset X$ and $t \geqslant 0$, we define

$$\tilde{\pi}^+(A) = \bigcup_{x \in A} \tilde{\pi}^+(x) \quad \text{and} \quad \tilde{\pi}(A, t) = \bigcup_{x \in A} \tilde{\pi}(x, t).$$

Analogously to the non-impulsive case, an impulsive semidynamical system satisfies the following standard properties: $\tilde{\pi}(x, 0) = x$ for all $x \in X$ and $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$, for all $x \in X$ and for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$. See [4] for a proof of it.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [3–8,10–12,17–19].

3. Additional definitions and results

Let us consider a metric space X with metric d. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and radius $\delta > 0$. Given $A \subset X$, let $B(A, \delta) = \{x \in X: d(x, A) < \delta\}$ where $d(x, A) = \inf\{d(x, y): y \in A\}$.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x $(x \in X)$ is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leqslant \mu < \nu \leqslant 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let $(X, \pi; M, I)$ be an impulsive semidynamical system. We now present the conditions *TC* and *STC* for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (*TC*), if there exists a *TC*-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (*STC*)), if there exists a *STC*-tube $F(L, [0, 2\lambda])$ through x. The theory about tube may be found in [10].

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition *TC*.

Theorem 3.1. *(See [10, Theorem 3.8].) Consider an impulsive semidynamical system $(X, \pi; M, I)$. Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition (TC). Then ϕ is continuous at x if and only if $x \notin M$.*

In the sequel, for each $x \in X$, we consider that the motion $\tilde{\pi}(x, t)$ is defined for every $t \geqslant 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\tilde{\pi}_x$.

Let $A \subset X$. If $\tilde{\pi}^+(A) \subset A$, we say that A is *positively $\tilde{\pi}$-invariant*. And, a set A is *I-invariant* if $I(x) \in A$ for all $x \in A \cap M$.

The *limit set* of a point $x \in X$ in $(X, \pi; M, I)$ is given by

$$\widetilde{L}^+(x) = \big\{ y \in X\colon \text{ there is a sequence } \{t_n\}_{n\geqslant 1} \subset \mathbb{R}_+ \\ \text{such that } t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x, t_n) \overset{n\to+\infty}{\longrightarrow} y \big\},$$

and the *prolongational limit set* of $x \in X$ is given by

$$\widetilde{J}^+(x) = \big\{ y \in X\colon \text{ there are sequences } \{x_n\}_{n\geqslant 1} \subset X \text{ and } \{t_n\}_{n\geqslant 1} \subset \mathbb{R}_+ \text{ such that} \\ x_n \overset{n\to+\infty}{\longrightarrow} x,\ t_n \overset{n\to+\infty}{\longrightarrow} +\infty \text{ and } \widetilde{\pi}(x_n, t_n) \overset{n\to+\infty}{\longrightarrow} y \big\}.$$

In the sequel, we mention some auxiliary results that will be very useful in the proof of the main results.

Lemma 3.1. *(See [4, Lemma 3.2].) Given an impulsive semidynamical system $(X, \pi; M, I)$, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geqslant 1}$ is a sequence in X which converges to w. Then, for any $t \geqslant 0$ there is a sequence $\{\epsilon_n\}_{n\geqslant 1} \subset \mathbb{R}$ such that $\epsilon_n \overset{n\to+\infty}{\longrightarrow} 0$ and $\widetilde{\pi}(z_n, t + \epsilon_n) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$.*

Lemma 3.2. *(See [5, Lemma 3.3].) Given an impulsive semidynamical system $(X, \pi; M, I)$, suppose $w \in X \setminus M$ and $\{z_n\}_{n\geqslant 1}$ is a sequence in X which converges to w. Then, for any $t \geqslant 0$ such that $t \neq \sum_{j=0}^{k} \phi(w_j^+)$, $k = 0, 1, 2, \ldots,$ we have $\widetilde{\pi}(z_n, t) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$.*

Lemma 3.3. *(See [8, Proposition 4.1].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Then $\widetilde{L}^+(x) \setminus M$ is positively $\widetilde{\pi}$-invariant for all $x \in X$.*

The next three definitions are established in [8].

Definition 3.1. A subset $A \subset X$ is *minimal* in $(X, \pi; M, I)$ if $A \setminus M \neq \emptyset$, A is closed, $A \setminus M$ is positively $\widetilde{\pi}$-invariant and A does not contain any proper subset satisfying these conditions.

Theorem 3.2. *(See [8, Theorem 4.1].) A subset $A \subset X$ is minimal in $(X, \pi; M, I)$ if and only if $A = \overline{\widetilde{\pi}^+(x)}$ for all $x \in A \setminus M$.*

Definition 3.2. A point $x \in X$ is said to be $\widetilde{\pi}$-*recurrent* if for every $\epsilon > 0$ there exists a number $T = T(\epsilon) > 0$, such that for every $t, s \geqslant 0$, the interval $[0, T]$ contains a number $\tau > 0$ such that

$$d\big(\widetilde{\pi}(x, t), \widetilde{\pi}(x, s + \tau)\big) < \epsilon.$$

A positive orbit $\widetilde{\pi}^+(x)$ is said to be $\widetilde{\pi}$-recurrent if $x \in X$ is $\widetilde{\pi}$-recurrent.

Definition 3.3. A subset $D \subset \mathbb{R}_+$ is called *relatively dense* if there is a number $L > 0$ such that $D \cap (\alpha, \alpha + L) \neq \emptyset$ for all $\alpha \geqslant 0$.

Theorem 3.3. *(See [8, Theorem 4.7].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be a compact minimal set. If $x \in A \setminus M$, then x is $\widetilde{\pi}$-recurrent.*

Theorem 3.4. *(See [8, Theorem 4.9].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $\overline{\widetilde{\pi}^+(x)}$ be compact for some $x \in X \setminus M$. The positive orbit $\widetilde{\pi}^+(x)$ is $\widetilde{\pi}$-recurrent if and only if for each $\epsilon > 0$ the set $K_\epsilon = \{t \in \mathbb{R}_+ \colon d(x, \widetilde{\pi}(x,t)) < \epsilon\}$ is relatively dense.*

Corollary 3.1. *(See [8, Corollary 4.2].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Suppose X complete and let $x \in X \setminus M$ be such that $\widetilde{L}^+(x) \setminus M \neq \emptyset$. If $\widetilde{\pi}^+(x) \cup \{x_k\}_{k \geqslant 1}$ is minimal then x is periodic.*

Next, we define the concept of region of uniform attraction. See Ref. [7].

Definition 3.4. Let $(X, \pi; M, I)$ be an impulsive semidynamical system and A be a subset of X. The set

$$\widetilde{P}_u^+(A) = \{x \in X \colon \text{ for every neighborhood } U \text{ of } A, \text{ there is a neighborhood } V \text{ of } x \text{ and a } T > 0 \text{ such that } \widetilde{\pi}(V,t) \subset U \text{ for all } t \geqslant T\},$$

is called the *region of uniform attraction* of the set A with respect to $\widetilde{\pi}$. If $x \in \widetilde{P}_u^+(A)$, we say that x is *uniformly $\widetilde{\pi}$-attracted* to A.

Proposition 3.1. *(See [7, Proposition 3.2].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $A \subset X$ be compact. Suppose X is locally compact. Then $\widetilde{P}_u^+(A) = \{x \in X \colon \widetilde{J}^+(x) \neq \emptyset$ and $\widetilde{J}^+(x) \subset A\}$.*

Lastly, we present another auxiliary result which will be used in the last theorem of this paper.

Lemma 3.4. *(See [6, Lemma 3.13].) Let $(X, \pi; M, I)$ be an impulsive semidynamical system. Let $x \notin M$ and $y \in \widetilde{L}^+(x)$. Then $\widetilde{J}^+(x) \subset \widetilde{J}^+(y)$.*

In the next two sections, we shall consider an impulsive semidynamical system $(X, \pi; M, I)$, where (X, d) is a metric space. Moreover, we shall assume the following additional hypotheses:

(H1) No initial point in (X, π) belongs to the impulsive set M and each element of M satisfies the condition (STC), consequently ϕ is continuous on $X \setminus M$ (see Theorem 3.1).
(H2) $M \cap I(M) = \emptyset$.
(H3) For each $x \in X$, the motion $\widetilde{\pi}(x,t)$ is defined for every $t \geqslant 0$, that is, $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$.

4. Recursive motions and Lyapunov stability

Given $x \in X$, we denote $\mathbb{N}(x)$ by

$$\mathbb{N}(x) = \mathbb{N} = \{0, 1, 2, 3, \dots\} \quad \text{if } x \in M$$

and

$$\mathbb{N}(x) = \mathbb{N}^* = \{1, 2, 3, \dots\} \quad \text{if } x \notin M.$$

Now, given $x \in X$ and $\epsilon > 0$, we consider the following set

$$L(x,\epsilon) = \{t \in \mathbb{R}_+ \colon |t - t_n(x)| > \epsilon \text{ for all } n \in \mathbb{N}(x)\},$$

where $t_n(x) = \sum_{j=0}^{n-1} \phi(x_j^+)$ for $n \geqslant 1$, $n \in \mathbb{N}$, and $t_0(x) = 0$.

Next, we present the concept of weakly almost $\tilde{\pi}$-periodic points. Such definition is due to Kaul [18].

Definition 4.1. A point $x \in X$ is called *weakly almost $\tilde{\pi}$-periodic*, if given $\epsilon > 0$ there is a number $T = T(\epsilon) > 0$, such that for any $\alpha \geqslant 0$, the interval $[\alpha, \alpha + T]$ contains a number $\tau = \tau(\alpha) > 0$ such that

$$d(\tilde{\pi}(x,t), \tilde{\pi}(x,t+\tau)) < \epsilon \quad \text{for all } t \in L(x,\epsilon).$$

In the sequel, we present conditions for a weakly almost $\tilde{\pi}$-periodic point to be $\tilde{\pi}$-recurrent.

Theorem 4.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X \setminus M$. Suppose that $\overline{\tilde{\pi}^+(x)}$ is compact. If x is weakly almost $\tilde{\pi}$-periodic then x is $\tilde{\pi}$-recurrent.*

Proof. Let $\epsilon > 0$ be given. We may assume that $0 \in L(x,\epsilon)$ since $x \in X \setminus M$. By hypothesis, there is $T = T(\epsilon) > 0$ such that for every $\alpha \geqslant 0$ the interval $[\alpha, \alpha + T]$ contains a number $\tau = \tau(\alpha) > 0$ such that

$$d(\tilde{\pi}(x,t), \tilde{\pi}(x,t+\tau)) < \epsilon \quad \text{for all } t \in L(x,\epsilon). \tag{4.1}$$

Consider $K_\epsilon = \{s \in \mathbb{R}_+ \colon d(x, \tilde{\pi}(x,s)) < \epsilon\}$. Hence by (4.1) we have $K_\epsilon \cap [\alpha, \alpha + T] \neq \emptyset$ for all $\alpha \geqslant 0$, because $0 \in L(x,\epsilon)$. It follows by Theorem 3.4 that x is $\tilde{\pi}$-recurrent. $\square$

If $x \in X \setminus M$, we can choose a sufficiently small $\epsilon_0 > 0$ such that $0 \in L(x,\epsilon)$ for all $\epsilon \in (0, \epsilon_0)$. Thus, provided x is weakly almost $\tilde{\pi}$-periodic one can obtain a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, such that $\pi(x,t_n) \overset{n\to+\infty}{\longrightarrow} x$. Hence, we have the following result.

Theorem 4.2. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X \setminus M$. If x is weakly almost $\tilde{\pi}$-periodic then $x \in \tilde{L}^+(x)$.*

In the continuous case, the converse of Theorem 4.2 does not hold in general. It also happens in the impulsive case, see Example 4.1. To present the example we recall two auxiliary definitions.

Remark 4.1. Let (X, σ) be a discrete dynamical system, see [20] for instance. A point $x \in X$ is called a recurrent point of σ if given $\epsilon > 0$ for any $N \in \mathbb{N}$ one can obtain a natural number $n > N$ such that $\sigma^n(x) \in B(x,\epsilon)$. A point $x \in X$ is called an almost periodic point of σ if given $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\{\sigma^{n+i}(x) \colon i = 0, 1, \ldots, N\} \cap B(x,\epsilon) \neq \emptyset$ for all $n \in \mathbb{N}$.

Example 4.1. Let (Y, d_Y) be a complete metric space and consider a discrete dynamical system (Y, σ) with σ continuous. Suppose that every point in Y is recurrent but is not almost periodic. The reader may find an example of such system in [20, Example 3.3]. Now, define $X = \mathbb{R} \times Y$

endowed with metric $d((t,x),(s,y)) = \max\{|t-s|, d_Y(x,y)\}$. Consider the semiflow $\pi : X \times \mathbb{R}_+ \to X$ given by

$$\pi\big((r,y),t\big) = (r+t, y).$$

Let $M = \{0\} \times Y$ and $I : M \to X$ given by

$$I(0,y) = \big(-1, \sigma(y)\big),$$

which is clearly continuous. Given $z = (-1, y) \in X$, we consider the following notations

$$z_{n+1} = \big(0, \sigma^n(y)\big) \quad \text{and} \quad I(z_{n+1}) = z_{n+1}^+ = \big(-1, \sigma^{n+1}(y)\big),$$

for $n = 0, 1, 2, \dots$, where $z_0^+ = z$.

Now, note that given $t, s \in \mathbb{R}_+$, then there are $k, m \in \mathbb{N}$ such that $t = k + t'$ with $0 \leqslant t' < 1$ and $s = m + s'$ with $0 \leqslant s' < 1$. Then

$$\begin{aligned} d\big(\widetilde{\pi}(z,t), \widetilde{\pi}(z,s)\big) &= d\big(\pi\big(z_k^+, t'\big), \pi\big(z_m^+, s'\big)\big) = d\big(\pi\big(\big(-1, \sigma^k(y)\big), t'\big), \pi\big(\big(-1, \sigma^m(y)\big), s'\big)\big) \\ &= d\big(\big(t'-1, \sigma^k(y)\big), \big(s'-1, \sigma^m(y)\big)\big) = \max\big\{\big|t'-s'\big|, d_Y\big(\sigma^k(y), \sigma^m(y)\big)\big\}. \end{aligned}$$

1) $z \in \widetilde{L}^+(z)$.

Indeed, let $\epsilon > 0$ be arbitrary. Since y is a recurrent point of σ, there is a sequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $n_k \overset{k \to +\infty}{\longrightarrow} +\infty$ and $d_Y(\sigma^{n_k}(y), y) < \epsilon$ for all $k \in \mathbb{N}$. Then

$$d\big(\widetilde{\pi}(z, n_k), z\big) = \max\big\{0, d_Y\big(\sigma^{n_k}(y), y\big)\big\} < \epsilon,$$

for all $k \in \mathbb{N}$. Therefore, $z \in \widetilde{L}^+(z)$.

2) z is not weakly almost $\widetilde{\pi}$-periodic.

Since y is not almost periodic, there is $\epsilon_0 > 0$ such that for all $N \in \mathbb{N}$ we may find $n_N \in \mathbb{Z}_+$ such that

$$d_Y\big(\sigma^{n_N+i}(y), y\big) \geqslant \epsilon_0, \tag{4.2}$$

for all $i \in \{0, 1, 2, \dots, N\}$.

Now, given an arbitrary $T > 0$, there is natural $N_T > 0$ such that $N_T \leqslant T < N_T + 1$. Let $n_{N_T} := n_T$ be given by (4.2). For any $s \in [n_T, n_T + T]$, we can write $s = n_T + i_0 + s'$ for some $i_0 \in \{0, 1, 2, \dots, N_T\}$ and $0 \leqslant s' < 1$. Then

$$d\big(z, \widetilde{\pi}(z,s)\big) = \max\big\{s', d_Y\big(y, \sigma^{n_T+i_0}(y)\big)\big\} \geqslant \epsilon_0.$$

Hence, z is not weakly almost $\widetilde{\pi}$-periodic.

If a continuous motion is positively Lyapunov stable and recurrent then it is almost periodic, see Theorem 8.11 in [21]. Next, we present the concept of Lyapunov stability for impulsive semidynamical systems and we obtain sufficient conditions for a $\tilde{\pi}$-recurrent point to be weakly almost $\tilde{\pi}$-periodic.

The concept of Lyapunov stability is established by Kaul in [19], see the next definition.

Definition 4.2. A point $x \in X$ is called *Lyapunov $\tilde{\pi}$-stable* with respect to a set $B \subset X$, if given $\epsilon > 0$, there is a $\delta > 0$ such that for any $y \in B$ with $d(y,x) < \delta$ we have

$$d\big(\tilde{\pi}(y,t), \tilde{\pi}(x,t)\big) < \epsilon \quad \text{for all } t \in L(x,\epsilon).$$

We have the following result.

Theorem 4.3. *Let $(X,\pi;M,I)$ be an impulsive semidynamical system and $x \in X \setminus M$. Suppose that $\overline{\tilde{\pi}^+(x)}$ is compact. If x is $\tilde{\pi}$-recurrent and Lyapunov $\tilde{\pi}$-stable with respect to $\tilde{\pi}^+(x)$, then x is weakly almost $\tilde{\pi}$-periodic.*

Proof. Let $\epsilon > 0$ be given. Since x is Lyapunov $\tilde{\pi}$-stable with respect to $\tilde{\pi}^+(x)$, then there is a $\delta > 0$ such that if $y \in \tilde{\pi}^+(x)$ and $d(x,y) < \delta$, we have

$$d\big(\tilde{\pi}(x,t), \tilde{\pi}(y,t)\big) < \epsilon \quad \text{for all } t \in L(x,\epsilon). \tag{4.3}$$

By the $\tilde{\pi}$-recurrence of x, we can use Theorem 3.4 and conclude that the set

$$K_\delta = \big\{r > 0\colon\ d\big(x, \tilde{\pi}(x,r)\big) < \delta\big\}$$

is relatively dense, that is, there is $T > 0$ such that for any $\alpha \geqslant 0$, the interval $[\alpha, \alpha+T]$ contains a number $\tau > 0$ such that $d(x, \tilde{\pi}(x,\tau)) < \delta$. Then by (4.3) we get

$$d\big(\tilde{\pi}(x,t), \tilde{\pi}(x,t+\tau)\big) < \epsilon \quad \text{for all } t \in L(x,\epsilon).$$

Hence, the point x is weakly almost $\tilde{\pi}$-periodic. □

Definition 4.3. An orbit $\tilde{\pi}^+(x)$, $x \in X$, is said to *approach uniformly* to a set $\mathcal{A}$, if for any $\epsilon > 0$, there exists a positive number $T = T(\epsilon) > 0$ such that

$$\mathcal{A} \subset B\big(\tilde{\pi}\big(x, [t, t+T]\big), \epsilon\big) \quad \text{for all } t \in \mathbb{R}_+.$$

Now, we intend to generalize the result presented in [3]. It is proved in Theorem 3.3, [3], that if $x \in X \setminus M$ is such that $\overline{\tilde{\pi}^+(x)}$ is compact and $\tilde{L}^+(x) \cap M = \emptyset$, then $\tilde{L}^+(x)$ is minimal if and only if $\tilde{\pi}^+(x)$ uniformly approximates its limit set $\tilde{L}^+(x)$. We shall show in Theorem 4.5 that this result holds if we drop the conditions $x \notin M$ and $\tilde{L}^+(x) \cap M = \emptyset$.

We start by presenting an auxiliary result.

Lemma 4.1. *Let A be a closed subset of X such that $A \setminus M$ is positively $\tilde{\pi}$-invariant. Let $a \in A \setminus M$ and $\{w_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $w_n \overset{n\to+\infty}{\longrightarrow} a$. If $\{\tau_n\}_{n\in\mathbb{N}}$ is a bounded sequence* *in* $\mathbb{R}_+$, *then given* $\epsilon > 0$ *there are a subsequence* $\{\tau_{n_\ell}\}_{\ell\in\mathbb{N}}$ *of* $\{\tau_n\}_{n\in\mathbb{N}}$, *a natural* $n_0 > 0$ *and* $\bar{a} \in A$ *such that* $d(\tilde{\pi}(w_{n_\ell}, \tau_{n_\ell}), \bar{a}) < \epsilon$ *for all* $n_\ell \geqslant n_0$.

Proof. Let $\epsilon > 0$ be given. First, note that there is a convergent subsequence of $\{\tau_n\}_{n\in\mathbb{N}}$, we say $\{\tau_{n_\ell}\}_{\ell\in\mathbb{N}}$ such that $\tau_{n_\ell} \overset{\ell\to+\infty}{\longrightarrow} \tau \in \mathbb{R}_+$. We have three cases to consider.

Case 1) $0 \leqslant \tau < \phi(a)$.

Since $a \notin M$ we have $\phi(w_n) \overset{n\to+\infty}{\longrightarrow} \phi(a)$. It follows that there is an integer $p_0 > 0$ such that

$$0 \leqslant \tau_{n_\ell} < \phi(w_{n_\ell})$$

for all $n_\ell \geqslant p_0$. Hence,

$$\tilde{\pi}(w_{n_\ell}, \tau_{n_\ell}) = \pi(w_{n_\ell}, \tau_{n_\ell}) \overset{\ell\to+\infty}{\longrightarrow} \pi(a, \tau) = \tilde{\pi}(a, \tau).$$

Set $\bar{a} = \tilde{\pi}(a, \tau)$. By the positive $\tilde{\pi}$-invariance of $A \setminus M$ we have $\bar{a} \in A$. Then, there is $n_0 > 0$ such that $d(\tilde{\pi}(w_{n_\ell}, \tau_{n_\ell}), \bar{a}) < \epsilon$ for all $n_\ell \geqslant n_0$.

Case 2) $\tau > \phi(a)$ and $\tau \neq \sum_{j=0}^{r} \phi(a_j^+)$ for all $r = 1, 2, \ldots$.

One can obtain a number $k \in \mathbb{N}$ such that

$$\tau = \sum_{j=0}^{k} \phi\big(a_j^+\big) + s \quad \text{with } 0 < s < \phi\big(a_{k+1}^+\big).$$

Note that $\tilde{\pi}(a, \tau) = \pi(a_{k+1}^+, s)$. By continuity of π and I we have $(w_n)_j^+ \overset{n\to+\infty}{\longrightarrow} a_j^+$ for all $j \in \mathbb{N}$. Since ϕ is continuous in $X \setminus M$, there is a sequence $\{s_{n_\ell}\}_{\ell\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau_{n_\ell} = \sum_{j=0}^{k} \phi\big((w_{n_\ell})_j^+\big) + s_{n_\ell} \quad \text{with } s_{n_\ell} \overset{\ell\to+\infty}{\longrightarrow} s.$$

Note that $\tilde{\pi}(w_{n_\ell}, \tau_{n_\ell}) = \pi((w_{n_\ell})_{k+1}^+, s_{n_\ell})$ for all $\ell \in \mathbb{N}$. Then

$$\tilde{\pi}(w_{n_\ell}, \tau_{n_\ell}) = \pi\big((w_{n_\ell})_{k+1}^+, s_{n_\ell}\big) \overset{\ell\to+\infty}{\longrightarrow} \pi\big(a_{k+1}^+, s\big) = \tilde{\pi}(a, \tau).$$

Set $\bar{a} = \tilde{\pi}(a, \tau) \in A$ and we have the result in this case.

Case 3) $\tau = \sum_{j=0}^{k} \phi(a_j^+)$ for some $k = 0, 1, 2, \ldots$.

Here, we consider two subcases: when $\{\tau_{n_\ell}\}_{n\in\mathbb{N}}$ admits a subsequence $\{\tau'_{n_\ell}\}_{\ell\in\mathbb{N}}$ such that $\tau'_{n_\ell} < \sum_{j=0}^{k} \phi(a_j^+)$ for all $\ell \in \mathbb{N}$ and when $\{\tau_{n_\ell}\}_{n\in\mathbb{N}}$ admits a subsequence $\{\tau''_{n_\ell}\}_{\ell\in\mathbb{N}}$ such that $\tau''_{n_\ell} \geqslant \sum_{j=0}^{k} \phi(a_j^+)$ for all $\ell \in \mathbb{N}$.

In the first subcase it is easy to see that

$$\tilde{\pi}\big(w_{n_\ell}, \tau'_{n_\ell}\big) \overset{\ell\to+\infty}{\longrightarrow} a_{k+1} \in A \quad \big(A \text{ is closed and } \tilde{\pi}^+(a) \subset A\big).$$

Thus, there is an integer $n'_0 > 0$ such that

$$d\big(\tilde{\pi}\big(w_{n_\ell}, \tau'_{n_\ell}\big), a_{k+1}\big) < \epsilon \quad \text{for all } n_\ell \geqslant n'_0.$$

In the second subcase we have $\widetilde{\pi}(w_{n_\ell}, \tau''_{n_\ell}) \stackrel{\ell\to+\infty}{\longrightarrow} a^+_{k+1} = I(a_{k+1}) \in A$ ($\widetilde{\pi}^+(a) \subset A$ by invariance). Then there is an integer $n''_0 > 0$ such that

$$d\big(\widetilde{\pi}\big(w_{n_\ell}, \tau''_{n_\ell}\big), a^+_{k+1}\big) < \epsilon \quad \text{for all } n_\ell \geqslant n''_0.$$

In conclusion, in all of the cases above we can find an element $\bar{a} \in A$ and a natural n_0 such that $d(\widetilde{\pi}(w_{n_\ell}, \tau_{n_\ell}), \bar{a}) < \epsilon$ for all $n_\ell \geqslant n_0$. $\square$

Theorem 4.4 gives sufficient conditions for a limit set to be minimal.

Theorem 4.4. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$ be such that $\overline{\widetilde{\pi}^+(x)}$ is compact. If $\widetilde{\pi}^+(x)$ approaches uniformly to $\widetilde{L}^+(x)$ then $\widetilde{L}^+(x)$ is a minimal set. Moreover, if a point $y \in \widetilde{L}^+(x) \setminus M$ is Lyapunov $\widetilde{\pi}$-stable with respect to $\widetilde{\pi}^+(y)$, then y is weakly almost $\widetilde{\pi}$-periodic.*

Proof. First, let us show that the limit set is minimal. Suppose to the contrary that there is a closed subset $A \subsetneq \widetilde{L}^+(x)$ such that $A \setminus M \neq \emptyset$ and $A \setminus M$ is positively $\widetilde{\pi}$-invariant. Let $w \in \widetilde{L}^+(x)$ be such that $w \notin A$ and consider $\epsilon = \dfrac{d(w, A)}{3} > 0$.

Since $\widetilde{\pi}^+(x)$ approaches uniformly to $\widetilde{L}^+(x)$, there is $T = T(\epsilon) > 0$ such that

$$\widetilde{L}^+(x) \subset B\big(\widetilde{\pi}\big(x, [t, t+T]\big), \epsilon\big) \quad \text{for all } t \in \mathbb{R}_+. \tag{4.4}$$

Let $a \in A \setminus M \subset \widetilde{L}^+(x)$, then there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, $t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\widetilde{\pi}(x, t_n) \stackrel{n\to+\infty}{\longrightarrow} a. \tag{4.5}$$

By (4.4) we have, in particular, that

$$\widetilde{L}^+(x) \subset B\big(\widetilde{\pi}\big(x, [t_n, t_n+T]\big), \epsilon\big) \quad \text{for all } n \in \mathbb{N}.$$

Thus

$$w \in B\big(\widetilde{\pi}\big(x, [t_n, t_n+T]\big), \epsilon\big) \quad \text{for all } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, there is $\tau_n \in [0, T]$ such that

$$d\big(w, \widetilde{\pi}(x, t_n + \tau_n)\big) < \epsilon.$$

We may assume without loss of generality that $\tau_n \stackrel{n\to+\infty}{\longrightarrow} \tau \in [0, T]$. Define $w_n = \widetilde{\pi}(x, t_n)$, $n \in \mathbb{N}$. By Lemma 4.1, there are $\bar{a} \in A$, a subsequence $\{\tau_{n_\ell}\}_{\ell\in\mathbb{N}}$ and an integer $n_0 > 0$ such that

$$d\big(\widetilde{\pi}(w_{n_\ell}, \tau_{n_\ell}), \bar{a}\big) < \epsilon,$$

for all $n_\ell \geqslant n_0$. Then

$$\begin{aligned}3\epsilon = d(w, A) &\leqslant d(w, \bar{a}) \\ &\leqslant d\big(w, \tilde{\pi}(w_{n_0}, \tau_{n_0})\big) + d\big(\tilde{\pi}(w_{n_0}, \tau_{n_0}), \bar{a}\big) \\ &< \epsilon + \epsilon = 2\epsilon\end{aligned}$$

which is a contradiction. Hence, we conclude that $\widetilde{L}^+(x)$ is minimal.

We also have that $\widetilde{L}^+(x)$ is compact since $\widetilde{L}^+(x) \subset \overline{\tilde{\pi}^+(x)}$. Then by Theorem 3.3 every point $y \in \widetilde{L}^+(x) \setminus M$ is $\tilde{\pi}$-recurrent. If $y \in \widetilde{L}^+(x) \setminus M$ is Lyapunov $\tilde{\pi}$-stable with respect to $\tilde{\pi}^+(y)$, it follows by Theorem 4.3 that $y \in \widetilde{L}^+(x) \setminus M$ is weakly almost $\tilde{\pi}$-periodic. □

Theorem 4.5. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$ be such that $\overline{\tilde{\pi}^+(x)}$ is compact. Then $\widetilde{L}^+(x)$ is a minimal set if and only if $\tilde{\pi}^+(x)$ approaches uniformly to its limit set $\widetilde{L}^+(x)$.*

Proof. The necessary condition follows by Theorem 4.4. Let us prove the sufficient condition. Suppose that $\tilde{\pi}^+(x)$ does not approach uniformly to $\widetilde{L}^+(x)$, that is, there are $\epsilon > 0$ and sequences $\{T_n\}_{n\in\mathbb{N}}, \{t_n\}_{n\in\mathbb{N}}, \subset \mathbb{R}_+$, $\{y_n\}_{n\in\mathbb{N}} \subset \widetilde{L}^+(x)$ such that $T_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and

$$d\big(y_n, \tilde{\pi}\big(x, [t_n, t_n + T_n]\big)\big) \geqslant \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{4.6}$$

We note that $\widetilde{L}^+(x)$ is compact since $\widetilde{L}^+(x) \subset \overline{\tilde{\pi}^+(x)}$, thus we may assume that there is $y \in \widetilde{L}^+(x)$ such that

$$y_n \overset{n\to+\infty}{\longrightarrow} y. \tag{4.7}$$

Moreover, since $y_n \in \widetilde{L}^+(x)$ for each $n \in \mathbb{N}$, then there is a sequence $\{\tau_m^n\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau_m^n \overset{m\to+\infty}{\longrightarrow} +\infty \quad \text{and} \quad \tilde{\pi}\big(x, \tau_m^n\big) \overset{m\to+\infty}{\longrightarrow} y_n \tag{4.8}$$

for each $n \in \mathbb{N}$. Thus from (4.8) it follows that there exists $m_n > n$ such that

$$d\big(\tilde{\pi}\big(x, \tau_{m_n}^n\big), y_n\big) < \frac{\epsilon}{2}, \tag{4.9}$$

for all $n \in \mathbb{N}$. Also, from (4.7) and (4.8) we obtain

$$\tilde{\pi}\big(x, \tau_{m_n}^n\big) \overset{n\to+\infty}{\longrightarrow} y. \tag{4.10}$$

On the other hand $\{\tilde{\pi}(x, t_n + \frac{T_n}{2})\}_{n\in\mathbb{N}} \subset \overline{\tilde{\pi}^+(x)}$, $(t_n + \frac{T_n}{2}) \overset{n\to+\infty}{\longrightarrow} +\infty$ and since $\overline{\tilde{\pi}^+(x)}$ is compact we may assume that there is $b \in \widetilde{L}^+(x)$ such that

$$\tilde{\pi}\left(x, t_n + \frac{T_n}{2}\right) \overset{n\to+\infty}{\longrightarrow} b. \tag{4.11}$$

In the sequel, we are going to show that there is an element $z \in \widetilde{L}^+(x) \setminus M$ such that $y \notin \overline{\tilde{\pi}^+(z)}$. It will contradict the minimality of $\widetilde{L}^+(x)$ and hence the theorem will be proved. We have two cases to consider: when $b \in M$ and otherwise.

Case 1) $b \notin M$. Let $t \geqslant 0$ be fixed and arbitrary such that $t \neq \sum_{j=0}^{k} \phi(b_j^+)$ for all $k = 0, 1, 2, \ldots$. From (4.11) and Lemma 3.2 we get

$$\tilde{\pi}\left(x, t_n + \frac{T_n}{2} + t\right) \overset{n\to+\infty}{\longrightarrow} \tilde{\pi}(b, t). \tag{4.12}$$

Since $T_n \overset{n\to+\infty}{\longrightarrow} +\infty$, there is an integer $n_0 > 0$ such that $\frac{T_n}{2} > t$ for $n \geqslant n_0$. Then $t_n < t_n + \frac{T_n}{2} + t < t_n + T_n$ for all $n \geqslant n_0$. Consequently, by using (4.6) and (4.9) we have

$$\begin{aligned} d\left(\tilde{\pi}\left(x, t_n + \frac{T_n}{2} + t\right), \tilde{\pi}\left(x, \tau_{m_n}^n\right)\right) &\geqslant d\left(y_n, \tilde{\pi}\left(x, t_n + \frac{T_n}{2} + t\right)\right) - d\big(y_n, \tilde{\pi}\left(x, \tau_{m_n}^n\right)\big) \\ &\geqslant \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}, \end{aligned}$$

for all $n \geqslant n_0$. As $n \to +\infty$ in the above inequality and using (4.10) and (4.12), we get $d(\tilde{\pi}(b, t), y) \geqslant \frac{\epsilon}{2}$. Since t is arbitrary, we conclude that

$$d\big(\tilde{\pi}(b, t), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } t \neq \sum_{j=0}^{k} \phi\big(b_j^+\big),\ k = 0, 1, 2, \ldots. \tag{4.13}$$

Now, let $t = \sum_{j=0}^{k} \phi(b_j^+)$ for some arbitrary $k \in \{0, 1, 2, \ldots\}$. We may take a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lambda_n \overset{n\to+\infty}{\longrightarrow} \sum_{j=0}^{k} \phi\big(b_j^+\big), \quad \text{with } \sum_{j=0}^{k} \phi\big(b_j^+\big) < \lambda_n < \sum_{j=0}^{k+1} \phi\big(b_j^+\big),\ n \in \mathbb{N}.$$

Then by (4.13) we have

$$d\big(\tilde{\pi}(b, \lambda_n), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

As $n \to +\infty$ we have

$$d\left(\tilde{\pi}\left(b, \sum_{j=0}^{k} \phi\big(b_j^+\big)\right), y\right) \geqslant \frac{\epsilon}{2},$$

since $\tilde{\pi}$ is continuous from the right. In conclusion, we have

$$d\big(\tilde{\pi}(b, t), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } t \geqslant 0,$$

and hence

$$y \notin \overline{\tilde{\pi}^+(b)}$$

which is a contradiction, because $b \in \widetilde{L}^+(x) \setminus M$ and then $\overline{\widetilde{\pi}^+(b)} = \widetilde{L}^+(x)$ since $\widetilde{L}^+(x)$ is minimal (see Theorem 3.2).

Case 2) $b \in M$. Since M satisfies the condition *STC*, there is a *STC*-tube $F(L,[0,2\lambda])$ through of b given by a section S. Moreover, the tube is a neighborhood of b, then there is an $\eta > 0$ such that

$$B(b,\eta) \subset F\big(L,[0,2\lambda]\big).$$

Denote $H_1 = F(L,[\lambda,2\lambda]) \cap B(b,\eta)$ and $H_2 = F(L,[0,\lambda]) \cap B(b,\eta)$. Also, denote $w_n = \widetilde{\pi}(x, t_n + \frac{T_n}{2})$, $n \in \mathbb{N}$. Then $w_n \stackrel{n\to+\infty}{\longrightarrow} b$ by (4.11).

We may assume without loss of generality (taking a subsequence, if necessary) that $\{w_n\}_{n\in\mathbb{N}} \subset H_1$ or $\{w_n\}_{n\in\mathbb{N}} \subset H_2$.

- First, suppose that $\{w_n\}_{n\in\mathbb{N}} \subset H_1$. We note that $\phi(w_n) \stackrel{n\to+\infty}{\longrightarrow} 0$. Then

$$\widetilde{\pi}\big(w_n, \phi(w_n)\big) \stackrel{n\to+\infty}{\longrightarrow} I(b),$$

that is,

$$\widetilde{\pi}\left(x, t_n + \frac{T_n}{2} + \phi(w_n)\right) \stackrel{n\to+\infty}{\longrightarrow} I(b). \tag{4.14}$$

In the sequel, we are going to show that $y \notin \overline{\widetilde{\pi}^+(I(b))}$. Note that $I(b) \in \widetilde{L}^+(x) \setminus M$, because $(t_n + \frac{T_n}{2} + \phi(w_n)) \stackrel{n\to+\infty}{\longrightarrow} +\infty$ and $I(M) \cap M = \emptyset$. Let $t \geqslant 0$ be such that $t \neq \sum_{j=0}^{k} \phi(I(b)_j^+)$ for all $k = 0, 1, 2, \ldots$. By (4.14) and Lemma 3.2 we get

$$\widetilde{\pi}\left(x, t_n + \frac{T_n}{2} + \phi(w_n) + t\right) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}\big(I(b), t\big). \tag{4.15}$$

There is $n_0' > 0$ such that $\frac{T_n}{2} > t + \phi(w_n)$ for all $n \geqslant n_0'$. Then, $t_n < t_n + \frac{T_n}{2} + \phi(w_n) + t < t_n + T_n$ for all $n \geqslant n_0'$. By (4.6) and (4.9), we obtain the following inequality

$$\begin{aligned}
&d\left(\widetilde{\pi}\left(x, t_n + \frac{T_n}{2} + \phi(w_n) + t\right), \widetilde{\pi}\big(x, \tau_{m_n}^n\big)\right) \\
&\quad \geqslant d\left(y_n, \widetilde{\pi}\left(x, t_n + \frac{T_n}{2} + \phi(w_n) + t\right)\right) - d\big(y_n, \widetilde{\pi}\big(x, \tau_{m_n}^n\big)\big) \geqslant \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},
\end{aligned}$$

for all $n \geqslant n_0'$. As $n \to +\infty$ in the above inequality and using (4.15) and (4.10) we have $d(\widetilde{\pi}(I(b), t), y) \geqslant \frac{\epsilon}{2}$. Since t is arbitrary, we get

$$d\big(\widetilde{\pi}\big(I(b), t\big), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } t \neq \sum_{j=0}^{k} \phi\big(I(b)_j^+\big),\ k = 0, 1, 2, \ldots. \tag{4.16}$$

Now, let $t = \sum_{j=0}^{k} \phi(I(b)_j^+)$ for some $k = 0, 1, 2, \ldots$. We consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lambda_n \overset{n\to+\infty}{\longrightarrow} \sum_{j=0}^{k} \phi\big(I(b)_j^+\big), \quad \text{with } \sum_{j=0}^{k} \phi\big(I(b)_j^+\big) < \lambda_n < \sum_{j=0}^{k+1} \phi\big(I(b)_j^+\big).$$

Thus, from (4.16) we have

$$d\big(\widetilde{\pi}\big(I(b), \lambda_n\big), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } n \in \mathbb{N}.$$

When $n \to +\infty$, it follows by the right continuity of $\widetilde{\pi}$ that

$$d\left(\widetilde{\pi}\left(I(b), \sum_{j=0}^{k} \phi\big(I(b)_j^+\big)\right), y\right) \geqslant \frac{\epsilon}{2}.$$

In this way, we conclude that

$$d\big(\widetilde{\pi}\big(I(b), t\big), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } t \geqslant 0.$$

Then $y \notin \overline{\widetilde{\pi}^+(I(b))}$, which is a contradiction because $I(b) \in \widetilde{L}^+(x) \setminus M$ and thus $\overline{\widetilde{\pi}^+(I(b))} = \widetilde{L}^+(x)$ since the limit set is minimal.

- Suppose now that $\{w_n\}_{n\in\mathbb{N}} \subset H_2$. Let $\beta \in \,]0, \phi(b)[$. Since $w_n \overset{n\to+\infty}{\longrightarrow} b$, we have

$$\widetilde{\pi}(w_n, \beta) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b, \beta),$$

that is,

$$\widetilde{\pi}\left(x, t_n + \frac{T_n}{2} + \beta\right) \overset{n\to+\infty}{\longrightarrow} \widetilde{\pi}(b, \beta).$$

We note that $b_1 = \widetilde{\pi}(b, \beta) \in \widetilde{L}^+(x) \setminus M$, since $(t_n + \frac{T_n}{2} + \beta) \overset{n\to+\infty}{\longrightarrow} +\infty$ and $0 < \beta < \phi(b)$. By following the same steps done above one can obtain the following inequality

$$d\big(\widetilde{\pi}(b_1, t), y\big) \geqslant \frac{\epsilon}{2} \quad \text{for all } t \geqslant 0.$$

Thus, $y \notin \overline{\widetilde{\pi}^+(b_1)}$ which is again a contradiction since $b_1 \in \widetilde{L}^+(x) \setminus M$ and $\widetilde{L}^+(x)$ is minimal.

Therefore, the result is proved. □

5. Zhukovskij quasi stability

In this section, we devote our study to the theory of Zhukovskij quasi stability. On impulsive semidynamical systems, this kind of stability was initially introduced by Changming Ding in [15]. We start by presenting the concept of reparametrization and then we define the concept of Zhukovskij quasi stability. The reader may find many results about Zhukovskij quasi stability in [15].

Definition 5.1. A *time reparametrization* is a homeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h(0) = 0$.

Definition 5.2. A point $x \in X \setminus M$ is called *Zhukovskij quasi $\tilde{\pi}$-stable* if given any $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that if $d(y, x) < \delta$, then one can find a time reparametrization τ_y such that

$$d\big(\tilde{\pi}(x,t), \tilde{\pi}\big(y, \tau_y(t)\big)\big) < \epsilon \quad \text{for all } t \geqslant 0.$$

Moreover, if there is $\lambda > 0$ such that $d(y, x) < \lambda$ implies $d(\tilde{\pi}(x,t), \tilde{\pi}(y, \tau_y(t))) \xrightarrow{t\to+\infty} 0$, then x is called *asymptotically Zhukovskij quasi $\tilde{\pi}$-stable*.

A subset $A \subset X \setminus M$ is Zhukovskij (asymptotically Zhukovskij) quasi $\tilde{\pi}$-stable if each point $y \in A$ possesses this property.

Definition 5.3. A point $x \in X \setminus M$ is *uniformly asymptotically Zhukovskij quasi $\tilde{\pi}$-stable* provided that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for each $s \geqslant 0$ and $y \in B(\tilde{\pi}(x, s), \delta)$ one can find a time reparametrization τ_y such that $d(\tilde{\pi}(x, s+t), \tilde{\pi}(y, \tau_y(t))) < \epsilon$ holds for all $t \geqslant 0$, and also $d(\tilde{\pi}(x, s+t), \tilde{\pi}(y, \tau_y(t))) \xrightarrow{t\to+\infty} 0$. A subset $A \subset X \setminus M$ is uniformly asymptotically Zhukovskij quasi $\tilde{\pi}$-stable if each point $y \in A$ possesses this property.

Remark 5.1. As sighted in [15], taking $s = \tau_y(t)$ and $h(s) = \tau_y^{-1}(s) = t$, we can write

$$d\big(\tilde{\pi}(x,t), \tilde{\pi}\big(y, \tau_y(t)\big)\big) = d\big(\tilde{\pi}\big(x, h(s)\big), \tilde{\pi}(y, s)\big),$$

where we get an equivalent definition for Zhukovskij quasi $\tilde{\pi}$-stability.

In [15], the author shows that a sufficient condition for the limit set of a point x to be minimal is that x is uniformly asymptotically Zhukovskij quasi $\tilde{\pi}$-stable, see Lemma 4.4 there. We show below that a limit set is minimal provided the points of this set which are not in M are Zhukovskij quasi $\tilde{\pi}$-stable.

Theorem 5.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. Suppose that the nonempty set $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$-stable. Then $\tilde{L}^+(x)$ is a minimal set.*

Proof. Let $y \in \tilde{L}^+(x) \setminus M$ be arbitrary. We need to show that $\tilde{L}^+(x) = \overline{\tilde{\pi}^+(y)}$. Suppose to the contrary that there is $z \in \tilde{L}^+(x)$ such that $z \notin \overline{\tilde{\pi}^+(y)}$. Let $\epsilon = d(z, \overline{\tilde{\pi}^+(y)}) > 0$.

Let $q \in \tilde{\pi}^+(y)$. Note that $\tilde{\pi}^+(y) \subset \tilde{L}^+(x)$ by Lemma 3.3. By hypothesis, there is $\delta = \delta(q) > 0$ such that if $d(v, q) < \delta$, we can find a reparametrization τ_v such that

$$d\big(\tilde{\pi}(q,t), \tilde{\pi}\big(v, \tau_v(t)\big)\big) < \frac{\epsilon}{3} \quad \text{for all } t \geqslant 0. \tag{5.1}$$

Moreover, there is $s > 0$ such that $d(\tilde{\pi}(x,s), q) < \delta$. By (5.1) we may find a reparametrization τ_w, $w = \tilde{\pi}(x,s)$, such that

$$d\big(\tilde{\pi}(q,t), \tilde{\pi}\big(w, \tau_w(t)\big)\big) < \frac{\epsilon}{3} \quad \text{for all } t \geqslant 0. \tag{5.2}$$

On the other hand, since $z \in \tilde{L}^+(x)$ then $z \in \tilde{L}^+(w)$. Thus there is a sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$ and $\tilde{\pi}(w,t_n) \overset{n\to+\infty}{\longrightarrow} z$. Let $n_0 > 0$ be such that $d(z, \tilde{\pi}(w, t_{n_0})) < \frac{\epsilon}{2}$. Let $s_n > 0$ be such that $t_n = \tau_w(s_n)$, $n \in \mathbb{N}$. Then

$$d\big(\tilde{\pi}(q, s_{n_0}), \tilde{\pi}\big(w, \tau_w(s_{n_0})\big)\big) \geqslant d\big(z, \tilde{\pi}(q, s_{n_0})\big) - d\big(z, \tilde{\pi}\big(w, \tau_w(s_{n_0})\big)\big) \geqslant \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2},$$

which contradicts (5.2). Therefore, $\tilde{L}^+(x)$ is minimal. □

Corollary 5.1. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system and $x \in X$. Suppose that $\tilde{L}^+(x)$ is compact and $\tilde{L}^+(x) \setminus M \neq \emptyset$.*

a) *If $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$-stable then $\tilde{L}^+(x) \setminus M$ is $\tilde{\pi}$-recurrent;*
b) *If $x \notin M$ and $\tilde{\pi}^+(x)$ is uniformly asymptotically Zhukovskij quasi $\tilde{\pi}$-stable then $\tilde{L}^+(x) \setminus M$ is $\tilde{\pi}$-recurrent.*

Proof. a) By Theorem 5.1 the limit set $\tilde{L}^+(x)$ is minimal. By Theorem 3.3 the points of $\tilde{L}^+(x) \setminus M$ are $\tilde{\pi}$-recurrent. b) In this case, the minimality of $\tilde{L}^+(x)$ follows by Lemma 4.4 of [15] and the conclusion follows by Theorem 3.3. □

Theorem 5.2. *Let $(X, \pi; M, I)$ be an impulsive semidynamical system, X complete and $x \in X \setminus M$. Assume $\tilde{L}^+(x) = \tilde{\pi}^+(x) \cup \{x_k\}_{k\geqslant 1}$.*

a) *If $\tilde{L}^+(x) \setminus M$ is Zhukovskij quasi $\tilde{\pi}$-stable then x is periodic;*
b) *If $\tilde{\pi}^+(x)$ is uniformly asymptotically Zhukovskij quasi $\tilde{\pi}$-stable then x is periodic.*

Proof. a) First, we note that $\tilde{L}^+(x) \setminus M \neq \emptyset$. By Theorem 5.1, the set $\tilde{L}^+(x)$ is minimal. Since $\tilde{L}^+(x) = \tilde{\pi}^+(x) \cup \{x_k\}_{k\geqslant 1}$, it follows by Corollary 3.1 that x is periodic. b) It is enough to use Lemma 4.4 of [15] in the proof of item a). □

In the sequel, we obtain some results about uniform attractors.

The first lemma concerns the relation between the sets $\tilde{L}^+(x)$ and $\tilde{J}^+(x)$, $x \in X$. A proof of it may be found in [15, Theorem 4.2].

Lemma 5.1. *If $x \in X \setminus M$ is Zhukovskij quasi $\tilde{\pi}$-stable, then $\tilde{L}^+(x) = \tilde{J}^+(x)$.*

By Lemma 5.1 and Proposition 3.1, we have the following result.

Lemma 5.2. *Suppose that X is locally compact, $x \in X \setminus M$ is Zhukovskij quasi $\tilde{\pi}$-stable and $\tilde{L}^+(x)$ is a nonempty compact set. Then $x \in \tilde{P}_u^+(\tilde{L}^+(x))$.*

Next, we present sufficient conditions for a limit set to belong to its uniform region of attraction.

Proposition 5.1. *Suppose that X is locally compact, $\widetilde{L}^+(x) \cap M = \emptyset$ and $\widetilde{L}^+(x)$ is Zhukovskij quasi $\widetilde{\pi}$-stable. If $\widetilde{L}^+(x)$ is compact then $\widetilde{L}^+(x) \subset \widetilde{P}_u^+(\widetilde{L}^+(x))$.*

Proof. Since $\widetilde{L}^+(x) \cap M = \emptyset$ we have that $\widetilde{L}^+(x)$ is positively $\widetilde{\pi}$-invariant by Lemma 3.3. Now, let $y \in \widetilde{L}^+(x)$. By the positive $\widetilde{\pi}$-invariance and closedness of $\widetilde{L}^+(x)$, we have $\overline{\widetilde{\pi}^+(y)} \subset \widetilde{L}^+(x)$. Consequently, $\widetilde{L}^+(y) \subset \widetilde{L}^+(x)$. Note that $\widetilde{L}^+(y) \neq \emptyset$ by compactness of $\widetilde{L}^+(x)$.

Since y is Zhukovskij quasi $\widetilde{\pi}$-stable it follows that $\widetilde{L}^+(y) = \widetilde{J}^+(y)$, see Lemma 5.1. Hence, $\widetilde{J}^+(y) \neq \emptyset$ and $\widetilde{J}^+(y) \subset \widetilde{L}^+(x)$. The result follows by Proposition 3.1. □

In the last result, we give conditions for a limit set to be a uniform $\widetilde{\pi}$-attractor.

Theorem 5.3. *Let X be locally compact and $\widetilde{L}^+(x) \cap M = \emptyset$ for some $x \in X$. If $\widetilde{L}^+(x)$ is a nonempty asymptotically Zhukovskij quasi $\widetilde{\pi}$-stable set with $\widetilde{L}^+(x)$ compact then $\widetilde{L}^+(x)$ is uniformly $\widetilde{\pi}$-attractor.*

Proof. For each $y \in \widetilde{L}^+(x)$, there is $\delta_y > 0$ such that if $d(z, y) < \delta_y$ then we may find a time reparametrization τ_z such that

$$d\big(\widetilde{\pi}(y,t), \widetilde{\pi}\big(z, \tau_z(t)\big)\big) \stackrel{t\to+\infty}{\longrightarrow} 0.$$

Since $\widetilde{L}^+(x) \cap M = \emptyset$ and $\widetilde{L}^+(x)$ is compact, there is $\beta > 0$ such that $B(\widetilde{L}^+(x), \beta) \cap M = \emptyset$.

By compactness of the limit set, there are $y_1, \ldots, y_n \in \widetilde{L}^+(x)$ such that $\widetilde{L}^+(x) \subset B(y_1, \frac{\delta_1}{2}) \cup \cdots \cup B(y_n, \frac{\delta_n}{2})$. Set $2\delta = \min\{\delta_{y_1}, \ldots, \delta_{y_n}, \beta\}$.

We claim that $B(\widetilde{L}^+(x), \delta) \subset \widetilde{P}_u^+(\widetilde{L}^+(x))$. In fact, let $w \in B(\widetilde{L}^+(x), \delta)$. Then $w \in B(y_j, \delta_j)$ for some $j \in \{1, \ldots, n\}$. Then

$$d\big(\widetilde{\pi}(y_j,t), \widetilde{\pi}\big(w, \tau_w(t)\big)\big) \stackrel{t\to+\infty}{\longrightarrow} 0,$$

or,

$$d\big(\widetilde{\pi}\big(y_j, h(t)\big), \widetilde{\pi}(w,t)\big) \stackrel{t\to+\infty}{\longrightarrow} 0,$$

where $h(t) = \tau_w^{-1}(t)$, see Remark 5.1. Since $\widetilde{\pi}^+(y_j) \subset \widetilde{L}^+(x)$ (because $\widetilde{L}^+(x)$ is positively $\widetilde{\pi}$-invariant) we have

$$d\big(\widetilde{L}^+(x), \widetilde{\pi}(w,t)\big) \stackrel{t\to+\infty}{\longrightarrow} 0.$$

Then $\widetilde{L}^+(w) \neq \emptyset$ and $\widetilde{L}^+(w) \subset \widetilde{L}^+(x)$. It is clear that $\widetilde{J}^+(w) \neq \emptyset$. Now, let us verify that $\widetilde{J}^+(w) \subset \widetilde{L}^+(x)$. Indeed, let $a \in \widetilde{L}^+(w)$. Since $w \notin M$ $(\delta < \beta)$ it follows by Lemma 3.4 that

$$\widetilde{J}^+(w) \subset \widetilde{J}^+(a).$$

Since $a \in \widetilde{L}^+(x)$, it is Zhukovskij quasi $\widetilde{\pi}$-stable and by Lemma 5.1 we have $\widetilde{J}^+(a) = \widetilde{L}^+(a)$. But $\widetilde{L}^+(x)$ is positively $\widetilde{\pi}$-invariant. Then $\overline{\widetilde{\pi}^+(a)} \subset \widetilde{L}^+(x)$, consequently, $\widetilde{L}^+(a) \subset \widetilde{L}^+(x)$. Then

$$\widetilde{J}^+(w) \subset \widetilde{L}^+(x).$$

Hence, by Proposition 3.1 we have $w \in \widetilde{P}_u^+(\widetilde{L}^+(x))$ and the result follows. □

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