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On the equations of thermally radiative magnetohydrodynamics

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Abstract

An initial–boundary value problem is considered for the viscous compressible thermally radiative magnetohydrodynamic (MHD) flows coupled to self-gravitation describing the dynamics of gaseous stars in a bounded domain of $\mathbb{R}^3$. The conservative boundary conditions are prescribed. Compared to Ducomet–Feireisl [13] (also see, for instance, Feireisl [18], Feireisl–Novotný [20]), a rather more general constitutive relationship is given in this paper. The analysis allows for the initial density with vacuum. Every transport coefficient admits a certain temperature scaling. The global existence of a variational (weak) solution with any finite energy and finite entropy data is established through a three-level approximation and methods of weak convergence.

Keywords: Magnetohydrodynamic (MHD) flows; Compressible; Thermally radiative; Global existence; Variational (weak) solution

1. Introduction

Magnetohydrodynamics (MHD) concerns the motion of conducting fluids (cf. gases) in an electromagnetic field with a very broad range of applications in physical areas from liquid metals to cosmic plasmas. In moving conducting magnetic fluids, magnetic fields can induce electric fields, and electric currents are developed, which create forces on the fluids and considerably

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affect changes in the magnetic fields. The dynamic motion of the fluids and the magnetic field interact strongly with each other and both the hydrodynamic and electrodynamic effects have to be taken into account. Except for this, considerable attention has been put to study the effects of thermal radiation recently, because the radiation field significantly affects the dynamics of fluids, for example, certain re-entry of space vehicles, astrophysical phenomena and nuclear fusion, and hydrodynamics with explicit account of radiation energy and momentum contribution constitutes the character of radiation hydrodynamics. In this paper, we consider the viscous compressible thermally radiative conducting fluids driven by the self-gravitation in the full magnetohydrodynamic setting. The equations to the three-dimensional full magnetohydrodynamic flows have the following form [3,13,29,30]:

$$
\begin{cases}
\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \quad \mathbf{x}\in\Omega\subset R^3,\ t>0,\\
(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p = \nabla\cdot\mathbb{S} + \rho\nabla\Psi + (\nabla\times\mathbf{H})\times\mathbf{H},\\
\mathcal{E}_t + \nabla\cdot\left(\left(\rho e + \frac{1}{2}\rho|\mathbf{u}|^2 + p\right)\mathbf{u}\right) + \nabla\cdot\mathbf{q}\\
\quad = \nabla\cdot\big((\mathbf{u}\times\mathbf{H})\times\mathbf{H} + \nu\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbb{S}\mathbf{u}\big) + \rho\nabla\Psi\cdot\mathbf{u},\\
\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}), \quad \nabla\cdot\mathbf{H} = 0,
\end{cases}
\tag{1.1}
$$

where $\rho\in\mathbb{R}$ denotes the density, $\mathbf{u}\in\mathbb{R}^3$ the fluid velocity and $\mathbf{H}\in\mathbb{R}^3$ the magnetic field, $p\in\mathbb{R}$ the pressure.

$$
\mathcal{E} = \rho e + \frac{1}{2}\big(\rho|\mathbf{u}|^2 + |\mathbf{H}|^2\big)
$$

is the total energy with e being the specific internal energy. $\mathbb{S}$ stands for the viscous stress tensor, given by Newton's law of viscosity:

$$
\mathbb{S} = \mu\big(\nabla\mathbf{u} + \nabla^\top\mathbf{u}\big) + \lambda(\nabla\cdot\mathbf{u})\mathbb{I}_3 \tag{1.2}
$$

with μ the shear viscosity coefficient and $\eta = \lambda + \frac{2}{3}\mu$ the bulk viscosity coefficient of the flow (while μ should be positive for any "genuinely" viscous fluid, η may vanish, e.g. for a monoatomic gas), $\mathbb{I}_3$ the 3×3 identity matrix and $\nabla^\top\mathbf{u}$ the transpose of the matrix $\nabla\mathbf{u}$. Note that

$$
\nabla\cdot\mathbb{S} = \left(\eta + \frac{1}{3}\mu\right)\nabla(\nabla\cdot\mathbf{u}) + \mu\Delta\mathbf{u},
$$

$$
\mathbb{S}:\nabla\mathbf{u} = \mu|\nabla\mathbf{u}|^2 + \mu\nabla\mathbf{u}:\nabla^\top\mathbf{u} + \left(\eta - \frac{2}{3}\mu\right)(\nabla\cdot\mathbf{u})^2.
$$

$\mathbf{q}$ is the heat flux obeying the classical Fourier's law:

$$
\mathbf{q} = -\kappa\nabla\vartheta, \quad \kappa\ge 0, \tag{1.3}
$$

where ϑ means the absolute temperature, κ is the heat conductivity coefficient. The term $\rho\nabla\Psi$ is the gravitational force where the potential Ψ obeys Poisson's equation on the whole physical space $\mathbb{R}^3$ which is

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$$-\Delta\Psi = G\rho \quad \text{with a constant } G > 0,$$

where ρ was extended to be zero outside Ω. The coefficient $\nu > 0$ is termed the magnetic diffusivity of the fluid. Usually, we refer to Eq. $(1.1)_1$ as the continuity equation (mass conservation equation), $(1.1)_2$ and $(1.1)_3$ as the momentum and the total energy conservation equation, respectively. It is well-known that the electromagnetic fields are governed by the Maxwell equations. In magnetohydrodynamics, the displacement currents can be neglected in the time dependent Maxwell equations (see [22,29,30]), which transforms the hyperbolic Maxwell's system into a parabolic equation from a mathematical viewpoint. Accordingly, Eq. $(1.1)_4$ is called the induction equation, and the electric field $\mathbf{E}$ is related to the magnetic induction vector $\mathbf{H}$ and the fluid velocity $\mathbf{u}$ via Ampère's law:

$$\nu\nabla \times \mathbf{H} = \mathbf{E} + \mathbf{u} \times \mathbf{H}.$$

As for the constraint $\nabla \cdot \mathbf{H} = 0$, it can be seen just as a restriction on the initial value $\mathbf{H}_0$, since $(\nabla \cdot \mathbf{H})_t \equiv 0$. The equations in (1.1) describe the macroscopic behavior of the magnetohydrodynamic flow with dissipative mechanisms. Magnetic reconnection is thought to be the mechanism responsible for the conversion of magnetic energy into heat and fluid motion (cf. [3,8]).

Next, we turn to the pressure–density–temperature (pdt) state equation. The well-known case is the ideal gas flow provided by Boyle's law:

$$p_G(\rho, \vartheta) = R\rho\vartheta,$$

where R is a constant inversely proportional to the mean molecular weight of the gas (cf. [18]). However, Boyle's law is definitely not satisfactory in the high temperature and density regime physically relevant to general viscous fluids in the full thermodynamical setting. For example, it is known the pressure of highly condensed cold matter is proportional to $\rho^{\frac{5}{3}}$ (see Chapters 3, 11 of [43]), also the isentropic state equation for a perfect monoatomic gas. In this paper, we will consider a much more general constitutive relationship than that introduced in [18], the so-called constitutive law for pressure, i.e., $p_G(\rho, \vartheta)$ will be determined via

$$p_G(\rho, \vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) \tag{1.4}$$

with the elastic pressure p_e and the thermal pressure components p_ϑ, p_{ϑ^2} being C^1 functions of the density. In particular, for the so-called electronic pressure, one has $p_G(\rho, \vartheta) = p_e(\rho) + R\rho\vartheta + \sqrt{\rho}\vartheta^2$ (cf. [43]). From the mathematical point of view, (1.4) can be understood as the first three terms in the Taylor expansion:

$$p_G(\rho, \vartheta) = p_G(\rho, \Theta) + (\vartheta - \Theta)\frac{\partial p_G}{\partial \vartheta}(\rho, \Theta) + \frac{(\vartheta - \Theta)^2}{2}\frac{\partial^2 p_G}{\partial^2 \vartheta}(\rho, \Theta) + \text{higher order terms}$$

for a given $\Theta > 0$.

In addition, it is worth-noting that the regularizing effect of radiation has been already observed in [9]. The radiation pressure is attributed to photons of very high energy, for example, the radiation energy associated with Planck distribution varies as the fourth power of the temperature, and the importance of the thermal radiation increases as the temperature is raised. Especially, at high temperatures, a completely different mechanism of heat energy transfer appears due to radiation, the energy and momentum densities of radiation field may become comparable to or even dominate the corresponding fluid quantities, for example, the heat conductivity coefficient κ becomes a rather sensitive function of temperature. As a consequence, the total pressure in fluid is augmented through the effect of high temperature radiation, by a radiation component $p_R(\vartheta)$ related to the absolute temperature through

$$p_R(\vartheta) = \frac{a}{3}\vartheta^4$$

with the Stefan–Boltzmann constant $a > 0$ (see [2,11,37], also see Chapter 15 in [15]).

To conclude, we have the equation of state

$$p = p(\rho, \vartheta) = p_G(\rho, \vartheta) + p_R(\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4 \tag{1.5}$$

in this paper, which relates the pressure with the density and the absolute temperature of the flow.

Given the (pdt) state equation discussed above, note that the basic principle of the second law of thermodynamics implies that the internal energy and pressure are interrelated through Maxwell's relationship, we define the specific entropy s, up to an additive constant, through the thermodynamics equation:

$$\vartheta Ds(\rho, \vartheta) = De(\rho, \vartheta) + p(\rho, \vartheta) D\left(\frac{1}{\rho}\right). \tag{1.6}$$

The quantity $\frac{1}{\theta}(De + pD(\frac{1}{\rho}))$ must be a perfect gradient, which is the well-known Gibbs' relation on p, e and s, implying that e and p are interrelated through

$$\frac{\partial e}{\partial \rho} = \frac{1}{\rho^2}\left(p - \vartheta\frac{\partial p}{\partial \vartheta}\right) = \frac{1}{\rho^2}\left(p_e(\rho) - \vartheta^2 p_{\vartheta^2}(\rho) - a\vartheta^4\right) \tag{1.7}$$

(see e.g. Chapter 3 in [1]). In fact, the equality (1.7) comes from $\frac{\partial^2 s}{\partial\rho\partial\vartheta} = \frac{\partial^2 s}{\partial\vartheta\partial\rho}$.

Accordingly, e can be written in the form:

$$e = P_e(\rho) - \vartheta^2 P_{\vartheta^2}(\rho) + \frac{a\vartheta^4}{\rho} + Q(\vartheta),$$

where

$$P_e(\rho) = \int_1^\rho \frac{p_e(z)}{z^2}\,dz \text{ is the elastic potential,}$$

$$P_{\vartheta^2}(\rho) = \int_1^\rho \frac{p_{\vartheta^2}(z)}{z^2}\,dz,$$

and the thermal energy contribution Q is a non-decreasing function of ϑ. Here

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$$Q(\vartheta) = \int_0^\vartheta c_v(\xi)\, d\xi,$$

where $c_v(\vartheta)$ denotes the specific heat at constant volume such that

$$c_v \in C^1\big([0,\infty)\big), \qquad \inf_{\vartheta\in[0,\infty)} c_v(\vartheta) > 0.$$

The subsequent analysis leans essentially on thermodynamic stability of the fluid system expressed through

$$\frac{\partial p}{\partial \rho} > 0, \qquad \frac{\partial e}{\partial \vartheta} > 0 \quad \text{for all } \rho, \vartheta > 0.$$

Taking the high temperature and density regime physically relevant to our model equation into account, we can suppose

$$\begin{cases} p_e(0) = p_\vartheta(0) = p_{\vartheta^2}(0) = 0, \\ p_e'(\rho) \geq a_1\rho^{\gamma-1} - b_1, \quad p_\vartheta'(\rho) \geq 0, \quad p_{\vartheta^2}'(\rho) \geq 0, \\ p_e(\rho) \leq a_2\rho^\gamma + b_2, \quad p_\vartheta(\rho) \leq a_3\rho^\varsigma + b_3, \quad p_{\vartheta^2}(\rho) \leq a_4\rho^\zeta + b_4, \end{cases} \tag{1.8}$$

with $a_1 > 0$, $\gamma \geq 2$, $\gamma > \frac{4}{3}\varsigma$, $\gamma > 2\zeta$. We remark here that p_e need not be a non-decreasing function of ρ.

Many theoretical studies have been devoted to the global-in-time existence of solutions with large data for the multidimensional continuum isothermal or isentropic fluid mechanics and electrodynamics (see [17,21,30,32,35,36,42]), especially for the magnetohydrodynamics because of its physical importance, complexity, rich phenomena and mathematical challenges; see [3,6,13,14,16,22,24,25,29,39,40] and the references cited therein. Note that the existence problem for a general full system including the energy equation is far from being solved. It is not know whether there is a classical (smooth) solution of system (1.1) with large initial data on an arbitrary time interval $(0, T)$ or not, even for the one-dimensional full perfect MHD equations with large data when all the viscosity, heat conductivity and magnetic diffusivity coefficients are constants, or for the three-dimensional Navier–Stokes equations describing the motion of compressible (incompressible) fluids. The simplest and most interesting case of the ideal gas flow with the viscosity coefficients and the heat conductivity coefficient being constants is completely open. P.-L. Lions [32] gives a formal proof of weak stability under the additional hypothesis of boundedness of ρ, $\mathbf{u}$ and ϑ in $L^\infty(\Omega \times (0, T))$. The corresponding problem for the one-dimensional Navier–Stokes equations was solved in [27] in the seventies last century. For the gases in one-dimension with small smooth initial data, the existence of global solutions was proved in [26], and the large-time behavior was studied in [33]. For large initial data, additional difficulties appear because of the presence of the magnetic field and its interaction with the hydrodynamic motion of the flow of large oscillation. Chen and Wang [5] investigated a free boundary problem for plane magnetohydrodynamic flows with general large initial data in 1-D and established the existence, uniqueness, and regularity of global solutions in H^1. Taking the effect of self-gravitation and the influence of high temperature radiation into account, global existence and uniqueness of a classical solution with large initial data was proved in [44] under a general assumption on the heat conductivity while all the viscosity, and magnetic diffusivity coefficients are constants. Based on

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the concept of variational (weak) solutions in the spirit of Leray's pioneering work (see [31]) in the context of incompressible, linearly viscous fluids, the existence theory was extended to the full Navier–Stokes system, including the thermal energy equation, under certain mostly technical hypothesis imposed on the quantities appearing in the constitutive equations (see Theorem 7.1 in [18]) by Ducomet and Feireisl. They first developed the global existence of variational solution in [18] under the assumptions that the viscosity coefficients μ and λ must be constant, while the heat conductivity coefficient κ depends on the temperature ϑ, and later they extended the result in [19] when μ and λ depend on ϑ. More complex, the effects of self-gravitation as well as the influence of radiation on the dynamics at high temperature regimes were included in [12]. Using the similar technique as in [12,18,19], Ducomet and Feireisl [13] studied the full compressible MHD equations while considering the effects of self-gravitation and the influence of radiations on the dynamics at high temperature regimes. Under the assumption that the viscosity coefficients depend on the temperature and the magnetic field, the pressure behaves like the power law $\rho^{\frac{5}{3}}$ for large density (reminiscent of the isentropic state equation for a perfect monoatomic gas), and all the transport coefficients satisfy certain $(1+\vartheta^\alpha)$-growth conditions for any $\alpha \in [1, \frac{65}{27})$, they introduced the total entropy balance as one of main field equations and proved the global existence of variational solution to any finite energy data on a bounded spatial domain in $\mathbb{R}^3$, supplemented with conservative boundary conditions. The reader is also referred to the monograph [20] for more details about the system and the methods. Hu and Wang [24] considered a 3-D model problem for full compressible MHD flows with more general pressure, by using the thermal equation as in [19] instead of the entropy equation employed in [13,20], they proved the existence of a global variational weak solution to the MHD equations with large data.

We shall study the global existence of the variational (weak) solutions to the real magnetohydrodynamic flows, with general pressure and internal energy while permitting the generation of heat by the magnetic field as well as its interaction with the fluid motion, in a bounded domain Ω in $\mathbb{R}^3$. In this paper, we supplement system (1.1) with the following initial and boundary conditions:

$$\begin{cases} (\rho, \rho\mathbf{u}, \vartheta, \mathbf{H})|_{t=0} = (\rho_0, \mathbf{m}_0, \vartheta_0, \mathbf{H}_0), \quad \text{for } \mathbf{x} \in \Omega, \\ \rho_0 \geq 0, \quad \operatorname*{ess\,inf}_{\Omega} \vartheta_0 > 0, \\ \rho_0 \in L^\gamma(\Omega), \quad \mathbf{m}_0 \in L^1(\Omega), \quad \dfrac{1}{\rho_0}|\mathbf{m}_0|^2 \in L^1(\Omega), \quad \big(\rho Q(\vartheta)\big)_0 = \rho_0 Q(\vartheta_0) \in L^1(\Omega), \\ \vartheta_0 \in L^\infty(\Omega), \quad \mathbf{H}_0 \in L^2(\Omega), \quad \nabla \cdot \mathbf{H}_0 = 0 \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{1.9}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \qquad \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad \text{and} \quad \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad (\nabla\times\mathbf{H})\times\mathbf{n}|_{\partial\Omega} = \mathbf{0}, \tag{1.10}$$

where $\mathbf{n}$ denotes the unit outward normal on $\partial\Omega$. The boundary condition prescribed on the velocity is the so-called non-slip boundary condition, on the temperature is the conservation boundary condition, which means the system is thermally insulated (isolated), and on the magnetic field is known as the perfectly conducting wall condition which describes the case where the wall of container is made of perfectly conductive materials. Such boundary conditions are classical in the theory of magnetohydrodynamics and conform to that the physical system (1.1) is energetically isolated. Note here

(i) $\mathcal{D}$ denotes C_0^∞, and $\mathcal{D}'$ for the sense of distributions;
(ii) $\frac{1}{\rho_0}|\mathbf{m}_0|^2 \in L^1(\Omega)$ indicates $\mathbf{m}_0 = \mathbf{0}$ a.a. $\mathbf{x} \in \{\rho_0 = 0\}$.

The problem considered in our paper seems more rational and physically valid in many astrophysical models, since it is well known that the dynamics of gaseous stars in astrophysics is dominated by intense magnetic fields, self-gravitation, and high temperature radiation (cf. [7,41]).

Given the rather poor *a priori* estimates (ensuring equi-integrability, or weak L^1 compactness of the quantities appearing in the corresponding balance laws) available for the MHD equations, approximate (or even exact) solutions are bounded only in the Lebesgue spaces of integrable functions, and, consequently, any existence theory must be built up on the methods of weak convergence. The idea of approximation was used in [12,20], where detailed existence proofs for simpler systems were given. In addition, the constitutive relations concerning the pressure in this paper are more general: we need to deal with the new terms in the (pdt) state equation, also for the general form of the thermal energy contribution $Q(\vartheta)$; and overcome the difficulty arising from the presence of the magnetic field and its coupling and interaction with the fluid variables. The heat conductivity is more complicated, not depending solely on the temperature. Except for the total energy conservation, we will formally obtain an entropy-type energy estimate involving the dissipative effects of viscosity, magnetic diffusion, and heat diffusion, which are essential to deduce the required available *a priori* estimates on the velocity, the magnetic induction vector and the temperature from boundedness of the initial total energy and the initial total entropy of the system by our careful analysis.

We introduce a suitable variational formulation of the problem and state the main existence result following a series of *a priori* estimates on the formal solution in Section 2, employ a three-level approximation scheme (see, for instance, [12,13,18–20,24]) to construct a sequence of approximation solutions in Section 3, and show the existence of global variational (weak) solution with large initial data in the last four sections. Our main result will be proved successively through the Galerkin method, a vanishing viscosity and vanishing artificial pressure limit passage using the methods of weak convergence.

2. Notations and results

2.1. Notations

(1) $\Omega_T = \Omega \times (0, T)$ for some fixed time $T > 0$.
(2) For $k \geq 1$ and $p \geq 1$, denote $W^{k,p} = W^{k,p}(\Omega)$ for the Sobolev space, whose norm is denoted as $\|\cdot\|_{W^{k,p}}$, and $H^k = W^{k,2}(\Omega)$. For $T > 0$ and a function space X, denote by $L^p(0, T; X)$ the set of Bochner measurable X-valued time dependent functions f such that $t \to \|f\|_X$ belongs to $L^p(0, T)$, and the corresponding Lebesgue norm is denoted by $\|\cdot\|_{L^p_T(X)}$.

Let us consider first a classical solution $(\rho, \mathbf{u}, \vartheta, \mathbf{H})$ of the problem (1.1), (1.9), (1.10) in Ω_T. Observe from the continuity equation that the total mass is a constant of motion, i.e., we obtain the conservation of mass in the integral form:

$$\int_\Omega \rho(t)\, d\mathbf{x} = \int_\Omega \rho_0\, d\mathbf{x} \quad \text{for all } t \in [0, T]. \tag{2.1}$$

Note that if we multiply the continuity equation by $b'(\rho)$, where $b \in C^1((0, \infty))$ and usually its derivative vanishes for large arguments (see, for instance, [10]), the renormalized continuity equation is obtained:

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$$\big(b(\rho)\big)_t + \nabla\cdot\big(b(\rho)\mathbf{u}\big) + \big(b'(\rho)\rho - b(\rho)\big)\nabla\cdot\mathbf{u} = 0. \tag{2.2}$$

Multiplying the momentum equation by $\mathbf{u}$, the induction equation by $\mathbf{H}$, and inserting the results into the total energy equation, we get the following internal energy balance

$$(\rho e)_t + \nabla\cdot(\rho e\mathbf{u}) + p\nabla\cdot\mathbf{u} = \mathbb{S}:\nabla\mathbf{u} - \nabla\cdot\mathbf{q} + \nu|\nabla\times\mathbf{H}|^2, \tag{2.3}$$

where $A:B$ denotes the scalar product of the two matrices A and B.

Recalling $\mathbf{q} = -\kappa\nabla\vartheta$ and the state equation

$$p(\rho,\vartheta) = p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4,$$
$$e(\rho,\vartheta) = P_e(\rho) - \vartheta^2 P_{\vartheta^2}(\rho) + \frac{a}{\rho}\vartheta^4 + Q(\vartheta),$$

we get the thermal energy equation

$$\begin{aligned}&\big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)_t + \nabla\cdot\big(\big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)\mathbf{u}\big) - \nabla\cdot(\kappa\nabla\vartheta)\\ &\quad= \mathbb{S}:\nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2 - \left(\vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4\right)\nabla\cdot\mathbf{u},\end{aligned} \tag{2.4}$$

where $\mathbb{S}:\nabla\mathbf{u}$ is termed the dissipation function responsible for the irreversible transfer of the mechanical energy into heat. Here we have used the fact that

$$\big(\rho P_e(\rho)\big)_t + \nabla\cdot\big(\rho P_e(\rho)\mathbf{u}\big) + p_e(\rho)\nabla\cdot\mathbf{u} = 0.$$

Moreover, if the temperature is strictly positive, multiplying (2.4) by $\frac{1}{\vartheta}$ and using the continuity equation, we obtain the entropy equation

$$(\rho s)_t + \nabla\cdot(\rho s\mathbf{u}) - \nabla\cdot\left(\frac{\kappa\nabla\vartheta}{\vartheta}\right) = \frac{\mathbb{S}:\nabla\mathbf{u} + \nu|\nabla\times\mathbf{H}|^2}{\vartheta} + \frac{\kappa|\nabla\vartheta|^2}{\vartheta^2}, \tag{2.5}$$

where the entropy

$$s = s(\rho,\vartheta) = \frac{4}{3}\frac{a\vartheta^3}{\rho} + \int_1^\vartheta \frac{c_v(\xi)}{\xi}\,d\xi - P_\vartheta(\rho) - 2\vartheta P_{\vartheta^2}(\rho)$$

with

$$P_\vartheta(\rho) = \int_1^\rho \frac{p_\vartheta(z)}{z^2}\,dz.$$

According to the Clausius–Duhem inequality (the second law of thermodynamics), the right-hand side of (2.5) must be non-negative for any possible motion, thus in particular, the viscosity coefficients μ, η for the Newtonian fluid and the magnetic diffusivity coefficient ν must be non-negative. Experiments show that the viscosity of fluids is quite sensitive to changes in temperature, for example, viscosity of gases increases with temperature, also of liquids decreases. The total heat-conductivity

$$\kappa := \kappa(\rho, \vartheta, \mathbf{H}) = \kappa_G(\rho, \vartheta, \mathbf{H}) + \kappa_R \vartheta^3$$

where $\kappa_R > 0$ is a constant (see [2]), and $\kappa_G > 0$ satisfies certain growth conditions. For the sake of simplicity, but not without certain physical background, also in agreement with numerous practical experiments, we shall assume all the transport coefficients admit some temperature scalings. Specifically, we assume

$$0 < c_1\big(1 + \vartheta^\alpha\big) \le \mu(\vartheta, \mathbf{H}) \le c_2(1 + \vartheta)^\alpha,$$
$$0 < c_3 \vartheta^\alpha \le \eta(\vartheta, \mathbf{H}) \le c_4(1 + \vartheta)^\alpha$$

for some constant $\alpha \ge \frac{1}{2}$, and set

$$0 < c_5\big(1 + \vartheta^\beta\big) \le \nu(\rho, \vartheta, \mathbf{H}), \qquad \kappa_G(\rho, \vartheta, \mathbf{H}) \le c_6\big(1 + \vartheta^\beta\big), \quad \text{and}$$
$$c_v(\vartheta) \le c_7\big(1 + \vartheta^{\frac{\beta}{2}-1}\big) \tag{2.6}$$

with $\beta \ge 1$ to be specified below. Note that we only consider the case when the viscosity coefficients are independent of the density (though being physically relevant) to avoid unsurmountable technical details in mathematics. The condition on $\kappa_G(\rho, \vartheta, \mathbf{H})$ is physically reasonable as experiments predict the value of $\beta \approx 4.5$–5.5 while Q should behave like $\vartheta^{1.5}$ for large arguments, which is in good agreement with the hypothesis on $c_v(\vartheta)$ (cf. [43]). We remark here that, if the magnetic field is absent, it has been shown by methods of statistical thermodynamics that $\mu = c\vartheta^{\frac{1}{2}}$ for a gas under normal conditions, and meanwhile, the coefficients of viscosity in gases show only little dependence on the density (see, for instance, Chapter 10 in [4]). The idea to impose several kinds of temperature scalings on the transport coefficients was inspired by [12, 13,18,23]. The effect of the magnetic field is indeed very complicated because the viscous stress becomes unisotropic depending effectively on the direction of $\mathbf{H}$ (see Section 19.44 in [4]).

Since the gravitational potential Ψ can be determined by the boundary value problem:

$$\begin{cases} -\Delta\Psi = G\rho, & (t, \mathbf{x}) \in \Omega_T, \\ \Psi|_{\partial\Omega} = 0, \end{cases} \tag{2.7}$$

by using the maximum principle, $\Psi \ge 0$ in Ω_T, and

$$\Psi = G(-\Delta)^{-1}[\rho] \quad \text{with } (-\Delta)^{-1}[\rho](\mathbf{x}) = \mathcal{F}_{\xi\to\mathbf{x}}\big[|\xi|^2 \mathcal{F}_{\mathbf{x}\to\xi}[\rho]\big], \tag{2.8}$$

where $\mathcal{F}$ stands for the Fourier transform.

Moreover, taking advantage of the continuity equation, we have

$$\int_\Omega \rho\nabla\Psi \cdot \mathbf{u}\, d\mathbf{x} = -\int_\Omega \Psi\nabla\cdot(\rho\mathbf{u})\, d\mathbf{x} = \int_\Omega \Psi\rho_t\, d\mathbf{x} = \frac{1}{2G}\frac{d}{dt}\int_\Omega |\nabla\Psi|^2\, d\mathbf{x}$$
$$= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho\Psi\, d\mathbf{x} = -\frac{G}{2}\frac{d}{dt}\int_\Omega \Delta^{-1}[\rho]\rho\, d\mathbf{x}.$$

From the total energy equation and the boundary conditions (1.10), we deduce that the total energy of the system is a constant of motion, i.e., the total energy is conserved,

$$\frac{d}{dt}\int_\Omega \left(\rho P_e(\rho) - \rho\vartheta^2 P_{\vartheta^2}(\rho) + a\vartheta^4 + \rho Q(\vartheta) + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2 + \frac{G}{2}\Delta^{-1}[\rho]\rho\right) d\mathbf{x} = 0,$$

$$\begin{aligned} E(t) &= \int_\Omega \left(\rho P_e(\rho) - \rho\vartheta^2 P_{\vartheta^2}(\rho) + a\vartheta^4 + \rho Q(\vartheta) + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2 + \frac{G}{2}\Delta^{-1}[\rho]\rho\right) d\mathbf{x} \\ &= E_0 \end{aligned} \tag{2.9}$$

for a.a. $t \in (0,T)$, where

$$\begin{aligned} E_0 = \int_\Omega & \rho_0 P_e(\rho_0) - \rho_0\vartheta_0^2 P_{\vartheta^2}(\rho_0) + a\vartheta_0^4 + \rho_0 Q(\vartheta_0) \\ & + \frac{1}{2\rho_0}|\mathbf{m}_0|^2 + \frac{1}{2}|\mathbf{H}_0|^2 + \frac{G}{2}\Delta^{-1}[\rho_0]\rho_0\, d\mathbf{x}. \end{aligned}$$

Note that p_e, p_{ϑ^2} are continuous functions vanishing at zero, thus

$$\rho \mapsto \rho P_e(\rho) \in C[0,\infty), \qquad \lim_{\rho\to 0+} \rho P_e(\rho) = 0,$$
$$\rho \mapsto \rho P_{\vartheta^2}(\rho) \in C[0,\infty), \qquad \lim_{\rho\to 0+} \rho P_{\vartheta^2}(\rho) = 0.$$

As for the energy contribution related to the term $\frac{G}{2}\int_\Omega \Delta^{-1}[\rho]\rho\, d\mathbf{x}$ is, in fact, negative. Using the fact that the total mass is a constant of motion, i.e., (2.1), the Hölder inequality and the classical elliptic estimate, we obtain

$$\begin{aligned} \frac{G}{2}\int_\Omega \left|\Delta^{-1}[\rho]\rho\right| d\mathbf{x} &\le \frac{G}{2}\|\rho\|_{L^\gamma}\left\|(-\Delta)^{-1}[\rho]\right\|_{L^{\frac{\gamma}{\gamma-1}}} \\ &\le C\|\rho\|_{L^\gamma}\|\rho\|_{L^1} \le C\|\rho\|_{L^\gamma}, \quad \gamma \ge 2. \end{aligned}$$

Next we shall obtain sufficient *a priori* estimates on the solution by virtue of the total energy conservation (2.9). Firstly, the assumption (1.8) implies that

$$p_e(\rho) \ge \frac{a_1}{\gamma}\rho^\gamma - b_1\rho.$$

Furthermore, there are two positive constants $\tilde{c_1}$ and $\tilde{c_2}$ such that

$$\rho P_e(\rho) \ge \tilde{c_1}\rho^\gamma - \tilde{c_2} \quad \text{for any } \rho \ge 0,$$

in particular,

$$\rho P_e(\rho) \ge \tilde{c_3}\left|p_e(\rho)\right| - \tilde{c_4} \quad \text{for } \rho \ge 0. \tag{2.10}$$

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By using the Cauchy–Schwarz inequality and the Hölder inequality, there are three positive constants $\tilde{c_5}(\le \tilde{c_1})$, $\tilde{c_6}(\le a)$ and $\tilde{c_7}$ such that

$$\left|\rho\vartheta^2 P_{\vartheta^2}(\rho)\right| \le \tilde{c_5}\rho^\gamma + \tilde{c_6}\vartheta^4 + \tilde{c_7}.$$

From (2.1), (2.9), we have

$$\rho^\gamma,\ \vartheta^4,\ \rho P_e(\rho),\ \rho\vartheta^2 P_{\vartheta^2}(\rho),\ \rho Q(\vartheta),\ \frac{1}{2}\rho|\mathbf{u}|^2,\ \frac{1}{2}|\mathbf{H}|^2 \in L^\infty\big(0,T;L^1(\Omega)\big). \tag{2.11}$$

Obviously, the elastic pressure component $p_e(\rho)$ is integrable as a result of (2.10). Moreover, by virtue of the Hölder inequality, $\rho\mathbf{u} \in L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))$.

Secondly, in order to get estimates on the temperature, we integrate (2.5) over Ω_T,

$$\int\limits_{\Omega_T}\left(\frac{\mathbb{S}:\nabla\mathbf{u}+\nu|\nabla\times\mathbf{H}|^2}{\vartheta}+\frac{\kappa|\nabla\vartheta|^2}{\vartheta^2}\right)d\mathbf{x}dt = S(T)-S_0, \tag{2.12}$$

where $\frac{\nabla\vartheta}{\vartheta}$ will be interpreted as $\nabla\ln\vartheta$ in the spirit of Lemma 5.3 in [12] (see also Lemma 2.1 in [13]), $S(t)=\int_\Omega \rho s\,d\mathbf{x}$, and

$$S_0 = \int\limits_\Omega \rho_0 s(\rho_0,\vartheta_0)\,d\mathbf{x} = \int\limits_\Omega\left(\frac{4}{3}a{\vartheta_0}^3 + \rho_0\int\limits_1^{\vartheta_0}\frac{c_v(\xi)}{\xi}d\xi - \rho_0 P_\vartheta(\rho_0) - 2\rho_0\vartheta_0 P_{\vartheta^2}(\rho_0)\right)d\mathbf{x}.$$

Moreover, the presence of ϑ in the denominator indicates that this quantity must be positive on a set of full measure for the above arguments to make sense.

It follows from (1.8) that for some certain $C>0$,

$$\left|\rho P_\vartheta(\rho)\right| \le C\big(1+\rho P_e(\rho)\big), \qquad \rho^2 P_{\vartheta^2}^2(\rho) \le C\big(1+\rho P_e(\rho)\big),$$

then

$$\begin{aligned}\rho s &\le \frac{4a}{3}\vartheta^3 + \rho Q(\vartheta) + \left|\rho P_\vartheta(\rho)\right| + 2\left|\rho\vartheta P_{\vartheta^2}(\rho)\right| \\ &\le \frac{4a}{3}\vartheta^3 + \rho Q(\vartheta) + \left|\rho P_\vartheta(\rho)\right| + \rho^2 P_{\vartheta^2}^2(\rho) + \vartheta^2 \\ &\le C\big(\rho Q(\vartheta) + \rho P_e(\rho) + \vartheta^4 + 1\big).\end{aligned}$$

Here we have also used

$$\int\limits_1^\vartheta \frac{c_v(\xi)}{\xi}d\xi \le 0, \quad 0<\vartheta\le 1,$$

$$\int\limits_1^\vartheta \frac{c_v(\xi)}{\xi}d\xi \le \int\limits_1^\vartheta c_v(\xi)\,d\xi = Q(\vartheta)-Q(1) \le Q(\vartheta), \quad \vartheta>1,$$

and hence

$$\int_1^{\vartheta} \frac{c_v(\xi)}{\xi}\, d\xi \le Q(\vartheta), \quad \text{for any } \vartheta > 0.$$

Then from (2.9), (2.12), we have

$$\int_{\Omega_T} \left(\frac{\mathbb{S} : \nabla \mathbf{u} + \nu |\nabla \times \mathbf{H}|^2}{\vartheta} + \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2} \right) d\mathbf{x}dt - \operatorname{ess} \inf_{t\in[0,T]} \int_{\Omega} \rho(t) \ln \vartheta(t)\, d\mathbf{x} \le C(E_0, T) - S_0.$$

Recalling the assumption (2.6) on $\kappa_G(\rho, \vartheta, \mathbf{H})$, we have

$$\begin{aligned} \int_{\Omega_T} \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2}\, d\mathbf{x}dt &= \int_{\Omega_T} \frac{(\kappa_R \vartheta^3 + \kappa_G(\rho, \vartheta, \mathbf{H}))|\nabla \vartheta|^2}{\vartheta^2}\, d\mathbf{x}dt \\ &\ge c \int_{\Omega_T} \frac{1 + \vartheta^{\beta} + \vartheta^3}{\vartheta^2} |\nabla \vartheta|^2\, d\mathbf{x}dt. \end{aligned}$$

Consequently, on the one hand,

$$\operatorname{ess} \sup_{t\in[0,T]} \int_{\Omega} \rho(t) \big|\ln \vartheta(t)\big|\, d\mathbf{x} + \int_{\Omega_T} \left(\big|\nabla \vartheta^{\frac{\beta}{2}}\big|^2 + \big|\nabla \vartheta^{\frac{3}{2}}\big|^2 + \left|\frac{\nabla \vartheta}{\vartheta}\right|^2 \right) d\mathbf{x}dt \le C(E_0, S_0, T), \quad (2.13)$$

which yields

$$\rho |\ln \vartheta| \in L^{\infty}\big(0, T; L^1(\Omega)\big), \qquad \frac{\nabla \vartheta}{\vartheta},\ \nabla \vartheta^{\frac{\beta}{2}},\ \nabla \vartheta^{\frac{3}{2}} \in L^2(\Omega_T).$$

First of all, $\nabla \vartheta^{\frac{3}{2}} \in L^2(\Omega_T)$ together with (2.11) give rise to

$$\vartheta^{\frac{3}{2}} \in L^2\big(0, T; H^1(\Omega)\big).$$

Next, since $\vartheta > 0$, then

$$\begin{aligned} \int_{\Omega} |\nabla \vartheta|^2\, d\mathbf{x} &= \int_{\Omega} \frac{\vartheta \nabla \vartheta}{\sqrt{\kappa}} \frac{\sqrt{\kappa} \nabla \vartheta}{\vartheta}\, d\mathbf{x} \\ &\le \left(\int_{\Omega} \frac{\vartheta^2 |\nabla \vartheta|^2}{\kappa}\, d\mathbf{x} \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2}\, d\mathbf{x} \right)^{\frac{1}{2}} \\ &\le C \left(\int_{\Omega} |\nabla \vartheta|^2\, d\mathbf{x} \right)^{\frac{1}{2}} \left(\int_{\Omega} \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2}\, d\mathbf{x} \right)^{\frac{1}{2}} \\ &\le \epsilon \int_{\Omega} |\nabla \vartheta|^2\, d\mathbf{x} + C_{\epsilon} \int_{\Omega} \frac{\kappa |\nabla \vartheta|^2}{\vartheta^2}\, d\mathbf{x}, \end{aligned}$$

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where $\epsilon > 0$ is small enough. Thus,

$$\nabla\vartheta \in L^2(\Omega_T).$$

Recalling $\vartheta \in L^\infty(0, T; L^4(\Omega))$ again, we have

$$\vartheta \in L^2\big(0, T; H^1(\Omega)\big).$$

Now, taking advantage of Lemma 5.3 in [12] (see also Lemma 2.1 in [13]) and the estimates in (2.13), we know

$$\ln\vartheta \in H^1(\Omega), \qquad \nabla\ln\vartheta = \frac{\nabla\vartheta}{\vartheta} \quad \text{a.e. on } \Omega,$$
$$\|\ln\vartheta\|_{L^2}^2 \le C\left(\|\rho\ln\vartheta\|_{L^1}^2 + \left\|\frac{\nabla\vartheta}{\vartheta}\right\|_{L^2}^2\right),$$

and furthermore,

$$\ln\vartheta \text{ is bounded in } L^2(\Omega_T)$$

by a constant depending only on the data and T. This estimate can be seen as "weak positivity" of the temperature ϑ. Finally, we conclude

$$\ln\vartheta \in L^2\big(0, T; H^1(\Omega)\big).$$

On the other hand, since

$$\begin{aligned}\frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} &= \frac{\mu(\vartheta,\mathbf{H})}{\vartheta}\left(|\nabla\mathbf{u}|^2 + \nabla\mathbf{u}:\nabla^\top\mathbf{u} - \frac{2}{3}(\nabla\cdot\mathbf{u})^2\right) + \frac{\eta(\vartheta,\mathbf{H})}{\vartheta}(\nabla\cdot\mathbf{u})^2\\ &= \frac{\mu(\vartheta,\mathbf{H})}{2\vartheta}\left|\nabla\mathbf{u} + \nabla^\top\mathbf{u} - \frac{2}{3}\nabla\cdot\mathbf{u}\mathbb{I}_3\right|^2 + \frac{\eta(\vartheta,\mathbf{H})}{\vartheta}(\nabla\cdot\mathbf{u})^2\\ &\ge c\vartheta^{\alpha-1}\left|\nabla\mathbf{u} + \nabla^\top\mathbf{u}\right|^2,\end{aligned}$$

and, by virtue of Young's inequality, it yields

$$\left|\nabla\mathbf{u} + \nabla^\top\mathbf{u}\right|^r \le C\big(\vartheta^{\alpha-1}\left|\nabla\mathbf{u} + \nabla^\top\mathbf{u}\right|^2 + \vartheta^4\big) \quad \text{with } r = \frac{8}{5-\alpha}.$$

Here we need $\alpha \le 1$. Hence $r \le 2$, and

$$\mathbf{u} \in L^r\big(0, T; W_0^{1,r}(\Omega)\big).$$

In view of the entropy equation (2.12) again, combining with the assumption on the magnetic diffusivity coefficient, we have

$$(1+\vartheta)^{\frac{\beta-1}{2}}\nabla\times\mathbf{H} \in L^2(\Omega_T).$$

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Bearing in mind the fact that $\|\nabla \times \mathbf{H}\|_{L^2} = \|\nabla \mathbf{H}\|_{L^2}$ when $\nabla \cdot \mathbf{H} = 0$, together with (2.11), we conclude

$$\mathbf{H} \in L^2\big(0, T; H^1(\Omega)\big).$$

Therefore, for the initial–boundary value problem, based on our assumptions on p_e, p_ϑ, p_{ϑ^2}, i.e., (1.8), on all the transport coefficients μ, η, ν, κ_G and on the initial data, we have *a priori* estimates resulting from boundedness of the initial total energy and the initial total entropy as follows:

$$\begin{gathered}
\rho^\gamma \in L^\infty\big(0, T; L^1(\Omega)\big), \qquad \rho\mathbf{u} \in L^\infty\big(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega)\big), \\
\rho P_e(\rho),\ \rho\vartheta^2 P_{\vartheta^2}(\rho),\ \rho Q(\vartheta) \in L^\infty\big(0, T; L^1(\Omega)\big), \\
\vartheta \in L^\infty\big(0, T; L^4(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big), \qquad \nabla\vartheta^{\frac{\beta}{2}} \in L^2(\Omega_T), \\
\vartheta^{\frac{3}{2}},\ \ln\vartheta \in L^2\big(0, T; H^1(\Omega)\big), \\
\mathbf{u} \in L^r\big(0, T; W_0^{1,r}(\Omega)\big), \qquad r = \frac{8}{5-\alpha}, \\
\mathbf{H} \in L^\infty\big(0, T; L^2(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big).
\end{gathered}$$

We remark that (i) the velocity gradient $\nabla\mathbf{u}$ is not known to be square integrable; (ii) a variational (weak) formulation of the momentum equation may not yield the full amount of mechanical energy dissipated by a (non-smooth) motion, then it may only satisfy the inequality

$$(\rho e)_t + \nabla \cdot (\rho e \mathbf{u}) + p \nabla \cdot \mathbf{u} \geq \mathbb{S} : \nabla\mathbf{u} - \nabla \cdot \mathbf{q} + \nu|\nabla \times \mathbf{H}|^2$$

instead of the internal energy balance (2.3). And consequently, Eq. (2.4) becomes

$$\begin{aligned}
&\big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)_t + \nabla \cdot \big(\big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)\mathbf{u}\big) - \nabla \cdot (\kappa\nabla\vartheta) \\
&\quad \geq \mathbb{S} : \nabla\mathbf{u} + \nu|\nabla \times \mathbf{H}|^2 - \left(\vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4\right)\nabla \cdot \mathbf{u}.
\end{aligned} \tag{2.14}$$

Using the same argument as the production of the entropy equation (2.5), the thermal energy inequality (2.14) can be "equivalently" expressed through the variational principle of entropy

$$\int\limits_{\Omega_T} \left(\rho s \varphi_t + \rho s \mathbf{u} \cdot \nabla\varphi - \frac{\kappa\nabla\vartheta}{\vartheta} \cdot \nabla\varphi\right) d\mathbf{x}dt \leq -\int\limits_{\Omega_T} \left(\frac{\mathbb{S} : \nabla\mathbf{u} + \nu|\nabla \times \mathbf{H}|^2}{\vartheta} + \frac{\kappa|\nabla\vartheta|^2}{\vartheta^2}\right)\varphi\, d\mathbf{x}dt,$$

for any $0 \leq \varphi \in \mathcal{D}(\Omega_T; \mathbb{R})$.

The above arguments suggest us what we mean by a variational (weak) solution of the system (1.1), (1.9), (1.10) based on the second law of thermodynamics and the integral representation of balance laws.

Definition 2.1. Given the initial distribution of the state variables

$$\rho|_{t=0} = \rho_0, \qquad \rho\mathbf{u}|_{t=0} = \mathbf{m}_0, \qquad \vartheta|_{t=0} = \vartheta_0, \qquad \mathbf{H}|_{t=0} = \mathbf{H}_0, \qquad \rho_0 \geq 0, \qquad \vartheta_0 > 0.$$

Let $T > 0$ be given; $(\rho, \mathbf{u}, \vartheta, \mathbf{H})$ is called a variational (weak) solution of (1.1), (1.9), (1.10), if

- $\rho \geq 0$, $\mathbf{u} \in L^r(0, T; W_0^{1,r}(\Omega))$ with $r > 1$ and $\mathbf{H} \in C([0, T]; L^2_{\text{weak}}(\Omega)) \cap L^2(0, T; H^1(\Omega))$ satisfy the continuity equation in $\mathcal{D}'(\mathbb{R}^3 \times [0, T))$, the momentum conservation equation and the induction equation in $\mathcal{D}'(\Omega \times [0, T))$, which are

$$\int\limits_{\Omega_T} \big(\rho\psi'\phi + \psi\rho\mathbf{u} \cdot \nabla\phi\big)\, d\mathbf{x}dt + \psi(0) \int\limits_{\Omega} \rho_0\phi\, d\mathbf{x} = 0,$$

for any $\psi \in C^\infty([0, T])$ with $\psi(T) = 0$ and $\phi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R})$, $\operatorname{ess\,lim}_{t\to 0+} \int_\Omega \rho\phi\, d\mathbf{x} = \int_\Omega \rho_0\phi\, d\mathbf{x}$;

$$\int\limits_{\Omega_T} \big(\psi'\rho\mathbf{u} \cdot \phi + \psi(\rho\mathbf{u} \otimes \mathbf{u}) : \nabla\phi + \psi p\nabla \cdot \phi\big)\, d\mathbf{x}dt$$
$$= \int\limits_{\Omega_T} \big(\psi\mathbb{S} : \nabla\phi - \rho\psi\nabla\Psi \cdot \phi - \psi\big((\nabla \times \mathbf{H}) \times \mathbf{H}\big) \cdot \phi\big)\, d\mathbf{x}dt - \psi(0) \int\limits_{\Omega} \mathbf{m}_0 \cdot \phi\, d\mathbf{x},$$

where $\Psi = G(-\Delta)^{-1}[1_\Omega\rho]$, and

$$\int\limits_{\Omega_T} \big(\psi'\mathbf{H} \cdot \phi + \psi(\mathbf{u} \times \mathbf{H}) \cdot (\nabla \times \phi) - \psi\nu(\nabla \times \mathbf{H}) \cdot (\nabla \times \phi)\big)\, d\mathbf{x}dt$$
$$+ \psi(0) \int\limits_{\Omega} \mathbf{H}_0 \cdot \phi\, d\mathbf{x} = 0,$$

for any $\psi \in C^\infty([0, T])$ with $\psi(T) = 0$ and $\phi \in \mathcal{D}(\Omega; \mathbb{R}^3)$, $\operatorname{ess\,lim}_{t\to 0+} \int_\Omega \rho\mathbf{u} \cdot \phi\, d\mathbf{x} = \int_\Omega \mathbf{m}_0 \cdot \phi\, d\mathbf{x}$, $\operatorname{ess\,lim}_{t\to 0+} \int_\Omega \mathbf{H} \cdot \phi\, d\mathbf{x} = \int_\Omega \mathbf{H}_0 \cdot \phi\, d\mathbf{x}$.
- The propagation of density oscillations is described by (2.2), i.e., the continuity equation is satisfied in the sense of renormalized solutions introduced in [10], that is, (2.2) holds in $\mathcal{D}'(\mathbb{R}^3 \times [0, T))$ with any $b \in C^1(\mathbb{R}^+)$ satisfying

$$b'(z) = 0 \quad \text{for all } z \in \mathbb{R}^+ \text{ large enough, e.g., } z \geq z_b, \tag{2.15}$$

where the constant z_b depends on the choice of function b, that means,

$$\int\limits_{\Omega_T} \big(b(\rho)\psi'\phi + \psi b(\rho)\mathbf{u} \cdot \nabla\phi + \psi\big(b(\rho) - b'(\rho)\rho\big)\nabla \cdot \mathbf{u}\phi\big)\, d\mathbf{x}dt$$
$$+ \psi(0) \int\limits_{\Omega} b(\rho_0)\phi\, d\mathbf{x} = 0, \tag{2.16}$$

for any $\psi \in C^\infty([0,T])$ with $\psi(T)=0$ and $\phi \in \mathcal{D}(\Omega;\mathbb{R})$, $\operatorname{ess\,lim}_{t\to 0+}\int_\Omega b(\rho)\phi\,d\mathbf{x} = \int_\Omega b(\rho_0)\phi\,d\mathbf{x}$.

- $\vartheta > 0$ satisfies the variational principle of entropy production

$$\int_{\Omega_T}\left(\rho s\psi'\phi + \psi\rho s\mathbf{u}\cdot\nabla\phi - \psi\frac{\kappa\nabla\vartheta}{\vartheta}\cdot\nabla\phi\right)d\mathbf{x}dt$$

$$\le -\int_{\Omega_T}\left(\frac{\mathbb{S}:\nabla\mathbf{u}+\nu|\nabla\times\mathbf{H}|^2}{\vartheta} + \frac{\kappa|\nabla\vartheta|^2}{\vartheta^2}\right)\psi\phi\,d\mathbf{x}dt - \psi(0)\int_\Omega \rho_0 s(\rho_0,\vartheta_0)\phi\,d\mathbf{x},$$

for any $0 \le \psi \in C^\infty([0,T])$ with $\psi(T)=0$ and $0\le\phi\in\mathcal{D}(\mathbb{R}^3;\mathbb{R})$, $\operatorname{ess\,lim}_{t\to 0+}\int_\Omega \rho s \times \phi d\mathbf{x} \ge \int_\Omega \rho_0 s(\rho_0,\vartheta_0)\phi\,d\mathbf{x}$.

- The total energy $E(t)$ defined in (2.9) is a constant of motion:

$$\int_0^T E\psi'\,dt = -E_0\psi(0),$$

for any $\psi\in C^\infty([0,T])$ with $\psi(T)=0$.

- $(\rho,\mathbf{u},\vartheta,\mathbf{H})$ satisfies (1.10) in the sense of trace a.a. in $(0,T)$.

Note that all the choices of the test functions agree with the boundary condition (1.10). We remark here, if the magnetic field $\mathbf{H}$ is absent, the system (1.1) with the constitutive relations (1.2), (1.3) is called the full Navier–Stokes–Fourier system, and a variational formulation of such a system with conservative boundary conditions was introduced in [12]. Now, we are ready to state our main theorem of this paper, which is the existence of global variational (weak) solutions for (1.1), (1.9), (1.10). More precisely, we prove

Theorem 2.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2+\iota}$, $\iota\in(0,1]$. Assume that the pressure p determined by (1.5), the internal energy e and the specific entropy s are interrelated by (1.6). Furthermore, suppose that the temperature scaling on μ, η satisfies $\frac{1}{2}\le\alpha\le 1$, on ν, κ_G and c_v satisfies $1\le\beta\le 4$. Then the system (1.1), (1.9), (1.10) has at least one global variational (weak) solution for all $T\in(0,\infty)$ such that*

$$\begin{cases} \rho\in C\big([0,T];L^1(\Omega)\big)\cap L^\infty\big(0,T;L^\gamma(\Omega)\big), \quad \rho\mathbf{u}\in C\big([0,T];L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(\Omega)\big),\\ \mathbf{u}\in L^r\big(0,T;W^{1,r}_0(\Omega)\big) \quad \text{with } r>1,\\ \vartheta\in L^\infty\big(0,T;L^4(\Omega)\big)\cap L^2\big(0,T;H^1(\Omega)\big), \quad \rho Q(\vartheta)\in L^\infty\big(0,T;L^1(\Omega)\big),\\ \vartheta^{\frac{3}{2}},\ \vartheta^{\frac{\beta}{2}},\ \ln\vartheta\in L^2\big(0,T;H^1(\Omega)\big),\\ \mathbf{H}\in C\big([0,T];L^2_{\text{weak}}(\Omega)\big)\cap L^2\big(0,T;H^1(\Omega)\big). \end{cases}$$

Remark 2.1. 1. $\rho\in L^\infty(0,T;L^\gamma(\Omega))$ can be strengthened to $\rho\in C([0,T];L^\gamma_{\text{weak}}(\Omega))$, in particular, $\rho(t)\rightharpoonup\rho_0$ in $L^\gamma(\Omega)$ as $t\to 0$, and $\int_\Omega\rho(t)\,d\mathbf{x} = \int_\Omega\rho_0\,d\mathbf{x}$ is a constant of motion (independent of $t\in[0,T]$).

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2. It will be shown that ϑ satisfies the initial condition

$$\operatorname{ess}\lim_{t\to 0+}\int_{\Omega}\vartheta\phi\,d\mathbf{x}=\int_{\Omega}\vartheta_0\phi\,d\mathbf{x},\quad \text{for any } \phi\in\mathcal{D}(\Omega;\mathbb{R})$$

if there exists a sequence of times $t_n\to 0$ such that $\{\vartheta_{t_n}\}$ is precompact in $L^1(\Omega)$.

3. Approximation scheme associated to (1.1)

We consider the following regularized problem

$$\begin{cases} \rho_t+\nabla\cdot(\rho\mathbf{u})=\varepsilon\Delta\rho, \\ (\rho\mathbf{u})_t+\nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u})+\nabla\Big(p_m(\rho)+p_b(\rho)+\vartheta p_\vartheta(\rho)+\vartheta^2 p_{\vartheta^2}(\rho)+\dfrac{a}{3}\vartheta^4\Big) \\ \quad +\delta\nabla\rho^\Gamma+\varepsilon\nabla\mathbf{u}\cdot\nabla\rho=\nabla\cdot\mathbb{S}+\rho\nabla\Psi+(\nabla\times\mathbf{H})\times\mathbf{H}, \\ \big(a\vartheta^4+\rho Q(\vartheta)-\rho\vartheta^2 P_{\vartheta^2}(\rho)\big)_t+\nabla\cdot\big(\big(a\vartheta^4+\rho Q(\vartheta)-\rho\vartheta^2 P_{\vartheta^2}(\rho)\big)\mathbf{u}\big) \\ \qquad -\nabla\cdot\big(\big(\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3\big)\nabla\vartheta\big) \\ \quad =\mathbb{S}:\nabla\mathbf{u}+\nu(\rho,\vartheta,\mathbf{H})|\nabla\times\mathbf{H}|^2+\varepsilon|\nabla\rho|^2\Big(\delta\Gamma\rho^{\Gamma-2}+\dfrac{p_m'(\rho)}{\rho}\Big) \\ \qquad -\Big(\vartheta p_\vartheta(\rho)+\vartheta^2 p_{\vartheta^2}(\rho)+\dfrac{a}{3}\vartheta^4\Big)\nabla\cdot\mathbf{u}, \\ \mathbf{H}_t-\nabla\times(\mathbf{u}\times\mathbf{H})=-\nabla\times\big(\nu(\rho,\vartheta,\mathbf{H})\nabla\times\mathbf{H}\big),\quad \nabla\cdot\mathbf{H}=0, \end{cases} \tag{3.1}$$

with the initial–boundary conditions

$$\begin{cases} \nabla\rho\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \rho|_{t=0}=\rho_{0,\delta}, \\ \mathbf{u}|_{\partial\Omega}=0,\quad (\rho\mathbf{u})|_{t=0}=\mathbf{m}_{0,\delta}=\begin{cases}\mathbf{m}_0, & \text{if } \rho_{0,\delta}\ge\rho_0, \\ 0, & \text{if } \rho_{0,\delta}<\rho_0,\end{cases} \\ \nabla\vartheta\cdot\mathbf{n}|_{\partial\Omega}=0\quad (\text{no-flux}),\quad \vartheta|_{t=0}=\vartheta_{0,\delta}, \\ \mathbf{H}\cdot\mathbf{n}|_{\partial\Omega}=(\nabla\times\mathbf{H})\times\mathbf{n}|_{\partial\Omega}=0,\quad \mathbf{H}|_{t=0}=\mathbf{H}_{0,\delta}, \end{cases} \tag{3.2}$$

where "the elastic pressure component" p_e has been decomposed as

$$p_e(\rho)=p_m(\rho)+p_b(\rho)$$

with $p_m, p_b\in C^1[0,\infty)$, $p_m'(\rho)\ge 0$, $|p_b|\le M$. The parameters $\varepsilon,\delta>0$ and $\delta\nabla\rho^\Gamma$ is the artificial pressure with $\Gamma>0$ (a constant to be determined when facilitating the limit passage $\varepsilon\to 0$).

Here the approximate initial density, temperature and magnetic induction vector distributions are chosen in such a way that

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$$\begin{cases} \rho_{0,\delta} \in C^{2+\iota}(\overline{\Omega}), \quad \nabla \rho_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \inf_{\mathbf{x}\in\Omega} \rho_{0,\delta} > 0, \\ \rho_{0,\delta} \to \rho_0 \quad \text{in } L^\gamma(\Omega), \quad \big|\{\mathbf{x} \in \Omega \mid \rho_{0,\delta} < \rho_0\}\big| \to 0 \quad \text{as } \delta \to 0; \\ \vartheta_{0,\delta} \in C^{2+\iota}(\overline{\Omega}), \quad \nabla \vartheta_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \inf_{\mathbf{x}\in\Omega} \vartheta_{0,\delta} > 0, \\ \vartheta_{0,\delta} \to \vartheta_0 \quad \text{in } L^1(\Omega) \text{ as } \delta \to 0; \\ \mathbf{H}_{0,\delta} \in \mathcal{D}\big(\Omega; \mathbb{R}^3\big), \quad \nabla \cdot \mathbf{H}_{0,\delta} = 0, \quad \mathbf{H}_{0,\delta} \cdot \mathbf{n}|_{\partial\Omega} = (\nabla \times \mathbf{H}_{0,\delta}) \times \mathbf{n}|_{\partial\Omega} = 0, \\ \mathbf{H}_{0,\delta} \to \mathbf{H}_0 \quad \text{in } L^2\big(\Omega; \mathbb{R}^3\big) \text{ as } \delta \to 0. \end{cases} \tag{3.3}$$

Taking $\varepsilon \to 0$ and $\delta \to 0$ in (3.1) will give the solution of system (1.1), (1.9), (1.10) in Theorem 2.1. Note that the most important principle we want to conform to is that the total energy is a constant of motion at every step of approximation. In particular, denoting

$$P_m(\rho) = \int_1^\rho \frac{p_m(z)}{z^2}\, dz,$$

the initial value of the regularized total energy

$$E_{0,\delta} = \int_\Omega \bigg(\rho_{0,\delta} P_m(\rho_{0,\delta}) - \rho_{0,\delta}\vartheta_{0,\delta}^2 P_{\vartheta^2}(\rho_{0,\delta}) + \rho_{0,\delta} Q(\vartheta_{0,\delta}) + a\vartheta_{0,\delta}^4 + \frac{1}{2\rho_{0,\delta}} |\mathbf{m}_{0,\delta}|^2 \\ + \frac{1}{2}|\mathbf{H}_{0,\delta}|^2 + \frac{\delta}{\Gamma - 1} \rho_{0,\delta}^\Gamma \bigg)\, d\mathbf{x}$$

is bounded by a constant independent of $\delta > 0$.

Moreover, it is easy to check that the corresponding approximate solutions satisfy the energy balance:

$$\frac{d}{dt} \int_\Omega \bigg(\rho P_m(\rho) - \rho\vartheta^2 P_{\vartheta^2}(\rho) + \rho Q(\vartheta) + \frac{1}{2}\rho|\mathbf{u}|^2 + a\vartheta^4 + \frac{1}{2}|\mathbf{H}|^2 + \frac{\delta}{\Gamma - 1}\rho^\Gamma \bigg)\, d\mathbf{x} \\ = \int_\Omega \rho \nabla\Psi \cdot \mathbf{u}\, d\mathbf{x} + \int_\Omega \nabla p_b \cdot \mathbf{u}\, d\mathbf{x} \quad \text{in } \mathcal{D}'(0, T).$$

Here we have used

$$\big(\rho P_m(\rho)\big)_t + \nabla \cdot \big(\rho P_m(\rho)\mathbf{u}\big) + p_m(\rho)\nabla \cdot \mathbf{u} = \varepsilon P_m(\rho)\Delta\rho + \varepsilon \frac{p_m(\rho)}{\rho}\Delta\rho$$

which leads to

$$\frac{d}{dt} \int_\Omega \rho P_m(\rho)\, d\mathbf{x} = \int_\Omega \nabla p_m(\rho) \cdot \mathbf{u}\, d\mathbf{x} - \varepsilon \int_\Omega \frac{p_m'(\rho)}{\rho} |\nabla\rho|^2\, d\mathbf{x},$$

and

$$
\begin{aligned}
&\delta\int_\Omega \nabla\rho^\Gamma\cdot\mathbf{u}\,d\mathbf{x}-\varepsilon\delta\Gamma\int_\Omega|\nabla\rho|^2\rho^{\Gamma-2}\,d\mathbf{x}\\
&\quad=\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}-\frac{\varepsilon\delta\Gamma}{\Gamma-1}\int_\Omega\nabla\rho\cdot\nabla\big(\rho^{\Gamma-1}\big)\,d\mathbf{x}\\
&\quad=\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}+\frac{\varepsilon\delta\Gamma}{\Gamma-1}\int_\Omega\rho^{\Gamma-1}\Delta\rho\,d\mathbf{x}\\
&\quad=\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}+\frac{\delta\Gamma}{\Gamma-1}\int_\Omega\rho^{\Gamma-1}\big(\rho_t+\nabla\cdot(\rho\mathbf{u})\big)\,d\mathbf{x}\\
&\quad=\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}+\frac{\delta}{\Gamma-1}\frac{d}{dt}\int_\Omega\rho^\Gamma\,d\mathbf{x}+\frac{\delta\Gamma}{\Gamma-1}\int_\Omega\rho^{\Gamma-1}\nabla\cdot(\rho\mathbf{u})\,d\mathbf{x}\\
&\quad=\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}+\frac{\delta}{\Gamma-1}\frac{d}{dt}\int_\Omega\rho^\Gamma\,d\mathbf{x}-\delta\Gamma\int_\Omega\rho^{\Gamma-1}\nabla\rho\cdot\mathbf{u}\,d\mathbf{x}\\
&\quad=\frac{\delta}{\Gamma-1}\frac{d}{dt}\int_\Omega\rho^\Gamma\,d\mathbf{x}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lim_{t\to0}\int_\Omega\left(\rho P_m(\rho)-\rho\vartheta^2P_{\vartheta^2}(\rho)+\rho Q(\vartheta)+a\vartheta^4+\frac12\rho|\mathbf{u}|^2+\frac12|\mathbf{H}|^2+\frac{\delta}{\Gamma-1}\rho^\Gamma\right)d\mathbf{x}\\
&=\int_\Omega\bigg(\rho_{0,\delta}P_m(\rho_{0,\delta})-\rho_{0,\delta}\vartheta_{0,\delta}^2P_{\vartheta^2}(\rho_{0,\delta})+\rho_{0,\delta}Q(\vartheta_{0,\delta})+a\vartheta_{0,\delta}^4+\frac{1}{2\rho_{0,\delta}}|\mathbf{m}_{0,\delta}|^2\\
&\qquad+\frac12|\mathbf{H}_{0,\delta}|^2+\frac{\delta}{\Gamma-1}\rho_{0,\delta}^\Gamma\bigg)\,d\mathbf{x}.
\end{aligned}
$$

After the above modification, the proof of Theorem 2.1 consists of the following steps:

Step 1: Solving problem for fixed parameters $\varepsilon>0$, $\delta>0$ and $\Gamma>0$ by the Galerkin method, deriving estimates independent of the dimension k of the Galerkin approximation and carrying out the limit as $k\to\infty$ provided Γ has been chosen large enough.

Step 2: Passing to the limit $\varepsilon\to0$.

Step 3: Letting $\delta\to0$.

4. Proof of Theorem 2.1

In this section we introduce the chain of approximations which we use to solve the original problem (1.1), (1.9), (1.10). At any level of approximations we formulate the statements about the existence of variational (weak) solutions and their properties which are needed to carry out the proof of existence for the original system.

To begin with, the goal proposed in Step 1 can be achieved via a Schauder–Tychonoff-type fixed point argument. More precisely, we first establish that ρ, Ψ, ϑ, and $\mathbf{H}$ can be computed

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successively from the first equation of (3.1), (2.8), and the last two equations of (3.1) as functions of $\mathbf{u}$, then the approximation problem for fixed parameters ε and δ can be easily solved by means of a modified Faedo–Galerkin method in the same way as in Chapter 7 in [18].

4.1. Solvability of continuity equation with dissipation

Given velocity field $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}, \mathbb{R}^3))$, the density $\rho := \rho[\mathbf{u}]$ is determined uniquely as the solution of the Neumann (suggested by the fact that conservation of mass in the form $(\int_\Omega \rho \, d\mathbf{x})_t = 0$ should hold) initial–boundary value problem:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = \varepsilon \Delta \rho, \quad \varepsilon > 0, \\ \rho|_{t=0} = \rho_{0,\delta}, \\ \nabla \rho \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

with $\rho_{0,\delta}$ satisfying (3.3). More precisely, since this is a linear parabolic Neumann problem in ρ, the existence and uniqueness of a classical solution

$$\rho \in C\big([0,T]; C^{2+\iota}(\overline{\Omega})\big), \qquad \rho_t \in C\big([0,T]; C^{\iota}(\overline{\Omega})\big),$$
$$\rho(t,\mathbf{x}) \ge \inf_{\mathbf{x}\in\Omega} \rho_{0,\delta}(\mathbf{x}) \exp\big(-\|\nabla \cdot \mathbf{u}\|_{L_t^1(L^\infty)}\big) > 0 \quad \text{on } \overline{\Omega}_T$$

can be obtained by the Galerkin method (Theorem 5.1.2 in [34], also see Section 7.6 in [38] for details), the solution mapping $\mathbf{u} \mapsto \rho[\mathbf{u}]$ is bounded and

$$\mathbf{u} \in C\big([0,T]; C_0^2\big(\overline{\Omega}, \mathbb{R}^3\big)\big) \mapsto \rho[\mathbf{u}] \in C^1(\overline{\Omega}_T)$$

is continuous (Proposition 7.1 in [18]). The gravitational potential Ψ will be solved by (2.7) and (2.8) by extending ρ to be zero outside of Ω.

4.2. Solvability of both the magnetic field and the temperature

In this section we show that the following system can be uniquely solved in terms of $\mathbf{u}$.

$$\begin{cases} \mathbf{H}_t - \nabla \times (\mathbf{u} \times \mathbf{H}) = -\nabla \times \big(\nu(\rho,\vartheta,\mathbf{H}) \nabla \times \mathbf{H}\big), \quad \nabla \cdot \mathbf{H} = 0, \\ \big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)_t + \nabla \cdot \big(\big(a\vartheta^4 + \rho Q(\vartheta) - \rho\vartheta^2 P_{\vartheta^2}(\rho)\big)\mathbf{u}\big) \\ \quad = \mathbb{S} : \nabla \mathbf{u} + \nu(\rho,\vartheta,\mathbf{H})|\nabla \times \mathbf{H}|^2 + \varepsilon|\nabla\rho|^2 \left(\delta \Gamma \rho^{\Gamma-2} + \dfrac{p_m'(\rho)}{\rho}\right) \\ \qquad + \nabla \cdot \big(\big(\kappa_G(\rho,\vartheta,\mathbf{H}) + \kappa_R \vartheta^3\big)\nabla\vartheta\big) - \left(\vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \dfrac{a}{3}\vartheta^4\right)\nabla \cdot \mathbf{u}, \\ \mathbf{H}|_{t=0} = \mathbf{H}_{0,\delta}, \quad \vartheta|_{t=0} = \vartheta_{0,\delta}, \\ \mathbf{H} \cdot \mathbf{n}|_{\partial\Omega} = (\nabla \times \mathbf{H}) \times \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla\vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{4.2}$$

with $\mathbf{H}_{0,\delta}$ and $\vartheta_{0,\delta}$ satisfying (3.3).

For given $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$, ρ has already been given by (4.1), Eq. $(4.2)_1$ is a quasi-linear parabolic-type structure in $\mathbf{H}$ and Eq. $(4.2)_2$ is indeed a non-degenerate parabolic-type system in terms of ϑ^4, since

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$$-\nabla\times\big(\nu(\rho,\vartheta,\mathbf{H})\nabla\times\mathbf{H}\big)=-\nabla\nu(\rho,\vartheta,\mathbf{H})\times(\nabla\times\mathbf{H})+\nu(\rho,\vartheta,\mathbf{H})\Delta\mathbf{H},$$

$$\nabla\cdot\big(\big(\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3\big)\nabla\vartheta\big)=\frac{\kappa_R}{4}\Delta\vartheta^4+\kappa_G(\rho,\vartheta,\mathbf{H})\Delta\vartheta+\nabla\kappa_G(\rho,\vartheta,\mathbf{H})\cdot\nabla\vartheta.$$

Thus, $\mathbf{H}$, ϑ can be solved by means of the standard Faedo–Galerkin methods. More explicitly, the boundary value problem of (4.2) has a unique solution ($\vartheta:=\vartheta[\mathbf{u}]$, $\mathbf{H}:=\mathbf{H}[\mathbf{u}]$) defined on the whole time interval $(0,T)$ satisfying the following properties:

- ϑ is a strong solution to (4.2) and strictly positive on $\overline{\Omega}_T$. In fact, the existence of a weak solution $\vartheta\in L^2(0,T;H^1(\Omega))$ can be obtained by the standard iterational process as in Chapter 1.2 in [28]. And the regularity of weak solutions (i.e. the Hölder continuity of weak solutions in a strictly interior subdomain) can be established as in Chapter 5.2 in [28]. As the first three terms on the right-hand side of $(4.2)_2$ are always non-negative, and the function $\vartheta=0$ is a subsolution, by using the comparison theorem, $\vartheta(t,\mathbf{x})\geq 0$ for all $t\in[0,T]$, $\mathbf{x}\in\Omega$. In agreement with the physical background and as required in the variational formulation introduced in Section 2, the absolute temperature must be positive a.a. on Ω_T.
- $\mathbf{H}\in C([0,T];L^2_{\mathrm{weak}}(\Omega))\cap L^2(0,T;H^1(\Omega))$.

5. The Faedo–Galerkin approximation scheme

In this section, we establish the existence of solutions to (3.1). Although $(3.1)_2$ and $(3.1)_3$ are of parabolic type, the unknowns $\mathbf{u}$ and ϑ appear to be multiplied by ρ in the leading terms, we have to use a more complicated approach based on the Faedo–Galerkin approximation technique to obtain the first level approximate solutions. In order to do this, assume the vector functions $\mathbf{w}_j=\mathbf{w}_j(\mathbf{x})$ ($j=1,2,\ldots$) are smooth, $\{\mathbf{w}_j\}_{j=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)$, and $\{\mathbf{w}_j\}_{j=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$. Define k-D Euclidean space $Y_k=\operatorname{span}\{\mathbf{w}_j\}_{j=1}^k$ with scalar product $\langle\mathbf{v},\mathbf{w}\rangle=\int_\Omega\mathbf{v}\cdot\mathbf{w}\,d\mathbf{x}$, $\mathbf{v},\mathbf{w}\in Y_k$ and let $P_k:(L^2(\Omega))^3\to Y_k$ be the orthonormal projection. The approximate velocity field $\mathbf{u}_k\in C([0,T];Y_k)$, we may write $\mathbf{u}_k(t,\mathbf{x})=\sum_{j=1}^k g_k^j(t)\mathbf{w}_j(\mathbf{x})$, satisfies

$$\begin{aligned}&\big\langle(\rho_k\mathbf{u}_k)_t,\mathbf{w}_j\big\rangle+\big\langle\nabla\cdot(\rho_k\mathbf{u}_k\otimes\mathbf{u}_k)+\nabla p_k+\delta\nabla\rho_k^\Gamma+\varepsilon\nabla\mathbf{u}_k\cdot\nabla\rho_k,\mathbf{w}_j\big\rangle\\&\quad=\big\langle\nabla\cdot\mathbb{S}_k+\rho_k\nabla\Psi_k+(\nabla\times\mathbf{H}_k)\times\mathbf{H}_k,\mathbf{w}_j\big\rangle\end{aligned}\tag{5.1}$$

with the initial conditions

$$\big\langle(\rho_k\mathbf{u}_k)(0),\mathbf{w}_j\big\rangle=\langle\mathbf{m}_{0,\delta},\mathbf{w}_j\rangle,$$

for all $t\in[0,T]$, $j=1,\ldots,k$, where $\{(\rho_k,\Psi_k,\mathbf{H}_k,\vartheta_k)\}_{k=1}^\infty$ are determined as the unique solution of (4.1), (2.8), (4.2) in terms of $\{\mathbf{u}_k\}_{k=1}^\infty$ on $[0,T]$. Here $\rho_k=\rho_{\delta,\varepsilon}[\mathbf{u}_k]$, etc., $p_k=p(\rho_k,\vartheta_k)$, $\Psi_k=G(-\Delta)^{-1}[\rho_k]$, and ε, δ, Γ are fixed positive parameters.

Given

$$f\in C\big([0,T];L^1(\Omega)\big),\qquad f_t\in L^1(\Omega_T),\qquad \operatorname*{ess\,inf}_{(t,\mathbf{x})\in\Omega_T}f(t,\mathbf{x})\geq a>0,$$

define an operator

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$$\mathcal{O}_{f(t)} : Y_k \to Y_k, \qquad \langle \mathcal{O}_{f(t)}\mathbf{u}, \mathbf{v}\rangle \equiv \int_\Omega f(t)\mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}, \quad \text{for } \mathbf{u}, \mathbf{v} \in Y_k,\ t \in [0, T].$$

It is easy to derive that $\mathcal{O}^{-1}_{f(t)}$ exists for all $t \in [0, T]$ and $\|\mathcal{O}^{-1}_{f(t)}\|_{\mathcal{L}(Y_k,Y_k)} \le \frac{1}{a}$.

Taking advantage of the operator $\mathcal{O}_{f(t)}$, and since

$$\rho_k(t) \ge \inf_{\mathbf{x}\in\Omega} \rho_{0,\delta}(\mathbf{x}) \exp\bigl(-\|\nabla\cdot\mathbf{u}_k\|_{L^1_t(L^\infty)}\bigr) > 0,$$

(5.1) can be rephrased as

$$\mathbf{u}_k(t) = \mathcal{O}^{-1}_{\rho_k(t)}\Bigg(P_k\mathbf{m}_{0,\delta} + \int_0^t P_k\bigl(-\nabla\cdot(\rho_k\mathbf{u}_k\otimes\mathbf{u}_k) - \nabla p_k - \delta\nabla\rho_k^\Gamma - \varepsilon\nabla\mathbf{u}_k\cdot\nabla\rho_k + \nabla\cdot\mathbb{S}_k + \rho_k\nabla\Psi_k + (\nabla\times\mathbf{H}_k)\times\mathbf{H}_k\bigr)\,d\tau\Bigg).$$

The local existence of the velocity $\mathbf{u}_k$ can be obtained by fixed point argument and the uniform estimates obtained from (5.2), (5.3) furnish the possibility of repeating the fixed point argument to extend the solution to the whole time interval $[0, T]$ (see [38, Chapter 7.7] for details). Thus, for any fixed $k = 1, 2, \ldots$, we solve first the regularized system for positive values of the parameters ε and δ, and the solution $(\rho_k, \mathbf{u}_k, \vartheta_k, \mathbf{H}_k)$ defined on the whole time interval.

Our plan is hereafter to send $k \to \infty$, and so we will need to obtain uniform estimates that are independent of the dimension k of Y_k. We start with the energy estimates which can be derived as follows: multiplying (5.1) by $g_k^j(t)$, summing $j = 1, \ldots, k$, and repeating the procedure for *a priori* estimates in Section 2. It yields the approximate kinetic energy and total energy balance:

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho_k\left(\frac{1}{2}|\mathbf{u}_k|^2 + \frac{\delta}{\Gamma-1}\rho_k^{\Gamma-1} + P_m(\rho_k)\right)d\mathbf{x} \\
&\quad + \int_\Omega \left(\mathbb{S}_k : \nabla\mathbf{u}_k + \varepsilon\left(\delta\Gamma\rho_k^{\Gamma-2} + \frac{p_m'(\rho_k)}{\rho_k}\right)|\nabla\rho_k|^2\right)d\mathbf{x} \\
&= \int_\Omega \bigl(\rho_k\nabla\Psi_k + (\nabla\times\mathbf{H}_k)\times\mathbf{H}_k\bigr)\cdot\mathbf{u}_k\,d\mathbf{x} \\
&\quad + \int_\Omega \left(p_b(\rho_k) + \vartheta_k p_\vartheta(\rho_k) + \vartheta_k^2 p_{\vartheta^2}(\rho_k) + \frac{a}{3}\vartheta_k^4\right)\nabla\cdot\mathbf{u}_k\,d\mathbf{x},
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\rho_k\left(\frac{1}{2}|\mathbf{u}_k|^2 + P_m(\rho_k) + Q(\vartheta_k) - \vartheta_k^2 P_{\vartheta^2}(\rho_k) + \frac{\delta}{\Gamma-1}\rho_k^{\Gamma-1}\right) + a\vartheta_k^4 + \frac{1}{2}|\mathbf{H}_k|^2\right)d\mathbf{x} \\
&= \int_\Omega \rho_k\nabla\Psi_k\cdot\mathbf{u}_k\,d\mathbf{x} + \int_\Omega p_b(\rho_k)\nabla\cdot\mathbf{u}_k\,d\mathbf{x},
\end{aligned} \tag{5.3}$$

where

$$\int_\Omega \mathbb{S}_k : \nabla \mathbf{u}_k \, d\mathbf{x} = \int_\Omega \left(\left(\eta(\vartheta_k, \mathbf{H}_k) + \frac{1}{3}\mu(\vartheta_k, \mathbf{H}_k) \right) (\nabla \cdot \mathbf{u}_k)^2 + \mu(\vartheta_k, \mathbf{H}_k) |\nabla \mathbf{u}_k|^2 \right) d\mathbf{x}.$$

Since ϑ_k is strictly positive on $\overline{\Omega}_T$, then multiplying the regularized thermal energy equation by $\frac{1}{\vartheta_k}$ and using the continuity equation with dissipation, we have

$$\begin{aligned}
&\left(\frac{4a}{3}\vartheta_k^3\right)_t + \nabla \cdot \left(\frac{4a}{3}\vartheta_k^3 \mathbf{u}_k\right) + \varepsilon \Delta\rho_k \left(\frac{Q(\vartheta_k)}{\vartheta_k} + \vartheta_k \big(P_{\vartheta^2}(\rho_k) + \rho_k P'_{\vartheta^2}(\rho_k)\big)\right) + p_\vartheta(\rho_k)\nabla \cdot \mathbf{u}_k \\
&\quad + \rho_k \frac{c_v(\vartheta_k)}{\vartheta_k}\vartheta_{kt} + \rho_k \frac{c_v(\vartheta_k)\nabla\vartheta_k \cdot \mathbf{u}_k}{\vartheta_k} - \big(2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k\big)_t - \nabla \cdot \big(2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k \mathbf{u}_k\big) \\
&= \frac{1}{\vartheta_k}\Bigg(\mathbb{S}_k : \nabla \mathbf{u}_k + \nabla \cdot \big((\kappa_G(\rho_k, \vartheta_k, \mathbf{H}_k) + \kappa_R \vartheta_k^3)\nabla\vartheta_k \big) + \nu(\rho_k, \vartheta_k, \mathbf{H}_k)|\nabla \times \mathbf{H}_k|^2 \\
&\quad + \varepsilon |\nabla\rho_k|^2 \left(\delta \Gamma \rho_k^{\Gamma-2} + \frac{p'_m(\rho_k)}{\rho_k} \right) \Bigg),
\end{aligned}$$

$$\int_\Omega \rho_k(t)\, d\mathbf{x} = \int_\Omega \rho_{0,\delta}\, d\mathbf{x} \quad \text{for any } t \geq 0, \tag{5.4}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_k^2 \, d\mathbf{x} + \varepsilon \int_\Omega |\nabla \rho_k|^2 \, d\mathbf{x} = -\frac{1}{2}\int_\Omega \rho_k^2 \nabla \cdot \mathbf{u}_k \, d\mathbf{x}, \tag{5.5}$$

and hence, on the one hand, the regularized thermal energy equation can be rewritten as an "entropy inequality":

$$\begin{aligned}
&\left(\frac{4a}{3}\vartheta_k^3 + \rho_k \int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi - 2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k \right)_t \\
&\quad + \nabla \cdot \left(\left(\frac{4a}{3}\vartheta_k^3 + \rho_k \int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi - 2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k \right) \mathbf{u}_k \right) \\
&\quad - \nabla \cdot \left(\frac{\kappa_G(\rho_k, \vartheta_k, \mathbf{H}_k) + \kappa_R \vartheta_k^3}{\vartheta_k} \nabla\vartheta_k \right) \\
&\geq \varepsilon \left(\int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi - \frac{Q(\vartheta_k)}{\vartheta_k} - \vartheta_k\big(P_{\vartheta^2}(\rho_k) + \rho_k P'_{\vartheta^2}(\rho_k)\big) \right) \Delta\rho_k - p_\vartheta(\rho_k)\nabla \cdot \mathbf{u}_k \\
&\quad + \frac{\kappa_G(\rho_k, \vartheta_k, \mathbf{H}_k) + \kappa_R \vartheta_k^3}{\vartheta_k^2}|\nabla\vartheta_k|^2 + \frac{\mathbb{S}_k : \nabla \mathbf{u}_k + \nu(\rho_k, \vartheta_k, \mathbf{H}_k)|\nabla \times \mathbf{H}_k|^2}{\vartheta_k}.
\end{aligned} \tag{5.6}$$

On the other hand, combining (5.3), (5.5), (5.6) yields

$$\begin{aligned}
&\frac{d}{dt}\int\limits_{\Omega}\left(\rho_k\left(\frac{1}{2}|\mathbf{u}_k|^2+P_m(\rho_k)+Q(\vartheta_k)-\vartheta_k^2 P_{\vartheta^2}(\rho_k)+\frac{\delta}{\Gamma-1}\rho_k^{\Gamma-1}\right)+a\vartheta_k^4+\frac{1}{2}|\mathbf{H}_k|^2\right)d\mathbf{x}\\
&\quad+\frac{d}{dt}\int\limits_{\Omega}\left(\frac{1}{2}\rho_k^2-\frac{4a}{3}\vartheta_k^3-\rho_k\int\limits_1^{\vartheta_k}\frac{c_v(\xi)}{\xi}\,d\xi+2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k\right)d\mathbf{x}\\
&\quad+\int\limits_{\Omega}\left(\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k^2}|\nabla\vartheta_k|^2+\varepsilon|\nabla\rho_k|^2\right.\\
&\quad\left.+\frac{\mathbb{S}_k:\nabla\mathbf{u}_k+\nu(\rho_k,\vartheta_k,\mathbf{H}_k)|\nabla\times\mathbf{H}_k|^2}{\vartheta_k}\right)d\mathbf{x}\\
&\le\varepsilon\int\limits_{\Omega}\left(\left(\frac{Q(\vartheta_k)}{\vartheta_k^2}+P_{\vartheta^2}(\rho_k)+\rho_k P'_{\vartheta^2}(\rho_k)\right)\nabla\vartheta_k\cdot\nabla\rho_k+\frac{p'_{\vartheta^2}(\rho_k)}{\rho_k}\vartheta_k|\nabla\rho_k|^2\right)d\mathbf{x}\\
&\quad+\int\limits_{\Omega}\rho_k\nabla\Psi_k\cdot\mathbf{u}_k\,d\mathbf{x}+\int\limits_{\Omega}\left(p_\vartheta(\rho_k)+p_b(\rho_k)-\frac{1}{2}\rho_k^2\right)\nabla\cdot\mathbf{u}_k\,d\mathbf{x}.
\end{aligned}\tag{5.7}$$

The classical elliptic estimate yields

$$\|\nabla\Psi_k\|_{L^\infty}\le C\left\|(-\Delta)^{-1}[\rho_k]\right\|_{W^{2,\Gamma}}\le C\|\rho_k\|_{L^\Gamma}$$

provided $\Gamma>3$. Taking advantage of (5.4), one has

$$\int\limits_{\Omega}\rho_k\nabla\Psi_k\cdot\mathbf{u}_k\,d\mathbf{x}\le C\|\rho_k\|_{L^\Gamma}\left(\int\limits_{\Omega}\rho_{0,\delta}\,d\mathbf{x}\right)^{\frac{1}{2}}\left(\int\limits_{\Omega}\rho_k|\mathbf{u}_k|^2\,d\mathbf{x}\right)^{\frac{1}{2}}.$$

By virtue of definition of $\mathbf{m}_{0,\delta}$ in (3.2), we have

$$\int\limits_{\Omega}\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,k}\,d\mathbf{x}\le\frac{1}{2}\int\limits_{\Omega}\left(\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}}+\rho_{0,\delta}|\mathbf{u}_{0,\delta,k}|^2\right)d\mathbf{x}=\frac{1}{2}\int\limits_{\Omega}\left(\frac{|\mathbf{m}_0|^2}{\rho_{0,\delta}}+\mathbf{m}_{0,\delta}\mathbf{u}_{0,\delta,k}\right)d\mathbf{x},$$

and

$$\int\limits_{\Omega}\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,k}\,d\mathbf{x}\le\int\limits_{\Omega}\frac{|\mathbf{m}_0|^2}{\rho_{0,\delta}}\,d\mathbf{x},$$

where the value of $\mathbf{u}_{0,\delta,k}\in Y_k$ is uniquely determined by

$$\int\limits_{\Omega}\rho_{0,\delta}\mathbf{u}_{0,\delta,k}\cdot\mathbf{w}_j\,d\mathbf{x}=\int\limits_{\Omega}\mathbf{m}_{0,\delta}\cdot\mathbf{w}_j\,d\mathbf{x},\qquad j=1,\dots,k.$$

Recalling the fact that

$$\frac{\mathbb{S}_k : \nabla \mathbf{u}_k}{\vartheta_k} \geq c\vartheta_k^{\alpha-1} \left|\nabla \mathbf{u}_k + \nabla^\top \mathbf{u}_k\right|^2,$$

and, by virtue of Young's inequality,

$$\left|\nabla \mathbf{u}_k + \nabla^\top \mathbf{u}_k\right|^r \leq C\left(\vartheta_k^{\alpha-1}\left|\nabla \mathbf{u}_k + \nabla^\top \mathbf{u}_k\right|^2 + \vartheta_k^4\right) \quad \text{with } r = \frac{8}{5-\alpha},$$

combining with

$$\int_\Omega \rho_{0,\delta} P_m(\rho_{0,\delta})\, d\mathbf{x} \leq \int_\Omega \left(\rho_{0,\delta} P_e(\rho_{0,\delta}) + C(1+\rho_{0,\delta})\right) d\mathbf{x},$$

the hypothesis (2.6) on c_v and κ_G, we are ready to apply Gronwall's lemma to (5.7) provided $\Gamma = \Gamma(r)$ is large enough to handle the last integral on the right-hand side of (5.7). Here the Cauchy–Schwarz inequality has been used repeatedly. Moreover,

$$\int_\Omega \left(|\nabla \ln \vartheta_k|^2 + \left|\nabla \vartheta_k^{\frac{3}{2}}\right|^2\right) d\mathbf{x} \leq C \int_\Omega \frac{\kappa_G(\rho_k, \vartheta_k, \mathbf{H}_k) + \kappa_R \vartheta_k^3}{\vartheta_k^2} |\nabla \vartheta_k|^2\, d\mathbf{x}.$$

Thus, the following estimates hold:

$$\sup_{t\in[0,T]} \left(\|\rho_k\|_{L^\Gamma} + \left\|\rho_k |\mathbf{u}_k|^2\right\|_{L^1}\right) \leq C(\delta), \tag{5.8}$$

$$\sup_{t\in[0,T]} \left(\left\|\rho_k Q(\vartheta_k)\right\|_{L^1} + \|\vartheta_k\|_{L^4} + \|\mathbf{H}_k\|_{L^2}\right) + \operatorname{ess}\sup_{t\in[0,T]} \int_\Omega \rho_k |\ln \vartheta_k|\, d\mathbf{x} \leq C(\delta), \tag{5.9}$$

$$\int_{\Omega_T} \frac{\mathbb{S}_k : \nabla \mathbf{u}_k + \nu |\nabla \times \mathbf{H}_k|^2}{\vartheta_k} + |\nabla \ln \vartheta_k|^2 + \left|\nabla \vartheta_k^{\frac{3}{2}}\right|^2 + \left|\nabla \vartheta_k^{\frac{\beta}{2}}\right|^2 + \varepsilon |\nabla \rho_k|^2\, d\mathbf{x}dt \leq C(\delta), \tag{5.10}$$

and

$$\begin{aligned} &\|\mathbf{u}_k\|_{L^r(0,T;W_0^{1,r})} \leq C(\delta) \quad \text{with } r = \frac{8}{5-\alpha}, \\ &\left\|(1+\vartheta_k)^{\frac{\beta-1}{2}} \nabla \times \mathbf{H}_k\right\|_{L^2(0,T;L^2)} \leq C(\delta). \end{aligned} \tag{5.11}$$

Note that all constants in (5.8)–(5.11) are independent of k and ε.

In order to identify a limit for $k \to \infty$ of the approximate solutions $\{(\rho_k, \mathbf{u}_k, \vartheta_k, \mathbf{H}_k)\}_{k=1}^\infty$ obtained above as a solution of problem (3.1), (3.2), additional estimates are needed. Firstly, we have the uniform estimates of the artificial pressure which is proportional to ρ^Γ and the density gradient estimates the same as in Section 5.2 and Section 5.3 in [9], we state without proof the following result:

Lemma 5.1. *Under the hypothesis of Theorem 2.1, let $\Gamma = \Gamma(r)$ be large enough, then the density sequence $\{\rho_k\}_{k=1}^\infty$ satisfies the following properties:*

$$\|\rho_k\|_{L^{\frac{\Gamma}{r'}}(0,T;L^{\frac{3\Gamma}{r'}})} \le C(\varepsilon,\delta), \quad r' := \frac{r}{r-1} = \frac{8}{3+\alpha},$$
$$\varepsilon\|\nabla\rho_k\|_{L^q(\Omega_T)} \le C(\delta) \quad \textit{for a certain } q > r',$$
$$\|\rho_{kt}\|_{L^\pi(\Omega_T)},\ \|\Delta\rho_k\|_{L^\pi(\Omega_T)} \le C(\varepsilon,\delta) \quad \textit{for a certain } \pi > 1.$$

Now, using the same arguments as in Section 2.4 in [21], for the sequences $\{\rho_k\}_{k=1}^\infty$ and $\{\mathbf{u}_k\}_{k=1}^\infty$, we have (at least for some chosen subsequences)

$$\rho_k \to \rho \quad \text{in } C\big([0,T]; L^\Gamma_{\text{weak}}\big) \cap L^1(\Omega_T),\ \rho \ge 0, \tag{5.12}$$
$$\rho_t,\ \Delta\rho \in L^\pi(\Omega_T) \quad \text{for a certain } \pi > 1,$$
$$\mathbf{u}_k \rightharpoonup \mathbf{u} \quad \text{in } L^r\big(0,T; W_0^{1,r}(\Omega)\big),\ r = \frac{8}{5-\alpha}, \tag{5.13}$$

where the limit velocity $\mathbf{u}$ satisfies the non-slip boundary condition in the sense of traces. And ρ is the unique strong solution to (4.1), i.e., the functions ρ, $\mathbf{u}$ satisfying the continuity equation with dissipation a.e. in Ω_T, the initial condition a.e. in Ω and the homogeneous Neumann boundary condition in the sense of traces a.e. in $(0,T)$. In particular,

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = \varepsilon\nabla\cdot(1_\Omega\nabla\rho) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^3 \times (0,T)\big)$$

provided ρ, $\mathbf{u}$ were extended to be zero outside Ω. Moreover, in accordance with (5.10) and Lemma 5.1, one has

$$\int_0^T \varepsilon\|\nabla\rho\|_{L^2}^2 + \varepsilon^q\|\nabla\rho\|_{L^q}^q\,dt \le C(\delta) \quad \text{for a certain } q > r'. \tag{5.14}$$

By using interpolation, the estimate (5.8) and Lemma 5.1 lead to

$$\rho_k \to \rho \quad \text{in } L^\pi(\Omega_T) \text{ for some } \pi > \Gamma. \tag{5.15}$$

Combining the estimate (5.8) and the strong convergence (5.15), we also have

$$\rho_k\mathbf{u}_k \overset{*}{\rightharpoonup} \rho\mathbf{u} \quad \text{in } L^\infty\big(0,T; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)\big).$$

The estimates obtained in Lemma 5.1 can be used to deduce from $(4.1)_1$ that the integral mean functions

$$t \mapsto \int_\Omega \rho_k\mathbf{u}_k\cdot\mathbf{w}_j\,d\mathbf{x} \text{ form a precompact system in } C\big([0,T]\big)$$

for any fixed j. This implies that

$$\rho_k\mathbf{u}_k \to \rho\mathbf{u} \quad \text{in } C\big([0,T]; L^{\frac{2\Gamma}{\Gamma+1}}_{\text{weak}}(\Omega)\big),$$

with the limit function satisfying $\rho\mathbf{u}(0) = \mathbf{m}_{0,\delta}$. As the space $L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)$ is compactly imbedded into the dual space $W^{-1,r'}(\Omega)$ for suitable (large) Γ, and, consequently,

$$\rho_k \mathbf{u}_k \otimes \mathbf{u}_k \rightharpoonup \rho\mathbf{u} \otimes \mathbf{u} \quad \text{in } L^r\big(0, T; L^\pi(\Omega)\big),\ 1 < \pi \le \frac{6\Gamma r}{3r + \Gamma r + 6\Gamma}. \tag{5.16}$$

In accordance with Lemma 5.1, it yields strong convergence

$$\nabla\rho_k \to \nabla\rho \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } r' \le \pi < q,$$

in particular,

$$\nabla\mathbf{u}_k \cdot \nabla\rho_k \to \nabla\mathbf{u} \cdot \nabla\rho \quad \text{in } \mathcal{D}'(\Omega_T).$$

Secondly, we need to show pointwise convergence of the sequence $\{\vartheta_k\}_{k=1}^\infty$. From estimates (5.9) and (5.10), repeating the procedure for *a priori* estimates in Section 2, we know

$$\{\vartheta_k\}_{k=1}^\infty \text{ is bounded in } L^\infty\big(0, T; L^4(\Omega)\big) \cap L^2\big(0, T; H^1(\Omega)\big), \tag{5.17}$$

$$\{\ln\vartheta_k\}_{k=1}^\infty \text{ is bounded in } L^2\big(0, T; H^1(\Omega)\big), \tag{5.18}$$

$$\big\{\vartheta_k^{\frac{3}{2}}\big\}_{k=1}^\infty \text{ is bounded in } L^2\big(0, T; H^1(\Omega)\big), \tag{5.19}$$

$$\big\{\nabla\vartheta_k^{\frac{\beta}{2}}\big\}_{k=1}^\infty \text{ is bounded in } L^2(\Omega_T). \tag{5.20}$$

This implies that, by selecting a subsequence if necessary, there exists a function ϑ such that

$$\vartheta_k \overset{*}{\rightharpoonup} \vartheta \quad \text{in } L^\infty\big(0, T; L^4(\Omega)\big), \quad \text{and} \quad \vartheta_k \rightharpoonup \vartheta \quad \text{in } L^2\big(0, T; H^1(\Omega)\big),$$
$$\ln\vartheta_k \rightharpoonup \overline{\ln\vartheta} \quad \text{in } L^2\big(0, T; H^1(\Omega)\big),$$

and

$$f(\vartheta_k) \rightharpoonup \overline{f(\vartheta)} \quad \text{in } L^2\big(0, T; H^1(\Omega)\big)$$

for any

$$f \in C^1(0, \infty), \qquad \big|f'(\xi)\big| \le C\left(\frac{1}{\xi} + \xi^{\frac{1}{2}}\right), \quad \xi > 0.$$

Here and in what follows, the symbol $\overline{F(v)}$ stands for a weak limit of a composition $\{F(v_k)\}_{k=1}^\infty$ in $L^1(\Omega_T)$.

Next, we claim that $\{\vartheta_{k_t}\}_{k=1}^\infty$ satisfies the "entropy inequality" (5.6). Note that, according to (5.8) and (5.18), by virtue of the Sobolev imbedding theorem $H^1(\Omega) \hookrightarrow L^6(\Omega)$, one has

$$\{\rho_k \ln\vartheta_k\}_{k=1}^\infty \text{ bounded in } L^2\big(0, T; L^{\frac{6\Gamma}{6+\Gamma}}(\Omega)\big).$$

Together with (5.9), we have

$$\rho_k \ln \vartheta_k \in L^\infty\big(0,T;L^1(\Omega)\big) \cap L^2\big(0,T;L^{\frac{6\Gamma}{6+\Gamma}}(\Omega)\big).$$

Similarly, (5.8) leads to

$$\sup_{t\in[0,T]} \|\rho_k \mathbf{u}_k\|_{L^{\frac{2\Gamma}{\Gamma+1}}} \le C(\delta). \tag{5.21}$$

Together with (5.18), it yields

$$\rho_k \mathbf{u}_k \ln \vartheta_k \in L^2\big(0,T;L^{\frac{6\Gamma}{3+4\Gamma}}(\Omega)\big).$$

Recalling the hypothesis (2.6) on c_v, we know, from (5.8) and (5.17),

$$\rho_k \vartheta_k^{\frac{\beta}{2}-1} \in L^\infty\big(0,T;L^1(\Omega)\big) \cap L^2\big(0,T;L^{\frac{6\Gamma}{6+\Gamma}}(\Omega)\big) \quad \text{for } \beta \le 4, \tag{5.22}$$

provided Γ large enough, say, $\Gamma > 3$, then

$$\rho_k \int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi \in L^\infty\big(0,T;L^1(\Omega)\big) \cap L^2\big(0,T;L^{\frac{6\Gamma}{6+\Gamma}}(\Omega)\big),$$

and

$$\rho_k \mathbf{u}_k \int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi \in L^2\big(0,T;L^{\frac{6\Gamma}{3+4\Gamma}}(\Omega)\big).$$

In accordance with hypothesis (1.8) and estimate (5.8),

$$\rho_k P_{\vartheta^2}(\rho_k) \text{ is bounded in } L^\infty\big(0,T;L^{\frac{\Gamma}{\varsigma}}(\Omega)\big),$$

and furthermore, together with (5.17),

$$\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k \in L^\infty\big(0,T;L^1(\Omega)\big) \cap L^2\big(0,T;L^{\frac{6\Gamma}{6\varsigma+\Gamma}}(\Omega)\big).$$

According to Lemma 6.3 of Chapter 6 in [18], we have

$$\begin{aligned} &\frac{4a}{3}\vartheta_k^3 + \rho_k \int_1^{\vartheta_k} \frac{c_v(\xi)}{\xi}\, d\xi - 2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k \\ &\quad \to \frac{4a}{3}\overline{\vartheta^3} + \overline{\rho \int_1^{\vartheta} \frac{c_v(\xi)}{\xi}\, d\xi} - 2\rho P_{\vartheta^2}(\rho)\vartheta \quad \text{in } L^2\big(0,T;H^{-1}(\Omega)\big). \end{aligned}$$

In addition, we employ (5.17) to get

$$\int_{\Omega_T}\left(\frac{4a}{3}\vartheta_k^3+\rho_k\int_1^{\vartheta_k}\frac{c_v(\xi)}{\xi}\,d\xi-2\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k\right)\vartheta_k\,d\mathbf{x}dt$$
$$\to\int_{\Omega_T}\left(\frac{4a}{3}\overline{\vartheta^3}+\overline{\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi}-2\rho P_{\vartheta^2}(\rho)\vartheta\right)\vartheta\,d\mathbf{x}dt. \tag{5.23}$$

Since

$$\lim_{k\to\infty}\int_{\Omega_T}\rho_k P_{\vartheta^2}(\rho_k)\vartheta_k^2\,d\mathbf{x}dt=\int_{\Omega_T}\rho P_{\vartheta^2}(\rho)\vartheta^2\,d\mathbf{x}dt,$$

then (5.23) reduces to

$$\int_{\Omega_T}\left(\frac{4a}{3}\vartheta_k^3+\rho_k\int_1^{\vartheta_k}\frac{c_v(\xi)}{\xi}\,d\xi\right)\vartheta_k\,d\mathbf{x}dt\to\int_{\Omega_T}\left(\frac{4a}{3}\overline{\vartheta^3}+\overline{\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi}\right)\vartheta\,d\mathbf{x}dt. \tag{5.24}$$

As the function $\xi\mapsto\frac{4a}{3}\xi^3+\rho\int_1^{\xi}\frac{c_v(y)}{y}\,dy$ is non-decreasing, we have, from (5.24), the following strong (pointwise) convergence

$$\vartheta_k\to\vartheta\quad\text{in } L^1(\Omega_T).$$

Employing (5.17) again, by virtue of a simple interpolation argument, we obtain

$$\vartheta_k\to\vartheta\quad\text{in } L^{\pi}(\Omega_T)\text{ for some }\pi>4.$$

Finally, using $\nabla\cdot\mathbf{H}_k=0$, the estimates (5.9), (5.11) on the magnetic field $\{\mathbf{H}_k\}_{k=1}^{\infty}$ imply that

$$\mathbf{H}_k\overset{*}{\rightharpoonup}\mathbf{H}\quad\text{in } L^{\infty}\big(0,T;L^2(\Omega)\big),\qquad \mathbf{H}_k\rightharpoonup\mathbf{H}\quad\text{in } L^2\big(0,T;H^1(\Omega)\big)\quad\text{and}$$
$$\mathbf{H}_k\to\mathbf{H}\quad\text{in } L^2(\Omega_T),$$

and consequently, via interpolation,

$$(\nabla\times\mathbf{H}_k)\times\mathbf{H}_k\rightharpoonup(\nabla\times\mathbf{H})\times\mathbf{H}\quad\text{in } L^{\pi}(\Omega_T)\text{ for a certain }\pi>1.$$

Moreover,

$$\mathbb{S}_k\rightharpoonup\mathbb{S}\quad\text{in } L^q(\Omega_T)\text{ for some }q>1,$$

where $\mathbb{S}=\mu(\vartheta,\mathbf{H})(\nabla\mathbf{u}+\nabla^{\top}\mathbf{u})+\eta(\vartheta,\mathbf{H})(\nabla\cdot\mathbf{u})\mathbb{I}_3$. We also have

$$\mathbf{u}_k\times\mathbf{H}_k\rightharpoonup\mathbf{u}\times\mathbf{H}\quad\text{in } L^{\pi}(\Omega_T)\text{ for a certain }\pi>1,$$

in accordance with (5.13).

Now, we can pass to the limit for $k \to \infty$ in (5.1) to get

$$(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4 + \delta\rho^\Gamma\right) + \varepsilon\nabla\mathbf{u}\cdot\nabla\rho$$
$$= \nabla\cdot\mathbb{S} + \rho\nabla\Psi + (\nabla\times\mathbf{H})\times\mathbf{H} \quad \text{in } \mathcal{D}'(\Omega_T), \tag{5.25}$$

where the potential Ψ satisfies (2.7).

Moreover, due to the estimates (5.10) and (5.17), as β satisfies the hypothesis in (5.22), we know

$$\nu(\rho_k,\vartheta_k,\mathbf{H}_k)\nabla\times\mathbf{H}_k = \sqrt{\vartheta_k\nu(\rho_k,\vartheta_k,\mathbf{H}_k)}\sqrt{\frac{\nu(\rho_k,\vartheta_k,\mathbf{H}_k)}{\vartheta_k}}\nabla\times\mathbf{H}_k$$

are bounded in $L^\pi(\Omega_T)$ for a certain $\pi > 1$.

Thus the limit quantities satisfy

$$\int_{\Omega_T}\big(\psi'\mathbf{H}\cdot\phi + \psi(\mathbf{u}\times\mathbf{H})\cdot(\nabla\times\phi) - \psi\nu(\nabla\times\mathbf{H})\cdot(\nabla\times\phi)\big)\,d\mathbf{x}dt + \psi(0)\int_\Omega \mathbf{H}_0\cdot\phi\,d\mathbf{x} = 0,$$

for any $\psi\in C^\infty([0,T])$ with $\psi(T)=0$ and $\phi\in\mathcal{D}(\Omega;\mathbb{R}^3)$.

In the same way, we let $k\to\infty$ in the energy equality (5.3) to get

$$-\int_{\Omega_T}\psi'\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2 + P_m(\rho) + Q(\vartheta) - \vartheta^2 P_{\vartheta^2}(\rho) + \frac{\delta}{\Gamma-1}\rho^{\Gamma-1}\right) + a\vartheta^4 + \frac{1}{2}|\mathbf{H}|^2\right)d\mathbf{x}dt$$
$$= \int_\Omega\left(\frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta}P_m(\rho_{0,\delta}) + \rho_{0,\delta}Q(\vartheta_{0,\delta}) - \rho_{0,\delta}\vartheta_{0,\delta}^2 P_{\vartheta^2}(\rho_{0,\delta}) + \frac{\delta}{\Gamma-1}\rho_{0,\delta}^\Gamma + a\vartheta_{0,\delta}^4\right.$$
$$\left. + \frac{1}{2}|\mathbf{H}_{0,\delta}|^2\right)d\mathbf{x} + \int_{\Omega_T}\psi\big(\rho\nabla\Psi\cdot\mathbf{u} + p_b(\rho)\nabla\cdot\mathbf{u}\big)d\mathbf{x}dt, \tag{5.26}$$

for any $\psi\in C^\infty([0,T])$ with $\psi(0)=1$, $\psi(T)=0$. Here we have used

$$\int_{\Omega_\tau}\big(\rho\nabla\Psi\cdot\mathbf{u} + p_b(\rho)\nabla\cdot\mathbf{u}\big)d\mathbf{x}dt = \lim_{k\to\infty}\int_{\Omega_\tau}\big(\rho_k\nabla\Psi_k\cdot\mathbf{u}_k + p_b(\rho_k)\nabla\cdot\mathbf{u}_k\big)d\mathbf{x}dt$$

for any $\tau\geq 0$ in accordance with (5.13), (5.15). By virtue of the hypothesis (2.6) on $c_v(\vartheta)$, (5.22) on β and the estimates (5.17), (5.20), we have

$$Q(\vartheta_k) \rightharpoonup \overline{Q(\vartheta)} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$

combining with (5.8),

$$\rho_k Q(\vartheta_k) \rightharpoonup \rho\overline{Q(\vartheta)} \quad \text{in } L^2\big(0,T;L^q(\Omega)\big),\ 1<q<\frac{6\Gamma}{\Gamma+6}.$$

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Now since $Q(\vartheta_k)$ tends to $Q(\vartheta)$ a.e. on Ω, we infer $Q(\vartheta) = \overline{Q(\vartheta)}$, and

$$\int_\Omega \rho Q(\vartheta)(\tau+)\,d\mathbf{x} = \lim_{h\to 0+}\frac{1}{h}\left(\lim_{k\to\infty}\int_\tau^{\tau+h}\int_\Omega \rho_k Q(\vartheta_k)\,d\mathbf{x}dt\right)$$

for a.e. $\tau \in [0, T]$. We also have used (5.16) to deduce

$$\begin{aligned}\int_\Omega \rho|\mathbf{u}|^2(\tau+)\phi\,d\mathbf{x} &= \lim_{h\to 0+}\frac{1}{h}\int_\tau^{\tau+h}\int_\Omega \rho|\mathbf{u}|^2\phi\,d\mathbf{x}dt\\ &= \lim_{h\to 0+}\frac{1}{h}\left(\lim_{k\to\infty}\int_\tau^{\tau+h}\int_\Omega \rho_k|\mathbf{u}_k|^2\phi\,d\mathbf{x}dt\right)\quad \text{for any } \phi\in\mathcal{D}(\Omega;\mathbb{R}),\end{aligned}$$

and, similarly,

$$\int_\Omega \rho P_m(\rho)(\tau+)\phi\,d\mathbf{x} = \lim_{h\to 0+}\frac{1}{h}\left(\lim_{k\to\infty}\int_\tau^{\tau+h}\int_\Omega \rho_k P_m(\rho_k)\phi\,d\mathbf{x}dt\right),$$

$$\int_\Omega \rho\vartheta^2 P_{\vartheta^2}(\rho)(\tau+)\phi\,d\mathbf{x} = \lim_{h\to 0+}\frac{1}{h}\left(\lim_{k\to\infty}\int_\tau^{\tau+h}\int_\Omega \rho_k\vartheta_k^2 P_{\vartheta^2}(\rho_k)\phi\,d\mathbf{x}dt\right),$$

$$\int_\Omega \rho^\Gamma(\tau+)\phi\,d\mathbf{x} = \lim_{h\to 0+}\frac{1}{h}\left(\lim_{k\to\infty}\int_\tau^{\tau+h}\int_\Omega \rho_k^\Gamma\phi\,d\mathbf{x}dt\right),$$

for a.e. $\tau \in [0, T]$.

The final aim in this section is to pass to the limit $k \to \infty$ in the "entropy inequality". Note it is enough to show that one can pass to the limit in all nonlinear terms contained in (5.6). To this end, we start with the observation that

$$\begin{aligned}\frac{\mathbb{S}_k : \nabla\mathbf{u}_k}{\vartheta_k} &= \frac{\mu(\vartheta_k, \mathbf{H}_k)}{\vartheta_k}\left(|\nabla\mathbf{u}_k|^2 + \nabla\mathbf{u}_k : \nabla^\top\mathbf{u}_k - \frac{2}{3}(\nabla\cdot\mathbf{u}_k)^2\right) + \frac{\eta(\vartheta_k, \mathbf{H}_k)}{\vartheta_k}(\nabla\cdot\mathbf{u}_k)^2\\ &= \left|\sqrt{\frac{\mu(\vartheta_k, \mathbf{H}_k)}{2\vartheta_k}}\left(\nabla\mathbf{u}_k + \nabla^\top\mathbf{u}_k - \frac{2}{3}\nabla\cdot\mathbf{u}_k\mathbb{I}_3\right)\right|^2 + \left(\sqrt{\frac{\eta(\vartheta_k, \mathbf{H}_k)}{\vartheta_k}}\nabla\cdot\mathbf{u}_k\right)^2,\end{aligned}$$

then, by the estimate (5.10) and the weak lower semi-continuity,

$$\int_{\Omega_\tau}\frac{\mathbb{S} : \nabla\mathbf{u}}{\vartheta}\,d\mathbf{x}dt \le \liminf_{k\to\infty}\int_{\Omega_\tau}\frac{\mathbb{S}_k : \nabla\mathbf{u}_k}{\vartheta_k}\,d\mathbf{x}dt$$

for any $\tau \ge 0$.

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Since one can estimate the entropy flux

$$\left|\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k}\nabla\vartheta_k\right| \le C\big(|\nabla\ln\vartheta_k| + \big(\vartheta_k^2+\vartheta_k^{\beta-1}\big)|\nabla\vartheta_k|\big),$$

where

$$\big(\vartheta_k^2+\vartheta_k^{\beta-1}\big)|\nabla\vartheta_k| = \frac{2}{3}\vartheta_k^{\frac{3}{2}}\big|\nabla\vartheta_k^{\frac{3}{2}}\big| + \frac{2}{\beta}\vartheta_k^{\frac{\beta}{2}}\big|\nabla\vartheta_k^{\frac{\beta}{2}}\big|,$$

by virtue of the hypothesis (5.22) on β and the estimates (5.17)–(5.20), one can show that

$$\left\{\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k}\nabla\vartheta_k\right\}_{k=1}^{\infty} \text{ is bounded in } L^{\pi}(\Omega_T) \text{ for a certain } \pi>1.$$

Furthermore,

$$\left\{\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k^2}|\nabla\vartheta_k|^2\right\}_{k=1}^{\infty} \text{ is bounded in } L^{\pi}(\Omega_T) \text{ for some } \pi>1.$$

And, consequently,

$$\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k}\nabla\vartheta_k \rightharpoonup \frac{\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3}{\vartheta}\nabla\vartheta \quad \text{in } L^{\pi}(\Omega_T) \text{ for a certain } \pi>1.$$

$$\frac{\kappa_G(\rho_k,\vartheta_k,\mathbf{H}_k)+\kappa_R\vartheta_k^3}{\vartheta_k^2}|\nabla\vartheta_k|^2 \rightharpoonup \frac{\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2 \quad \text{in } L^{\pi}(\Omega_T) \text{ for a certain } \pi>1.$$

Making use of Lemmas 5.3, 5.4 of Chapter 5 in [12], the estimates (5.8)–(5.11), the above relations together with (5.6) yields the desired variational form of the entropy inequality:

$$\begin{aligned}
&\int_{\Omega_T}\psi'\left(\frac{4a}{3}\vartheta^3+\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi-2\rho P_{\vartheta^2}(\rho)\vartheta\right)\phi\,d\mathbf{x}dt\\
&\quad+\int_{\Omega_T}\psi\left(\frac{4a}{3}\vartheta^3+\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi-2\rho P_{\vartheta^2}(\rho)\vartheta\right)\mathbf{u}\cdot\nabla\phi\,d\mathbf{x}dt\\
&\quad-\int_{\Omega_T}\psi\left(\frac{\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3}{\vartheta}\nabla\vartheta\right)\cdot\nabla\phi\,d\mathbf{x}dt\\
&\le\int_{\Omega_T}\varepsilon\nabla\left(\psi\phi\left(\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi-\frac{Q(\vartheta)}{\vartheta}-\vartheta\big(P_{\vartheta^2}(\rho)+\rho P'_{\vartheta^2}(\rho)\big)\right)\right)\cdot\nabla\rho\\
&\quad+\psi\phi p_{\vartheta}(\rho)\nabla\cdot\mathbf{u}\,d\mathbf{x}dt
\end{aligned}$$

$$-\int_{\Omega_T} \psi\phi\left(\frac{\kappa_G(\rho,\vartheta,\mathbf{H})+\kappa_R\vartheta^3}{\vartheta^2}|\nabla\vartheta|^2+\frac{\mathbb{S}:\nabla\mathbf{u}+\nu(\rho,\vartheta,\mathbf{H})|\nabla\times\mathbf{H}|^2}{\vartheta}\right)d\mathbf{x}dt$$

$$-\psi(0)\int_{\Omega}\phi\left(\frac{4a}{3}\vartheta_{0,\delta}^3+\rho_{0,\delta}\int_1^{\vartheta_{0,\delta}}\frac{c_v(\xi)}{\xi}\,d\xi\right)d\mathbf{x} \tag{5.27}$$

for any $0 \le \psi \in C^\infty([0,T])$ with $\psi(T)=0$ and $0 \le \phi \in \mathcal{D}(\Omega;\mathbb{R})$.

6. Passing to the limit $\varepsilon \to 0$

Our goal in this section is to take the vanishing limit of the artificial viscosity $\varepsilon \to 0$ for the family of approximate solutions $\{(\rho_{\delta,\varepsilon}, \Psi_{\delta,\varepsilon}, \mathbf{H}_{\delta,\varepsilon}, \vartheta_{\delta,\varepsilon})\}$ constructed in Section 5, i.e., to get rid of the artificial viscosity in $(3.1)_1$. In other words, we are to show the weak sequential stability (compactness) for the approximate solutions. Denote $\rho_\varepsilon = \rho_{\delta,\varepsilon}$, etc. in this section. Due to the bounds of the density estimates in Lemma 5.1 depending on ε, we definitely loose boundedness of $\nabla\rho_\varepsilon$ and, consequently, strong compactness of the sequence of $\{\rho_\varepsilon\}_{\varepsilon>0}$ in $L^1(\Omega_T)$ becomes a central issue now, so more refined estimates are needed to make sure the limit passage. At this stage, we first point out that it is easy to check that the sequences $\{\rho_\varepsilon\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, $\{\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\}_{\varepsilon>0}$, $\{\rho_\varepsilon\nabla\Psi_\varepsilon\}_{\varepsilon>0}$, $\{(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^\pi(\Omega_T)$ for a certain $\pi>1$ because of the estimates (5.8), (5.11). Moreover, since

$$\mathbb{S}_\varepsilon = \sqrt{\vartheta_\varepsilon\mu(\vartheta_\varepsilon,\mathbf{H}_\varepsilon)}\sqrt{\frac{\mu(\vartheta_\varepsilon,\mathbf{H}_\varepsilon)}{\vartheta_\varepsilon}}\left(\nabla\mathbf{u}_\varepsilon+\nabla^\top\mathbf{u}_\varepsilon-\frac{2}{3}(\nabla\cdot\mathbf{u}_\varepsilon)\mathbb{I}_3\right)$$
$$+\sqrt{\vartheta_\varepsilon\eta(\vartheta_\varepsilon,\mathbf{H}_\varepsilon)}\sqrt{\frac{\eta(\vartheta_\varepsilon,\mathbf{H}_\varepsilon)}{\vartheta_\varepsilon}}(\nabla\cdot\mathbf{u}_\varepsilon)\mathbb{I}_3,$$

by virtue of (5.9), (5.10), we know that $\{\mathbb{S}_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\pi(\Omega_T)$ for a certain $\pi>1$. As already pointed out in [18], both classical stumbling blocks of this approach – the phenomena of oscillations and concentrations – are likely to appear. In order to deal with the non-linear constitutive equations for the pressure and other quantities, the density oscillations as well as concentrations in the temperature must be excluded. Therefore, we have to find a bound (independent of ε) for the the pressure term in a reflexive space $L^\pi(\Omega_T)$ with $\pi>1$. To this end, similar as in Section 6.1 in [13], let us introduce an operator $\mathcal{B}=(\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3)$ with the following properties:

- $$\mathcal{B}:\left\{f\in L^\pi(\Omega)\,\middle|\,\int_\Omega f\,d\mathbf{x}=0\right\}\mapsto W^{1,\pi}(\Omega)^3$$

  is a bounded linear operator, i.e.,

$$\left\|\mathcal{B}[f]\right\|_{W^{1,\pi}(\Omega)^3}\le C(\pi)\|f\|_{L^\pi}\quad\text{for any } 1<\pi<\infty; \tag{6.1}$$

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- the function $\mathbf{v} = \mathcal{B}[f]$ solves the BVP:

$$\nabla \cdot \mathbf{v} = f \quad \text{in } \Omega, \qquad \mathbf{v}|_{\partial\Omega} = \mathbf{0}; \tag{6.2}$$

  Usually, the symbol $\mathcal{B} \approx (\nabla \cdot)^{-1}$ is called the Bogovskii operator.
- if $f \in L^\pi(\Omega)$ can be written as $f = \nabla \cdot \mathbf{g}$ with $\mathbf{g} \in L^q(\Omega)^3$, $\mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}$, then

$$\left\| \mathcal{B}[f] \right\|_{L^\pi} \le C(\pi) \|\mathbf{g}\|_{L^q} \quad \text{for any } 1 < \pi < \infty.$$

Considering the regularity of the approximate density functions, we can use the quantities

$$\left\{ \psi(t)\mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] \right\}_{\varepsilon > 0}, \qquad \psi \in \mathcal{D}(0, T), \ 0 \le \psi \le 1$$

as test functions in the momentum equation (5.25). Bearing in mind property (6.2) of the linear operator $\mathcal{B}$, we have the following integral identity:

$$\begin{aligned} &\int_0^T \psi \left(\int_\Omega \left(p_e(\rho_\varepsilon) + \vartheta_\varepsilon p_\vartheta(\rho_\varepsilon) + \vartheta_\varepsilon^2 p_{\vartheta^2}(\rho_\varepsilon) + \frac{a}{3}\vartheta_\varepsilon^4 + \delta \nabla \rho_\varepsilon^\Gamma \right) \rho_\varepsilon \, d\mathbf{x} \right) dt \\ &\quad = I + II + \cdots + IX, \end{aligned} \tag{6.3}$$

where

$$I = \frac{\int_\Omega \rho_\varepsilon \, d\mathbf{x}}{|\Omega|} \int_0^T \psi \left(\int_\Omega p_e(\rho_\varepsilon) + \vartheta_\varepsilon p_\vartheta(\rho_\varepsilon) + \vartheta_\varepsilon^2 p_{\vartheta^2}(\rho_\varepsilon) + \frac{a}{3}\vartheta_\varepsilon^4 + \delta \nabla \rho_\varepsilon^\Gamma \, d\mathbf{x} \right) dt,$$

$$II = \int_0^T \psi \left(\int_\Omega \mathbb{S}_\varepsilon : \nabla \mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] d\mathbf{x} \right) dt,$$

$$III = -\int_0^T \psi \left(\int_\Omega (\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla \mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] d\mathbf{x} \right) dt,$$

$$IV = \varepsilon \int_0^T \psi \int_\Omega \left(\nabla \mathbf{u}_\varepsilon \cdot \nabla \rho_\varepsilon \cdot \mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] d\mathbf{x} \right) dt,$$

$$V = -\int_0^T \psi \int_\Omega \left(\rho_\varepsilon \nabla \Psi_\varepsilon \cdot \mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] d\mathbf{x} \right) dt,$$

$$VI = -\int_0^T \psi' \int_\Omega \left(\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\left[\rho_\varepsilon - \frac{1}{|\Omega|} \int_\Omega \rho_\varepsilon \, d\mathbf{x} \right] d\mathbf{x} \right) dt,$$

$$VII = -\varepsilon \int_0^T \psi \int_\Omega \big(\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\big[\nabla \cdot (\nabla \rho_\varepsilon)\big]\, d\mathbf{x}\big)\, dt,$$

$$VIII = \int_0^T \psi \int_\Omega \big(\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}\big[\nabla \cdot (\rho_\varepsilon \mathbf{u}_\varepsilon)\big]\, d\mathbf{x}\big)\, dt,$$

$$IX = -\int_0^T \psi \bigg(\int_\Omega (\nabla \times \mathbf{H}) \times \mathbf{H} \cdot \mathcal{B}\bigg[\rho_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \rho_\varepsilon\, d\mathbf{x}\bigg]\, d\mathbf{x}\bigg)\, dt.$$

By virtue of the hypothesis (1.8), the fact that

$$\int_\Omega \rho_\varepsilon\, d\mathbf{x} = \int_\Omega \rho_{0,\delta}\, d\mathbf{x} \quad \text{independent of } t,$$

and the property (6.1), one has the integrals I, II, III, V and IX bounded uniformly with respect to $\varepsilon > 0$.

Using the same argument as in Section 6.1 in [13], we know that the left terms IV, VI, VII and $VIII$ are bounded uniformly for any small ε, and furthermore, from (6.3), the resulting estimate reads

$$\int_{\Omega_T} \rho_\varepsilon^{\Gamma+1}\, d\mathbf{x}\, dt \le C(\delta), \quad \text{with } C(\delta) \text{ independent of } \varepsilon.$$

Now, we have, at least for some subsequences

$$\rho_\varepsilon \to \rho \quad \text{in } C\big([0,T]; L^\Gamma_{\text{weak}}(\Omega)\big), \tag{6.4}$$

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^r\big(0,T; W_0^{1,r}(\Omega)\big),\ r = \frac{8}{5-\alpha}, \tag{6.5}$$

and

$$\rho_\varepsilon \mathbf{u}_\varepsilon \to \rho \mathbf{u} \quad \text{in } C\big([0,T]; L^{\frac{2\Gamma}{\Gamma+1}}_{\text{weak}}(\Omega)\big), \tag{6.6}$$

as $L^\Gamma_{\text{weak}}(\Omega)$ is continuously imbedded into the dual space $W^{-1,r'}(\Omega)$ for suitable (large) $\Gamma (> \frac{24}{17+3\alpha})$;

$$\begin{gathered} \vartheta_\varepsilon \overset{*}{\rightharpoonup} \vartheta \quad \text{in } L^\infty\big(0,T; L^4(\Omega)\big), \quad \text{and} \quad \vartheta_k \rightharpoonup \vartheta \quad \text{in } L^2\big(0,T; H^1(\Omega)\big), \\ \ln \vartheta_\varepsilon \rightharpoonup \overline{\ln \vartheta} \quad \text{in } L^2\big(0,T; H^1(\Omega)\big), \\ \vartheta_\varepsilon^{\frac{\beta}{2}} \rightharpoonup \overline{\vartheta^{\frac{\beta}{2}}} \quad \text{in } L^2\big(0,T; H^1(\Omega)\big), \end{gathered} \tag{6.7}$$

and

$$f(\vartheta_\varepsilon) \rightharpoonup \overline{f(\vartheta)} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big)$$

for any

$$f \in C^1(0,\infty), \qquad \big|f'(\xi)\big| \le C\left(\frac{1}{\xi}+\xi^{\frac{1}{2}}\right), \quad \xi > 0,$$

$$\mathbf{H}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{H} \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big), \qquad \mathbf{H}_\varepsilon \rightharpoonup \mathbf{H} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$
$$\mathbf{H}_\varepsilon \to \mathbf{H} \quad \text{in } L^2\big(0,T;L^2(\Omega)\big),$$
$$(\nabla\times\mathbf{H}_\varepsilon)\times\mathbf{H}_\varepsilon \rightharpoonup (\nabla\times\mathbf{H})\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi > 1,$$
$$\mathbf{u}_\varepsilon\times\mathbf{H}_\varepsilon \rightharpoonup \mathbf{u}\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi > 1,$$

and

$$\nu(\rho_\varepsilon,\vartheta_\varepsilon,\mathbf{H}_\varepsilon)\nabla\times\mathbf{H}_\varepsilon \rightharpoonup \nu(\rho,\vartheta,\mathbf{H})\nabla\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi > 1.$$

Furthermore, based on (6.4) and (6.5), if $\Gamma > r'$, we can show that the density $\rho \ge 0$ and the velocity $\mathbf{u}$ solve the original continuity equation $(1.1)_1$ on the whole space $\mathbb{R}^3\times[0,T)$ via extending both of them to be zero outside Ω, where, in fact,

$$\rho_\varepsilon\mathbf{u}_\varepsilon \to \rho\mathbf{u} \quad \text{in } C\big([0,T];L^{\frac{2\Gamma}{\Gamma+1}}_{\text{weak}}(\mathbb{R}^3)\big)$$

provided ρ_ε, $\mathbf{u}_\varepsilon$ were extended to be zero outside Ω. Thus ρ, $\mathbf{u}$ satisfy the integral identity:

$$\int_{\Omega_T}\big(\rho\psi'\phi+\psi\rho\mathbf{u}\cdot\nabla\phi\big)\,d\mathbf{x}dt+\psi(0)\int_\Omega \rho_0\phi\,d\mathbf{x}=0,$$

for any $\psi \in C^\infty([0,T])$ with $\psi(T)=0$ and $\phi\in\mathcal{D}(\mathbb{R}^3;\mathbb{R})$. Meanwhile, by using the celebrated regularization technique of R.J. DiPerna and P.-L. Lions [10], ρ, $\mathbf{u}$ (being extended to be zero outside Ω) solve the renormalized continuity equation (2.2) in $\mathcal{D}'(\mathbb{R}^3\times[0,T))$ for any continuously differentiable function b satisfying (2.15).

By the Hölder inequality, together with (6.5) and (6.6), we have

$$\rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \rightharpoonup \rho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^r\big(0,T;L^\pi(\Omega)\big),\ 1<\pi\le\frac{6\Gamma r}{3r+\Gamma r+6\Gamma}.$$

Here we have used the continuous imbedding

$$L^{\frac{2\Gamma}{\Gamma+1}}_{\text{weak}}(\Omega)\subset W^{-1,r'}(\Omega) \quad \text{provided } \Gamma\ge\frac{12}{5+3\alpha}.$$

Furthermore, by the same token, in accordance with (6.6) and (6.7), we have

$$\rho_\varepsilon\mathbf{u}_\varepsilon\ln\vartheta_\varepsilon \rightharpoonup \rho\mathbf{u}\overline{\ln\vartheta} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi>1.$$

Using the same argument as in Section 5, we know

$$\rho_\varepsilon \mathbf{u}_\varepsilon \int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi \rightharpoonup \rho \mathbf{u} \overline{\int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi > 1.$$

By virtue of the Hölder inequality and (5.14), we have

$$\varepsilon \int_\Omega \nabla\left(\varpi\left(\int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi - \frac{Q(\vartheta_\varepsilon)}{\vartheta_\varepsilon} - \vartheta_\varepsilon\big(P_{\vartheta^2}(\rho_\varepsilon) + \rho_\varepsilon P'_{\vartheta^2}(\rho_\varepsilon)\big)\right)\right) \cdot \nabla\rho_\varepsilon\, d\mathbf{x} \to 0$$
$$\text{in } L^\pi(0,T) \tag{6.8}$$

for any $\pi \geq 1$ and any $\varpi \in C^1(\overline{\Omega})$.

Hence, for the quantities $\frac{4a}{3}\vartheta_\varepsilon^3 + \rho_\varepsilon \int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi - 2\rho_\varepsilon P_{\vartheta^2}(\rho_\varepsilon)\vartheta_\varepsilon$ in (5.27), by the same argument as in Section 5,

$$\int_{\Omega_T} \left(\frac{4a}{3}\vartheta_\varepsilon^3 + \rho_\varepsilon \int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi - 2\rho_\varepsilon P_{\vartheta^2}(\rho_\varepsilon)\vartheta_\varepsilon\right)\vartheta_\varepsilon\, d\mathbf{x}dt$$
$$\to \int_{\Omega_T} \left(\frac{4a}{3}\overline{\vartheta^3} + \rho \overline{\int_1^{\vartheta} \frac{c_v(\xi)}{\xi}\, d\xi} - 2\overline{\rho P_{\vartheta^2}(\rho)}\vartheta\right)\vartheta\, d\mathbf{x}dt.$$

Note that, in contrast to Section 5, we do not know yet if the densities $\{\rho_\varepsilon\}_{\varepsilon>0}$ converge strongly in $L^1(\Omega_T)$. Fortunately, it holds that

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} \rho_\varepsilon P_{\vartheta^2}(\rho_\varepsilon)\vartheta_\varepsilon^2\, d\mathbf{x}dt = \int_{\Omega_T} \overline{\rho P_{\vartheta^2}(\rho)}\vartheta^2\, d\mathbf{x}dt.$$

Thus, exactly as in Section 5,

$$\int_{\Omega_T} \left(\frac{4a}{3}\vartheta_\varepsilon^3 + \rho_\varepsilon \int_1^{\vartheta_\varepsilon} \frac{c_v(\xi)}{\xi}\, d\xi\right)\vartheta_\varepsilon\, d\mathbf{x}dt \to \int_{\Omega_T} \left(\frac{4a}{3}\overline{\vartheta^3} + \rho \overline{\int_1^{\vartheta} \frac{c_v(\xi)}{\xi}\, d\xi}\right)\vartheta\, d\mathbf{x}dt,$$

which implies strong convergence of $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, and especially, $\vartheta_\varepsilon \to \vartheta$ in $L^2(\Omega_T)$. Then it comes from Lemma 5.4 in [12] that ϑ is strictly positive a.e. on Ω_T, and $\ln\vartheta = \overline{\ln\vartheta}$.

Letting $\varepsilon \to 0$ in (5.25), due to the estimate (5.14), the extra terms

$$\varepsilon\nabla\mathbf{u}_\varepsilon \cdot \nabla\rho_\varepsilon \to 0 \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi > 1,$$

indeed note that

$$\varepsilon \int_\Omega \nabla\mathbf{u}_\varepsilon \cdot \nabla\rho_\varepsilon \cdot \phi\, d\mathbf{x} \to 0 \quad \text{for any } \phi \in C\big(\overline{\Omega}; \mathbb{R}^3\big) \text{ as } \varepsilon \to 0$$

uniformly in $t \in [0, T]$, we know the limit quantities satisfy an "averaged" momentum equation

$$(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(\overline{p_e(\rho)} + \vartheta\overline{p_\vartheta(\rho)} + \vartheta^2\overline{p_{\vartheta^2}(\rho)} + \frac{a}{3}\vartheta^4 + \delta\overline{\rho^\Gamma}\right)$$
$$= \nabla\cdot\mathbb{S} + \rho\nabla\Psi + (\nabla\times\mathbf{H})\times\mathbf{H} \quad \text{in } \mathcal{D}'(\Omega_T), \tag{6.9}$$

where the potential Ψ satisfies (2.7).

In order to commute the limits with the composition operators in the "averaged" momentum equation (6.9), we need to show strong (pointwise) convergence of the sequence $\{\rho_\varepsilon\}_{\varepsilon>0}$, i.e.,

$$\rho_\varepsilon \to \rho \quad \text{in } L^1(\Omega_T).$$

This is a lengthy but nowadays formal procedure, making good use of the special function $b(\rho) = \rho\ln\rho$ in the renormalized continuity equation and the weak continuity of the effective viscous pressure "$p - (\lambda+2\mu)\nabla\cdot\mathbf{u}$". We omit it here (readers can refer to Section 6.3 in [9] for the details). Consequently, the limit functions ρ, $\mathbf{u}$ satisfy the momentum equation $(1.1)_2$, where p is replaced by $p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4 + \delta\rho^\Gamma$, in $\mathcal{D}'(\Omega_T)$.

The magnetic induction vector $\mathbf{H}$ satisfies

$$\int_{\Omega_T} \left(\psi'\mathbf{H}\cdot\phi + \psi(\mathbf{u}\times\mathbf{H})\cdot(\nabla\times\phi) - \psi\nu(\nabla\times\mathbf{H})\cdot(\nabla\times\phi)\right) d\mathbf{x}dt$$
$$+ \psi(0)\int_\Omega \mathbf{H}_0\cdot\phi\, d\mathbf{x} = 0, \tag{6.10}$$

for any $\psi \in C^\infty([0,T])$ with $\psi(T) = 0$ and $\phi \in \mathcal{D}(\Omega;\mathbb{R}^3)$.

To conclude, we have to let $\varepsilon \to 0$ in the energy equality (5.26) and the entropy inequality (5.27). Since ρ, $\mathbf{u}$ satisfy the renormalized continuity equation in $\mathcal{D}'(\mathbb{R}^3\times[0,T))$, then

$$\int_{\Omega_T} \psi p_b(\rho)\nabla\cdot\mathbf{u}\, d\mathbf{x}dt = \int_{\Omega_T} \psi'\rho P_b(\rho)\, d\mathbf{x}dt + \int_\Omega \rho_{0,\delta}P_b(\rho_{0,\delta})\, d\mathbf{x}$$

for any $\psi \in C^\infty([0,T])$ with $\psi(0) = 1$, $\psi(T) = 0$. Here $P_b(\rho) = \int_1^\rho \frac{p_b(z)}{z^2}\, dz$.

Thus passing the limit for $\varepsilon \to 0$ in (5.26), one has the total energy balance:

$$-\int_{\Omega_T} \psi'\left(\rho\left(\frac{1}{2}|\mathbf{u}|^2 + P_e(\rho) + Q(\vartheta) - \vartheta^2 P_{\vartheta^2}(\rho) + \frac{G}{2}\Delta^{-1}[\rho] + \frac{\delta}{\Gamma-1}\rho^{\Gamma-1}\right)\right.$$
$$\left. + a\vartheta^4 + \frac{1}{2}|\mathbf{H}|^2\right) d\mathbf{x}dt$$
$$= \int_\Omega \left(\rho_{0,\delta}\left(\frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}^2} + P_e(\rho_{0,\delta}) + Q(\vartheta_{0,\delta}) - \vartheta_{0,\delta}^2 P_{\vartheta^2}(\rho_{0,\delta}) + \frac{G}{2}\Delta^{-1}[\rho_{0,\delta}] + \frac{\delta}{\Gamma-1}\rho_{0,\delta}^{\Gamma-1}\right)\right.$$
$$\left. + a\vartheta_{0,\delta}^4 + \frac{1}{2}|\mathbf{H}_{0,\delta}|^2\right) d\mathbf{x} \triangleq E_{0,\delta} \tag{6.11}$$

for any $\psi \in C^\infty([0,T])$ with $\psi(0)=1$, $\psi(T)=0$. Note that,

$$\Delta^{-1}[\rho_\varepsilon] \to \Delta^{-1}[\rho] \quad \text{in } C(\overline{\Omega}_T)$$

by combining (6.4) with the standard elliptic theory. Here we have used the fact that

$$\begin{aligned}\int_\Omega \rho_\varepsilon \nabla \Psi_\varepsilon \cdot \mathbf{u}_\varepsilon \, d\mathbf{x} &= -\int_\Omega \Psi_\varepsilon \nabla \cdot (\rho_\varepsilon \mathbf{u}_\varepsilon)\, d\mathbf{x} = \int_\Omega \Psi_\varepsilon (\rho_{\varepsilon t} - \varepsilon \Delta \rho_\varepsilon)\, d\mathbf{x} \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_\varepsilon \Psi_\varepsilon \, d\mathbf{x} + \varepsilon G \int_\Omega \rho_\varepsilon^2 \, d\mathbf{x} \\ &= \frac{G}{2}\frac{d}{dt}\int_\Omega \rho_\varepsilon (-\Delta)^{-1}[\rho_\varepsilon]\, d\mathbf{x} + \varepsilon G \int_\Omega \rho_\varepsilon^2 \, d\mathbf{x}.\end{aligned}$$

The limit entropy inequality reads

$$\begin{aligned}&\int_{\Omega_T} \psi' \left(\frac{4a}{3}\vartheta^3 + \rho \int_1^\vartheta \frac{c_v(\xi)}{\xi}\, d\xi - \rho P_\vartheta(\rho) - 2\rho\vartheta P_{\vartheta^2}(\rho) \right) \phi \, d\mathbf{x}dt \\ &\quad + \int_{\Omega_T} \psi \left(\frac{4a}{3}\vartheta^3 + \rho \int_1^\vartheta \frac{c_v(\xi)}{\xi}\, d\xi - \rho P_\vartheta(\rho) - 2\rho\vartheta P_{\vartheta^2}(\rho) \right) \mathbf{u} \cdot \nabla\phi \, d\mathbf{x}dt \\ &\quad - \int_{\Omega_T} \psi \left(\frac{\kappa_G(\rho,\vartheta,\mathbf{H}) + \kappa_R \vartheta^3}{\vartheta} \nabla\vartheta \right) \cdot \nabla\phi \, d\mathbf{x}dt \\ &\le -\int_{\Omega_T} \psi\phi \left(\frac{\kappa_G(\rho,\vartheta,\mathbf{H}) + \kappa_R \vartheta^3}{\vartheta^2} |\nabla\vartheta|^2 + \frac{\mathbb{S} : \nabla\mathbf{u} + \nu(\rho,\vartheta,\mathbf{H})|\nabla\times\mathbf{H}|^2}{\vartheta} \right) d\mathbf{x}dt \\ &\quad - \psi(0)\int_\Omega \phi \left(\frac{4a}{3}\vartheta_{0,\delta}^3 + \rho_{0,\delta} \int_1^{\vartheta_{0,\delta}} \frac{c_v(\xi)}{\xi}\, d\xi - \rho_{0,\delta} P_\vartheta(\rho_{0,\delta}) - 2\rho_{0,\delta}\vartheta_{0,\delta} P_{\vartheta^2}(\rho_{0,\delta}) \right) d\mathbf{x}\end{aligned} \tag{6.12}$$

for any $0 \le \psi \in C^\infty([0,T])$ with $\psi(T)=0$ and $0 \le \phi \in \mathcal{D}(\Omega;\mathbb{R})$. Here we have used (6.8), the fact that

$$\begin{aligned}&\int_{\Omega_T} \psi\phi p_\vartheta(\rho) \nabla \cdot \mathbf{u} \, d\mathbf{x}dt \\ &\quad = \int_{\Omega_T} \psi' \phi \rho P_\vartheta(\rho)\, d\mathbf{x}dt + \psi(0)\int_\Omega \phi \rho_{0,\delta} P_\vartheta(\rho_{0,\delta})\, d\mathbf{x} + \int_{\Omega_T} \psi \rho P_\vartheta(\rho) \mathbf{u} \cdot \nabla\phi \, d\mathbf{x}dt.\end{aligned}$$

7. Passing to the limit $\delta \to 0$

Our final task is to carry out the limit process when the parameter $\delta \to 0$ to recover the original system (1.1) by evoking the full strength of the pressure and temperature estimates, in other words, we will establish the weak sequential stability property for the approximate solutions set $\{\rho_\delta, \mathbf{u}_\delta, \vartheta_\delta, \mathbf{H}_\delta\}_{\delta>0}$ constructed in Section 6. To begin with, note that, from hypothesis (3.3) on the approximate initial data, we have

$$E_{0,\delta} \to E_0 \quad \text{as } \delta \to 0,$$

and

$$\int_\Omega \phi\left(\frac{4a}{3}\vartheta_{0,\delta}^3 + \rho_{0,\delta}\int_1^{\vartheta_{0,\delta}} \frac{c_v(\xi)}{\xi}\,d\xi - \rho_{0,\delta}P_\vartheta(\rho_{0,\delta}) - 2\rho_{0,\delta}\vartheta_{0,\delta}P_{\vartheta^2}(\rho_{0,\delta})\right)d\mathbf{x}$$
$$\to \int_\Omega \phi\left(\frac{4a}{3}\vartheta_0^3 + \rho_0\int_1^{\vartheta_0} \frac{c_v(\xi)}{\xi}\,d\xi - \rho_0P_\vartheta(\rho_0) - 2\rho_0\vartheta_0P_{\vartheta^2}(\rho_0)\right)d\mathbf{x} \quad \text{as } \delta \to 0$$

for any $\phi \in C^\infty(\overline{\Omega})$, $\phi \geq 0$.

In light with the total energy balance (6.11), the following uniform bounds by a constant depending only on E_0 hold:

$$\rho_\delta \text{ bounded in } L^\infty\big(0,T;L^\gamma(\Omega)\big), \tag{7.1}$$
$$\rho_\delta|\mathbf{u}_\delta|^2,\ \rho_\delta Q(\vartheta_\delta) \text{ bounded in } L^\infty\big(0,T;L^1(\Omega)\big),$$

and, consequently, by Hölder's inequality,

$$\rho_\delta\mathbf{u}_\delta \text{ bounded in } L^\infty\big(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega)\big); \tag{7.2}$$
$$\vartheta_\delta \text{ bounded in } L^\infty\big(0,T;L^4(\Omega)\big),$$
$$\mathbf{H}_\delta \text{ bounded in } L^\infty\big(0,T;L^2(\Omega)\big), \tag{7.3}$$

and

$$\delta\int_{\Omega_T}\rho_\delta^\Gamma\,d\mathbf{x}dt \leq C \quad \text{uniformly with respect to } \delta > 0.$$

As for the term $\int_\Omega \Delta^{-1}[\rho]\rho\,d\mathbf{x}$ in the energy equality (6.11) related to the gravitational potential, from the elliptic estimates and the fact that

$$\int_\Omega \rho_\delta(t)\,d\mathbf{x} = \int_\Omega \rho_{0,\delta}\,d\mathbf{x} = M_0 \quad \text{for a.a. } t \in (0,T),$$

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we know, by virtue of Hölder's inequality,

$$\int_{\Omega} \left|\Delta^{-1}[\rho]\rho\right| d\mathbf{x} \le \|\rho\|_{L^2} \left\|\Delta^{-1}[\rho]\right\|_{L^2} \le M_0 \|\rho\|_{L^2}.$$

Similarly, the entropy production inequality (6.12) gives rise to

$$\operatorname*{ess\ sup}_{t\in[0,T]} \int_{\Omega} \rho_\delta |\ln \vartheta_\delta| \, d\mathbf{x} \le C(S_0),$$

$$\int_{\Omega_T} \frac{\mathbb{S}_\delta : \nabla \mathbf{u}_\delta + \nu |\nabla \times \mathbf{H}_\delta|^2}{\vartheta_\delta} + |\nabla \ln \vartheta_\delta|^2 + \left|\nabla \vartheta_\delta^{\frac{3}{2}}\right|^2 + \left|\nabla \vartheta_\delta^{\frac{\beta}{2}}\right|^2 d\mathbf{x}dt \le C(S_0), \tag{7.4}$$

together with

$$\int_{\Omega_T} \frac{\kappa_G(\rho_\delta, \vartheta_\delta, \mathbf{H}_\delta) + \kappa_R \vartheta_\delta^3}{\vartheta_\delta^2} |\nabla \vartheta_\delta|^2 \, d\mathbf{x}dt \le C(S_0),$$

and, consequently, by using the same arguments in Section 2 or in Section 5,

$$\left\|(1+\vartheta_\delta)^{\frac{\beta-1}{2}} \nabla \times \mathbf{H}_\delta\right\|_{L^2(L^2)} \le C(S_0),$$

$$\|\vartheta_\delta\|_{L^2(H^1)} + \|\ln \vartheta_\delta\|_{L^2(H^1)} + \left\|\vartheta_\delta^{\frac{3}{2}}\right\|_{L^2(H^1)} + \left\|\vartheta_\delta^{\frac{\beta}{2}}\right\|_{L^2(H^1)} \le C(E_0, S_0),$$

$$\|\mathbf{u}_\delta\|_{L^r(W_0^{1,r})} \le C(E_0, S_0), \quad r = \frac{8}{5-\alpha},$$

furthermore, utilizing estimates (7.1), (7.2), (7.3) and the hypothesis (1.8),

$$\vartheta_\delta p_\vartheta(\rho_\delta),\ \vartheta_\delta^2 p_{\vartheta^2}(\rho_\delta),\ (\nabla \times \mathbf{H}_\delta) \times \mathbf{H}_\delta,\ \mathbf{u}_\delta \times \mathbf{H}_\delta,\ \nu(\rho_\delta, \vartheta_\delta, \mathbf{H}_\delta) \nabla \times \mathbf{H}_\delta \text{ bounded in } L^\pi(\Omega_T)$$

for some certain $\pi > 1$,

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \text{ bounded in } L^r\left(0, T; L^{\frac{6\gamma r}{3r+\gamma r+6\gamma}}(\Omega)\right).$$

And the bounds are independent of $\delta > 0$. Here we have used $\vartheta_\delta^{\frac{3}{2}} \in L^2(0, T; H^1(\Omega))$, and a simple interpolation argument

$$L^\infty\left(0, T; L^4(\Omega)\right) \cap L^3\left(0, T; L^9(\Omega)\right) \subset L^\pi(\Omega_T), \quad \text{with } \pi = \frac{17}{3}.$$

Note that $\ln \vartheta_\delta$ is bounded in $L^2(\Omega_T)$ by a constant independent of $\delta > 0$, which implies the strict positivity of the temperature.

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Since

$$\mathbb{S}_\delta = \sqrt{\vartheta_\delta \mu(\vartheta_\delta, \mathbf{H}_\delta)} \sqrt{\frac{\mu(\vartheta_\delta, \mathbf{H}_\delta)}{\vartheta_\delta}} \left(\nabla \mathbf{u}_\delta + \nabla^\top \mathbf{u}_\delta - \frac{2}{3} (\nabla \cdot \mathbf{u}_\delta) \mathbb{I}_3 \right) + \sqrt{\vartheta_\delta \eta(\vartheta_\delta, \mathbf{H}_\delta)} \sqrt{\frac{\eta(\vartheta_\delta, \mathbf{H}_\delta)}{\vartheta_\delta}} (\nabla \cdot \mathbf{u}_\delta) \mathbb{I}_3,$$

the estimate (7.4) yields

$$\mathbb{S}_\delta \text{ bounded in } L^2\big(0, T; L^{\frac{4}{3}}(\Omega)\big)$$

uniformly with respect to δ.

Moreover, we can repeat step by step the proof of the refined pressure estimates in [12]. The resulting estimate reads

$$\int_{\Omega_T} \left(p_e(\rho_\delta) + \vartheta_\delta p_\vartheta(\rho_\delta) + \vartheta_\delta^2 p_{\vartheta^2}(\rho_\delta) + \frac{a}{3} \vartheta_\delta^4 + \delta \rho_\delta^\Gamma \right) \rho_\delta^\omega \, d\mathbf{x} dt \le C(E_0, S_0),$$

in particular,

$$\{\rho_\delta^{\gamma+\omega}\}_{\delta>0}, \ \{\delta \rho_\delta^{\Gamma+\omega}\}_{\delta>0} \text{ are bounded in } L^1(\Omega_T). \tag{7.5}$$

In view of the above estimates, we may assume that, up to a subsequence,

$$\rho_\delta \to \rho \quad \text{in } C\big([0, T]; L^\gamma_{\mathrm{weak}}(\Omega)\big),$$
$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in } L^r\big(0, T; W_0^{1,r}(\Omega)\big), \ r = \frac{8}{5-\alpha},$$

and

$$\rho_\delta \mathbf{u}_\delta \to \rho \mathbf{u} \quad \text{in } C\big([0, T]; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{weak}}(\Omega)\big),$$

as $L^\gamma_{\mathrm{weak}}(\Omega)$ is continuously imbedded into the dual space $W^{-1,r'}(\Omega)$ since $\gamma \ge 2$. Moreover, due to the choice of initial data $\rho_{0,\delta}$ and $\mathbf{m}_{0,\delta}$,

$$\rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \quad \text{a.e. on } \Omega,$$
$$\frac{\delta}{\Gamma - 1} \int_\Omega \rho_{0,\delta}^\Gamma \, d\mathbf{x} \to 0 \quad \text{as } \delta \to 0$$

and

$$\rho \mathbf{u}(0, \mathbf{x}) = \mathbf{m}_0(\mathbf{x}) \quad \text{a.e. on } \Omega.$$

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Consequently, in accordance with the hypothesis $\gamma \geq 2$, the space $L^{\frac{2\gamma}{\gamma+1}}(\Omega)$ is compactly imbedded into $W^{-1,r'}(\Omega)$, which yields compactness of the convective terms:

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho \mathbf{u} \otimes \mathbf{u} \quad \text{in } L^r\big(0,T;L^{\frac{6\gamma r}{3r+\gamma r+6\gamma}}(\Omega)\big).$$

Here we have used the fact that ρ_δ, $\mathbf{u}_\delta$ satisfy (1.1)$_1$ and $(\rho_\delta \mathbf{u}_\delta)_t$ can be expressed by (6.9). Since $\gamma \geq 2$, we can use the regularization technique developed in [10] to show that ρ, $\mathbf{u}$ satisfy the (2.16), and furthermore, $\rho \in C([0,T];L^1(\Omega))$.

$$\vartheta_\delta \overset{*}{\rightharpoonup} \vartheta \quad \text{in } L^\infty\big(0,T;L^4(\Omega)\big), \quad \text{and} \quad \vartheta_\delta \rightharpoonup \vartheta \quad \text{in } L^2\big(0,T;H^1(\Omega)\big), \tag{7.6}$$
$$\ln\vartheta_\delta \rightharpoonup \overline{\ln\vartheta} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$

furthermore,

$$\rho_\delta \ln\vartheta_\delta \rightharpoonup \rho\overline{\ln\vartheta} \quad \text{in } L^2\big(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega)\big),$$
$$\rho_\delta \ln\vartheta_\delta \mathbf{u}_\delta \rightharpoonup \rho\overline{\ln\vartheta}\mathbf{u} \quad \text{in } L^2\big(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega)\big),$$
$$\vartheta_\delta^{\frac{\beta}{2}} \rightharpoonup \overline{\vartheta^{\frac{\beta}{2}}} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$
$$Q(\vartheta_\delta) \rightharpoonup \overline{Q(\vartheta)} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),$$
$$\rho_\delta Q(\vartheta_\delta) \rightharpoonup \rho\overline{Q(\vartheta)} \quad \text{in } L^2\big(0,T;L^{\frac{6\gamma}{6+\gamma}}(\Omega)\big),$$
$$\rho_\delta \mathbf{u}_\delta Q(\vartheta_\delta) \rightharpoonup \rho\mathbf{u}\overline{Q(\vartheta)} \quad \text{in } L^2\big(0,T;L^{\frac{6\gamma}{3+4\gamma}}(\Omega)\big),$$

and

$$f(\vartheta_\delta) \rightharpoonup \overline{f(\vartheta)} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big)$$

for any

$$f \in C^1(0,\infty), \qquad \big|f'(\xi)\big| \leq C\left(\frac{1}{\xi} + \xi^{\frac{1}{2}}\right), \quad \xi > 0,$$
$$\mathbf{H}_\delta \overset{*}{\rightharpoonup} \mathbf{H} \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big),$$

which can be improved to

$$\mathbf{H}_\delta \to \mathbf{H} \quad \text{in } C\big([0,T];L^2_{\text{weak}}(\Omega)\big),$$

since $\mathbf{H}_{\delta t}$ can be expressed through Eq. (6.10);

$$\mathbf{H}_\delta \rightharpoonup \mathbf{H} \quad \text{in } L^2\big(0,T;H^1(\Omega)\big), \qquad \mathbf{H}_\delta \to \mathbf{H} \quad \text{in } L^2(\Omega_T),$$
$$(\nabla\times\mathbf{H}_\delta)\times\mathbf{H}_\delta \rightharpoonup (\nabla\times\mathbf{H})\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi>1,$$
$$\mathbf{u}_\delta\times\mathbf{H}_\delta \rightharpoonup \mathbf{u}\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi>1,$$
$$\nu(\rho_\delta,\vartheta_\delta,\mathbf{H}_\delta)\nabla\times\mathbf{H}_\delta \rightharpoonup \nu(\rho,\vartheta,\mathbf{H})\nabla\times\mathbf{H} \quad \text{in } L^\pi(\Omega_T) \text{ for a certain } \pi>1,$$

and

$$\mathbf{H}(0,\mathbf{x})=\mathbf{H}_0(\mathbf{x}) \quad \text{a.e. on } \Omega.$$

As usual as in Section 5 and Section 6, our next duty is to show strong (pointwise) convergence of the temperature. By virtue of hypothesis (1.8), we have

$$\rho_\delta P_\vartheta(\rho_\delta) \text{ is bounded in } L^\infty\big(0,T;L^{\frac{\gamma}{\varsigma}}(\Omega)\big), \text{ with } \frac{\gamma}{\varsigma}>\frac{4}{3},$$
$$\rho_\delta P_{\vartheta^2}(\rho_\delta) \text{ is bounded in } L^\infty\big(0,T;L^{\frac{\gamma}{\zeta}}(\Omega)\big), \text{ with } \frac{\gamma}{\zeta}>2,$$

and

$$\rho_\delta\vartheta_\delta P_{\vartheta^2}(\rho_\delta) \text{ is bounded in } L^\infty\big(0,T;L^{\frac{4\gamma}{\gamma+4\zeta}}(\Omega)\big), \text{ with } \frac{4\gamma}{\gamma+4\zeta}>\frac{4}{3},$$

in accordance with (7.6).

Using the entropy inequality (6.12) together with Lemma 6.3 of Chapter 6 in [18], we have

$$\frac{4a}{3}\vartheta_\delta^3+\rho_\delta\int_1^{\vartheta_\delta}\frac{c_v(\xi)}{\xi}\,d\xi-\rho_\delta P_\vartheta(\rho_\delta)-2\rho_\delta\vartheta_\delta P_{\vartheta^2}(\rho_\delta)$$
$$\to \frac{4a}{3}\overline{\vartheta^3}+\overline{\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi}-\overline{\rho P_\vartheta(\rho)}-2\vartheta\overline{\rho P_{\vartheta^2}(\rho)} \quad \text{in } L^2\big(0,T;H^{-1}(\Omega)\big),$$

and, in particular,

$$\int_{\Omega_T}\left(\frac{4a}{3}\vartheta_\delta^3+\rho_\delta\int_1^{\vartheta_\delta}\frac{c_v(\xi)}{\xi}\,d\xi-\rho_\delta P_\vartheta(\rho_\delta)-2\rho_\delta\vartheta_\delta P_{\vartheta^2}(\rho_\delta)\right)\vartheta_\delta\,d\mathbf{x}dt$$
$$\to \int_{\Omega_T}\left(\frac{4a}{3}\overline{\vartheta^3}+\overline{\rho\int_1^{\vartheta}\frac{c_v(\xi)}{\xi}\,d\xi}-\overline{\rho P_\vartheta(\rho)}-2\vartheta\overline{\rho P_{\vartheta^2}(\rho)}\right)\vartheta\,d\mathbf{x}dt. \tag{7.7}$$

Since ρ_δ satisfies the renormalized equation (2.16), we have

$$b(\rho_\delta)\to\overline{b(\rho)} \quad \text{in } C\big([0,T];L^\gamma_{\text{weak}}(\Omega)\big)$$

provided b is a bounded and continuously differential function. Thus, a simple approximation argument yields

$$\rho_\delta P_\vartheta(\rho_\delta) \to \overline{\rho P_\vartheta(\rho)} \quad \text{in } C\big([0,T]; L^{\frac{\gamma}{\xi}}_{\text{weak}}(\Omega)\big),$$
$$\rho_\delta P_{\vartheta^2}(\rho_\delta) \to \overline{\rho P_{\vartheta^2}(\rho)} \quad \text{in } C\big([0,T]; L^{\frac{\gamma}{\xi}}_{\text{weak}}(\Omega)\big),$$

whence, using (7.6) again,

$$\lim_{\delta\to 0} \int_{\Omega_T} \rho_\delta P_\vartheta(\rho_\delta)\vartheta_\delta \, d\mathbf{x}dt = \lim_{\delta\to 0} \int_{\Omega_T} \overline{\rho P_\vartheta(\rho)}\vartheta \, d\mathbf{x}dt,$$
$$\lim_{\delta\to 0} \int_{\Omega_T} \rho_\delta P_{\vartheta^2}(\rho_\delta)\vartheta_\delta^2 \, d\mathbf{x}dt = \lim_{\delta\to 0} \int_{\Omega_T} \overline{\rho P_\vartheta(\rho)}\vartheta^2 \, d\mathbf{x}dt.$$

Consequently, (7.7) reduces to

$$\int_{\Omega_T} \left(\frac{4a}{3}\vartheta_\delta^3 + \rho_\delta \int_1^{\vartheta_\delta} \frac{c_v(\xi)}{\xi}\, d\xi \right) \vartheta_\delta \, d\mathbf{x}dt \to \int_{\Omega_T} \left(\frac{4a}{3}\overline{\vartheta^3} + \overline{\rho \int_1^{\vartheta} \frac{c_v(\xi)}{\xi}\, d\xi} \right) \vartheta \, d\mathbf{x}dt.$$

Exactly as in Section 6, we obtain

$$\vartheta_\delta \to \vartheta \quad \text{in } L^2(\Omega_T).$$

Thanks to Lemma 5.4 in [12], ϑ is positive a.a. on Ω_T and $\ln\vartheta = \overline{\ln\vartheta}$.

In order to establish strong convergence of the sequence $\{\rho_\delta\}_{\delta>0}$, we pursue the approach similarly as in Sections 7.5, 7.6 in [12]. The results on propagation of oscillations stated in Section 7.6 in [12] yield

$$\rho_\delta \to \rho \quad \text{in } L^1(\Omega_T),$$

which can be strengthened to

$$\rho_\delta \to \rho \quad \text{in } C\big([0,T]; L^1(\Omega)\big)$$

(see [10] or Section 6.7 in [18]).

Consequently, the limit function ρ, $\mathbf{u}$ satisfy the continuity equation

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^3 \times [0,T)\big)$$

as well as its renormalized version (2.16).

Similarly, the momentum equation

$$(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\left(p_e(\rho) + \vartheta p_\vartheta(\rho) + \vartheta^2 p_{\vartheta^2}(\rho) + \frac{a}{3}\vartheta^4\right)$$
$$= \nabla\cdot\mathbb{S} + \rho\nabla\Psi + (\nabla\times\mathbf{H})\times\mathbf{H}$$

is satisfied in $\mathcal{D}'(\Omega_T)$. One can handle the induction equation as well, i.e.,

$$\mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H})$$

is satisfied in $\mathcal{D}'(\Omega_T)$.

By the same token, we can pass to the limit in the energy equality (6.11) in order to obtain (6.11). Note that

$$\delta\rho_\delta^\Gamma \to 0 \quad \text{in } L^1(\Omega_T)$$

as a consequence of (7.5). Hence, the regularizing δ-dependent terms on the left-hand side disappear. And it is a routine matter to deal with the entropy inequality (6.12) based on the above estimates. We remark here that it is standard to pass to the limit in the production rate keeping the correct sense of the inequality as all terms are convex with respect to the spatial gradients of $\mathbf{u}$, ϑ and $\mathbf{H}$.

Last but not least, we need to show that the temperature ϑ tends to its prescribed initial distribution ϑ_0 for $t\to 0$. Since the total energy of the system is conserved, we have

$$E(t) = \int_\Omega \left(\rho P_e(\rho) - \rho\vartheta^2 P_{\vartheta^2}(\rho) + a\vartheta^4 + \rho Q(\vartheta) + \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}|\mathbf{H}|^2 + \frac{G}{2}\Delta^{-1}[1_\Omega\rho]\rho\right)d\mathbf{x}$$
$$= \int_\Omega \left(\rho_0 P_e(\rho_0) - \rho_0\vartheta_0^2 P_{\vartheta^2}(\rho_0) + a\vartheta_0^4 + \rho_0 Q(\vartheta_0) + \frac{1}{2\rho_0}|\mathbf{m}_0|^2 + \frac{1}{2}|\mathbf{H}_0|^2 - \frac{1}{2}\rho_0\Psi_0\right)d\mathbf{x},$$

with $\Psi_0 = G(-\Delta)^{-1}[1_\Omega\rho_0]$.

Bearing in mind,

$$\rho(t) \rightharpoonup \rho_0 \quad \text{in } L^\gamma(\Omega) \text{ for } t\to 0,$$
$$\rho\mathbf{u}(t) \rightharpoonup \mathbf{m}_0 \quad \text{in } L^{\frac{2\gamma}{\gamma+1}}(\Omega) \text{ for } t\to 0,$$
$$\mathbf{H}(t) \rightharpoonup \mathbf{H}_0 \quad \text{in } L^2(\Omega) \text{ for } t\to 0,$$

using the same argument as in Section 7.7 in [12] to derive

$$\operatorname*{ess\,lim}_{t\to 0+} \int_\Omega \left(\frac{4a}{3}\vartheta^3 + \rho\int_1^\vartheta \frac{c_v(\xi)}{\xi}\,d\xi - \rho P_\vartheta(\rho) - 2\rho\vartheta P_{\vartheta^2}(\rho)\right)\phi\,d\mathbf{x}$$
$$\geq \int_\Omega \left(\frac{4a}{3}\vartheta_0^3 + \rho_0\int_1^{\vartheta_0} \frac{c_v(\xi)}{\xi}\,d\xi - \rho_0 P_\vartheta(\rho_0) - 2\rho_0\vartheta_0 P_{\vartheta^2}(\rho_0)\right)\phi\,d\mathbf{x}$$

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for any $0 \le \phi \in \mathcal{D}(\Omega; \mathbb{R})$. And

$$\vartheta(t) \to \vartheta_0 \quad \text{in } L^4(\Omega) \text{ as } t \to 0+.$$

Theorem 2.1 has been proved.

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