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J. Differential Equations 257 (2014) 4108–4132

Journal of Differential Equations

www.elsevier.com/locate/jde

Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity

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Received 8 January 2014; revised 12 July 2014

Available online 18 August 2014

Abstract

In this paper, we consider the initial–boundary value problem of the 3D primitive equations for oceanic and atmospheric dynamics with only horizontal diffusion in the temperature equation. Global well-posedness of strong solutions are established with H^2 initial data.

MSC: 35Q35; 76D03; 86A10

Keywords: Well-posedness; Strong solution; Primitive equation

1. Introduction

The primitive equations derived from the Boussinesq system of incompressible flow are fundamental models for weather prediction, see, e.g., Lewandowski [12], Majda [16], Pedlosky [17], Vallis [21], and Washington and Parkinson [22]. Due to their importance, the primitive equations has been studied analytically by many authors, see, e.g., [13,14,20,18,16] and the references therein.

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http://dx.doi.org/10.1016/j.jde.2014.08.003
0022-0396/

In this paper, we consider the viscous primitive equations with only horizontal diffusion in the temperature equation:

$$\partial_t v + (v \cdot \nabla_H)v + w\partial_z v + \nabla_H p - \Delta v + f_0 k \times v = 0, \tag{1.1}$$

$$\partial_z p + T = 0, \tag{1.2}$$

$$\nabla_H \cdot v + \partial_z w = 0, \tag{1.3}$$

$$\partial_t T + v \cdot \nabla_H T + w\partial_z T - \Delta_H T = 0, \tag{1.4}$$

where the horizontal velocity $v = (v^1, v^2)$, the vertical velocity w, the temperature T and the pressure p are the unknowns and f_0 is the Coriolis parameter. In this paper, we use the notations $\nabla_H = (\partial_x, \partial_y)$ and $\Delta_H = \partial_x^2 + \partial_y^2$ to denote the horizontal gradient and the horizontal Laplacian, respectively. The dominant horizontal eddy diffusivity in this model is justified by some geophysicists due to the strong horizontal turbulent mixing.

In 1990s, Lions, Temam and Wang [13–15] initialed the mathematical studies on the primitive equations, where among other issues they established the global existence of weak solutions. The uniqueness of weak solutions for 2D case was later proven by Bresch, Guillén-González, Masmoudi and Rodríguez-Bellido [1]; however, the uniqueness of weak solutions in the three-dimensional case is still unclear. Local existence of strong solutions was obtained by Guillén-González, Masmoudi and Rodríguez-Bellido [8]. Global existence of strong solutions for 2D case was established by Bresch, Kazhikhov and Lemoine in [2] and Temam and Ziane in [20], while the 3D case was established by Cao and Titi [6]. Global strong solutions for 3D case were also obtained by Kobelkov [9] later by using a different approach, see also the subsequent articles Kukavica and Ziane [10,11]. The systems considered in all the papers [6,9–11] are assumed to have diffusion in all directions. It is proven by Cao and Titi [7] that these global existence results still hold true for the system with only vertical diffusion, provided the local in time strong solutions exist. As the complement of [7], local existence results for the system with only vertical diffusion are recently given by Cao, Li and Titi [4] with H^2 initial data. Notably, the inviscid primitive equation, with or without coupling to the heat equation has been shown by Cao et al. [3] to blow up in finite time (see also [23]).

In this paper, we consider the primitive equations with only horizontal diffusion in the temperature equation, i.e. system (1.1)–(1.4). The aim of this paper is to show that the strong solutions exist globally for system (1.1)–(1.4) subject to some initial and boundary conditions. More precisely, we consider the problem in the domain $\Omega_0 = M \times (-h, 0)$, with $M = (0, 1) \times (0, 1)$, and supplement system (1.1)–(1.4) with the following boundary and initial conditions:

$$v, w \text{ and } T \text{ are periodic in } x \text{ and } y, \tag{1.5}$$

$$(\partial_z v, w)|_{z=-h,0} = (0, 0), \qquad T|_{z=-h} = 1, \qquad T|_{z=0} = 0, \tag{1.6}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.7}$$

We note that, since there is no normal flow on the boundaries, the boundary conditions for T as in (1.6), i.e. $T|_{z=-h} = 1$ and $T|_{z=0} = 0$, are preserved, as long as they are satisfied initially. In fact, with the aid of the boundary condition $w|_{z=-h,0} = 0$, the temperature T satisfies the following equation

$$\partial_t T + v \cdot \nabla_H T - \Delta_H T = 0,$$

on the boundaries, i.e. $z=-h$ or $z=0$, form which, one deduces $T|_{z=-h}=1$ and $T|_{z=0}=0$, as long as these are satisfied initially. On account of this, we put the boundary conditions for T in (1.6), and suppose that $T_0|_{z=-h}=1$ and $T_0|_{z=0}=0$.

Replacing T and p by $T+\frac{z}{h}$ and $p-\frac{z^2}{2h}$, respectively, then system (1.1)–(1.4) with (1.5)–(1.7) is equivalent to the following system

$$\partial_t v+(v\cdot\nabla_H)v+w\partial_z v+\nabla_H p-\Delta v+f_0 k\times v=0, \tag{1.8}$$

$$\partial_z p+T=0, \tag{1.9}$$

$$\nabla_H\cdot v+\partial_z w=0, \tag{1.10}$$

$$\partial_t T+(v\cdot\nabla_H)T+w\left(\partial_z T+\frac{1}{h}\right)-\Delta_H T=0, \tag{1.11}$$

subject to the boundary and initial conditions

$$v, w, T \text{ are periodic in } x \text{ and } y, \tag{1.12}$$

$$(\partial_z v, w)|_{z=-h,0}=0, \qquad T|_{z=-h,0}=0, \tag{1.13}$$

$$(v, T)|_{t=0}=(v_0, T_0). \tag{1.14}$$

Here, for simplicity, we still use T_0 to denote the initial temperature in (1.14), though it is now different from that in (1.7).

Notice that the periodic subspace $\mathcal{H}$, given by

$$\begin{aligned}\mathcal{H}:=\big\{(v,w,p,T)\big|&v, w, p \text{ and } T \text{ are spatially periodic in all three variables}\\ &\text{and even, odd, even and odd in } z \text{ variable, respectively}\big\},\end{aligned}$$

is invariant under the dynamics system (1.8)–(1.11). That is if the initial data satisfy the properties stated in the definition of $\mathcal{H}$, then, as we will see later (see Theorem 1.1), the solutions to system (1.8)–(1.11) will obey the same symmetry as the initial data. This motivated us to consider the following system

$$\partial_t v+(v\cdot\nabla_H)v+w\partial_z v+\nabla_H p-\Delta v+f_0 k\times v=0, \tag{1.15}$$

$$\partial_z p+T=0, \tag{1.16}$$

$$\nabla_H\cdot v+\partial_z w=0, \tag{1.17}$$

$$\partial_t T+(v\cdot\nabla_H)T+w\left(\partial_z T+\frac{1}{h}\right)-\Delta_H T=0, \tag{1.18}$$

in $\Omega:=M\times(-h,h)$, subject to the boundary and initial conditions

$$v, w, p \text{ and } T \text{ are periodic in } x, y, z, \tag{1.19}$$

$$v \text{ and } p \text{ are even in } z, \text{ and } w \text{ and } T \text{ are odd in } z, \tag{1.20}$$

$$(v, T)|_{t=0}=(v_0, T_0). \tag{1.21}$$

One can easily check that the restriction on the sub-domain Ω_0 of a solution (v,w,p,T) to system (1.15)–(1.21) is a solution to the original system (1.8)–(1.14). Because of this, throughout this paper, we mainly concern on the study of system (1.15)–(1.21) defined on Ω, while the well-posedness results for system (1.8)–(1.14) defined on Ω_0 follow as a corollary of those for system (1.15)–(1.21).

For any function $\phi(x, y, z)$ defined on Ω, we define functions $\bar{\phi}$ and $\tilde{\phi}$ as follows

$$\bar{\phi}(x, y) = \frac{1}{2h}\int_{-h}^{h} \phi(x, y, z)dz, \qquad \tilde{\phi} = \phi - \bar{\phi}.$$

Using these notations, system (1.15)–(1.21) is equivalent to (see [7] for example)

$$\begin{aligned} &\partial_t v - \Delta v + (v \cdot \nabla_H)v - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\partial_z v \\ &\quad + f_0 k \times v + \nabla_H\left(p_s(x, y, t) - \int_{-h}^{z} T(x, y, \xi, t)d\xi\right) = 0, \end{aligned} \tag{1.22}$$

$$\nabla_H \cdot \bar{v} = 0, \tag{1.23}$$

$$\partial_t T - \Delta_H T + v \cdot \nabla_H T - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\left(\partial_z T + \frac{1}{h}\right) = 0, \tag{1.24}$$

complemented with the following boundary and initial conditions

$$v \text{ and } T \text{ are periodic in } x, y, z, \tag{1.25}$$

$$v \text{ and } T \text{ are even and odd in } z, \text{ respectively}, \tag{1.26}$$

$$(v, T)|_{t=0} = (v_0, T_0). \tag{1.27}$$

Throughout this paper, we use $L^q(\Omega)$, $L^q(M)$ and $W^{m,q}(\Omega)$, $W^{m,q}(M)$ to denote the standard Lebesgue and Sobolev spaces, respectively. For $q = 2$, we use H^m instead of $W^{m,2}$. We use $W^{m,q}_{\mathrm{per}}(\Omega)$ and H^m_{per} to denote the spaces of periodic functions in $W^{m,q}(\Omega)$ and $H^m(\Omega)$, respectively. For simplicity, we use the same notations L^p and H^m to denote the N product spaces $(L^p)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_p$ to denote the L^p norm of u.

Definitions of strong solution, maximal existence time and global strong solution to system (1.22)–(1.27) are given by the following three definitions, respectively.

Definition 1.1. Given a positive number t_0. Let $v_0 \in H^2(\Omega)$ and $T_0 \in H^2(\Omega)$ be two periodic functions, such that they are even and odd in z, respectively. A couple (v, T) is called a strong solution to system (1.22)–(1.27) (or equivalently (1.15)–(1.21)) on $\Omega \times (0, t_0)$ if

(i) v and T are periodic in x, y, z, and they are even and odd in z, respectively;
(ii) v and T have the regularities

$$\begin{aligned} &v \in L^\infty\big(0, t_0; H^2(\Omega)\big) \cap C\big([0, t_0]; H^1(\Omega)\big) \cap L^2\big(0, t_0; H^3(\Omega)\big), \\ &T \in L^\infty\big(0, t_0; H^2(\Omega)\big) \cap C\big([0, t_0]; H^1(\Omega)\big), \qquad \nabla_H T \in L^2\big(0, t_0; H^2(\Omega)\big), \end{aligned}$$

$$\partial_t v \in L^2\big(0, t_0; H^1(\Omega)\big), \qquad \partial_t T \in L^2\big(0, t_0; H^1(\Omega)\big);$$

(iii) v and T satisfy Eqs. (1.22)–(1.24) a.e. in $\Omega \times (0, t_0)$ and the initial condition (1.27).

Definition 1.2. A finite positive number $\mathcal{T}^*$ is called the maximal existence time of a strong solution (v, T) to system (1.22)–(1.27) if (v, T) is a strong solution to the system on $\Omega \times (0, t_0)$ for any $t_0 < \mathcal{T}^*$ and $\lim_{t\to\mathcal{T}^*_-}(\|v\|^2_{H^2} + \|T\|^2_{H^2}) = \infty$.

Definition 1.3. A couple (v, T) is called a global strong solution to system (1.22)–(1.27) if it is a strong solution on $\Omega \times (0, t_0)$ for any $t_0 < \infty$.

The main result of this paper is the following global existence result.

Theorem 1.1. *Suppose that the periodic functions $v_0, T_0 \in H^2(\Omega)$ are even and odd in z, respectively. Then system (1.22)–(1.27) (or equivalently (1.15)–(1.21)) has a unique global strong solution (v, T).*

Local existence of strong solutions are obtained by a regularization mechanism. More precisely, we add the vertical diffusion term, with a diffusion coefficient $\varepsilon > 0$, in the temperature equation to obtain a regularized system. We then establish uniform estimates, in ε, for strong solutions of the regularized system, over a short interval of time independent of ε, and then take the limit, as ε goes to zero, to obtain local strong solutions to system (1.22)–(1.27). To obtain the global strong solutions, from the local existence results, we need to establish the a priori estimates on the derivatives of the solution, up to the second order. The first crucial estimate is the L^6 estimate on v, which has been originally obtained by Cao and Titi in [6,7]. Next, we establish the estimates on the derivatives. Resulting from the lack of sufficient information on the equation for the vertical velocity w, and the absence of the vertical diffusion in the temperature equation, the treatments of different derivatives will vary. In particular, when we deal with the derivatives of v of the same order, we first work on the vertical derivatives and then the horizontal ones. The reason for this is due to the fact that we need the estimates on the vertical derivatives to handle the term of the form $(\int_{-h}^z \nabla_H \cdot v d\xi)\partial_z v$, which has "stronger nonlinearity" than the term of the form $(v \cdot \nabla_H)v$. Keeping this in mind and making use of the L^6 estimates for v, we successfully obtain the estimates on $\partial_z^2 v$, then on $\nabla_H \partial_z v$ and finally on $\nabla^2 v$ and $\nabla^2 T$. As a result, we obtain the a priori estimates which guarantee the global existence of strong solutions.

As a corollary of Theorem 1.1, we have the following theorem, which states the well-posedness of strong solutions to system (1.8)–(1.14). The strong solutions to system (1.8)–(1.14) are defined in the similar way as before.

Theorem 1.2. *Let v_0 and T_0 be two functions such that they are periodic in x and y. Denote by v_0^{ext} and T_0^{ext} the even and odd extensions in z of v_0 and T_0, respectively. Suppose that $v_0^{\text{ext}}, T_0^{\text{ext}} \in H^2_{\text{per}}(\Omega)$. Then system (1.8)–(1.14) has a unique global strong solution (v, T).*

The existence part follows directly by applying Theorem 1.1 with initial data $(v_0^{\text{ext}}, T_0^{\text{ext}})$ and restricting the solution on the sub-domain Ω_0. While the uniqueness part can be proven in the same way as that for Theorem 1.1. It should be pointed out that, due to the same reason as stated in Remark 1.1 in [4], the condition that $v_0^{\text{ext}}, T_0^{\text{ext}} \in H^2_{\text{per}}(\Omega)$ in the above theorem is necessary for the existence of strong solutions to system (1.8)–(1.14).

The rest of the paper is organized as follows: in the next section, Section 2, we prove the local existence of strong solutions; in Section 3, by establishing the necessary a priori estimates, we show that the local strong solution can be extended to be a global one, and thus obtain a global strong solution.

Throughout this paper, we use C to denote a general constant which may be different from line to line.

2. Local existence of strong solutions

In this section, we establish the local existence of strong solutions to system (1.15)–(1.21), or equivalently system (1.22)–(1.27).

We first cite the following proposition on the local existence of strong solutions to the system with full diffusion (see Proposition 2.1 of [4]).

Proposition 2.1. *Let $v_0 \in H^2(\Omega)$ and $T_0 \in H^2(\Omega)$ be two periodic functions, such that they are even and odd in z, respectively. Then for any given $\varepsilon > 0$, there is a $t_\varepsilon > 0$, depending on ε, and a unique strong solutions $(v_\varepsilon, T_\varepsilon)$, with $(v_\varepsilon, T_\varepsilon) \in L^\infty(0, t_\varepsilon; H^2(\Omega)) \cap C([0, t_\varepsilon]; H^1(\Omega)) \cap L^2(0, t_\varepsilon; H^3(\Omega))$ and $(\partial_t v_\varepsilon, \partial_t T_\varepsilon) \in L^2(0, t_\varepsilon; H^1(\Omega))$, to the following system*

$$\partial_t v - \Delta v + (v \cdot \nabla_H)v - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\partial_z v$$
$$+ f_0 k \times v + \nabla_H\left(p_s(x, y, t) - \int_{-h}^{z} T(x, y, \xi, t)d\xi\right) = 0, \tag{2.1}$$

$$\nabla_H \cdot \bar{v} = 0, \tag{2.2}$$

$$\partial_t T - \Delta_H T - \varepsilon\partial_z^2 T + v \cdot \nabla_H T - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\left(\partial_z T + \frac{1}{h}\right) = 0, \tag{2.3}$$

subject to the boundary and initial conditions (1.25)–(1.27).

The following lemma will be used to obtain a uniform lower bound of the existence time, independent of ε, and the uniform in ε estimates on the local strong solution $(v_\varepsilon, T_\varepsilon)$ obtained in Proposition 2.1. It also plays an important role in proving the uniqueness of strong solutions.

Lemma 2.1. *(See [5].) The following inequalities hold true*

$$\int_M \left(\int_{-h}^{h} f(x, y, z)dz\right)\left(\int_{-h}^{h} g(x, y, z)h(x, y, z)dz\right)dxdy$$
$$\le C\|f\|_2^{1/2}\left(\|f\|_2^{1/2} + \|\nabla_H f\|_2^{1/2}\right)\|g\|_2\|h\|_2^{1/2}\left(\|h\|_2^{1/2} + \|\nabla_H h\|_2^{1/2}\right),$$

and

$$\int_M \left(\int_{-h}^{h} f(x,y,z)dz \right) \left(\int_{-h}^{h} g(x,y,z)h(x,y,z)dz \right) dxdy$$
$$\le C\|f\|_2 \|g\|_2^{1/2} \left(\|g\|_2^{1/2} + \|\nabla_H g\|_2^{1/2} \right) \|h\|_2^{1/2} \left(\|h\|_2^{1/2} + \|\nabla_H h\|_2^{1/2} \right),$$

for every f, g, h such that the right hand sides make sense and are finite.

We also need the following lemma on differentiation under the integral sign and integration by parts.

Lemma 2.2. *(See [4].) Let f and g be two spatial periodic functions such that*

$$f \in L^2(0, t_0; H^3(\Omega)), \qquad \partial_t f \in L^2(0, t_0; H^1(\Omega)),$$
$$g \in L^2(0, t_0; H^2(\Omega)), \qquad \partial_t g \in L^2(0, t_0; L^2(\Omega)).$$

Then it follows that

$$\frac{d}{dt} \int_\Omega |\Delta f|^2 dxdydz = -2 \int_\Omega \nabla \partial_t f \nabla \Delta f dxdydz,$$
$$\int_\Omega \nabla \partial_{x^i}^2 f \nabla \Delta f dxdydz = \int_\Omega |\partial_{x^i} \Delta f|^2 dxdydz$$

and

$$\frac{d}{dt} \int_\Omega |\partial_{x^i} g|^2 dxdydz = -2 \int_\Omega \partial_t g \partial_{x^i}^2 g dxdydz,$$
$$\int_\Omega \partial_{x^i}^2 g \partial_{x^j}^2 g dxdydz = \int_\Omega |\partial_{x^i} \partial_{x^j} g|^2 dxdydz,$$

for a.e. $t \in (0, t_0)$, where $x^i, x^j \in \{x, y, z\}$.

The next lemma is a version of the Aubin–Lions lemma.

Lemma 2.3 *(Aubin–Lions lemma). (See Simon [19], Corollary 4.) Assume that X, B and Y are three Banach spaces, with $X \hookrightarrow\hookrightarrow B \hookrightarrow Y$. Then it holds that*

(i) If F is a bounded subset of $L^p(0, T; X)$ where $1 \le p < \infty$, and $\frac{\partial F}{\partial t} = \{\frac{\partial f}{\partial t} | f \in F\}$ is bounded in $L^1(0, T; Y)$, then F is relatively compact in $L^p(0, T; B)$;

(ii) If F is bounded in $L^\infty(0, T; X)$ and $\frac{\partial F}{\partial t}$ is bounded in $L^r(0, T; Y)$ where $r > 1$, then F is relatively compact in $C([0, T]; B)$.

Now we provide a lower bound, in dependent of ε, for the existence time and establish the uniform, in ε, estimates for the solution $(v_\varepsilon, T_\varepsilon)$ obtained in Proposition 2.1. We have the following:

Proposition 2.2. *The local strong solution $(v_\varepsilon, T_\varepsilon)$ given by Proposition 2.1 can be established on the interval $(0, t_0)$, such that*

$$\sup_{0\le t\le t_0}\left(\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2\right)+\int_0^{t_0}\left(\|\nabla v_\varepsilon\|_{H^2}^2+\|\nabla_H T_\varepsilon\|_{H^2}^2+\varepsilon\|\partial_z T_\varepsilon\|_{H^2}^2\right)dt\le C$$

and

$$\int_0^{t_0}\left(\|\partial_t v_\varepsilon\|_{H^1}^2+\|\partial_t T_\varepsilon\|_{H^1}^2\right)dt\le C,$$

where t_0 and C are two positive constants independent of ε.

Proof. Suppose $(0, t_\varepsilon^*)$ is the maximal interval of existence of the local strong solution $(v_\varepsilon, T_\varepsilon)$. We are going to show that $t_\varepsilon^* > t_0$, for some positive number t_0 independent of ε.

We focus in our analysis on the interval $(0, t_\varepsilon^*)$. Multiplying (2.1) by v_ε and (2.3) by T_ε, respectively, and summing the resulting equations up, then it follows from integrating by parts and using (2.2) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|v_\varepsilon|^2+|T_\varepsilon|^2\right)dxdydz+\int_\Omega\left(|\nabla v_\varepsilon|^2+|\nabla_H T_\varepsilon|^2+\varepsilon|\partial_z T|^2\right)dxdydz\\&=\int_\Omega\left[\nabla_H\left(\int_{-h}^z T_\varepsilon d\xi\right)v_\varepsilon+\frac{1}{h}\left(\int_{-h}^z\nabla_H\cdot v_\varepsilon d\xi\right)T_\varepsilon\right]dxdydz.\end{aligned}$$

Applying the operator ∇ to Eqs. (2.1) and (2.3), multiplying the resulting equations by $-\nabla\Delta v_\varepsilon$ and $-\nabla\Delta T_\varepsilon$, respectively, summing these equalities up and integrating over Ω, it follows from integrating by parts and Lemma 2.2 that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|\Delta v_\varepsilon|^2+|\Delta T_\varepsilon|^2\right)dxdydz+\int_\Omega\left(|\nabla\Delta v_\varepsilon|^2+|\nabla_H\Delta T_\varepsilon|^2+\varepsilon|\partial_z\Delta T_\varepsilon|^2\right)dxdydz\\&=\int_\Omega\Bigg\{\nabla\left[(v_\varepsilon\cdot\nabla_H)v_\varepsilon-\left(\int_{-h}^z\nabla_H\cdot v_\varepsilon d\xi\right)\partial_z v_\varepsilon-\nabla_H\left(\int_{-h}^z T_\varepsilon d\xi\right)\right]:\nabla\Delta v_\varepsilon\\&\quad-\Delta\left[v_\varepsilon\cdot\nabla_H T_\varepsilon-\left(\int_{-h}^z\nabla_H\cdot v_\varepsilon d\xi\right)\left(\partial_z T_\varepsilon+\frac{1}{h}\right)\right]\Delta T_\varepsilon\Bigg\}dxdydz.\end{aligned}$$

Summing the above two equalities up yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\left(|v_\varepsilon|^2+|\Delta v_\varepsilon|^2+|T_\varepsilon|^2+|\Delta T_\varepsilon|^2\right)dxdydz\\&\quad+\int_\Omega\left(|\nabla v_\varepsilon|^2+|\nabla\Delta v_\varepsilon|^2+|\nabla_H T_\varepsilon|^2+|\nabla_H\Delta T_\varepsilon|^2+\varepsilon|\partial_z T_\varepsilon|^2+\varepsilon|\partial_z\Delta T_\varepsilon|^2\right)dxdydz\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Bigg\{ \nabla_H \Bigg(\int_{-h}^{z} T_\varepsilon d\xi \Bigg) v_\varepsilon + \frac{1}{h} \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) T_\varepsilon \\
&\quad + \nabla \Bigg[(v_\varepsilon \cdot \nabla_H) v_\varepsilon - \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) \partial_z v_\varepsilon - \nabla_H \Bigg(\int_{-h}^{z} T_\varepsilon d\xi \Bigg) \Bigg] : \nabla \Delta v_\varepsilon \\
&\quad - \Delta \Bigg[v_\varepsilon \cdot \nabla_H T_\varepsilon - \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) \Bigg(\partial_z T_\varepsilon + \frac{1}{h} \Bigg) \Bigg] \Delta T_\varepsilon \Bigg\} dxdydz \\
&= \int_\Omega \Bigg\{ \Bigg(\int_{-h}^{z} \nabla_H T_\varepsilon d\xi \Bigg) v_\varepsilon + \frac{1}{h} \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) T_\varepsilon + \Bigg[\nabla v_\varepsilon \cdot \nabla_H v_\varepsilon \\
&\quad + v_\varepsilon \cdot \nabla_H \nabla v_\varepsilon - \nabla \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) \partial_z v_\varepsilon - \Bigg(\int_{-h}^{z} \nabla_H \cdot v_\varepsilon d\xi \Bigg) \nabla \partial_z v_\varepsilon \\
&\quad - \nabla_H \nabla \Bigg(\int_{-h}^{z} T_\varepsilon d\xi \Bigg) \Bigg] : \nabla \Delta v_\varepsilon - \Bigg[\Delta v_\varepsilon \cdot \nabla_H T_\varepsilon + 2 \nabla v_\varepsilon \cdot \nabla_H \nabla T_\varepsilon \\
&\quad - \Bigg(\int_{-h}^{z} \nabla_H \cdot \Delta v_\varepsilon d\xi \Bigg) \Bigg(\partial_z T_\varepsilon + \frac{1}{h} \Bigg) - 2 \Bigg(\int_{-h}^{z} \nabla \nabla_H \cdot v_\varepsilon d\xi \Bigg) \nabla \partial_z T_\varepsilon \Bigg] \Delta T_\varepsilon \Bigg\} dxdydz.
\end{aligned}
$$

We estimate the integral on the right hand side of the above equality, denoted by I_1 as follows. By Lemma 2.1, and using the Hölder, Young, Sobolev and Poincaré inequalities, we have

$$
\begin{aligned}
I_1 &\le C \int_\Omega \Bigg\{ \Bigg(\int_{-h}^{h} |\nabla_H T_\varepsilon| d\xi \Bigg) |v_\varepsilon| + \Bigg(\int_{-h}^{h} |\nabla v_\varepsilon| d\xi \Bigg) |T_\varepsilon| + \Bigg[|\nabla v_\varepsilon|^2 + |v_\varepsilon| \big| \nabla^2 v_\varepsilon \big| \\
&\quad + \Bigg(\int_{-h}^{h} \big| \nabla^2 v_\varepsilon \big| d\xi \Bigg) |\partial_z v_\varepsilon| + \Bigg(\int_{-h}^{h} |\nabla v_\varepsilon| d\xi \Bigg) \big| \nabla^2 v_\varepsilon \big| + \Bigg(\int_{-h}^{h} \big| \nabla^2 T_\varepsilon \big| d\xi \Bigg) \Bigg] |\nabla \Delta v_\varepsilon| \\
&\quad + \Bigg[|\Delta v_\varepsilon| |\nabla_H T_\varepsilon| + |\nabla v_\varepsilon| |\nabla_H \nabla T_\varepsilon| + \Bigg(\int_{-h}^{h} |\nabla_H \Delta v_\varepsilon| d\xi \Bigg) \Bigg] |\Delta T_\varepsilon| \Bigg\} dxdydz \\
&\quad + C \int_\Omega \int_M \Bigg[\Bigg(\int_{-h}^{h} |\nabla \Delta v_\varepsilon| d\xi \Bigg) \Bigg(\int_{-h}^{h} |\partial_z T_\varepsilon| |\Delta T_\varepsilon| d\xi \Bigg) \\
&\quad + \Bigg(\int_{-h}^{h} \big| \nabla^2 v_\varepsilon \big| d\xi \Bigg) \Bigg(\int_{-h}^{h} \big| \nabla^2 T_\varepsilon \big|^2 d\xi \Bigg) \Bigg] dxdydz
\end{aligned}
$$

$$
\begin{aligned}
&\le C\big[\|\nabla_H T_\varepsilon\|_2\|v_\varepsilon\|_2+\|\nabla v_\varepsilon\|_2\|T_\varepsilon\|_2+\big(\|\nabla v_\varepsilon\|_4^2+\|v_\varepsilon\|_\infty\big\|\nabla^2 v_\varepsilon\big\|_2 \\
&\quad+\big\|\nabla^2 v_\varepsilon\big\|_3\|\nabla v_\varepsilon\|_6+\big\|\nabla^2 T_\varepsilon\big\|_2\big)\|\nabla\Delta v_\varepsilon\|_2+\big(\big\|\nabla^2 v_\varepsilon\big\|_3\big\|\nabla_H T_\varepsilon\big\|_6 \\
&\quad+\|\nabla v_\varepsilon\|_3\|\nabla_H\nabla T_\varepsilon\|_6+\|\nabla\Delta v_\varepsilon\|_2\big)\|\Delta T_\varepsilon\|_2\big]+C\|\nabla\Delta v_\varepsilon\|_2\|\partial_z T_\varepsilon\|_2^{1/2} \\
&\quad\times\big(\|\partial_z T_\varepsilon\|_2^{1/2}+\|\nabla_H\partial_z T_\varepsilon\|_2^{1/2}\big)\|\Delta T_\varepsilon\|_2^{1/2}\big(\|\Delta T_\varepsilon\|_2^{1/2}+\|\nabla_H\Delta T_\varepsilon\|_2^{1/2}\big) \\
&\quad+C\big\|\nabla^2 v_\varepsilon\big\|_2\big\|\nabla^2 T_\varepsilon\big\|_2\big(\big\|\nabla^2 T_\varepsilon\big\|_2+\big\|\nabla_H\nabla^2 T_\varepsilon\big\|_2\big) \\
&\le C\big[\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2+\big(\|v_\varepsilon\|_{H^2}^2+\|\Delta v_\varepsilon\|_2^{1/2}\|\nabla\Delta v_\varepsilon\|_2^{1/2}\|v_\varepsilon\|_{H^2}+\|T_\varepsilon\|_{H^2}^2\big)\|\nabla\Delta v_\varepsilon\|_2 \\
&\quad+\big(\|\Delta v_\varepsilon\|_2^{1/2}\|\nabla\Delta v_\varepsilon\|_2^{1/2}\|T_\varepsilon\|_{H^2}+\|v_\varepsilon\|_{H^2}\|\nabla_H\Delta T_\varepsilon\|_2+\|\nabla\Delta v_\varepsilon\|_2\big) \\
&\quad\times\|\Delta T_\varepsilon\|_2\big]+C\|\nabla\Delta v_\varepsilon\|_2\big(\|T_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^{3/2}\|\nabla_H\Delta T_\varepsilon\|_2^{1/2}\big) \\
&\quad+C\|v_\varepsilon\|_{H^2}\big(\|T_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}\|\nabla_H\Delta T_\varepsilon\|_2\big) \\
&\le\frac{1}{2}\big(\|\nabla\Delta v_\varepsilon\|_2^2+\|\nabla_H\Delta T_\varepsilon\|_2^2\big)+C\big(1+\|v_\varepsilon\|_{H^2}^6+\|T_\varepsilon\|_{H^2}^6\big).
\end{aligned}
$$

Therefore, we obtain, for any $t\in(0,t_\varepsilon^*)$,

$$
\begin{aligned}
&\sup_{0\le s\le t}\big(\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2\big)+\int_0^t\big(\|\nabla v_\varepsilon\|_{H^2}^2+\|\nabla_H T_\varepsilon\|_{H^2}^2+\varepsilon\|\partial_z T_\varepsilon\|_{H^2}^2\big)ds \\
&\quad\le CC_0+C\int_0^t\big(1+\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2\big)^3 ds,
\end{aligned}
$$

where $C_0=\|v_0\|_{H^2}^2+\|T_0\|_{H^2}^2+1$.

Setting

$$
f(t)=\sup_{0\le s\le t}\big(\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2+1\big)+\int_0^t\big(\|\nabla v_\varepsilon\|_{H^2}^2+\|\nabla_H T_\varepsilon\|_{H^2}^2+\varepsilon\|\partial_z T_\varepsilon\|_{H^2}^2\big)ds
$$

for $t\in[0,t_\varepsilon^*)$. Then one has

$$
f(t)\le CC_0+C\int_0^t\big(f(s)\big)^3 ds,\quad t\in\big[0,t_\varepsilon^*\big).
$$

Set $F(t)=\int_0^t(f(t)^3)ds+1$, then we have

$$
F'(t)=\big(f(t)\big)^3\le C_1\big(F(t)\big)^3,\quad\forall t\in\big[0,t_\varepsilon^*\big),
$$

where C_1 is a positive constant depending only on h and (v_0,T_0). This inequality implies

$$
F(t)\le\frac{1}{\sqrt{1-2C_1t}},\quad\forall t\in\big[0,t_\varepsilon^*\big)\cap\left[0,\frac{1}{2C_1}\right),
$$

and thus

$$\sup_{0\le s\le t}\big(\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2\big)+\int_0^t\big(\|\nabla v_\varepsilon\|_{H^2}^2+\|\nabla_H T_\varepsilon\|_{H^2}^2+\varepsilon\|\partial_z T_\varepsilon\|_{H^2}^2\big)ds$$
$$\le CC_0+CF(t)\le CC_0+\frac{C}{\sqrt{1-2C_1t}}\le C(C_0+\sqrt{2}),$$

for any $t\in[0,t_\varepsilon^*)\cap[0,\frac{1}{4C_1}]$. Recalling that t_ε^* is the maximal existence time, the above inequality implies that $t_\varepsilon^*>\frac{1}{4C_1}$. Thus we can take $t_0=\frac{1}{4C_1}$.

Thanks to the estimates we have just proved, one can use the same argument as in the last paragraph of the proof of Proposition 3.1 in [4] to obtain the estimates on $\partial_t v_\varepsilon$ and $\partial_t T_\varepsilon$, and thus we omit the details here. This completes the proof. □

Now we can prove the local well-posedness of strong solutions to system (1.15)–(1.21), or equivalently system (1.22)–(1.27).

Proposition 2.3. *Let $v_0\in H^2(\Omega)$ and $T_0\in H^2(\Omega)$ be two periodic functions, such that they are even and odd in z, respectively. Then system (1.22)–(1.27) has a unique strong solution (v,T) in $\Omega\times(0,t_0)$, where t_0 is the same positive time stated in Proposition 2.2. Moreover, the strong solution depends continuously on the initial data.*

Proof. By Proposition 2.1 and Proposition 2.2, for any given $\varepsilon>0$, system (2.1)–(2.3), subject to the boundary and initial conditions (1.25)–(1.27), has a unique strong solution $(v_\varepsilon,T_\varepsilon)$ in $\Omega\times(0,t_0)$ such that

$$\sup_{0\le t\le t_0}\big(\|v_\varepsilon\|_{H^2}^2+\|T_\varepsilon\|_{H^2}^2\big)+\int_0^{t_0}\big(\|\nabla v_\varepsilon\|_{H^2}^2+\|\nabla_H T_\varepsilon\|_{H^2}^2+\varepsilon\|\partial_z T_\varepsilon\|_{H^2}^2\big)dt\le C$$

and

$$\int_0^{t_0}\big(\|\partial_t v_\varepsilon\|_{H^1}^2+\|\partial_t T_\varepsilon\|_{H^1}^2\big)dt\le C,$$

where C is a constant independent of ε. On account of these estimates and applying Lemma 2.3, there is a subsequence, still denoted by $\{(v_\varepsilon,T_\varepsilon)\}$, and (v,T), such that

$$(v_\varepsilon,T_\varepsilon)\to(v,T),\quad \text{in } C\big([0,t_0];H^1(\Omega)\big),$$
$$(\nabla v_\varepsilon,\nabla_H T_\varepsilon)\to(\nabla v,\nabla_H T),\quad \text{in } L^2\big(0,t_0;H^1(\Omega)\big),$$
$$(v_\varepsilon,T_\varepsilon)\overset{*}{\rightharpoonup}(v,T),\quad \text{in } L^\infty\big(0,t_0;H^2(\Omega)\big),$$
$$(\nabla v_\varepsilon,\nabla_H T_\varepsilon)\rightharpoonup(\nabla v,\nabla_H T),\quad \text{in } L^2\big(0,t_0;H^2(\Omega)\big),$$
$$(\partial_t v_\varepsilon,\partial_t T_\varepsilon)\rightharpoonup(\partial_t v,\partial_t T),\quad \text{in } L^2\big(\Omega\times(0,t_0)\big),$$

where $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ are the weak and weak-$*$ convergence, respectively. Thanks to these convergence, one can easily show that (v, T) is a strong solution to system (1.22)–(1.27), or equivalently to system (1.15)–(1.21). The continuous dependence on the initial data, in particular the uniqueness, are straightforward corollary of Proposition 2.4 below, see Corollary 2.1. $\square$

For the continuous dependence on the initial data, the solutions are not required to have so high regularities as stated in Definition 1.1. In fact, we have the following:

Proposition 2.4. *Let (v_1, T_1) and (v_2, T_2) be two spatially periodic functions, satisfying the following regularity properties*

$$(v_i, T_i) \in L^\infty\big(0, t_0; H^1(\Omega)\big) \cap C\big([0, t_0]; L^2(\Omega)\big),$$
$$\big(\partial_t v_i, \partial_t T_i, \delta\partial_z^2 v_i\big) \in L^2\big(\Omega \times (0, t_0)\big), \qquad (\nabla_H v_i, \nabla_H T_i) \in L^2\big(0, t_0; H^1(\Omega)\big),$$

$i = 1, 2$, where $\delta \geq 0$ is a given constant. Set

$$\begin{aligned}\phi(t) = 1 &+ \|v_2(t)\|_2^4 + \|\partial_z v_2(t)\|_2^4 + \|v_2(t)\|_2^2\|\nabla_H v_2(t)\|_2^2 \\ &+ \|\partial_z v_2(t)\|_2^2\|\nabla_H \partial_z v_2(t)\|_2^2 + \|T_2(t)\|_2^4 + \|\partial_z T_2(t)\|_2^4 \\ &+ \|T_2(t)\|_2^2\|\nabla_H T_2(t)\|_2^2 + \|\partial_z T_2(t)\|_2^2\|\nabla_H \partial_z T_2(t)\|_2^2,\end{aligned} \tag{2.4}$$

for any $t \in (0, t_0)$. Suppose that both (v_1, T_1) and (v_2, T_2) satisfy the following system

$$\begin{aligned}&\partial_t v - \Delta_H v - \delta\partial_z^2 v + (v \cdot \nabla_H)v - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\partial_z v \\ &\quad + f_0 k \times v + \nabla_H\left(p_s(x, y, t) - \int_{-h}^{z} T(x, y, \xi, t)d\xi\right) = 0,\end{aligned} \tag{2.5}$$

$$\nabla_H \cdot \bar{v} = 0, \tag{2.6}$$

$$\partial_t T - \Delta_H T + v \cdot \nabla_H T - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)d\xi\right)\left(\partial_z T + \frac{1}{h}\right) = 0, \tag{2.7}$$

in $\Omega \times (0, t_0)$.

Setting $(v, T) = (v_1 - v_2, T_1 - T_2)$, then it follows that

$$\begin{aligned}&\sup_{0\leq s\leq t}\big(\|v\|_2^2 + \|T\|_2^2\big) + \int_0^t \big(\|\nabla v\|_2^2 + \|\nabla_H T\|_2^2 + \delta\|\partial_z v\|_2^2\big)ds \\ &\quad \leq Ce^{C\int_0^t \phi(s)ds}\big(\|(v_1)_0 - (v_2)_0\|_2^2 + \|(T_1)_0 - (T_2)_0\|_2^2\big),\end{aligned}$$

for any $t \in (0, t_0)$, where $((v_i)_0, (T_i)_0)$, $i = 1, 2$, are the initial values of (v_i, T_i).

Proof. One can easily check that (v, T) satisfies

$$\partial_t v - \Delta_H v - \delta\partial_z^2 v + (v_1 \cdot \nabla_H)v + (v \cdot \nabla_H)v_2 - \left(\int_{-h}^{z} \nabla_H \cdot v_1 d\xi\right)\partial_z v - \left(\int_{-h}^{z} \nabla_H \cdot v d\xi\right)\partial_z v_2 + f_0 k \times v + \nabla_H p_s(x, y, t) - \nabla_H\left(\int_{-h}^{z} T(x, y, \xi, t)d\xi\right) = 0, \tag{2.8}$$

$$\nabla_H \cdot \bar{v} = 0, \tag{2.9}$$

$$\partial_t T - \Delta_H T + v_1 \cdot \nabla_H T + v \cdot \nabla_H T_2 - \left(\int_{-h}^{z} \nabla_H \cdot v_1 d\xi\right)\partial_z T - \left(\int_{-h}^{z} \nabla_H \cdot v d\xi\right)\left(\partial_z T_2 + \frac{1}{h}\right) = 0. \tag{2.10}$$

Multiplying (2.8) by v and integrating over Ω, then it follows from integrating by parts and (2.9) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 dxdydz + \int_\Omega \left(|\nabla_H v|^2 + \delta|\partial_z v\|_2^2\right)dxdydz = \int_\Omega \left\{\left[\left(\int_{-h}^{z} \nabla_H \cdot v d\xi\right)\partial_z v_2 - (v \cdot \nabla_H)v_2\right] \cdot v - \left(\int_{-h}^{z} T d\xi\right)\nabla_H \cdot v\right\}dxdydz. \tag{2.11}$$

By Lemma 2.1, and using Young's inequality, we have the following estimates

$$\begin{aligned}
&\left|\int_\Omega \left(\int_{-h}^{z} \nabla_H \cdot v d\xi\right)\partial_z v_2 \cdot v dxdydz\right| \\
&\le \int_M \left(\int_{-h}^{h} |\nabla_H v|dz\right)\left(\int_{-h}^{h} |\partial_z v_2||v|dz\right)dxdy \\
&\le C\|\nabla_H v\|_2 \|\partial_z v_2\|_2^{1/2}\left(\|\partial_z v_2\|_2^{1/2} + \|\nabla_H \partial_z v_2\|_2^{1/2}\right)\|v\|_2^{1/2}\left(\|v\|_2^{1/2} + \|\nabla_H v\|_2^{1/2}\right) \\
&\le \frac{1}{8}\|\nabla_H v\|_2^2 + C\left(1 + \|\partial_z v_2\|_2^4 + \|\partial_z v_2\|_2^2\|\partial_z \nabla_H v_2\|_2^2\right)\|v\|_2^2 \\
&\le \frac{1}{8}\|\nabla_H v\|_2^2 + C\phi(t)\|v\|_2^2,
\end{aligned}$$

for any $t \in (0, t_0)$. Noticing that $|v_2(z)| \le \frac{1}{2h}\int_{-h}^h |v_2(z)|dz + \int_{-h}^h |\partial_z v_2|dz$, it follows from integrating by parts, applying Lemma 2.1, and using Young's inequality that

$$\begin{aligned}
&\left|\int_\Omega (v\cdot\nabla_H)v_2\cdot v\,dxdydz \le \int_\Omega |\nabla_H v||v||v_2|dxdydz\right| \\
&\le C\int_M \left(\int_{-h}^h \big(|v_2|+|\partial_z v_2|\big)dz\right)\left(\int_{-h}^h |\nabla_H v||v|dz\right)dxdy \\
&\le C\|\partial_z v_2\|_2^{1/2}\big(\|\partial_z v_2\|_2^{1/2}+\|\nabla_H\partial_z v_2\|_2^{1/2}\big)\|\nabla_H v\|_2\|v\|_2^{1/2}\big(\|v\|_2^{1/2}+\|\nabla_H v\|_2^{1/2}\big) \\
&\quad + C\|v_2\|_2^{1/2}\big(\|v_2\|_2^{1/2}+\|\nabla_H v_2\|_2^{1/2}\big)\|\nabla_H v\|_2\|v\|_2^{1/2}\big(\|v\|_2^{1/2}+\|\nabla_H v\|_2^{1/2}\big) \\
&\le \frac18\|\nabla_H v\|_2^2 + C\big(1+\|v_2\|_2^4+\|v_2\|_2^2\|\nabla_H v_2\|_2^2+\|\partial_z v_2\|_2^4+\|\partial_z v_2\|_2^2\|\partial_z\nabla_H v_2\|_2^2\big)\|v\|_2^2 \\
&\le \frac18\|\nabla_H v\|_2^2 + C\phi(t)\|v\|_2^2, \qquad (2.12)
\end{aligned}$$

for any $t \in (0, t_0)$. The above two inequalities, substituted into (2.11), imply

$$\frac{d}{dt}\big\|v(t)\big\|_2^2 + \frac32\big\|\nabla_H v(t)\big\|^2 + 2\delta\big\|\partial_z v(t)\big\|_2^2 \le C\phi(t)\big(\big\|v(t)\big\|_2^2 + \big\|T(t)\big\|_2^2\big), \qquad (2.13)$$

for any $t \in (0, t_0)$.

Multiplying (2.10) by T and integrating by parts yield

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_\Omega |T|^2 dxdydz + \int_\Omega |\nabla_H T|^2 dxdydz \\
&\quad = -\int_\Omega \left[v\cdot\nabla_H T_2 - \left(\int_{-h}^z \nabla_H\cdot v\,d\xi\right)\left(\partial_z T_2 + \frac1h\right)\right] T\,dxdydz. \qquad (2.14)
\end{aligned}$$

Using the fact that $|T_2(z)| \le \frac{1}{2h}\int_{-h}^h |T_2(z)|dz + \int_{-h}^h |\partial_z T_2|dz$, the same argument as that for (2.12) gives

$$\begin{aligned}
&\left|\int_\Omega v\cdot\nabla_H T_2 T\,dxdydz\right| \\
&\quad = \left|\int_\Omega T_2(\nabla_H\cdot vT + v\cdot\nabla_H T)dxdydz\right| \\
&\quad \le \left|\int_\Omega |\nabla_H v||T||T_2|dxdydz\right| + \left|\int_\Omega |\nabla_H T||v||T_2|dxdydz\right| \\
&\quad \le C\left|\int_M \left(\int_{-h}^h (|T_2|+|\partial_z T_2|)dz\right)\left(\int_{-h}^h |\nabla_H v||T|dz\right)dxdy\right|
\end{aligned}$$

$$
\begin{aligned}
&+C\left|\int_M\left(\int_{-h}^{h}\big(|T_2|+|\partial_z T_2|\big)dz\right)\left(\int_{-h}^{h}|v||\nabla_H T|dz\right)dxdy\right|\\
&\le\frac14\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2\big)+C\big(1+\|T_2\|_2^4+\|T_2\|_2^2\|\nabla_H T_2\|_2^2+\|\partial_z T_2\|_2^4\\
&\quad+\|\partial_z T_2\|_2^2\|\partial_z\nabla_H T_2\|_2^2\big)\big(\|v\|_2^2+\|T\|_2^2\big)\\
&\le\frac14\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2\big)+C\phi(t)\big(\|v\|_2^2+\|T\|_2^2\big),
\end{aligned}
$$

for any $t\in(0,t_0)$. Applying Lemma 2.1 again, it follows from the Young inequality that

$$
\begin{aligned}
&\left|\int_\Omega\left(\int_{-h}^{z}\nabla_H\cdot v\,d\xi\right)\left(\partial_z T_2+\frac1h\right)T\,dxdydz\right|\\
&\le C\int_M\left(\int_{-h}^{h}|\nabla_H v|dz\right)\left(\int_{-h}^{h}\big(|\partial_z T_2|+1\big)|T|dz\right)dxdy\\
&\le C\|\nabla_H v\|_2\|\partial_z T_2\|_2^{1/2}\big(\|\partial_z T_2\|_2^{1/2}+\|\nabla_H\partial_z T_2\|_2^{1/2}\big)\|T\|_2^{1/2}\\
&\quad\times\big(\|T\|_2^{1/2}+\|\nabla_H T\|_2^{1/2}\big)+C\|\nabla_H v\|_2\|T\|_2\\
&\le\frac14\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2\big)+C\big(1+\|\partial_z T_2\|_2^4+\|\partial_z T_2\|_2^2\|\nabla_H\partial_z T_2\|_2^2\big)\|T\|_2^2\\
&\le\frac14\big(\|\nabla_H v\|_2^2+\|\nabla_H T\|_2^2\big)+C\phi(t)\|T\|_2^2,
\end{aligned}
$$

for any $t\in(0,t_0)$. Substituting the above two estimates into (2.14), one has

$$
\frac{d}{dt}\|T(t)\|_2^2+\|\nabla_H T(t)\|^2\le\|\nabla_H v(t)\|_2^2+C\phi(t)\big(\|v(t)\|_2^2+\|T(t)\|_2^2\big),
$$

for any $t\in(0,t_0)$.

Summing the above inequality up with (2.13) leads to

$$
\begin{aligned}
&\frac{d}{dt}\big(\|v(t)\|_2^2+\|T(t)\|_2^2\big)+\frac12\big(\|\nabla_H v(t)\|_2^2+\|\nabla_H T(t)\|_2^2+\delta\|\partial_z v(t)\|_2^2\big)\\
&\quad\le C\phi(t)\big(\|v(t)\|_2^2+\|T(t)\|_2^2\big),
\end{aligned}
$$

for any $t\in(0,t_0)$, which, by Gronwall's inequality, implies the conclusion. □

As a corollary of Proposition 2.4, we have the following corollary, which guarantees the uniqueness and continuous dependence on initial data of strong solutions to system (1.22)–(1.27), or equivalently to system (1.15)–(1.21).

Corollary 2.1. *Strong solution to system (1.22)–(1.27) depends continuously on initial data, and in particular is unique.*

Proof. Let (v_1, T_1) and (v_2, T_2) be two strong solutions to system (1.22)–(1.27) on $\Omega \times (0, t_0)$. Recalling the regularity properties of strong solution (v_2, T_2), the function $\phi(t)$ defined by (2.4) is integrable on time interval $(0, t_0)$. By virtue of this fact, one can apply Proposition 2.4 to obtain the continuous dependence on initial data, and in particular the uniqueness. This completes the proof. □

Remark 2.1. By Proposition 2.4, neither the vertical viscosity, i.e. one can take $\delta = 0$, nor the vertical diffusion, which is zero in our case, are necessary to guarantee the uniqueness, and continuous dependence on initial data, of the solutions that enjoy the regularity properties stated in the proposition.

3. Global existence of strong solutions

In this section, we show that the local strong solution established for short time in Section 2 can be in fact extended to be a global one.

Let (v, T) be the unique strong solution obtained in Proposition 2.3. Suppose that $(0, \mathcal{T}^*)$ is the maximal interval of existence. If $\mathcal{T}^* = \infty$ there is nothing to prove. Therefore, for the next analysis, we assume by contradiction that $\mathcal{T}^* < \infty$, and we will focus our analysis on $(0, \mathcal{T}^*)$. We have the following three propositions which will provide the needed a priori estimates on (v, T).

Proposition 3.1. *There is a bounded continuously increasing function $K_1(t)$, on $[0, \mathcal{T}^*)$, such that*

$$\sup_{0\le s\le t}\left(\|v\|_2^2 + \|T\|_\infty^2 + \|v\|_6^2 + \|\nabla_H \bar{v}\|_{L^2(M)}^2 + \|\partial_z v\|_6^2\right) + \int_0^t \left(\|\nabla v\|_2^2 + \|\nabla_H T\|_2^2 + \|\Delta_H \bar{v}\|_{L^2(M)}\right) ds \le K_1(t),$$

for any $t \in [0, \mathcal{T}^*)$.

Proof. The conclusion follows directly from inequalities (59), (69), (91) and (103) in [7], with slight modifications. Thus we omit the proof here. □

Set $u = \partial_z v$, then it satisfies

$$\partial_t u - \Delta u + (v \cdot \nabla_H)u - \left(\int_{-h}^{z} \nabla_H \cdot v(x, y, \xi, t)\, d\xi\right)\partial_z u + (u \cdot \nabla_H)v - (\nabla_H \cdot v)u + f_0 k \times u - \nabla_H T = 0, \tag{3.1}$$

on $(0, \mathcal{T}^*)$.

Proposition 3.2. *There is a bounded continuously increasing function $K_2(t)$, on $[0, \mathcal{T}^*)$, such that*

$$\sup_{0\le s\le t} \|\nabla u\|_2^2 + \int_0^t \left\|\nabla^2 u\right\|_2^2 ds \le K_2(t),$$

for any $t \in [0, \mathcal{T}^*)$.

Proof. By the boundary conditions (1.25) and (1.26), $u = \partial_z v$ is $2h$ periodic and odd in the vertical variable z, and thus $u(x, y, -h, t) = -u(x, y, h, t) = -u(x, y, -h, t)$, which implies $u|_{z=-h} = 0$ and $\nabla_H u|_{z=-h} = 0$, for any $t \in (0, \mathcal{T}^*)$. Multiplying Eq. (3.1) by $-\partial_z^2 u$ and integration by parts, using Lemma 2.2 and the fact that $|\nabla_H u(x, y, z, t)| \le \int_{-h}^h |\nabla_H \partial_z u(x, y, \xi, t)| d\xi$, then it follows from the Hölder, Sobolev and Poincaré inequalities that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_z u|^2 dxdydz + \int_\Omega |\partial_z \nabla u|^2 dxdydz \\
&= \int_\Omega \left[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H \cdot v d\xi\right)\partial_z u\right]\cdot \partial_z^2 u dxdydz \\
&\quad + \int_\Omega \left[(u\cdot\nabla_H)v - (\nabla_H\cdot v)u + f_0 k\times u - \nabla_H T\right]\cdot\partial_z^2 u dxdydz \\
&= -\int_\Omega \big\{\big[2(u\cdot\nabla_H)u - 2(\nabla_H\cdot v)\partial_z u + (\partial_z u\cdot\nabla_H)v \\
&\quad - (\nabla_H\cdot u)u\big]\cdot\partial_z u + \nabla_H T\cdot\partial_z^2 u\big\} dxdydz \\
&= -\int_\Omega \big\{\big[2(u\cdot\nabla_H)u - (\nabla_H\cdot u)u\big]\cdot\partial_z u + 2v\cdot\nabla_H\big(|\partial_z u|^2\big) \\
&\quad - \nabla_H\cdot\partial_z u v\cdot\partial_z u - (\partial_z u\cdot\nabla_H)\partial_z u\cdot v + \nabla_H T\cdot\partial_z^2 u\big\} dxdydz \\
&\le C\int_\Omega \left[|u|\left(\int_{-h}^h |\nabla_H \partial_z u| d\xi\right)|\partial_z u| + |v||\nabla_H\partial_z u||\partial_z u| + |\nabla_H T|\left|\partial_z^2 u\right|\right] dxdydz \\
&\le C\big[\big(\|u\|_6 + \|v\|_6\big)\|\nabla_H\partial_z u\|_2\|\partial_z u\|_3 + \|\nabla_H T\|_2\left\|\partial_z^2 u\right\|_2\big] \\
&\le C\big[\big(\|u\|_6 + \|v\|_6\big)\|\nabla_H\partial_z u\|_2\|\partial_z u\|_2^{1/2}\|\nabla\partial_z u\|_2^{1/2} + \|\nabla_H T\|_2\left\|\partial_z^2 u\right\|_2\big] \\
&\le \frac{1}{2}\|\nabla\partial_z u\|_2^2 + C\big(\|u\|_6^4 + \|v\|_6^4\big)\|\partial_z u\|_2^2 + C\|\nabla_H T\|_2^2.
\end{aligned}$$

Thanks to Proposition 3.1, this inequality gives

$$\sup_{0\le s\le t}\|\partial_z u\|_2^2 + \int_0^t \|\nabla\partial_z u\|_2^2 ds$$

$$\le e^{C\int_0^t(\|u\|_6^4+\|v\|_6^4)ds}\left(\|\partial_z u_0\|_2^2 + C\int_0^t \|\nabla_H T\|_2^2 ds\right)$$
$$\le Ce^{K_1^2(t)t}\big(\|v_0\|_{H^2}^2 + K_1(t)\big) =: K_2'(t), \tag{3.2}$$

for every $t\in[0,\mathcal{T}^*)$.

Multiplying (3.1) by $-\Delta_H u$ and integrating by parts, using Lemma 2.2 and the fact that $|\nabla_H v(x,y,z,t)| \le |\nabla_H \bar v(x,y,t)| + \int_{-h}^h |\nabla_H u(x,y,\xi,t)|d\xi$, then it follows from the Hölder, Sobolev, Poincaré inequality and Lemma 2.1 that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla_H u|^2 dxdydz + \int_\Omega |\nabla_H\nabla u|^2 dxdydz\\
&= \int_\Omega \Bigg[(v\cdot\nabla_H)u - \left(\int_{-h}^z \nabla_H\cdot v d\xi\right)\partial_z u + (u\cdot\nabla_H)v\\
&\quad - (\nabla_H\cdot v)u + f_0 k\times u - \nabla_H T\Bigg]\cdot\Delta_H u dxdydz\\
&\le C\int_\Omega\Bigg[|v||\nabla_H u| + \left(\int_{-h}^h |\nabla_H u|dz + |\nabla_H\bar v|\right)\big(|\partial_z u| + |u|\big)\\
&\quad + |\nabla_H T|\Bigg]|\Delta_H u|dxdydz\\
&\le C\big[\|v\|_6\|\nabla_H u\|_3 + \|u\|_6\big(\|\nabla_H u\|_3 + \|\nabla_H\bar v\|_3\big) + \|\nabla_H T\|_2\big]\|\Delta_H u\|_2\\
&\quad + C\int_M\left[\left(\int_{-h}^h |\nabla_H u|d\xi\right) + |\nabla_H\bar v|\right]\left(\int_{-h}^h |\partial_z u||\Delta_H u|d\xi\right)dxdy\\
&\le C\big[\big(\|u\|_6 + \|v\|_6\big)\|\nabla_H u\|_3 + \|u\|_6\|\nabla_H\bar v\|_3 + \|\nabla_H T\|_2\big]\|\Delta_H u\|_2\\
&\quad + C\big[\|\nabla_H u\|_2^{1/2}\big(\|\nabla_H u\|_2^{1/2} + \|\nabla_H^2 u\|_2^{1/2}\big) + \|\nabla_H\bar v\|_2^{1/2}\big(\|\nabla_H\bar v\|_2^{1/2} + \|\nabla_H^2\bar v\|_2^{1/2}\big)\big]\\
&\quad \times\|\partial_z u\|_2^{1/2}\big(\|\partial_z u\|_2^{1/2} + \|\nabla_H\partial_z u\|_2^{1/2}\big)\|\Delta_H u\|_2\\
&\le C\big[\big(\|u\|_6 + \|v\|_6\big)\|\nabla_H u\|_2^{1/2}\|\nabla\nabla_H u\|_2^{1/2} + \|u\|_6\|\nabla_H\bar v\|_{L^2(M)}^{2/3}\|\Delta_H\bar v\|_{L^2(M)}^{1/3}\\
&\quad + \|\nabla_H T\|_2\big]\|\Delta_H u\|_2 + C\big(\|\nabla_H\bar v\|_{L^2(M)}^{1/2}\|\Delta_H\bar v\|_{L^2(M)}^{1/2} + \|\nabla_H u\|_2^{1/2}\|\nabla\nabla_H u\|_2^{1/2}\big)\\
&\quad \times\|\partial_z u\|_2^{1/2}\|\nabla\partial_z u\|_2^{1/2}\|\Delta_H u\|_2\\
&\le \frac{1}{2}\|\nabla_H\nabla u\|_2^2 + C\big(\|u\|_6^4 + \|v\|_6^4\big)\|\nabla_H u\|_2^2 + C\big(\|u\|_6^2\|\Delta_H\bar v\|_{L^2(M)}^2 + \|\nabla_H T\|_{L^2(M)}^2\big)\\
&\quad + C\|\partial_z u\|_2^2\|\nabla\partial_z u\|_2^2\|\nabla_H u\|_2^2 + C\|\nabla_H\bar v\|_{L^2(M)}\|\Delta_H\bar v\|_{L^2(M)}\|\partial_z u\|_2\|\nabla\partial_z u\|_2
\end{aligned}$$

$$\leq \frac{1}{2}\|\nabla_H \nabla u\|_2^2 + C\big(\|u\|_6^4 + \|v\|_6^4 + \|\partial_z u\|_2^2\|\nabla\partial_z u\|_2^2\big)\|\nabla_H u\|_2^2$$
$$+ C\big(\|u\|_6^2\|\Delta_H \bar{v}\|_{L^2(M)}^2 + \|\nabla_H T\|_2^2 + \|\nabla_H \bar{v}\|_{L^2(M)}^2\|\Delta_H \bar{v}\|_{L^2(M)}^2 + \|\partial_z u\|_2^2\|\nabla\partial_z u\|_2^2\big).$$

Therefore, by Proposition 3.1, and using (3.2), it follows from the above inequality that for any $t \in [0, \mathcal{T}^*)$

$$\sup_{0\leq s\leq t}\|\nabla_H u\|_2^2 + \int_0^t \|\nabla\nabla_H u\|_2^2 ds$$
$$\leq e^{C\int_0^t(\|u\|_6^4+\|v\|_6^4+\|\partial_z u\|_2^2\|\nabla\partial_z u\|_2^2)ds}\Bigg[\|\nabla_H u_0\|_2^2 + C\int_0^t\big(\|u\|_6^2\|\Delta_H\bar{v}\|_{L^2(M)}^2$$
$$+ \|\nabla_H T\|_2^2 + \|\nabla_H\bar{v}\|_{L^2(M)}^2\|\Delta_H\bar{v}\|_{L^2(M)}^2 + \|\partial_z u\|_2^2\|\nabla\partial_z u\|_2^2\big)ds\Bigg]$$
$$\leq Ce^{C(K_1^2(t)t + K_2'^2(t))}\big(\|v_0\|_{H^2}^2 + K_1^2(t) + K_1(t) + K_2'^2(t)\big) =: K_2''(t).$$

Combining this inequality with (3.2), we have

$$\sup_{0\leq s\leq t}\|\nabla u\|_2^2 + \int_0^t \|\nabla^2 u\|_2^2 ds \leq K_2'(t) + K_2''(t) =: K_2(t),$$

for any $t \in [0, \mathcal{T}^*)$, completing the proof. □

Proposition 3.3. *There is a bounded continuously increasing function $K_3(t)$, on $[0, \mathcal{T}^*)$, such that*

$$\sup_{0\leq s\leq t}\big(\|\Delta_H v\|_2^2 + \|\Delta T\|_2^2\big) + \int_0^t\big(\|\nabla\Delta_H v\|_2^2 + \|\nabla_H\Delta T\|_2^2\big) \leq K_3(t),$$

for any $t \in [0, \mathcal{T}^*)$.

Proof. It follows from integrating by parts, the Sobolev embedding inequality and the Poincaré inequality that

$$\int_\Omega |\nabla_H v|^4 dxdydz = -\int_\Omega \nabla_H\cdot\big(|\nabla_H v|^2\nabla_H v\big)v\,dxdydz$$
$$\leq C\int_\Omega |\nabla_H v|^2|v|\big|\nabla_H^2 v\big|dxdydz \leq C\|\nabla_H v\|_4^2\|v\|_6\big\|\nabla_H^2 v\big\|_3$$

and

$$\begin{aligned}\int_\Omega \left|\nabla_H^2 v\right|^3 dxdydz &= -\int_\Omega \nabla_H \cdot \left(\left|\nabla_H^2 v\right|\nabla_H^2 v\right)\nabla_H v dxdydz \\ &\le C\int_\Omega \left|\nabla_H^2 v\right|\left|\nabla_H^3 v\right||\nabla_H v| dxdydz \le C\left\|\nabla_H^2 v\right\|_3 \left\|\nabla_H^3 v\right\|_2 \|\nabla_H v\|_6 \\ &\le C\left\|\nabla_H^2 v\right\|_3 \left\|\nabla_H^3 v\right\|_2 \|\nabla_H \nabla v\|_2 = C\left\|\nabla_H^2 v\right\|_3 \|\nabla_H \Delta_H v\|_2 \|\nabla_H \nabla v\|_2.\end{aligned}$$

The above two inequalities imply

$$\|\nabla_H v\|_4^2 \le C\|v\|_6 \left\|\nabla_H^2 v\right\|_3, \qquad \|\nabla_H v\|_3 \le C\|\nabla_H \Delta_H v\|_2^{1/2} \|\nabla_H \nabla v\|_2^{1/2},$$

and thus

$$\|\nabla_H v\|_4^2 \le C\|v\|_6 \|\nabla_H \nabla v\|_2^{1/2} \|\nabla_H \Delta_H v\|_2^{1/2}. \tag{3.3}$$

Applying the operator ∇_H to Eq. (1.22), multiplying the resulting equation by $-\nabla_H \Delta_H v$ and integrating over Ω, then it follows from Lemma 2.2, (1.23), (3.3), the Hölder, Sobolev and Poincaré inequalities that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta_H v|^2 dxdydz + \int_\Omega |\nabla \Delta_H v|^2 dxdydz \\ &= \int_\Omega \nabla_H\left[(v\cdot\nabla_H)v - \left(\int_{-h}^z \nabla_H \cdot v d\xi\right)\partial_z v - \nabla_H\left(\int_{-h}^z T d\xi\right)\right] : \nabla_H \Delta_H v dxdydz \\ &\le C\int_\Omega \left[|v|\left|\nabla_H^2 v\right| + |\nabla_H v|^2 + \left(\int_{-h}^h \left|\nabla_H^2 v\right| d\xi\right)|\partial_z v| + \left(\int_{-h}^h |\nabla_H \cdot v| d\xi\right)|\partial_z \nabla_H v| \right. \\ &\quad \left. + \left(\int_{-h}^h \left|\nabla_H^2 T\right| d\xi\right)\right]|\nabla_H \Delta_H v| dxdydz \\ &\le C\left(\|v\|_6 \left\|\nabla_H^2 v\right\|_3 + \|\nabla_H v\|_4^2 + \left\|\nabla_H^2 v\right\|_3 \|\partial_z v\|_6 + \|\nabla_H v\|_3 \|\nabla_H u\|_6 \right. \\ &\quad \left. + \left\|\nabla_H^2 T\right\|_2\right)\|\nabla_H \Delta_H v\|_2 \\ &\le C\left[\left(\|u\|_6 + \|v\|_6\right)\left\|\nabla_H^2 v\right\|_2^{1/2}\left(\left\|\nabla_H^2 v\right\|_2^{1/2} + \left\|\nabla\nabla_H^2 v\right\|_2^{1/2}\right) \right. \\ &\quad \left. + \|v\|_6 \|\nabla_H \nabla v\|_2^{1/2} \|\nabla_H \Delta_H v\|_2^{1/2} + \|\nabla\nabla_H v\|_2 \|\nabla\nabla_H u\|_2 + \|\Delta_H T\|_2\right]\|\nabla_H \Delta_H v\|_2 \\ &\le C\left[\left(\|v\|_6 + \|u\|_6\right)\|\Delta_H v\|_2^{1/2} \|\nabla \Delta_H v\|_2^{1/2} + \|v\|_6\left(\|\Delta_H v\|_2^{1/2} + \|\nabla_H u\|_2^{1/2}\right) \right. \\ &\quad \left. \times \|\nabla_H \Delta_H v\|_2^{1/2} + \left(\|\Delta_H v\|_2 + \|\nabla_H u\|_2\right)\left\|\nabla^2 u\right\|_2 + \|\Delta_H T\|_2\right]\|\nabla_H \Delta_H v\|_2 \\ &\le \frac{1}{2}\|\nabla \Delta_H v\|_2^2 + C\left[\left(\|u\|_6^4 + \|v\|_6^4 + \left\|\nabla^2 u\right\|_2^2\right)\left(\|\Delta_H v\|_2^2 + \|\nabla_H u\|_2^2\right) + \|\Delta_H T\|_2^2\right],\end{aligned}$$

and thus

$$\begin{aligned}
&\frac{d}{dt}\|\Delta_H v\|_2^2+\|\nabla\Delta_H v\|_2^2\\
&\quad\le C\big[\big(\|u\|_6^4+\|v\|_6^4+\big\|\nabla^2 u\big\|_2^2\big)\big(\|\Delta_H v\|_2^2+\|\nabla_H u\|_2^2\big)+\|\Delta_H T\|_2^2\big].
\end{aligned}\tag{3.4}$$

Applying the operator ∇ to Eq. (1.24), multiplying the resulting equation by $-\nabla\Delta T$ and integrating over Ω, using the facts that

$$\big|\Delta v(x,y,z,t)\big|\le\big|\Delta_H\bar{v}(x,y,t)\big|+\int_{-h}^{h}\big|\Delta u(x,y,\xi,t)\big|d\xi,$$

$$\big|\nabla v(x,y,z,t)\big|\le\big|\nabla_H\bar{v}(x,y,t)\big|+\int_{-h}^{h}\big|\nabla u(x,y,\xi,t)\big|d\xi,$$

$$\big|\nabla\nabla_H\cdot v(x,y,z,t)\big|\le\int_{-h}^{h}\big|\nabla\nabla_H\cdot u(x,y,\xi,t)\big|d\xi,$$

then it follows from integrating by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\Delta T|^2dxdydz+\int_\Omega|\nabla_H\Delta T|^2dxdydz\\
&=-\int_\Omega\Delta\left[v\cdot\nabla_H T-\left(\int_{-h}^{z}\nabla_H\cdot v d\xi\right)\left(\partial_z T+\frac{1}{h}\right)\right]\Delta T dxdydz\\
&=-\int_\Omega\left[\Delta v\cdot\nabla_H T+2\nabla v:\nabla_H\nabla T-\left(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\right)\left(\partial_z T+\frac{1}{h}\right)\right.\\
&\qquad\left.-2\left(\int_{-h}^{z}\nabla\nabla_H\cdot v d\xi\right)\cdot\nabla\partial_z T\right]\Delta T dxdydz\\
&=-\int_\Omega\left[\Delta v\cdot\nabla_H T+2\nabla v:\nabla_H\nabla T-\frac{1}{h}\left(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\right)\right.\\
&\qquad\left.-2\left(\int_{-h}^{z}\nabla\nabla_H\cdot v d\xi\right)\nabla\partial_z T\right]\Delta T dxdydz\\
&\qquad+\int_\Omega\left(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\right)\partial_z T\big(\Delta_H T+\partial_z^2 T\big)dxdydz
\end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \bigg[\Delta v\cdot\nabla_H T + 2\nabla v : \nabla_H\nabla T - \frac{1}{h}\bigg(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\bigg) \\
&\quad - 2\bigg(\int_{-h}^{z}\nabla\nabla_H\cdot v d\xi\bigg)\cdot\nabla\partial_z T\bigg]\Delta T\,dxdydz - \int_\Omega\bigg[\Delta\nabla_H\cdot vT\Delta_H T \\
&\quad + \bigg(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\bigg)T\Delta_H\partial_z T + \frac{1}{2}\Delta\nabla_H\cdot v|\partial_z T|^2\bigg]dxdydz \\
&= -\int_\Omega \bigg[\Delta v\cdot\nabla_H T + 2\nabla v : \nabla_H\nabla T - \frac{1}{h}\bigg(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\bigg) \\
&\quad - 2\bigg(\int_{-h}^{z}\nabla\nabla_H\cdot v d\xi\bigg)\nabla\partial_z T\bigg]\Delta T\,dxdydz - \int_\Omega\bigg[\Delta\nabla_H\cdot vT\Delta_H T \\
&\quad + \bigg(\int_{-h}^{z}\Delta\nabla_H\cdot v d\xi\bigg)T\Delta_H\partial_z T - \Delta v\cdot\nabla_H\partial_z T\partial_z T\bigg]dxdydz.
\end{aligned}
$$

We estimate the quantity on the right hand side of the above equality, denoted by I_2, as follows. By Lemma 2.1, Lemma 2.2, and using the Hölder, Sobolev and Poincaré inequalities, we deduce

$$
\begin{aligned}
I_2 &\le C\int_\Omega\bigg[\bigg(\int_{-h}^{h}|\Delta u|d\xi + |\Delta_H\bar v|\bigg)|\nabla_H T| + \bigg(\int_{-h}^{h}|\nabla u|d\xi + |\nabla_H\bar v|\bigg)|\nabla_H\nabla T| \\
&\quad + \bigg(\int_{-h}^{h}|\nabla_H\Delta v|d\xi\bigg) + \bigg(\int_{-h}^{h}|\nabla\nabla_H\cdot u|d\xi\bigg)|\nabla\partial_z T|\bigg]|\Delta T|dxdydz \\
&\quad + C\int_\Omega\bigg[|\Delta\nabla_H v||T||\Delta_H T| + \bigg(\int_{-h}^{h}|\Delta\nabla_H v|d\xi\bigg)|T||\Delta_H\partial_z T|\bigg]dxdydz \\
&\quad + C\int_\Omega\bigg(\int_{-h}^{h}|\Delta u|d\xi + |\Delta_H\bar v|\bigg)|\partial_z T||\nabla_H\partial_z T|dxdydz \\
&\le C\int_M\bigg[\int_{-h}^{h}\big(|\nabla u| + |\nabla^2 u|\big)d\xi\bigg]\bigg[\int_{-h}^{h}\big(|\nabla T| + |\nabla^2 T|\big)|\nabla^2 T|d\xi\bigg]dxdy \\
&\quad + C\int_M\big(|\nabla_H\bar v| + |\Delta_H\bar v|\big)\bigg(\int_{-h}^{h}\big(|\nabla T| + |\nabla^2 T|\big)|\nabla^2 T|d\xi\bigg)dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_M \left(\int_{-h}^{h} |\nabla_H \Delta v| d\xi\right) |\Delta T| dxdydz + C\int_\Omega \Bigg[|\Delta \nabla_H v||T||\Delta_H T| \\
&+ \left(\int_{-h}^{h} |\Delta \nabla_H v| d\xi\right) |T||\Delta_H \partial_z T| \Bigg] dxdydz \\
\le{}& C\Big[\big(\|\nabla u\|_2 + \big\|\nabla^2 u\big\|_2 + \|\nabla_H \bar{v}\|_{L^2(M)} + \|\Delta_H \bar{v}\|_{L^2(M)}\big)\big(\|\nabla T\|_2^{1/2} + \big\|\nabla^2 T\big\|_2^{1/2}\big) \\
&\times \big(\|\nabla T\|_2^{1/2} + \big\|\nabla^2 T\big\|_2^{1/2} + \|\nabla_H \nabla T\|_2^{1/2} + \big\|\nabla_H \nabla^2 T\big\|_2^{1/2}\big) \\
&\times \big\|\nabla^2 T\big\|_2^{1/2}\big(\big\|\nabla^2 T\big\|_2^{1/2} + \big\|\nabla_H \nabla^2 T\big\|_2^{1/2}\big) + \|\nabla_H \Delta v\|_2 \|\Delta T\|_2 \\
&+ \|\nabla_H \Delta v\|_2 \|T\|_\infty \|\Delta_H T\|_2 + \|\nabla_H \Delta v\|_2 \|T\|_\infty \|\Delta_H \partial_z T\|_2 \Big] \\
\le{}& C\Big[\big(\big\|\nabla^2 u\big\|_2 + \|\Delta_H \bar{v}\|_{L^2(M)}\big) \|\Delta T\|_2 \big(\|\Delta T\|_2 + \|\nabla_H \Delta T\|_2\big) + \|\nabla_H \Delta v\|_2 \|\Delta T\|_2 \\
&+ \|\Delta \nabla_H v\|_2 \|T\|_\infty \|\Delta_H T\|_2 + \|\nabla_H \Delta v\|_2 \|T\|_\infty \|\nabla_H \Delta T\|_2 \Big] \\
\le{}& C\big(1 + \big\|\nabla^2 u\big\|_2^2 + \|\Delta_H \bar{v}\|_2^2\big) \|\Delta T\|_2^2 + \frac{1}{2}\|\nabla_H \Delta T\|_2^2 + C\big(1 + \|T\|_\infty^2\big) \|\nabla_H \Delta v\|_2^2.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta T\|_2^2 + \|\nabla_H \Delta T\|_2^2 \\
&\quad \le C\big(1 + \|T\|_\infty^2\big)\|\nabla_H \Delta v\|_2^2 + C\big(1 + \big\|\nabla^2 u\big\|_2^2 + \|\Delta_H \bar{v}\|_2^2\big)\|\Delta T\|_2^2.
\end{aligned} \tag{3.5}
$$

For any given $t \in [0, \mathcal{T}^*)$, recalling that $K_1(t)$ is a bounded continuously increasing function, it follows from Proposition 3.1 that

$$
\sup_{0 \le s \le t} \|T\|_\infty^2 \le \sup_{0 \le s \le t} K_1(s) \le K_1(t).
$$

Therefore it follows from (3.5) that

$$
\begin{aligned}
&\frac{d}{ds}\big\|\Delta T(s)\big\|_2^2 + \big\|\nabla_H \Delta T(s)\big\|_2^2 \\
&\quad \le C\big(1 + \big\|\nabla^2 u(s)\big\|_2^2 + \big\|\Delta_H \bar{v}(s)\big\|_{L^2(M)}^2\big)\big\|\Delta T(s)\big\|_2^2 \\
&\qquad + C\big(1 + K_1(t)\big)\big\|\nabla_H \Delta v(s)\big\|_2^2,
\end{aligned} \tag{3.6}
$$

for all $s \in (0, t)$, with $t \in (0, \mathcal{T}^*)$. On the other hand, by (3.4), it holds that

$$
\begin{aligned}
&\frac{d}{ds}\big\|\Delta_H v(s)\big\|_2^2 + \big\|\nabla \Delta_H v(s)\big\|_2^2 \\
&\quad \le C\big(1 + \big\|u(s)\big\|_6^4 + \big\|v(s)\big\|_6^4 + \big\|\nabla^2 u(s)\big\|_2^2\big)\big(\big\|\Delta_H v(s)\big\|_2^2 + \big\|\Delta_H T(s)\big\|_2^2\big) \\
&\qquad + C\big(\big\|u(s)\big\|_6^4 + \big\|v(s)\big\|_6^4 + \big\|\nabla^2 u(s)\big\|_2^2\big)\big\|\nabla_H u(s)\big\|_2^2.
\end{aligned} \tag{3.7}
$$

Choose a sufficiently big positive constant α. Multiplying (3.7) by $\alpha(1+K_1(t))$ and summing the resulting inequality up with (3.6), then we obtain

$$\begin{aligned}
&\frac{d}{ds}\left[\alpha\big(1+K_1(t)\big)\left\|\Delta_H v(s)\right\|_2^2+\left\|\Delta T(s)\right\|_2^2\right]+\left(\left\|\nabla\Delta_H v(s)\right\|_2^2+\left\|\nabla_H\Delta T(s)\right\|_2^2\right)\\
&\quad\le C\big(1+K_1(t)\big)\left(1+\|u\|_6^4+\|v\|_6^4+\|\Delta_H\bar v\|_{L^2(M)}^2+\left\|\nabla^2 u\right\|_2^2\right)(s)\\
&\qquad\times\left(\|\Delta_H v\|_2^2+\|\Delta T\|_2^2\right)(s)+C\big(1+K_1(t)\big)\left(\|u\|_6^4+\|v\|_6^4+\left\|\nabla^2 u\right\|_2^2\right)(s)\left\|\nabla_H u(s)\right\|_2^2
\end{aligned}$$

for any $0\le s\le t<\mathcal{T}^*$. By Proposition 3.1 and Proposition 3.2, it follows from this inequality that

$$\begin{aligned}
&\sup_{0\le s\le t}\left(\|\Delta_H v\|_2^2+\|\Delta T\|_2^2\right)+\int_0^t\left(\|\nabla\Delta_H v\|_2^2+\|\nabla_H\Delta T\|_2^2\right)\\
&\quad\le Ce^{C(1+K_1(t))\int_0^t(1+\|u\|_6^4+\|v\|_6^4+\|\Delta_H\bar v\|_{L^2(M)}^2+\|\nabla^2 u\|_2^2)ds}\\
&\qquad\times\left[\|v_0\|_{H^2}^2+\|T_0\|_{H^2}^2+\big(1+K_1(t)\big)\int_0^t\left(\|u\|_6^4+\|v\|_6^4+\left\|\nabla^2 u\right\|_2^2\right)\|\nabla_H u\|_2^2 ds\right]\\
&\quad\le Ce^{C(1+K_1(t))(t+K_1^2(t)t+K_1(t)+K_2(t))}\left[\|v_0\|_{H^2}^2+\|T_0\|_{H^2}^2\right.\\
&\qquad\left.+\big(1+K_1(t)\big)\big(K_1^2(t)t+K_2(t)\big)K_2(t)\right]=:K_3(t),
\end{aligned}$$

for every $t\in[0,\mathcal{T}^*)$, completing the proof. $\square$

With these a priori estimates in hand, we are now ready to prove the global existence of strong solutions as follows.

Proof of Theorem 1.1. By Proposition 2.3, there is a unique strong solution (v,T) in $\Omega\times(0,t_0)$. We consider the solution on the maximal interval of existence $(0,\mathcal{T}^*)$. We need to prove that $\mathcal{T}^*=\infty$. Recall that we have assumed by contradiction that $\mathcal{T}^*<\infty$. By Proposition 3.1, Proposition 3.2 and Proposition 3.3, we have the following estimate

$$\sup_{0\le s\le t}\left(\left\|v(s)\right\|_{H^2}^2+\left\|T(s)\right\|_{H^2}^2\right)+\int_0^t\left(\|\nabla v\|_{H^2}^2+\|\nabla_H T\|_{H^2}^2\right)ds\le CK(t),$$

for any $t\in(0,\mathcal{T}^*)$, where

$$K(t)=K_1(t)+K_2(t)+K_3(t).$$

Note that $K(t)$ is a bounded continuously increasing function, on $(0,\mathcal{T}^*)$, the above inequality implies that

$$\sup_{0\le t<\mathcal{T}^*}\left(\left\|v(t)\right\|_{H^2}^2+\left\|T(t)\right\|_{H^2}^2\right)+\int_0^{\mathcal{T}^*}\left(\|\nabla v\|_{H^2}^2+\|\nabla_H T\|_{H^2}^2\right)dt\le CK\big(\mathcal{T}^*\big),$$

and thus, by Proposition 2.3, we can extend such strong solution beyond $\mathcal{T}^*$, contradicting to the definition of $\mathcal{T}^*$. This contradiction implies that $\mathcal{T}^* = \infty$, and thus completes the proof of Theorem 1.1. □

Acknowledgments

The work of C.C. work is supported in part by NSF grant DMS-1109022. The work of E.S.T. is supported in part by the Minerva Stiftung/Foundation, and by the NSF grants DMS-1009950, DMS-1109640 and DMS-1109645.

References

[1] D. Bresch, F. Guillén-González, N. Masmoudi, M.A. Rodríguez-Bellido, On the uniqueness of weak solutions of the two-dimensional primitive equations, Differential Integral Equations 16 (2003) 77–94.
[2] D. Bresch, A. Kazhikhov, J. Lemoine, On the two-dimensional hydrostatic Navier–Stokes equations, SIAM J. Math. Anal. 36 (2004) 796–814.
[3] C. Cao, S. Ibrahim, K. Nakanishi, E.S. Titi, Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics, Comm. Math. Phys. (2013), in press.
[4] C. Cao, J. Li, E.S. Titi, Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity, Arch. Ration. Mech. Anal. 214 (2014) 35–76.
[5] C. Cao, E.S. Titi, Global well-posedness and finite-dimensional global attractor for a 3-D planetary geostrophic viscous model, Comm. Pure Appl. Math. 56 (2003) 198–233.
[6] C. Cao, E.S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Ann. of Math. 166 (2007) 245–267.
[7] C. Cao, E.S. Titi, Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion, Comm. Math. Phys. 310 (2012) 537–568.
[8] F. Guillén-González, N. Masmoudi, M.A. Rodríguez-Bellido, Anisotropic estimates and strong solutions of the primitive equations, Differential Integral Equations 14 (2001) 1381–1408.
[9] G.M. Kobelkov, Existence of a solution in the large for the 3D large-scale ocean dynamics equations, C. R. Math. Acad. Sci. Paris 343 (2006) 283–286.
[10] I. Kukavica, M. Ziane, On the regularity of the primitive equations of the ocean, C. R. Math. Acad. Sci. Paris 345 (2007) 257–260.
[11] I. Kukavica, M. Ziane, On the regularity of the primitive equations, Nonlinearity 20 (2007) 2739–2753.
[12] R. Lewandowski, Analyse Mathématique et Océanographie, Masson, Paris, 1997.
[13] J.L. Lions, R. Temam, S. Wang, New formulations of the primitive equations of the atmosphere and applications, Nonlinearity 5 (1992) 237–288.
[14] J.L. Lions, R. Temam, S. Wang, On the equations of the large-scale ocean, Nonlinearity 5 (1992) 1007–1053.
[15] J.L. Lions, R. Temam, S. Wang, Mathematical study of the coupled models of atmosphere and ocean (CAO III), J. Math. Pures Appl. 74 (1995) 105–163.
[16] A. Majda, Introduction to PDEs and Waves for the Atmosphere and Ocean, New York University, Courant Institute of Mathematical Sciences/American Mathematical Society, New York/Providence, RI, 2003.
[17] J. Pedlosky, Geophysical Fluid Dynamics, 2nd edition, Springer, New York, 1987.
[18] M. Petcu, R. Temam, M. Ziane, Some mathematical problems in geophysical fluid dynamics, in: Handbook of Numerical Analysis, vol. 14, Elsevier, 2009, pp. 577–750.
[19] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (1987) 65–96.
[20] R. Temam, M. Ziane, Some mathematical problems in geophysical fluid dynamics, in: Handbook of Mathematical Fluid Dynamics, vol. 3, Elsevier, 2004, pp. 535–657.
[21] G.K. Vallis, Atmospheric and Oceanic Fluid Dynamics, Cambridge Univ. Press, 2006.
[22] W.M. Washington, C.L. Parkinson, An Introduction to Three Dimensional Climate Modeling, Oxford University Press, Oxford, 1986.
[23] T.K. Wong, Blowup of solutions of the hydrostatic Euler equations, Proc. Amer. Math. Soc. (2013), in press.