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Regularity for an obstacle problem of Hessian equations on Riemannian manifolds

Gejun Bao, Weisong Dong, Heming Jiao *

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

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Abstract

In this paper, we study the regularity for solutions to an obstacle problem of Hessian type fully nonlinear equations on Riemannian manifolds. As an application, the existence of a $C^{1,1}$ solution is proved.

Keywords: Regularity; Obstacle problem; Hessian equations; Riemannian manifolds

1. Introduction

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 2$ with smooth boundary ∂M and $\bar{M} := M \cup \partial M$. In this paper we study the obstacle problem

$$\max\left\{u - h, -\left(f\left(\lambda\left(\nabla^2 u + A[u]\right)\right) - \psi(x, u, \nabla u)\right)\right\} = 0 \tag{1.1}$$

in M with boundary condition

$$u = \varphi \quad \text{on } \partial M, \tag{1.2}$$

where $h \in C^3(\bar{M})$, $\varphi \in C^4(\partial M)$, $h > \varphi$ on ∂M, f is a symmetric function of $\lambda \in \mathbb{R}^n$, $\nabla^2 u$ denotes the Hessian of a function u on M, $A[u] = A(x, u, \nabla u)$ is a smooth $(0, 2)$ tensor which may

* Corresponding author.
E-mail addresses: baogj@hit.edu.cn (G. Bao), dweeson@gmail.com (W. Dong), jiaoheming@163.com (H. Jiao).

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depend on u and ∇u, and for a $(0, 2)$ tensor X on M, $\lambda(X)$ denotes the eigenvalues of X with respect to the metric g.

Our motivation to study the obstacle problem for Hessian equations comes in part from their applications in differential geometry such as finding the greatest hypersurface with an obstacle whose curvature (for example the Weingarten curvature) is bounded below by a nonnegative function. For other applications please see [17] where Liu and Zhou considered an obstacle problem for Monge–Ampère equations (when $f = \sigma_n^{1/n}$, see (2.5)) arising from the study of affine maximal surfaces, and [6] in which Gerhardt studied hypersurfaces of prescribed mean curvature bounded from below by an obstacle, while Kinderlehrer [13] treated minimal surfaces over obstacles.

The interest to consider (1.1) is also from its connection to the problem of optimal transportation (see [21] for example). In [2], Caffarelli and McCann studied a class of optimal transportation problems which is equivalent to a double obstacle problem for Monge–Ampère equations. An interesting result of Oberman [19] states that the convex envelope is the viscosity solution of a nonlinear obstacle problem which is essentially an obstacle problem for Monge–Ampère equations (see [20] also).

When $A \equiv \kappa u g$, the obstacle problem for Hessian equations on Riemannian manifolds was studied by Jiao and Wang [12] under various conditions which exclude the case that $f = (\sigma_k/\sigma_l)^{1/(k-l)}$. For Monge–Ampère equations, Xiong and Bao [24] treated the case that $A \equiv 0$ and Lee [15] considered similar problem when $\psi \equiv 1$, $\varphi \equiv 0$ in a strictly convex domain in $\mathbb{R}^n$. For the study of Hessian equations on Riemannian manifolds, the reader is referred to [8–10,16,23] and their references.

The rest of this paper is organized as follows. In Section 2 we discuss the assumptions of this work and state our main results. In Section 3 we introduce some notations and an approximating Dirichlet problem. The C^0 estimates and gradient estimates on the boundary for solutions to the approximating problem are treated in Section 4 while in Section 5 and Section 6, the gradient and second derivative estimates are established respectively. In Section 7, we will prove the existence of a smooth solution to the approximating problem to finish our proof of Theorem 2.2.

2. Assumptions and main results

In this section, we discuss the assumptions of this work and state our main results. Following Caffarelli, Nirenberg and Spruck [4], the function $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ is assumed to be defined in an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin,

$$\Gamma_n \equiv \left\{\lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0\right\} \subseteq \Gamma \neq \mathbb{R}^n$$

and to satisfy the fundamental structure conditions

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma, \quad 1 \leq i \leq n, \tag{2.1}$$

$$f \text{ is a concave function in } \Gamma, \tag{2.2}$$

and

$$f > 0 \quad \text{in } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma. \tag{2.3}$$

A function $u \in C^2(M)$ is called admissible at $x_0 \in M$ if $\lambda(\nabla^2 u + A[u])(x_0) \in \overline{\Gamma}$ and we call it admissible in M if it is admissible at each $x \in M$. It is shown in [4] that (2.1) implies that (1.1) is degenerate elliptic for admissible solutions. While (2.2), which is a type of growth condition essentially, ensures that F defined by $F(r) = f(\lambda(r))$ for $r = \{r_{ij}\} \in \mathcal{S}^{n\times n}$ with $\lambda(r) \in \Gamma$ is concave, where $\mathcal{S}^{n\times n}$ is the set of $n \times n$ symmetric matrices.

We recall the notion of viscosity solution to (1.1) and (1.2) (see [22] and [5]).

Definition 2.1. We define a function $u \in C^0(\bar{M})$ to be a viscosity subsolution of (1.1) and (1.2) if for any function $\phi \in C^2(M)$ and point $x_0 \in M$ satisfying $u(x_0) = \phi(x_0)$, $u \le \phi$ in M we have

$$\max\bigl\{\phi(x_0) - h(x_0), -\bigl(f\bigl(\lambda\bigl(\nabla^2\phi(x_0) + A[\phi](x_0)\bigr)\bigr) - \psi[\phi](x_0)\bigr)\bigr\} \le 0,$$

where $\psi[\phi](x_0) = \psi(x_0, \phi(x_0), \nabla\phi(x_0))$, and $u \le \varphi$ on ∂M. While u is called a viscosity supersolution of (1.1) and (1.2) if for any function $\phi \in C^2(M)$ and point $x_0 \in M$ at which ϕ is admissible, satisfying $u(x_0) = \phi(x_0)$, $u \ge \phi$ in M we have

$$\max\bigl\{\phi(x_0) - h(x_0), -\bigl(f\bigl(\lambda\bigl(\nabla^2\phi(x_0) + A[\phi](x_0)\bigr)\bigr) - \psi[\phi](x_0)\bigr)\bigr\} \ge 0,$$

and $u \ge \varphi$ on ∂M. The function u is a viscosity solution of (1.1) and (1.2) if it is both a viscosity subsolution and supersolution.

In this paper, we shall prove the existence of a viscosity solution in $C^{1,1}(\bar{M})$ to (1.1) and (1.2). Our strategy is to use a penalization technique for which we consider a singular perturbation problem (see (3.4)). We shall use the methods in [9] and [10], where the authors studied the corresponding fully nonlinear elliptic equations on general Riemannian manifolds, to establish the *a priori* C^2 estimates independent of the perturbation for solutions to (3.4). After establishing the C^2 estimates, (2.1) and (2.3) ensure that Eq. (3.4) is uniformly elliptic and the $C^{2,\alpha}$ estimates follow by the Evans–Krylov theory. Next, the existence of smooth solutions to (3.4) can be derived using the method of continuity and degree theory. As is well known, the concavity condition (2.2) which is crucial to the Evans–Krylov theory as well as the second order estimates, plays an extremely important role in the theory of fully nonlinear equations. So conditions (2.1)–(2.3) are standard and fundamental in the study of Hessian equations.

The ideas proposed in [9] and [10] allow us to consider various classes of fully nonlinear equations under conditions which are nearly optimal. In order to state our main results let us introduce some notations adopted from [9].

For $\sigma > 0$ let $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$ and $\partial\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) = \sigma\}$ which is a smooth and convex hypersurface in $\mathbb{R}^n$ by assumptions (2.1) and (2.2). We shall only consider the case $\Gamma^\sigma \ne \emptyset$. For $\lambda \in \Gamma$ we use $T_\lambda = T_\lambda \partial\Gamma^{f(\lambda)}$ to denote the tangent plane at λ to the level surface $\partial\Gamma^{f(\lambda)}$.

The following condition is essential to our work in this paper:

$$\partial\Gamma^\sigma \cap T_\lambda \partial\Gamma^{f(\lambda)} \text{ is nonempty and compact}, \quad \forall\sigma > 0,\ \lambda \in \Gamma^\sigma. \tag{2.4}$$

Condition (2.4) means that the level set of f would not be too "straight" when $|\lambda|$ is large. So assumption (2.4) excludes linear elliptic equations but is satisfied by a very general class of functions f. In particular, (2.4) holds for those f whose level set is strictly convex. Thus, (2.4) holds for $f = \sigma_k^{1/k}$, $k \ge 2$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ which we recall was not covered by the work of Jiao and Wang [12], $1 \le l < k \le n$, defined on the cone $\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \dots, k\}$, where $\sigma_k(\lambda)$ are the elementary symmetric functions

$$\sigma_k(\lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \dots \lambda_{i_k}, \quad k = 1, \dots, n. \tag{2.5}$$

Another example satisfying (2.4) is $f = \log P_k$, where

$$P_k(\lambda) := \prod_{i_1 < \dots < i_k} (\lambda_{i_1} + \dots + \lambda_{i_k}), \quad 1 \le k \le n$$

defined in the cone

$$\mathcal{P}_k := \{\lambda \in \mathbb{R}^n : \lambda_{i_1} + \dots + \lambda_{i_k} > 0\}.$$

The following condition is used to overcome the difficulty caused by the presence of curvature in the boundary estimates for second order derivatives (see [9] or [10]):

$$\sum f_i(\lambda)\lambda_i \ge 0, \quad \forall \lambda \in \Gamma. \tag{2.6}$$

Finally, note that for fixed $x \in \bar{M}$, $z \in \mathbb{R}$ and $p \in T_x^*M$,

$$A(x, z, p) : T_x^*M \times T_x^*M \to \mathbb{R}$$

is a symmetric bilinear map. We shall use the notation

$$A^{\xi\eta}(x, \cdot, \cdot) := A(x, \cdot, \cdot)(\xi, \eta), \quad \xi, \eta \in T_x^*M \tag{2.7}$$

and, for a function $v \in C^2(M)$, $A[v] := A(x, v, \nabla v)$, $A^{\xi\eta}[v] := A^{\xi\eta}(x, v, \nabla v)$.

Throughout the paper we assume $\psi \in C^3(T^*M \times \mathbb{R})$ (for convenience we shall write $\psi = \psi(x, z, p)$ for $(x, p) \in T^*M$ and $z \in \mathbb{R}$ though), $\psi > 0$, and that there exists an admissible subsolution $\underline{u} \in C^2(\bar{M})$ satisfying

$$\begin{aligned} f\big(\lambda\big(\nabla^2 \underline{u} + A[\underline{u}]\big)\big) &\ge \psi(x, \underline{u}, \nabla \underline{u}) \quad \text{in } M, \\ \underline{u} &= \varphi \quad \text{on } \partial M \end{aligned} \tag{2.8}$$

and $\underline{u} \le h$ in M.

The reader is referred to Theorem 1.3 of [12] in which the third author and Wang constructed some subsolutions satisfying (2.8) in some special cases. By (2.3), we can see that $\lambda(\nabla^2 \underline{u} + A[\underline{u}])(x) \in K$ for all $x \in \bar{M}$, where K is a compact subset of Γ, since $\psi(x, \underline{u}, \nabla \underline{u}) \ge \delta_0 > 0$ for some constant δ_0.

As in [10], we make the following technical assumptions:

$$-\psi(x, z, p) \quad \text{and} \quad A^{\xi\xi}(x, z, p) \text{ are concave in } p, \tag{2.9}$$

and

$$-\psi_z, A_z^{\xi\xi} \geq 0, \quad \forall \xi \in T_x M. \tag{2.10}$$

For the gradient estimates, we usually need some growth conditions and in this paper, we assume that (see [8])

$$\begin{cases} p \cdot \nabla_x A^{\xi\xi}(x,z,p) + |p|^2 A_z^{\xi\xi}(x,z,p) \leq \bar{\omega}_1(x,z)|\xi|^2\big(1+|p|^{\gamma_1}\big), \\ p \cdot \nabla_x \psi(x,z,p) + |p|^2 \psi_z(x,z,p) \geq -\bar{\omega}_2(x,z)\big(1+|p|^{\gamma_2}\big), \end{cases} \tag{2.11}$$

for some continuous functions $\bar{\omega}_1, \bar{\omega}_2 \geq 0$ and constants $\gamma_1, \gamma_2 > 0$.

We shall establish the gradient estimates under two groups of conditions. Firstly we use the following condition that there exists $\bar{c} > 0$ such that

$$A^{\xi\xi}_{p_k p_l}(x,z,p)\eta_k\eta_l \leq -\bar{c}|\xi|^2|\eta|^2 + \bar{c}\big|g(\xi,\eta)\big|^2, \quad \forall \xi, \eta \in T_x M \tag{2.12}$$

and

$$\lim_{t\to\infty} f(t\mathbf{1}) = +\infty \tag{2.13}$$

where $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^n$ which holds for $f = \sigma_k^{1/k}$ and $f = (\sigma_k/\sigma_l)^{1/(k-l)}$ obviously.

We remark that the condition (2.12) implies the following MTW condition which was introduced by Ma, Trudinger and Wang in [18] to establish interior regularity for potential functions of the optimal transportation problem:

$$A^{\xi\xi}_{p_k p_l}(x,z,p)\eta_k\eta_l \leq -\bar{c}|\xi|^2|\eta|^2, \quad \forall \xi, \eta \in T_x M,\ \xi \perp \eta.$$

An alternative assumption

$$f_j(\lambda) \geq \nu_0\Big(1 + \sum f_i(\lambda)\Big) \quad \text{for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{2.14}$$

where ν_0 is a uniform positive constant, is commonly used in deriving gradient estimates, see [8,11,22] and [23] for example. Together with (2.14), we also need (2.6) and the following growth conditions (see [8]):

$$p \cdot D_p\psi(x,z,p), -p \cdot D_p A^{\xi\xi}(x,z,p)/|\xi|^2 \leq \bar{\omega}(x,z)\big(1+|p|^\gamma\big) \tag{2.15}$$

and

$$\big|A^{\xi\eta}(x,z,p)\big| \leq \bar{\omega}(x,z)|\xi||\eta|\big(1+|p|^\gamma\big), \quad \forall \xi, \eta \in T_x \bar{M},\ \xi \perp \eta, \tag{2.16}$$

for some continuous function $\bar{\omega} \geq 0$ and constant $\gamma \in (0,2)$. Our main results are stated in the next theorem.

Theorem 2.2. *Suppose that (2.1)–(2.4), (2.6) and (2.8)–(2.10) hold. Assume that* $A(x, z, p) \equiv A(x, p)$ *or* $\operatorname{tr} A(x, z, 0) \le 0$ *when* z *is sufficiently large and*

$$\left|A^{\xi\xi}(x, z, p)\right| \le \bar{\omega}(x, z)|\xi|^2\left(1 + |p|^2\right) \tag{2.17}$$

for any $\xi \in T_x M$ *when* $|p|$ *is sufficiently large, where* $\bar{\omega} \ge 0$ *is a continuous function. Then there exists a viscosity solution* $u \in C^{1,1}(\bar{M})$ *to (1.1) and (1.2) under any of the following additional assumptions: (i) (2.11)–(2.13) hold for* $\gamma_1 < 4$, $\gamma_2 = 2$ *in (2.11); (ii) (2.11) and (2.14)–(2.16) hold for* $\gamma_1, \gamma_2 < 4$ *in (2.11).*

Furthermore, $u \in C^{3,\alpha}(E)$ *for any* $\alpha \in (0, 1)$, *and*

$$f\left(\lambda\left(\nabla^2 u + A(x, u, \nabla u)\right)\right) = \psi(x, u, \nabla u) \quad \text{in } E,$$

where $E \equiv \{x \in M : u(x) < h(x)\}$.

3. Preliminaries

Throughout the paper ∇ denotes the Levi-Civita connection of (M^n, g). The curvature tensor is defined by

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Let $e_1, \ldots, e_n$ be local frames on M^n. We denote $g_{ij} = g(e_i, e_j)$, $\{g^{ij}\} = \{g_{ij}\}^{-1}$. Define the Christoffel symbols Γ_{ij}^k by $\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$ and the curvature coefficients

$$R_{ijkl} = g\left(R(e_k, e_l)e_j, e_i\right), \qquad R^i_{jkl} = g^{im} R_{mjkl}.$$

We shall use the notation $\nabla_i = \nabla_{e_i}$, $\nabla_{ij} = \nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k$, etc.

For a differentiable function v defined on M^n, we usually identify ∇v with the gradient of v, and use $\nabla^2 v$ to denote the Hessian of v which is locally given by $\nabla_{ij} v = \nabla_i(\nabla_j v) - \Gamma_{ij}^k \nabla_k v$. We recall that $\nabla_{ij} v = \nabla_{ji} v$ and

$$\nabla_{ijk} v - \nabla_{jik} v = R^l_{kij} \nabla_l v, \tag{3.1}$$

$$\begin{aligned} \nabla_{ijkl} v - \nabla_{klij} v = {} & R^m_{ljk} \nabla_{im} v + \nabla_i R^m_{ljk} \nabla_m v + R^m_{lik} \nabla_{jm} v \\ & + R^m_{jik} \nabla_{lm} v + R^m_{jil} \nabla_{km} v + \nabla_k R^m_{jil} \nabla_m v. \end{aligned} \tag{3.2}$$

By direct calculation, we see, for each $1 \le i, j, k \le n$,

$$\nabla_{ijk} v = \nabla^3 v(e_i, e_j, e_k) = \nabla_i(\nabla_{jk} v) - \nabla_{lk} v \Gamma^l_{ji} - \nabla_{jl} v \Gamma^l_{ki}$$

and

$$\nabla_{ikj} v = \nabla^3 v(e_i, e_k, e_j) = \nabla_i(\nabla_{kj} v) - \nabla_{lj} v \Gamma^l_{ki} - \nabla_{kl} v \Gamma^l_{ji}.$$

Therefore, by the symmetry of $\{\nabla_{ij}v\}$, we have

$$\nabla_{ijk}v = \nabla_{ikj}v.$$

It follows that, by (3.1),

$$\nabla_{ikj}v - \nabla_{jik}v = R^l_{kij}\nabla_l v. \tag{3.3}$$

We shall use a penalization technique to consider the following singular perturbation problem

$$\begin{cases} f\big(\lambda\big(\nabla^2 u + A(x,u,\nabla u)\big)\big) = \psi(x,u,\nabla u) + \beta_\varepsilon(u-h) & \text{in } M, \\ u = \varphi & \text{on } \partial M, \end{cases} \tag{3.4}$$

where the penalty function β_ε is defined by

$$\beta_\varepsilon(z) = \begin{cases} 0, & z \le 0, \\ z^3/\varepsilon, & z > 0, \end{cases}$$

for $\varepsilon \in (0,1)$. Obviously, $\beta_\varepsilon \in C^2(\mathbb{R})$ satisfies

$$\begin{aligned} &\beta_\varepsilon, \beta'_\varepsilon, \beta''_\varepsilon \ge 0; \\ &\beta_\varepsilon(z) \to \infty \quad \text{as } \varepsilon \to 0^+, \text{ whenever } z > 0; \\ &\beta_\varepsilon(z) = 0, \quad \text{whenever } z \le 0. \end{aligned}$$

(See [24].) Obviously, $\underline{u}$ is also a subsolution to (3.4) since $\underline{u} \le h$. Let $u_\varepsilon \in C^3(\bar{M}) \cap C^4(M)$ be an admissible solution to (3.4) with $u_\varepsilon \ge \underline{u}$. We show that there exists a constant C independent of ε such that

$$|u_\varepsilon|_{C^2(\bar{M})} \le C \tag{3.5}$$

for small ε.

From now on, we may drop the subscript ε when there is no possible confusion. For simplicity we shall denote $U := \nabla^2 u + A(x,u,\nabla u)$ and, under a local frame $e_1, \dots, e_n$,

$$\begin{aligned} U_{ij} &\equiv U(e_i,e_j) = \nabla_{ij}u + A^{ij}(x,u,\nabla u), \\ \nabla_k U_{ij} &\equiv \nabla U(e_k,e_i,e_j) = \nabla_{kij}u + \nabla_k A^{ij}(x,u,\nabla u) \\ &\equiv \nabla_{kij}u + \nabla'_k A^{ij}(x,u,\nabla u) + A^{ij}_z(x,u,\nabla u)\nabla_k u \\ &\quad + A^{ij}_{p_l}(x,u,\nabla u)\nabla_{kl}u \end{aligned} \tag{3.6}$$

where $A^{ij} = A^{e_ie_j}$ and $\nabla'_k A^{ij}$ denotes the *partial* covariant derivative of A when viewed as depending on $x \in M$ only, while the meanings of A^{ij}_z and $A^{ij}_{p_l}$, etc. are obvious. Similarly we can calculate $\nabla_{kl}U_{ij} = \nabla_k\nabla_l U_{ij} - \Gamma^m_{kl}\nabla_m U_{ij}$, etc.

Let F be the function defined by

$$F(r) = f\big(\lambda(r)\big)$$

for a symmetric matrix r with $\lambda(r) \in \Gamma$. Throughout the paper we shall use the notation

$$F^{ij} = \frac{\partial F}{\partial r_{ij}}(U), \qquad F^{ij,kl} = \frac{\partial^2 F}{\partial r_{ij} \partial r_{kl}}(U).$$

By (2.1), the matrix $\{F^{ij}\}$ has positive eigenvalues $f_1, \dots, f_n$. Moreover, when U_{ij} is diagonal so is $\{F^{ij}\}$, and the following identities hold

$$F^{ij} U_{ij} = \sum f_i \lambda_i, \qquad F^{ij} U_{ik} U_{kj} = \sum f_i \lambda_i^2$$

where $\lambda(U) = (\lambda_1, \dots, \lambda_n)$.

Our main tool is the following theorem proved in [10].

Theorem 3.1. *Assume that (2.1), (2.2) and (2.4) hold. Let K be a compact subset of Γ and $0 < a \le b < \sup_\Gamma f$. There exist positive constants $\theta = \theta(K, [a,b])$ and $R = R(K, [a,b])$ such that for any $\lambda \in \Gamma^{[a,b]} = \overline{\Gamma^a} \setminus \Gamma^b$, when $|\lambda| \ge R$,*

$$\sum f_i(\lambda)(\mu_i - \lambda_i) \ge \theta + \theta \sum f_i(\lambda) + f(\mu) - f(\lambda), \quad \forall \mu \in K. \tag{3.7}$$

4. C^0 estimates

In this section, we consider the C^0 estimates and gradient estimates on the boundary for u_ε. Actually, we can prove

Theorem 4.1. *There exists a constant C independent of ε such that*

$$\sup_{\bar{M}} |u| + \sup_{\partial M} |\nabla u| \le C, \tag{4.1}$$

provided (i) $A(x,z,p) \equiv A(x,p)$ and $A^{\xi\xi}(x,p)$ is concave in p for each $\xi \in T_x M$ or

(ii) $\operatorname{tr} A(x,z,0) \le 0$ when z is sufficiently large and (2.17) holds.

Proof. (4.1) is clear under the assumption (i), so we just prove (4.1) when (ii) is assumed. Note that

$$\Delta u + \operatorname{tr} A(x,u,\nabla u) > 0$$

since $\Gamma \subset \{\lambda \in \mathbb{R}^n : \sum \lambda_i > 0\}$. Suppose $\sup_{\bar{M}} u$ is achieved at $x_0 \in M$. Thus, at x_0, $\nabla u = 0$ and $\Delta u \le 0$. We have, at x_0,

$$0 < \Delta u + \operatorname{tr} A\big(x_0, u(x_0), 0\big) \le \operatorname{tr} A\big(x_0, u(x_0), 0\big).$$

Therefore, there exists a positive constant c_0 under control such that $u(x_0) \le c_0$. Then we obtain

$$\sup_M |u| \le \max\left\{\sup_M |\underline{u}|, \sup_{\partial M} |\varphi|, c_0\right\} \le C.$$

Now let $v = e^{au}$, where a is a constant sufficiently large to be chosen later. We see that

$$\triangle v = ae^{au}\big(\triangle u + a|\nabla u|^2\big).$$

Let $\mu \equiv \sup_M |u|$. It follows that, by (2.17),

$$\triangle v + a^2 e^{a\mu} \ge ae^{au}\big(\triangle u + a|\nabla u|^2 + a\big) \ge ae^{au}\big(\triangle u + \operatorname{tr} A(x, u, \nabla u)\big) > 0,$$

when a is sufficiently large. Let ϕ be the solution to

$$\begin{cases} \triangle \phi + a^2 e^{a\mu} = 0 & \text{in } M, \\ \phi = e^{a\varphi} & \text{on } \partial M. \end{cases}$$

Then $e^{-a\mu} \le v \le \phi$ on $\bar{M}$ by the maximum principle. Furthermore,

$$\nabla_\nu v \le \nabla_\nu \phi \quad \text{on } \partial M$$

where ν is the interior unit normal to ∂M. Therefore, we get

$$\nabla_\nu u \le \frac{\nabla_\nu \phi}{av} \le C \quad \text{on } \partial M.$$

It follows that

$$\sup_{\partial M} |\nabla_\nu u| \le C.$$

Then we get (4.1) since $\nabla_\xi u(x_0) = \nabla_\xi \varphi(x_0)$ for any $x_0 \in \partial M$ and $\xi \in T_{x_0}\partial M$. $\square$

Remark 4.2. We can see from the proof that (4.1) holds for any admissible function $u \in C^2(\bar{M})$ satisfying $u \ge \underline{u}$ in M and $u = \varphi$ on ∂M.

5. The interior gradient estimates

In this section we establish the interior gradient estimates of u_ε. Similar to Lemma 3.2 of [24] (see [12] also), we can prove the following lemma which is crucial to establish both the gradient estimates and second derivative estimates.

Lemma 5.1. *There exists a positive constant c_0 independent of ε such that*

$$0 \le \beta_\varepsilon(u - h) \le c_0 \quad \text{in } M. \tag{5.1}$$

Proof. We consider the maximum of $u-h$ on $\bar{M}$, and we may assume it is achieved at an interior point $x_0 \in M$ since $u-h=\varphi-h<0$ on ∂M. We have, at x_0, $\nabla(u-h)=0$ and $\nabla^2 u \le \nabla^2 h$. Therefore, at x_0,

$$\begin{aligned}\beta_\varepsilon(u-h) &= f\big(\lambda\big(\nabla^2 u + A(x,u,\nabla u)\big)\big) - \psi(x,u,\nabla u)\\ &\le f\big(\lambda\big(\nabla^2 h + A(x,u,\nabla h)\big)\big) - \psi(x,u,\nabla h) \le c_0\end{aligned}$$

for some uniform constant $c_0>0$ independent of ε by (4.1). Hence (5.1) holds. □

Theorem 5.2. *Assume that (2.1), (2.2), (2.11), (2.12) and (2.13) hold for $\gamma_1<4$, $\gamma_2=2$ in (2.11). Then for ε sufficiently small,*

$$\max_{\bar{M}} |\nabla u| \le C\Big(1+\max_{\partial M} |\nabla u|\Big), \tag{5.2}$$

where C is a positive constant depending on $|u|_{C^0(\bar{M})}$, $|\underline{u}|_{C^2(\bar{M})}$ and other known data.

Proof. To prove (5.2), we set $w=|\nabla u|$ and suppose the function $w\phi^{-a}$ achieves a positive maximum at an interior point $x_0\in M$, where ϕ is a positive function to be determined and $0<a<1$ is a constant. Choose a smooth orthonormal local frame $e_1,\dots,e_n$ about x_0 such that $\nabla_{e_i}e_j=0$ at x_0 and $\{U_{ij}(x_0)\}$ is diagonal. The function $\log w - a\log\phi$ attains its maximum at x_0 where

$$\frac{\nabla_i w}{w} - \frac{a\nabla_i\phi}{\phi} = 0, \tag{5.3}$$

$$\frac{\nabla_{ii} w}{w} + \frac{(a-a^2)|\nabla_i\phi|^2}{\phi^2} - \frac{a\nabla_{ii}\phi}{\phi} \le 0 \tag{5.4}$$

for $i=1,\dots,n$. Note that for each fixed $1\le i\le n$,

$$w\nabla_i w = \nabla_l u \nabla_{il} u$$

and, by (3.3) and (5.3),

$$\begin{aligned}w\nabla_{ii} w &= \nabla_l u\nabla_{iil}u + \nabla_{il}u\nabla_{il}u - \nabla_i w\nabla_i w\\ &= \big(\nabla_{lii}u + R^k_{iil}\nabla_k u\big)\nabla_l u + \left(\delta_{kl} - \frac{\nabla_k u\nabla_l u}{w^2}\right)\nabla_{ik}u\nabla_{il}u\\ &\ge \big(\nabla_l U_{ii} - A^{ii}_{p_k}\nabla_{lk}u - A^{ii}_u\nabla_l u - \nabla'_l A^{ii}\big)\nabla_l u - C|\nabla u|^2\\ &= \nabla_l u\nabla_l U_{ii} - \frac{w^2}{\phi}\big(aA^{ii}_{p_k}\nabla_k\phi + \phi A^{ii}_u\big) - \nabla_l u\nabla'_l A^{ii} - Cw^2.\end{aligned} \tag{5.5}$$

Here we have used the Einstein summation convention over the l and the k indices.

Differentiating Eq. (3.4), by (5.3),

$$\begin{aligned} F^{ii}\nabla_l u \nabla_l U_{ii} &= \nabla_l u \nabla_l' \psi + \psi_u |\nabla u|^2 + \psi_{p_k} \nabla_l u \nabla_{lk} u \\ &\quad + \beta_\varepsilon'(u-h)\big(|\nabla u|^2 - \nabla u \cdot \nabla h\big) \\ &= \nabla_l u \nabla_l' \psi + \psi_u |\nabla u|^2 + \frac{aw^2}{\phi}\psi_{p_k}\nabla_k \phi \\ &\quad + \beta_\varepsilon'(u-h)\big(|\nabla u|^2 - \nabla u \cdot \nabla h\big). \end{aligned} \tag{5.6}$$

Let $\phi = (u - \underline{u}) + b > 0$, where $b = 1 + \sup_M(\underline{u} - u)$. By (2.12) we find

$$\begin{aligned} -A_{p_k}^{ii}\nabla_k \phi &= A_{p_k}^{ii}(x, u, \nabla u)\nabla_k(\underline{u} - u) \\ &\geq A^{ii}(x, u, \nabla \underline{u}) - A^{ii}(x, u, \nabla u) + \bar{c}\big(|\nabla \phi|^2 - |\nabla_i \phi|^2\big) \\ &\geq A^{ii}(x, \underline{u}, \nabla \underline{u}) - A^{ii}(x, u, \nabla u) + \bar{c}\big(|\nabla \phi|^2 - |\nabla_i \phi|^2\big) - C. \end{aligned} \tag{5.7}$$

Because of the convexity of ψ in p, we see

$$\psi_{p_k}\nabla_k \phi = \psi_{p_k}(x, u, \nabla u)\nabla_k(u - \underline{u}) \geq \psi(x, u, \nabla u) - \psi(x, u, \nabla \underline{u}). \tag{5.8}$$

By (5.4), (5.5) and (5.7), we have

$$\begin{aligned} 0 \geq{}& \frac{\nabla_l u}{w^2}F^{ii}\nabla_l U_{ii} + \frac{a}{\phi}F^{ii}(\underline{U}_{ii} - U_{ii}) + \frac{a\bar{c}|\nabla\phi|^2}{\phi}\sum F^{ii} \\ &+ \frac{a - a^2 - \bar{c}a\phi}{\phi^2}F^{ii}|\nabla_i \phi|^2 - F^{ii}A_u^{ii} - \frac{\nabla_l u}{w^2}F^{ii}\nabla_l' A^{ii} - C\sum F^{ii}. \end{aligned} \tag{5.9}$$

Without loss of generality, we assume $\bar{c}$ is sufficiently small such that $\bar{c}\phi < \frac{1}{2}$ and thus we can guarantee that

$$\frac{a - a^2 - \bar{c}a\phi}{\phi^2} > \frac{\frac{1}{2}a - a^2}{\phi^2} > 0$$

by choosing a sufficiently small.

By the concavity of F, we derive for B sufficiently large

$$\begin{aligned} F^{ii}(\underline{U}_{ii} - U_{ii}) &= F^{ii}(\underline{U}_{ii} + 2Bg_{ii} - U_{ii} - 2Bg_{ii}) \\ &\geq F(\underline{U} + 2Bg) - \psi(x, u, \nabla u) - \beta_\varepsilon(u-h) - 2B\sum F^{ii} \\ &\geq F(Bg) - \psi(x, u, \nabla u) - \beta_\varepsilon(u-h) - 2B\sum F^{ii}. \end{aligned} \tag{5.10}$$

We may assume that $|\nabla u|$ is sufficiently large to make

$$|\nabla u|^2 - \nabla u \cdot \nabla h > \frac{1}{2}|\nabla u|^2.$$

Therefore, by (2.11), (5.6), (5.8), (5.9) and (5.10), we have

$$
\begin{aligned}
0 \geq\ & \frac{\phi}{w^2}\big(\nabla_l u \nabla'_l \psi + \psi_u |\nabla u|^2\big) + a\big(\psi(x,u,\nabla u) - \psi(x,u,\nabla \underline{u})\big) \\
& + \frac{\phi}{w^2}\beta'_\varepsilon(u-h)\big(|\nabla u|^2 - \nabla u \cdot \nabla h\big) + \bar{c}'|\nabla\phi|^2 \sum F^{ii} \\
& - \frac{\phi}{w^2}\big(\nabla_l u F^{ii} \nabla'_l A^{ii} + F^{ii} A^{ii}_u |\nabla u|^2\big) - C\phi \sum F^{ii} \\
& + a\Big(F(Bg) - \psi(x,u,\nabla u) - \beta_\varepsilon(u-h) - 2B\sum F^{ii}\Big) \\
\geq\ & aF(Bg) - a\psi(x,u,\nabla \underline{u}) - C\phi|\nabla u|^{\gamma_2 - 2} \\
& + \big(\bar{c}'\phi|\nabla\phi|^2 - C\phi|\nabla u|^{\gamma_1-2} - C\phi - 2aB\big)\sum F^{ii} \\
& + \frac{\phi}{2}\beta'_\varepsilon(u-h) - a\beta_\varepsilon(u-h),
\end{aligned} \tag{5.11}
$$

where $\bar{c}' = a\bar{c}$.

Now by Lemma 5.1, we find that

$$
u - h \leq (c_0\varepsilon)^{1/3} \quad \text{in } M. \tag{5.12}
$$

It follows that

$$
\begin{aligned}
\frac{\phi}{2}\beta'_\varepsilon(u-h) - a\beta_\varepsilon(u-h) &= \frac{(u-h)^2}{\varepsilon}\left(\frac{3\phi}{2} - a(u-h)\right) \\
&\geq \frac{(u-h)^2}{\varepsilon}\big(1 - a(c_0\varepsilon)^{1/3}\big) > 0
\end{aligned}
$$

provided $\varepsilon < \frac{1}{c_0 a^3}$. Thus, we see

$$
\begin{aligned}
0 \geq\ & aF(Bg) - a\psi(x,u,\nabla \underline{u}) - C\phi|\nabla u|^{\gamma_2-2} \\
& + \big(\bar{c}'\phi|\nabla\phi|^2 - C\phi|\nabla u|^{\gamma_1-2} - C\phi - 2aB\big)\sum F^{ii}.
\end{aligned}
$$

By (2.13), choosing B sufficiently large, we may assume $aF(Bg) - a\psi(x,u,\nabla\underline{u}) - C\phi|\nabla u|^{\gamma_2-2} \geq 0$ and we obtain

$$
\bar{c}'\phi|\nabla\phi|^2 - C\phi|\nabla u|^{\gamma_1-2} - C\phi - 2aB \leq 0,
$$

from which we can get a bound for $|\nabla u(x_0)|$. The proof of (5.2) is completed. □

Theorem 5.3. *Assume that (2.1), (2.2), (2.6), (2.11), (2.14), (2.15) and (2.16) hold for $\gamma_1, \gamma_2 < 4$ in (2.11). Then (5.2) holds.*

Proof. By (5.6), we have

$$F^{ii}\nabla_l u\nabla_l U_{ii} = \nabla_l u\psi_{x_l} + \psi_u|\nabla u|^2 + \frac{aw^2}{\phi}\psi_{p_k}\nabla_k\phi, \tag{5.13}$$

provided $|\nabla u|$ is sufficiently large.

In the proof of Theorem 5.2, let $\phi = -u + \sup_M u + 1$. By the concavity of $A^{ii}(x,z,p)$ in p,

$$A^{ii} = A^{ii}(x,u,\nabla u) \le A^{ii}(x,u,0) + A^{ii}_{p_k}(x,u,0)\nabla_k u. \tag{5.14}$$

By (2.6) and (5.14), we have

$$\begin{aligned} -F^{ii}\nabla_{ii}\phi &= F^{ii}\nabla_{ii}u = F^{ii}U_{ii} - F^{ii}A^{ii} \\ &\ge -F^{ii}A^{ii} \ge -C\big(1+|\nabla u|\big)\sum F^{ii}. \end{aligned} \tag{5.15}$$

Thus, from (5.4), (5.5), (5.13), (5.15), (2.11) and (2.15) we drive for $a<1$,

$$\begin{aligned} 0 \ge{}& \frac{(a-a^2)}{\phi^2}F^{ii}|\nabla_i u|^2 + \frac{\nabla_l u\nabla_l'\psi}{w^2} + \psi_u - \frac{a}{\phi}\psi_{p_k}\nabla_k u \\ &+ \frac{a}{\phi}F^{ii}A^{ii}_{p_k}\nabla_k u - F^{ii}A^{ii}_u - F^{ii}\frac{\nabla_l u\nabla_l' A^{ii}}{w^2} - C\big(1+|\nabla u|\big)\sum F^{ii} \\ \ge{}& c_1F^{ii}|\nabla_i u|^2 - C\big(|\nabla u|^{\gamma_2-2} + |\nabla u|^{\gamma} + 1\big) \\ &- C\big(1+|\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma_1-2}\big)\sum F^{ii}, \end{aligned} \tag{5.16}$$

where $c_1 = \min_{\bar M}\frac{(a-a^2)}{\phi^2} > 0$.

Without loss of generality, we assume $\nabla_1 u(x_0) \ge \frac{1}{n}|\nabla u(x_0)| > 0$. Note that $U_{ij}(x_0)$ is diagonal. By (5.3), (5.14) and (2.16) we find

$$\begin{aligned} U_{11} &= -\frac{a}{\phi}|\nabla u|^2 + A^{11} + \frac{1}{\nabla_1 u}\sum_{k\ge 2}\nabla_k u A^{1k} \\ &\le -\frac{a}{\phi}|\nabla u|^2 + C\big(1+|\nabla u| + |\nabla u|^{\gamma-2}\big) < 0 \end{aligned} \tag{5.17}$$

provided $|\nabla u|$ is sufficiently large. Therefore, by (2.14),

$$f_1 \ge \nu_0\left(1+\sum_i^n f_i\right).$$

Thus, by (5.16), we have

$$\begin{aligned}
0 &\geq c_1 F^{11}|\nabla_1 u|^2 - C\big(|\nabla u|^{\gamma_2-2} + |\nabla u|^{\gamma} + 1\big) \\
&\quad - C\big(1 + |\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma_1-2}\big)\sum F^{ii} \\
&\geq \frac{c_1\nu_0}{n^2}\Big(1 + \sum F^{ii}\Big)|\nabla u|^2 - C\big(|\nabla u|^{\gamma_2-2} + |\nabla u|^{\gamma} + 1\big) \\
&\quad - C\big(1 + |\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma_1-2}\big)\sum F^{ii} \\
&= \frac{c_1\nu_0}{n^2}|\nabla u|^2 - C\big(|\nabla u|^{\gamma_2-2} + |\nabla u|^{\gamma} + 1\big) \\
&\quad + \left(\frac{c_1\nu_0}{n^2}|\nabla u|^2 - C\big(1 + |\nabla u| + |\nabla u|^{\gamma} + |\nabla u|^{\gamma_1-2}\big)\right)\sum F^{ii}.
\end{aligned} \tag{5.18}$$

Then we can get a bound $|\nabla u(x_0)| \leq C$ from (5.18). $\square$

Since we have obtained a bound $|u|_{C^1(\bar{M})} \leq C$, there exist uniform constants $\psi_1 > \psi_0 > 0$ independent of ε such that

$$\psi_0 \leq \psi(x, u, \nabla u) \leq \psi_1. \tag{5.19}$$

Let $\mathcal{L}$ be the linear operator locally defined by

$$\mathcal{L}v := F^{ij}\nabla_{ij}v + \big(F^{ij}A^{ij}_{p_k} - \psi_{p_k}\big)\nabla_k v, \quad v \in C^2(M) \tag{5.20}$$

where $A^{ij}_{p_k} \equiv A^{ij}_{p_k}[u] \equiv A^{ij}_{p_k}(x, u, \nabla u)$, $\psi_{p_k} \equiv \psi_{p_k}[u] \equiv \psi_{p_k}(x, u, \nabla u)$. Then in our case, Proposition 2.2 in [10] becomes:

Lemma 5.4. *There exist uniform positive constants R, θ depending only on $\lambda(\nabla^2\underline{u} + A[\underline{u}])$, ψ_0 and $\psi_1 + c_0$ such that*

$$\mathcal{L}(\underline{u} - u) \geq \theta\Big(1 + \sum F^{ii}\Big) - \beta_\varepsilon(u - h) \quad \textit{whenever } \big|\lambda(U)\big| \geq R.$$

The proof is the same as in [10] by using Theorem 3.1, so we omit it here.

6. Estimates for second order derivatives

In this section we will consider the estimates for second order derivative of u_ε and we also drop the subscript ε as usual. Note that there exists a uniform constant C independent of ε such that $\mathrm{tr}(A[u]) \leq C$ on $\bar{M}$. Let ζ be the solution to

$$\triangle\zeta + C = 0$$

in M with $\zeta = \varphi$ on ∂M. Then we get $u \leq \zeta$ in M by the maximum principle since $\triangle u + C > 0$ in M. Since $h > \varphi$ on ∂M we have $h > \zeta \geq u$ in a neighborhood of ∂M in which $\beta_\varepsilon(u - h) \equiv 0$. Therefore, by the arguments of Section 5 in [10], we can obtain a constant C independent of ε such that

$$\left|\nabla^2 u\right| \le C \quad \text{on } \partial M.$$

Set

$$W = \max_{x\in\bar{M},\, \xi\in T_x M,\, |\xi|=1} \big(A^{\xi\xi}(x,u,\nabla u) + \nabla_{\xi\xi}u\big)e^{\phi},$$

where ϕ is a C^2 function to be determined. It suffices to estimate W. We may assume W is achieved at an interior point $x_0 \in M$ and for some unit vector $\xi \in T_{x_0}M$. Choose a smooth orthonormal local frame $e_1, \ldots, e_n$ about x_0 such that $\xi = e_1$, $\nabla_i e_j(x_0) = 0$ and that $U_{ij}(x_0)$ is diagonal. We need only estimate $U_{11}(x_0) > 0$ from above.

At the point x_0 where the function $\log U_{11} + \phi$ (defined near x_0) attains its maximum, we have

$$\frac{\nabla_i U_{11}}{U_{11}} + \nabla_i\phi = 0 \quad \text{for each } i = 1, \cdots, n \tag{6.1}$$

and

$$\frac{\nabla_{ii}U_{11}}{U_{11}} - \left(\frac{\nabla_i U_{11}}{U_{11}}\right)^2 + \nabla_{ii}\phi \le 0. \tag{6.2}$$

Differentiating Eq. (3.4) twice, we obtain at x_0, by (6.1),

$$\begin{aligned} F^{ii}\nabla_{11}U_{ii} + F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} \ge{}& \psi_{p_j}\nabla_j U_{11} + \psi_{p_l p_k}\nabla_{1l}u\nabla_{1k}u + \beta''_\varepsilon(u-h)\big(\nabla_1(u-h)\big)^2 \\ &+ \beta'_\varepsilon(u-h)\nabla_{11}(u-h) - CU_{11} \\ \ge{}& -U_{11}\psi_{p_j}\nabla_j\phi - CU_{11} + \psi_{p_1p_1}U_{11}^2 \\ &+ (U_{11}-C)\beta'_\varepsilon(u-h) \end{aligned} \tag{6.3}$$

provided U_{11} is sufficiently large.

In addition, we have,

$$(\nabla_i U_{11})^2 \le (\nabla_1 U_{1i})^2 + CU_{11}^2, \tag{6.4}$$

$$\nabla_{ii}U_{11} \ge \nabla_{11}U_{ii} + \nabla_{ii}A^{11} - \nabla_{11}A^{ii} - CU_{11}, \tag{6.5}$$

and

$$\begin{aligned} F^{ii}\big(\nabla_{ii}A^{11} - \nabla_{11}A^{ii}\big) \ge{}& F^{ii}\big(A^{11}_{p_j}\nabla_{iij}u - A^{ii}_{p_j}\nabla_{11j}u\big) - CU_{11}\sum F^{ii} \\ &+ F^{ii}\big(A^{11}_{p_ip_i}U_{ii}^2 - A^{ii}_{p_1p_1}U_{11}^2\big) - C\sum F^{ii} \\ \ge{}& U_{11}F^{ii}A^{ii}_{p_j}\nabla_j\phi - CU_{11}\Big(\sum F^{ii} + 1\Big) \\ &- C\sum_{i\ge 2}F^{ii}U_{ii}^2 - U_{11}^2\sum_{i\ge 2}F^{ii}A^{ii}_{p_1p_1} - C\beta'_\varepsilon(u-h). \end{aligned} \tag{6.6}$$

Therefore, by (6.2), (6.3), (6.5) and (6.6), we get

$$\begin{aligned}\mathcal{L}\phi \le{}& E - \psi_{p_1p_1}U_{11} + \frac{C}{U_{11}}\sum F^{ii}U_{ii}^2 + U_{11}\sum_{i\ge 2}F^{ii}A^{ii}_{p_1p_1}\\ &+\left(\frac{C}{U_{11}}-1\right)\beta_\varepsilon'(u-h) + C\sum F^{ii} + C \qquad (6.7)\end{aligned}$$

where

$$E = \frac{1}{U_{11}^2}F^{ii}(\nabla_i U_{11})^2 + \frac{1}{U_{11}}F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl}.$$

Let

$$\phi = \frac{\delta|\nabla u|^2}{2} + b\eta$$

where b, δ are undetermined constants, $0<\delta<1\le b$, and η is a C^2 function which may depend on u but not on its derivatives. We have

$$\begin{aligned}\nabla_i\phi &= \delta\nabla_j u\nabla_{ij}u + b\nabla_i\eta\\ &= \delta\nabla_j u\big(U_{ij} - A^{ij}\big) + b\nabla_i\eta \qquad (6.8)\end{aligned}$$

and

$$\begin{aligned}\nabla_{ii}\phi &= \delta(\nabla_{ij}u)^2 + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta\\ &\ge \frac{\delta}{2}U_{ii}^2 - C\delta + \delta\nabla_j u\nabla_{iij}u + b\nabla_{ii}\eta. \qquad (6.9)\end{aligned}$$

By (6.8),

$$(\nabla_i\phi)^2 \le C\delta^2\big(1+U_{ii}^2\big) + Cb^2 \le C\delta^2U_{ii}^2 + Cb^2. \qquad (6.10)$$

Using (3.3), (6.8) and the equality in (5.6), we have

$$\begin{aligned}F^{ii}\nabla_{iij}u\nabla_j u \ge{}& F^{ii}\nabla_j u\big(\nabla_j U_{ii} - \nabla_j A^{ii}\big) - C|\nabla u|^2\sum F^{ii}\\ \ge{}& \psi_{p_k}\nabla_{jk}u\nabla_j u + \beta_\varepsilon'(u-h)\nabla_j u\nabla_j(u-h) - C\sum F^{ii}\\ &- C|\nabla u|^2\Big(\sum F^{ii}+1\Big) - F^{ii}A^{ii}_{p_k}\nabla_j u\nabla_{jk}u\\ \ge{}& \big(\psi_{p_k} - F^{ii}A^{ii}_{p_k}\big)\nabla_{jk}u\nabla_j u - C\Big(\sum F^{ii}+1\Big) - C\beta_\varepsilon'(u-h). \qquad (6.11)\end{aligned}$$

Therefore, we see

$$\mathcal{L}\phi \ge b\mathcal{L}\eta + \frac{\delta}{2} F^{ii} U_{ii}^2 - C\delta\beta_\varepsilon'(u-h) - C\sum F^{ii} - C. \tag{6.12}$$

Now we estimate E following [9] (see [23] also) by using an inequality shown by Andrews [1] and Gerhardt [7]. For fixed $0 < s \le 1/3$, let

$$J = \{i : U_{ii} \le -sU_{11}\}, \qquad K = \{i : U_{ii} > -sU_{11}\}.$$

We have (see [9])

$$\begin{aligned}
-F^{ij,kl}\nabla_1 U_{ij}\nabla_1 U_{kl} &\ge \sum_{i\neq j} \frac{F^{ii} - F^{jj}}{U_{jj} - U_{ii}} (\nabla_1 U_{ij})^2 \\
&\ge 2\sum_{i\ge 2} \frac{F^{ii} - F^{11}}{U_{11} - U_{ii}} (\nabla_1 U_{i1})^2 \\
&\ge \frac{2}{(1+s)U_{11}} \sum_{i\in K} \big(F^{ii} - F^{11}\big)(\nabla_1 U_{i1})^2 \\
&\ge \frac{2(1-s)}{(1+s)U_{11}} \sum_{i\in K} \big(F^{ii} - F^{11}\big)\big((\nabla_i U_{11})^2 - CU_{11}^2/s\big).
\end{aligned} \tag{6.13}$$

By (6.13), (6.10) and (6.1),

$$\begin{aligned}
E &\le \frac{1}{U_{11}^2} \sum_{i\in J} F^{ii}(\nabla_i U_{11})^2 + C\sum_{i\in K} F^{ii} + \frac{CF^{11}}{U_{11}^2} \sum_{i\notin J} (\nabla_i U_{11})^2 \\
&\le \sum_{i\in J} F^{ii}(\nabla_i \phi)^2 + C\sum F^{ii} + CF^{11}\sum (\nabla_i\phi)^2 \\
&\le Cb^2 \sum_{i\in J} F^{ii} + C\delta^2 \sum F^{ii}U_{ii}^2 + C\sum F^{ii} + C\big(\delta^2 U_{11}^2 + b^2\big)F^{11}.
\end{aligned} \tag{6.14}$$

Therefore, by (6.7), (6.12) and (6.14), we derive

$$\begin{aligned}
b\mathcal{L}\eta \le{}& \left(C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}\right) F^{ii}U_{ii}^2 + Cb^2 \sum_{i\in J} F^{ii} + C\Big(\sum F^{ii} + 1\Big) \\
&+ Cb^2 F^{11} + \left(\frac{C}{U_{11}} + C\delta - 1\right)\beta_\varepsilon'(u-h).
\end{aligned} \tag{6.15}$$

By doing a minimization over δ, we can guarantee that

$$\max\left\{C\delta^2 - \frac{\delta}{2}, C\delta - 1\right\}$$

is negative. We choose this δ and then let

$$c_1 := -\frac{1}{2}\max\left\{C\delta^2 - \frac{\delta}{2}, C\delta - 1\right\}$$

so that $c_1 > 0$. Then we may assume

$$\max\left\{C\delta^2 - \frac{\delta}{2} + \frac{C}{U_{11}}, \frac{C}{U_{11}} + C\delta - 1\right\} \le -c_1$$

for otherwise we have $U_{11} \le \frac{C}{c_1}$ and we are done.

Now let $\eta = \underline{u} - u$, by Lemma 5.4, we have, when $U_{11} \ge R$,

$$(b\theta - C)\Big(1 + \sum F^{ii}\Big) \le -c_1 F^{ii} U_{ii}^2 + Cb^2 F^{11} + Cb^2 \sum_{i \in J} F^{ii} + b\beta_\varepsilon(u-h) - c_1\beta'_\varepsilon(u-h).$$

Choosing b sufficiently large such that $b\theta - C > 0$, and we then have

$$0 \le -c_1 F^{ii} U_{ii}^2 + Cb^2 F^{11} + Cb^2 \sum_{i \in J} F^{ii} + b\beta_\varepsilon(u-h) - c_1\beta'_\varepsilon(u-h). \tag{6.16}$$

By (5.12), we have

$$b\beta_\varepsilon(u-h) - c_1\beta'_\varepsilon(u-h) = \frac{(u-h)^2}{\varepsilon}\big(b(u-h) - 3c_1\big) \le 0 \tag{6.17}$$

provided $\varepsilon \le \frac{1}{c_0}(\frac{3c_1}{b})^3$. It follows from (6.16) and (6.17) that

$$-c_1 F^{ii} U_{ii}^2 + Cb^2 \sum_{i \in J} F^{ii} + Cb^2 F^{11} \ge 0$$

when ε is small. Note that $|U_{ii}| \ge sU_{11}$ for $i \in J$. It follows that

$$\big(Cb^2 - c_1 s^2 U_{11}^2\big) \sum_{i \in J} F^{ii} + \big(Cb^2 - c_1 U_{11}^2\big) F^{11} \ge 0.$$

This implies a bound $U_{11}(x_0) \le \frac{Cb^2}{c_1 s^2}$ or $U_{11}(x_0) \le \frac{Cb^2}{c_1}$.

7. Existence of smooth solution to (3.4)

In this section, we prove the existence of smooth solution to (3.4) by using the method of continuity and a degree theory argument based on the estimates we have established. The proof is standard so we only provide a sketch here. For more details we refer the readers to [3] and [8]. First we note that by the Evans–Krylov theory (see [14] for example) and Schauder theory we can get higher estimates of u_ε which may depend on ε. For example, we can obtain

$$|u_\varepsilon|_{C^{5,\alpha}(\bar{M})} \le C = C(\varepsilon), \tag{7.1}$$

where $0 < \alpha < 1$.

Case 1. $A = A(x, p)$, $\psi = \psi(x, p)$.

For each fixed $t \in [0, 1]$, consider the Dirichlet problem

$$\begin{aligned} f\big(\lambda(U)\big) &= t\psi(x, \nabla u) + (1-t) f\big(\lambda(\underline{U})\big) + \beta_\varepsilon(u - h) \quad \text{in } M, \\ u &= \varphi \quad \text{on } \partial M, \end{aligned} \tag{7.2}$$

where $U = \nabla^2 u + A(x, \nabla u)$ and $\underline{U} = \nabla^2 \underline{u} + A(x, \nabla \underline{u})$. Note that $\underline{u}$ is a subsolution to (7.2). Similar to (7.1), any admissible solution $u_\varepsilon^t \in C^\infty(\bar{M})$ satisfies the *a priori* estimates

$$|u_\varepsilon^t|_{C^{5,\alpha}(\bar{M})} \le C = C(\varepsilon)$$

since $u_\varepsilon^t \ge \underline{u}$ by the maximum principle. Obviously, $\underline{u}$ is the unique solution to (7.2) when $t = 0$. By the method of continuity, for each $t \in [0, 1]$, there exists a unique admissible solution to (7.2) in $C^\infty(\bar{M})$.

Case 2. The general case: $A = A(x, z, p)$, $\psi = \psi(x, z, p)$.

For $R > 0$, let

$$\mathcal{Q}_R = \big\{ v \in C^{5,\alpha}(\bar{M}) : |v|_{C^{5,\alpha}(\bar{M})} < R,\ v > 0 \text{ in } M, v|_{\partial M} = 0 \text{ and } \nabla_\nu v > 0 \text{ on } \partial M \big\},$$

where $\alpha \in (0, 1)$, and ν is the unit interior normal to ∂M. For $t \in [0, 1]$ and fixed $v \in \overline{\mathcal{Q}}_R$, consider the Dirichlet problem

$$\begin{aligned} f\big(\lambda\big(\nabla^2 u + A^t(x, \nabla u)\big)\big) &= \psi^t(x, \nabla u) \quad \text{in } M, \\ u &= \varphi \quad \text{on } \partial M, \end{aligned} \tag{7.3}$$

where

$$A^t(x, \nabla u) = tA(x, \underline{u} + v, \nabla u) + (1-t)A(x, \underline{u}, \nabla u)$$

and

$$\psi^t(x, \nabla u) = t\psi(x, \underline{u} + v, \nabla u) + \frac{1-t}{2} f\big(\lambda(\underline{U})\big) + \beta_\varepsilon(u - h).$$

We see that $\underline{u}$ is a subsolution of (7.3) by (2.10) and according to Case 1, there exists a unique solution $u^t \in C^{5,\alpha}(\bar{M})$ satisfying $u^t \ge \underline{u}$ in M to (7.3) for each $t \in [0, 1]$.

Consider the map $T^t v = u^t - \underline{u}$. We see that

$$\big|u^t\big|_{C^{5,\alpha}(\bar{M})} \le C = C(R),$$

and

$$u^t > \underline{u} \quad \text{in } M, \qquad \nabla_\nu u^t > \nabla_\nu \underline{u} \quad \text{on } \partial M$$

by the maximum principle and the Hopf lemma.

Let $v \in \mathcal{Q}_R$ be an arbitrary solution to $T^t v = v$ which means that $\underline{u} + v$ is the unique solution to (7.3) and $\lambda[\nabla^2(\underline{u} + v) + A^t(x, \nabla(\underline{u} + v))] \in \Gamma$.

We claim that $\underline{u} + v$ is admissible in M. Indeed, for any $x \in M$, we may assume that $A(x, \underline{u} + v, \nabla(\underline{u} + v)) - A(x, \underline{u}, \nabla(\underline{u} + v))$ is diagonal at x by choosing a smooth local frame $e_1, \dots, e_n$ about x. We can derive from (2.10) that $A^{ii}(x, \underline{u} + v, \nabla(\underline{u} + v)) \geq A^{ii}(x, \underline{u}, \nabla(\underline{u} + v))$ for each $i = 1, \dots, n$ since $v > 0$ in M. It follows that, at x,

$$\begin{aligned}
&\big\{\nabla_{ij}(\underline{u} + v) + A^{ij}\big(x, \underline{u} + v, \nabla(\underline{u} + v)\big)\big\} \\
&\quad - \big\{\nabla_{ij}(\underline{u} + v) + \big(A^t\big)^{ij}\big(x, \nabla(\underline{u} + v)\big)\big\} \\
&= (1 - t)\big\{A^{ij}\big(x, \underline{u} + v, \nabla(\underline{u} + v)\big) - A^{ij}\big(x, \underline{u}, \nabla(\underline{u} + v)\big)\big\} \geq 0,
\end{aligned}$$

where $(A^t)^{ij}(x, \nabla(\underline{u} + v)) = A^t(x, \nabla(\underline{u} + v))(e_i, e_j)$ (see (2.7)). Therefore, $\nabla^2(\underline{u} + v) + A(x, \underline{u} + v, \nabla(\underline{u} + v)) \geq \nabla^2(\underline{u} + v) + A^t(x, \nabla(\underline{u} + v))$ and $\underline{u} + v$ is admissible in M.

By Remark 4.2 and the arguments in Section 4, there exists a positive constant C independent of R such that

$$|\underline{u} + v|_{C^1(\bar{M})} \leq C.$$

Note that the constants in the second derivative estimates depend only on $|\underline{u} + v|_{C^1(\bar{M})}$ and other known data. We then obtain a positive constant C_0 independent of R such that

$$|\underline{u} + v|_{C^2(\bar{M})} \leq C_0$$

and thus $|v|_{C^{5,\alpha}(\bar{M})} \leq C_1$ independent of R. It follows that the equation $T^t v - v = 0$ admits no solution on the boundary of $\mathcal{Q}_R$ when R is sufficiently large.

Thus the degree

$$\deg\big(I - T^t, \mathcal{Q}_R, 0\big) = \gamma \tag{7.4}$$

is well defined and independent of t for R sufficiently large. When $t = 0$ there exists a unique function $v^0 = u^0 - \underline{u}$ satisfying $v^0 - T^0 v^0 = 0$ which is a regular point of $I - T^0$. Consequently $\gamma = \pm 1$, and $T^t v - v = 0$ has a solution $v^t \in \mathcal{Q}_R$ for all $t \in [0, 1]$. The function $u^1 = \underline{u} + v^1$ is then a solution of (3.4). Then we obtain a smooth solution u_ε to (3.4). Furthermore, (3.5) holds.

Thus, there exist a subsequence u_{ε_k} and a function $u \in C^{1,1}(\bar{M})$ such that

$$u_{\varepsilon_k} \to u \text{ in } C^{1,\alpha}(\bar{M}), \quad \forall \alpha \in (0, 1), \quad \text{as } \varepsilon_k \to 0.$$

Similar to [24], we can see that $\underline{u} \leq u \leq h$ and u is a (viscosity) solution of (1.1) and (1.2). Furthermore, $u \in C^{3,\alpha}(E)$ for any $\alpha \in (0, 1)$, by the Evans–Krylov theory and Schauder theory. We then complete the proof of Theorem 2.2.

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