ARTICLE IN PRESS YJDEQ:8851

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Journal of Differential Equations

www.elsevier.com/locate/jde

Nonsimultaneous blowup for a complex valued semilinear heat equation

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Received 2 May 2016; revised 21 November 2016

Abstract

This paper is concerned with blow-up solutions for a complex valued semilinear heat equation. Nonsimultaneous blow-up solutions predicted in our previous work are constructed by a fixed point argument.

Keywords: Semilinear heat equation; Simultaneous blow-up

1. Introduction

We study blow-up solutions of a one dimensional complex-valued semilinear heat equation.

$$z_t = z_{xx} + z^2, \tag{1}$$

where $z(x,t)$ is a complex valued function and $x \in \mathbb{R}$. This equation is a special case of Constantin–Lax–Majda equation with a viscosity term, which is a one dimensional model for the 3D Navier–Stokes equations (see [3,9–11,4]). If we write $z = a + ib$ ($a, b \in \mathbb{R}$), (1) is rewritten as a parabolic system.

$$a_t = a_{xx} + a^2 - b^2, \qquad b_t = b_{xx} + 2ab.$$

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http://dx.doi.org/10.1016/j.jde.2017.05.024
0022-0396/

Please cite this article in press as: J. Harada, Nonsimultaneous blowup for a complex valued semilinear heat equation, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.05.024

When z is real-valued (i.e. $b \equiv 0$), this system is reduced to a single parabolic equation $a_t = a_{xx} + a^2$. For such a single case, blow-up problems have been extensively studied by many authors. In this paper, we focus on blow-up solutions of (1) for the case where z is not real-valued. It is known that this system possesses finite time blow-up solutions (see [4,8]). Particularly Nouaili–Zaag [8] show the existence of simultaneous blow-up solutions of (1). Following their work, the author [6] extends their results and discuss the possibilities of nonsimultaneous blow-up.

Let $z(x,t)$ be a blow-up solution of (1) and $\xi \in \mathbb{R}$ be its blow-up point. We introduce the self-similar transformation.

$$U(y,s) = (T-t)z(\xi + e^{-\frac{s}{2}}y, t), \qquad t = T - e^{-s}. \tag{2}$$

This function U satisfies

$$U_s = AU - U + U^2, \qquad y \in \mathbb{R},\ s > -\log T,$$

where $AU = U_{yy} - \frac{y}{2}U_y$. We define $L^2_\rho(\mathbb{R}) = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R});\ \|f\|^2_\rho = (f,f)_\rho < \infty \right\}$, where the inner product of $L^2_\rho(\mathbb{R})$ is given by $(f_1, f_2)_\rho = \int_{-\infty}^{\infty} f_1(y) f_2(y) e^{-|y|^2/4} dy$. The operator $A = \frac{d^2}{dy^2} - \frac{y}{2}\frac{d}{dy}$ is self-adjoint in $L^2_\rho(\mathbb{R})$. We denote by H_n the nth eigenfunction of $-AH = \lambda H$ in $L^2_\rho(\mathbb{R})$. Its eigenvalue is given by $\frac{n}{2}$. We now recall our previous work.

Theorem 1.1 *(see Theorem 1.4 [6]). Let $z(x,t)$ be a blow-up solution of (1) and $\xi \in \mathbb{R}$ be its blow-up point. Define $U(y,s)$ as (2) and put $U = u + iv$. If two conditions*:

$$\text{(A1)} \qquad \sup_{0<t<T} (T-t)\|z(t)\|_\infty < \infty \quad \textit{(Type I)},$$

$$\text{(A2)} \qquad \lim_{s\to\infty} \|v(s)\|_\rho = 0$$

are satisfied, one of the following cases occurs.

$$\text{(C1)} \quad \begin{cases} u = 1 - \mathsf{c}_0 s^{-1} H_2 + O\left(s^{-2}\log s\right) & \text{in } L^2_\rho(\mathbb{R}), \\ v = \mathsf{c}_2 s^{-m} e^{-\frac{(m-2)s}{2}} H_m + O\left(s^{-(m+1)} e^{-\frac{(m-2)s}{2}} \log s\right) & \text{in } L^2_\rho(\mathbb{R}), \end{cases} \quad (m \ge 2)$$

$$\text{(C2)} \quad \begin{cases} u = 1 - \mathsf{c}_1 e^{-(k-1)s} H_{2k} + O\left(e^{-\frac{(2k-1)s}{2}}\right) & \text{in } L^2_\rho(\mathbb{R}), \\ v = \mathsf{c}_2 e^{-\frac{(m-2)s}{2}} H_m + O\left(e^{-\frac{(m-1)s}{2}}\right) & \text{in } L^2_\rho(\mathbb{R}), \end{cases} \quad (m \ge 2k,\ k \ge 2)$$

where $\mathsf{c}_0 = \frac{\sqrt[4]{\pi}}{2}$, $\mathsf{c}_1 > 0$, $\mathsf{c}_2 \neq 0$.

Theorem 1.2 *(see Theorem 1.7 [6]). If* (C1) *with $m \ge 4$ or* (C2) *with $m \ge 4k$ in Theorem 1.1 occurs, b is bounded for $(x,t) \in (\xi - \epsilon, \xi + \epsilon) \times (0,T)$ for some $\epsilon > 0$.*

The case (C1) with $m = 2$ in Theorem 1.1 corresponds to blow-up solutions constructed in [8]. A goal of this paper is to construct all blow-up solutions described in Theorem 1.1, which proves the existence of nonsimultaneous blow-up solutions of (1) from Theorem 1.2.

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Theorem 1.3. *All blow-up solutions described in Theorem 1.1 exist.*

2. Preliminary

Let $z(x,t)$ be a blow-up solution of (1) and $\xi \in \mathbb{R}$ be its blow-up set. Without loss of generality, we can assume $\xi = 0$. Furthermore define $U(y,s)$ as (2). It satisfies

$$U_s = AU - U + U^2, \qquad y \in \mathbb{R},\ s > s_1, \tag{3}$$

where $s_1 = -\log T$ and $AU = U_{yy} - \frac{y}{2}U_y$. Put $\rho(y) = e^{-y^2/4}$ and $L^2_\rho(\mathbb{R}) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R});\ \|f\|^2_\rho = (f,f)_\rho < \infty\}$, where the inner product of $L^2_\rho(\mathbb{R})$ is defined by

$$(f_1, f_2)_\rho = \int_{-\infty}^{\infty} f_1(y) f_2(y) \rho(y) dy.$$

The eigenfunction H_n and the eigenvalue λ_n of $-AH = \lambda H$ in $L^2_\rho(\mathbb{R})$ are given by

$$H_n(y) = \mathsf{C}_n h_n\left(\frac{y}{2}\right), \qquad \lambda_n = \frac{n}{2} \qquad (n = 0, 1, 2, \cdots),$$

where h_n is the Hermite polynomial given by

$$h_n(y) = (-1)^n e^{y^2}\left(\frac{d^n}{dy^n} e^{-y^2}\right)$$

and C_n is the normalization constant such that $\|H_n\|_\rho = 1$. The functional space $L^2_\rho(\mathbb{R})$ is spanned by eigenfunctions $\{H_n\}_{n \geq 0}$. Without loss of generality, we can assume that the coefficient of y^n in $H_n(y)$ is positive. Here we recall the following inequality (see (6) [6] p. 4218).

$$\int_{-\infty}^{\infty} y^2 f^2 \rho dy < c\left(\|f\|^2_\rho + \|f_y\|^2_\rho\right). \tag{4}$$

3. Construction of blow-up solutions

The proof of Theorem 1.3 is based on ideas in [1], which are developed in [2,7] for a single case. Following [8], we apply this method to (1). To construct all blow-up solutions described in Theorem 1.1, we introduce new functional spaces (see Definition 3.1, Definition 3.2) and provide corresponding a priori estimates. By the use of comparison arguments (see Lemma 3.3) and the inequality (4), we can provide simpler and more elementary proof than that of [8]. The proof in this section is also valid for a single case.

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3.1. Strategy of the proof

Our proof uses the construction method in [1] (see also [2,7,8]). For simplicity, we consider the case (C1). Let

$$f=\frac{1}{1+\mathsf{c}_0 s^{-1}H_2},\qquad g=\mathsf{c}_2\left(\frac{s^{-m}e^{-\frac{(m-2)s}{2}}}{(1+\mathsf{c}_0 s^{-1}H_2)^2}\right)H_m, \tag{5}$$

where $\mathsf{c}_0=\frac{\sqrt[4]{\pi}}{2}$ and c_2 is a nonzero constant. These are global profile functions obtained in Theorem 1.5 in [6]. A goal is to construct a solution $U=u+iv$ of (3) satisfying

$$\lim_{s\to\infty}s\|u(s)-f\|_\rho=0,\qquad \lim_{s\to\infty}s^m e^{\frac{(m-2)s}{2}}\|v(s)-g\|_\rho=0.$$

Since $f=1-\mathsf{c}_0 s^{-1}H_2+O(s^{-2})$ in $L^2_\rho(\mathbb{R})$ and $g=\mathsf{c}_2 s^{-m}e^{-(m-2)s/2}H_m+O(s^{-(m+1)}e^{-(m-2)s/2})$ in $L^2_\rho(\mathbb{R})$, this proves Theorem 1.3. To construct such solutions, we study the behavior of $(p,q)=(u-f,v-g)$. Then p and q satisfy

$$p_s=Ap+p+N_1,\qquad q_s=Aq+q+N_2, \tag{6}$$

where N_1 and N_2 are negligible terms. We expand p and q by the eigenfunctions $\{H_n\}_{n\ge0}$ of $-AH=\lambda H$.

$$p(s)=\sum_{n=0}^{\infty}p_i(s)H_i,\qquad q(s)=\sum_{n=0}^{\infty}q_i(s)H_i.$$

We here consider only $p_0(s)$, $p_1(s)$, $q_0(s)$, $\cdots$, $q_{m-1}(s)$. Since other components $\{p_i(s)\}_{i\ge2}$ and $\{q_j(s)\}_{j\ge m}$ are controlled by the spectrum properties of A (see Lemma 3.4 and Lemma 3.5) and the maximum principle (see Lemma 3.3), it is enough to control those $(m+2)$ components. Finally this $(m+2)$ dimensional problem is solved by a fixed point argument (see Section 3.3).

3.2. A functional setting for the case (C1)

Let f,g be given by (5). It is known that those global profile functions f,g give better approximations than expansions in Theorem 1.1 (see Theorem 1.5 [6]). These satisfy

$$f_s=Af-f+f^2+\Lambda_1,\qquad g_s=Ag-g+2fg+\Lambda_2,$$

where Λ_1,Λ_2 are

$$\Lambda_1=\mathsf{c}_0 s^{-2}H_2\left(1+\mathsf{c}_0 s^{-1}H_2\right)^{-2}-2\mathsf{c}_0^2 s^{-2}(H_2')^2\left(1+\mathsf{c}_0 s^{-1}H_2\right)^{-3},$$
$$\Lambda_2=-\frac{m}{s}g-6\left(\mathsf{c}_0 s^{-1}H_2'\right)^2 f^2g+2\left(\mathsf{c}_0 s^{-2}H_2\right)fg+4(\mathsf{c}_0 s^{-1}H_2')\left(\frac{H_m'}{H_m}\right)fg.$$

Since H_n is the nth degree polynomial, it holds that

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$$|\Lambda_1| < cs^{-2}\left(1+y^2\right), \qquad |\Lambda_2| < cs^{-(m+1)}e^{-\frac{(m-2)s}{2}}\left(1+|y|^m+s^{-1}|y|^{m+2}\right).$$

Furthermore we define $p, q, p_i, q_i, p_\perp$ and $q_\perp$ by

$$p = u - f = \sum_{i=0}^{2} p_i H_i + p_\perp, \qquad q = v - g = \sum_{j=0}^{m} q_j H_j + q_\perp.$$

For $\Omega \subset \mathbb{R}^2$, we define $BC(\Omega) = L^\infty(\Omega) \cap C(\Omega)$.

Definition 3.1. Let $s_2 \geq s_1$ and $V(s_1, s_2)$ be a subset in $BC(\mathbb{R} \times [s_1, s_2])^2$ whose elements consist of functions (p, q) satisfying the following inequalities for $s \in (s_1, s_2)$.

$$s^{\frac{3}{2}}\left(|p_0| + |p_1|\right), \ s^{\frac{5}{4}}|p_2|, \ s^{\frac{3}{2}}\|p_\perp\|_\rho < 1,$$
$$s^{\frac{1}{4}}\left(|q_0| + \cdots + |q_{m-1}|\right), \ |q_2|, \ s^{\frac{1}{4}}\|q_\perp\|_\rho < s^{-m}e^{-\frac{(m-2)s}{2}}.$$

We remark that $V(s_1, s_2)$ does not require any restrictions on L^∞-norms of (p, q). This is the difference between this definition and Definition 3.1 in [8]. We will see that L^∞-norms of (p, q) are automatically controlled as far as the solution belongs to $V(s_1, s_2)$ (see Lemma 3.3). This simplifies the proof. If we can find a solution in $V(s_1, s_2)$ with $s_2 = \infty$, this gives the desired blow-up solution which behaves like (C1) in Theorem 1.1. To construct such solutions, we consider the following form of initial data.

$$\begin{aligned} u|_{s=s_1} &= f|_{s=s_1} + \left(\frac{d_0 H_0 + d_1 s_1^{-1} H_1}{1 + s_1^{-2} H_2}\right), \\ v|_{s=s_1} &= \frac{g|_{s=s_1}}{1 + s_1^{-2m} H_{2m}} + \left(\frac{b_0 H_0 + b_1 s_1^{-1} H_1 + \cdots + b_{m-1} s_1^{-(m-1)} H_{m-1}}{1 + s_1^{-2m} H_{2m}}\right), \end{aligned} \tag{7}$$

where $d = (d_0, d_1) \in \mathbb{R}^2$ and $b = (b_0, b_1, \cdots, d_{m-1}) \in \mathbb{R}^m$ are parameters.

3.3. *Proof of Theorem 1.3 for the case (C1)*

For given $(d, b) \in \mathbb{R}^2 \times \mathbb{R}^m$, we denote by $U = u + iv$ a solutions of (3) with initial data (7). Let

$$p = u - f, \qquad q = v - g.$$

From this definition, p and q depend on $(d, b) \in \mathbb{R}^2 \times \mathbb{R}^m$. We denote by $(d, b) \in D(s_1, s_2)$ if $(p, q) \in V(s_1, s_2)$. Therefore it is sufficient to prove the existence of parameters $(d, b) \in D(s_1, s_2)$ with $s_2 = \infty$. To find such parameters, we apply a fixed point argument. From Lemma 3.1, we find that $D(s_1, s_1)$ is an open star-shaped set with respect to the origin. Furthermore since $(d, b) = (0, 0) \in D(s_1, s_1)$, it is clear that $D(s_1, s_2)$ is not empty if s_2 is sufficiently close to s_1. To derive a contradiction, we assume that $D(s_1, s_2) = \emptyset$ for some $s_2 > s_1$. Then there exists $s_3 \in (s_1, s_2]$ such that $\overline{D(s_1, \tau)} \neq \emptyset$ for $\tau \in [s_1, s_3)$ and $D(s_1, \tau) = \emptyset$ for $\tau > s_3$. In this situation, for any $(d, b) \in \overline{D(s_1, s_1)}$, there exists $\tau \in [s_1, s_3]$ such that $(p(\tau), q(\tau)) \in$

$\partial V(s_1,\tau)$, namely $|p_i(\tau)| = \tau^{-3/2}$ for some $i \in \{0,1\}$ or $|q_j(\tau)| = \tau^{-m-\frac{1}{4}}e^{-(m-2)\tau/2}$ for some $j \in \{0,\cdots,m-1\}$. Other possibilities are excluded from Lemmas 3.4–3.5. We now define a mapping $P : \overline{D(s_1,s_1)} \to \partial([-1,1]^2 \times [-1,1]^m)$ by

$$P(d,b) = \left(\tau^{\frac{3}{2}}d(\tau), \tau^{m+\frac{1}{4}}e^{\frac{(m-2)\tau}{2}}b(\tau)\right).$$

From Lemma 3.1 and Lemma 3.6, we find that this mapping P is continuous on $\overline{D(s_1,s_1)}$. However since $D(s_1,s_1)$ is a star-shaped set, it is impossible. Therefore we conclude that $D(s_1,s_2) \neq \emptyset$ for any $s_2 > s_1$. As a consequence, we can choose $(d_n,b_n) \in \mathbb{R}^2 \times \mathbb{R}^m$ such that $(d_n,b_n) \in D(s_1,s_2)$ with $s_2 = s_1 + n$. Let $U_n = u_n + iv_n$ be a corresponding solution of (3). Since $p_n = u_n - f$ and $q_n = v_n - g$ satisfy estimates in Definition 3.1 for $s \in (s_1,s_2)$, by taking a subsequence, we obtain $U = \lim_{n\to\infty} U_n$, which is the desired solution. This proves Theorem 1.3 for the case (C1).

3.4. A priori estimates for the case (C1)

In this subsection, we use the same notations as in Section 3.3. For given $(d,b) \in \mathbb{R}^2 \times \mathbb{R}^m$, we denote by $U = u + iv$ a solutions of (3) with initial data (7) and put $p = u - f$, $q = v - g$. However (u,v) and (p,q) depend on $(d,b) \in \mathbb{R}^2 \times \mathbb{R}^m$, we do not write their dependence for simplicity.

Lemma 3.1. *It holds that*

$$(p|_{s=s_1}, H_i)_\rho = (1+o(1))d_i s_1^{-i} \quad (i=0,1),$$
$$|(p|_{s=s_1}, H_2)_\rho| + \|p_\perp|_{s=s_1}\|_\rho < cs_1^{-2}(|d_0| + |d_1|s_1^{-1}),$$
$$(q|_{s=s_1}, H_j)_\rho = b_j s_1^{-j} + O\left(s_1^{-3m}e^{-\frac{(m-2)s}{2}}\right) + O\left(s_1^{-2m}\sum_{j=0}^{m-1}|b_j|s_1^{-j}\right) \quad (j=0,\cdots m-1),$$
$$|(q|_{s=s_1}, H_m)_\rho| + \|q_\perp|_{s=s_1}\|_\rho < cs_1^{-3m}e^{-\frac{(m-2)s}{2}} + cs_1^{-2m}\sum_{j=0}^{m-1}|b_j|s_1^{-j}.$$

Proof. We note from (7) that

$$p|_{s=s_1} = \frac{d_0H_0 + d_1s_1^{-1}H_1}{1+s_1^{-2}H_2},$$
$$q|_{s=s_1} = -\left(\frac{s_1^{-2m}H_{2m}}{1+s_1^{-2m}H_{2m}}\right)g|_{s=s_1} + \frac{b_0H_0 + \cdots + b_{m-1}s_1^{-(m-1)}H_{m-1}}{1+s_1^{-2m}H_{2m}}.$$

Since $(H_{n_1}, H_{n_2}) = 0$ if $n_1 \neq n_2$ and $1 + s_1^{-2n}H_{2n} > \frac{1}{2}$ on $\mathbb{R}$ for any fixed $n \in \mathbb{N}$, we easily obtain conclusion. $\square$

The following lemma follows from the definition of $V(s_1,s_2)$ and Lemma 3.1. We skip the proof.

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Lemma 3.2. *Let* $(p, q) \in V(s_1, s_1)$. *Then it holds that*

$$|d_i| < 2s_1^{-(\frac{3}{2}-i)} \quad (i = 0, 1), \qquad |b_j| < 2s_1^{-(m+\frac{1}{4}-j)} e^{-\frac{(m-2)s}{2}} \quad (j = 0, \cdots m-1).$$

Lemma 3.3. *Let* $(p, q) \in V(s_1, s_2)$. *Then it holds that for* $s \in (s_1, s_2)$ *and* $y \in \mathbb{R}$

$$\|p(s)\|_\infty < 2, \qquad |q(y, s)| < ce^{-\frac{(m-2)s}{2}}(1 + |y|^m).$$

Proof. From the choice of initial data, we easily see that

$$1 - u|_{s=s_1} = \frac{\mathsf{c}_0 s^{-1} H_2(1 + s_1^{-2} H_2 + d_0 H_0 + d_1 s_1^{-1} H_1) - (d_0 H_0 + d_1 s_1^{-1} H_1))}{(1 + \mathsf{c}_0 s_1^{-1} H_2)(1 + s_1^{-2} H_2)}.$$

Since $H_n(y)$ is the n the polynomial $H_n(y) = a_{nn} y^n + \cdots + a_{n0}$ with $a_{nn} > 0$ (see Section 2), we get from Lemma 3.2 that for large $y > 0$

$$\begin{aligned} 1 + s_1^{-2} H_2 + d_0 H_0 + d_1 s_1^{-1} H_1 &> 1 + \frac{a_{22}}{2} s_1^{-2} y^2 - 2a_{00}|d_0| - 2a_{11}|d_1| s_1^{-1} y \\ &> 1 + \frac{a_{22}}{2} s_1^{-2} y^2 - 4a_{00} s_1^{-\frac{3}{2}} - 4a_{11} s_1^{-\frac{3}{2}} y \\ &> \frac{a_{22}}{2}\left(\frac{y}{s_1} - 4\left(\frac{a_{11}}{a_{22}}\right) s_1^{-\frac{1}{2}}\right)^2 + 1 - 4a_{00} s_1^{-\frac{3}{2}} - 8\left(\frac{a_{11}^2}{a_{22}}\right) s_1^{-1}. \end{aligned}$$

Therefore there exists $y_1 > 0$ such that $1 - u|_{s=s_1} > 0$ for $y > y_1$. Furthermore since $u = 1 - \frac{\sqrt[4]{\pi}}{2} s^{-1} H_2(1 + o(1))$ in $L^2_\rho(\mathbb{R})$, it holds that $u|_{y=y_1} < 1$ for $s \in (s_1, s_2)$. Since $\bar{u} = 1$ becomes a super-solution, a comparison argument shows $u < 1$ for $y > y_1$, $s \in (s_1, s_2)$. By the same way, we can show that $u < 1$ for $y < -y_1$, $s \in (s_1, s_2)$. Since $\|p\|_\rho < cs^{-5/4}$, the first estimate is derived. Next we prove the second estimate. Let $y_2 > y_1$ be a point such that $H_m(y) > 1$ for $y > y_2$. From the definition of $V(s_1, s_2)$, we find that $v|_{y=y_2} < cs^{-m} e^{-(m-2)s/2}$ for $s \in (s_1, s_2)$. Furthermore from the definition of g and Lemma 3.2, we easily see that $v|_{s=s_1} < cs^{-m} e^{-(m-2)s/2} H_m$ for $y > y_2$. Let $\bar{v} = e^{-(m-2)s/2} H_m$. Since $|u| < 1$ for $|y| > y_1$, we find that $\bar{v}_s - A\bar{v} + \bar{v} - 2u\bar{v} > 0$ for $y > y_2$. Therefore a comparison argument shows that $v < e^{-(m-2)s/2} H_m$ for $y > y_2$, $s \in (s_1, s_2)$. By the same way as the above, we obtain $|v| < ce^{-(m-2)s/2}(1 + |y|^m)$ for $y \in \mathbb{R}$, $s \in (s_1, s_2)$. □

Next we derive a priori estimates of solutions in $L^2_\rho(\mathbb{R})$.

Lemma 3.4. *Let* $(p, q) \in V(s_1, s_2)$. *Then it holds that for* $s \in (s_1, s_2)$

$$\|p_\perp\|_\rho < cs^{-\frac{7}{4}}, \qquad |p_2| < cs^{-\frac{3}{2}}.$$

Proof. Let $(p, q) \in V(s_1, s_2)$. We first estimate $p_\perp$. Multiplying $p_s = Ap + p - 2(1 - f)p + p^2 - \Lambda_1 - v^2$ by $p_\perp$ and integrating over $\mathbb{R}$, we see that

$$\frac{1}{2}\frac{d}{ds}\|p_\perp\|_\rho^2 < -\|\nabla p_\perp\|_\rho^2 + \|p_\perp\|_\rho^2 + c\int_{\mathbb{R}} |(1-f)|\left(\sum_{i=0}^{2}|p_iH_i|^2 + p_\perp^2\right)\rho\, dy$$
$$+\int_{\mathbb{R}}\left(\sum_{i=0}^{2}|p_iH_i|^2 + p_\perp^2\right)|p_\perp|\rho\, dy + \int_{\mathbb{R}}(|\Lambda_1| + v^2)|p_\perp|\rho\, dy.$$

From Lemmas 3.1–3.3 and $|\Lambda_1| < cs^{-2}(1+y^2)$, we get from (4) that

$$\frac{1}{2}\frac{d}{ds}\|p_\perp\|_\rho^2 < -\frac{1}{4}\|p_\perp\|_\rho^2 + c\left(s^{-1}\sum_{i=0}^{2}|p_i|^2 + \sum_{i=0}^{2}|p_i|^4\right) + cs^{-4} < -\frac{1}{4}\|p_\perp\|_\rho^2 + cs^{-\frac{7}{2}}.$$

Therefore it follows from Lemma 3.1 and Lemma 3.2 that

$$\|p_\perp\|_\rho^2 < e^{-\frac{s-s_1}{2}}\|p_\perp|_{s=s_1}\|_\rho^2 + cs^{-\frac{7}{2}} < c\left(\left(\frac{s}{s_1}\right)^7 e^{-\frac{s_1}{2}(\frac{s}{s_1}-1)}\right)s^{-7} + cs^{-\frac{7}{2}} < cs^{-\frac{7}{2}}.$$

Next we provide estimates of p_2. Since $\int_{\mathbb{R}} H_2^3\rho\, dy = \mathsf{c}_0^{-1}$ (see p. 829 in [5]) and

$$\left|\int_{\mathbb{R}}(1-f)pH_2\rho\, dy - p_2\mathsf{c}_0 s^{-1}\left(\int_{\mathbb{R}} H_2^3\rho\, dy\right)\right| < cs^{-1}(|p_0| + |p_1| + \|p_\perp\|_\rho) + cs^{-2}|p_2|,$$

we get

$$|\dot{p}_2 + 2s^{-1}p_2| < cs^{-1}(|p_0| + |p_1| + \|p_\perp\|_\rho) + cs^{-2}|p_2| + |(p^2, H_2)_\rho| + |(\Lambda_1, H_2)_\rho|$$
$$+ |(v^2, H_2)_\rho|.$$

By the explicit form of H_2, we find that $(\mathsf{c}_0H_2 - 2\mathsf{c}_0(H_2')^2, H_2)_\rho = 0$. This implies $|(\Lambda_1, H_2)| < cs^{-3}$. Furthermore we easily see that $|(p^2, H_2)_\rho| < c(|p_0|^2 + |p_1|^2 + |p_2|^2) + |(p_\perp^2, H_2)_\rho|$ and

$$|(p_\perp^2, H_2)_\rho| < \|p_\perp\|_\rho^{\frac{7}{4}}\left(\int_{\mathbb{R}} p_\perp^2 H_2^8\rho\, dy\right)^{\frac{1}{8}} < c\|p_\perp\|_\rho^{\frac{7}{4}}. \tag{8}$$

The last inequality follows from Lemma 3.3. Therefore we get

$$|\dot{p}_2 + 2s^{-1}p_2| < cs^{-\frac{5}{2}} + cs^{-\frac{21}{8}} < cs^{-\frac{5}{2}},$$

which implies

$$|p_2| < \left(\frac{s_1}{s}\right)^2|p_2|_{s=s_1}| + cs^{-\frac{3}{2}}.$$

From Lemma 3.1 and Lemma 3.2, we conclude $|p_2| < cs_1^{-\frac{3}{2}}s^{-2} + cs^{-\frac{3}{2}} < cs^{-\frac{3}{2}}$. □

Lemma 3.5. *Let $(p, q) \in V(s_1, s_2)$. Then it holds that for $s \in (s_1, s_2)$*

$$|q_m|,\ s^{\frac{1}{4}}\|q_\perp\|_\rho < \frac{1}{2}s^{-m}e^{-\frac{(m-2)s}{2}}.$$

Proof. Multiplying $q_s = Aq + q - 2(1-f)q + 2pq + 2pg - \Lambda_2$ by $q_\perp$ and integrating over $\mathbb{R}$, we see that

$$\frac{1}{2}\frac{d}{ds}\|q_\perp\|_\rho^2 < -\|\nabla q_\perp\|_\rho^2 + \|q_\perp\|_\rho^2 + c\int_{\mathbb{R}} |(1-f)|\left(\sum_{j=0}^{m}|q_jH_j|^2 + q_\perp^2\right)\rho\, dy$$
$$+ c\int_{\mathbb{R}}|p|\left(\sum_{j=0}^{m}|q_jH_j|^2 + q_\perp^2\right)\rho\, dy + (2\|pg\|_\rho + \|\Lambda_2\|_\rho)\|q_\perp\|_\rho.$$

The term $\|pg\|_\rho$ is estimated as (8).

$$\int_{\mathbb{R}} p^2g^2\rho\, dy < \left(\int_{\mathbb{R}} p^2\rho\, dy\right)^{\frac{7}{8}}\left(\int_{\mathbb{R}} p^2g^{16}\rho\, dy\right)^{\frac{1}{8}} < c\|p\|_\rho^{\frac{7}{4}}\left(\int_{\mathbb{R}} g^{16}\rho\, dy\right)^{\frac{1}{8}}.$$

By the same way as in the proof of the previous lemma, we get

$$\frac{1}{2}\frac{d}{ds}\|q_\perp\|_\rho^2 < -\left(\frac{m-\frac{3}{2}}{2}\right)\|q_\perp\|_\rho^2 + cs^{-1}\sum_{j=0}^{m}q_j^2 + c\|p\|_\rho\sum_{j=0}^{m}q_j^2 + c\left(\|pg\|_\rho^2 + \|\Lambda_2\|_\rho^2\right)$$
$$< -\left(\frac{m-\frac{3}{2}}{2}\right)\|q_\perp\|_\rho^2 + cs^{-2m-1}e^{-(m-2)s}.$$

From Lemma 3.1 and Lemma 3.2, it holds that

$$\begin{aligned}\|q_\perp\|_\rho^2 &< e^{-(m-\frac{3}{2})(s-s_1)}\|q_\perp|_{s=s_1}\|_\rho^2 + cs^{-2m-1}e^{-(m-2)s}\\ &< ce^{-(m-\frac{3}{2})(s-s_1)}s_1^{-3m}e^{-(m-2)s_1} + cs^{-2m-1}e^{-(m-2)s}\\ &< cs_1^{-m}s^{-2m-\frac{1}{2}}e^{-(m-2)s}\left(\frac{s}{s_1}\right)^{2m+\frac{1}{2}}e^{-\frac{s_1}{2}(\frac{s}{s_1}-1)} + cs^{-2m-1}e^{-(m-2)s}\\ &< cs_1^{-m}s^{-2m-\frac{1}{2}}e^{-(m-2)s} + cs^{-2m-1}e^{-(m-2)s}.\end{aligned}$$

Next we estimate q_m. Since $\int_{\mathbb{R}} H_2H_m^2\rho\, dy = m/\mathsf{c}_0$ (see proof of Lemma 3.4 in [6]), we get

$$\left|\dot{q}_m + \left(\frac{(m-2)}{2} + ms^{-1}\right)q_m\right| < cs^{-1}\left(\sum_{j=0}^{m-1}|q_j| + \|q_\perp\|_\rho\right)$$
$$+ cs^{-2}|q_2| + |(pq, H_m)_\rho| + \|p\|_\rho\|gH_m\|_\rho + \|\Lambda_2\|_\rho.$$

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Since $|(pq,H_m)_\rho|<c\|p\|_\rho\|q\|_\rho+|(p_\perp q_\perp,H_m)_\rho|$, by the same way as (8), it holds that

$$\begin{aligned}|(pq,H_m)_\rho|&<cs^{-m-\frac{5}{4}}e^{-\frac{(m-2)s}{2}}+c\|q_\perp\|_\rho\|p_\perp\|_\rho^{\frac{3}{4}}\left(\int_{\mathbb{R}}p_\perp^2H_m^8\rho\,dy\right)^{\frac{1}{8}}\\&<cs^{-m-\frac{5}{4}}e^{-\frac{(m-2)s}{2}}+cs^{-m-\frac{11}{8}}e^{-\frac{(m-2)s}{2}}.\end{aligned}$$

Therefore we obtain

$$\left|\dot{q}_m+\left(\frac{(m-2)}{2}+ms^{-1}\right)q_m\right|<cs^{-m-\frac{5}{4}}e^{-\frac{(m-2)s}{2}}.$$

From Lemma 3.1 and Lemma 3.2, we conclude

$$\begin{aligned}|q_m|&<\left(\frac{s_1}{s}\right)^me^{-\frac{(m-2)(s-s_1)}{2}}|q_m|_{s=s_1}|+cs^{-m-\frac{5}{4}}e^{-\frac{(m-2)s}{2}}\\&<cs_1^{-2m}s^{-m}e^{-\frac{(m-2)s}{2}}+cs^{-m-\frac{5}{4}}e^{-\frac{(m-2)s}{2}}.\end{aligned}$$

The proof is completed. □

As a consequence of the above lemmas, we obtain the transverse property of solutions in $V(s_1,s_2)$.

Lemma 3.6. *Let $(p,q)\in V(s_1,s_2)$. If $|p_i(s_2)|=s_2^{-3/2}$ for some $i\in\{0,1\}$, there exists $\epsilon>0$ such that $|p_i|>s^{-3/2}$ for $s\in(s_2,s_2+\epsilon)$. If $|q_j(s_2)|=s_2^{-m-\frac{1}{4}}e^{-(m-s)s_2}$ for some $j\in\{0,\cdots,m-1\}$, there exists $\epsilon>0$ such that $|q_j|>s^{-m-\frac{1}{4}}e^{-(m-s)s}$ for $s\in(s_2,s_2+\epsilon)$.*

Proof. Since $p_s=Ap+p-2(1-f)p+p^2-\Lambda_1-v^2$, we get

$$\begin{aligned}\left|\frac{d}{ds}(s^{\frac{3}{2}}p_i)-\left(1-\frac{i}{2}\right)s^{\frac{3}{2}}p_i\right|&<cs^{\frac{1}{2}}\|p\|_\rho+cs^{\frac{3}{2}}|(p^2,H_2)_\rho|+s^{\frac{3}{2}}\|\Lambda_1\|_\rho+s^{\frac{3}{2}}|(v^2,H_2)_\rho|\\&<cs^{-\frac{3}{4}}.\end{aligned}$$

Therefore if $p_i(s_2)=s_2^{-3/2}$ for some $i\in\{0,1\}$, it holds that

$$\frac{d}{ds}(s^{\frac{3}{2}}p_i)|_{s=s_2}>1-\frac{i}{2}-cs_2^{-\frac{3}{4}}>0.$$

For the case $p_i(s_2)=-s_2^{-3/2}$ for some $i\in\{0,1\}$, by the same reason, we obtain $\frac{d}{ds}(s^2p_i)|_{s=s_2}<0$. Therefore the first part is proved. Since $q_s=Aq+q-2(1-f)q+2pq+2pg-\Lambda_2$, we see that

$$\left|\frac{d}{ds}(s^{m+\frac{1}{4}}e^{\frac{(m-2)s}{2}}q_j)-\left(\frac{m}{2}-\frac{j}{2}\right)s^{m+\frac{1}{4}}e^{\frac{(m-2)s}{2}}q_j\right|<cs^{-\frac{3}{4}}.$$

The second part follows from this estimate. □

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3.5. Proof of Theorem 1.3 for the case (C2)

Since the argument for the case (C1) can be directly extended to that for the case (C2), we skip the proofs except for Lemma 3.10. The only difference between two cases is whether global profile functions f, g include the polynomial decay s^{-1} in their expression or not (see (5), (9)). We first define global profile functions f, g by

$$f = \frac{1}{1+\mathsf{c}_1 e^{-(k-1)s} H_{2k}}, \qquad g = \frac{\mathsf{c}_2 e^{-\frac{(m-2)s}{2}} H_m}{(1+\mathsf{c}_1 e^{-(k-1)s} H_{2k})^2} \quad (m \geq 2k,\ k \geq 2), \tag{9}$$

where $\mathsf{c}_1 > 0$ and $\mathsf{c}_2 \neq 0$. These satisfy

$$f_s = Af - f + f^2 + \Lambda_1, \qquad g_s = Ag - g + 2fg + \Lambda_2,$$

where Λ_1, Λ_2 are given by

$$\begin{aligned}
\Lambda_1 &= -2(1+\mathsf{c}_1 e^{-(k-1)s} H_{2k})^{-3} (\mathsf{c}_1 e^{-(k-1)s} H'_{2k})^2, \\
\Lambda_2 &= 4\left(c_1 e^{-(k-1)s} H'_{2k}\right)\left(\frac{H'_m}{H_m}\right) fg - 6\left(c_1 e^{-(k-1)s} H'_{2k}\right)^2 f^2 g.
\end{aligned}$$

We easily see that

$$\begin{aligned}
|\Lambda_1| &< ce^{-2(k-1)s}(1+y^{4k-2}), \\
|\Lambda_2| &< ce^{-(k+\frac{m}{2}-2)s}(1+y^{2k+m-2}) + ce^{-(2k+\frac{m}{2}-3)s}(1+y^{4k+m-2}).
\end{aligned}$$

We define $p, q, p_i, q_i, p_\perp$ and $q_\perp$ by

$$p = u - f = \sum_{i=0}^{2k-1} p_i H_i + p_{2k} H_{2k} + p_\perp, \qquad q = v - g = \sum_{j=0}^{m-1} q_j H_j + q_m H_m + q_\perp.$$

Definition 3.2. Let $s_2 \geq s_1$ and $V(s_1, s_2)$ be a subset in $BC(\mathbb{R} \times [s_1, s_2])^2$ whose elements consist of functions (p, q) satisfying the following inequalities for $s \in (s_1, s_2)$.

$$\|p\|_\rho < e^{-(k-1)s}, \qquad \|q\|_\rho < e^{-\frac{(m-2)s}{2}}.$$

Initial data is defined by

$$u|_{s=s_1} = f|_{s=s_1} + \left(\frac{d_0 H_0 + d_1 e^{-\frac{s_1}{4}} H_1 + \cdots + d_{2k-1} e^{-\frac{(2k-1)s_1}{4}} H_{2k-1}}{1+e^{-ks} H_{4k}}\right),$$

$$v|_{s=s_1} = \frac{g|_{s=s_1}}{1+e^{-(m-2)s_1} H_{2m}} + \left(\frac{b_0 H_0 + b_1 e^{-\frac{s_1}{4}} H_1 + \cdots + b_{m-1} e^{-\frac{(m-1)s_1}{4}} H_{m-1}}{1+e^{-ms} H_{4m}}\right).$$

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Lemma 3.7. *It holds that*

$$(p|_{s=s_1}, H_i)_\rho = d_i e^{-\frac{is_1}{4}} + O\left(e^{-ks}\sum_{i=0}^{2k-1}|d_i|e^{-\frac{is_1}{4}}\right) \quad (i=0,\cdots,2k-1),$$

$$|(p|_{s=s_1}, H_2)_\rho| + \|p_\perp|_{s=s_1}\|_\rho < ce^{-ks_1}\sum_{i=0}^{2k-1}|d_i|e^{-\frac{is_1}{4}},$$

$$(q|_{s=s_1}, H_j)_\rho = b_j e^{-\frac{js_1}{4}} + O\left(e^{-\frac{3(m-2)s_1}{2}}\right) + O\left(e^{-ms_1}\sum_{j=0}^{m-1}|b_j|e^{-\frac{js_1}{4}}\right) \quad (j=0,\cdots m-1),$$

$$|(q|_{s=s_1}, H_m)_\rho| + \|q_\perp|_{s=s_1}\|_\rho < ce^{-\frac{3(m-2)s_1}{2}} + ce^{-ms_1}\sum_{j=0}^{m-1}|b_j|e^{-\frac{js_1}{4}}.$$

Lemma 3.8. *Let $(p,q)\in V(s_1,s_1)$. Then it holds that*

$$|d_i| < 2e^{-(k-1-\frac{i}{4})s_1} \quad (i=0,\cdots,2k-1), \qquad |b_j| < 2e^{-(\frac{m-2}{2}-\frac{j}{4})s_1} \quad (j=0,\cdots m-1).$$

Lemma 3.9. *Let $(p,q)\in V(s_1,s_2)$. Then it holds that for $s\in(s_1,s_2)$ and $y\in\mathbb{R}$*

$$\|p(s)\|_\infty < 2, \qquad |q(y,s)| < ce^{-\frac{(m-2)s}{2}}(1+|y|^m).$$

Lemma 3.10. *Let $(p,q)\in V(s_1,s_2)$. Then it holds that for $s\in(s_1,s_2)$*

$$|p_{2k}| + \|p_\perp\|_\rho < \frac{1}{2}e^{-(k-1)s}.$$

Proof. Let $(p,q)\in V(s_1,s_2)$. We first estimate $p_\perp$. Since p satisfies $p_s = Ap + p - 2(1-f)p + p^2 - \Lambda_1 - v^2$, we get

$$\frac{1}{2}\frac{d}{ds}\|p_\perp\|_\rho^2 < -\|\nabla p_\perp\|_\rho^2 + \|p_\perp\|_\rho^2 + c\int_{\mathbb{R}}|1-f|\left(\sum_{i=0}^{k}|p_iH_i|^2 + p_\perp^2\right)\rho\, dy$$
$$+ \int_{\mathbb{R}}\left(\sum_{i=0}^{k}|p_iH_i|^2 + p_\perp^2\right)|p_\perp|\rho\, dy + \int_{\mathbb{R}}(|\Lambda_1| + v^2)|p_\perp|\rho\, dy.$$

By the explicit form of f and $|p|<2$, it holds that

$$\int_{\mathbb{R}}|1-f|p_\perp^2\rho\, dy < ce^{-(k-1)s}\left(\int_{|y|<s} + \int_{|y|>s}\right)\frac{(1+y^{2k})p_\perp^2}{1+\mathsf{c}_1e^{-(k-1)s}H_{2k}}\rho\, dy$$
$$< cs^{2k}e^{-(k-1)s}\|p_\perp\|_\rho^2 + ce^{-(k-1)s}\int_{|y|>s}\left(p - \sum_{i=0}^{2k-1}p_iH_i\right)^2 y^{2k}\rho\, dy$$

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$$
\begin{aligned}
&< cs^{2k} e^{-(k-1)s} \|p_\perp\|_\rho^2 + ce^{-(k-1)s} \int\limits_{|y|>s} y^{6k-2} \rho \, dy \\
&< cs^{2k} e^{-(k-1)s} \|p_\perp\|_\rho^2 + cs^{6k-3} e^{-(k-1)s} e^{-\frac{s^2}{4}}.
\end{aligned}
$$

From Lemma 3.9, we easily see that $\int_{\mathbb{R}} v^4 \rho \, dy < ce^{-2(m-2)s}$. Therefore since $(p, q) \in V(s_1, s_2)$ and $m \geq 2k$, it holds that

$$
\frac{1}{2} \frac{d}{ds} \|p_\perp\|_\rho^2 < -\left(k - \frac{3}{4}\right) \|p_\perp\|_\rho^2 + ce^{-3(k-1)}.
$$

As a consequence, from Lemma 3.7 and Lemma 3.8, we obtain

$$
\begin{aligned}
\|p_\perp\|_\rho^2 &< e^{-2(k-\frac{3}{4})(s-s_1)} \|p_\perp|_{s=s_1}\|_\rho^2 + ce^{-(k-\frac{3}{2})s_1} e^{-2(k-\frac{3}{4})s} \\
&< ce^{-2ks_1} e^{-2(k-1)s} + ce^{-(k-\frac{3}{2})s_1} e^{-2(k-\frac{3}{4})s}.
\end{aligned}
$$

Next we estimate p_{2k}. Since $p_s = Ap + p + 2(-1 + f)p + p^2 - \Lambda_1 - v^2$ and $m \geq 2k$, we get

$$
|\dot{p}_{2k} + (k-1)p_{2k}| < ce^{-(k-1)s} \|p\|_\rho + \left| \int\limits_{\mathbb{R}} p^2 H_{2k} \rho \, dy \right| + c\|\Lambda_1\|_\rho + \left| \int\limits_{\mathbb{R}} v^2 H_{2k} \rho \, dy \right|.
$$

Here we note that

$$
\left| \int\limits_{\mathbb{R}} p^2 H_{2k} \rho \, dy \right| < \left(\int\limits_{\mathbb{R}} p^2 \rho \, dy \right)^{\frac{3}{4}} \left(\int\limits_{\mathbb{R}} p^2 H_{2k}^2 \rho \, dy \right)^{\frac{1}{4}} < c\|p\|_\rho^{\frac{3}{2}}.
$$

Therefore since $m \geq 2k$, we obtain

$$
|\dot{p}_{2k} + (k-1)p_{2k}| < ce^{-\frac{3(k-1)s}{2}}.
$$

Integrating this inequality, we conclude

$$
\begin{aligned}
|p_{2k}| &< e^{-(k-1)(s-s_1)} |p_{2k}|_{s=s_1}| + ce^{-\frac{1}{2}(k-1)s_1} e^{-(k-1)s} \\
&< ce^{-ks_1} e^{-(k-1)s} + ce^{-\frac{1}{2}(k-1)s_1} e^{-(k-1)s}. \quad \square
\end{aligned}
$$

Lemma 3.11. *Let $(p, q) \in V(s_1, s_2)$. Then it holds that for $s \in (s_1, s_2)$*

$$
|q_m| + \|q_\perp\|_\rho < \frac{1}{2} e^{-\frac{(m-2)s}{2}}.
$$

Lemma 3.12. *Let $(p, q) \in V(s_1, s_2)$. If $|p_i(s_2)| = e^{-(k-1)s_2}$ for some $i \in \{0 \cdots, 2k-1\}$, there exists $\epsilon > 0$ such that $|p_i| > e^{-(k-1)s}$ for $s \in (s_2, s_2 + \epsilon)$. If $|q_j(s_2)| = e^{-(m-2)s_2/2}$ for some $j \in \{0, \cdots, m-1\}$, there exists $\epsilon > 0$ such that $|q_j| > e^{-(m-2)s/2}$ for $s \in (s_2, s_2 + \epsilon)$.*

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Acknowledgments

The author would like to thank the referee for his/her careful reading of this manuscript. The author is partly supported by Grant-in-Aid for Young Scientists (B) No. 26800065.

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