



Wong–Zakai approximations and attractors for stochastic reaction–diffusion equations on unbounded domains[☆]

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Abstract

In this paper, we study the Wong–Zakai approximations given by a stationary process via the Wiener shift and their associated long term behavior of the stochastic reaction–diffusion equation driven by a white noise. We first prove the existence and uniqueness of tempered pullback attractors for the Wong–Zakai approximations of stochastic reaction–diffusion equation. Then, we show that the attractors of Wong–Zakai approximations converges to the attractor of the stochastic reaction–diffusion equation for both additive and multiplicative noise.

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1. Introduction

In this paper, we study the Wong–Zakai approximations given by a stationary process via the Wiener shift and their associated long term behavior of the stochastic reaction–diffusion equation driven by white noise:

$$du = (\Delta u - \lambda u + f(t, x, u) + g(t, x))dt + h(t, x, u) \circ dW, \quad t > \tau, x \in \mathbb{R}^n \quad (1.1)$$

where λ is a positive constant, τ is the initial time, f and h are nonlinear terms, $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$, $W = W(t, \omega)$ is a one-dimensional two-sided Brownian motion, the symbol \circ indicates that the equation is understood in the sense of Stratonovich integration.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the classical Wiener probability space, where

$$\Omega = C_0(\mathbb{R}, \mathbb{R}) := \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$$

with the open compact topology, \mathcal{F} is its Borel σ -algebra, and \mathbb{P} is the Wiener measure. The Brownian motion has the form $W(t, \omega) = \omega(t)$. Consider the Wiener shift θ_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

It is known that the probability measure \mathbb{P} is an ergodic invariant measure for θ_t . $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ forms a metric dynamical system, see Arnold [1].

For each $\delta \in \mathbb{R}$, let $\mathcal{G}_\delta : \Omega \rightarrow \mathbb{R}$ denote the random variable

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta} \omega(\delta).$$

Then we have

$$\mathcal{G}_\delta(\theta_t \omega) = \frac{1}{\delta} (\omega(t + \delta) - \omega(t)). \quad (1.2)$$

From the properties of Brownian motions, it follows that $\mathcal{G}_\delta(\theta_t \omega)$ is a stationary stochastic process with a normal distribution and is unbounded in t for almost all ω . $\mathcal{G}_\delta(\theta_t \omega)$ may be viewed as an approximation of white noise in the sense

$$\lim_{\delta \rightarrow 0} \sup_{t \in [0, T]} \left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - W(t, \omega) \right| = 0, a.s.$$

for each $T > 0$, see [26]. This approximation was used in [27, 34] to study chaotic behavior of random dynamical systems.

We consider the following Wong–Zakai approximations of equation (1.1) driven by a multiplicative noise of $\mathcal{G}_\delta(\theta_t \omega)$:

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + h(t, x, u) \mathcal{G}_\delta(\theta_t \omega), \quad t > \tau, x \in \mathbb{R}^n, \quad (1.3)$$

which is a random partial differential equation driven by a stationary process.

As demonstrated in later sections, the random system (1.3) possesses some striking advantages over the stochastic system (1.1). For instance, we will show, in this paper, the random equation (1.3) generates a random dynamical system (continuous cocycle) for a wide class of nonlinearity h and it has a tempered pullback random attractor in $L^2(\mathbb{R}^n)$. In contrast, for the stochastic equation (1.1), one can only prove it generates a continuous cocycle when h is either u or independent of u . Of course, the existence of pullback random attractors for (1.1) can only be established for such a particular h (see, e.g., [3,45] for the autonomous stochastic case and [44,43] for the non-autonomous stochastic case). Recently, some progress on the existence of random dynamical systems has been made for random differential equations by using rough path analysis [4], and for a class of stochastic PDEs driven by a fractional Brownian motion by using fractional calculus [13]. In general, the existence of random dynamical systems is unknown for stochastic PDEs (see, e.g., [10]), let alone the existence of pullback random attractors. Despite the significant difference of equations (1.1) and (1.3), we are still able to find close relations between their solutions under certain conditions. Actually, for linear multiplicative noise and additive white noise, we will prove the solutions of (1.3) converge to that of (1.1) in $L^2(\mathbb{R}^n)$ when $\delta \rightarrow 0$. Based on this result, we will further prove the convergence of pullback random attractors of (1.3) as $\delta \rightarrow 0$, and show their limit is the attractor of (1.1) in an appropriate sense.

Using deterministic differential equations to approximate stochastic differential equations was introduced by Wong and Zakai in their pioneer work [46,47] in which they studied both piecewise linear approximations and piecewise smooth approximations for one-dimensional Brownian motions. Their work was later extended to stochastic differential equations of higher dimension, for example, by McShane [28], Stook and Varadhan [35], Sussmann [36,37], Ikeda, Nakao and Yamato [19], Ikeda and Watanabe [20], and recently by Kelly and Melbourne [21], and Shen and Lu [34] in which the same approximations as this paper were studied. The results of the Wong–Zakai approximations have also been generalized to stochastic differential equations driven by martingales and semimartingales, see for example, Konecny [22], Kurtz–Protter [24,25], Nakao [29], Nakao–Yamato [30] and Protter [32].

There are also a large number of publications on Wong–Zakai approximations of solutions for stochastic partial differential equations, see for example, Bally–Millet–Sanz-Sole [2], Brzezniak–Capinski–Flandoli [5], Brzezniak–Flandoli [6], Deya–Jolis–Quer-Sardanyons [9], Ganguly [12], Grecksch–Schmalfuss [14], Gyongy [15,16], Gyongy–Shmatkov [17], Hairer–Pardoux [18], Nowak [31], Schmalfus [33], Tessitore–Zabczyk [38], and Twardowska [39–42].

2. Theory of non-autonomous random attractors

In this section, we recall some results on pullback random attractors for non-autonomous random dynamical systems from [44,43]. The reader is referred to [7,8,11,33] for the theory of pullback random attractors for autonomous random dynamical systems.

Let $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be a metric dynamical system, (X, d) a complete separable metric space, and \mathcal{D} a collection of some families of nonempty subsets of X .

Definition 2.1. A mapping $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \rightarrow X$ is called a continuous cocycle on X over $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

- (i) $\Phi(\cdot, \tau, \cdot, \cdot): \mathbb{R}^+ \times \Omega \times X \rightarrow X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable;
- (ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X ;

- (iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;
- (iv) $\Phi(t, \tau, \omega, \cdot) : X \rightarrow X$ is continuous.

If, in addition, there exists a positive number T such that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot),$$

then Φ is called a periodic cocycle on X with period T .

Definition 2.2. A mapping $\psi : \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow X$ is called a complete orbit of Φ if for every $t \in \mathbb{R}^+$, $\tau, s \in \mathbb{R}$, and $\omega \in \Omega$, the following holds:

$$\Phi(t, \tau + s, \theta_s \omega, \psi(s, \tau, \omega)) = \psi(t + s, \tau, \omega). \quad (2.1)$$

If, in addition, there exists $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ such that $\psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_t \omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$, then ψ is called a \mathcal{D} -complete orbit of Φ .

Definition 2.3. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a \mathcal{D} -pullback absorbing set for Φ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for every $D \in \mathcal{D}$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T. \quad (2.2)$$

If, in addition, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $K(\tau, \omega)$ is a closed nonempty subset of X and K is measurable in ω with respect to \mathcal{F} , then we say K is a closed measurable \mathcal{D} -pullback absorbing set for Φ .

Definition 2.4. The cocycle Φ is said to be \mathcal{D} -pullback asymptotically compact in X if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X \quad (2.3)$$

whenever $t_n \rightarrow +\infty$, and $x_n \in B(\tau - t_n, \theta_{-t_n} \omega)$ with $\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.5. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is called a \mathcal{D} -pullback random attractor for Φ if for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

- (i) \mathcal{A} is measurable in ω with respect to \mathcal{F} and $\mathcal{A}(\tau, \omega)$ is compact in X .
- (ii) \mathcal{A} is invariant: $\Phi(t, \tau, \omega, \mathcal{A}(\tau, \omega)) = \mathcal{A}(\tau + t, \theta_t \omega)$, $\forall t \geq 0$.
- (iii) \mathcal{A} attracts every member of \mathcal{D} : for every $D \in \mathcal{D}$,

$$\lim_{t \rightarrow +\infty} d_X(\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)), \mathcal{A}(\tau, \omega)) = 0,$$

where d_X is the Hausdorff semi-distance in X . If, in addition, there exists $T > 0$ such that

$$\mathcal{A}(\tau + T, \omega) = \mathcal{A}(\tau, \omega), \quad \forall \tau \in \mathbb{R}, \forall \omega \in \Omega,$$

then we say \mathcal{A} is periodic with period T .

We borrow the following result for non-autonomous random dynamical systems from [43]. Similar results can be found in [7,8,11,33] for autonomous random systems.

Proposition 2.1. *Let \mathcal{D} be an inclusion closed collection of some families of nonempty subsets of X , and Φ be a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. Then Φ has a \mathcal{D} -pullback random attractor \mathcal{A} in \mathcal{D} if Φ is \mathcal{D} -pullback asymptotically compact in X and Φ has a closed measurable \mathcal{D} -pullback absorbing set K in \mathcal{D} . The \mathcal{D} -pullback random attractor \mathcal{A} is unique and is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\mathcal{A}(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D \in \mathcal{D}} \Omega(D, \tau, \omega) = \{\psi(0, \tau, \omega) : \psi \text{ is a } \mathcal{D}\text{-complete orbit of } \Phi\}, \quad (2.4)$$

where $\Omega(K)$ and $\Omega(D)$ are the omega-limit sets of K and D , respectively. If, in addition, both Φ and K are T -periodic, then so is the attractor \mathcal{A} .

Note that the measurability of the attractor \mathcal{A} with respect to \mathcal{F} can be found in [44]. Next, we recall a result regarding upper semicontinuity of non-autonomous pullback random attractors. Suppose Λ is an interval such that $\lambda_0 \in \Lambda$ and for each $\lambda \in \Lambda$, Φ_λ is a continuous cocycle on X over \mathbb{R} and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$. Assume that for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\lambda_n \in \Lambda$ with $\lambda_n \rightarrow \lambda_0$, and $x_n, x \in X$ with $x_n \rightarrow x$,

$$\lim_{n \rightarrow \infty} \Phi_{\lambda_n}(t, \tau, \omega, x_n) = \Phi_{\lambda_0}(t, \tau, \omega, x). \quad (2.5)$$

For each $\lambda \in \Lambda$, let \mathcal{D}_λ be a collection of some families of subsets of X . Suppose for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exists $R_{\lambda_0}(\tau, \omega) > 0$ such that

$$D = \{D(\tau, \omega) = \{x \in X : \|x\|_X \leq R_{\lambda_0}(\tau, \omega)\} : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_{\lambda_0}. \quad (2.6)$$

Given $\lambda \in \Lambda$, let $\mathcal{A}_\lambda \in \mathcal{D}$ and $K_\lambda \in \mathcal{D}$ be a \mathcal{D}_λ -pullback random attractor and a \mathcal{D}_λ -pullback absorbing set of Φ_λ , respectively, such that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{\lambda \rightarrow \lambda_0} \|K_\lambda(\tau, \omega)\|_X \leq R_{\lambda_0}(\tau, \omega), \quad (2.7)$$

where $R_{\lambda_0}(\tau, \omega)$ is as in (2.6). We finally assume that for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\text{if } \lambda_n \rightarrow \lambda_0 \text{ and } x_n \in \mathcal{A}_{\lambda_n}(\tau, \omega), \text{ then } \{x_n\}_{n=1}^\infty \text{ is precompact in } X. \quad (2.8)$$

The following result on upper semicontinuity of non-autonomous pullback random attractors was proved in [44].

Proposition 2.2. *Suppose that (2.5) and (2.7)–(2.8) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$d_X(\mathcal{A}_\lambda(\tau, \omega), \mathcal{A}_{\lambda_0}(\tau, \omega)) \rightarrow 0 \quad \text{as } \lambda \rightarrow \lambda_0.$$

3. Random dynamical systems for Wong–Zakai approximations

In this section, we first define a continuous cocycle for random reaction–diffusion equations driven by approximate white noise (Wong–Zakai approximations), and then prove the existence of pullback random attractors.

3.1. Continuous cocycles

Let $\tau, \delta \in \mathbb{R}$ with $\delta \neq 0$. Consider the following Wong–Zakai approximation of the non-autonomous stochastic reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + h(t, x, u)\mathcal{G}_\delta(\theta_t \omega), \quad t > \tau, x \in \mathbb{R}^n, \quad (3.1)$$

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \quad (3.2)$$

where $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$. The nonlinearity function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t, x, s) \leq -\alpha_1 |s|^p + \psi_1(t, x), \quad (3.3)$$

$$|f(t, x, s)| \leq \alpha_2 |s|^{p-1} + \psi_2(t, x), \quad (3.4)$$

$$\frac{\partial}{\partial s} f(t, x, s) \leq -\alpha_3 |s|^{p-2} + \psi_3(t, x), \quad (3.5)$$

where $p > 2$, α_1, α_2 and α_3 are positive constants, $\psi_1 \in L^1_{loc}(\mathbb{R}, L^1(\mathbb{R}^n))$, $\psi_2 \in L^{p_1}_{loc}(\mathbb{R}, L^{p_1}(\mathbb{R}^n))$ with $\frac{1}{p_1} + \frac{1}{p} = 1$, and $\psi_3 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n))$. Let $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$|h(t, x, s)| \leq \beta_1(t, x)|s|^{q-1} + \beta_2(t, x), \quad (3.6)$$

$$\left| \frac{\partial}{\partial s} h(t, x, s) \right| \leq \beta_3(t, x)|s|^{q-2} + \beta_4(t, x), \quad (3.7)$$

where $2 \leq q < p$, $\beta_1 \in L^{\frac{p}{p-q}}_{loc}(\mathbb{R}, L^{\frac{p}{p-q}}(\mathbb{R}^n))$ and $\beta_2 \in L^{p_1}_{loc}(\mathbb{R}, L^{p_1}(\mathbb{R}^n))$ and $\beta_3, \beta_4 \in L^\infty_{loc}(\mathbb{R}, L^\infty(\mathbb{R}^n))$.

Let $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ be the metric dynamical system mentioned in the introduction. Then there exists a $\{\theta_t\}_{t \in \mathbb{R}}$ -invariant subset $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$,

$$\frac{\omega(t)}{t} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty. \quad (3.8)$$

For the sake of convenience, from now on, we will abuse the notation slightly and write the space $\tilde{\Omega}$ as Ω . Given $\delta \neq 0$, recall that the random variable \mathcal{G}_δ is defined by

$$\mathcal{G}_\delta(\omega) = \frac{\omega(\delta)}{\delta}, \quad \text{for all } \omega \in \Omega. \quad (3.9)$$

From (3.9) we find

$$\mathcal{G}_\delta(\theta_t \omega) = \frac{\omega(t + \delta) - \omega(t)}{\delta} \quad \text{and} \quad \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds = \int_t^{t+\delta} \frac{\omega(s)}{\delta} ds + \int_\delta^0 \frac{\omega(s)}{\delta} ds. \quad (3.10)$$

By (3.10) and the continuity of ω , one can prove the uniform convergence of \mathcal{G}_δ on any finite interval as stated below,

Lemma 3.1 ([26]). *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then for every $\varepsilon > 0$, there exists $\delta_0 = \delta_0(\varepsilon, \tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$,*

$$\left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) \right| < \varepsilon.$$

By Lemma 3.1 we find that there exist $\tilde{\delta} = \tilde{\delta}(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \tilde{\delta}$ and $t \in [\tau, \tau + T]$,

$$\left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds \right| \leq \left| \int_0^t \mathcal{G}_\delta(\theta_s \omega) ds - \omega(t) \right| + |\omega(t)| \leq c. \quad (3.11)$$

We now consider the existence and uniqueness of solutions of equation (3.1) supplemented with initial condition (3.2). To this end, we set $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| < k\}$ for each $k \in \mathbb{N}$ and consider the following equation defined in \mathcal{O}_k :

$$\frac{\partial u_k}{\partial t} = \Delta u_k - \lambda u_k + f(t, x, u_k) + g(t, x) + h(t, x, u_k) \mathcal{G}_\delta(\theta_t \omega), \quad t > \tau, x \in \mathcal{O}_k, \quad (3.12)$$

with boundary condition

$$u_k(t, x) = 0, \quad \forall t > \tau, |x| = k, \quad (3.13)$$

with the initial condition

$$u_k(\tau, x) = u_\tau(x), \quad x \in \mathcal{O}_k. \quad (3.14)$$

Under assumptions (3.3)–(3.7), we can show that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^n)$, problem (3.12)–(3.14) is well-posed in $L^2(\mathcal{O}_k)$. In addition, the solution is $(\mathcal{F}, \mathcal{B}(L^2(\mathcal{O}_k)))$ -measurable with respect to $\omega \in \Omega$. Then by examining the limiting behavior of u_k as $k \rightarrow \infty$, we can prove the following result for problem (3.1)–(3.2).

Lemma 3.2. *Suppose (3.3)–(3.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^n)$, problem (3.1)–(3.2) has a unique solution $u(\cdot, \tau, \omega, u_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^n)) \cap L^2_{loc}((\tau, \infty), H^1(\mathbb{R}^n))$. This solution is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$ -measurable in ω and continuous in initial data u_τ in $L^2(\mathbb{R}^n)$.*

Proof. The proof consists of several steps. We first derive uniform estimates on the solution u_k of (3.12)–(3.14). By (3.12) we obtain for $t > \tau$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_k\|^2 + \|\nabla u_k\|^2 + \lambda \|u_k\|^2 &= \int_{\mathcal{O}_k} f(t, x, u_k) u_k dx + \int_{\mathcal{O}_k} g(t, x) u_k dx \\ &\quad + \mathcal{G}_\delta(\theta_t \omega) \int_{\mathcal{O}_k} h(t, x, u_k) u_k dx. \end{aligned} \quad (3.15)$$

For the first term on the right-hand side of (3.15), by (3.3) we obtain that

$$\int_{\mathcal{O}_k} f(t, x, u_k) u_k dx \leq -\alpha_1 \int_{\mathcal{O}_k} |u_k|^p dx + \int_{\mathcal{O}_k} \psi_1(t, x) dx. \quad (3.16)$$

By (3.6), we get

$$\begin{aligned} \mathcal{G}_\delta(\theta_t \omega) \int_{\mathcal{O}_k} h(t, x, u_k) u_k dx &\leq |\mathcal{G}_\delta(\theta_t \omega)| \int_{\mathcal{O}_k} (\beta_1(t, x) |u_k|^q + \beta_2(t, x) |u_k|) dx \\ &\leq \frac{\alpha_1}{2} \int_{\mathcal{O}_k} |u_k|^p dx + c_1 \int_{\mathcal{O}_k} |\beta_1 \mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} dx + c_2 \int_{\mathcal{O}_k} |\beta_2 \mathcal{G}_\delta(\theta_t \omega)|^{p_1} dx \\ &\leq \frac{\alpha_1}{2} \int_{\mathcal{O}_k} |u_k|^p dx + c_1 |\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} \|\beta_1(t)\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \\ &\quad + c_2 |\mathcal{G}_\delta(\theta_t \omega)|^{p_1} \|\beta_2(t)\|_{L^{p_1}}^{p_1}. \end{aligned} \quad (3.17)$$

Finally, Young's inequality implies that

$$\int_{\mathcal{O}_k} g(t, x) u_k dx \leq \frac{\lambda}{4} \|u_k\|^2 + \frac{1}{\lambda} \|g(t)\|^2. \quad (3.18)$$

Thus it follows from (3.15)–(3.18) that for $t > \tau$

$$\begin{aligned} \frac{d}{dt} \|u_k\|^2 + 2 \|\nabla u_k\|^2 + \frac{3\lambda}{2} \|u_k\|^2 + \alpha_1 \|u_k\|_{L^p}^p &\leq \frac{2}{\lambda} \|g(t)\|^2 + 2 \|\psi_1(t)\|_{L^1} \\ &\quad + c_1 |\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} \|\beta_1(t)\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} + c_2 |\mathcal{G}_\delta(\theta_t \omega)|^{p_1} \|\beta_2(t)\|_{L^{p_1}}^{p_1}. \end{aligned} \quad (3.19)$$

Multiplying (3.19) by $e^{\frac{3}{2}\lambda t}$ and then integration over (τ, t) with $t \geq \tau$, we get for every $\omega \in \Omega$

$$\|u_k(t, \tau, \omega, u_\tau)\|^2 + 2 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} \|\nabla u_k(s, \tau, \omega, u_\tau)\|^2 ds$$

$$\begin{aligned}
& + \alpha_1 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} \|u_k(s, \tau, \omega, u_{\tau})\|_{L^p}^p ds \\
& \leq e^{\frac{3}{2}\lambda(\tau-t)} \|u_{\tau}\|^2 + \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} \left(\frac{2}{\lambda} \|g(s)\|^2 + 2 \|\psi_1(s)\|_{L^1} \right) ds \\
& + c_1 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} |\mathcal{G}_{\delta}(\theta_s \omega)|^{\frac{p}{p-q}} \|\beta_1(s)\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} ds + c_2 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} |\mathcal{G}_{\delta}(\theta_s \omega)|^{p_1} \|\beta_2(s)\|_{L^{p_1}}^{p_1} ds.
\end{aligned} \tag{3.20}$$

Note that $\mathcal{G}_{\delta}(\theta_t \omega)$ is continuous in t for fixed ω . Therefore, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > \tau$, we obtain from (3.20) that

$$\{u_k\}_{k=1}^{\infty} \text{ is bounded in } L^{\infty}(\tau, T; L^2(\mathcal{O}_k)) \cap L^p(\tau, T; L^p(\mathcal{O}_k)) \cap L^2(\tau, T; H_0^1(\mathcal{O}_k)). \tag{3.21}$$

By (3.4) and (3.6) we have

$$\int_{\tau}^T \int_{\mathcal{O}_k} |f(t, x, u_k)|^{p_1} dx dt \leq c \int_{\tau}^T \int_{\mathcal{O}_k} |u_k|^p dx dt + c \int_{\tau}^T \int_{\mathcal{O}_k} |\psi_2(t, x)|^{p_1} dx dt, \tag{3.22}$$

and

$$\begin{aligned}
\int_{\tau}^T \int_{\mathcal{O}_k} |h(t, x, u_k)|^{p_1} dx dt & \leq c \int_{\tau}^T \int_{\mathcal{O}_k} \left(|\beta_1(t, x)|^{p_1} |u_k|^{p_1(q-1)} + |\beta_2(t, x)|^{p_1} \right) dx dt, \\
& \leq c \int_{\tau}^T \int_{\mathcal{O}_k} \left(|\beta_1(t, x)|^{\frac{p}{p-q}} + |u_k|^p + |\beta_2(t, x)|^{p_1} \right) dx dt
\end{aligned} \tag{3.23}$$

where $\frac{1}{p_1} + \frac{1}{p} = 1$. Thus (3.22) and (3.23) along with (3.21) imply that

$$\{f(t, x, u_k)\}_{k=1}^{\infty} \text{ and } \{h(t, x, u_k) \mathcal{G}_{\delta}(\theta_t \omega)\}_{k=1}^{\infty} \text{ are bounded in } L^{p_1}(\tau, T; L^{p_1}(\mathcal{O}_k)). \tag{3.24}$$

By (3.21) and (3.24) we obtain that

$$\left\{ \frac{du_k}{dt} \right\}_{k=1}^{\infty} \text{ is bounded in } L^2(\tau, T; H^{-1}(\mathcal{O}_k)) + L^{p_1}(\tau, T; L^{p_1}(\mathcal{O}_k)). \tag{3.25}$$

Extend u_k to the entire space \mathbb{R}^n by setting $u_k = 0$ on $\mathbb{R}^n \setminus \mathcal{O}_k$ and denote the extension still by u_k . It follows from (3.21), (3.24) and (3.25) that there exist $\hat{u} \in L^2(\mathbb{R}^n)$, $u \in L^{\infty}(\tau, T; L^2(\mathbb{R}^n)) \cap L^p(\tau, T; L^p(\mathbb{R}^n)) \cap L^2(\tau, T; H^1(\mathbb{R}^n))$, $\chi_1 \in L^{p_1}(\tau, T; L^{p_1}(\mathbb{R}^n))$, $\chi_2 \in L^2(\tau, T; H^{-1}(\mathbb{R}^n))$ such that, up to a subsequence,

$$u_k \rightarrow u \quad \text{weak-star in } L^\infty(\tau, T; L^2(\mathbb{R}^n)), \quad (3.26)$$

$$u_k \rightarrow u \quad \text{weakly in } L^p(\tau, T; L^p(\mathbb{R}^n)) \text{ and } L^2(\tau, T; H^1(\mathbb{R}^n)), \quad (3.27)$$

$$f(t, x, u_k) + \mathcal{G}_\delta(\theta_t \omega) h(t, x, u_k) \rightarrow \chi_1 \quad \text{weakly in } L^{p_1}(\tau, T; L^{p_1}(\mathbb{R}^n)), \quad (3.28)$$

$$\frac{du_k}{dt} \rightarrow \chi_2 \quad \text{weakly in } L^2(\tau, T; H^{-1}(\mathbb{R}^n)) + L^{p_1}(\tau, T; L^{p_1}(\mathbb{R}^n)), \quad (3.29)$$

$$u_k(t_0, \tau, \omega, u_\tau) \rightarrow \hat{u} \quad \text{weakly in } L^2(\mathbb{R}^n) \text{ for a fixed } t_0 \in [\tau, T]. \quad (3.30)$$

By a standard procedure (see [23]), we can check that $\chi_1 = f(t, x, u) + \mathcal{G}_\delta(\theta_t \omega) h(t, x, u)$, $\chi_2 = \frac{du}{dt}$ and $\hat{u} = u(t_0)$. Therefore, u is a weak solution of (3.1) supplemented with initial condition (3.2).

Next, we prove the uniqueness of solutions. Suppose u_1 and u_2 are two solutions of (3.1) with initial conditions $u_1(\tau, x) = u_{1,\tau}$ and $u_2(\tau, x) = u_{2,\tau}$, respectively. Then $\tilde{u} = u_1 - u_2$ satisfies

$$\frac{d\tilde{u}}{dt} = \Delta \tilde{u} - \lambda \tilde{u} + l_1(t) \tilde{u} + \mathcal{G}_\delta(\theta_t \omega) l_2(t) \tilde{u}, \quad (3.31)$$

where $l_1(t) = \int_0^1 \frac{\partial f}{\partial s}(t, x, \xi u_1 + (1 - \xi) u_2) d\xi$ and $l_2(t) = \int_0^1 \frac{\partial h}{\partial s}(t, x, \xi u_1 + (1 - \xi) u_2) d\xi$. If $q > 2$, by (3.5) and (3.7), we obtain for $t \in [\tau, T]$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{u}\|^2 &\leq \int_{\mathbb{R}^n} l_1(t) |\tilde{u}|^2 dx + \mathcal{G}_\delta(\theta_t \omega) \int_{\mathbb{R}^n} l_2(t) |\tilde{u}|^2 dx \\ &\leq -\alpha_3 \int_{\mathbb{R}^n} \left(\int_0^1 |\xi u_1 + (1 - \xi) u_2|^{p-2} d\xi + \psi_3(t, x) \right) |\tilde{u}|^2 dx \\ &\quad + |\mathcal{G}_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} (\beta_3 \int_0^1 |\xi u_1 + (1 - \xi) u_2|^{q-2} d\xi + \beta_4) |\tilde{u}|^2 dx \\ &\leq -\alpha_3 \int_{\mathbb{R}^n} \int_0^1 |\xi u_1 + (1 - \xi) u_2|^{p-2} |\tilde{u}|^2 d\xi dx + \int_{\mathbb{R}^n} \psi_3 |\tilde{u}|^2 dx \\ &\quad + c \int_{\mathbb{R}^n} |\beta_3 \mathcal{G}_\delta(\theta_t \omega)|^{\frac{p-2}{p-q}} |\tilde{u}|^2 dx + \alpha_3 \int_0^1 \int_{\mathbb{R}^n} |\xi u_1 + (1 - \xi) u_2|^{p-2} |\tilde{u}|^2 d\xi dx \\ &\quad + |\mathcal{G}_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} \beta_4 |\tilde{u}|^2 dx \\ &\leq c_1 \|\tilde{u}\|^2, \end{aligned} \quad (3.32)$$

where c_1 is a positive constant depending on τ, T and ω . Note that if $q = 2$, then (3.32) holds obviously. Therefore, for all $t \in [\tau, T]$, we have

$$\|u_1(t, \tau, \omega, u_{1,\tau}) - u_2(t, \tau, \omega, u_{2,\tau})\|^2 \leq e^{c_1(t-\tau)} \|u_{1,\tau} - u_{2,\tau}\|^2, \quad (3.33)$$

which implies the uniqueness and continuous dependence of solutions on initial data in $L^2(\mathbb{R}^n)$. By (3.30) and the uniqueness of solutions, we infer that for every $\omega \in \Omega$, the whole sequence $u_k(t_0, \tau, \omega, u_\tau) \rightarrow u(t_0, \tau, \omega, u_\tau)$ weakly in $L^2(\mathbb{R}^n)$ for any fixed $t_0 \in [\tau, T]$ and $\omega \in \Omega$. Then the measurability of $u(t, \tau, \omega, u_\tau)$ follows from that of $u_k(t, \tau, \omega, u_\tau)$. This completes the proof. \square

The following result is useful when proving the asymptotic compactness of solutions.

Lemma 3.3. Suppose (3.3)–(3.7) hold and $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $L^2(\mathbb{R}^n)$. Then for every $\tau \in \mathbb{R}$, $t > \tau$ and $\omega \in \Omega$, there exist $u_0 \in L^2(\tau, t; L^2(\mathbb{R}^n))$ and a subsequence $\{u(\cdot, \tau, \omega, u_{n_m})\}_{m=1}^\infty$ of $\{u(\cdot, \tau, \omega, u_n)\}_{n=1}^\infty$ such that $u(s, \tau, \omega, u_{n_m}) \rightarrow u_0(s)$ in $L^2(\mathcal{O}_k)$ as $m \rightarrow \infty$ for every $k \in \mathbb{N}$ and for almost all $s \in (\tau, t)$.

Proof. Let T be a sufficiently large number such that $t \in (\tau, T]$. Note that the embedding $H_0^1(\mathcal{O}_k) \hookrightarrow L^2(\mathcal{O}_k)$ is compact, by (3.21) and (3.25), we can show that there is $\hat{u} \in L^2(\tau, T; L^2(\mathbb{R}^n))$ such that, up to a subsequence,

$$u(\cdot, \tau, \omega, u_n) \rightarrow \hat{u}(\cdot) \text{ in } L^2(\tau, T; L^2(\mathcal{O}_k)) \text{ for every } k \in \mathbb{N}.$$

Then for each $k \in \mathbb{N}$, there exists a subset $I_k \subseteq [\tau, T]$ of measure zero and a subsequence $u_{n_m}^k$ of u_n such that

$$u(s, \tau, \omega, u_{n_m}^k) \rightarrow \hat{u}(s) \text{ in } L^2(\mathcal{O}_k), \quad \text{for all } s \in [\tau, T] \setminus I_k.$$

Then by a diagonal process, we find that there exists a subset $I \subseteq [\tau, T]$ of measure zero and a subsequence (which we do not relabel) such that

$$u(s, \tau, \omega, u_n) \rightarrow \hat{u}(s) \text{ in } L^2(\mathcal{O}_k), \quad \text{for all } s \in [\tau, T] \setminus I \text{ and } k \in \mathbb{N}. \quad (3.34)$$

This completes the proof. \square

We now define a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ such that for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $u_\tau \in L^2(\mathbb{R}^n)$,

$$\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), \quad (3.35)$$

where u is a solution of problem (3.1)–(3.2). Then Φ is a continuous cocycle on $L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

We will study tempered pullback attractors for Φ in $L^2(\mathbb{R}^n)$. To this end, we first need to specify a collection of families of subsets of $L^2(\mathbb{R}^n)$. Given a bounded nonempty subset D of $L^2(\mathbb{R}^n)$, the Hausdorff semi-distance between D and the origin in $L^2(\mathbb{R}^n)$ is denoted by $\|D\| = \sup_{\psi \in D} \|\psi\|$. Let \mathcal{D} be the collection of all families of tempered nonempty subsets of $L^2(\mathbb{R}^n)$, i.e.,

$$\mathcal{D} = \{D = D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega : D \text{ is tempered}\}.$$

The following condition will be needed for g and ψ_1 when deriving uniform estimates of solutions:

$$\int_{-\infty}^{\tau} e^{\lambda s} (\|g(s)\|^2 + \|\psi_1(s)\|_{L^1}) ds < \infty, \quad \forall \tau \in \mathbb{R}. \quad (3.36)$$

When constructing tempered pullback attractors, we will assume that

$$\lim_{t \rightarrow -\infty} e^{ct} \int_{-\infty}^0 e^{\lambda s} (\|g(s+t)\|^2 + \|\psi_1(s+t)\|_{L^1}) ds = 0, \quad \forall c > 0. \quad (3.37)$$

3.2. Pullback random attractors

This subsection is devoted to the proof of existence of pullback random attractors for the random reaction–diffusion equation (3.1)–(3.2) in $L^2(\mathbb{R}^n)$. We first show that problem (3.1)–(3.2) has a tempered pullback absorbing set in $L^2(\mathbb{R}^n)$, then derive uniform estimates on the tails of solutions and the asymptotic compactness of the solutions. We finally prove the existence and uniqueness of tempered random attractors for problem (3.1)–(3.2). Throughout this subsection, we assume $\beta_1 \in L^\infty(\mathbb{R}, L^{\frac{p}{p-q}}(\mathbb{R}^n))$ and $\beta_2 \in L^\infty(\mathbb{R}, L^{p_1}(\mathbb{R}^n))$.

Lemma 3.4. Suppose (3.3)–(3.7), and (3.36) hold. Then for every $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\sigma, \tau, \omega, D) > 0$ such that for all $t \geq T$, the solution u of equation (3.1) with ω replaced by $\theta_{-\tau}\omega$ satisfies

$$\begin{aligned} & \|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\ & \leq M \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1} + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds, \end{aligned} \quad (3.38)$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and M is a positive constant independent of σ, τ, ω and D .

Proof. It follows from (3.1) that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 + \lambda \|u\|^2 = \int_{\mathbb{R}^n} (f(t, x, u) + g(t, x)) u dx + \mathcal{G}_\delta(\theta_t\omega) \int_{\mathbb{R}^n} h(t, x, u) u dx. \quad (3.39)$$

We now estimate each term on the right-hand side (3.39). By (3.3) we obtain that

$$\int_{\mathbb{R}^n} f(t, x, u) u dx \leq -\alpha_1 \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} \psi_1(t, x) dx. \quad (3.40)$$

By (3.6), we get

$$\begin{aligned} \mathcal{G}_\delta(\theta_t \omega) \int_{\mathbb{R}^n} h(t, x, u) u dx &\leq |\mathcal{G}_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} (\beta_1(t, x) |u|^q + \beta_2(t, x) |u|) dx \\ &\leq \frac{\alpha_1}{2} \int_{\mathbb{R}^n} |u|^p dx + c |\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} \|\beta_1(t)\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \\ &\quad + c |\mathcal{G}_\delta(\theta_t \omega)|^{p_1} \|\beta_2(t)\|_{L^{p_1}}^{p_1}. \end{aligned} \quad (3.41)$$

Finally, Young's inequality implies that

$$\int_{\mathbb{R}^n} g(t, x) u dx \leq \frac{\lambda}{4} \|u\|^2 + \frac{1}{\lambda} \|g(t)\|^2. \quad (3.42)$$

Thus it follows from (3.39)–(3.42) that

$$\begin{aligned} \frac{d}{dt} \|u\|^2 + 2 \|\nabla u\|^2 + \frac{\lambda}{2} \|u\|^2 + \alpha_1 \|u\|_{L^p}^p &\leq -\lambda \|u\|^2 + \frac{2}{\lambda} \|g(t)\|^2 \\ &\quad + 2 \|\psi_1(t)\|_{L^1} + c (|\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_t \omega)|^{p_1}). \end{aligned} \quad (3.43)$$

Multiplying (3.43) by $e^{\lambda t}$ and then integration over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$, we get for every $\omega \in \Omega$

$$\begin{aligned} \|u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 + 2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla u(s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 ds \\ + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|^2 ds \\ \leq e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left(\frac{2}{\lambda} \|g(s)\|^2 + 2 \|\psi_1(s)\|_{L^1} \right) ds \\ + c \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} (|\mathcal{G}_\delta(\theta_{s-\tau} \omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s-\tau} \omega)|^{p_1}) ds \\ \leq e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 + \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \left(\frac{2}{\lambda} \|g(s+\tau)\|^2 \right. \\ + 2 \|\psi_1(s+\tau)\|_{L^1} + c \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} (|\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} \\ + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1}) ds \Big) ds. \end{aligned} \quad (3.44)$$

Here the last two integrals in (3.44) are well defined due to (3.8), (3.10) and (3.36). Since $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and D is tempered, we find that

$$\limsup_{t \rightarrow +\infty} e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq \limsup_{t \rightarrow +\infty} e^{\lambda(\tau-t-\sigma)} \|D(\tau-t, \theta_{-t}\omega)\|^2 = 0,$$

which shows that there exists $T = T(\sigma, \tau, \omega, D) > 0$ such that for all $t \geq T$,

$$e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds. \quad (3.45)$$

Therefore (3.38) follows from (3.44) and (3.45). This ends the proof. \square

As an immediate consequence of Lemma 3.4, we obtain the existence of \mathcal{D} -pullback absorbing sets for system (3.1)–(3.2).

Corollary 3.1. Suppose (3.3)–(3.7), (3.36) and (3.37) hold. Then the continuous cocycle Φ of problem (3.1)–(3.2) has a closed measurable \mathcal{D} -pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$K(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq R(\tau, \omega)\}, \quad (3.46)$$

where $R(\tau, \omega)$ is given by

$$R(\tau, \omega) = M \int_{-\infty}^0 e^{\lambda s} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1} + |\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds, \quad (3.47)$$

with M being as in (3.38).

Proof. By Lemma 3.4, it is clear that K pullback attracts all elements in \mathcal{D} . In what follows, we prove that K given by (3.46) is tempered. Let γ be an arbitrary positive number. Then for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have by (3.47)

$$\begin{aligned} e^{\gamma t} \|K(\tau+t, \theta_t\omega)\|^2 &\leq e^{\gamma t} R(\tau+t, \theta_t\omega) \\ &= M e^{\gamma t} \int_{-\infty}^0 e^{\lambda s} (\|g(s+\tau+t)\|^2 + \|\psi_1(s+\tau+t)\|_{L^1}) ds \\ &\quad + M e^{\gamma t} \int_{-\infty}^0 e^{\lambda s} (|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}) ds. \end{aligned} \quad (3.48)$$

First of all, it follows from (3.37) that

$$\lim_{t \rightarrow -\infty} e^{\gamma(\tau+t)} \int_{-\infty}^0 e^{\lambda s} \left(\|g(s + \tau + t)\|^2 + \|\psi_1(s + \tau + t)\|_{L^1} \right) ds = 0. \quad (3.49)$$

Let $\tilde{\gamma} = \min\{\gamma, \lambda\}$. Then for the last term in (3.48) we have for every $t \leq 0$,

$$\begin{aligned} & e^{\gamma t} \int_{-\infty}^0 e^{\lambda s} (|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}) ds \\ & \leq \int_{-\infty}^0 e^{\tilde{\gamma}(s+t)} (|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}) ds \\ & \leq \int_{-\infty}^t e^{\tilde{\gamma}s} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds. \end{aligned} \quad (3.50)$$

By (3.8) and (3.10), we know that

$$\int_{-\infty}^0 e^{\tilde{\gamma}s} (|\mathcal{G}_\delta(\theta_s\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s\omega)|^{p_1}) ds < \infty,$$

and hence we have by (3.50)

$$\lim_{t \rightarrow -\infty} e^{\gamma t} \int_{-\infty}^0 e^{\lambda s} (|\mathcal{G}_\delta(\theta_{s+t}\omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_{s+t}\omega)|^{p_1}) ds = 0. \quad (3.51)$$

It follows from (3.48), (3.49) and (3.51) that K belongs to \mathcal{D} . Moreover, since for each $\tau \in \mathbb{R}$, $R(\tau, \cdot) : \Omega \rightarrow \mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$ -measurable, then K given by (3.46) is also measurable. Thus, $K \in \mathcal{D}$ is a closed measurable \mathcal{D} -pullback absorbing set for Φ , as desired. \square

Next, we derive uniform estimates on the tails of solutions for large space and time variables, which will play an important role for proving the asymptotic compactness of solutions.

Lemma 3.5. Suppose (3.3)–(3.7) and (3.36) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for any $\varepsilon > 0$, there exists $T = T(\tau, \omega, D, \varepsilon) \geq 1$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$ and $\sigma \in [\tau - 1, \tau]$, the solution u of problem (3.1)–(3.2) with ω replaced by $\theta_{-\tau}\omega$ satisfies

$$\int_{|x| \geq N} |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \leq \varepsilon, \quad (3.52)$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Let ρ be a smooth function defined on \mathbb{R}^+ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0, & \text{for } 0 \leq s \leq 1, \\ 1, & \text{for } s \geq 2. \end{cases} \quad (3.53)$$

Let k be a fixed positive integer which will be specified later. First multiplying (3.1) by $\rho(\frac{|x|^2}{k^2})u$ and then taking the integral over \mathbb{R}^n , we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^2 dx - 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta u u dx &= -2\lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^2 dx \\ &+ 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, u) u dx + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t, x) u dx \\ &+ 2\mathcal{G}_\delta(\theta_t \omega) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) h(t, x, u) u dx. \end{aligned} \quad (3.54)$$

We now estimate the terms in (3.54) as follows. First we have

$$- \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta u u dx = \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla u|^2 dx + \int_{k \leq |x| \leq \sqrt{2}k} u \rho'\left(\frac{|x|^2}{k^2}\right) \frac{2x}{k^2} \cdot \nabla u dx. \quad (3.55)$$

Note that the second term on the right-hand side of (3.55) is bounded by

$$\begin{aligned} \left| \int_{k \leq |x| \leq \sqrt{2}k} u \rho'\left(\frac{|x|^2}{k^2}\right) \frac{2x}{k^2} \cdot \nabla u dx \right| &\leq \frac{2\sqrt{2}}{k} \int_{k \leq |x| \leq \sqrt{2}k} |\rho'\left(\frac{|x|^2}{k^2}\right)| |u \nabla u| dx \\ &\leq \frac{c}{k} (\|\nabla u\|^2 + \|u\|^2). \end{aligned} \quad (3.56)$$

Using (3.3), we have

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, u) u dx \leq -\alpha_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^p dx + \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\psi_1(t, x)| dx. \quad (3.57)$$

By Young's inequality, we find

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t, x) u dx \leq \frac{\lambda}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^2 dx + \frac{1}{2\lambda} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |g(t, x)|^2 dx. \quad (3.58)$$

For the last term in (3.54), we have from (3.6)

$$\begin{aligned}
 \mathcal{G}_\delta(\theta_t \omega) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) h(t, x, u) u dx &\leq |\mathcal{G}_\delta(\theta_t \omega)| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) (\beta_1(t, x) |u|^q + \beta_2(t, x) |u|) dx \\
 &\leq \frac{\alpha_1}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^p dx + c |\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\beta_1(t, x)|^{\frac{p}{p-q}} dx \\
 &\quad + c |\mathcal{G}_\delta(\theta_t \omega)|^{p_1} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\beta_2(t, x)|^{p_1} dx.
 \end{aligned} \tag{3.59}$$

Therefore, by (3.55)–(3.59), there is a $N_1 = N_1(\varepsilon) > 0$ such that for all $k \geq N_1$

$$\begin{aligned}
 \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^2 dx + \alpha_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^p dx + \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u|^2 dx &\leq \varepsilon \|u\|_{H^1(\mathbb{R}^n)}^2 \\
 + c \int_{|x| \geq k} (|g(t, x)|^2 + |\psi_1(t, x)|) dx + c |\mathcal{G}_\delta(\theta_t \omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\beta_1(t, x)|^{\frac{p}{p-q}} dx \\
 + c |\mathcal{G}_\delta(\theta_t \omega)|^{p_1} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\beta_2(t, x)|^{p_1} dx.
 \end{aligned} \tag{3.60}$$

Given $t \geq 1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, multiplying (3.60) by $e^{\lambda t}$ and then integration over $(\tau - t, \sigma)$ where $\sigma \in [\tau - 1, \tau]$, we get

$$\begin{aligned}
 &\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 dx \\
 &\leq e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 + \varepsilon \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\
 &\quad + \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \int_{|x| \geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds \\
 &\quad + c \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \left(|\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} \int_{|x| \geq k} |\beta_1(s, x)|^{\frac{p}{p-q}} dx + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1} \int_{|x| \geq k} |\beta_2(s, x)|^{p_1} dx \right) ds.
 \end{aligned} \tag{3.61}$$

By the fact that $u_{\tau-t} \in D(\tau - t, \theta_{-t} \omega)$ and D is tempered, we have

$$\limsup_{t \rightarrow +\infty} e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq e^{\lambda} \limsup_{t \rightarrow +\infty} e^{-\lambda t} \|D(\tau - t, \theta_{-t} \omega)\|^2 = 0,$$

which means that we can choose $T_1 = T_1(\tau, \omega, D, \varepsilon) \geq 1$ such that for all $t \geq T_1$,

$$e^{\lambda(\tau-t-\sigma)} \|u_{\tau-t}\|^2 \leq \varepsilon. \quad (3.62)$$

By (3.36), there is a $N_2 = N_2(\tau, \lambda, \varepsilon) \geq N_1$ such that for all $k \geq N_2$

$$\begin{aligned} & \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \int_{|x| \geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds \\ & \leq e^{\lambda} \int_{-\infty}^0 e^{\lambda s} \int_{|x| \geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds \leq \varepsilon. \end{aligned} \quad (3.63)$$

By (3.8) and (3.10), we find

$$\begin{aligned} & \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \left(|\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} \int_{\mathbb{R}^n} |\beta_1(s, x)|^{\frac{p}{p-q}} dx + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1} \int_{\mathbb{R}^n} |\beta_2(s, x)|^{p_1} dx \right) ds \\ & \leq c \int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \left(|\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1} \right) ds < \infty, \end{aligned}$$

which implies that there is a $N_3 = N_3(\tau, \lambda, \omega, \varepsilon) \geq N_2$ such that for all $k \geq N_3$

$$\int_{-\infty}^{\sigma-\tau} e^{\lambda(s+\tau-\sigma)} \left(|\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} \int_{|x| \geq k} |\beta_1(s, x)|^{\frac{p}{p-q}} dx + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1} \int_{|x| \geq k} |\beta_2(s, x)|^{p_1} dx \right) ds \leq \varepsilon. \quad (3.64)$$

Therefore it follows from Lemma 3.4 and (3.61)–(3.64) that for all $\sigma \in [\tau - 1, \tau]$, $t \geq T_1(\tau, \omega, D, \varepsilon)$ and $k \geq N_3$

$$\begin{aligned} & \int_{|x| \geq \sqrt{2}k} |u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 dx \leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u(\sigma, \tau - t, \theta_{-\tau} \omega, u_{\tau-t})|^2 dx \\ & \leq c\varepsilon \left(1 + \int_{-\infty}^0 e^{\lambda s} (\|g(s+\tau, x)\|^2 + \|\psi_1(s+\tau, x)\|_{L^1} + |\mathcal{G}_\delta(\theta_s \omega)|^{\frac{p}{p-q}} + |\mathcal{G}_\delta(\theta_s \omega)|^{p_1}) ds \right), \end{aligned}$$

which concludes the proof. \square

Lemma 3.6. Suppose (3.3)–(3.7) and (3.36) hold. Then the continuous cocycle Φ of problem (3.1)–(3.2) is D -pullback asymptotically compact in $L^2(\mathbb{R}^n)$.

Proof. We need to show that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, $t_n \rightarrow +\infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$. By Lemma 3.4, we find that there exist $T = T(\tau, \omega, D) > 0$ and $c = c(\tau, \omega) > 0$ such that for all $t \geq T$ and $u_0 \in D(\tau - t, \theta_{-t}\omega)$,

$$\|u(\tau - 1, \tau - t, \theta_{-\tau}\omega, u_0)\| \leq c(\tau, \omega). \quad (3.65)$$

Since $t_n \rightarrow +\infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega)$, by (3.65) we find that there is $N_1 = N_1(\tau, \omega, D) > 0$ such that for all $n \geq N_1$,

$$\|u(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\| \leq c(\tau, \omega).$$

This shows that

$$\{u(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})\}_{n=1}^{\infty} \text{ is bounded in } L^2(\mathbb{R}^n). \quad (3.66)$$

Thus we get from (3.66) and Lemma 3.3 that there exist $s \in (\tau - 1, \tau)$, $u_0 \in L^2(\mathbb{R}^n)$ and a subsequence (not relabeled) such that for every $k \in \mathbb{N}$ as $n \rightarrow \infty$

$$u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) = u(s, \tau - 1, \theta_{-\tau}\omega, u(\tau - 1, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})) \rightarrow u_0 \text{ in } L^2(\mathcal{O}_k). \quad (3.67)$$

Since $u_0 \in L^2(\mathbb{R}^n)$, for any given $\epsilon > 0$, there exists $K_1 = K_1(\epsilon) > 0$ such that for all $k \geq K_1$,

$$\int_{|x| \geq k} |u_0|^2 dx \leq \epsilon. \quad (3.68)$$

On the other hand, by Lemma 3.5, there exist $N_2 = N_2(\tau, \omega, D, \epsilon) \geq 1$ and $K_2 = K_2(\tau, \omega, \epsilon) \geq K_1$ such that for all $n \geq N_2$ and $k \geq K_2$,

$$\int_{|x| \geq k} |u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})|^2 dx \leq \epsilon. \quad (3.69)$$

Finally, by (3.67), we find that there exists $N_3 = N_3(\tau, \omega, D, \epsilon) \geq N_2$ such that for all $n \geq N_3$,

$$\int_{|x| < K_2} |u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) - u_0|^2 dx \leq \epsilon. \quad (3.70)$$

By (3.33), we have

$$\begin{aligned} & \|u(\tau, s, \theta_{-\tau}\omega, u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})) - u(\tau, s, \theta_{-\tau}\omega, u_0)\|^2 \\ & \leq c \left\{ \int_{|x| < K_2} + \int_{|x| \geq K_2} \right\} |u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) - u_0|^2 dx \end{aligned}$$

$$\begin{aligned} &\leq c \int_{|x| < K_2} |u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n}) - u_0|^2 dx \\ &\quad + c \int_{|x| \geq K_2} (|u(s, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})|^2 + |u_0|^2) dx, \end{aligned}$$

which together with (3.68)–(3.70) implies that $u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})$ converges to $u(\tau, s, \theta_{-\tau}\omega, u_0)$ in $L^2(\mathbb{R}^n)$ as $n \rightarrow +\infty$. Since $\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) = u(\tau, \tau - t_n, \theta_{-\tau}\omega, u_{0,n})$, the desired result follows immediately. \square

We are now ready to present the existence of \mathcal{D} -pullback attractors for Φ .

Theorem 3.1. *Suppose (3.3)–(3.7), (3.36) and (3.37) hold. Then the continuous cocycle Φ associated with problem (3.1) and (3.2) has a unique \mathcal{D} -pullback random attractor, which is characterized by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\begin{aligned} \mathcal{A}(\tau, \omega) &= \Omega(K, \tau, \omega) = \bigcup_{B \in \mathcal{D}} \Omega(B, \tau, \omega) \\ &= \{\Gamma(0, \tau, \omega) : \Gamma \text{ is a } \mathcal{D}\text{-complete orbit of } \Phi\}, \end{aligned}$$

where K is given by (3.46). If, in addition, f , h , g and ψ_1 are T -periodic functions with respect to t , then the \mathcal{D} -pullback random attractor $\mathcal{A}(\tau, \omega)$ is also T -periodic.

Proof. Note that Corollary 3.1 shows that Φ has a closed measurable \mathcal{D} -pullback absorbing set K as given by (3.46), and Lemma 3.6 implies that Φ is asymptotically compact in $L^2(\mathbb{R}^n)$ with respect to \mathcal{D} . Therefore, the existence of \mathcal{D} -pullback random attractor $\mathcal{A}(\tau, \omega)$ follows from Proposition 2.1 immediately. Moreover, this attractor is unique and its structure is given as above. If, in addition, f , h , g and ψ_1 are T -periodic functions with respect to t , then the cocycle Φ and the \mathcal{D} -pullback absorbing set K are also T -periodic. As a consequence, the periodicity of \mathcal{A} follows from Proposition 2.1. \square

4. Reaction–diffusion equations driven by multiplicative noise

In this section, we consider reaction–diffusion equation (1.1) driven by multiplicative noise: for any given $\tau \in \mathbb{R}$

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + u \circ \frac{dw(t)}{dt}, \quad t > \tau, x \in \mathbb{R}^n \quad (4.1)$$

which is supplemented with initial condition (3.2). To discuss random attractors in this case, we need to convert (4.1) into a pathwise deterministic equation that can be done by the standard transformation $v(t, \tau, \omega) = e^{-\omega(t)}u(t, \tau, \omega)$. Then v satisfies

$$\frac{dv}{dt} = \Delta v - \lambda v + e^{-\omega(t)}f(t, x, e^{\omega(t)}v) + e^{-\omega(t)}g(t, x), \quad t > \tau, \quad (4.2)$$

with the initial conditions

$$v(\tau, x) = v_\tau(x), \quad (4.3)$$

where $v_\tau(x) = e^{-\omega(\tau)}u_\tau(x)$. Given $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathbb{R}^n)$, system (4.2)–(4.3) is a deterministic system. Similar to Lemma 3.2 in the previous section, one can prove that if f satisfies all the assumptions in the previous section then system (4.2)–(4.3) has a unique solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^n)) \cap L^2_{loc}([\tau, \infty), H^1(\mathbb{R}^n))$. In addition $v(\cdot, \tau, \omega, v_\tau)$ is continuous in v_τ with respect to the norm of $L^2(\mathbb{R}^n)$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$ -measurable in $\omega \in \Omega$. This enables us to define a continuous cocycle $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ over \mathbb{R} and $(\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ in the following way

$$\Phi_0(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = e^{\omega(t) - \omega(-\tau)}v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau), \quad (4.4)$$

where $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

We first will show that system (4.1) and (3.2) has a \mathcal{D} -pullback random attractors in $L^2(\mathbb{R}^n)$. To this end, we must derive uniform estimates of the solutions which are given below.

Lemma 4.1. *Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\sigma, \tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\sigma, \tau, \omega, D) > 0$ such that for all $t \geq T$, the solution u of system (4.1) and (3.2) satisfies*

$$\begin{aligned} e^{2\omega(\sigma-\tau)} \|u(\sigma, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 &+ \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1}^2 ds \\ &\leq M \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s+\sigma-\tau)} (\|g(s+\sigma)\|^2 + \|\psi_1(s+\sigma)\|_{L^1}) ds, \end{aligned} \quad (4.5)$$

where $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and M is a positive constant independent of σ, τ, ω and D .

Proof. From (4.2), it follows that

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 + \|\nabla v\|^2 + \lambda \|v\|^2 = e^{-\omega(t)} \int_{\mathbb{R}^n} f(t, x, u) v dx + e^{-\omega(t)} \int_{\mathbb{R}^n} g(t, x) v dx. \quad (4.6)$$

By (3.3) we obtain that

$$e^{-\omega(t)} \int_{\mathbb{R}^n} f(t, x, u) v dx \leq -\alpha_1 e^{-2\omega(t)} \|u\|_{L^p}^p + e^{-2\omega(t)} \|\psi_1(t)\|_{L^1}. \quad (4.7)$$

For the last term in (4.6) Young's inequality implies that

$$e^{-\omega(t)} \int_{\mathbb{R}^n} g(t, x) v dx \leq \frac{\lambda}{8} \|v\|^2 + \frac{2}{\lambda} e^{-2\omega(t)} \|g(t)\|^2. \quad (4.8)$$

Thus it follows from (4.6)–(4.8) that

$$\begin{aligned} & \frac{d}{dt} \|v\|^2 + 2\|\nabla v\|^2 + \frac{\lambda}{4} \|v\|^2 + 2\alpha_1 e^{(p-2)\omega(t)} \|v\|_{L^p}^p \\ & \leq -\frac{3}{2}\lambda \|v\|^2 + \frac{4}{\lambda} e^{-2\omega(t)} \|g(t)\|^2 + 2e^{-2\omega(t)} \|\psi_1(t)\|_{L^1}. \end{aligned} \quad (4.9)$$

Multiplying (4.9) by $e^{\frac{3}{2}\lambda t}$ and then integrating over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$, we get for every $\omega \in \Omega$

$$\begin{aligned} & \|v(\sigma, \tau - t, \omega, v_{\tau-t})\|^2 + 2 \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)} \|\nabla v(s, \tau - t, \omega, v_{\tau-t})\|^2 ds \\ & + \frac{\lambda}{4} \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)} \|v(s, \tau - t, \omega, v_{\tau-t})\|^2 ds \\ & \leq e^{\frac{3}{2}\lambda(\tau-t-\sigma)} \|v_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s)} \left(\frac{4}{\lambda} \|g(s)\|^2 + 2\|\psi_1(s)\|_{L^1} \right) ds. \end{aligned} \quad (4.10)$$

Replacing ω by $\theta_{-\tau}\omega$ in (4.10), by the fact that for any $s \geq \tau - t$

$$u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) = e^{\omega(s-\tau)-\omega(-\tau)} v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}), \quad (4.11)$$

we get

$$\begin{aligned} & e^{2\omega(\sigma-\tau)} \|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + 2 \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \|\nabla u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\ & + \frac{\lambda}{4} \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\ & \leq e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-\tau)} \|u_{\tau-t}\|^2 + \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \left(\frac{4}{\lambda} \|g(s)\|^2 + 2\|\psi_1(s)\|_{L^1} \right) ds \\ & \leq e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-\tau)} \|u_{\tau-t}\|^2 + \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s+\sigma-\tau)} \left(\frac{4}{\lambda} \|g(s+\sigma)\|^2 \right. \\ & \quad \left. + 2\|\psi_1(s+\sigma)\|_{L^1} \right) ds. \end{aligned} \quad (4.12)$$

By (3.8) and (3.36), we see the following integral is convergent:

$$\int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s+\sigma-\tau)} \left(\frac{4}{\lambda} \|g(s+\sigma)\|^2 + 2\|\psi_1(s+\sigma)\|_{L^1} \right) ds < \infty. \quad (4.13)$$

Since $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ and D is tempered, we find that

$$\limsup_{t \rightarrow \infty} e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|u_{\tau-t}\|^2 \leq \limsup_{t \rightarrow \infty} e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|D(\tau-t, \theta_{-t}\omega)\|^2 = 0,$$

which shows that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,

$$e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|u_{\tau-t}\|^2 \leq \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s+\sigma-\tau)} \left(\frac{4}{\lambda} \|g(s+\sigma)\|^2 + 2\|\psi_1(s+\sigma)\|_{L^1} \right) ds,$$

which together with (4.12) implies the desired estimates. \square

From (4.9) we immediately get the following estimates.

Corollary 4.1. *Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, there exists $c = c(\tau, \omega, T) > 0$ such that for all $t \in [\tau, \tau + T]$, the solution v of equation (4.2) satisfies*

$$\|v(t, \tau, \omega, v_\tau)\|^2 + \int_{\tau}^t \|v(s, \tau, \omega, v_\tau)\|_{L^p}^p ds \leq c\|v_\tau\|^2 + c \int_{\tau}^t (\|g(s)\|^2 + \|\psi_1(s)\|_{L^1}) ds. \quad (4.14)$$

To prove the asymptotic compactness of solutions of (4.1) on unbounded domains, we first have to establish such compactness on bounded domains as given below.

Lemma 4.2. *Suppose (3.3)–(3.7) hold and $\{u_n\}_{n=1}^\infty$ be a bounded sequence in $L^2(\mathbb{R}^n)$. Then for every $\tau \in \mathbb{R}$, $t > \tau$ and $\omega \in \Omega$, there exist $u_0 \in L^2(\tau, t; L^2(\mathbb{R}^n))$ and a subsequence $\{u(\cdot, \tau, \omega, u_{n_m})\}_{m=1}^\infty$ of $\{u(\cdot, \tau, \omega, u_n)\}_{n=1}^\infty$ such that $u(s, \tau, \omega, u_{n_m}) \rightarrow u_0(s)$ in $L^2(\mathcal{O}_k)$ as $m \rightarrow +\infty$ for every $k \in \mathbb{N}$ and for almost all $s \in (\tau, t)$.*

Proof. The proof is quite similar to Lemma 3.3 and hence omitted here. \square

Lemma 4.3. *Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for any $\varepsilon > 0$, there exists $T = T(\tau, \omega, D, \varepsilon) \geq 1$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$ and $\sigma \in [\tau-1, \tau]$, the solution u of system (4.1)–(3.2) satisfies*

$$\int_{|x| \geq N} |u(\sigma, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \leq \varepsilon, \quad (4.15)$$

where $u_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$.

Proof. Let ρ be the smooth function as defined in Lemma 3.5, let k be a fixed positive integer which will be specified later. Multiplying (4.2) by $\rho(\frac{|x|^2}{k^2})v$ and then taking the integral over \mathbb{R}^n , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx &= \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta v v dx - \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx \\ &\quad + e^{-\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, e^{\omega(t)} v) v dx + e^{-\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t, x) v dx. \end{aligned} \quad (4.16)$$

We now estimate the terms in (4.16) as follows. First similar to (3.55) and (3.56), we have

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta v v dx \leq - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\nabla u|^2 dx + \frac{c}{k} (\|\nabla v\|^2 + \|v\|^2). \quad (4.17)$$

Using (3.3), we have

$$\begin{aligned} e^{-\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, e^{\omega(t)} v) v dx &\leq -\alpha_1 e^{(p-2)\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^p dx \\ &\quad + e^{-2\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\psi_1(t, x)| dx. \end{aligned} \quad (4.18)$$

By Young's inequality, we find

$$e^{-\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t, x) v dx \leq \frac{\lambda}{4} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{1}{\lambda} e^{-2\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |g(t, x)|^2 dx. \quad (4.19)$$

Then, it follows from (4.16)–(4.19) that

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{3}{2} \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx \\ &\leq \frac{c}{k} \|v\|_{H^1(\mathbb{R}^n)}^2 + c e^{-2\omega(t)} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) (|g|^2 + |\psi_1|) dx. \end{aligned} \quad (4.20)$$

Therefore by (4.20), there is a $N_1 = N_1(\varepsilon) > 0$ such that for all $k \geq N_1$

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{3}{2} \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx \leq \varepsilon \|v\|_{H^1(\mathbb{R}^n)}^2 + c e^{-2\omega(t)} \int_{|x| \geq k} (|g|^2 + |\psi_1|) dx. \quad (4.21)$$

Given $t \geq 1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, multiplying (4.21) by $e^{\frac{3}{2}\lambda t}$ and then integrating over $(\tau - t, \sigma)$ where $\sigma \in [\tau - 1, \tau]$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})|^2 dx &\leq e^{\frac{3}{2}\lambda(\tau-t-\sigma)} \|v_{\tau-t}\|^2 \\
&+ \varepsilon \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)} \|v(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\
&+ ce^{2\omega(-\tau)} \int_{-\infty}^{\sigma-\tau} e^{\frac{3}{2}\lambda(s+\tau-\sigma)-2\omega(s)} \int_{|x|\geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds.
\end{aligned} \tag{4.22}$$

It follows from (4.11) and (4.22) that

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx &\leq e^{2\omega(\sigma-\tau)} e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|u_{\tau-t}\|^2 \\
&+ \varepsilon e^{2\omega(\sigma-\tau)} \int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\
&+ ce^{2\omega(\sigma-\tau)} \int_{-\infty}^{\sigma-\tau} e^{\frac{3}{2}\lambda(s+\tau-\sigma)-2\omega(s)} \int_{|x|\geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds.
\end{aligned} \tag{4.23}$$

By the fact that $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$ and D is tempered, we have

$$\limsup_{t \rightarrow \infty} e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|u_{\tau-t}\|^2 \leq e^{\frac{3}{2}\lambda} \limsup_{t \rightarrow \infty} e^{-\frac{3}{2}\lambda t - 2\omega(-t)} \|D(\tau - t, \theta_{-t}\omega)\|^2 = 0,$$

which means that we can choose $T_1 = T_1(\tau, \omega, D, \varepsilon) \geq 1$ such that for all $t \geq T_1$,

$$e^{2\omega(\sigma-\tau)} e^{\frac{3}{2}\lambda(\tau-t-\sigma)-2\omega(-t)} \|u_{\tau-t}\|^2 \leq \varepsilon. \tag{4.24}$$

By (4.13) and Lemma 4.1, there is a $T_2 = T_2(\tau, \omega, D) \geq T_1$ such that for all $t \geq T_2$,

$$\begin{aligned}
&\int_{\tau-t}^{\sigma} e^{\frac{3}{2}\lambda(s-\sigma)-2\omega(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\
&\leq e^{\frac{3}{2}\lambda} \int_{\tau-t}^{\tau} e^{\frac{3}{2}\lambda(s-\tau)-2\omega(s-\tau)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{H^1(\mathbb{R}^n)}^2 ds \\
&\leq c \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1}) ds < \infty.
\end{aligned} \tag{4.25}$$

By (3.36), there is a $N_2 = N_2(\tau, \varepsilon) \geq N_1$ such that for all $k \geq N_2$

$$\begin{aligned}
& \int_{-\infty}^{\sigma-\tau} e^{\frac{3}{2}\lambda(s+\tau-\sigma)-2\omega(s)} \int_{|x|\geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds \\
& \leq e^{\frac{3}{2}\lambda} \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s)} \int_{|x|\geq k} (|g(s+\tau, x)|^2 + \psi_1(s+\tau, x)) dx ds \leq \varepsilon. \quad (4.26)
\end{aligned}$$

Therefore it follows from (4.23)–(4.26) that for all $\sigma \in [\tau - 1, \tau]$, $t \geq T_2$ and $k \geq N_2$

$$\begin{aligned}
& \int_{|x|\geq\sqrt{2}k} |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \\
& \leq c\varepsilon(1 + \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1}) ds),
\end{aligned}$$

which concludes the proof. \square

We are now ready to present the existence of \mathcal{D} -pullback attractors for Φ_0 .

Theorem 4.1. Suppose (3.3)–(3.5), (3.36) and (3.37) hold. Then the continuous cocycle Φ_0 of equation (4.2) has a unique \mathcal{D} -pullback attractor $\mathcal{A}_0 = \{\mathcal{A}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $L^2(\mathbb{R}^n)$. If, in addition, f , g and ψ_1 are T -periodic functions with respect to t , then the \mathcal{D} -pullback attractor \mathcal{A}_0 is also T -periodic.

Proof. For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote $K_0(\tau, \omega)$ by

$$K_0(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq R_0(\tau, \omega)\}, \quad (4.27)$$

where $R_0(\tau, \omega)$ is given by

$$R_0(\tau, \omega) = M \int_{-\infty}^0 e^{\frac{3}{2}\lambda s - 2\omega(s)} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1}) ds. \quad (4.28)$$

Here M is a positive constant as in (4.5). For every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, it follows from Lemma 4.1 that there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$

$$\Phi_0(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_0(\tau, \omega). \quad (4.29)$$

By (3.8) and (3.37), we can verify that K_0 given by (4.27) is tempered. Consequently, $K_0 \in \mathcal{D}$ is a closed measurable \mathcal{D} -pullback absorbing set for Φ_0 . Similar to Lemma 3.6, one can show that Φ_0 is \mathcal{D} -pullback asymptotically compact in $L^2(\mathbb{R}^n)$ by using Lemmas 4.2 and 4.3. Then the desired result follows from Proposition 2.1 immediately. \square

We now propose to approximate the solution of equation (4.1) by the following pathwise deterministic equation: for any $\tau \in \mathbb{R}$,

$$\frac{du_\delta}{dt} = \Delta u_\delta - \lambda u_\delta + f(t, x, u_\delta) + g(t, x) + u_\delta \mathcal{G}_\delta(\theta_t \omega), \quad t > \tau, \quad (4.30)$$

along with the initial condition

$$u_\delta(\tau, x) = u_{\delta, \tau}(x), \quad x \in \mathbb{R}^n. \quad (4.31)$$

To indicate the dependence of solutions on the parameter δ , we write the solution of equation (4.30) as u_δ . As we know, for every $\delta \neq 0$, equation (4.30) defines a continuous cocycle Φ_δ in $L^2(\mathbb{R}^n)$ which possesses a unique \mathcal{D} -pullback attractor \mathcal{A}_δ . We now want to study the limiting behavior of solutions of (4.30) as $\delta \rightarrow 0$. We will first prove the convergence of solutions of (4.30) to that of the stochastic equation (4.1) in $L^2(\mathbb{R}^n)$ as $\delta \rightarrow 0$. We then prove the upper semicontinuity of attractors \mathcal{A}_δ as $\delta \rightarrow 0$.

To prove the convergence of solutions, we introduce the transformation:

$$v_\delta(t, \tau, \omega, v_{\delta, \tau}) = e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} u_\delta(t, \tau, \omega, u_{\delta, \tau}) \text{ and } v_{\delta, \tau} = e^{-\int_0^\tau \mathcal{G}_\delta(\theta_r \omega) dr} u_{\delta, \tau}. \quad (4.32)$$

Then we get from (4.30) that

$$\frac{dv_\delta}{dt} - \Delta v_\delta + \lambda v_\delta = e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} f(t, x, u_\delta) + e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} g(t, x), \quad t > \tau, \quad (4.33)$$

with the initial conditions

$$v_\delta(\tau, x) = v_{\delta, \tau}(x), \quad x \in \mathbb{R}^n. \quad (4.34)$$

For the solutions of (4.33)–(4.34) we have

Lemma 4.4. *Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $T > 0$, there exist $\delta_0 = \delta_0(\tau, \omega, T) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$, the solution v_δ of system (4.33)–(4.34) satisfies*

$$\begin{aligned} \|v_\delta(t, \tau, \omega, v_{\delta, \tau})\|^2 &+ \int_\tau^t \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{H^1}^2 ds + \int_\tau^t \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{L^p}^p ds \\ &\leq c \|v_{\delta, \tau}\|^2 + c \int_\tau^t (\|g(s)\|^2 + \|\psi_1(s)\|_{L^1}) ds. \end{aligned} \quad (4.35)$$

Proof. By (4.33), we get for every $\omega \in \Omega$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_\delta\|^2 + \lambda \|v_\delta\|^2 + \|\nabla v_\delta\|^2 &= e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} f(t, x, u_\delta) v_\delta dx \\ &\quad + e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} g(t, x) v_\delta dx. \end{aligned} \quad (4.36)$$

By (4.36) we obtain that

$$\begin{aligned} e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} f(t, x, u_\delta) v_\delta dx &= e^{-2 \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} f(t, x, u_\delta) u_\delta dx \\ &\leq -\alpha_1 e^{(p-2) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|v_\delta\|_{L^p}^p + e^{-2 \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|\psi_1(t)\|_{L^1}. \end{aligned} \quad (4.37)$$

For the last term in (4.36) Young's inequality implies that

$$e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} g(t, x) v_\delta dx \leq \frac{\lambda}{8} \|v_\delta\|^2 + \frac{2}{\lambda} e^{-2 \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|g(t)\|^2. \quad (4.38)$$

Thus it follows from (4.36)–(4.38) that

$$\begin{aligned} \frac{d}{dt} \|v_\delta\|^2 + \frac{\lambda}{4} \|v_\delta\|^2 + 2 \|\nabla v_\delta\|^2 &\leq -\frac{3}{2} \lambda \|v_\delta\|^2 - 2\alpha_1 e^{(p-2) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|v_\delta\|_{L^p}^p \\ &\quad + \frac{4}{\lambda} e^{-2 \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|g(t)\|^2 + 2e^{-2 \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \|\psi_1(t)\|_{L^1}. \end{aligned} \quad (4.39)$$

Multiplying (4.39) by $e^{\frac{3}{2}\lambda t}$ and then integrating over (τ, t) with $t \geq \tau$, we get for every $\omega \in \Omega$

$$\begin{aligned} \|v_\delta(t, \tau, \omega, v_{\delta, \tau})\|^2 + \frac{\lambda}{4} \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|^2 ds &+ 2 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t)} \|\nabla v_\delta(s, \tau, \omega, v_{\delta, \tau})\|^2 ds \\ &+ 2\alpha_1 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t) + (p-2) \int_0^s \mathcal{G}_\delta(\theta_r \omega) dr} \|v_\delta(s, \tau, \omega, v_{\delta, \tau})\|_{L^p}^p ds \\ &\leq e^{-\frac{3}{2}\lambda(t-\tau)} \|v_{\delta, \tau}\|^2 + 2 \int_{\tau}^t e^{\frac{3}{2}\lambda(s-t) - 2 \int_0^s \mathcal{G}_\delta(\theta_r \omega) dr} \left(\frac{2}{\lambda} \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} \right) ds. \end{aligned} \quad (4.40)$$

Then by (3.11) and (4.40) we get (4.35). \square

Lemma 4.5. Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\delta \neq 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T$, the solution u_δ of system (4.30) satisfies

$$\begin{aligned}
 & \|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 + \frac{\lambda}{4} \int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r\omega) dr} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
 & + 2 \int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r\omega) dr} \|\nabla u_\delta(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
 & \leq 4 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r\omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds,
 \end{aligned} \tag{4.41}$$

where $u_{\delta, \tau-t} \in D(\tau - t, \theta_{-\tau}\omega)$.

Proof. It follows from (4.32) and (4.39) that

$$\begin{aligned}
 & \|u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 + \frac{\lambda}{4} \int_{\tau-t}^\tau e^{\frac{3}{2}\lambda(s-\tau) - 2 \int_s^\tau \mathcal{G}_\delta(\theta_{r-\tau}\omega) dr} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
 & + 2 \int_{\tau-t}^\tau e^{\frac{3}{2}\lambda(s-\tau) - 2 \int_s^\tau \mathcal{G}_\delta(\theta_{r-\tau}\omega) dr} \|\nabla u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
 & \leq e^{-\frac{3}{2}\lambda t + 2 \int_{\tau-t}^\tau \mathcal{G}_\delta(\theta_{r-\tau}\omega) dr} \|u_{\delta, \tau-t}\|^2 + 2 \int_{\tau-t}^\tau e^{\frac{3}{2}\lambda(s-\tau) + 2 \int_s^\tau \mathcal{G}_\delta(\theta_{r-\tau}\omega) dr} \left(\frac{2}{\lambda} \|g(s)\|^2 \right. \\
 & \quad \left. + \|\psi_1(s)\|_{L^1} \right) ds \\
 & \leq e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r\omega) dr} \|u_{\delta, \tau-t}\|^2 + 2 \int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r\omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 \right. \\
 & \quad \left. + \|\psi_1(s + \tau)\|_{L^1} \right) ds,
 \end{aligned} \tag{4.42}$$

which implies the desired estimates. \square

Note that Lemma 4.5 shows that problem (4.30)–(4.31) has a tempered pullback absorbing set.

Lemma 4.6. Suppose (3.3)–(3.5), (3.36) and (3.37) hold. Then the continuous cocycle Φ_δ of system (4.30)–(4.31) has a closed measurable \mathcal{D} -pullback absorbing set $K_\delta = \{K_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$K_\delta(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq R_\delta(\tau, \omega)\}, \tag{4.43}$$

where $R_\delta(\tau, \omega)$ is given by

$$R_\delta(\tau, \omega) = 4 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds. \quad (4.44)$$

In addition, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{\delta \rightarrow 0} R_\delta(\tau, \omega) = R_0(\tau, \omega), \quad (4.45)$$

where $R_0(\tau, \omega)$ is defined in (4.28).

Proof. Note K_δ given by (4.43) is a closed measurable random set in $L^2(\mathbb{R}^n)$. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D \in \mathcal{D}$, it follows from Lemma 4.5 that there exists $T_0 = T_0(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T_0$

$$\Phi_\delta(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K_\delta(\tau, \omega). \quad (4.46)$$

This shows that K_δ pullback attracts all elements in \mathcal{D} . By (3.8) one may verify that K_δ is tempered. The convergence (4.45) can be obtained by the Lebesgue's dominated convergence theorem as in [26]. The details are omitted here. \square

For later purpose, we need uniform estimates on the tails of solutions with respect to δ .

Lemma 4.7. Suppose (3.3)–(3.5) and (3.36) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varepsilon > 0$, there exist $\delta_0 = \delta_0(\omega) > 0$, $T = T(\tau, \omega, \varepsilon) > 0$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$ and $0 < |\delta| < \delta_0$, the solution u_δ of (4.30) satisfies

$$\int_{|x| \geq N} |u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_{\delta, \tau-t})|^2 dx \leq \varepsilon, \quad (4.47)$$

where $u_{\delta, \tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$ with K_δ given by (4.43).

Proof. Let ρ be the function defined in Lemma 3.5. By (4.33) we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_\delta|^2 - \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta v_\delta v_\delta dx + \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v_\delta|^2 \\ &= e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, u_\delta) v_\delta dx + e^{-\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) v_\delta g(t, x) dx. \end{aligned} \quad (4.48)$$

By (4.48) and the process to derive (4.23) we obtain

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u_\delta(\tau, \tau - t, \theta_{-t}\omega, u_{\delta, \tau-t})|^2 dx \leq e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u_{\delta, \tau-t}|^2 dx$$

$$\begin{aligned}
 & + \frac{2c_0}{k} \int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})\|_{H^1}^2 ds \\
 & + 2 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{|x| \geq k} \left(\frac{1}{\lambda} |g(s + \tau)|^2 + |\psi_1(s + \tau)| \right) ds.
 \end{aligned} \tag{4.49}$$

Note that

$$2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr = -2 \int_s^{s+\delta} \frac{\omega(r)}{\delta} dr + 2 \int_0^\delta \frac{\omega(r)}{\delta} dr. \tag{4.50}$$

By the continuity of ω we have $\lim_{\delta \rightarrow 0} \int_0^\delta \frac{\omega(r)}{\delta} dr = 0$, and hence there exists $\delta_1 = \delta_1(\omega) > 0$ such that for all $0 < |\delta| < \delta_1$,

$$\left| 2 \int_0^\delta \frac{\omega(r)}{\delta} dr \right| \leq 1. \tag{4.51}$$

Similarly, there exists r_1 between s and $s + \delta$ such that $2 \int_s^{s+\delta} \frac{\omega(s)}{\delta} ds = 2\omega(r_1)$, which along with (3.8) implies that there exists $T_1 = T_1(\omega) < 0$ such that for all $s \leq T_1$ and $|\delta| \leq 1$,

$$2 \left| \int_s^{s+\delta} \frac{\omega(s)}{\delta} ds \right| \leq \frac{\lambda}{8} - \frac{\lambda}{8} s. \tag{4.52}$$

Let $\delta_2 = \min\{\delta_1, 1\}$. By (4.50)–(4.52) we get for all $0 < |\delta| < \delta_2$ and $s \leq T_1$,

$$2 \left| \int_s^0 \mathcal{G}_\delta(\theta_s \omega) ds \right| < \frac{\lambda}{8} - \frac{\lambda}{8} s + 1. \tag{4.53}$$

On the other hand, as in (3.11), there exist $\delta_0 = \delta_0(\omega) \in (0, \delta_2)$ and $c_1(\omega) > 0$ such that for all $0 < |\delta| < \delta_0$ and $T_1 \leq s \leq 0$,

$$2 \left| \int_s^0 \mathcal{G}_\delta(\theta_s \omega) ds \right| \leq c_1(\omega),$$

which along with (4.53) implies that for all $0 < |\delta| < \delta_0$ and $s \leq 0$,

$$2 \left| \int_s^0 \mathcal{G}_\delta(\theta_s \omega) ds \right| < \frac{\lambda}{8} - \frac{\lambda}{8} s + c_2(\omega), \tag{4.54}$$

where $c_2(\omega) = 1 + c_1(\omega)$. By (4.44) and the assumption $u_{\delta, \tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$, we get

$$\begin{aligned} e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_{\delta, \tau-t}\|^2 &\leq e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|K_\delta(\tau - t, \theta_{-t}\omega)\|^2 \\ &\leq 4e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_{r-t}\omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau - t)\|^2 + \|\psi_1(s + \tau - t)\|_{L^1} \right) ds \\ &\leq 4e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{-\infty}^{-t} e^{\frac{3}{2}\lambda(s+t) + 2 \int_s^{-t} \mathcal{G}_\delta(\theta_r \omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds. \end{aligned} \quad (4.55)$$

By (4.54) we find that for all $0 < |\delta| < \delta_0$, $s \leq 0$ and $t \geq 0$,

$$2 \left| \int_s^{-t} \mathcal{G}_\delta(\theta_r \omega) dr \right| \leq 2 \left| \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr \right| + 2 \left| \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr \right| \leq \frac{\lambda}{4} + 2c_2 + \frac{\lambda}{8}t - \frac{\lambda}{8}s,$$

which along with (3.36) and (4.54)–(4.55) shows that for all $0 < |\delta| < \delta_0$,

$$\begin{aligned} &e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_{\delta, \tau-t}\|^2 \\ &\leq 4e^{\frac{3}{8}\lambda + 3c_2} e^{-\frac{11}{8}\lambda t} \int_{-\infty}^{-t} e^{\frac{3}{2}\lambda(s+t) + \frac{1}{8}\lambda t - \frac{1}{8}\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds \\ &\leq 4e^{\frac{3}{8}\lambda + 3c_2} e^{-\frac{11}{8}\lambda t} \int_{-\infty}^{-t} e^{\frac{9}{8}\lambda(s+t) + \frac{1}{8}\lambda t - \frac{1}{8}\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds \\ &\leq 4e^{\frac{3}{8}\lambda + 3c_2} e^{-\frac{1}{8}\lambda t} \int_{-\infty}^{-t} e^{\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (4.56)$$

Thus, there exists $T_2 = T_2(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T_2$ and $0 < |\delta| < \delta_0$,

$$e^{-\frac{3}{2}\lambda t + 2 \int_{-t}^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_{\delta, \tau-t}\|^2 \leq \frac{\varepsilon}{3}. \quad (4.57)$$

By (4.42) and (4.54), there exists $T_3 = T_3(\tau, \omega) > 0$ such that for all $t \geq T_3$ and $0 < |\delta| < \delta_0$,

$$\begin{aligned} &\int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|_{H^1}^2 ds \\ &\leq 1 + 2 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds \end{aligned}$$

$$\leq 1 + 2e^{\frac{\lambda}{8} + c_2} \int_{-\infty}^0 e^{\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds,$$

which implies that there exists $N_1 = N_1(\tau, \omega, \varepsilon) > 0$ such that for all $k \geq N_1$, $t \geq T_3$ and $0 < |\delta| < \delta_0$,

$$\frac{2c_0}{k} \int_{-t}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \|u_\delta(s + \tau, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})\|_{H^1}^2 ds \leq \frac{\varepsilon}{3}. \quad (4.58)$$

By (4.54) we get for all $0 < |\delta| \leq \delta_0$,

$$\begin{aligned} & 2 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{|x| \geq k} \left(\frac{1}{\lambda} |g(s + \tau)|^2 + |\psi_1(s + \tau)| \right) ds \\ & \leq 2e^{\frac{\lambda}{8} + c_2(\omega)} \int_{-\infty}^0 e^{\lambda s} \int_{|x| \geq k} \left(\frac{1}{\lambda} |g(s + \tau)|^2 + |\psi_1(s + \tau)| \right) dx ds, \end{aligned}$$

which implies that there exists $N_2 = N_2(\tau, \omega, \varepsilon) > 0$ such that for all $k \geq N_2$ and $0 < |\delta| < \delta_0$,

$$2 \int_{-\infty}^0 e^{\frac{3}{2}\lambda s + 2 \int_s^0 \mathcal{G}_\delta(\theta_r \omega) dr} \int_{|x| \geq k} \left(\frac{1}{\lambda} |g(s + \tau)|^2 + |\psi_1(s + \tau)| \right) dx ds \leq \frac{\varepsilon}{3}. \quad (4.59)$$

Let $N = \max\{N_1, N_2\}$ and $T = \max\{T_1, T_2\}$, it follows from (4.49) and (4.57)–(4.59) that for all $t \geq T$, $k \geq N$ and $0 < |\delta| < \delta_0$,

$$\int_{|x| \geq \sqrt{2}k} |u_\delta(\tau, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})|^2 \leq \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |u_\delta(\tau, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})|^2 \leq \varepsilon.$$

This completes the proof. \square

For the attractor \mathcal{A}_δ of Φ_δ , we have the following uniform compactness.

Lemma 4.8. Suppose (3.3)–(3.5), (3.36) and (3.37) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, if $\delta_n \rightarrow 0$ and $u_n \in \mathcal{A}_{\delta_n}(\tau, \omega)$, then $\{u_n\}_{n=1}^\infty$ is precompact in $L^2(\mathbb{R}^n)$.

Proof. Let $\delta_0 = \delta_0(\omega)$ be the number in Lemma 4.7. Given $\varepsilon > 0$, we will show that $\bigcup_{0 < |\delta| < \delta_0} \mathcal{A}_\delta(\tau, \omega)$ has a finite covering of balls of radius less than ε . By (4.44) and (4.54) we find that for all $0 < |\delta| < \delta_0$,

$$R_\delta(\tau, \omega) \leq 4e^{\frac{1}{8}\lambda + c_2} \int_{-\infty}^0 e^{\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds. \quad (4.60)$$

Denote

$$B(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq R(\tau, \omega)\}, \quad (4.61)$$

where $R(\tau, \omega)$ is given by

$$R(\tau, \omega) = 4e^{\frac{1}{8}\lambda + c_2} \int_{-\infty}^0 e^{\lambda s} \left(\frac{2}{\lambda} \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} \right) ds. \quad (4.62)$$

By (4.60)–(4.62) we find that $K_\delta(\tau, \omega) \subseteq B(\tau, \omega)$ for all $0 < |\delta| < \delta_0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Therefore, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\bigcup_{0 < |\delta| < \delta_0} \mathcal{A}_\delta(\tau, \omega) \subseteq \bigcup_{0 < |\delta| < \delta_0} K_\delta(\tau, \omega) \subseteq B(\tau, \omega). \quad (4.63)$$

By Lemma 4.7, there exist $T = T(\tau, \omega, \varepsilon) > 0$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$ and $0 < |\delta| < \delta_0$,

$$\int_{|x| \geq N} |u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})|^2 dx \leq \frac{\varepsilon}{2} \quad (4.64)$$

for any $u_{\delta, \tau-t} \in K_\delta(\tau - t, \theta_{-t}\omega)$. By (4.64) and the invariance of \mathcal{A}_δ , we get

$$\int_{|x| \geq N} |u|^2 dx \leq \frac{\varepsilon}{2}, \quad \text{for all } u \in \bigcup_{0 < |\delta| < \delta_0} \mathcal{A}_\delta(\tau, \omega). \quad (4.65)$$

On the other hand, from [26], we find that the sequence $\{u_n\}_{n=1}^\infty$ is precompact in $L^2(\mathcal{O}_N)$ with $\mathcal{O}_N = \{x \in \mathbb{R}^n : |x| \leq N\}$. This together with (4.65) completes the proof. \square

Next, we establish the convergence of solutions of (4.30) as $\delta \rightarrow 0$. For that purpose, we further assume the following condition on f : there exists $\psi_4(t, x) \in L_{loc}^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))$ such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$

$$\left| \frac{\partial f}{\partial s}(t, x, s) \right| \leq \psi_4(t, x)(1 + |s|^{p-2}). \quad (4.66)$$

Lemma 4.9. Suppose (3.3)–(3.5) and (4.66) hold. Let u and u_δ are the solutions of (4.1) and (4.30), respectively. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $\varepsilon \in (0, 1)$, there exists $\delta_0 = \delta_0(\tau, \omega, T, \varepsilon) > 0$ and $c = c(\tau, \omega, T) > 0$ such that for all $0 < |\delta| < \delta_0$ and $t \in [\tau, \tau + T]$,

$$\|u_{\delta}(t, \tau, \omega, u_{\delta, \tau}) - u(t, \tau, \omega, u_{\tau})\|^2 \leq c \|u_{\delta, \tau} - u_{\tau}\|^2 + c\varepsilon \left(1 + \|u_{\delta, \tau}\|^2 + \|u_{\tau}\|^2 + \int_{\tau}^t (\|\psi_2(s)\|_{L^{p_1}}^{p_1} + \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} ds)\right). \quad (4.67)$$

Proof. The proof is quite similar to the case of bounded domains [26] based on Corollary 4.1 and Lemma 4.4, and hence is omitted here. \square

The main result of this section is given below.

Theorem 4.2. Suppose (3.3)–(3.5), (3.36), (3.37) and (4.66) hold. Then for every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \rightarrow 0} d_{L^2(\mathbb{R}^n)}(\mathcal{A}_{\delta}(\tau, \omega), \mathcal{A}_0(\tau, \omega)) = 0. \quad (4.68)$$

Proof. Let $\delta_n \rightarrow 0$ and $u_{\delta_n, \tau} \rightarrow u_{\tau}$ in $L^2(\mathbb{R}^n)$. Then by Lemma 4.9, we find that for all $\tau \in \mathbb{R}$, $t \geq 0$ and $\omega \in \Omega$,

$$\Phi_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \rightarrow \Phi_0(t, \tau, \omega, u_{\tau}) \quad \text{in } L^2(\mathbb{R}^n). \quad (4.69)$$

By (4.45) and (4.60) we have for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{\delta \rightarrow 0} \|K_{\delta}(\tau, \omega)\| \leq R_0(\tau, \omega), \quad (4.70)$$

which along with (4.69) and Lemma 4.8 shows that all conditions (2.5) and (2.7)–(2.8) in Proposition 2.2 are fulfilled, and thus (4.68) follows. \square

5. Reaction–diffusion equations driven by additive noise

In this section, we discuss approximations of the stochastic equation with additive white noise:

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + h(x) \frac{dw(t)}{dt}, \quad t > \tau, \quad (5.1)$$

where h belongs to $H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. Consider the following pathwise deterministic equation:

$$\frac{\partial u}{\partial t} = \Delta u - \lambda u + f(t, x, u) + g(t, x) + h(x) \mathcal{G}_{\delta}(\theta_t \omega), \quad t > \tau. \quad (5.2)$$

We will establish the relations between the solutions of (5.1) and (5.2). To that end, we need to transform the stochastic equation (5.1) into a pathwise deterministic one. Let $v(t, \tau, \omega) = u(t, \tau, \omega) - h(x)\omega(t)$. By (5.1) we have

$$\frac{\partial v}{\partial t} - \Delta v + \lambda v = f(t, x, v + h(x)\omega(t)) + g(t, x) - \lambda h(x)\omega(t) + \omega(t) \Delta h(x), \quad t > \tau, \quad (5.3)$$

with initial condition

$$v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n, \quad (5.4)$$

where $v_\tau = u_\tau - h(x)\omega(\tau)$. Given $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $v_\tau \in L^2(\mathbb{R}^n)$, by (3.3)–(3.5), one can show that system (5.3)–(5.4) has a unique solution $v(\cdot, \tau, \omega, v_\tau) \in C([\tau, \infty), L^2(\mathbb{R}^n))$. In addition $v(\cdot, \tau, \omega, v_\tau)$ is continuous in v_τ with respect to the norm of $L^2(\mathbb{R}^n)$ and is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$ -measurable in $\omega \in \Omega$. Thus, we may define a continuous cocycle $\Psi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ for system (5.1) and (3.2) by

$$\Psi_0(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau) = v(t + \tau, \tau, \theta_{-\tau}\omega, v_\tau) + h(x)(\omega(t) - \omega(-\tau)), \quad (5.5)$$

where $v_\tau = u_\tau + h(x)\omega(-\tau)$. Hereafter, we assume the following conditions for the function ψ_2 in (3.4): for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\lambda s} \|\psi_2(s)\|_{L^{p_1}}^{p_1} ds < \infty \quad (5.6)$$

and for any $c > 0$

$$\lim_{t \rightarrow -\infty} e^{ct} \int_{-\infty}^0 e^{\lambda s} \|\psi_2(s+t)\|_{L^{p_1}}^{p_1} ds = 0. \quad (5.7)$$

Lemma 5.1. Suppose (3.3)–(3.5), (3.36), (3.37), (5.6) and (5.7) hold. Then the continuous cocycle Ψ_0 has a closed measurable \mathcal{D} -pullback absorbing set $B_0 = \{B_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$B_0(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq M_0(\tau, \omega)\}, \quad (5.8)$$

where $M_0(\tau, \omega)$ is given by

$$\begin{aligned} M_0(\tau, \omega) = & c + c \int_{-\infty}^0 e^{\lambda s} (\|g(s+\tau)\|^2 + \|\psi_1(s+\tau)\|_{L^1} + \|\psi_2(s+\tau)\|_{L^{p_1}}^{p_1}) ds \\ & + c \int_{-\infty}^0 e^{\lambda s} |\omega(s) - \omega(-\tau)|^p ds + c|\omega(-\tau)|^2, \end{aligned} \quad (5.9)$$

with c being a positive number independent of τ , ω and \mathcal{D} .

Proof. By (5.3) we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v\|^2 + \|\nabla v\|^2 + \lambda \|v\|^2 \\ = & (f(t, x, v + h(x)\omega(t)), v) + (g(t, x), v) - \lambda \omega(t)(h(x), v) + (\Delta h, v)\omega(t). \end{aligned} \quad (5.10)$$

By (3.3) and (3.4) we obtain that there is a $c_1 > 0$ such that

$$\begin{aligned} (f(v + a\omega(t), t), v) &= (f(t, x, v + h(x)\omega(t)), v + h\omega) - \omega(f(t, x, v + h\omega), h) \\ &\leq -\frac{1}{2}\alpha_1 \|v + h\omega\|_{L^p}^p + c_1 |\omega(t)|^p \|h\|_{L^p}^p + \|\psi_1(t)\|_{L^1} + \|\psi_2(t)\|_{L^{p_1}}^{p_1}. \end{aligned} \quad (5.11)$$

By Young's inequality, the last three terms in (5.10) is bounded by

$$\frac{\lambda}{4} \|v\|^2 + \frac{2}{\lambda} \|g(t)\|^2 + 2\lambda |\omega(t)|^2 \|h\|^2 + \frac{1}{2} \|\nabla v\|^2 + \frac{1}{2} |\omega(t)|^2 \|\nabla h\|^2. \quad (5.12)$$

Thus it follows from (5.10)–(5.12) that

$$\begin{aligned} &\frac{d}{dt} \|v\|^2 + \frac{\lambda}{2} \|v\|^2 + \|\nabla v\|^2 + \alpha_1 \|u\|_{L^p}^p \\ &\leq -\lambda \|v\|^2 + \frac{4}{\lambda} \|g(t)\|^2 + 2\|\psi_1(t)\|_{L^1} + 2\|\psi_2(t)\|_{L^{p_1}}^{p_1} + c_2(1 + |\omega(t)|^p), \end{aligned} \quad (5.13)$$

where c_2 is a positive constant. Multiplying (5.13) by $e^{\lambda t}$, replacing ω by $\theta_{-\tau}\omega$ and then integrating over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$, we get

$$\begin{aligned} &\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\ &+ \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|_{L^p}^p ds \\ &\leq e^{\lambda(\tau-t-\sigma)} \|v_{\tau-t}\|^2 + \frac{c_2}{\lambda} + c_2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} |\omega(s - \tau) - \omega(-\tau)|^p ds \\ &+ 2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left(\frac{2}{\lambda} \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} + \|\psi_2(s)\|_{L^{p_1}}^{p_1} \right) ds. \end{aligned} \quad (5.14)$$

Note that for any $s \geq \tau - t$,

$$u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) = v(s, \tau - t, \theta_{-\tau}\omega, v_{\tau-t}) + h(\omega(s - \tau) - \omega(-\tau)),$$

which together with (5.14) implies

$$\|u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds$$

$$\begin{aligned}
& + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|_{L^p}^p ds \\
& \leq 2\|v(\sigma, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 + \lambda \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds \\
& + 2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla v(s, \tau-t, \theta_{-\tau}\omega, v_{\tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|_{L^p}^p ds \\
& + 2\|h\|^2 |\omega(\sigma-\tau) - \omega(-\tau)|^2 + \lambda \|h\|^2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} |\omega(s-\tau) - \omega(-\tau)|^2 ds \\
& + 2\|\nabla h\|^2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} |\omega(s-\tau) - \omega(-\tau)|^2 ds \\
& \leq 4e^{\lambda(\tau-t-\sigma)} (\|u_{\tau-t}\|^2 + \|h\|^2 |\omega(-t) - \omega(-\tau)|^2) + c_3 + c_3 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} |\omega(s-\tau) - \omega(-\tau)|^p ds \\
& + 4 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left(\frac{2}{\lambda} \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} + \|\psi_2(s)\|_{L^{p_1}}^{p_1} \right) ds + 2\|h\|^2 |\omega(\sigma-\tau) - \omega(-\tau)|^2
\end{aligned} \tag{5.15}$$

From (5.15) we find that there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that for all $t \geq T_1$,

$$\begin{aligned}
& \|u(\sigma, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds \\
& + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u(s, \tau-t, \theta_{-\tau}\omega, u_{\tau-t})\|_{L^p}^p ds \\
& \leq c + c \int_{-\infty}^0 e^{\lambda s} |\omega(s+\sigma-\tau) - \omega(-\tau)|^p ds \\
& + c \int_{-\infty}^0 e^{\lambda s} (\|g(s+\sigma)\|^2 + \|\psi_1(s+\sigma)\|_{L^1} + \|\psi_2(s+\sigma)\|_{L^{p_1}}^{p_1}) ds + c|\omega(\sigma-\tau) - \omega(-\tau)|^2,
\end{aligned} \tag{5.16}$$

where c is a positive constant independent of τ , ω and D . Note that (5.16) implies that for all $t \geq T_1$

$$u(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq B_0(\tau, \omega), \quad (5.17)$$

where $B_0(\tau, \omega)$ is given by (5.8). By using (3.8), (3.37) and (5.7), one can easily check that B_0 is tempered in $L^2(\mathbb{R}^n)$, which along with (5.17) completes the proof. \square

We now deal with the uniform estimates on the tails of solutions of (5.1) based on the estimates (5.16).

Lemma 5.2. *Suppose (3.3)–(3.5), (3.36), (3.37), (5.6) and (5.7) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ and for any $\varepsilon > 0$, there exists $T = T(\tau, \omega, D, \varepsilon) > 0$ and $N = N(\tau, \omega, \varepsilon) > 0$ such that for all $t \geq T$ and $\sigma \in [\tau - 1, \tau]$, the solution u of equation (5.1) satisfies*

$$\int_{|x| \geq N} |u(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\tau-t})|^2 dx \leq \varepsilon, \quad (5.18)$$

where $u_{\tau-t} \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Let ρ be the smooth function defined in Lemma 3.5. It follows from (5.3) and (4.17) that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx &\leq \frac{c}{k} \|v\|_{H^1(\mathbb{R}^n)} - 2\lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + h\omega) v dx \\ &\quad + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) g(t) v dx - 2\lambda \omega(t) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) h v dx \\ &\quad + 2\omega(t) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) \Delta h v dx. \end{aligned} \quad (5.19)$$

Using (3.3) and (3.4), we get

$$\begin{aligned} &2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + h\omega) v dx \\ &= 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + h\omega) (v + h\omega) dx - 2\omega(t) \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) f(t, x, v + h\omega) h dx \\ &\leq -2\alpha_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v + h\omega|^p dx + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\psi_1(t, x)| dx \\ &\quad + 2|\omega(t)| \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |h| (\alpha_2 |v + h\omega|^{p-1} + \psi_2(t)) dx \\ &\leq -\alpha_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v + h\omega|^p dx + 2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\psi_1(t, x)| dx \end{aligned}$$

$$+ c_1 |\omega(t)|^p \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |h|^p dx + c_1 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |\psi_2(t, x)|^{p_1} dx, \quad (5.20)$$

where c_1 is a positive constant. The last three terms in (5.19) are bounded by

$$\frac{\lambda}{2} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + c_2 |\omega(t)|^2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) (|h|^2 + |\Delta h|^2) dx + c_2 \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |g(t)|^2 dx. \quad (5.21)$$

Then, it follows from (5.19)–(5.21) that there is a positive constant c such that ,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx + \frac{3}{2} \lambda \int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v|^2 dx &\leq \frac{c}{k} \|v\|_{H^1(\mathbb{R}^n)}^2 + c |\omega(t)|^p \int_{|x| \geq k} |h|^p dx \\ &+ c |\omega(t)|^2 \int_{|x| \geq k} (|h|^2 + |\Delta h|^2) dx + c \int_{|x| \geq k} (|g(t)|^2 + |\psi_1(t)| + |\psi_2(t)|^{p_1}) dx. \end{aligned} \quad (5.22)$$

Then (5.18) follows from (5.16), (5.22) and the arguments of Lemma 4.3. The details are omitted. \square

As a consequence of Lemmas 5.1 and 5.2, we obtain the existence of \mathcal{D} -pullback random attractors for Ψ_0 .

Theorem 5.1. *Suppose (3.3)–(3.5), (3.36), (3.37), (5.6) and (5.7) hold. Then the continuous cocycle Ψ_0 associated with equation (5.1) has a unique \mathcal{D} -pullback random attractor $\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \in \mathcal{D}$ in $L^2(\mathbb{R}^n)$. If, in addition, f, g, ψ_1 and ψ_2 are T -periodic function with respect to t , then the \mathcal{D} -pullback random attractor $\tilde{\mathcal{A}}_0(\tau, \omega)$ is also T -periodic.*

As we proved in the previous sections, for every $\delta \neq 0$, equation (5.2) defines a continuous cocycle Ψ_δ in $L^2(\mathbb{R}^n)$ which possesses a unique \mathcal{D} -pullback random attractor $\tilde{\mathcal{A}}_\delta(\tau, \omega)$. In what follows, we will discuss the convergence of these attractors as $\delta \rightarrow 0$. Denote

$$\begin{aligned} v_\delta(t, \tau, \omega, v_{\delta, \tau}) &= u_\delta(t, \tau, \omega, u_{\delta, \tau}) - h(x) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr \\ \text{with } v_{\delta, \tau} &= u_{\delta, \tau} - h(x) \int_0^\tau \mathcal{G}_\delta(\theta_r \omega) dr. \end{aligned} \quad (5.23)$$

Then we get from (5.2) and (5.23) that

$$\begin{aligned} \frac{dv_\delta}{dt} + \Delta v_\delta + \lambda v_\delta &= f(t, x, v_\delta + h(x) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr) + g(t, x) \\ &\quad + \Delta h \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr - \lambda h(x) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr \end{aligned} \quad (5.24)$$

with initial condition

$$v_\delta(\tau, x) = v_{\delta, \tau}(x), \quad x \in \mathbb{R}^n. \quad (5.25)$$

Lemma 5.3. Suppose (3.3)–(3.5), (3.36), (3.37), (5.6) and (5.7) hold. Then the continuous cocycle Ψ_δ associated with system (5.2) and (3.2) has a closed measurable \mathcal{D} -pullback absorbing set $B_\delta = \{B_\delta(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, which is given by for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$B_\delta(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq M_\delta(\tau, \omega)\}, \quad (5.26)$$

where $M_\delta(\tau, \omega)$ is given by

$$\begin{aligned} M_\delta(\tau, \omega) &= c \int_{-\infty}^0 e^{\lambda s} (\|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_{L^1} + \|\psi_2(s + \tau)\|_{L^{p_1}}^{p_1}) ds \\ &\quad + c \int_{-\infty}^0 e^{\lambda s} \left| \int_{-\tau}^s \mathcal{G}_\delta(\theta_r \omega) dr \right|^p ds + c \left| \int_{-\tau}^0 \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 + c. \end{aligned} \quad (5.27)$$

Here c is a positive constant independent of τ, ω and δ . In addition, we have for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$

$$\lim_{\delta \rightarrow 0} M_\delta(\tau, \omega) = M_0(\tau, \omega), \quad (5.28)$$

where $M_0(\tau, \omega)$ is defined as in (5.9) with a different constant c .

Proof. By (5.24), we get for every $\omega \in \Omega$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_\delta\|^2 + \|\nabla v_\delta\|^2 + \lambda \|v_\delta\|^2 &= \int_{\mathbb{R}^n} f(t, x, v_\delta + h(x) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr) v_\delta dx \\ &\quad + (g(t), v_\delta) - \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr (\nabla h, \nabla v_\delta) + \lambda \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr (h, v_\delta). \end{aligned} \quad (5.29)$$

By (3.3) we obtain that

$$\int_{\mathbb{R}^n} f(t, x, v_\delta + h(x)) \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr v_\delta dx$$

$$\leq -\frac{\alpha_1}{2} \|u_\delta\|_{L^p}^p + \|\psi_1(t)\|_{L^1} + \|\psi_2(t)\|_{L^{p_1}}^{p_1} + c_1 \left| \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr \right|^p. \quad (5.30)$$

Young's inequality implies that

$$(g(t), v_\delta) \leq \frac{\lambda}{8} \|v_\delta\|^2 + \frac{2}{\lambda} \|g(t)\|^2. \quad (5.31)$$

The last two terms in (5.29) is bounded by

$$\frac{\lambda}{8} \|v_\delta\|^2 + \frac{1}{2} \|\nabla v_\delta\|^2 + c_2 \left(\int_0^t \mathcal{G}_\delta(\theta_r \omega) dr \right)^2. \quad (5.32)$$

Thus it follows from (5.29)–(5.32) that

$$\begin{aligned} & \frac{d}{dt} \|v_\delta\|^2 + \frac{3}{2} \lambda \|v_\delta\|^2 + \|\nabla v_\delta\|^2 + \alpha_1 \|u_\delta\|_p^p \\ & \leq \frac{4}{\lambda} \|g(t)\|^2 + 2 \|\psi_1(t)\|_{L^1} + 2 \|\psi_2(t)\|_{L^{p_1}}^{p_1} + c_3 \left| \int_0^t \mathcal{G}_\delta(\theta_r \omega) dr \right|^p + c_3. \end{aligned} \quad (5.33)$$

Multiplying (5.33) by $e^{\lambda t}$, replacing ω by $\theta_{-\tau} \omega$ and then integrating over $(\tau - t, \sigma)$ with $\sigma \geq \tau - t$, we get

$$\begin{aligned} & \|v_\delta(\sigma, \tau - t, \theta_{-\tau} \omega, v_{\delta, \tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^\sigma e^{\lambda(s-\sigma)} \|v_\delta(s, \tau - t, \theta_{-\tau} \omega, v_{\delta, \tau-t})\|^2 ds \\ & + \int_{\tau-t}^\sigma e^{\lambda(s-\sigma)} \|\nabla v_\delta(s, \tau - t, \theta_{-\tau} \omega, v_{\delta, \tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^\sigma e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau} \omega, u_{\delta, \tau-t})\|_{L^p}^p ds \\ & \leq e^{\lambda(\tau-t-\sigma)} \|v_{\delta, \tau-t}\|^2 + \frac{c_3}{\lambda} + c_3 \int_{\tau-t}^\sigma e^{\lambda(s-\sigma)} \left| \int_{-\tau}^{s-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^p ds \\ & + 2 \int_{\tau-t}^\sigma e^{\lambda(s-\sigma)} \left(\frac{2}{\lambda} \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} + \|\psi_2(s)\|_{L^{p_1}}^{p_1} \right) ds. \end{aligned} \quad (5.34)$$

By (5.23) and (5.34) we have

$$\begin{aligned}
& \|u_\delta(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
& + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|_{L^p}^p ds \\
& \leq 2 \|v_\delta(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\delta, \tau-t})\|^2 + \lambda \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|v_\delta(s, \tau - t, \theta_{-\tau}\omega, v_{\delta, \tau-t})\|^2 ds \\
& + 2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla v_\delta(s, \tau - t, \theta_{-\tau}\omega, v_{\delta, \tau-t})\|^2 ds \\
& + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|_{L^p}^p ds \\
& + 2 \|h\|^2 \left| \int_{-\tau}^{\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 + \lambda \|h\|^2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left| \int_{-\tau}^{s-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 ds \\
& + 2 \|\nabla h\|^2 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left| \int_{-\tau}^{s-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 ds \\
& \leq 4e^{\lambda(\tau-t-\sigma)} (\|u_{\delta, \tau-t}\|^2 + \|h\|^2 \left| \int_{-\tau}^{-t} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2) + c_4 + c_4 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left| \int_{-\tau}^{s-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^p ds \\
& + 4 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \left(\frac{2}{\lambda} \|g(s)\|^2 + \|\psi_1(s)\|_{L^1} + \|\psi_2(s)\|_{L^{p_1}}^{p_1} \right) ds + c_4 \left| \int_{-\tau}^{\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 \\
& \leq 4e^{\lambda(\tau-t-\sigma)} (\|u_{\delta, \tau-t}\|^2 + \|h\|^2 \left| \int_{-\tau}^{-t} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2) + c_4 + c_4 \int_{-\infty}^0 e^{\lambda s} \left| \int_{-\tau}^{s+\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^p ds \\
& + 4 \int_{-\infty}^0 e^{\lambda s} \left(\frac{2}{\lambda} \|g(s+\sigma)\|^2 + \|\psi_1(s+\sigma)\|_{L^1} + \|\psi_2(s+\sigma)\|_{L^{p_1}}^{p_1} \right) ds + c_4 \left| \int_{-\tau}^{\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2,
\end{aligned} \tag{5.35}$$

where c_4 is a positive constant independent of τ, ω and D . It follows from (5.35) that for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $u_{\delta, \tau-t} \in D(\tau - t, \theta_{-\tau}\omega) \in \mathcal{D}$, there exists $T_1 = T_1(\tau, \omega, D, \delta) > 0$ such that for all $t \geq T_1$,

$$\begin{aligned}
& \|u_\delta(\sigma, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds \\
& + \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|\nabla u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|^2 ds + \alpha_1 \int_{\tau-t}^{\sigma} e^{\lambda(s-\sigma)} \|u_\delta(s, \tau - t, \theta_{-\tau}\omega, u_{\delta, \tau-t})\|_{L^p}^p ds \\
& \leq c + c \int_{-\infty}^0 e^{\lambda s} \left| \int_{-\tau}^{s+\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^p ds + c \left| \int_{-\tau}^{\sigma-\tau} \mathcal{G}_\delta(\theta_r \omega) dr \right|^2 \\
& \quad + c \int_{-\infty}^0 e^{\lambda s} (\|g(s+\sigma)\|^2 + \|\psi_1(s+\sigma)\|_{L^1} + \|\psi_2(s+\sigma)\|_{L^{p_1}}^{p_1}) ds, \tag{5.36}
\end{aligned}$$

where c is a positive constant independent of τ , ω and D . Therefore, we get that for all $t \geq T_1$,

$$u_\delta(\tau, \tau - t, \theta_{-\tau}\omega, D(\tau - t, \theta_{-\tau}\omega)) \subseteq B_\delta(\tau, \omega), \tag{5.37}$$

where $B_\delta(\tau, \omega)$ is given by (5.26). In addition, B_δ is tempered due to (3.8), (3.10), (3.37) and (5.7). The proof of (5.28) is similar to that of (4.45) and the details are omitted here. \square

Lemma 5.4. Suppose (3.3)–(3.5) hold and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Let u and u_{δ_n} be the solutions of (5.1) and (5.2) with initial data u_τ and $u_{\delta_n, \tau}$, respectively. If $u_{\delta_n, \tau} \rightarrow u_\tau$ in $L^2(\mathbb{R}^n)$ as $n \rightarrow \infty$, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$ and $t \in [\tau, \tau + T]$,

$$u_{\delta_n}(t, \tau, \omega, u_{\delta_n, \tau}) \rightarrow u(t, \tau, \omega, u_\tau) \quad \text{in } L^2(\mathbb{R}^n) \text{ as } n \rightarrow \infty.$$

Proof. The proof is quite similar to Lemma 4.9 and the details are left to the reader. \square

To establish the upper semicontinuity of pullback random attractors $\tilde{\mathcal{A}}_\delta$ for Ψ_δ , we also need to prove the uniform compactness of these attractors for small δ . To this end, we must establish the uniform smallness of tails of solutions with respect to small δ , which can be achieved by the arguments of Lemma 4.7. Actually, in the present case, the proof is much simpler and hence we will not repeat the details again. Based on the uniform estimates on the tails of solutions, we can obtain the following uniform compactness of random attractors as in Lemma 4.8.

Lemma 5.5. Suppose (3.3)–(3.5), (3.36), (3.37), (5.6) and (5.7) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, if $\delta_n \rightarrow 0$ and $u_n \in \tilde{\mathcal{A}}_{\delta_n}(\tau, \omega)$, then $\{u_n\}_{n=1}^\infty$ is precompact in $L^2(\mathbb{R}^n)$.

Finally, we are ready to present our main result of this section.

Theorem 5.2. Suppose (3.3)–(3.5), (3.36), (3.37), (4.66), (5.6) and (5.7) hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{\delta \rightarrow 0} d_{L^2(\mathbb{R}^n)}(\tilde{\mathcal{A}}_\delta(\tau, \omega), \tilde{\mathcal{A}}_0(\tau, \omega)) = 0. \tag{5.38}$$

Proof. The proof is similar to that of [Theorem 4.2](#) in terms of [Lemmas 5.3–5.5](#). The details are omitted here. \square

Remark 5.1. Note that there are total eight conditions on the nonlinear drift term f in [\(3.1\)](#) from Section 3 to Section 5. We here give an example of such f which satisfies all these assumptions [\(3.3\)–\(3.5\)](#), [\(3.36\)–\(3.37\)](#), [\(4.66\)](#) and [\(5.6\)–\(5.7\)](#). This example is quite similar to the one discussed in [\[26\]](#) in the case of bounded domains.

Given $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, let $\phi(t, x) = |t|^r |\phi_0(x)|$ for some $r > 0$ and $\phi_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Denote

$$f(t, x, s) = -\alpha_1 |s|^{p-2} s + \frac{\phi(t, x)}{1 + s^2}, \quad \text{for all } t, s \in \mathbb{R} \text{ and } x \in \mathbb{R}^n,$$

where $\alpha_1 > 0$ and $p > 2$ are constants. By Young's inequality, one can verify that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t, x, s)s = -\alpha_1 |s|^p + \frac{s\phi(t, x)}{1 + s^2} \leq -\alpha_1 |s|^p + \phi(t, x),$$

$$|f(t, x, s)| \leq \alpha_1 |s|^{p-1} + \phi(t, x),$$

$$\frac{\partial f}{\partial s}(t, x, s) = -\alpha_1(p-1)|s|^{p-2} - \frac{2s}{(1+s^2)^2}\phi(t, x) \leq -\alpha_1(p-1)|s|^{p-2} + \phi(t, x),$$

and

$$\left| \frac{\partial f}{\partial s}(t, x, s) \right| \leq \alpha_1(p-1)|s|^{p-2} + \phi(t, x) \leq (\alpha_1(p-1) + \phi(t, x))(1 + |s|^{p-2}).$$

Let $g(t, x) = |t|^{r_0} g_0(x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, where $r_0 > 0$ is a constant and $g_0 \in L^2(\mathbb{R}^n)$. Based on the above inequalities, we find that f and g satisfy all the conditions [\(3.3\)–\(3.5\)](#), [\(3.36\)–\(3.37\)](#), [\(4.66\)](#) and [\(5.6\)–\(5.7\)](#) with

$$\psi_1(t, x) = \psi_2(t, x) = \psi_3(t, x) = \phi(t, x), \quad \psi_4(t, x) = (p-1)\alpha_1 + \phi(t, x),$$

and $\alpha_2 = \alpha_1$ and $\alpha_3 = (p-1)\alpha_1$.

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