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Convergence of the solution of the stochastic 3D globally modified Cahn–Hilliard–Navier–Stokes equations

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Abstract

We study in this article the stochastic 3D globally modified Cahn–Hilliard–Navier–Stokes model in a 3D dimensional bounded domain. We prove the existence and uniqueness of strong solutions. Furthermore, we discuss the relation of the stochastic 3D globally modified Cahn–Hilliard–Navier–Stokes equations to the stochastic 3D Cahn–Hilliard–Navier–Stokes equations by proving a convergence theorem, that as the parameter N tends to infinity, a subsequence of solutions of the stochastic 3D globally modified Cahn–Hilliard–Navier–Stokes equations converges to a weak martingale solution of the stochastic 3D Cahn–Hilliard–Navier–Stokes equations.

MSC: 35R60; 35Q35; 60H15; 76M35; 86A05

Keywords: Cahn–Hilliard–Navier–Stokes; Globally modified; Stochastic; Galerkin scheme

1. Introduction

It is well accepted that the incompressible Navier–Stokes (NS) equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult

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problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [16]. For instance, this approach is used in [2] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, [5,16,15,17]. In the isothermal compressible case, the existence of a global weak solution is proved in [13]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity v and the order parameter ϕ. This system can be written as a NS equation coupled with a convective Allen–Cahn equation, [16]. The associated initial and boundary value problem was studied in [16] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [16] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [32]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at it infancy as noted in [16]. As noted in [15], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in which these patters organize themselves into parallel layers (see, e.g. [26] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn–Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier–Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [19]). This is a system of equations where an incompressible Navier–Stokes equation for the (mean) velocity v is coupled with a convective Cahn–Hilliard equation for the order parameter ϕ, which represents the relative concentration of one of the fluids.

Many challenges in the mathematical and numerical analysis of the Allen–Cahn–Navier–Stokes equations (AC-NSE) or the Cahn–Hilliard–Navier–Stokes equations CH-NSE) are related to the fact that the full mathematical theory for the 3D Navier–Stokes equation (NSE) is still lacking at present. Since the uniqueness theorem for the global weak solutions (or the global existence of strong solutions) of the initial-value problem of the 3D Navier–Stokes system is not proved yet, the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system. Using regular approximation equations to study the classical 3D Navier–Stokes systems has become an effective tool both from the numerical and the theoretical point of views. As noted in [34], it was demonstrated analytically and numerically in many works that the LANS-α model gives a good approximation in the study of many problems related to turbulence flows. In particular, it was found that the explicit steady analytical solution of the LANS-α model compare successfully with empirical and numerical experiment data for a wide range of Reynolds numbers in turbulent channel and pipe flows, [34]. Let us recall that the inviscid 3D LANS-α equations was first proposed in [21,20]. As described in [24], the 3D LANS-α equations are a systems of partial differential equations for the mean velocity in which a nonlinear dispersive mechanism filters the small scales. As such, the 3D LANS-α equations serve as an appropriate model for turbulent flows and a suitable approximation of the 3D NS as documented in [7,9,8,10].

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In [6], the authors proposed a three dimensional system of a globally modified Navier–Stokes equations (GMNSE). They studied the existence and uniqueness of strong solutions and established the existence of global V-attractors. As noted in [6], the GMNSE prevents large gradients dominating the dynamic and leading to explosion. Motivated by the results given in [6], we studied in [25] a three-dimensional system of a globally modified CH-NS equations (GMCHNSE) and proved the existence and uniqueness of strong solutions as well as the existence of $\mathcal{U}$-attractors. Let us note that the coupling between the Navier–Stokes and the Cahn–Hilliard equations introduces in the coupled model a highly nonlinear term that makes the analysis more involved. In this article, we consider a stochastic version of the model studied in [25].

Let us recall that stochastic partial differential equations (SPDE) are sometimes used to model physical systems subjected to the influence of internal, external or environmental noises. As noted in [4,3], SPDE can also be used to describe systems that are too complex to be described deterministically, e.g., a flow of a chemical substance in a river subjected to wind and rain, an airflow around an airplane wing perturbed by the random state of the atmosphere and weather, etc. With the development of the theory of stochastic processes, systems such as the Navier–Stokes perturbed by noises have been widely investigated with the goal to better understand the complex phenomena of turbulent flow. The mathematical theory of the stochastic Navier–Stokes equation is very rich, covering a broad area of deep results on existence of solutions, dynamical system feature, ergodicity, and many more. The presence of noise in a model can lead to new and important phenomena. For instance, contrary to the deterministic case, it is known that the 2D Navier–Stokes system driven with a sufficiently degenerate noise has a unique invariant measure and hence exhibits ergodic behavior in the sense that the time average of a solution is equal to the average over all possible initial data, [4].

The aim of this article is to investigate the stochastic version of the GMCHNSE studied in [25]. The model includes an abstract and general form of random external forces depending eventually on the velocity v of the fluid and the phase function ϕ. We prove the existence and uniqueness of a strong solution in a three dimensional bounded domain. Here the word "strong" means "strong" both in the sense of the theory of partial differential equations and the theory of stochastic analysis. The proof of the existence relies on the Galerkin approximation, the local monotonicity of the coefficients and some compactness results. Moreover we investigate the asymptotic behavior of the unique solution when the parameter N tends to infinity. This gives the existence of a weak martingale solution for the stochastic 3D CH-NSE.

The article is organized as follows. In the next section we present the stochastic 3D GMCHNSE model and its mathematical setting. The existence and uniqueness a solution is given in Section 3. The asymptotic behavior of the solution is investigated in Section 4. Finally in the Appendix for the reader's convenience, we recall two compacts embedding theorems, a convergence theorem for the stochastic integral and a stochastic Gronwall lemma.

2. A stochastic GMCHNSE model and its mathematical setting

2.1. Governing equations

In this article, we consider a stochastic version of the GMCHNSE a three-dimensional domain. We assume that the domain $\mathcal{M}$ of the fluid is a bounded domain in $\Re^3$. We first recall the following 3D stochastic CH-NSE

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$$\begin{cases} dv(t) = [\nu_1 \Delta v - (v.\nabla)v + \nabla p - g_1(v,\phi) + \mathcal{K}\mu\nabla\phi]\,dt + \displaystyle\sum_{k=1}^{\infty} \tilde{\sigma}_k^1(v,\phi)(t))dW_k^1(t), \\ d\phi(t) = [\nu_3 \Delta\mu - v.\nabla\phi - g_2(v,\phi)]\,dt + \displaystyle\sum_{k=1}^{\infty} \tilde{\sigma}_k^2(v,\phi)(t))dW_k^2(t), \\ \mu = -\nu_2\Delta\phi + \alpha f(\phi), \\ \nabla.v = 0, \end{cases} \tag{2.1}$$

in $\mathcal{M} \times [0,T]$, where $W_t^k = (W_k^1, W_k^2)(t), t \geq 0, k = 1, 2, ...$ is a sequence of independent one dimensional standard Brownian motions on some complete filtration probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in(0,T)})$. If $(e_k)_{k\geq 1}$ is an orthonormal basis of l^2, we may formally define W by taking $W = \sum_k W_k e_k$. As such W is a cylindrical Brownian motion evolving over l^2. We recall that l^2 is the Hilbert space consisting of all sequences of square summable real numbers. We define the auxiliary space $\mathcal{U}_0 \supset l^2$ via $\mathcal{U}_0 = \left\{ v = \sum_{k=1}^{\infty} \alpha_k e_k : \sum_{k=1}^{\infty} \alpha_k^2 k^{-2} < \infty \right\}$ endowed with the norm $|v|^2_{\mathcal{U}_0} := \sum_{k=1}^{\infty} \frac{\alpha_k^2}{k^2}$ for $v = \sum_{k=1}^{\infty} \alpha_k e_k$. Note that the embedding of $l^2 \subset \mathcal{U}_0$ is Hilbert–Schmidt. Moreover, using standard martingale arguments with the fact that each W_k is almost surely continuous (see [28]), we have that for almost every $\omega \in \Omega$, $W(\omega) \in C([0,T]; \mathcal{U}_0)$. The external volume force $(g_1(v,\phi), g_2(v,\phi)$ are given. The terms $\sum_{k=1}^{\infty} \tilde{\sigma}_k^1(v,\phi)(t))dW_k^1(t)$, $\sum_{k=1}^{\infty} \tilde{\sigma}_k^2(v,\phi)(t))dW_k^2(t)$ represent random external forces depending eventually on (v,ϕ). See Section 3 for the precise assumptions on the coefficients $g = (g_1, g_2)$ and $\{\tilde{\sigma}_k = (\tilde{\sigma}_k^1, \tilde{\sigma}_k^2); k = 1, ..., \infty\}$.

In (2.1), the unknown functions are the velocity $v = (v_1, v_2, v_3)$ of the fluid, its pressure p and the order (phase) parameter ϕ. The quantity μ is the variational derivative of the following free energy functional

$$\mathcal{F}_1(\phi) = \int_{\mathcal{M}} \left(\frac{\nu_2}{2} |\nabla\phi|^2 + \alpha F(\phi) \right) ds, \tag{2.2}$$

where, e.g., $F(x) = \int_0^x f(\zeta)d\zeta$. Here, the constants $\nu_1 > 0$, $\nu_3 > 0$ and $\mathcal{K} > 0$ correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here ν_2, $\alpha > 0$ are two physical parameters describing the interaction between the two phases. In particular, ν_2 is related with the thickness of the interface separating the two fluids. Hereafter, as in [16] we assume that $\nu_2 \leq \alpha$. A typical example of potential F is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type $F(x) = \gamma_1 x^4 - \gamma_2 x^2$, γ_1, γ_2 being positive constants. As noted in [15], $(2.1)_1$ can be replaced by

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$$dv(t) = [\nu_1 \Delta v - (v.\nabla)v - \nabla p - g_1(v,\phi) - \mathcal{K}\text{div}\,(\nabla\phi \otimes \nabla\phi)]\,dt + \sum_{k=1}^{\infty} \tilde{\sigma}_k^1(v,\phi)(t))dW_k^1(t), \tag{2.3}$$

where $\tilde{p} = p - \mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))$, since $\mathcal{K}\mu\nabla\phi = \nabla(\mathcal{K}(\frac{\nu_2}{2}|\nabla\phi|^2 + \alpha F(\phi))) - \mathcal{K}\text{div}\,(\nabla\phi \otimes \nabla\phi)$. The stress tensor $\nabla\phi \otimes \nabla\phi$ is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for these models, as in [15] we assume that the boundary conditions for ϕ are the natural no-flux condition

$$\partial_\eta \phi = \partial_\eta \mu = 0 \ \text{ on } \partial\mathcal{M} \times (0,\infty), \tag{2.4}$$

where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and η is the outward normal to $\partial\mathcal{M}$. For ϕ a scalar function defined on $\mathcal{M}$, we denote

$$\langle\phi(t)\rangle = \frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} \phi(x,t)dx, \tag{2.5}$$

where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. Concerning the boundary condition for v, we assume the Dirichlet (no-slip) boundary condition

$$v = 0 \ \text{ on } \partial\mathcal{M} \times (0,\infty). \tag{2.6}$$

Therefore we assume that there is no relative motion at the fluid–solid interface.

The initial condition is given by

$$(v,\phi)(0) = (v_0,\phi_0). \tag{2.7}$$

Now, we define the function $F_N : \Re^+ \to \Re^+$ by

$$F_N(r) = \min\{1, N/r\}, \ r \in \Re^+, \tag{2.8}$$

for some (fixed) $N \in \Re^+$.

The following lemma gives some important properties of the map F_N (see [6,31] for the proof)

Lemma 1. *For* $\forall p_1, p_2, M, N \in \Re^+, \ p_2 \neq 0$, *we have*

$$1)\ \ 0 \le F_N(p_2) \le \frac{N}{p_2}, \tag{2.9}$$

$$2)\ \ |F_N(p_1) - F_N(p_2)| \le \frac{1}{N} F_N(p_1)F_N(p_2)|p_1 - p_2|, \tag{2.10}$$

$$3)\ \ |F_M(p_1) - F_N(p_2)| \le \frac{|M-N|}{p_2} + \frac{|p_1 - p_2|}{p_2}. \tag{2.11}$$

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Now we consider the following stochastic 3D GMCHNSE

$$\begin{cases} dv(t) = [\nu_1 \Delta v - F_N(\|v\|)[(v.\nabla)v] - \nabla p - g_1(v,\phi) + \mu \nabla \phi]\, dt \\ + \displaystyle\sum_{k=1}^{\infty} \tilde{\sigma}_k^1(v,\phi)(t) dW_k^1(t), \\ d\phi(t) = [\nu_3 \Delta \mu - v.\nabla\phi - g_2(v,\phi)]\, dt + \displaystyle\sum_{k=1}^{\infty} \tilde{\sigma}_k^2(v,\phi)(t) dW_k^2(t), \\ \mu = -\nu_2 \Delta\phi + \alpha f(\phi), \\ \nabla.v = 0, \end{cases} \tag{2.12}$$

in $\mathcal{M} \times (0, +\infty)$, where $\|v\|$ is a norm defined below.

As noted in [6] in the case of the GMNSE, the GMCHNSE are indeed globally modified. The factor $F_N(\|v\|)$ depends on the norm $\|v\|$. It prevents large values of $\|v\|$ dominating the dynamics. Just like the GMNSE, the GMCHNSE violates the basic laws of mechanics, but mathematically the model is well defined, [25].

2.2. Mathematical setting

We first recall from [15] a weak formulation of (2.1), (2.4), (2.6)–(2.7). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial\mathcal{M}$ (e.g., of class $\mathcal{C}^2$). We also assume that $f \in \mathcal{C}^3(\Re)$ satisfies

$$\begin{cases} \displaystyle\lim_{|x| \to +\infty} f'(x) > 0, \\ |f^{(i)}(x)| \le c_f(1 + |x|^{2-i}),\ \forall x \in \Re,\ i = 0, 1, 2, \end{cases} \tag{2.13}$$

where c_f is some positive constant.

If X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, we will denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. We set

$$\mathcal{V}_1 = \{u \in \mathcal{C}_c^\infty(\mathcal{M}) :\ \operatorname{div} u = 0 \text{ in } \mathcal{M}\}.$$

We denote by H_1 and V_1 the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^3$ and $(H_0^1(\mathcal{M}))^3$ respectively. The scalar product in H_1 is denoted by $(\cdot,\cdot)_{L^2}$ and the associated norm by $|\cdot|_{L^2}$. Moreover, the space V_1 is endowed with the scalar product

$$((u,v)) = \sum_{i=1}^{3} (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \|u\| = ((u,u))^{1/2}.$$

We now define the operator A_0 by

$$A_0 v = -\mathcal{P}\Delta v,\ \forall v \in D(A_0) = H^2(\mathcal{M}) \cap V_1,$$

where $\mathcal{P}$ is the Leray–Helmotz projector in $L^2(\mathcal{M})$ onto H_1. Then, A_0 is a self-adjoint positive unbounded operator in H_1 which is associated with the scalar product defined above. Furthermore, A_0^{-1} is a compact linear operator on H_1 and $|A_0 \cdot|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the H^2-norm.

Hereafter, we set

$$H_2 = L^2(\mathcal{M}),\ V_2 = H^1(\mathcal{M}),\ H = H_1 \times H_2,\ V = V_1 \times V_2. \tag{2.14}$$

We introduce the linear non-negative unbounded operator on $L^2(\mathcal{M})$

$$A_1\phi = -\Delta\phi,\ \forall\phi \in D(A_1) = \{\phi \in H^2(\mathcal{M}),\ \partial_\eta\phi = 0,\ \text{ on } \partial\mathcal{M}\}, \tag{2.15}$$

and we endow $D(A_1)$ with the norm $|A_1 \cdot|_{L^2} + |\langle\cdot\rangle|_{L^2}$, which is equivalent to the H^2-norm. Also we define the linear positive unbounded operator on the Hilbert space $L_0^2(\mathcal{M})$ of the L^2-functions with null mean

$$B_n\phi = -\Delta\phi,\ \forall\phi \in D(B_n) = D(A_1) \cap L_0^2(\mathcal{M}). \tag{2.16}$$

Note that B_n^{-1} is a compact linear operator on $L_0^2(\mathcal{M})$. More generally, we can define B_n^s for any $s \in \Re$, noting that $|B_n^{s/2} \cdot|_{L^2},\ s > 0$, is an equivalent norm to the canonical H^s-norm on $D(B_n^{s/2}) \subset H^s(\mathcal{M}) \cap L_0^2(\mathcal{M})$. Also note that $A_1 = B_n$ on $D(B_n)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_n^{s/2})$, we have that $|B_n^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|_{L^2}$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$. Now we define the Hilbert spaces $\mathcal{H}$ and $\mathcal{U}$ by

$$\mathcal{H} = H_1 \times H^1(\mathcal{M}),\ \mathcal{U} = V_1 \times D(A_1),\ \ \mathcal{Z} = V_1 \times D(A_1^{3/2}), \tag{2.17}$$

endowed with the scalar products whose associated norms are respectively

$$\begin{aligned}
&|(v,\phi)|_{\mathcal{H}}^2 = |v|_{L^2}^2 + \nu_2(|\nabla\phi|_{L^2}^2 + \gamma|\phi|_{L^2}^2),\ \ \|(v,\phi)\|_{\mathcal{U}}^2 = \|v\|^2 + |A_1\phi|_{L^2}^2, \\
&\|(v,\phi)\|_{\mathcal{Z}}^2 = \|v\|^2 + |A_1^{3/2}\phi|_{L^2}^2.
\end{aligned} \tag{2.18}$$

We will also use the following notation:

$$\begin{aligned}
&\langle u_1, u_2\rangle_{\mathcal{Z}} = \langle A_0 v_1, v_2\rangle + \langle A_1^2\phi_1, A_1\phi_2\rangle,\ \forall u_1 = (v_1,\phi_1), u_2 = (v_2,\phi_2) \in \mathcal{Z}, \\
&\langle u_1, u_2\rangle_{\mathcal{U}} = \langle A_0 v_1, v_2\rangle + \langle A_1^2\phi_1, \phi_2\rangle,\ \forall u_1 = (v_1,\phi_1), u_2 = (v_2,\phi_2) \in \mathcal{U}.
\end{aligned} \tag{2.19}$$

It follows that

$$\langle u_1, u_1\rangle_{\mathcal{U}} = \|u_1\|_{\mathcal{U}}^2,\ \forall u_1 \in \mathcal{U},\ \ \langle u_1, u_1\rangle_{\mathcal{Z}} = \|u_1\|_{\mathcal{Z}}^2,\ \forall u_1 \in \mathcal{Z}. \tag{2.20}$$

We introduce the bilinear operators B^0, B^1 (and their associated trilinear forms b^0, b^1) as well as the coupling mapping R^0, which are defined from $D(A_0) \times D(A_0)$ into H_1, $D(A_0) \times D(A_1)$ into $L^2(\mathcal{M})$, and $L^2(\mathcal{M}) \times D(A_1^2)$ into H_1, respectively. More precisely, we set

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$$
\begin{aligned}
&(B^0(u,v),w) = \int_{\mathcal{M}} [(u\cdot\nabla)v]\cdot w dx = b^0(u,v,w),\ \forall u,v,w\in D(A_0),\\
&(B^1(u,\phi),\rho) = \int_{\mathcal{M}} [u\cdot\nabla\phi]\rho dx = b^1(u,\phi,\rho),\ \forall u\in D(A_0),\ \phi,\rho\in D(A_1),\\
&(R^0(\mu,\phi),w) = \int_{\mathcal{M}} \mu[\nabla\phi\cdot w]dx = b^1(w,\phi,\mu),\ \forall w\in D(A_0),\ (\mu,\phi)\in L^2(\mathcal{M})\times D(A_1^2).
\end{aligned} \tag{2.21}
$$

Note that

$$R^0(\mu,\phi) = \mathcal{P}\mu\nabla\phi.$$

We recall from [16,15,17] the following properties of B^0, B^1 and R^0.

$$
\begin{aligned}
&|b^0(u,v,w)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_0v|_{L^2}|w|_{L^2},\ \forall u\in V_1, v\in D(A_0), w\in H_1,\\
&|B^0(u,v)|_{L^2} \le c\|u\|\|v\|^{1/2}|A_0v|_{L^2}^{1/2},\ \forall u\in V_1, v\in D(A_0),
\end{aligned} \tag{2.22}
$$

$$
\begin{aligned}
&|b^1(u,\phi,\psi)| \le c|u|_{L^2}^{1/2}\|u\|^{1/2}|A_1\phi|_{L^2}|\psi|_{L^2},\ \forall u\in V_1, \phi\in D(A_1), \psi\in H_2,\\
&|B^1(v,\phi)|_{L^2} \le c\|v\|\|\phi\|^{1/2}|A_1\phi|_{L^2}^{1/2},\ \forall v\in V_1, \phi\in D(A_1),
\end{aligned} \tag{2.23}
$$

$$|R^0(A_1\phi,\rho)|_{L^2} \le c|A_1\phi|_{L^2}|A_1\rho|_{L^2}^{1/2}|A_1^{3/2}\rho|_{L^2}^{1/2},\ \forall\phi\in D(A_1), \rho\in D(A_1^{3/2}). \tag{2.24}$$

Hereafter we set

$$b_N^0(u,v,w) = F_N(\|v\|)b^0(u,v,w),\quad \langle B_N^0(u,v),w\rangle = b_N^0(u,v,w),\ \forall u,v,w\in V_1. \tag{2.25}$$

We note that

$$
\begin{aligned}
&b_N^0(u,v,v) = 0,\ \forall u,v\in V_1,\\
&b^1(v,\phi,A_1\phi) = \langle R^0(A_1\phi,\phi),v\rangle,\ \forall (v,\phi)\in V_1\times D(A_1).
\end{aligned} \tag{2.26}
$$

To further simplify the presentation, we define the operators $A:\mathcal{U}\to\mathcal{U}^*$, $B_N:\mathcal{U}\times\mathcal{U}\to\mathcal{U}^*$, $B:\mathcal{U}\times\mathcal{U}\to\mathcal{U}^*$, $R:\mathcal{U}\times\mathcal{U}\to\mathcal{U}^*$, and $E:\mathcal{U}\to\mathcal{U}^*$ as follows.

$$\langle Au_1,u_2\rangle = \langle A_0v_1,v_2\rangle + \langle {A_1}^2\phi_1,\phi_2\rangle, \tag{2.27}$$

for $u_1=(v_1,\phi_1), u_2=(v_2,\phi_2)\in\mathcal{U}$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\mathcal{U}$ and $\mathcal{U}^*$ or between V_i and V_i^*, $i=1,2$.

Note that

$$\langle Au,u\rangle_{\mathcal{U}} \equiv \langle A_0v,A_0v\rangle + \langle A_1^2\phi,A_1^2\phi\rangle = |Au|_{L^2}^2,\ \forall u=(v,\phi)\in\mathcal{U}. \tag{2.28}$$

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We also set

$$\begin{aligned}&\langle B_N(u_1,u_2),u_3\rangle \equiv b_N(u_1,u_2,u_3)=b_N^0(v_1,v_2,v_3)+b^1(v_1,\phi_2,\phi_3),\\&\forall u_1=(v_1,\phi_1),\ u_2=(v_2,\phi_2),\ u_3=(v_3,\phi_3)\in\mathcal{U},\\[1ex]&\langle B(u_1,u_2),u_3\rangle \equiv b(u_1,u_2,u_3)=b^0(v_1,v_2,v_3)+b^1(v_1,\phi_2,\phi_3),\\&\forall u_1=(v_1,\phi_1),\ u_2=(v_2,\phi_2),\ u_3=(v_3,\phi_3)\in\mathcal{U},\end{aligned}\tag{2.29}$$

$$\begin{aligned}&R(u_1,u_2)=(R^0(A_1\phi_1,\phi_2),0),\\&\forall u_1=(v_1,\phi_1),\ u_2=(v_2,\phi_2)\in\mathcal{U},\end{aligned}\tag{2.30}$$

$$E(u_1)\equiv(E_1(u_1),E_2(u_1))=(0,A_1f(\phi_1)),\ \forall u_1=(v_1,\phi_1),u_2=(v_2,\phi_2)\in V_2.\tag{2.31}$$

For simplicity we will also set

$$B_N(u_1)=B_N(u_1,u_1),\quad R(u)=R(u,u).\tag{2.32}$$

We also set

$$D(A)=\{u=(v,\phi)\in H,\ Au\in\mathcal{H}\}\equiv D(A_0)\times D(A_1^2).$$

Without loss of generality, we set $\nu_1=\nu_2=\nu_3=\alpha=\mathcal{K}=1$.

With the above notations, if we set $u=(v,\phi)$, $u_0=(v_0,\phi_0)$, $G(u)\equiv g(v,\phi)=(g_1(v,\phi), g_2(v,\phi))$, $\sigma_k(u)=(\tilde{\sigma}_k^1(v,\phi),\sigma_k^2(v,\phi))$ then we can rewrite (2.12) as:

$$du(t)=[-Au-B_N(u)-Eu+R(u)+G(u)]\,dt+\sum_{k=1}^{\infty}\sigma_k(u(t))dW_k(t),\quad u(0)=u_0.\tag{2.33}$$

Here $G=(g_1,g_2)$ is a mapping from $\mathcal{U}$ (resp. $\mathcal{H}$) into $\mathcal{U}$ (resp. $\mathcal{H}$) and $\sigma_k(.)$, $k\geq 1$ is a sequence of mappings from $\mathcal{U}$ (resp. $\mathcal{H}$) into $\mathcal{U}$ (resp. $\mathcal{H}$).

Consider the following hypothesis.

$$\begin{aligned}&|G(u)-G(v)|_{\mathcal{H}}^2\leq c|u-v|_{\mathcal{H}}^2,\ \forall u,v\in\mathcal{H},\ \ \|G(u)-G(v)\|_{\mathcal{U}}^2\leq c\|u-v\|_{\mathcal{U}}^2,\ \forall u,v\in\mathcal{U},\\&\sum_{k=1}^{\infty}\|\sigma_k(u)-\sigma_k(v)\|_{\mathcal{U}}^2\leq c\|u-v\|_{\mathcal{U}}^2,\ \sum_{k=1}^{\infty}\|\sigma_k(u)\|_{\mathcal{U}}^2<\infty,\ \ \forall u,v\in\mathcal{U},\\&\sum_{k=1}^{\infty}|\sigma_k(u)|_{\mathcal{H}}^2<\infty,\ \sum_{k=1}^{\infty}|\sigma_k(u)-\sigma_k(v)|_{\mathcal{H}}^2\leq c|u-v|_{\mathcal{H}}^2,\ \ \forall u,v\in\mathcal{H}.\end{aligned}\tag{2.34}$$

Remark 2.1. The hypothesis (2.34) imply that for every $u\in\mathcal{U}$ $(\mathcal{H}\ resp.)$ the map $\sigma(u):=(\sigma_k(u))_{k\in\mathbb{N}}:l^2\to\mathcal{U}$ $(\mathcal{H}\ resp.)$ defined by

$$\sigma(u)h:=\sum_{k=1}^{\infty}\sigma_k(u)h_k,\quad h=(h_k)_{k\in\mathbb{N}}\in l^2,$$

is in $L_2(l^2, \mathcal{U}(\mathcal{H}\, resp.))$ = Hilbert–Schmidt operators from l^2 to $\mathcal{U}$ ($\mathcal{H}\, resp.$), and that $u \mapsto \sigma(u)$ is Lipschitz.

Hereafter, we will denote by c a generic positive constant that depends on the domain $\mathcal{M}$. The next result will be useful in our study of the stochastic 3D GMCHNSE.

Lemma 2. *Let $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in \mathcal{U}$ and $(w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)$. Let $\mathcal{NL}^1$ defined by:*

$$\mathcal{NL}^1(u_1, u_2) = \langle F_N(\|v_1\|)B^0(v_1, v_1) - F_N(\|v_2\|)B^0(v_2, v_2), w\rangle. \tag{2.35}$$

There exists a constant $c_3 > 0$ independent of u_1 and u_2 such that

$$\mathcal{NL}^1(u_1, u_2) \le \tfrac{1}{8}\|u_1 - u_2\|_{\mathcal{U}}^2 + c_3 N^4 |u_1 - u_2|_{\mathcal{H}}^2. \tag{2.36}$$

Proof. See [6,31]. □

The next lemma shows that B_N^0, B^1, R^0 and E are locally Lipschitz.

Lemma 3. *The maps B_N^0, B^1 and R^0 defined from $\mathcal{U} \to \mathcal{U}^*$ are locally Lipschitz continuous i.e. for every $r > 0$, there exists a constant L_r such that*

$$\begin{aligned}
&\|B_N^0(u_1) - B_N^0(u_2)\|_{\mathcal{U}^*} \le L_r \|u_1 - u_2\|_{\mathcal{U}},\\
&\|R^0(u_1) - R^0(u_2)\|_{\mathcal{U}^*} \le L_r \|u_1 - u_2\|_{\mathcal{U}},\\
&\|B^1(u_1) - B^1(u_2)\|_{\mathcal{U}^*} \le L_r \|u_1 - u_2\|_{\mathcal{U}},\\
&\|E(u_1) - E(u_2)\|_{\mathcal{U}^*} \le L_r \|u_1 - u_2\|_{\mathcal{U}},
\end{aligned} \tag{2.37}$$

for $u_1, u_2 \in \mathcal{U}$ with $\|u_1\|_{\mathcal{U}}, \|u_2\|_{\mathcal{U}} \le r$.

Proof. Let $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in \mathcal{U}$ and $(w, \psi) \in \mathcal{U}$. We assume that $\|u_1\| \le r$ and $\|u_2\| \le r$. To prove $(2.37)_1$, we note that

$$\begin{aligned}
&\langle B_N^0(v_1) - B_N^0(v_2), w\rangle = b_N^0(v_1, v_1, w) - b_N^0(v_2, v_2, w)\\
&= F_N(\|v_2\|)b^0(v_1 - v_2, v_2, w) + (F_N(\|v_2\|) - F_N(\|v_1\|))\, b^0(v_1, v_2, w)\\
&+ F_N(\|v_1\|)b^0(v_1, v_1 - v_2, w)\\
&\le \frac{c}{\|v_2\|} N\|v_1 - v_2\|\|v_2\|\|w\| + \frac{c}{\|v_2\|\|v_1\|} N\|v_1 - v_2\|\|v_1\|\|v_2\|\|w\|\\
&+ \frac{c}{\|v_1\|} N\|v_1\|\|v_1 - v_2\|\|w\|,\\
&\le cN\|v_1 - v_2\|\|w\|.
\end{aligned} \tag{2.38}$$

From this, we deduce $(2.37)_1$.

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For (2.37)$_2$, we note that

$$\begin{aligned}
&|\langle R^0(A_1\phi_1,\phi_1)-R^0(A_1\phi_2,\phi_2),w\rangle| \le |b^1(w,\phi_1,A_1(\phi_1-\phi_2))|+|b^1(w,\phi_1-\phi_2,A_1\phi_2)|\\
&\le c\|w\||A_1\phi_1|_{L^2}|A_1(\phi_1-\phi_2)|_{L^2}+c\|w\||A_1\phi_2|_{L^2}|A_1(\phi_1-\phi_2)|_{L^2}\\
&\le cr\|w\||A_1\phi_1|_{L^2}|A_1(\phi_1-\phi_2)|_{L^2}.
\end{aligned}\tag{2.39}$$

It follows that

$$\|R^0(A_1\phi_1,\phi_1)-R^0(A_1\phi_2,\phi_2)\|_{V_1^*}\le L_r|A_1(\phi_1-\phi_2)|_{L^2}\le L_r\|u_1-u_2\|_{\mathcal{U}}.\tag{2.40}$$

We also have

$$\begin{aligned}
&|\langle B^1(v_1,\phi_1)-B^1(v_2,\phi_2),A_1\psi\rangle| \le |b^1(v_1-v_2,\phi_1,A_1\psi)|+|b^1(v_2,\phi_1-\phi_2,A_1\psi)|\\
&\le c\|v_1-v_2\||A_1\phi_1|_{L^2}|A_1\psi|_{L^2}+c\|v_2\||A_1\psi|_{L^2}|A_1(\phi_1-\phi_2)|_{L^2}\\
&\le cr(\|v_1-v_2\|+|A_1(\phi_1-\phi_2)|_{L^2})|A_1\psi|_{L^2}\\
&\le cr\|u_1-u_2\|_{\mathcal{U}}|A_1\psi|_{L^2}.
\end{aligned}\tag{2.41}$$

It follows that

$$\|\langle B^1(v_1,\phi_1)-B^1(v_2,\phi_2)\|_{D(A_1^{-1})}\le L_r(|A_1(\phi_1-\phi_2)|_{L^2}+\|v_1-v_2\|)\le L_r\|u_1-u_2\|_{\mathcal{U}}.\tag{2.42}$$

For (2.37)$_4$, we note that (see [15])

$$\begin{aligned}
\alpha|\langle A_1f(\phi_1)-A_1f(\phi_2),A_1\psi\rangle| &\le Q_1(|A_1\phi_1|_{L^2},|A_1\phi_2|_{L^2})|A_1(\phi_1-\phi_2)|_{L^2}|A_1\psi|_{L^2}\\
&\le Q_1(r,r)|A_1(\phi_1-\phi_2)|_{L^2}|A_1\psi|_{L^2},
\end{aligned}\tag{2.43}$$

where $Q_1=Q_1(x_1,x_2)$ is a monotone non-decreasing function of x_1 and x_2.

We derive that

$$\|A_1f(\phi_1)-A_1f(\phi_2)\|_{D(A_1^{-1})}\le Q_1(r,r)|A_1(\phi_1-\phi_2)|_{L^2}\le L_r\|u_1-u_2\|_{\mathcal{U}}.\tag{2.44}$$

Therefore (2.37)$_4$ follows. □

Now for $u=(v,\phi)\in\mathcal{U}$, we set

$$\mathcal{G}(u):=-Au-B_N(u)-G(u)+R(u)-E(u)\equiv(\mathcal{G}_1(u),\mathcal{G}_2(u)),$$

where

$$\begin{aligned}
\mathcal{G}_1(u)&=-A_0v-B_N^0(v,v)+R^0(A_1\phi,\phi)-g_1(v,\phi),\\
\mathcal{G}_2(u)&=A_1^2\phi-B^1(v,\phi)-E_2(\phi)-g_2(v,\phi).
\end{aligned}$$

With these notations, it is clear that (2.33) can be written as

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$$du(t) = \mathcal{G}(u)dt + \sum_{k=1}^{\infty} \sigma_k(u(t))dW_k(t), \quad u(0) = u_0. \tag{2.45}$$

Hereafter, we assume that the function f satisfies the additional conditions:

$$\begin{aligned} &\langle A_1 f(\phi_1) - A_1 f(\phi_2), A_1(\phi_1 - \phi_2)\rangle \geq -\alpha_1 |A_1^{3/2}(\phi_1 - \phi_2)|_{L^2}^2 - \alpha_0 \|\phi_1 - \phi_2\|^2, \\ &\langle A_1 f(\phi_1), A_1\phi_1\rangle \geq -\alpha_1 |A_1^{3/2}\phi_1|_{L^2}^2 - \alpha_0 \|\phi_1\|^2, \ \forall \phi_1 \in D(A_1^{3/2}), \end{aligned} \tag{2.46}$$

where $\alpha_0 > 0$ and $\alpha_1 > 0$ are positive constants independent of ϕ_1, ϕ_2 with $0 < \alpha_1 < \frac{1}{4}$. The next lemma shows some local monotonicity of the operator $\mathcal{G}$.

Lemma 4. *For $u_1, u_2 \in D(A) \subset \mathcal{U}$, we have*

$$\langle \mathcal{G}(u_1) - \mathcal{G}(u_2), u_1 - u_2\rangle \leq -\frac{1}{2}\|u_1 - u_2\|_{\mathcal{Z}}^2 + c(1 + \|u_2\|_{\mathcal{U}}^2))|u_1 - u_2|_{\mathcal{H}}^2, \tag{2.47}$$

where the constant $c > 0$ is independent of u_1, u_2.

Proof. Let $u_1 = (v_1, \phi_1),\ u_2 = (v_2, \phi_2),\ (w, \psi) = (v_1, \phi_1) - (v_2, \phi_2)$. Note that

$$\begin{aligned} &\langle \mathcal{G}(u_1) - \mathcal{G}(u_2), u_1 - u_2\rangle = \langle \mathcal{G}_1(u_1) - \mathcal{G}_1(u_2), w\rangle + \langle \mathcal{G}_2(u_1) - \mathcal{G}_2(u_2), A_1\psi\rangle \\ &= -\|u_1 - u_1\|_{\mathcal{Z}}^2 - \langle B_N^0(v_1, v_1) - B_N^0(v_2, v_2), w\rangle + \langle R^0(A_1\phi_1, \phi_1) - R^0(A_1\phi_2, \phi_2), w\rangle \\ &-\langle B^1(v_1, \phi_1) - B^1(v_2, \phi_2), A_1\psi\rangle - \langle E_2(\phi_1) - E_2(\phi_2), A_1\psi\rangle - \langle G(u_1) - G(u_2), u_1 - u_2\rangle. \end{aligned} \tag{2.48}$$

From [6,31], we have

$$\langle B_N^0(v_1, v_1) - B_N^0(v_2, v_2), w\rangle \leq \epsilon_1\|w\|^2 + c_3 N^4 |w|_{L^2}^2, \tag{2.49}$$

for $\epsilon_1 > 0$ to be chosen later.

We also have

$$\begin{aligned} &\langle R^0(A_1\phi_1, \phi_1) - R^0(A_1\phi_2, \phi_2), w\rangle = b_1(w, \phi_1, A_1\psi) + b_1(w, \psi, A_1\phi_2), \\ &-\langle B^1(v_1, \phi_1) - B^1(v_2, \phi_2), A_1\psi\rangle = -b_1(w, \phi_1, A_1\psi) - b_1(v_2, \psi, A_1\psi), \end{aligned} \tag{2.50}$$

which gives

$$\begin{aligned} &\langle R^0(A_1\phi_1, \phi_1) - R^0(A_1\phi_2, \phi_2), w\rangle - \langle B^1(v_1, \phi_1) - B^1(v_2, \phi_2), A_1\psi\rangle \\ &= b_1(w, \psi, A_1\phi_2) - b_1(v_2, \psi, A_1\psi). \end{aligned} \tag{2.51}$$

From (2.22), we have

$$\begin{aligned}|b_1(w,\psi,A_1\phi_2)| &\le c|w|_{L^2}^{1/2}\|w\|^{1/2}|A_1\psi|_{L^2}|A_1\phi_2|_{L^2}\\ &\le c|w|_{L^2}^{1/2}\|w\|^{1/2}\|\psi\|^{1/2}|A_1^{3/2}\psi|_{L^2}^{1/2}|A_1\phi_2|_{L^2}\\ &\le \epsilon_1(\|w\|^2+|A_1^{3/2}\psi|_{L^2}^2)+c|A_1\phi_2|_{L^2}^2(|w|_{L^2}^2+\|\psi\|^2),\end{aligned} \tag{2.52}$$

$$\begin{aligned}|b_1(v_2,\psi,A_1\psi)| &\le c\|v_2\||A_1\psi|_{L^2}^2\\ &\le \epsilon_1|A_1^{3/2}\psi|_{L^2}^2+c\|v_2\|^2\|\psi\|^2.\end{aligned} \tag{2.53}$$

From $(2.34)_1$ and (2.46), we also have

$$\begin{aligned}-\langle G(u_1)-G(u_2),u_1-u_2\rangle &\le c|u_1-u_2|_{\mathcal{H}}^2,\\ -\langle E_2(\phi_1)-E_2(\phi_2),A_1\psi\rangle &\equiv -\langle A_1f(\phi_1)-A_1f(\phi_2),A_1\psi\rangle\\ &\le \alpha_0|u_1-u_2|_{\mathcal{H}}^2+\alpha_1\|u_1-u_2\|_{\mathcal{Z}}^2.\end{aligned} \tag{2.54}$$

It follows from (2.48)–(2.54) that

$$\langle \mathcal{G}(u_1)-\mathcal{G}(u_2),u_1-u_2\rangle \le -\frac{1}{2}\|u_1-u_2\|_{\mathcal{Z}}^2+c(1+\|u_2\|_{\mathcal{U}}^2)|u_1-u_2|_{\mathcal{H}}^2, \tag{2.55}$$

where $c>0$ is independent of u_1,u_2 and $\epsilon_1>0$ is chosen such that $3\epsilon_1+\alpha_1<1/2$. Therefore (2.47) is proved. □

3. Existence and uniqueness of solutions

In this section, we present one of the main results of the paper.

Theorem 1. *Assume that the hypotheses (2.34), (2.46) hold and $u_0\in L^2(\Omega,\mathcal{F}_0;\mathcal{U})$. Then there exists a unique solution to the stochastic 3D GMCHNSE (2.45) that satisfies the following energy inequality*

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u(t)\|_{\mathcal{U}}^2\right)+\mathbb{E}\int_0^T|Au(t)|_{L^2}^2dt<\infty.$$

Proof. • I) **Uniqueness**

Let X and $\tilde{X}$ two solutions to (2.45) starting from the same initial value u_0. For any $\xi>0$, define the stopping time

$$\tau_\xi := \inf\{t\in[0,T] : \|X(t)\|_{\mathcal{U}}\vee\|\tilde{X}(t)\|_{\mathcal{U}}|\ge\xi\}.$$

Set $\Theta(t)=X(t)-\tilde{X}(t)$. Then by Itô's formula, we have

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$$
\begin{aligned}
|\Theta(t)|^2_{\mathcal{H}} = 2\int_0^t \langle \mathcal{G}(X(s)) - \mathcal{G}(\tilde{X}(s)), \Theta(s)\rangle ds \\
+ 2\sum_{k=1}^{\infty}\int_0^t (\sigma_k(X(s)) - \sigma_k(\tilde{X}(s)), \Theta(s)) dW_k(s) \\
+ \sum_{k=1}^{\infty}\int_0^t |\sigma_k(X(s)) - \sigma_k(\tilde{X}(s))|^2_{\mathcal{H}} ds. \qquad (3.1)
\end{aligned}
$$

From Lemma 4, there exists a constant $c_0 > 0$ such that

$$\langle \mathcal{G}(X(s)) - \mathcal{G}(\tilde{X}(s), \Theta(s)\rangle \le -\frac{1}{2}\|\Theta(s)\|^2_{\mathcal{Z}} + c(1 + \|\tilde{X}(t)\|^2_{\mathcal{U}})|\Theta(s)|^2_{\mathcal{H}}. \qquad (3.2)$$

Combining the estimates (3.2) with (3.1), we get

$$\mathbb{E}|\Theta(t\wedge\tau_\xi)|^2_{\mathcal{H}} \le C_{\xi,T}\int_0^t \mathbb{E}|\Theta(s\wedge\tau_\xi)|^2_{\mathcal{H}} ds.$$

By Gronwall's inequality, we get for any $t \in [0, T]$

$$\mathbb{E}|\Theta(t\wedge\tau_\xi)|^2_{\mathcal{H}} = 0.$$

And the uniqueness follows by letting $\xi \to \infty$ and Fatou's lemma.

- II) **Existence**
  We will use the Galerkin approximation combined with the strong monotonicity of the stochastic 3D GMCHNSE. We shall do this in two steps:
  Step 1: Assume $u_0 \in L^6(\Omega, \mathcal{F}_0; \mathcal{U})$.
  Let $\{e_i : i \ge 1\} \subset D(A)$ be a fixed orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of A, so that it is also orthogonal in $\mathcal{U}$. Denote π_n the orthogonal projection from $\mathcal{H}$ onto the finite dimensional space $\mathcal{H}_n := span\{e_1, e_2, ..., e_n\}$:

$$\pi_n v := \sum_{i=1}^{n} (v, e_i)e_i.$$

Thus π_n is also the orthogonal projection onto $\mathcal{H}_n$ in $\mathcal{U}$.
Consider the following finite dimensional stochastic differential equations in $\mathcal{H}_n$:

$$du_n(t) = [\pi_n\mathcal{G}(u_n(t))]dt + \sum_{k=1}^{\infty}\pi_n\sigma_k(u_n(t))dW_k(t), \quad u_n(0) = \pi_n u_0. \qquad (3.3)$$

For $u \in \mathcal{H}_n$, we have

$$\langle u, \pi_n \mathcal{G}(u)\rangle \leq C_N(1+|u|^2_{\mathcal{H}_n}) \text{ and } \sum_{k=1}^{\infty} \|\pi_n \sigma_k(u)\|^2_{\mathcal{H}_n} \leq c(1+|u|^2_{\mathcal{H}_n}). \tag{3.4}$$

Moreover by Lemma 3, and (2.34), the maps

$$u \in \mathcal{H}_n \rightarrowtail \pi_n \mathcal{G}(u) \in \mathcal{H}_n \text{ and } u \in \mathcal{H}_n \rightarrowtail \pi_n \sigma$$

are respectively locally Lipschitz continuous and Lipschitz continuous. Then by the theory of stochastic differential equations (see [23,22]), there exists a unique continuous $(\mathcal{F}_t)$-adapted process $u_n(t)$ satisfying

$$u_n(t) = u_n(0) + \int_0^t \pi_n \mathcal{G}(u_n(s))ds + \sum_{k=1}^{\infty} \int_0^t \pi_n \sigma_k(u_n(s))dW_s^k$$

and for any $n \geq i$,

$$\langle u_n(t), e_i\rangle_{\mathcal{U}} = \langle u_0, e_i\rangle_{\mathcal{U}} + \int_0^t \langle \pi_n \mathcal{G}(u_n(s)), e_i\rangle_{\mathcal{U}} ds + \sum_{k=1}^{\infty} \int_0^t \langle \pi_n \sigma_k(u_n(s)), e_i\rangle_{\mathcal{U}} dW_s^k.$$

We now prove some a priori estimates of the approximated solution.

Lemma 5. *There exists a constant C independent of n such that*

$$1) \sup_{t\in[0,T]} \mathbb{E}\left(|u_n(t)|^2_{\mathcal{H}}\right) + \mathbb{E}\int_0^T \|u_n(s)\|^2_{\mathcal{Z}} ds \leq C\mathbb{E}\left(1+|u_0|^2_{\mathcal{H}}\right) \leq C, \tag{3.5}$$

$$2) \mathbb{E}\left(\sup_{t\in[0,T]} |u_n(t)|^p_{\mathcal{H}}\right) \leq C\left(1+\mathbb{E}(|u_0|^p_{\mathcal{H}})\right), \ \forall p \geq 2. \tag{3.6}$$

Proof. The proof is similar to that of Lemma 7 given in Section 4. □

Lemma 6. *There exists a constant C independent of n such that*

$$1) \sup_{t\in[0,T]} \mathbb{E}\left(\|u_n(t)\|^2_{\mathcal{U}}\right) + \mathbb{E}\int_0^T |Au_n(s)|^2_{L^2} ds \leq c\left(1+\mathbb{E}(\|u_0\|^2_{\mathcal{U}})\right), \tag{3.7}$$

$$2) \mathbb{E}\left(\sup_{t\in[0,T]} \|u_n(t)\|^2_{\mathcal{U}}\right) \leq c\left(1+\mathbb{E}(\|u_0\|^2_{\mathcal{U}})\right), \tag{3.8}$$

$$3) \sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|^6_{\mathcal{U}} dt + \mathbb{E}\int_0^T \|u_n(s)\|^4_{\mathcal{U}} |Au_n(s)|^2_{L^2} ds \leq c\left(1+\mathbb{E}(\|u_0\|^6_{\mathcal{U}})\right), \tag{3.9}$$

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$$4)\ \sup_n \int_0^T \mathbb{E}|\pi_n \mathcal{G}(u_n(t))|^2_{\mathcal{H}} dt < \infty. \tag{3.10}$$

Proof. 1) By Itô's formula, we have

$$\begin{aligned}\|u_n(t)\|^2_{\mathcal{U}} &= \|u_n(0)\|^2_{\mathcal{U}} + 2\int_0^t \langle \mathcal{G}(u_n(s), u_n(s)\rangle_{\mathcal{U}} ds \\ &+ 2\sum_{k=1}^{\infty}\int_0^t \langle \sigma_k(u_n(s)), u_n(s)\rangle_{\mathcal{U}} dW_k(s) + \sum_{k=1}^{\infty}\int_0^t \|\sigma_k(u_n(s))\|^2_{\mathcal{U}} ds.\end{aligned} \tag{3.11}$$

Note that for $u_n = (v_n, \phi_n)$, we have

$$\begin{aligned}&\langle \mathcal{G}(u_n), u_n\rangle_{\mathcal{U}} = \langle -Au_n, u_n\rangle_{\mathcal{U}} - \langle B^0_N(v_n, v_n), A_0 v_n\rangle - \langle B^1(v_n, \phi_n), A_1^2\phi_n\rangle \\ &+\langle R^0(A_1\phi_n\phi_n), A_0 v_n\rangle - \langle A_1 f(\phi_n), A_1^2\phi_n\rangle - \langle G(u_n), u_n\rangle_{\mathcal{U}}.\end{aligned} \tag{3.12}$$

But we have

$$-\langle Au_n, u_n\rangle_{\mathcal{U}} = -|Au_n|^2_{L^2}. \tag{3.13}$$

As noted in [6], we have

$$|b^0_N(v_n, v_n, A_0 v_n)| \le \frac{1}{8}|A_0 v_n|^2_{L^2} + cN^4\|v_n\|^2. \tag{3.14}$$

We also have

$$\begin{aligned}|\langle B^1(v_n, \phi_n), A_1^2\phi_n\rangle| &= |\langle b^1(v_n, \phi_n, A_1^2\phi_n) \\ &\le c\|v_n\|^{1/2}|A_0 v_n|^{1/2}_{L^2}\|\phi_n\||A_1^2\phi_n|_{L^2} \\ &\le \frac{1}{8}(|A_0 v_n|^2_{L^2} + |A_1^2\phi_n|^2_{L^2}) + c\|v_n\|^2\|\phi_n\|^4,\end{aligned} \tag{3.15}$$

where we have used the Agmon's inequality in the second line: $|v_n|_{L^\infty} \le \|v_n\|^{\frac{1}{2}}|A_0 v_n|^{\frac{1}{2}}_{L^2}$, see [33],

$$\begin{aligned}|\langle R^0(A_1\phi_n, \phi_n), A_0 v_n\rangle| &= |b^1(A_0 v_n, \phi_n, A_1\phi_n)| \\ &\le c|A_0 v_n|_{L^2}|A_1\phi_n|^{3/2}_{L^2}|A_1^{3/2}\phi_n|^{1/2}_{L^2} \\ &\le |A_0 v_n|_{L^2}|A_1^2\phi_n|^{5/6}_{L^2}\|\phi_n\|^{7/6} \\ &\le \frac{1}{8}(|A_0 v_n|^2_{L^2} + |A_1^2\phi_n|^2_{L^2}) + c\|\phi_n\|^{14}.\end{aligned} \tag{3.16}$$

In the second line of these inequalities, we have used the inequality (2.24). We note that

$$|\langle A_1 f(\phi_n), A_1^2\phi_n\rangle| = |\langle f''\phi_n)(A_1^{1/2}\phi_n)^2 + f'(\phi_n)A_1\phi_n, A_1^2\phi_n\rangle| \le J_1 + J_2. \tag{3.17}$$

Using the properties (2.13) of f, we can check that

$$\begin{aligned} J_1 &\equiv |\langle f''(\phi_n)(A_1^{1/2}\phi_n)^2, A_1^2\phi_n\rangle| \\ &\le c\int_{\mathcal{M}} |A_1^{1/2}\phi_n|^2|A_1^2\phi_n|dx \le c|A_1^{1/2}\phi_n|^2_{L^4}|A_1^2\phi_n|_{L^2} \\ &\le c\|\phi_n\|^{1/2}|A_1\phi_n|^{3/2}_{L^2}|A_1^2\phi_n|_{L^2} \le c\|\phi_n\|^{5/4}|A_1^{3/2}\phi_n|^{3/4}_{L^2}|A_1^2\phi_n|_{L^2} \\ &\le \frac{1}{8}|A_1^2\phi_n|^2_{L^2} + c\|\phi_n\|^{10}. \end{aligned} \tag{3.18}$$

The second line of the preceding inequalities uses the following Ladyzhenkaya's inequality in 3D: $\|\phi_n\|_{L^4} \le c|\phi_n|^{\frac{1}{4}}\|\phi_n\|^{\frac{3}{4}}$, see [29] for the proof.
Similarly, we have

$$\begin{aligned} J_2 &\equiv |\langle f'(\phi_n)A_1\phi_n, A_1^2\phi_n\rangle| \\ &\le c\int_{\mathcal{M}} (1+|\phi_n|)|A_1\phi_n||A_1^2\phi_n|dx \\ &\le c|A_1\phi_n|_{L^2}|A_1^2\phi_n|_{L^2} + c\|\phi_n\||A_1\phi_n|_{L^3}|A_1^2\phi_n|_{L^2} \\ &\le c\|\phi_n\|^{1/2}|A_1^2\phi_n|^{3/2}_{L^2} + c\|\phi_n\|\|\phi_n\|^{1/2}|A_1^2\phi_n|^{7/4}_{L^2} \\ &\le \frac{1}{8}|A_1^2\phi_n|^2_{L^2} + c\|\phi_n\|^2 + c\|\phi_n\|^8\|\phi_n\|^{10}. \end{aligned} \tag{3.19}$$

Using the Young's inequality, it follows that

$$|-\langle E(u_n), u_n\rangle_{\mathcal{U}}| \le \tfrac{1}{8}|Au_n|^2_{L^2} + c|u_n|^{18}_{\mathcal{H}} + c. \tag{3.20}$$

The properties of G give

$$|-\langle G(u_n), u_n\rangle_{\mathcal{U}}| \le c(1+\|u_n\|^2_{\mathcal{U}}). \tag{3.21}$$

The estimates (3.13)–(3.20), and (3.11) yield

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$$\|u_n(t)\|_{\mathcal{U}}^2 \leq \|u_0\|_{\mathcal{U}}^2 - \int_0^t |Au_n(s)|_{L^2}^2 ds + c\int_0^t (1+\|u_n(s)\|_{\mathcal{U}}^2 + |u_n(s)|_{\mathcal{H}}^{18}) ds$$
$$+ 2\sum_{k=1}^{\infty}\int_0^t ((\sigma_k(u_n(s)), u_n(s)))_{\mathcal{U}} dW_k(s). \tag{3.22}$$

Taking expectation, we get

$$\mathbb{E}\left[\|u_n(t)\|_{\mathcal{U}}^2\right] \leq \mathbb{E}\|u_0\|_{\mathcal{U}}^2 - \mathbb{E}\int_0^t |Au_n(s)|_{L^2}^2 ds + C\mathbb{E}\int_0^t \left(1+\|u_n(s)\|_{\mathcal{U}}^2 + |u_n(s)|_{\mathcal{H}}^{18}\right) ds. \tag{3.23}$$

Hence by Gronwall's inequality and Lemma 5, we have for any $T > 0$,

$$\sup_{t\in[0,T]} \mathbb{E}\|u_n(t)\|_{\mathcal{U}}^2 + \mathbb{E}\int_0^T |Au_n(s)|_{L^2}^2 ds \leq C\mathbb{E}\left(1+\|u_0\|_{\mathcal{U}}^2 + |u_0|_{\mathcal{H}}^{18}\right) \leq C. \tag{3.24}$$

This proves 1).

2) Taking the supremum over $[0, T]$ with (3.22), we get

$$\sup_{t\in[0,T]} \|u_n(t)\|_{\mathcal{U}}^2 + \int_0^T |Au_n(s)|_{L^2}^2 ds \leq \|u_0\|_{\mathcal{U}}^2 + c\int_0^T (1+\|u_n(s)\|_{\mathcal{U}}^2 + |u_n(s)|_{\mathcal{H}}^{18}) ds$$
$$+ \sup_{t\in[0,T]} 2\sum_{k=1}^{\infty}\int_0^t ((\sigma_k(u_n(s)), u_n(s)))_{\mathcal{U}} dW_k(s). \tag{3.25}$$

Applying the Burkholder's inequality, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\sum_{k=1}^{\infty}\int_0^t \langle\sigma_k(u_n(s)), u_n(s)\rangle_{\mathcal{U}} dW_k(s)\right|\right)$$
$$\leq c\mathbb{E}\left(\int_0^T \sum_{k=1}^{\infty}\langle\sigma_k(u_n(s)), u_n(s)\rangle_{\mathcal{U}}^2 ds\right)^{\frac{1}{2}},$$
$$\leq c\mathbb{E}\left(\sup_{s\in[0,T]}\|u_n(s)\|_{\mathcal{U}}^2\right)^{\frac{1}{2}}\left(\int_0^T (1+\|u_n(s)\|_{\mathcal{U}}^2) ds\right)^{\frac{1}{2}}$$

$$\leq \epsilon\mathbb{E}\left(\sup_{s\in[0,T]}\|u_n(s)\|_{\mathcal{U}}^2\right)+C_\epsilon\mathbb{E}\int_0^T(1+\|u_n(s)\|_{\mathcal{U}}^2)ds.$$

Combining (3.25), (3.24) and the above inequalities, we get

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u_n(t)\|_{\mathcal{U}}^2\right)\leq C\left(1+\mathbb{E}\|u_0\|_{\mathcal{U}}^2\right)+\epsilon\mathbb{E}\left(\sup_{t\in[0,T]}\|u_n(t)\|_{\mathcal{U}}^2\right)$$

Choosing ϵ small enough, we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|u_n(t)\|_{\mathcal{U}}^2\right)\leq C\left(1+\mathbb{E}\|u_0\|_{\mathcal{U}}^2\right).$$

This ends the proof of 2).

3) We apply Itô's formula the function $f(x)=x^3$ and the real-valued process $Y(t)=\|u_n(t)\|_{\mathcal{U}}^2$ and we get

$$\begin{aligned}\|u_n(t)\|_{\mathcal{U}}^6=&\|u_n(0)\|_{\mathcal{U}}^6+6\int_0^t\|u_n(s)\|_{\mathcal{U}}^4((\mathcal{G}(u_n(s)),u_n(s)))_{\mathcal{U}}ds\\&+3\sum_{k=1}^\infty\int_0^t\|u_n(s)\|_{\mathcal{U}}^4\|\pi_n\sigma_k(u_n(s))\|_{\mathcal{U}}^2ds\\&+12\sum_{k=1}^\infty\int_0^t\|u_n(s)\|_{\mathcal{U}}^2((\pi_n\sigma_k(u_n(s)),u_n(s)))^2ds\\&+6\sum_{k=1}^\infty\int_0^t\|u_n(s)\|_{\mathcal{U}}^4((\pi_n\sigma_k(u_n(s)),u_n(s)))_{\mathcal{U}}dW_k(s).\end{aligned}$$

Note that

$$\langle\mathcal{G}(u_n(s)),u_n(s)\rangle_{\mathcal{U}}\leq-|Au_n(s)|_{L^2}^2+c\|u_n(s)\|_{\mathcal{U}}^2+c|u_n(s)|_{\mathcal{H}}^{q_0}. \tag{3.26}$$

It follows that

$$\begin{aligned}\|u_n(t)\|_{\mathcal{U}}^6\leq\|u_n(0)\|_{\mathcal{U}}^6-6\int_0^t\|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds+c\int_0^t\|u_n(s)\|_{\mathcal{U}}^6ds\\+c\int_0^t|u_n(s)|_{\mathcal{H}}^{q_0}ds+12c\int_0^t\|u_n(s)\|_{\mathcal{U}}^4(1+\|u_n(s)\|_{\mathcal{U}}^2)ds\end{aligned}$$

$$+6\sum_{k=1}^{\infty}\int_0^t \|u_n(s)\|_{\mathcal{U}}^4((\pi_n\sigma_k(u_n(s)),u_n(s)))_{\mathcal{U}}dW_k(s). \tag{3.27}$$

Using the inequality $|x|^{2p-2} \le 1+|x|^{2p}$ for $p\ge 1$, we get

$$\|u_n(t)\|_{\mathcal{U}}^6+6\int_0^t \|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds \le \|u_0\|_{\mathcal{U}}^6+c\int_0^t(1+\|u_n(s)\|_{\mathcal{U}}^6+|u_n(s)|_{\mathcal{H}}^{q_0})ds$$

$$+6\sum_{k=1}^{\infty}\int_0^t \|u_n(s)\|_{\mathcal{U}}^4((\pi_n\sigma_k(u_n(s)),u_n(s)))_{\mathcal{U}}dW_k(s).$$

Taking the supremum over $[0,t]$, we obtain

$$\begin{aligned}&\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6+6\int_0^t \|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds\\
&\le \|u_0\|_{\mathcal{U}}^6+c\int_0^t(1+\|u_n(s)\|_{\mathcal{U}}^6+|u_n(s)|_{\mathcal{H}}^{q_0})ds\\
&+6\sup_{s'\in[0,t]}\left(\sum_{k=1}^{\infty}\int_0^{s'}\|u_n(s)\|_{\mathcal{U}}^4((\pi_n\sigma_k(u_n(s)),u_n(s)))_{\mathcal{U}}dW_k(s)\right).\end{aligned} \tag{3.28}$$

By Burkholder's inequality, we have

$$\begin{aligned}&\mathbb{E}\sup_{s'\in[0,t]}\left(\sum_{k=1}^{\infty}\int_0^{s'}\|u_n(s)\|_{\mathcal{U}}^4((\pi_n\sigma_k(u_n(s)),u_n(s)))dW_k(s)\right)\\
&\le c\mathbb{E}\left(\int_0^t\sum_{k=1}^{\infty}\|u_n(s)\|_{\mathcal{U}}^8((\pi_n\sigma_k(u_n(s)),u_n(s)))^2ds\right)^{\frac{1}{2}}\\
&\le c\mathbb{E}\left(\int_0^t\sum_{k=1}^{\infty}\|u_n(s)\|_{\mathcal{U}}^8\|\pi_n\sigma_k(u_n(s))\|_{\mathcal{U}}^2\|u_n(s)\|_{\mathcal{U}}^2ds\right)^{\frac{1}{2}}\\
&\le \mathbb{E}\left(\int_0^t\|u_n(s)\|_{\mathcal{U}}^{10}(1+\|u_n(s)\|_{\mathcal{U}}^2)ds\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq c\mathbb{E}\left(\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^4(1+\|u_n(s)\|_{\mathcal{U}}^2)ds\right)^{\frac{1}{2}}$$

$$\leq \frac{c}{2}\epsilon\mathbb{E}\left(\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6\right)+\frac{c}{2\epsilon}\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^4ds+\frac{c}{2\epsilon}\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^6ds$$

$$\leq \frac{c}{2}\epsilon\mathbb{E}\left(\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6\right)+\frac{c}{2\epsilon}T+\frac{c}{\epsilon}\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^6ds. \tag{3.29}$$

Taking the expectation in (3.28) and using (3.29), we obtain

$$\mathbb{E}\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6+6\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds$$

$$\leq \mathbb{E}\|u_0\|_{\mathcal{U}}^6+CT+\int_0^t\mathbb{E}\sup_{s'\in[0,s]}\|u_n(s')\|_{\mathcal{U}}^6ds+\frac{c}{2\epsilon}T$$

$$+\frac{c}{2}\epsilon\mathbb{E}\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6+\frac{c}{\epsilon}\int_0^t\mathbb{E}\sup_{s'\in[0,s]}\|u_n(s')\|_{\mathcal{U}}^6ds+c\int_0^t\mathbb{E}\sup_{s'\in[0,s]}|u_n(s')|_{\mathcal{H}}^{q_0}ds.$$

Taking ϵ sufficiently small, we get

$$\mathbb{E}\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6+6\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds$$

$$\leq \mathbb{E}\|u_0\|_{\mathcal{U}}^6+(CT+\frac{c}{2\epsilon}T)+(1+\frac{c}{\epsilon})\int_0^t\mathbb{E}\sup_{s'\in[0,s]}\|u_n(s')\|_{\mathcal{U}}^6ds$$

$$+c\int_0^t\mathbb{E}\sup_{s'\in[0,s]}|u_n(s')|_{\mathcal{H}}^{q_0}ds. \tag{3.30}$$

Applying Gronwall's inequality, we obtain

$$\mathbb{E}\sup_{s\in[0,t]}\|u_n(s)\|_{\mathcal{U}}^6+6\mathbb{E}\int_0^t\|u_n(s)\|_{\mathcal{U}}^4|Au_n(s)|_{L^2}^2ds\leq C. \tag{3.31}$$

This ends the proof of 3).

4) We also have

$$|\pi_n \mathcal{G}(u_n(t)|_{L^2}^2 \leq c\left(|Au_n|_{L^2}^2 + |B_N(u_n)|_{L^2}^2 + |G(u_n)|_{L^2}^2 + |R^0(u_n)|_{L^2}^2 + |E(u_n)|_{L^2}^2\right) \tag{3.32}$$

We can easily check that

$$|B_N^0(v_n, v_n)|_{L^2}^2 \leq c\|v_n\||A_0 v_n|_{L^2} \leq \frac{1}{8}|Au_n|_{L^2}^2 + c\|u_n\|_{\mathcal{U}}^2, \tag{3.33}$$

$$|B^1(v_n, \phi_n)|_{L^2}^2 \leq c\|v_n\||A_0 v_n|_{L^2}\|\phi_n\| \leq \frac{1}{8}|Au_n|_{L^2}^2 + c\|u_n\|_{\mathcal{U}}^2 |u_n|_{\mathcal{H}}^2, \tag{3.34}$$

$$\begin{aligned} |R^0(A_1\phi_n, \phi_n)|_{L^2}^2 &\leq c|A_1\phi_n|_{L^2}^3 |A_1^{3/2}\phi_n|_{L^2} \\ &\leq c|A_1^2\phi_n|_{L^2}^{3/2} |\phi_n|_{L^2}^{5/2} \leq \tfrac{1}{8}|Au_n|_{L^2}^2 + |u_n|_{\mathcal{H}}^{10}, \end{aligned} \tag{3.35}$$

$$\begin{aligned} |E(u_n)|_{L^2}^2 &= \alpha^2 |A_1 f(\phi_n)|_{L^2}^2 \leq \frac{1}{8}|A_1^2\phi_n|_{L^2}^2 + \|\phi_n\|^{q_0} \\ &\leq \tfrac{1}{8}|Au_n|_{L^2}^2 + |u_n|_{\mathcal{H}}^{q_0}. \end{aligned} \tag{3.36}$$

From (2.34), we also have

$$|G(u_n)|_{\mathcal{H}}^2 \leq c(1 + |u_n|_{\mathcal{H}}^2) \leq c(1 + \|u_n\|_{\mathcal{H}}^2). \tag{3.37}$$

It follows that

$$|\pi_n \mathcal{G}(u_n(t))|_{L^2}^2 \leq c(|Au_n|_{L^2}^2 + |u_n|_{\mathcal{H}}^{q_0} + 1 + \|u_n\|_{\mathcal{U}}^2)$$

and

$$\mathbb{E}\int_0^T |\pi_n \mathcal{G}(u_n(t))|_{L^2}^2 < \infty.$$

This ends the proof of the lemma. □

Let $\Omega_T = \Omega \times [0, T]$. Using the energy estimates (3.7)–(3.10) along with the Banach–Alaoglu theorem, we can extract a subsequence of $\{u_n\}$ still denoted by $(u_n)_n$ and processes $\tilde{u} \in L^2(\Omega_T; \mathcal{H}) \cap L^2(\Omega; L^\infty([0, T]; \mathcal{U}))$, $\mathcal{G} \in L^2(\Omega_T; \mathcal{H})$ and $\tilde{\sigma} := (\tilde{\sigma_k})_{k\in\mathbb{N}} \in L_2(l_2; \mathcal{U})$ for which the following hold:

i) $u_n \rightharpoonup \tilde{u}$ weakly in $L^2(\Omega_T; D(A))$, hence weakly in $L^2(\Omega_T; \mathcal{U})$,
ii) $u_n \rightharpoonup \tilde{u}$ in $L^2(\Omega, L^\infty([0, T]; \mathcal{U}))$ with respect to the weak star topology,
iii) $\pi_n \mathcal{G}(u_n) \rightharpoonup \mathcal{G}$ weakly in $L^2(\Omega_T; H)$,
iv) $\pi_n \sigma(u_n) \rightharpoonup \tilde{\sigma}$ weakly in $L^2(\Omega_T; L_2(l_2; \mathcal{U}))$,
v) $u_n \rightharpoonup \tilde{u}$ weakly also in $L^6(\Omega_T; \mathcal{U})$.

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For $0 \leq t \leq T$, define

$$u(t) = u_0 + \int_0^t \mathcal{G}(s)ds + \sum_{k=1}^{\infty} \int_0^t \tilde{\sigma}_k(s)dW_k(s).$$

It follows from [27] that $u = \tilde{u}\ dt \times \mathbb{P}$-a.e. and u has continuous paths in $\mathcal{U}$. To complete the proof of the theorem, we need to show that

$$\mathcal{G}(s) = \mathcal{G}(\tilde{u}(s)) \text{ and } \tilde{\sigma_k}(s) = \sigma_k(\tilde{u}(s)) - a.e. \text{ on } \Omega_T.$$

The proof follows the same steps as in [30]. Fix an integer K. Take $\vartheta \in L^2(\Omega_T, \mathcal{H}_K)$ where $\mathcal{H}_K$ is the linear span of $e_1, e_2, ..., e_K$. By Itô's formula, writing $u = u - \vartheta + \vartheta$, we have

$$\begin{aligned}
&\mathbb{E}\left[|u(t)|^2_{\mathcal{H}} e^{-r(t)}\right] - \mathbb{E}\left[|u_0|^2_{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\int_0^t 2e^{-r(s)}\langle \mathcal{G}(s), u(s)\rangle ds\right] + \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s)|^2_{\mathcal{H}} ds\right] \\
&- \mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)|u(s)|^2_{\mathcal{H}} ds\right] \\
&= \mathbb{E}\left[\int_0^t 2e^{-r(s)}\langle g(s), u(s)\rangle ds\right] + \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s)|^2_{\mathcal{H}} ds\right] \\
&- \mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)\{|u(s) - \vartheta(s)|^2_{\mathcal{H}} + 2\langle u(s) - \vartheta(s), \vartheta(s)\rangle + |\vartheta(s)|^2_{\mathcal{H}} ds\} ds\right],
\end{aligned} \tag{3.38}$$

where $r(t)$ is a non-negative stochastic process which is absolutely continuous and to be chosen later. A similar expression also holds for $\mathbb{E}\left[|u_n(t)|^2_{\mathcal{H}} e^{-r(t)}\right] - \mathbb{E}\left[|u_0|^2_{\mathcal{H}}\right]$, that is

$$\begin{aligned}
&\mathbb{E}\left[|u_n(t)|^2_{\mathcal{H}} e^{-r(t)}\right] - \mathbb{E}\left[|u_0|^2_{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\int_0^t 2e^{-r(s)}\langle \mathcal{G}(u_n(s)), u_n(s)\rangle ds\right] \\
&+ \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(u_n(s))|^2_{\mathcal{H}} ds\right] - \mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)|u_n(s)|^2_{\mathcal{H}} ds\right]
\end{aligned}$$

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$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^t 2e^{-r(s)}\langle \mathcal{G}(u_n(s)), u_n(s)\rangle ds\right] + \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(u_n(s))|_{\mathcal{H}}^2 ds\right] \\
&- \mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)\{|u_n(s)-\vartheta(s)|_{\mathcal{H}}^2 + 2\langle u_n(s)-\vartheta(s), \vartheta(s)\rangle + |\vartheta(s)|_{\mathcal{H}}^2 ds\} ds\right].
\end{aligned}
\tag{3.39}
$$

For any non-negative $\psi \in L^\infty([0,T],\mathbb{R})$, the weak convergence implies that

$$
\begin{aligned}
&\int_0^T \psi(t)dt\mathbb{E}\left[|u(t)|_{\mathcal{H}}^2 e^{-r(t)}\right] - \mathbb{E}\left[|u_0|_{\mathcal{H}}^2\right] \\
&= \int_0^T \psi(t)dt\mathbb{E}\left[|\tilde{u}(t)|_{\mathcal{H}}^2 e^{-r(t)}\right] - \mathbb{E}\left[|u_0|_{\mathcal{H}}^2\right] \\
&\le \liminf_{n\to\infty} \int_0^T \psi(t)dt\mathbb{E}\left[|u_n(t)|_{\mathcal{H}}^2 e^{-r(t)}\right] - \mathbb{E}\left[|u_0|_{\mathcal{H}}^2\right].
\end{aligned}
\tag{3.40}
$$

By substituting the corresponding expressions, (3.40) becomes

$$
\begin{aligned}
&\int_0^T \psi(t)dt \left\{\mathbb{E}\left[\int_0^t 2e^{-r(s)}\langle \mathcal{G}(s), u(s)\rangle ds\right] + \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s)|_{\mathcal{H}}^2 ds\right]\right\} \\
&- \int_0^T \psi(t)dt \left\{\mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)\{|u(s)-\vartheta(s)|_{\mathcal{H}}^2 + 2\langle u(s)-\vartheta(s), \vartheta(s)\rangle\} ds\right]\right\} \\
&\le \liminf_{n\to\infty} \int_0^T \psi(t)dt \left[\mathbb{E}\int_0^t 2e^{-r(s)}\langle \mathcal{G}(u_n(s)), u_n(s)\rangle ds\right] \\
&+ \liminf_{n\to\infty} \int_0^T \psi(t)dt\ \mathbb{E}\left[\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\pi_n\sigma_k(u_n(s))|_{\mathcal{H}}^2 ds\right] \\
&- \liminf_{n\to\infty} \int_0^T \psi(t)dt\ \mathbb{E}\left[\int_0^t e^{-r(s)} r'(s)\{|u_n(s)-\vartheta(s)|_{\mathcal{H}}^2 + 2\langle u_n(s)-\vartheta(s), \vartheta(s)\rangle\} ds\right] \\
&:= \liminf_{n\to\infty} Z_n,
\end{aligned}
\tag{3.41}
$$

where $Z_n = Z_n^1 + Z_n^2 + Z_n^3$ with

$$Z_n^1 = \int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\Big\{-r'(s)|u_n(s)-\vartheta(s)|^2_{\mathcal{H}}$$
$$+2\langle \mathcal{G}(u_n(s))-\mathcal{G}(\vartheta(s)), u_n(s)-\vartheta(s)\rangle\Big\}ds$$
$$+\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t \sum_{k=1}^{\infty}|\pi_n\sigma_k(u_n(s))-\pi_n\sigma_k(\vartheta(s))|^2_{\mathcal{H}}ds, \tag{3.42}$$

$$Z_n^2 = \int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\left\{-2r'(s)\langle u_n(s)-\vartheta(s),\vartheta(s)\rangle + 2\langle \mathcal{G}(u_n(s)),\vartheta(s)\rangle\right\}ds$$
$$+2\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\{\langle \mathcal{G}(\vartheta(s)), u_n(s)\rangle - 2\langle \mathcal{G}(\vartheta(s)),\vartheta(s)\rangle\}ds$$
$$+2\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\sum_{k=1}^{\infty}\langle \pi_n\sigma_k(u_n(s)),\sigma_k(\vartheta(s))\rangle ds, \tag{3.43}$$

$$Z_n^3 = \int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\left\{2\sum_{k=1}^{\infty}\langle \pi_n\sigma_k(u_n(s)),\pi_n\sigma_k(\vartheta(s))-\sigma_k(\vartheta(s))\rangle\right\}ds$$
$$-\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\sum_{k=1}^{\infty}|\pi_n\sigma_k(\vartheta(s))|^2_{\mathcal{H}}ds. \tag{3.44}$$

Set $r'(s) = C_1(1+\|\vartheta(s)\|^2_{\mathcal{U}})+c$. In view of (2.47) and (2.34) we see that $Z_n^1 \le 0$. By the weak convergence, it is clear that $Z_n^2 \to Z^2$, where

$$Z^2 = \int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\left\{-2r'(s)\langle u(s)-\vartheta(s),\vartheta(s)\rangle + 2\langle \mathcal{G}(s),\vartheta(s)\rangle\right\}ds$$
$$+2\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\{\langle \mathcal{G}(\vartheta(s)), u(s)\rangle - 2\langle \mathcal{G}(\vartheta(s)),\vartheta(s)\rangle\}ds$$
$$+2\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\sum_{k=1}^{\infty}\langle \tilde{\sigma}_k(s),\sigma_k(\vartheta(s))\rangle ds. \tag{3.45}$$

Also

$$Z_n^3 \to Z^3 := -\int_0^T \psi(t)dt \sum_{k=1}^{\infty} \mathbb{E}\left[\int_0^t |\sigma_k(\vartheta(s))|_{\mathcal{H}}^2 ds\right]. \tag{3.46}$$

Combining (3.41)–(3.46) after cancellations it turns out that

$$\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)}\left\{-r'(s)|u(s)-\vartheta(s)|_{\mathcal{H}}^2 + 2\langle \mathcal{G}(s) - \mathcal{G}(\vartheta(s)), u(s)-\vartheta(s)\rangle\right\} ds$$
$$+\int_0^T \psi(t)dt\ \mathbb{E}\int_0^t e^{-r(s)} \sum_{k=1}^{\infty} |\tilde{\sigma}_k(s) - \sigma_k(\vartheta(s))|_{\mathcal{H}}^2 ds \le 0. \tag{3.47}$$

As K is arbitrary, by approximation it is seen that (3.47) holds true for every $\vartheta \in L^2(\Omega_T, D(A))$. In particular, take $\vartheta(s) = u(s)$ in (3.47) to obtain $\tilde{\sigma}_k(s) = \sigma_k(u(s))$ for every $k \ge 1$. For $\lambda \in [-1,1]$, $\tilde{\vartheta} \in L^\infty(\Omega_T, D(A))$, set $\vartheta_\lambda(s) = u(s) - \lambda\tilde{\vartheta}(s)$. Replace ϑ by ϑ_λ in (3.47) to get

$$\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)}\left\{-\lambda^2 r'_\lambda(s)|\tilde{\vartheta}(s)|_{\mathcal{H}}^2 + 2\lambda\langle \mathcal{G}(s) - \mathcal{G}(\vartheta_\lambda(s)), \tilde{\vartheta}(s)\rangle\right\} ds\right] \le 0, \tag{3.48}$$

where $r_\lambda(s)$ is defined as $r(s)$ with ϑ replaced by ϑ_λ. Dividing (3.48) by λ we obtain

$$\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)}\left\{-\lambda r'_\lambda(s)|\tilde{\vartheta}(s)|_{\mathcal{H}}^2 + 2\langle \mathcal{G}(s) - \mathcal{G}(\vartheta_\lambda(s)), \tilde{\vartheta}(s)\rangle\right\} ds\right] \le 0, \tag{3.49}$$

for $\lambda > 0$, and

$$\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)}\left\{-\lambda r'_\lambda(s)|\tilde{\vartheta}(s)|_{\mathcal{H}}^2 + 2\langle \mathcal{G}(s) - \mathcal{G}(\vartheta_\lambda(s)), \tilde{\vartheta}(s)\rangle\right\} ds\right] \ge 0 \tag{3.50}$$

for $\lambda < 0$.
By (2.47), we have

$$|\langle \mathcal{G}(u(s)) - \mathcal{G}(\vartheta_\lambda(s)), \tilde{\vartheta}(s)\rangle|$$
$$\le \frac{|\lambda|}{2}\|\tilde{\vartheta}(s)\|_{\mathcal{U}}^2 + C_1|\lambda||\tilde{\vartheta}(s)|_{\mathcal{H}}^2\|u(s)\|_{\mathcal{U}}^4 + C_1|\lambda||\tilde{\vartheta}(s)|_{\mathcal{H}}^2. \tag{3.51}$$

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Therefore by the dominated convergence theorem, we get

$$\lim_{\lambda\to 0}\mathbb{E}\left[\int_0^T e^{-r_\lambda(s)}\langle\mathcal{G}(s)-\mathcal{G}(\vartheta_\lambda(s)),\tilde{\vartheta}(s)\rangle ds\right]$$
$$=\mathbb{E}\left[\int_0^T e^{-r_0(s)}\langle\mathcal{G}(s)-\mathcal{G}(u(s)),\tilde{\vartheta}(s)\rangle ds\right]. \tag{3.52}$$

Letting $\lambda\to 0^+$ in (3.49) and $\lambda\to 0^-$ in (3.50), we obtain

$$\mathbb{E}\left[\int_0^T e^{-r_0(s)}\langle\mathcal{G}(s)-\mathcal{G}(u(s)),\tilde{\vartheta}(s)\rangle ds\right]=0.$$

As $\tilde{\vartheta}$ is arbitrary, we conclude that $\mathcal{G}(s)=\mathcal{G}(u(s))$ a.e. on Ω_T. Then

$$u(t)=u_0+\int_0^t \mathcal{G}(u(s))ds+\sum_{k=1}^{\infty}\int_0^t \sigma_k(u(s))dW_k(s). \tag{3.53}$$

Step 2: General case: $\mathbb{E}\left(\|u_0\|_{\mathcal{U}}^2\right)<\infty$.
Let $X_n(0)\in L^6(\Omega;\mathcal{F}_0,\mathcal{U})$ such that $\mathbb{E}|A(X_n(0)-u_0)|_{L^2}^2\to 0$.
Let $X_n(t)=(\upsilon_n,\varphi_n)(t), t\geq 0$ be the solution of the following equation

$$dX_n(t)=\mathcal{G}(X_n(t))dt+\sum_{k=1}^{\infty}\sigma_k(X_n(t))dW_k(t),\;\; X_n(0)=X_n(0)\in\mathcal{U}. \tag{3.54}$$

The existence of X_n is guaranteed by Step 1. As in the proof of (3.7), we can show that

$$\begin{aligned}&\sup_n\left\{\mathbb{E}\sup_{t\in[0,T]}\|X_n(t)\|_{\mathcal{U}}^2+\mathbb{E}\sup_{t\in[0,T]}|X_n(t)|_{\mathcal{H}}^{q_0}+\mathbb{E}\int_0^T|AX_n(t)|_{L^2}^2dt\right\}\\ &\leq c\sup_n\mathbb{E}\|X_n(0)\|_{\mathcal{U}}^2+c\sup_n\mathbb{E}|X_n(0)|_{\mathcal{H}}^{q_0}<\infty.\end{aligned} \tag{3.55}$$

This implies that there exist a subsequence of X_n still denoted by the same symbol and a process $X=(\upsilon,\varphi)\in L^2(\Omega;L^\infty([0,T];\mathcal{U}))\cap L^2(\Omega_T;D(A))$ such that
i) $X_n\to X$ weakly in $L^2(\Omega_T;D(A))$,
ii) $X_n\to X$ in $L^2(\Omega;L^\infty([0,T];\mathcal{U}))$ equipped with the weak star topology.
Next, we show that X_n also converges to X in probability in $L^\infty([0,T];\mathcal{U})$.

For $\xi > 0$, define the stopping time

$$\tau_\xi^n := \inf\{t \in [0,\infty) : \|X_n(t)\|_{\mathcal{U}} > \xi \text{ or } \int_0^t |AX_n(s)|_{L^2}^2 ds \geq \xi \text{ or } |X_n(t)|_{\mathcal{H}}^{q_0} > \xi\}.$$

τ_ξ^n is a stopping time since X_n is continuous in $\mathcal{U}$. It follows from (3.58) that there exists a constant M, independent of n, ξ so that

$$\begin{aligned}
\mathbb{P}(\tau_\xi^n \leq T) &\leq \mathbb{P}\left(\sup_{t\in[0,T]} \|X_n(t)\|_{\mathcal{U}} > \xi\right) + \mathbb{P}\left(\int_0^T |AX_n(t)|_{L^2}^2 dt > \xi\right) \\
&+ \mathbb{P}\left(\sup_{t\in[0,T]} |X_n(t)|_{\mathcal{H}}^{q_0} > \xi\right) \\
&\leq \frac{M}{\xi^2} + \frac{M}{\xi} + \frac{M}{\xi}. \qquad (3.56)
\end{aligned}$$

We are going to prove that

$$\mathbb{E}\left(\sup_{0\leq t\leq \tau_\xi^n\wedge\tau_\xi^m} \|X_n(t) - X_m(t)\|_{\mathcal{U}}^2\right) \leq C_{\xi,T}\, \mathbb{E}\|X_n(0) - X_m(0)\|_{\mathcal{U}}^2. \qquad (3.57)$$

Let $\Theta(t) \equiv (w, \psi)(t) = X_n(t) - X_m(t)$ and $\tau_\xi^{m,n} = \tau_\xi^n \wedge \tau_\xi^m$.
By Itô's formula, we have

$$\begin{aligned}
d\|\Theta\|_{\mathcal{U}}^2 &= 2\langle \mathcal{G}(X_n) - \mathcal{G}(X_m), \Theta\rangle_{\mathcal{U}} dt + 2\sum_{k=1}^\infty \langle \sigma_k(X_n) - \sigma_k(X_m), \Theta(t)\rangle_{\mathcal{U}} dW_k(t) \\
&+ \sum_{k=1}^\infty \|\sigma_k(X_n) - \sigma_k(X_m)\|_{\mathcal{U}}^2 dt
\end{aligned}$$

Note that

$$\begin{aligned}
&\langle \mathcal{G}(X_n) - \mathcal{G}(X_m), \Theta\rangle_{\mathcal{U}} = -\langle A\Theta, A\Theta\rangle - \langle B_N^0(\upsilon_n) - B_N^0(\upsilon_m), A_0 w\rangle \\
&-\langle B^1(\upsilon_n, \varphi_n) - B^1(\upsilon_m, \varphi_m), A_1^2\psi\rangle + \langle R^0(A_1\varphi_n, \varphi_n) - R^0(A_1\varphi_m, \varphi_m), A_0 w\rangle \qquad (3.58) \\
&-\alpha\langle A_1 f(\varphi_n) - A_1 f(\varphi_m), A_1^2\psi\rangle.
\end{aligned}$$

We now estimate each term of (3.58).

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Let us set

$$K_1 = B_N^0(\upsilon_n, \upsilon_n) - B_N^0(\upsilon_m, \upsilon_m),$$
$$K_2 = R^0(A_1\varphi_n, \varphi_n) - R^0(A_1\varphi_m, \varphi_m) = R^0(A_1(\varphi_n - \varphi_m), \varphi_n) + R^0(A_1\varphi_m, \varphi_n - \varphi_m),$$
$$K_3 = B^1(\upsilon_n, \varphi_n) - B^1(\upsilon_m, \varphi_m) = B^1(\upsilon_n - \upsilon_m, \varphi_n) + B^1(\upsilon_m, \varphi_n - \varphi_m). \tag{3.59}$$

We can easily check that (see [6])

$$|\langle K_1, A_0 w\rangle| \leq \frac{1}{8}|A_0 w|_{L^2}^2 + CN^4\|w\|^2 + c|A_0 v_2|_{L^2}^2\|w\|^2 + c|A_0 v_2|_{L^2}^2\|w\|^2. \tag{3.60}$$

We also note that

$$\begin{aligned}
&|\langle K_2, A_0 w\rangle| \leq |b_1(A_0 w, \varphi_n, A_1\psi, \varphi_n))| + |b_1(A_0 w, \varphi_n - \varphi_m, A_1\varphi_m)| \\
&\leq c|A_0 w|_{L^2}|A_1\varphi_n|_{L^2}|A_1 w|_{L^2}^{1/2}|A_1^{3/2}\psi|_{L^2}^{1/2} + c|A_0 w|_{L^2}|A_1\varphi_m|_{L^2}|A_1\psi|_{L^2}^{1/2}|A_1^{3/2}\psi|_{L^2}^{1/2} \\
&\leq \frac{1}{8}(|A_0 w|_{L^2}^2 + |A_1^2\psi|_{L^2}^2) + c(|A_1\varphi_n|_{L^2}^4 + |A_1\varphi_m|_{L^2}^4)|A_1\psi|_{L^2}^2 \\
&\leq \frac{1}{8}(|A_0 w|_{L^2}^2 + |A_1^2\psi|_{L^2}^2) + c(|A_1^2\varphi_n|_{L^2}^{2/3}\|\varphi_n\|^{4/3} + |A_1^2\varphi_m|_{L^2}^{2/3}\|\varphi_m\|^{4/3})|A_1\psi|_{L^2}^2,
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
&|\langle K_3, A_1^2\psi\rangle| \leq |b_1(w, \varphi_n, A_1^2\psi)| + |b_1(\upsilon_m, \psi, A_1^2\psi\rangle| \\
&\leq c|w|_{L^2}^{1/2}\|w\|^{1/2}|A_1\varphi_n|_{L^2}|A_1^2\psi|_{L^2} + c|\upsilon_m|_{L^2}^{1/2}\|\upsilon_m\|^{1/2}|A_1\psi|_{L^2}|A_1^2\psi|_{L^2} \\
&\leq \frac{1}{8}(|A_0 w|_{L^2}^2 + |A_1^2\psi|_{L^2}^2) + c|A_1\varphi_n|_{L^2}^4\|w\|^2 + c|\upsilon_m|_{L^2}\|\upsilon_m\||A_1\psi|_{L^2}^2 \\
&\leq \frac{1}{8}(|A_0 w|_{L^2}^2 + |A_1^2\psi|_{L^2}^2) + c|A_1^2\varphi_n|_{L^2}^{2/3}\|\varphi_n\|^{4/3}\|w\|^2 + c|\upsilon_m|_{L^2}\|\upsilon_m\||A_1\psi|_{L^2}^2.
\end{aligned} \tag{3.62}$$

We also have

$$\alpha|\langle A_1 f(\varphi_n) - A_1 f(\varphi_m), A_1^2\psi\rangle| \leq Q_1(\|\varphi_n\|, \|\varphi_m\|)|A_1\psi|_{L^2}^2 + \frac{1}{8}|A_1^2\psi|_{L^2}^2, \tag{3.63}$$

where $Q_1(\|\varphi_n\|, \|\varphi_m\|) \leq c(1 + \|\varphi_n\|^{q_0} + \|\varphi_m\|^{q_0})$ for some integer $q_0 > 1$.

From the properties G given in (2.34), we also have $|\langle G(X_n) - G(X_m), \Theta\rangle_{\mathcal{U}}| \leq c\|\Theta\|_{\mathcal{U}}^2$.

By (3.58), for any pair of stopping times $0 \leq \sigma_a \leq \sigma_b \leq \tau_\xi^n \wedge \tau_\xi^n$, we have

$$\mathbb{E}\left(\sup_{t\in[\sigma_a,\sigma_b]}\|\Theta\|_{\mathcal{U}}^2 + \int_{\sigma_a}^{\sigma_b}|A\Theta(t)|_{L^2}^2 dt\right)$$
$$\leq c\mathbb{E}\left(\|\Theta(\tau_a)\|_{\mathcal{U}}^2 + \int_{\tau_a}^{\tau_b}(1 + |AX_m(s)|_{L^2}^2)|\Theta(s)|_{\mathcal{H}}^2 ds\right)$$

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$$+\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\left|2\sum_{k=1}^{\infty}\int_{\tau_a}^{t}\langle\sigma_k(X_n)-\sigma_k(X_m),\Theta\rangle_{\mathcal{U}}dW_k\right|\right) \tag{3.64}$$

For the last term in (3.64), the Burkholder–Davis–Gundy inequality implies

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]}\left|2\sum_{k=1}^{\infty}\int_{\tau_a}^{t}\langle\sigma_k(X_n)-\sigma_k(X_m),\Theta\rangle_{\mathcal{U}}dW_k\right|\right)\\
&\leq c\mathbb{E}\left(\sum_{k=1}^{\infty}\int_{\tau_a}^{\tau_b}\langle\sigma_k(X_n)-\sigma_k(X_m),\Theta\rangle_{\mathcal{U}}^2dt\right)^{\frac{1}{2}}\\
&\leq\frac{1}{2}\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\|\Theta(t)\|_{\mathcal{U}}^2+C\mathbb{E}\int_{\tau_a}^{\tau_b}\|\Theta(s)\|_{\mathcal{U}}^2ds.
\end{aligned} \tag{3.65}$$

Combining (3.64) and (3.65), we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[\sigma_a,\sigma_b]}\|\Theta(t)\|_{\mathcal{U}}^2+\int_{\sigma_a}^{\sigma_b}|A\Theta(t)|_{L^2}^2dt\right)\\
&\leq c\,\mathbb{E}\left(\|\Theta(\tau_a)\|_{\mathcal{U}}^2+c\int_{\tau_a}^{\tau_b}(1+|AX_m(s)|_{L^2}^2+|X_n|_{\mathcal{H}}^{q_0}+|X_m|_{\mathcal{H}}^{q_0})\|\Theta(s)\|_{\mathcal{U}}^2ds\right),
\end{aligned} \tag{3.66}$$

where c is a constant independent of the choice of τ_a,τ_b.

By definition of τ_ξ^m, we have

$$\int_0^{\tau_\xi^n\wedge\tau_\xi^m}(1+|AX_m|_{L^2}^2+|X_n|_{\mathcal{H}}^{q_0}+|X_m|_{\mathcal{H}}^{q_0})\leq(\xi+1)\ \mathbb{P}-a.s.$$

Then by application of the Gronwall lemma for stochastic processes (see Lemma 15), we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau_\xi^n\wedge\tau_\xi^m}\|\Theta(t)\|_{\mathcal{U}}^2\right)\leq C_{\xi,T}\ \mathbb{E}\|\Theta(0)\|_{\mathcal{U}}^2,$$

and this proves (3.57).

For $\eta>0$ and any $\xi>0$, we get

$$\mathbb{P}\left(\sup_{t\in[0,T]}\|\Theta(t)\|_{\mathcal{U}}>\eta\right)\leq\mathbb{P}(\tau_R^n\leq T)+\mathbb{P}(\tau_\xi^m\leq T)+\mathbb{P}\left(\sup_{t\in[0,\tau_\xi^n\wedge\tau_\xi^m]}\|\Theta(t)\|_{\mathcal{U}}>\eta\right). \tag{3.67}$$

Given an arbitrary small constant $\delta > 0$, in view of (3.56), one can choose ξ such that $\mathbb{P}(\tau_\xi^n \leq T) \leq \frac{\delta}{4}$ and $\mathbb{P}(\tau_\xi^m \leq T) \leq \frac{\delta}{4}$. For such ξ, by (3.57) there exists N_0 such that for $m, n \geq N_0$,

$$\mathbb{P}\left(\sup_{t\in[0,\tau_\xi^n \wedge \tau_\xi^m]} \|\Theta(t)\|_{\mathcal{U}} > \eta \right) \leq \frac{\delta}{4}.$$

Therefore

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|\Theta(t)\|_{\mathcal{U}} > \eta \right) \leq \delta \text{ and } \lim_{n,m\to\infty} \mathbb{P}\left(\sup_{t\in[0,T]} \|\Theta(t)\|_{\mathcal{U}} > \eta \right) = 0$$

This proves that X_n converges to X in probability in $L^\infty([0,T];\mathcal{U})$. Finally we want to show that X solves (2.45). To this end, it suffices to prove that for $\vartheta \in \mathcal{V}_1 \times \mathcal{C}_c(\mathcal{M})$,

$$\langle X(t), \vartheta \rangle = \langle u_0, \vartheta \rangle + \int_0^T \langle \mathcal{G}(X(s)), \vartheta \rangle ds = \sum_{k=1}^{\infty} \int_0^t (\sigma_k(X(s)), \vartheta) dW_k(s). \tag{3.68}$$

But for every $n \geq 1$, we know that

$$\langle X_n(t), \vartheta \rangle = \langle u_0, \vartheta \rangle + \int_0^T \langle \mathcal{G}(X_n(s)), \vartheta \rangle ds = \sum_{k=1}^{\infty} \int_0^t (\sigma_k(X_n(s)), \vartheta) dW_k(s). \tag{3.69}$$

Since X_n converges to X in probability in $L^\infty([0,T];\mathcal{U})$, there exists a subsequence of X_n (still denoted by the same symbol) such that X_n converges to X in $\mathcal{U}$ for almost all $t \in [0,T]$, that is

$$X_n \to X \text{ in } \mathcal{U} \ p.p.\ t \in [0,T]. \tag{3.70}$$

Using Vitali's theorem, we can prove as in [12] that

$$\int_0^t \langle \mathcal{G}(X_n(s)), \vartheta \rangle ds \to \int_0^t \rangle \mathcal{G}(X(s)), \vartheta \rangle ds \text{ in } L^2(\Omega).$$

We also have

$$\sum_{k=1}^{\infty} \int_0^t \langle \sigma_k(X_n(s)), \vartheta \rangle dW_k(s) \to \sum_{k=1}^{\infty} \int_0^t \langle \sigma_k(X(s)), \vartheta \rangle dW_k(s) \text{ in } L^2(\Omega).$$

We conclude that X satisfies (3.68) and this ends the proof of the existence. □

4. Convergence to martingale solutions of the stochastic 3D CH-NSE

Let μ_0 be a probability measure on $\mathcal{U}$ such that $\int_{\mathcal{U}} \|U\|^6 d\mu_0(U) < \infty$. Let u_0 be an $\mathcal{F}_0$-random variable in $\mathcal{U}$ with distribution μ_0. Let u^N be the unique strong solution of the stochastic 3D GMCHNSE. In this section, we are going to study the asymptotic behavior of u^N when $N \to \infty$.

4.1. Some a priori estimates

Lemma 7. *We have the following a priori estimates on* u^N

$$\mathbb{E} \sup_{s\in[0,T]} |u^N(s)|^2_{\mathcal{H}} \le c_1, \tag{4.1}$$

$$\mathbb{E} \int_0^T \|u^N(s)\|^2_{\mathcal{Z}} ds \le c_2, \tag{4.2}$$

$$\mathbb{E} \sup_{s\in[0,T]} |u^N(s)|^4_{\mathcal{H}} \le c_3, \tag{4.3}$$

$$\mathbb{E} \left(\int_0^T \|u^N(s)\|^2_{\mathcal{Z}} ds \right)^2 \le c_4, \tag{4.4}$$

where the constants c_1, c_2, c_3 *and* c_4 *are independent of* N.

Proof. By Itô's formula, we get

$$\begin{aligned} |u^N(t)|^2_{\mathcal{H}} =& |u_0|^2_{\mathcal{H}} + 2\int_0^t \langle \mathcal{G}(u^N), u^N\rangle ds + 2\sum_{k=1}^{\infty} \int_0^t \langle \sigma_k(u^N), u^N\rangle dW_k(s) \\ &+ \sum_{k=1}^{\infty} \int_0^t |\sigma_k(u^N(s))|^2_{\mathcal{H}} ds. \end{aligned} \tag{4.5}$$

Note that for $u^N = (v^N, \phi^N) \in \mathcal{U}$, we also have (see (2.26), (2.34), (2.28) and (2.46))

$$-\langle Au^N, u^N\rangle = -\|u^N\|^2_{\mathcal{Z}}, \tag{4.6}$$

$$\begin{aligned} &-\langle B_N(u^N), u^N\rangle + \langle R^0(u^N), u^N\rangle \\ &\equiv -b^0_N(v^N, v^N, v^N) - b^1(v^N, \phi^N, A_1\phi^N) + b^1(v^N, \phi^N, A_1\phi^N) = 0, \end{aligned} \tag{4.7}$$

and

$$-\langle E(u^N), u^N\rangle \le \alpha_0 |u^N|^2_{\mathcal{H}} + \alpha_1 \|u^N\|^2_{\mathcal{Z}}, \tag{4.8}$$

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$$\sum_{k=1}^{\infty} |\sigma_k(u^N)|^2_{\mathcal{H}} \leq c(1+|u^N|^2_{\mathcal{H}}). \tag{4.9}$$

The properties of G give

$$|-\langle G(u^N), u^N\rangle| \leq c(1+|u^N|^2_{\mathcal{H}}). \tag{4.10}$$

It follows that

$$\langle \mathcal{G}(u^N), u^N\rangle \leq -\tfrac{1}{2}\|u^N\|^2_{\mathcal{Z}} + c(1+|u^N|^2_{\mathcal{H}}). \tag{4.11}$$

Using the estimates (4.5)–(4.11), we arrive at

$$|u^N|^2_{\mathcal{H}} + \int_0^t \|u^N(s)\|^2_{\mathcal{Z}} ds \leq |u_0|^2_{\mathcal{H}} + c\int_0^t (1+|u^N|^2_{\mathcal{H}})ds + 2\sum_{k=1}^{\infty}\int_0^t \langle \sigma_k(u^N), u^N\rangle dW_k(s). \tag{4.12}$$

Taking the supremum over $[0, T]$, we get

$$\begin{aligned} \sup_{s\in[0,T]} |u^N(s)|^2_{\mathcal{H}} + \int_0^T \|u^N(s)\|^2_{\mathcal{Z}} ds \; &\leq |u_0|^2_{\mathcal{H}} + c\textstyle\int_0^T (1+|u^N(s)|^2_{\mathcal{H}})ds \\ &+2\sup_{t\in[0,T]}\left(\sum_{k=1}^{\infty}\int_0^t (\sigma_k(u^N), u^N)dW_k(s)\right). \end{aligned} \tag{4.13}$$

Raising both sides to the power $\frac{p}{2}$ for $p \geq 2$, then taking expectations, we obtain with the Minkowski inequality and Fubini's theorem

$$\begin{aligned} \mathbb{E}\sup_{s\in[0,T]} |u^N(s)|^p_{\mathcal{H}} &\leq \mathbb{E}|u_0|^p + c\mathbb{E}\int_0^T (1+|u^N|^p_{\mathcal{H}})ds \\ &+2^{\frac{p}{2}}\mathbb{E}\sup_{t\in[0,T]}\left(\left|\sum_{k=1}^{\infty}\int_0^t (\sigma_k(u^N), u^N)dW_k(s)\right|\right)^{\frac{p}{2}}. \end{aligned} \tag{4.14}$$

For the stochastic term, we use the Burkhölder–Davis–Gundy inequality

$$\mathbb{E}\sup_{t\in[0,T]}\left(\left|\sum_{k=1}^{\infty}\int_0^t (\sigma_k(u^N), u^N)dW_k(s)\right|\right)^{\frac{p}{2}}$$

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$$\leq c\mathbb{E}\left(\sum_{k=1}^{\infty}\int_0^T |u^N|^2_{\mathcal{H}}|\sigma_k(u^N)|^2_{\mathcal{H}}ds\right)^{\frac{p}{4}}$$

$$\leq c\mathbb{E}\left(\sup_{s\in[0,T]}|u^N|^2_{\mathcal{H}}\int_0^T(1+|u^N|^2_{\mathcal{H}})dt\right)^{\frac{p}{4}}$$

$$\leq \frac{1}{2}\mathbb{E}\sup_{s\in[0,T]}|u^N|^p_{\mathcal{H}}+c'\mathbb{E}\int_0^T|u^N|^p_{\mathcal{H}}dt+c'. \tag{4.15}$$

Applying the above estimate to (4.14), we obtain

$$\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}|u^N(s)|^p_{\mathcal{H}}\leq \mathbb{E}|u_0|^p_{\mathcal{H}}+c'\mathbb{E}\int_0^T|u^N(s)|^p_{\mathcal{H}}ds+c'. \tag{4.16}$$

Since

$$\mathbb{E}\int_0^T|u^N(t)|^p_{\mathcal{H}}dt\leq\int_0^T\mathbb{E}\sup_{s\in[0,t]}|u^N(s)|^p_{\mathcal{H}}dt,$$

the deterministic Gronwall lemma implies that

$$\mathbb{E}\sup_{t\in[0,T]}|u^N(t)|^p_{\mathcal{H}}\leq\mathbb{E}|u_0|^p_{\mathcal{H}}+c'. \tag{4.17}$$

Letting $p=4$ and $p=2$, we obtain the estimates (4.1) and (4.3).

The estimate (4.12) implies

$$2\int_0^t\|u^N(s)\|^2_{\mathcal{Z}}ds\leq|u_0|^2_{\mathcal{H}}+c\int_0^t(1+|u^N|^2_{\mathcal{H}})ds+2\sum_{k=1}^{\infty}\int_0^t(\sigma_k(u^N),u^N)dW_k(s). \tag{4.18}$$

Taking the supremum over $[0,T]$, raising both sides to the power 2 then taking expectation, we obtain with Minkowski's inequality and Fubini's theorem

$$\mathbb{E}\left(\int_0^T\|u^N(s)\|^2_{\mathcal{Z}}ds\right)^2\leq c\mathbb{E}|u_0|^4_{\mathcal{H}}+c\mathbb{E}\int_0^T|u^N(s)|^4_{\mathcal{H}}ds \tag{4.19}$$

$$+4\mathbb{E}\sup_{t\in[0,T]}\left(\sum_{k=1}^{\infty}\int_0^t(\sigma_k(u^N),u^N)dW_k(s)\right)^2. \tag{4.20}$$

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For the stochastic term, we have

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\left(\sum_{k=1}^{\infty}\int_0^t(\sigma_k(u^N),u^N)dW_k(s)\right)^2\\
&\le c\mathbb{E}\sum_{k=1}^{\infty}\int_0^T|u^N|_{\mathcal{H}}^2|\sigma_k(u^N)|_{\mathcal{H}}^2ds\\
&\le\frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}|u^N(t)|_{\mathcal{H}}^4+c'\int_0^T|u^N(s)|_{\mathcal{H}}^4ds+c'.
\end{aligned}$$

This together with (4.20) implies the estimate (4.4). The proof of Lemma 7 is complete. □

4.2. *Estimates in fractional Sobolev spaces*

We will apply the compactness result based on fractional Sobolev spaces in Lemma 13 (of the Appendix) with

$$\mathcal{Y}=L^2(0,T;\mathcal{U})\cap W^{\gamma,2}(0,T;D(A^{-1})),\quad 0<\gamma<\frac{1}{2}. \tag{4.21}$$

For this purpose, we will need the following estimates on fractional derivatives of u^N.

Lemma 8.

$$\mathbb{E}|u^N|_{\mathcal{Y}}\le k_1, \tag{4.22}$$

$$\mathbb{E}\left|u^N-\int_0^t\sigma(u^N)dW(s)\right|^2_{H^1(0,T;\mathcal{U}')}\le k_2, \tag{4.23}$$

$$\mathbb{E}\left|\int_0^t\sigma(u^N)dW(s)\right|^2_{W^{\gamma,6}(0,T;\mathcal{H})}\le k_3,\quad \forall\gamma<\frac{1}{2}, \tag{4.24}$$

where the constants k_1, k_2 and k_3 are independent of N.

Proof. Note that $u^N=(v^N,\phi^N)$ can be written as

$$u^N(t)=u_0+\int_0^t\mathcal{G}(u^N(s))ds+\sum_{k=1}^{\infty}\int_0^t\sigma_k(u^N(s))dW_k(s):=J_1+J_2+J_3. \tag{4.25}$$

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Note that for $\vartheta = (\upsilon, \varphi)$, we have

$$\begin{aligned}
&\langle J_2, \vartheta\rangle = \langle \mathcal{G}(u^N), \vartheta\rangle = \langle -Au^N, \vartheta\rangle - \langle B_N^0(v^N), \upsilon\rangle - \langle B^1(v^N, \phi^N), \varphi\rangle \\
&+\langle R^0(A_1\phi^N, \phi^N), \upsilon\rangle - \alpha\langle A_1 f(\phi^N), \varphi\rangle - \langle G(u^N), \vartheta\rangle \qquad (4.26)\\
&:= \langle J_2^1 + J_2^2 + J_2^3 + j_2^4 + J_2^5 + J_2^6, \vartheta\rangle.
\end{aligned}$$

For J_2^1, we note that

$$|Au^N|_{\mathcal{U}^*} \le c\|u^N\|_{\mathcal{U}}. \qquad (4.27)$$

With (4.2), we obtain

$$\mathbb{E}|J_2^1|^2_{W^{1,2}(0,T;\mathcal{U}^*)} \text{ is bounded independently of } N. \qquad (4.28)$$

For J_2^2, we observe that for $\upsilon \in D(A_0)$

$$\begin{aligned}
\langle B_N^0(v^N), \upsilon\rangle &= b_N^0(v^N, v^N, \upsilon)\\
&= F_N(\|v^N\|)b^0(v^N, v^N, \upsilon)\\
&=\le c|v^N|\|v^N\||\upsilon|_{L^\infty}\\
&\le c|v^N|_{L^2}\|v^N\||A_0\upsilon|_{L^2}. \qquad (4.29)
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{E}|B_N^0(v^N)|^2_{L^2(0,T;D(A_0^{-1}))} &\le c\mathbb{E}\left(\sup_{s\in[0,T]} |v^N(s)|^2_{L^2}\int_0^T \|v^N(s)\|^2 ds\right)\\
&\le c\left(\mathbb{E}\sup_{s\in[0,T]} |v^N(s)|^4_{L^2}\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T \|v^N(s)\|^2 ds\right)^2\right)^{\frac{1}{2}}. \qquad (4.30)
\end{aligned}$$

This along with (4.3) and (4.4) conclude that

$$\mathbb{E}|J_2^2|^2_{W^{1,2}(0,T;D(A_0^{-1}))} \text{ is bounded independently of } N. \qquad (4.31)$$

For J_2^4, we observe that for $\upsilon \in D(A_0)$

$$\begin{aligned}
\langle R^0(A_1\phi^N, \phi^N), \upsilon\rangle &= b_N^1(\upsilon, \phi^N, A_1\phi^N)\\
&\le c|A_1\phi^N|_{L^2}\|\phi^N\||A_0\upsilon|_{L^2}. \qquad (4.32)
\end{aligned}$$

This implies that

$$\begin{aligned}\mathbb{E}|R^0(A_1\phi^N,\phi^N)|^2_{L^2(0,T;D(A_0^{-1}))} &\leq c\mathbb{E}\left(\sup_{s\in[0,T]}\|\phi^N(s)\|^2\int_0^T|A_1\phi^N(s)|^2_{L^2}ds\right)\\ &\leq c\left(\mathbb{E}\sup_{s\in[0,T]}\|\phi^N(s)\|^4\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_0^T|A_1\phi^N(s)|^2_{L^2}ds\right)^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.33}$$

This along with (4.3) and (4.4) conclude that

$$\mathbb{E}|J_2^4|^2_{W^{1,2}(0,T;D(A_0^{-1}))}\ \text{ is bounded independently of } N.\tag{4.34}$$

For J_2^6, using the estimate (4.1), we have

$$\mathbb{E}|J_2^5|^2_{W^{1,2}(0,T;\mathcal{U}')}\ \text{ is bounded independently of } N.\tag{4.35}$$

The other terms J_2^3, J_2^5 and J_2^6 are estimates similarly. More precisely, we can check that $\mathbb{E}|J_2^3|^2_{W^{1,2}(0,T;D(A_1^{-1}))}$, $\mathbb{E}|J_2^5|^2_{W^{1,2}(0,T;A_1^{-1})}$ and $\mathbb{E}|J_2^6|^2_{W^{1,2}(0,T;\mathcal{U}')}$ are also bounded independently of N.

For the term J_3, Lemma 13 implies that, $\forall\gamma<\frac{1}{2}$

$$\begin{aligned}\mathbb{E}|J_3|^6_{W^{\gamma,6}(0,T;\mathcal{H})} &= \mathbb{E}\left|\sum_{k=1}^{\infty}\int_0^t\sigma_k(u^N(s))dW_k(s)\right|^6_{W^{\gamma,6}(0,T;\mathcal{H})}\\ &\leq C(\gamma)\mathbb{E}\int_0^T\sum_{k=1}^{\infty}|\sigma_k(u^N(s))|^6_{\mathcal{H}}ds\\ &\leq C(\gamma)\mathbb{E}\int_0^T(1+|u^N|^6_{\mathcal{H}})ds.\end{aligned}$$

This together with (4.17) imply that

$$\mathbb{E}|J_3|^2_{W^{\gamma,6}(0,T;\mathcal{H})}\ \text{ is bounded independently of } N,\ \forall\gamma<\frac{1}{2}.\tag{4.36}$$

Indeed by Hölder's inequality, we have

$$\mathbb{E}|J_3|^2_{W^{\gamma,6}(0,T;\mathcal{H})}\leq c\left(\mathbb{E}|J_3|^6_{W^{\gamma,6}(0,T;\mathcal{H})}\right)^{\frac{1}{3}}<\infty.$$

Hence we obtain (4.24).

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Collecting the estimates (4.28)–(4.36), we obtain

$$\mathbb{E}|u^N|_{W^{\gamma,2}(0,T;D(A^{-1}))} \text{ is bounded independently of } N. \tag{4.37}$$

By (4.2), we deduce

$$\mathbb{E}|u^N|_{L^2(0,T;\mathcal{U})} \text{ is bounded independently of } N. \tag{4.38}$$

From (4.37) and (4.38), we obtain (4.22).

We observe from (4.25) that $u^N(t) - \int_0^t \sigma(u^N)dW(s) = J_1 + J_2$ combined with the estimates (4.28)–(4.35), we obtain (4.23) as desired. □

4.3. Compactness arguments for $\{(u^N, W)\}_N$

With the estimates independent of N, we can establish the compactness of the family (u^N, W). For this purpose, we consider the following phase spaces:

$$\mathcal{X}_u = L^2(0,T;\mathcal{H}) \cap C(0,T;D(A^{-1})), \quad \mathcal{X}_W = C(0,T;\mathcal{U}_0), \quad \mathcal{X} = \mathcal{X}_u \times \mathcal{X}_W. \tag{4.39}$$

We then define the probability laws of u^N and W respectively in the corresponding phase spaces:

$$\mu_u^N(.) = \mathbb{P}(u^N \in .), \tag{4.40}$$

and

$$\mu_W(.) = \mathbb{P}(W \in .). \tag{4.41}$$

This defines a family of probability measures $\mu^N = \mu_u^N \times \mu_W$ on the phase space $\mathcal{X}$. We now prove that this family is tight in N. More precisely:

Lemma 9. *Consider the measures μ^N on $\mathcal{X}$ defined according to (4.40) and (4.41). Then the family $\{\mu^N\}_N$ is tight and therefore weakly compact over the phase space $\mathcal{X}$.*

Proof. We can use the same technic as in the proof of Lemma 4.1 in [11]. The main idea is to apply Lemma 12 (of the Appendix) and Chebychev's inequality to (4.22)–(4.24). □

Strong convergence as $N \to \infty$. Since the family of measures $\{\mu^N\}$ associated with the family (u^N, W) is weakly compact on $\mathcal{X}$, we deduce that μ^N converges weakly to a probability μ on $\mathcal{X}$ up to a subsequence. We can apply the Skorokhod embedding theorem (see Theorem 2.4 in [28]) to deduce the strong convergence of a further subsequence, that is:

Proposition 1. *There exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and a subsequence N_k of random vectors $(\tilde{u}^{N_k}, \tilde{W}^{N_k}$ with values in $\mathcal{X}$ such that*

(i) *$(\tilde{u}^{N_k}, \tilde{W}^{N_k}$ have the same probability distributions as (u^{N_k}, W^{N_k}).*

(ii) $(\tilde{u}^{N_k}, \tilde{W}^{N_k})$ *converges almost surely as* $N_k \to \infty$*, in the topology of* $\mathcal{X}$*, to an element* $(\tilde{u}, \tilde{W}) \in \mathcal{X}$*, i.e.*

$$\tilde{u}^{N_k} \to \tilde{u} \text{ strongly in } L^2(0,T;\mathcal{H}) \cap \mathcal{C}([0,T]; D(A^{-1}))\ a.s., \tag{4.42}$$

$$\tilde{W}^{N_k} \to \tilde{W} \text{ strongly in } \mathcal{C}([0,T];\mathcal{U}_0)\ a.s., \tag{4.43}$$

where $(\tilde{u}, \tilde{W})$ *has distribution* μ.

(iii) $\tilde{W}^{N_k}$ *is a cylindrical Wiener process, relative to the filtration* $\tilde{\mathcal{F}}_t^{N_k}$*, given by the completion of the* σ*-algebra generated by* $\{(\tilde{u}^{N_k}(s), \tilde{W}^{N_k}(s)); s \le t\}$.

(iv) *All the statistical estimates on* u^{N_k} *are valid for* $\tilde{u}^{N_k}$*, in particular, the estimates (4.1)–(4.4) hold.*

(v) *Each pair* $(\tilde{u}^{N_k}, \tilde{W}^{N_k})$ *satisfies (2.45) as an equation in* $\mathcal{H}$*, that is*

$$d\tilde{u}^{N_k}(t) = \mathcal{G}(\tilde{u}^{N_k})dt + \sum_{l=1}^{\infty} \sigma_l(\tilde{u}^{N_k}(t))d\tilde{W}_l^{N_k}(t), \quad \tilde{u}^{N_k}(0) = \tilde{u}_0^{N_k}. \tag{4.44}$$

The following lemma proves that $\tilde{u}^{N_k}$, $\tilde{u}$ is weakly continuous with value in $\mathcal{H}$

Lemma 10. *The stochastic processes* $\tilde{u}^{N_k}$ *and* $\tilde{u} \in C([0,T]; \mathcal{H}_w)$ $\tilde{\mathbb{P}}$*-a.s.*

Proof. The proof follows from the fact that $\tilde{u}^{N_k} \in L^\infty(0,T;\mathcal{H}) \cap \mathcal{C}([0,T], D(A^{-1}))$ a.s., hence $\tilde{u}^{N_k}$ is weakly continuous with values in $\mathcal{H}$ a.s. □

4.4. Passage to the limit

With the strong convergence in (4.42), we can pass to the limit in (4.44). Thanks to (4.3) and (4.2), we deduce the existence of an element

$$\tilde{u} \in L^4(\tilde{\Omega}; L^\infty(0,T;\mathcal{H})) \cap L^2(\tilde{\Omega}; L^2(0,T;\mathcal{U})),$$

and a subsequence still denoted as N_k such that

$$\tilde{u}^{N_k} \rightharpoonup \tilde{u} \text{ weak star in } L^4(\tilde{\Omega}; L^\infty(0,T;\mathcal{H})), \tag{4.45}$$

and

$$\tilde{u}^{N_k} \rightharpoonup \tilde{u} \text{ weakly in } L^2(\tilde{\Omega}; L^2(0,T;\mathcal{U})). \tag{4.46}$$

Combining the strong convergence (4.42), the estimate (4.3) and the Vitali convergence theorem, we get

$$\tilde{u}^{N_k} \to \tilde{u} \text{ strongly in } L^2(\tilde{\Omega}; L^2(0,T;\mathcal{H})), \tag{4.47}$$

and, thus possibly extracting a new subsequence denoted in the same way to save notation, one has also

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$$\tilde{u}^{N_k} \to \tilde{u} \text{ for almost all } \omega,\ t \text{ with respect to the measure } d\tilde{\mathbb{P}} \otimes dt. \tag{4.48}$$

Fix $\vartheta \in D(A)$. Using the weak convergence (4.46), we can pass to the limit in the linear term.
We are going to prove that

$$\begin{aligned}
&\int_0^t F_{N_k}(\|\tilde{u}^{N_k}(s)\|_{\mathcal{U}}) b(\tilde{u}^{N_k}(s), \tilde{u}^{N_k}(s), \vartheta) ds \to \int_0^t b(\tilde{u}(s), \tilde{u}(s), \vartheta) ds \text{ in } L^1(\tilde{\Omega} \times (0,T)),\\
&\int_0^t \langle R(\tilde{u}^{N_k}(s), \tilde{u}^{N_k}(s)), \vartheta\rangle ds \to \int_0^t \langle R(\tilde{u}(s), \tilde{u}(s)), \vartheta\rangle ds \text{ in } L^1(\tilde{\Omega} \times (0,T)),\\
&\int_0^t \langle E(\tilde{u}^{N_k}(s)), \vartheta\rangle ds \to \int_0^t \langle E(\tilde{u}(s)), \vartheta\rangle ds \text{ in } L^1(\tilde{\Omega} \times (0,T)).
\end{aligned} \tag{4.49}$$

The following lemma will be crucial for the proof of (4.49).

Lemma 11. *We have*

$$F_N(\|\tilde{u}^N(s)\|_{\mathcal{U}}) \to 1 \text{ in } L^p(\tilde{\Omega}; L^p(0,T;\mathbb{R})) \text{ as } N \to \infty \text{ and } p > 1. \tag{4.50}$$

Proof. From the estimate (4.2), we have

$$\tilde{E} \int_0^T \|\tilde{u}^N(s)\|_{\mathcal{U}}^2 ds \le k_1.$$

Let

$$O_N = \{s \in (0,T), \|\tilde{u}^N(s)\|_{\mathcal{U}} \ge N \ a.s.\}$$

and $|O_N|$ the Lebesgue measure of O_N. Then

$$N^2 \tilde{E}|O_N| \le \tilde{E} \int_0^T \tilde{u}^N(s)\|_{\mathcal{U}}^2 ds \le k_1,$$

and so

$$\tilde{E}|O_N| \le \frac{k}{N^2} \to 0 \text{ as N } \to \infty.$$

Observing that

$$T - |O_N| = \int_{[0,T]-O_N} F_N(\|u^N(s)\|_{\mathcal{U}}) ds,$$

we deduce that

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$$T - \tilde{E}|O_N| \le \tilde{E}\int_0^T F_N(\|\tilde{u}^N(s)\|_{\mathcal{U}})ds \le T.$$

These inequalities show that

$$\tilde{E}\int_0^T F_N(\|\tilde{u}^N(s)\|_{\mathcal{U}})ds \to \int_0^T 1ds \text{ as N } \to \infty.$$

But as $0 \le F_N(\|\tilde{u}^N(s)\|_{\mathcal{U}}) \le 1$, we get

$$\tilde{E}\int_0^T |1 - F_N(\|u^N(s)\|_{\mathcal{U}})|ds = \tilde{E}\int_0^T (1 - F_N(\|u^N(s)\|_{\mathcal{U}}))ds \to 0 \text{ as N } \to \infty.$$

Finally since $|1 - F_N(\|\tilde{u}^N(s)\|_{\mathcal{U}})| \le 1$, we arrive at

$$\begin{aligned}\tilde{E}\int_0^T |1 - F_N(\|u^N(s)\|_{\mathcal{U}})|^p ds =&\tilde{E}\int_0^T |1 - F_N(\|u^N(s)\|_{\mathcal{U}})||1 - F_N(\|u^N(s)\|_{\mathcal{U}})|_{\mathcal{U}}^{p-1} ds \\ &\le \tilde{E}\int_0^T |1 - F_N(\|u^N(s)\|_{\mathcal{U}})|ds \to 0 \text{ as N } \to \infty.\end{aligned}$$

This ends the proof of the lemma. □

For the proof of (4.49), we introduce the abbreviations as in [6],

$$\begin{aligned}F_{N_k}(s) &= F_{N_k}(\|\tilde{u}^{N_k}(s)\|_{\mathcal{U}}),\\ b^{N_k}(s) &= b(\tilde{u}^{N_k}(s), \tilde{u}^{N_k}(s), w),\\ b(s) &= b(\tilde{u}(s), \tilde{u}(s), w).\end{aligned}$$

To prove (4.49), we write

$$\begin{aligned}\tilde{E}\int_0^T \left(\int_0^t (F_{N_k}(s)b_{N_k}(s) - b(s))ds\right)dt =& \tilde{E}\int_0^T \left(\int_0^t (F_{N_k}(s) - 1)b_{N_k}(s)ds\right)dt \\ &+ \tilde{E}\int_0^T \left(\int_0^t (b_{N_k}(s) - b(s))ds\right)dt. \qquad (4.51)\end{aligned}$$

Reasoning as in the proof of the convergence of the 3D globally modified Navier–Stokes equations studied in [12], the second term of this equality tends to 0, that is

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$$\int_0^t b_{N_k}(s)ds \to \int_0^t b(s)ds \text{ in } L^1(\tilde{\Omega}\times(0,T)). \tag{4.52}$$

For the first term, we get

$$\begin{aligned}
&\tilde{E}\int_0^T \left(\int_0^t (F_{N_k}(s)-1)b_{N_k}(s)ds\right)dt \\
&\leq \left(\tilde{E}\int_0^T\int_0^t |F_{N_k}(s)-1|^2 dsdt\right)^{\frac{1}{2}} \left(\tilde{E}\int_0^T\int_0^t |b_{N_k}(s)|^2 dsdt\right)^{\frac{1}{2}} \\
&\leq T\left(\tilde{E}\int_0^T |F_{N_k}(s)-1|^2 ds\right)^{\frac{1}{2}} \left(\tilde{E}\int_0^T |b_{N_k}(s)|^2 ds\right)^{\frac{1}{2}},
\end{aligned}$$

and

$$\int_0^t (F_{N_k}(s)-1)b_{N_k}(s)ds \to 0 \text{ in } L^1(\tilde{\Omega}\times(0,T)),$$

since

$$\begin{aligned}
\tilde{E}\int_0^T |b_{N_k}(s)|^2 ds &\leq c_2|Aw|^2_{L^2}\tilde{E}\int_0^T |\tilde{u}^{N_k}(s)|^2_{\mathcal{H}}\|\tilde{u}^{N_k}(s)\|^2_{\mathcal{U}}ds \\
&\leq c_2|Aw|^2_{L^2}\left(\tilde{E}\sup_{s\in[0,T]}|\tilde{u}^{N_k}(s)|^4_{\mathcal{H}}\right)^{\frac{1}{2}}\left(\tilde{E}\left(\int_0^T \|\tilde{u}^{N_k}(s)\|^2_{\mathcal{U}}ds\right)^2\right)^{\frac{1}{2}} \\
&< \infty,
\end{aligned}$$

and Lemma 11 shows that

$$\tilde{E}\int_0^T |F_{N_k}(s)-1|^2 ds \to 0.$$

This proves $(4.49)_1$. The proofs of $(4.49)_2$ and $(4.49)_3$ are similar.

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The convergence

$$\int_0^t (\mathcal{G}(\tilde{u}^{N_k}(s), w)ds \to \int_0^t (\mathcal{G}(u), w)ds \text{ in } L^1(\tilde{\Omega} \times (0, T)) \tag{4.53}$$

follows from estimate (4.3), the Lipschitz condition on $\mathcal{G}$ and the Vitali convergence theorem.

For the stochastic term, by (4.48), we obtain

$$|\tilde{u}^{N_k} - \tilde{u}|^2_{\mathcal{H}} \to 0, \text{ for a.e. } (\omega, t) \in \tilde{\Omega} \times (0, T).$$

Thus, along with Lipschitz condition on σ, we deduce

$$|\sigma(\tilde{u}^{N_k}) - \sigma(\tilde{u})|_{L_2(l^2,\mathcal{H})} \to 0 \text{ for a.e. } (\omega, t) \in \tilde{\Omega} \times (0, T).$$

On the other hand

$$\begin{aligned} &\sup_{N_k} \tilde{E}\left(\int_0^T |\sigma(\tilde{u}^{N_k})|^4_{L_2(l^2,\mathcal{H})} ds\right) \\ &\leq \sup_{N_k} \tilde{E}\left(\int_0^T (1 + |\tilde{u}^{N_k}(s)|^4_{\mathcal{H}}) ds\right). \end{aligned}$$

We therefore infer from (4.3) that $|\sigma(\tilde{u}^{N_k}|_{L_2(l^2,\mathcal{H})}$ is uniformly integrable for N_k in $L^q(\tilde{\Omega} \times (0, T))$ for any $q \in [1, 4)$.

With the Vitali convergence theorem, we deduce that for all such $q \in [1, 4)$,

$$\sigma(\tilde{u}^{N_k}) \to \sigma(\tilde{u}) \text{ in } L^q(\tilde{\Omega}, L^q(0, T; L_2(l^2, \mathcal{H}))). \tag{4.54}$$

In particular, we get the convergence in probability of $\sigma(\tilde{u}^{N_k})$ in $L^2(0, T; L_2(l_2, \mathcal{H}))$.

Thus along with the convergence (4.43), we apply Lemma 14 (of the Appendix) and deduce that

$$\int_0^t \sigma(\tilde{u}^{N_k}) d\tilde{W}^{N_k} \to \int_0^t \sigma(\tilde{u}) d\tilde{W} \;\text{ in probability in } L^2(0, T; \mathcal{H}). \tag{4.55}$$

By (4.55) and Vitali convergence theorem, we infer a stronger convergence result:

$$\int_0^t \sigma(\tilde{u}^{N_k}) d\tilde{W}^{N_k} \to \int_0^t \sigma(\tilde{u}) d\tilde{W} \text{ in } L^2(\tilde{\Omega}; L^2(0, T; \mathcal{H})). \tag{4.56}$$

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Collecting all the convergence results, we obtain

$$(\tilde{u}(t), \vartheta) + \int_0^t \langle \mathcal{G}(\tilde{u}(s)), \vartheta \rangle ds = (\tilde{u}(0), \vartheta) + \int_0^t (\sigma(\tilde{u}(s), \vartheta) d\tilde{W}(s), \tag{4.57}$$

for all $\vartheta \in D(A)$ and for a.e. $\omega \in \tilde{\Omega}, t \in (0, T)$. The equality (4.57) is also valid for $\vartheta \in \mathcal{U}$ by density argument.

We have then proved the following result.

Proposition 2. *The pair* $(\tilde{S}, \tilde{u})$ *where* $\tilde{S} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}}, \tilde{W})$ *is a martingale solution of the stochastic 3D GMCHNSE.*

We now summarize the result obtained in the following theorem which says that, up to a subsequence the solution u^N of the stochastic 3D GMCHNSE converges in law to a martingale solution of the original 3D stochastic CH-NSE when N tends to infinity.

Theorem 2 *(Convergence of the stochastic 3D GMCHNSE). There exists a martingale weak solution* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{u})$ *of the stochastic 3D CH-NSE and a sequence* $(\tilde{u}^N)$ *of random processes defined on* $\tilde{\Omega}$*, with the same law as* u^N*, so that up to a subsequence, the following convergence holds:*

$$\tilde{u}^N = (\tilde{v}^N, \tilde{\phi}^N) \to \tilde{u} = (\tilde{v}, \tilde{\phi}) \text{ in } L^2(\tilde{\Omega} \times [0, T] \times \mathcal{M})^3 \times L^2(\tilde{\Omega} \times [0, T] \times \mathcal{M}).$$

5. Appendix

In section 5.1 we recall some results of deterministic nature. In Section 5.2 we present a result of probabilistic nature.

5.1. Compact embedding theorems

We recall the theorems from [14] (see also [11] for Lemma 12)

Definition 5.1 *(The fractional derivative space).* We assume that H is a separable Hilbert space. Given $p \geq 2$, $\gamma \in (0, 1)$, $W^{\gamma,p}(0, T; H)$ denotes the Sobolev space of all $h \in L^p(0, T; H)$ such that

$$\int_0^T \int_0^T \frac{|h(t) - h(s)|_H^p}{|t - s|^{1+\gamma p}} dt ds$$

which is endowed with the Banach norm

$$|h|_{W^{\gamma,p}(0,T;H)} = \left(\int_0^T |h(t)|_H^p dt + \int_0^T \int_0^T \frac{|h(t) - h(s)|_H^p}{|t - s|^{1+\gamma p}} dt ds \right)^{\frac{1}{p}} < \infty.$$

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Lemma 12.

(i) *Let $\mathcal{E}_0 \subset \mathcal{E} \subset \mathcal{E}_1$ be Banach spaces, $\mathcal{E}_0$ and $\mathcal{E}_1$ reflexive, with continuous injections and a compact embedding of $\mathcal{E}_0$ in $\mathcal{E}$. Let $1 < p < \infty$ and $\gamma \in (0,1)$ be given. Let $\mathcal{Y}$ be the space*

$$\mathcal{Y} = L^p(0,T;\mathcal{E}_0) \cap W^{\gamma,p}(0,T;\mathcal{E}_1),$$

endowed with the natural norm. Then the embedding of $\mathcal{Y}$ in $L^p(0,T;\mathcal{E})$ is compact.

(ii) *If $\mathcal{E} \subset \tilde{\mathcal{E}}$ are two Banach spaces with $\mathcal{E}$ compactly in $\tilde{\mathcal{E}}$, $1 < p < \infty$ and $\gamma \in (0,1)$ satisfy*

$$\gamma p > 1,$$

then the space $W^{\gamma,p}(0,T;\mathcal{E})$ is compactly embedded into $\mathcal{C}([0,T];\tilde{\mathcal{E}})$.

The following lemma is based on the Burkholder–Davis–Gundy inequality and the notion of fractional derivatives (see [14] for the proof).

Lemma 13. *Let $q \geq 2$, $\gamma < \frac{1}{2}$ be given. Then, for any progressively measurable process $h \in L^q(\Omega \times (0,T); L_2(\mathcal{U},H))$, we have*

$$\int_0^t h(s)dW(s) \in L^q(\Omega, W^{\gamma,q}(0,T;H)),$$

and there exists a constant $c' = c'(q,\gamma) \geq 0$ independent of h such that

$$\mathbb{E}\left|\int_0^t h(s)dW(s)\right|^q_{W^{\gamma,q}(0,T;H)} \leq c'(q,\gamma)\mathbb{E}\int_0^t |h(s)|^q_{L_2(\mathcal{U},H)}ds. \tag{5.1}$$

5.2. Convergence theorem for the noise term

The following convergence theorem for the stochastic integral is used to facilitate the passage to the limit. The statements and proofs can be found in [1], [11].

Lemma 14. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a fixed probability space, and $\mathcal{X}$ a separable Hilbert space. Consider a sequence of stochastic bases $S_n := (\Omega,\mathcal{F},\{\mathcal{F}_t^n\}_{t\geq 0},\mathbb{P},W^n)$, such that each W^n is a cylindrical Brownian motion (over $\mathcal{U}$) with respect to $\{\mathcal{F}_t^n\}_{t\geq 0}$. We suppose that the $\{G^n\}_{n\geq 1}$ are a sequence of $\mathcal{X}$-valued $\mathcal{F}_t^n$ predictable processes so that $G^n \in L^2((0,T);L_2(\mathcal{U},\mathcal{X})$ a.s. Finally consider $\mathcal{S} := (\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P},W)$ and a function $G \in L^2((0,T);L_2(\mathcal{U},\mathcal{X}))$, which is $\mathcal{F}_t$ predictable. If*

$$W^n \to W \text{ in probability in } C([0,T];\mathcal{U}_0), \tag{5.2}$$

$$G^n \to G \text{ in probability in } L^2((0,T);L_2(\mathcal{U},\mathcal{X})), \tag{5.3}$$

then

$$\int_0^t G^n dW^n \to \int_0^t G dW \text{ in probability in } L^2((0,T);\mathcal{X}). \tag{5.4}$$

5.3. A stochastic Gronwall lemma

The following Gronwall lemma for stochastic processes is useful to prove the existence of strong solution for the stochastic 3D globally modified Navier–Stokes equations. See [18] for the proof.

Lemma 15. *Fix $T > 0$. Assume that $X, Y, Z, K : [0,T] \times \Omega \to \mathbb{R}$ are real-valued, non-negative stochastic processes. Let $\tau < T$ be a stopping time so that*

$$\mathbb{E}\int_0^\tau (KX + Z)ds < \infty.$$

Assume, moreover that for some fixed constant k,

$$\int_0^\tau K ds < k, a.s.$$

Suppose that for all stopping times $0 \le \tau_a < \tau_b \le \tau$

$$\mathbb{E}\left(\sup_{t\in[\tau_a,\tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds\right) \le c_0 \mathbb{E}\left(X(\tau_a) + \int_{\tau_a}^{\tau_b} (KX + Z)ds\right),$$

where c_0 is a constant independent of the choice of τ_a, τ_b. Then

$$\mathbb{E}\left(\sup_{t\in[0,\tau]} X + \int_0^\tau Y ds\right) \le c\mathbb{E}\left(X(0) + \int_0^\tau Z ds\right),$$

where $c = c(c_0, T, k)$.

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