

An alternative approach to norm bound computation for inverses of linear operators in Hilbert spaces

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Abstract

In the present paper, we propose a computer-assisted procedure to prove the invertibility of a linear operator in a Hilbert space and to compute a verified norm bound of its inverse. A number of the authors have previously proposed two verification approaches that are based on projection and constructive a priori error estimates. The approach of the present paper is expected to bridge the gap between the two previous procedures in actual numerical verifications. Several verification examples that confirm the actual effectiveness of the proposed procedure are reported.

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1. Introduction

Let X and Y be complex Hilbert spaces endowed with inner products $(u, v)_X$ and $(u, v)_Y$ and their respective norms $\|u\|_X = \sqrt{(u, u)_X}$ and $\|u\|_Y = \sqrt{(u, u)_Y}$, and let $D(\mathcal{A})$ be a complex Banach space. We assume that

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$$D(\mathcal{A}) \subset X \subset Y \quad (1)$$

and that the embedding $D(\mathcal{A}) \hookrightarrow X$ is compact. We also assume that there exists a numerically evaluated embedding constant $C_p > 0$ such that

$$\|u\|_Y \leq C_p \|u\|_X, \quad \forall u \in X. \quad (2)$$

Let linear operators $\mathcal{A} : D(\mathcal{A}) \rightarrow Y$ and $\mathcal{Q} : X \rightarrow Y$ be given, and consider a linear operator defined by

$$\mathcal{L} := \mathcal{A} + \mathcal{Q} : D(\mathcal{A}) \rightarrow Y. \quad (3)$$

The present paper introduces procedures for proving by computer the invertibility of \mathcal{L} and for computing a constant $M > 0$ satisfying

$$\|\mathcal{L}^{-1}\phi\|_X \leq M \|\phi\|_Y, \quad \forall \phi \in Y \quad (4)$$

which represents a mathematically rigorous bound for the operator norm of $\mathcal{L}^{-1} : Y \rightarrow X$.

There are at least two applications for determining the existence and norm bounds for \mathcal{L}^{-1} : first, eigenvalue enclosure for self-adjoint or non-self-adjoint eigenvalue problems in Hilbert spaces; second, computer-assisted proofs for nonlinear equations in Hilbert spaces.

Eigenvalue enclosure means the determination of subsets of the complex field which do not contain eigenvalues of the given problem, and it should provide us with important information about eigenvalue distribution [22,45]. For example, when we apply an eigenvalue-excluding procedure proposed by [45] (see Section 5.2), the eigenvalue excluding area is proportional to $1/M$; thus, it is desirable that we should obtain $M > 0$ in (4) as small as possible.

In addition, in the context of computer-assisted proofs for nonlinear equations in Hilbert spaces, the operator \mathcal{L} defined by (3) represents the linearization of a given nonlinear problem, and the verification of the invertibility of \mathcal{L} and the computable norm bound for \mathcal{L}^{-1} play an essential role in, for example, Newton-type or Newton–Kantorovich-type formulations, which aim at proving the existence of a solution of a nonlinear problem with a guaranteed error bound [34,35,47]. Let us take an example: consider the problem of finding $u \in X$ satisfying

$$\mathcal{A}u - f(u) = 0. \quad (5)$$

Here $f : X \rightarrow Y$ is a nonlinear Fréchet differentiable operator which maps bounded sets in X into bounded sets in Y . Let $\hat{u} \in X$ be a computable approximate solution of (5) and, for simplicity, assume $\mathcal{A}\hat{u} \in Y$; then the problem can be rewritten as that of finding the residual $w := u - \hat{u}$ satisfying $\mathcal{A}w = f(\hat{u} + w) - \mathcal{A}\hat{u}$. Denoting $g(w) := f(w + \hat{u}) - \mathcal{A}\hat{u} : X \rightarrow Y$, we obtain the residual form

$$\mathcal{A}w - g(w) = 0. \quad (6)$$

Now we apply a Newton-type formulation to (6). By using a Fréchet derivative $g'(0) = f'(\hat{u})$, the linearized operator \mathcal{L} can be taken as $\mathcal{L} = \mathcal{A} - g'(0)$ and under the assumption of invert-

ibility of \mathcal{L} along with the condition (4), we can find that $F(w) := \mathcal{L}^{-1}(g(w) - g'(0)w)$ is a compact operator on X because of the bounded assumption on f and (4), and (6) is rewritten as fixed-point equation $w = F(w)$. Then, from the Schauder fixed-point theorem, for the nonempty closed convex subset of X defined by $W = \{w \in X \mid \|w\|_X \leq \alpha\}$ with $\alpha > 0$, the computable condition

$$M \sup_{w \in W} \|g(w) - g'(0)w\|_Y \leq \alpha \quad (7)$$

implies the existence of the fixed point $w \in W$ of F with an explicit error bound. In (7), the estimation of the Y -norm includes a bound of the defect (residual) of $\|\mathcal{A}\hat{u} - f(\hat{u})\|_Y$ and other bounds that depend on α and \hat{u} . We note that the most important information in (7) is the invertibility of \mathcal{L} and it is also desirable to obtain M as small as possible. For the case that $\mathcal{A}\hat{u}$ does not belong to Y , see [47], and concerning nonlinear problems in Hilbert spaces, we can refer to powerful computer-assisted proofs for the ODEs and PDEs [1–3,5–9,12,38,39] which are essentially based on fixed-point methods.

We previously proposed two numerical approaches [45–47] to verify the invertibility of \mathcal{L} defined by (3) and to bound M satisfying (4).

These approaches are based on orthogonal projections to Galerkin approximations in Hilbert spaces with constructive a priori error estimations and can be applied to the case in which the operator \mathcal{L} is non-self-adjoint. The first approach in [45, Theorem 4.1] transforms the problem $\mathcal{L}u = \phi$ for $\phi \in Y$ into an equivalent fixed-point problem on X and constructs a validated bound M as well as verifying the invertibility of \mathcal{L} . Although many computer-assisted proofs have shown the effectiveness of the first approach [29,21,32,45,33], this approach has a restriction such that the lower bound of M is not less than C_p and does not converge to the exact operator norm of \mathcal{L} . In the second approach in [45, Theorem 5.1] and [46,47], some of the authors considered another estimation for M and showed that M is expected to converge, as the Galerkin space increases, to its exact operator norm of \mathcal{L}^{-1} . However, it has been reported that, for some linear differential operators, computing the criterion of invertibility for the second approach is more difficult as compared to that of the first approach, and the criterion for the second approach requires an additional computational cost [45,16].

The goal of the present paper is to bridge the gap between the two previous approaches. We propose an alternative computer-assisted procedure to compute a verified bound of the norm for inverses of linear operators in Hilbert spaces. The proposed procedure is in fact a generalization of the methods presented in [16] for second-order linear elliptic operators. Even though the criterion for the invertibility of \mathcal{L} is the same as [45, Theorem 4.1], the criterion has no limitation, such as the requirement that the lower bound of M not be less than C_p . The proposed procedure would not converge to its exact operator norm, as in [46]. However, according to the conditions for Galerkin approximation subspaces of X , verification examples reveal that the proposed approach has a better bound than the approach in [46].

The remainder of the present paper is organized as follows. The next section describes the assumptions on the given linear operator and introduces some finite-dimensional approximation subspaces with related constants. Section 3 describes known results based on [45–47]. Section 4 proposes an alternative approach involving the theorems described in the previous section and presents various considerations. Several verification results of the procedures are reported in Section 5.

2. Galerkin approximation and related constants

This section describes assumptions on the linear operator \mathcal{L} and introduces a finite-dimensional approximation subspace with related constants. Assume that the operator \mathcal{A} has the following properties: **A1** and **A2**.

A1. \mathcal{A} is bijective with bounded inverse $\mathcal{A}^{-1} : Y \rightarrow D(\mathcal{A})$.

The operator $\mathcal{A}^{-1} := I_{D(\mathcal{A}) \hookrightarrow X} \circ \mathcal{A}^{-1} : Y \rightarrow X$ is therefore compact due to the compactness of the embedding $D(\mathcal{A}) \hookrightarrow X$.

A2. The operator \mathcal{A} satisfies

$$(u, v)_X = (\mathcal{A}u, v)_Y, \quad \forall u \in D(\mathcal{A}), \quad \forall v \in X. \quad (8)$$

For example, if \mathcal{A} is the Laplacian, **A2** is based solely on partial integration. By virtue of **A1**, **A2**, and (2), we obtain

$$\|\mathcal{A}^{-1}u\|_X \leq C_p \|u\|_Y, \quad \forall u \in Y, \quad (9)$$

because

$$\begin{aligned} \|\mathcal{A}^{-1}u\|_X^2 &= (\mathcal{A}^{-1}u, \mathcal{A}^{-1}u)_X = (u, \mathcal{A}^{-1}u)_Y \\ &\leq \|u\|_Y \|\mathcal{A}^{-1}u\|_Y \\ &\leq C_p \|u\|_Y \|\mathcal{A}^{-1}u\|_X. \end{aligned}$$

Next, let X_h be a finite-dimensional approximation subspace of X dependent on the parameter $h > 0$. For example, in the case of a PDE problem, X_h can be taken to be a finite element subspace with mesh size h . The orthogonal projection $P_h : X \rightarrow X_h$ is defined by

$$(v - P_h v, v_h)_X = 0, \quad \forall v_h \in X_h, \quad \forall v \in X. \quad (10)$$

Since X_h is a closed subspace of X , by defining $X_* := (I - P_h)X$, any element $u \in X$ can be uniquely decomposed as $u = u_h + u_*$ for $u_h \in X_h$ and $u_* \in X_*$. We assume that P_h and \mathcal{Q} have the properties **A3**, **A4**, and **A5**.

A3. There exists $C(h) > 0$ such that

$$\|(I - P_h)u\|_X \leq C(h) \|\mathcal{A}u\|_Y, \quad \forall u \in D(\mathcal{A}). \quad (11)$$

A4. $\mathcal{Q} : X \rightarrow Y$ is bounded and there exist $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$\|\mathcal{Q}u\|_Y \leq \tau_1 \|P_h u\|_X + \tau_2 \|(I - P_h)u\|_X, \quad \forall u \in X. \quad (12)$$

A5. There exists $\tau_3 > 0$ such that

$$\|P_h \mathcal{A}^{-1} \mathcal{Q} u_*\|_X \leq \tau_3 \|u_*\|_X, \quad \forall u_* \in X_*. \quad (13)$$

Assumption **A3** corresponds to error estimation of the orthogonal projection P_h . We emphasize that the estimate (11) is indispensable in our argument, and the compactness of the embedding $D(\mathcal{A}) \hookrightarrow X$ is essential in obtaining the constant $C(h)$ with the desired properties. Assumptions **A4** and **A5** provide more detailed information about the boundedness of the operator $\mathcal{Q} : X \rightarrow Y$. In order for our approach to succeed, concrete values of the constants $C(h)$ and τ_i ($i = 1, 2, 3$) must be known and must be evaluated in the rigorous mathematical sense, and $C(h)$ should have the property that $C(h) \rightarrow 0$ as $h \rightarrow 0$. Moreover, the constants τ_i depend on \mathcal{Q} and P_h . Concrete examples of $C(h)$ and τ_i are shown in Section 5.

Let $\{\phi_i\}_{i=1}^N$ be a basis function of X_h with $N := \dim X_h$, and let G , A_1 , A_2 , L_1 , and L_2 be $N \times N$ matrices defined by

$$G_{mn} := (\phi_n, \phi_m)_X + (\mathcal{Q}\phi_n, \phi_m)_Y, \quad (14)$$

$$[A_1]_{mn} := (\phi_n, \phi_m)_X, \quad (15)$$

$$[A_2]_{mn} := (\phi_n, \phi_m)_Y, \quad (16)$$

for $1 \leq m, n \leq N$, and L_i ($i = 1, 2$) such that $A_i = L_i L_i^H$. Usually, L_i is taken to be the Cholesky factor of A_i because A_i is positive definite, i.e., L_i is a lower triangular matrix. Here, H represents conjugate transposition. Now, let $\rho > 0$ and $\hat{\rho} > 0$ be upper bounds satisfying

$$\|L_1^H G^{-1} L_1\|_2 \leq \rho, \quad (17)$$

$$\|L_1^H G^{-1} L_2\|_2 \leq \hat{\rho}, \quad (18)$$

respectively, which requires invertibility of matrix G . Evaluation of ρ and $\hat{\rho}$, including a proof of the invertibility of G , can be reduced to the verified computation of the maximum singular value of a matrix [37].

3. Known results

In this section, we introduce two computer-assisted proofs of the invertibility of \mathcal{L} as well as the computable bound of M .

The following theorem is a first approach, the essential proof of which can be found in [45, Theorem 4.1].

Theorem 1. *If*

$$\kappa := C(h)(\rho \tau_1 \tau_3 + \tau_2) < 1 \quad (19)$$

then \mathcal{L} defined by (3) has an inverse and $M > 0$ for (4) is obtained by

$$M = \frac{C_p}{1 - \kappa} \left\| \begin{bmatrix} \rho(1 - \tau_2 C(h)) & \rho \tau_3 \\ \rho \tau_1 C(h) & 1 \end{bmatrix} \right\|_2. \quad (20)$$

Proof. Since

$$\mathcal{L}\psi = (\mathcal{A} + \mathcal{Q})\psi = \mathcal{A}(I + \mathcal{A}^{-1}\mathcal{Q})\psi, \quad \psi \in D(\mathcal{A}), \quad (21)$$

setting $\tilde{\mathcal{L}} := I + \mathcal{A}^{-1}\mathcal{Q}$ on X , the criterion $\kappa < 1$ assures the invertibility of $\tilde{\mathcal{L}}$ by [45, Theorem 3.5], and if $\tilde{\mathcal{L}}$ is invertible, \mathcal{L} is also invertible according to [45, Lemma 3.1]. Therefore, under the condition $\kappa < 1$, for each $\phi \in Y$, setting $\psi = \mathcal{L}^{-1}\phi \in D(\mathcal{A})$, we have $\psi = (\tilde{\mathcal{L}})^{-1}\mathcal{A}^{-1}\phi$ from (21). Therefore, [45, Theorem 4.1] and (9) derive the estimation (20). \square

Theorem 1 is based on a fixed-point problem for $\tilde{\mathcal{L}}$ on X for constructing a validated bound M , and the estimate (20) could converge to $C_p \max\{\rho, 1\}$ as $h \rightarrow 0$. As a result, Theorem 1 fails to converge to its exact operator norm. In order to overcome this overestimation, the second approach for \mathcal{L}^{-1} was proposed [46,15,32,43]. The proof is presented in [45, Theorem 5.1].

Theorem 2. *If*

$$\hat{\kappa} := C(h)\tau_2(\hat{\rho}\tau_1 + 1) < 1, \quad (22)$$

then \mathcal{L} defined by (3) has an inverse and $M > 0$ for (4) is obtained by

$$M = \frac{\sqrt{\hat{\rho}^2 + C(h)^2(1 + \hat{\rho}\tau_1)^2}}{1 - \hat{\kappa}}. \quad (23)$$

Roughly speaking, G is a finite-dimensional projection of \mathcal{L} , and (18) reflects the invertibility of this projection. In (23), $\hat{\kappa} \rightarrow 0$ and $C(h) \rightarrow 0$ as $h \rightarrow 0$ implies $M/\rho \rightarrow 1$. Therefore, in Theorem 2, the estimation (23) is expected to converge to the exact operator norm of \mathcal{L}^{-1} , as $h \rightarrow 0$.

However, sometimes the criterion $\hat{\kappa} < 1$ is harder than $\kappa < 1$ for fixed h , experimentally, and in Theorem 2 the computation of $\hat{\rho}$ generally requires an additional matrix decomposition $A_2 = L_2 L_2^H$ with more computational cost than Theorem 1.

4. An alternative approach

In this section, we propose an alternative approach to Theorem 1 or Theorem 2 for \mathcal{L}^{-1} and present some consideration.

Theorem 3. *If $\kappa < 1$, then \mathcal{L} defined by (3) has an inverse and $M > 0$ for (4) is obtained by*

$$M = \frac{\sqrt{(\rho(C_p + C(h)(\tau_3 - C_p\tau_2)))^2 + (C(h)(1 + \rho C_p\tau_1))^2}}{1 - \kappa}. \quad (24)$$

Proof. We will show

$$\|u\|_X \leq M \|\mathcal{L}u\|_Y, \quad \forall u \in D(\mathcal{A}) \quad (25)$$

for M defined by (24). The inequality (25) shows that \mathcal{L} is a one-to-one function. Furthermore, for any given $\phi \in Y$, the equation

$$\psi \in D(\mathcal{A}), \quad \mathcal{L}\psi = \phi \quad (26)$$

is equivalent to

$$\psi \in X, \quad \tilde{\mathcal{L}}\psi = (I + \mathcal{A}^{-1}\mathcal{Q})\psi = \mathcal{A}^{-1}\phi. \quad (27)$$

Since $\mathcal{A}^{-1}\mathcal{Q} : X \rightarrow X$ is compact, the Fredholm alternative holds for equation (27), whereby \mathcal{L} being one-to-one implies that (27), and hence (26), is uniquely solvable. Hence, \mathcal{L} is bijective, and (25) yields (4), with M from (24).

Since each $u \in D(\mathcal{A})$ can be decomposed into

$$u = u_h + u_*, \quad u_h = P_h u \in X_h, \quad u_* = (I - P_h)u \in X_*,$$

setting $\psi := \mathcal{L}u \in L^2(\Omega)$, we will estimate $\|u_h\|_X$ and $\|u_*\|_X$ by $\|\psi\|_Y$.

For the finite-dimensional part, $\|u_h\|_X$, because $\psi = \mathcal{A}u + \mathcal{Q}u$, using **A2** we have

$$(\psi, v)_Y = (u, v)_X + (\mathcal{Q}u, v)_Y, \quad \forall v \in X. \quad (28)$$

In (28), by taking v as $v_h \in X_h$ and using $(u, v_h)_X = (u_h, v_h)_X$, it holds that

$$\begin{aligned} (u_h, v_h)_X + (\mathcal{Q}u_h, v_h)_Y &= (-\mathcal{Q}u_* + \psi, v_h)_Y \\ &= (\mathcal{A}\mathcal{A}^{-1}(-\mathcal{Q}u_* + \psi), v_h)_Y \\ &= (\mathcal{A}^{-1}(-\mathcal{Q}u_* + \psi), v_h)_X \\ &= (P_h\mathcal{A}^{-1}(-\mathcal{Q}u_* + \psi), v_h)_X \end{aligned} \quad (29)$$

from **A2** and the orthogonality of the projection of P_h . Now, setting

$$u_h = \sum_{i=1}^N a_i \phi_i, \quad \mathbf{a} = [a_1, \dots, a_N]^T \in \mathbb{C}^N,$$

$$P_h\mathcal{A}^{-1}(-\mathcal{Q}u_* + \psi) = \sum_{i=1}^N b_i \phi_i, \quad \mathbf{b} = [b_1, \dots, b_N]^T \in \mathbb{C}^N,$$

and noting that (29) holds for each $v_h \in S_h$, from (14) and (15), we obtain

$$\mathbf{a} = G^{-1}A_1\mathbf{b}. \quad (30)$$

Therefore, using (30), (17), A5, and (9), it can be confirmed that

$$\begin{aligned}
 \|u_h\|_X &= \|L_1^H \mathbf{a}\|_2 \\
 &= \|L_1^H G^{-1} A_1 \mathbf{b}\|_2 \\
 &= \|L_1^H G^{-1} L_1 L_1^H \mathbf{b}\|_2 \\
 &\leq \|L_1^H G^{-1} L_1\|_2 \|L_1^H \mathbf{b}\|_2 \\
 &\leq \rho \|P_h \mathcal{A}^{-1} (-\mathcal{Q}u_* + \psi)\|_X \\
 &\leq \rho \left(\|P_h \mathcal{A}^{-1} \mathcal{Q}u_*\|_X + \|P_h \mathcal{A}^{-1} \psi\|_X \right) \\
 &\leq \rho \tau_3 \|u_*\|_X + \rho C_p \|\psi\|_Y.
 \end{aligned} \tag{31}$$

For the infinite-dimensional part, $\|u_*\|_X$, using A3 and A4, we obtain

$$\begin{aligned}
 \|u_*\|_X &= \|(I - P_h)u\|_X \leq C(h) \|\mathcal{A}u\|_Y \\
 &= C(h) \|\mathcal{Q}u + \psi\|_Y \\
 &= C(h) (\|\mathcal{Q}u\|_Y + \|\psi\|_Y) \\
 &\leq C(h) (\tau_1 \|u_h\|_X + \tau_2 \|u_*\|_X + \|\psi\|_Y).
 \end{aligned} \tag{32}$$

Substituting (31) into (32), we obtain

$$\|u_*\|_X \leq C(h) (\rho \tau_1 \tau_3 + \tau_2) \|u_*\|_X + (1 + \rho C_p \tau_1) \|\psi\|_Y;$$

then (19) and the condition $\kappa < 1$ ensure that

$$\|u_*\|_X \leq \frac{1}{1 - \kappa} C(h) (1 + \rho C_p \tau_1) \|\psi\|_Y. \tag{33}$$

Moreover, substituting (33) into (32), we obtain

$$\|u_h\|_X \leq \frac{\rho}{1 - \kappa} (C_p + C(h)(\tau_3 - C_p \tau_2)) \|\psi\|_Y \tag{34}$$

and the desired conclusion from $\|u\|_X^2 = \|u_h\|_X^2 + \|u_*\|_X^2$. \square

Remark 1. Theorem 3 is a generalization of the methods presented in [16] for the second-order linear elliptic operators based on perturbation theory. While the criterion $\kappa < 1$ for the invertibility of \mathcal{L} in Theorem 3 is the same as Theorem 1, the estimation of M (25) by (24) in Theorem 3 indicates a direct norm upper bound for \mathcal{L}^{-1} , which is connected to Theorem 2. Therefore, the approach of Theorem 3 is expected to bridge the gap between the two previous procedures, especially in actual numerical verifications.

Remark 2. In Theorem 3, assuming that κ and $C(h)$ converge to 0 as $h \rightarrow 0$ implies that M for (24) converges to $C_p \rho$. Generally, $M \rightarrow C_p \rho$ does not indicate convergence to the exact operator norm. However, as we can show that $\hat{\rho} \leq C_p \rho$ in Section 4.1 when $\hat{\rho} \sim C_p \rho$, it is expected that we can obtain a sufficiently “good” M with low computational cost.

4.1. A consideration for theorems

In Theorems 2 and 3, the obtained upper bounds for M are expected to converge to $\hat{\rho}$ and ρC_p , respectively, as $h \rightarrow 0$. The following result assures Theorem 2 has an advantage over Theorem 3 from the *ideal* point of view.

Lemma 1. *It holds that*

$$\hat{\rho} \leq \rho C_p. \quad (35)$$

Proof. For each $u_h = \sum_{i=1}^N u_i \phi \in S_h$, by setting $\mathbf{u} = [u_1, \dots, u_N]^T \in \mathbb{C}^N$, the inequality (2) shows that

$$\|L_2^H \mathbf{u}\|_2 \leq C_p \|L_1^H \mathbf{u}\|_2. \quad (36)$$

Therefore, using (36), it holds that

$$\begin{aligned} \hat{\rho} &= \|L_1^H G^{-1} L_2\|_2 = \|L_1^H G^{-1} L_1 L_1^{-1} L_2\|_2 \\ &\leq \rho \|L_1^{-1} L_2\|_2 = \rho \|L_2^H L_1^{-H}\|_2 \\ &= \rho \sup_{\mathbf{x}^H \mathbf{x} = 1} \|L_2^H L_1^{-H} \mathbf{x}\|_2 \\ &\leq \rho C_p \sup_{\mathbf{x}^H \mathbf{x} = 1} \|L_1^H L_1^{-H} \mathbf{x}\|_2 = \rho C_p. \quad \square \end{aligned}$$

Nevertheless, from an *actual computational* point of view, since the criterion $\hat{\kappa} < 1$ is sometimes more difficult to experimentally investigate than $\kappa < 1$ for fixed h , Theorem 3 may be effective. We present some comparisons in the next section.

5. Verification examples

In this section, we report several verified computation results of the three theorems. We use the interval arithmetic toolbox INTLAB [36] Version 6 with MATLAB 7.14.0.739 (R2012a) on a Fujitsu PRIMERGY TX300 S5 (CPU: Intel Xeon E5520 2.27 GHz, OS: Red Hat Enterprise Linux Server release 5.6), for Section 5.1, and INTLAB Version 10 with MATLAB 2016b run on macOS 10.12.6 for Section 5.2.

5.1. Second-order elliptic operators

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal or polyhedral domain ($d = 1, 2, 3$), and for some integer m , let $H^m(\Omega)$ denote the complex L^2 -Sobolev space of order m on Ω . We define the Hilbert space $H_0^1(\Omega) := \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega\}$ with the inner product $(\nabla u, \nabla v)_{L^2(\Omega)}$

and the norm $\|u\|_{H_0^1(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$, where $(u, v)_{L^2(\Omega)}$ implies L^2 -inner product on Ω . Let $H(\Delta; L^2(\Omega)) := \{u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega)\}$ be a Banach space with respect to the graph norm $\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}$ and assume that the embedding $H(\Delta; L^2(\Omega)) \hookrightarrow H_0^1(\Omega)$ is compact.

Consider the linear elliptic operator

$$\mathcal{L}u := -\Delta u + b \cdot \nabla u + cu : H(\Delta; L^2(\Omega)) \rightarrow L^2(\Omega) \quad (37)$$

for $b \in L^\infty(\Omega)^d$ and $c \in L^\infty(\Omega)$ with norms

$$\|b\|_{L^\infty(\Omega)^d} = \operatorname{ess\,sup}_{x \in \Omega} \sqrt{|b_1(x)|^2 + \cdots + |b_d(x)|^2}, \quad \|c\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_{x \in \Omega} |c(x)|,$$

respectively. We can set

$$\begin{aligned} D(\mathcal{A}) &= H(\Delta; L^2(\Omega)), & X &= H_0^1(\Omega), & Y &= L^2(\Omega), & \mathcal{A} &= -\Delta, \\ \mathcal{Q} &= b \cdot \nabla + c, & (u, v)_X &= (\nabla u, \nabla v)_{L^2(\Omega)}, & (u, v)_Y &= (u, v)_{L^2(\Omega)}. \end{aligned} \quad (38)$$

It is well known that **A1** holds [10], and **A2** is an immediate consequence of partial integration. For **A3**, we note that P_h is now the usual H_0^1 -projection, and (11) holds for many finite element subspaces of $H_0^1(\Omega)$ [4,23,26–28] or function spaces of Fourier series with finite truncation [40]. For example, for bilinear and biquadratic elements, $C(h) = h/\pi$ and $h/(2\pi)$, respectively, for the rectangular mesh on the square domain [23], and $C(h) = 0.493h$ for the linear and uniform triangular mesh of the convex polygonal domain [17,19,20]. Here, $h > 0$ represents the element side length for a given finite element mesh. Furthermore, a constructive a priori L^∞ error estimate for the projection P_h can also be obtained [24,25]. In the case of a nonconvex polygonal domain, there are some useful techniques and considerations to obtain mathematically rigorous upper bounds for the constant $C(h)$ satisfying (10) with adequate order for such nonconvex domains [11,18,49–51].

Concerning **A4** and **A5**, we can take

$$\tau_1 = \|b\|_{L^\infty(\Omega)^d} + C_p \|c\|_{L^\infty(\Omega)}, \quad \tau_2 = \|b\|_{L^\infty(\Omega)^d} + C(h) \|c\|_{L^\infty(\Omega)}, \quad \tau_3 = C_p \tau_2$$

for the Poincaré or Rayleigh–Ritz constant satisfying

$$\|u\|_{L^2(\Omega)} \leq C_p \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \quad (39)$$

For example, if $\Omega = (0, 1) \times (0, 1)$, C_p can be taken as $1/(\pi\sqrt{2})$. Note that if P_h and $(-\Delta)^{-1}$ commute [40], or b is differentiable [29], we can derive more accurate estimates for τ_i ($i = 1, 2, 3$) [43,33,16]. In previous studies, some of the authors also showed that it is possible to obtain a similar kind of accurate estimate for τ_i , even though b is not differentiable [13,30,31].

5.1.1. One-dimensional problem

Consider a one-dimensional problem on $\Omega = (0, 1)$ with $b(x) = r \sin(\pi x)$ for $r \in \mathbb{R}$ and $c \in \mathbb{R}$. We divide the interval $(0, 1)$ by partitions of equal size $h > 0$ and take X_h as the set of piecewise linear functions on each subinterval. Here, the constants defined in (39) and (10) can

Table 1

Verification results for the invertibility and bound of M for $b = 2.5 \sin(\pi x)$ and $c = -10$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
10	12.6637	3.6970	0.6865	12.4285	1.9761	–	0.6865	12.2786
30	12.9669	3.8003	0.0956	4.4655	0.6249	10.1500	0.0956	4.4598
50	12.9916	3.8084	0.0409	4.2504	0.3696	6.0452	0.0409	4.2485
100	13.0020	3.8119	0.0142	4.1667	0.1827	4.6645	0.0142	4.1663
200	13.0047	3.8128	0.0056	4.1465	0.0908	4.1936	0.0056	4.1464
500	13.0054	3.8130	0.0019	4.1409	0.0362	3.9561	0.0019	4.1409
1,000	13.0055	3.8131	0.0009	4.1401	0.0181	3.8832	0.0009	4.1401

Table 2

Verification results for the invertibility and bound of M for $b = -20 \sin(\pi x)$ and $c = -20$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
10	2.6420	0.3552	3.9293	–	6.8074	–	3.9293	–
30	2.5044	0.3542	0.5592	1.8684	2.2167	–	0.5592	1.5439
50	2.4950	0.3542	0.2518	1.0293	1.3246	–	0.2518	0.9502
100	2.4911	0.3542	0.0948	0.8417	0.6603	1.0469	0.0948	0.8249
200	2.4911	0.3542	0.0396	0.8040	0.3296	0.5289	0.0396	0.8002
500	2.4899	0.3542	0.0140	0.7943	0.1318	0.4080	0.0140	0.7938
1,000	2.4899	0.3542	0.0067	0.7930	0.0659	0.3792	0.0067	0.7929

be taken as $C(h) = h/\pi$, $C_p = 1/\pi$, respectively. Tables 1 and 2 show the verification results obtained with various h for $(b, c) = (2.5 \sin(\pi x), -10)$ and $(b, c) = (-20 \sin(\pi x), -20)$. The “–” symbol indicates that the invertibility criterion does not hold. Bold numbers indicate the “best” estimate for M . For $1/h = 100$, the former case of (b, c) is $\hat{\rho}/(C_p \rho) \sim 0.9210$ and the latter case is $\hat{\rho}/(C_p \rho) \sim 0.4466$. It can be confirmed that when $1/h$ is small, Theorem 3 gives a more accurate bound M than Theorem 1 or Theorem 2, and when $1/h$ tends to be large, Theorem 2 is expected to converge to the exact operator norm because of the tendency for $\hat{\rho}/M \rightarrow 1$.

5.1.2. Two-dimensional problem

The next example is a two-dimensional non-self-adjoint operator \mathcal{L} on $\Omega = (0, 1) \times (0, 1)$ with

$$b = R \begin{bmatrix} -y + 1/2 \\ x - 1/2 \end{bmatrix}, \quad c \in \mathbb{C}$$

which originates from a stationary convection–diffusion equation. Here, R is a positive constant. We take linear and uniform triangular meshes on Ω with element side length $h > 0$ for a given finite element mesh. It is well known that (10) holds for $C(h) = 0.493h$ and $C_p = 1/(\pi\sqrt{2})$.

Tables 3 and 4 show the verification results of the invertibility and bound of M for $(R, c) = (10, -10 - 5i)$ and $(R, c) = (10, 15)$, respectively. For $1/h = 50$, the former case of (b, c) is $\hat{\rho}/(C_p \rho) \sim 0.9481$ and the latter case is $\hat{\rho}/(C_p \rho) \sim 0.5682$.

It can be confirmed that when $1/h$ is small, Theorem 3 gives a more accurate bound M than Theorem 1 or Theorem 2, and when $1/h$ tends to be large, Theorem 2 is expected to converge to the exact operator norm because of the tendency for $\hat{\rho}/M \rightarrow 1$.

Table 3

Verification results for the invertibility and bound of M for $R = 10$ and $c = -10 - 5i$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
5	1.7039	0.3656	2.3287	–	3.6305	–	2.3287	–
10	1.7751	0.3946	0.7724	1.8734	1.7974	–	0.7724	1.6510
20	1.7941	0.4025	0.2814	0.5384	0.8798	3.4926	0.2814	0.5033
50	1.7995	0.4047	0.0869	0.4222	0.3456	0.6227	0.0869	0.4174
100	1.8001	0.4050	0.0392	0.4092	0.1716	0.4897	0.0392	0.4082

Table 4

Verification results for the invertibility and bound of M for $R = 10$ and $c = 15$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
5	0.9732	0.1270	1.8758	–	1.9610	–	1.8758	–
8	0.9903	0.1276	0.9032	3.3368	1.1493	–	0.9032	2.6387
10	0.9939	0.1277	0.6488	0.8671	0.8987	1.6951	0.6488	0.6589
20	0.9986	0.1279	0.2497	0.3543	0.4284	0.2453	0.2497	0.2760
50	0.9999	0.1279	0.0818	0.2632	0.1663	0.1559	0.0818	0.2316
100	1.0001	0.1279	0.0379	0.2426	0.0823	0.1400	0.0379	0.2267

Table 5

Verification results for the invertibility and bound of M for $a = 0.001$ (lower solution).

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
10	1.0586	0.2356	0.0030	0.2391	0.0030	0.2421	0.0030	0.2447
20	1.0599	0.2379	0.0008	0.2388	0.0008	0.2395	0.0008	0.2402
30	1.0601	0.2383	0.0004	0.2387	0.0004	0.2391	0.0004	0.2394
40	1.0602	0.2385	0.0002	0.2387	0.0002	0.2389	0.0002	0.2391
50	1.0603	0.2386	0.0002	0.2387	0.0002	0.2388	0.0002	0.2389

5.1.3. The case of $b = 0$

The next example is the case of a linearized equation of semilinear PDEs:

$$\begin{cases} -\Delta u = 1 + u + u^2 - au^3 & \text{in } (0, 1) \times (0, 1), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (40)$$

Equation (40) has at least two positive solutions, which we refer to as the upper and lower solutions. Let u_h be the finite element solutions obtained by the Newton–Raphson method using usual floating point arithmetic. Then, the linearized operator at u_h is defined by $\mathcal{L} = -\Delta - 1 - 2u_h + 3au_h^2$. Tables 5 and 6 show the verification results of the linearized operator with $a = 0.001$. For $1/h = 50$, in the case of the lower solution, $\hat{\rho}/(C_p \rho) \sim 0.9995$, and in the case of the upper solution, $\hat{\rho}/(C_p \rho) \sim 0.6040$. For the case of the lower solution, a solution of (40) can be enclosed by simple infinite-dimensional sequential iteration [44]. Therefore, the three theorems yield the same bound M . For the case of the upper solution, the effectiveness of the method, and the validity of Theorem 2 is shown.

Table 6

Verification results for the invertibility and bound of M for $a = 0.001$ (upper solution).

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
10	2.5948	0.3545	1.1823	–	0.7722	1.9668	1.1823	–
20	2.6622	0.3624	0.2856	0.9204	0.1861	0.4756	0.2856	0.8883
30	2.6758	0.3640	0.1262	0.7216	0.0822	0.4087	0.1262	0.7074
40	2.6807	0.3645	0.0709	0.6671	0.0461	0.3887	0.0709	0.6590
50	2.6830	0.3648	0.0453	0.6438	0.0295	0.3800	0.0453	0.6386

5.2. Fourth-order differential operators

The Orr–Sommerfeld equation

$$\begin{cases} (-D^2 + a^2)^2 u + ia \operatorname{Re}[V(-D^2 + a^2) + V'']u = \lambda(-D^2 + a^2)u & \text{in } (x_1, x_2), \\ u(x_1) = u(x_2) = u'(x_1) = u'(x_2) = 0, \end{cases} \quad (41)$$

is one of the central equations governing the linearized stability theory of incompressible flows. Here, $D = d/dx$, i , $a > 0$, and $\operatorname{Re} > 0$ denote, respectively, the derivative, the imaginary unit, the wave number of the perturbation, and the Reynolds number of an underlying fluid moving in a stationary flow with a given real-valued flow profile $V \in C^2(x_1, x_2)$. The Orr–Sommerfeld equation (41) is a non-self-adjoint eigenvalue problem for the eigenpair $[u, \lambda]^t$. We focus on the case of the plane Poiseuille flow $V = 1 - x^2$ for $x_1 = -1$ and $x_2 = 1$. Let $\mu \in \mathbb{C}$ be a point for which we want to prove that no eigenvalue of (41) is close to μ . For the shifted operator

$$\mathcal{L}u = (-D^2 + a^2)^2 u + ia \operatorname{Re}[V(-D^2 + a^2) + V'']u - \mu(-D^2 + a^2)u,$$

under the invertibility of the \mathcal{L} and the norm bound of its inverse, [45, Theorem 2.1] states that there is no eigenvalue of (41) in the open disk

$$\{z \in \mathbb{C} \mid |z - \mu| < 1/(C_p M)\} \quad (42)$$

for the Orr–Sommerfeld equation (41).

For the subspace

$$H_0^2(\Omega) := \left\{ v \in H^2(\Omega) \mid v(-1) = v'(-1) = v(1) = v'(1) = 0 \right\},$$

for $\Omega = (-1, 1)$, we can take $D(\mathcal{A}) = H^4(\Omega) \cap H_0^2(\Omega)$, $X = H_0^2(\Omega)$, $Y = L^2(\Omega)$, with inner products

$$(u, v)_X = ((-D^2 + a^2)u, (-D^2 + a^2)v)_{L^2(\Omega)}, \quad (u, v)_Y = (u, v)_{L^2(\Omega)},$$

and $\mathcal{A} = (-D^2 + a^2)^2$, $\mathcal{Q} = ia \operatorname{Re}[V(-D^2 + a^2) + V''] - \mu(-D^2 + a^2)$. Moreover, we can take $C_p = 1/(\pi^2/4 + a^2)$.

Table 7

Verification results for the invertibility and bound of M for $\mu_1 = -100 + 1552.59i$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
100	6.1158	0.0293	1.4125	–	33.7429	–	1.4125	–
120	6.1163	0.0293	0.7129	6.5290	23.4326	–	0.7129	5.5598
150	6.1184	0.0293	0.3160	2.5766	14.9968	–	0.3160	2.4010
200	6.1310	0.0293	0.1165	1.9538	8.4358	–	0.1165	1.9100
400	8.0981	0.0293	0.0159	2.3287	2.1119	–	0.0159	2.3259
500	14.7946	0.0294	0.0109	4.2434	1.3570	–	0.0109	4.2434

Table 8

Verification results for the invertibility and bound of M for $\mu_2 = -200 + 1552.59i$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
100	2.6778	0.0148	0.7029	3.1815	17.0894	–	0.7029	2.2902
120	2.6781	0.0148	0.3708	1.3431	11.8681	–	0.3708	1.1107
150	2.6791	0.0148	0.1759	0.9524	7.5957	–	0.1759	0.8725
200	2.6848	0.0148	0.0721	0.8198	4.2727	–	0.0721	0.7964
400	3.5695	0.0148	0.0123	1.0229	1.0690	–	0.0123	1.0213
500	6.5817	0.0148	0.0082	1.8827	0.6856	0.0470	0.0082	1.8816

Table 9

Verification results for the invertibility and bound of M for $\mu_3 = -500 + 1552.59i$.

$1/h$	ρ	$\hat{\rho}$	Theorem 1		Theorem 2		Theorem 3	
			κ	M	$\hat{\kappa}$	M	κ	M
100	0.9971	0.0062	0.3573	0.6593	7.2514	–	0.3573	0.3941
120	0.9983	0.0062	0.2043	0.4768	5.0361	–	0.2043	0.3273
150	0.9995	0.0062	0.1078	0.3862	3.2232	–	0.1078	0.3006
200	1.0020	0.0062	0.0506	0.3355	1.8131	–	0.0506	0.2904
400	1.2767	0.0062	0.0105	0.3665	0.4535	0.0113	0.0105	0.3646
500	2.2102	0.0062	0.0068	0.6315	0.2905	0.0087	0.0068	0.6310

For the case of equation (41), when we introduce a finite-dimensional approximation subspace $X_h \subset H_0^2(\Omega)$, using base functions constructed from piecewise cubic Hermite interpolating polynomials with uniform partition size h , we can take

$$C(h) = \frac{\sqrt{3}}{p} h^2 \left(1 + \frac{a^2}{p} h^2 \right), \quad v_1 = C(h)s_1, \quad v_2 = s_2 + \tau_3 C_p, \quad v_3 = s_2 + C(h)s_3,$$

where $s_1 := 2s_3 C_p + s_2 + Re\|V'\|_\infty$, $s_2 := \|-iaReV + \mu\|_\infty$, $s_3 := aRe\|V''\|_\infty$, and $p = 6\sqrt{70}/\sqrt{4+\sqrt{5}}$ [41,42]. In particular, for $C(h)$, we use the interpolation error estimates in $H^4(\Omega) \cap H_0^2(\Omega)$ [14].

Tables 7, 8, and 9 show the verification results of the linearized operator for $Re = 5776$, $a = 1.019$, $\mu_1 = -100 + 1552.59i$, $\mu_2 = -200 + 1552.59i$, and $\mu_3 = -500 + 1552.59i$. The “–” symbol indicates that the assumptions of the theorems did not hold.

When the candidate excluding point μ is taken far from the eigenvalues of problem (41), Theorem 2 gives a better result (smaller M) than Theorem 1 or Theorem 3. When μ is closer to an eigenvalue, the criterion $\hat{\kappa} < 1$ could not be satisfied, whereas Theorem 1 or Theorem 3 works. For example, when we take $\mu = -500 + 1552.59i$ and $h = 1/100$, we obtain the eigenvalue excluding radii for (42) by Theorem 1 and Theorem 3 as 5.3180 and 8.8977, respectively, while the invertibility verification of \mathcal{L} by Theorem 2 fails. Thus, for such μ and h , our proposed estimation is better than those of other estimation methods. Note that, especially for h smaller than $1/200$, the round-off error for M tends to be a serious problem. Moreover, the round-off error for M is expected to be larger for a larger partition number. See reference [42] for a stability proof based on eigenvalue closures for the Orr–Sommerfeld equation (41).

Finally, let us remark that, by using Theorem 3 (reported first in the present paper) and the proposed Newton-type fixed-point formulation described in Section 1, we have succeeded in enclosing the critical Reynolds number R_c which means the largest R such that the flow is stable of the Orr–Sommerfeld problem (41) within the interval $[5772.22181620969, 5772.22181620970]$. We believe this constitutes the first strict mathematical proof for the critical Reynolds number R_c . We also emphasize that the verification of the invertibility of linearized operator \mathcal{L} and the computable norm bound for \mathcal{L}^{-1} play an essential role and that the above narrow bound could not be obtained by our previous approaches (Theorem 1 and Theorem 2). We will report on a detailed computer-assisted proof for enclosing R_c in our upcoming paper [48].

6. Conclusion

We propose a new computer-assisted procedure to prove the invertibility of a linear operator in a Hilbert space and to compute a verified norm bound of its inverse. The criterion for the invertibility of \mathcal{L} is the same as that in [45, Theorem 4.1] but has no limitation such as that the lower bound of M must not be less than C_p . Although the proposed procedure would not converge to its exact operator norm, as in [46], some verification examples reveal that the proposed procedure has a better bound than the approach in Theorem 2. We conclude that the proposed method can bridge the gap between the two previous approaches, and an appropriate procedure can be selected taking into consideration the particular problem or the computational cost, for example.

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