

Positive radial solutions for multiparameter Dirichlet systems with mean curvature operator in Minkowski space and Lane–Emden type nonlinearities

Daniela Gurban, Petru Jebelean *

Department of Mathematics, West University of Timișoara, Blvd. V. Pârvan, no. 4, 300223 Timișoara, Romania

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Abstract

We deal with Dirichlet systems involving the mean curvature operator in Minkowski space

$$\mathcal{M}(w) = \operatorname{div} \left(\frac{\nabla w}{\sqrt{1 - |\nabla w|^2}} \right)$$

in a ball in $\mathbb{R}^N$. Using the fixed point index and the lower and upper solutions method, we first obtain the existence of positive solutions for a class of differential systems with a singular φ -Laplacian, subjected to homogeneous mixed boundary conditions. The main application of the general results concerns Dirichlet systems with Lane–Emden type nonlinearities in a superlinear case, depending on two parameters λ_1, λ_2 . So, we prove that for such a system there exists a continuous curve Γ which separates the first quadrant in two disjoint unbounded open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system has zero, at least one or at least two radial positive solutions, according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$.

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* Corresponding author.

E-mail addresses: gurbandaniela@yahoo.com (D. Gurban), petru.jebelean@e-uvt.ro (P. Jebelean).

1. Introduction

In this paper we study the existence and multiplicity of positive solutions for non-variational radial systems of type

$$\begin{cases} \mathcal{M}(u) + g_1(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + g_2(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)} \end{cases} \quad (1.1)$$

where $\mathcal{M}$ stands for the mean curvature operator in Minkowski space

$$\mathcal{M}(w) = \operatorname{div} \left(\frac{\nabla w}{\sqrt{1 - |\nabla w|^2}} \right),$$

$\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$ is an integer and the functions $g_1, g_2 : [0, R] \times [0, \infty)^2 \rightarrow [0, \infty)$ are continuous.

In recent years, many papers were devoted to the study of the Dirichlet problem for a single equation, with operator $\mathcal{M}$, either in a general bounded domain [2,3,13–15,29] or in a ball [4,5,12,30] – when the problem has a radial structure. This type of problems are originated in the study – in differential geometry or special relativity, of maximal or constant mean curvature hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1} = \{(x, t) : x \in \mathbb{R}^N, t \in \mathbb{R}\}$ endowed with the Lorentzian metric $\sum_{j=1}^N (dx_j)^2 - (dt)^2$), with the property that their mean extrinsic curvature is respectively zero or constant [1,10,22,32]. On the other hand, it is well known that for the classical Laplacian and other more general nonlinear elliptic operators, Dirichlet problems in the case of a single equation were widely studied. These results were extended to elliptic systems which describe new and specific phenomena and require different approaches (see e.g. [6,16,18–20,33] and the references therein). An important type of nonlinearities involved in these studies concerning the systems are the so-called “Lane–Emden nonlinearities”, having either the additive form $k_1 u^p + k_2 v^q$ [17,31,34] or the multiplicative one $k_3 u^\alpha v^\beta$ [18,21,27]. In the case of a single equation, such nonlinearities (now, having the form ku^γ) were already considered in Dirichlet problems with operator $\mathcal{M}$, in both radial and non-radial cases [5,12,15,30,35]. The study of systems with operator $\mathcal{M}$ seems to be at the beginning; we notice the recent papers [25] for the non-radial case and [24] for the radial one. In [24], the authors consider Lane–Emden systems in both the superlinear and sublinear cases. Actually, the main goal therein is to extend a result of non-existence, existence and multiplicity from [5] for a single equation to a one parameter systems of type

$$\begin{cases} \mathcal{M}(u) + \lambda \mu(|x|)(p+1)u^p v^{q+1} = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + \lambda \mu(|x|)(q+1)u^{p+1} v^q = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)} \end{cases} \quad (1.2)$$

where $\mu : [0, R] \rightarrow [0, \infty)$ is a continuous function, with $\mu(r) > 0$ for all $r \in (0, R]$. Precisely, according to [5, Corollary 1] one has that, if $\gamma > 1$, then there exists $\Lambda > 0$ such that problem

$$\mathcal{M}(u) + \lambda \mu(|x|)u^\gamma = 0 \text{ in } \mathcal{B}(R) \quad u|_{\partial\mathcal{B}(R)} = 0 \quad (1.3)$$

has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$, while in [24, Corollary 5.2] it is shown that the same holds true for (1.2) provided that $\max\{p, q\} > 1$. It is worth to point out that the presence of the weight function μ in (1.2) and (1.3) allows to consider the so-called Hénon-Lane-Emden nonlinearities, when $\mu(|x|) = |x|^\sigma$ (see [7,8,26]). The approach in [24] relies on Leray-Schauder degree, the lower and upper solutions method and a key critical point argument – of course, this being possible due to the variational structure of the problem.

In view of the above it appears as being natural to extend the previous study to more general, multiparameter systems with Lane-Emden nonlinearities which do not have a variational structure. So, the main goal of this work is firstly to prove by non-variational methods the existence of positive radial solutions of problem (1.1) in order to study then the more general two-parameters Lane-Emden system of the form

$$\begin{cases} \mathcal{M}(u) + \lambda_1 \mu_1(|x|) u^{p_1} v^{q_1} = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + \lambda_2 \mu_2(|x|) u^{p_2} v^{q_2} = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial \mathcal{B}(R)} = 0 = v|_{\partial \mathcal{B}(R)}, \end{cases} \quad (1.4)$$

where p_1, q_2 are nonnegative, while q_1, p_2 are positive exponents – meaning that (1.4) is coupled. The choice of this type of exponents is inspired by the work [11], where the case of a non parameterized p and q -Laplacian system has been studied. The weight functions $\mu_1, \mu_2 : [0, R] \rightarrow [0, \infty)$ are assumed to be continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$. By using the fixed point index we prove that there exist $\lambda_1^* > 0 < \lambda_2^*$, such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, system (1.4) has a solution (u, v) with both u and v positive in $\mathcal{B}(R)$ and radially strictly decreasing. If in addition it holds $\min\{p_1, q_2\} > 1$, then we show that there exists a continuous curve Γ splitting the first quadrant into two disjoint unbounded, open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system (1.4) has zero, at least one or at least two positive radial solutions according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$, respectively. The set $\mathcal{O}_1$ is adjacent to the coordinates axes $0\lambda_1$ and $0\lambda_2$ and the curve Γ approaches asymptotically to two lines parallel to the axes $0\lambda_1$ and $0\lambda_2$. The proof of this result mainly relies on the existence of a lower solution in the cone of positive functions and fixed point index estimations.

As usual, when we are seeking for radial solutions of (1.1), by setting $r = |x|$ and $u(x) = u(r)$, $v(x) = v(r)$, the Dirichlet problem (1.1) reduces to the homogeneous mixed boundary value problem:

$$\begin{cases} [r^{N-1} \varphi(u')] + r^{N-1} g_1(r, u, v) = 0, \\ [r^{N-1} \varphi(v')] + r^{N-1} g_2(r, u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \quad (1.5)$$

where

$$\varphi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in \mathbb{R}, |y| < 1).$$

By a *solution* of (1.5) we mean a couple of nonnegative functions $(u, v) \in C^1[0, R] \times C^1[0, R]$ with $\|u'\|_\infty < 1$, $\|v'\|_\infty < 1$ and $r \mapsto r^{N-1} \varphi(u'(r))$, $r \mapsto r^{N-1} \varphi(v'(r))$ of class C^1 on $[0, R]$, which satisfies problem (1.5). Here and below, $\|\cdot\|_\infty$ stands for the usual sup-norm on

$C := C[0, R]$. We say that $u \in C$ is *positive* if $u > 0$ on $[0, R]$. By a *positive solution* of (1.5) we understand a solution (u, v) with both u and v positive.

The rest of the paper is organized as follows. In Section 2 we present some preliminary results concerning the reformulation of system (1.5) as a fixed point problem as well as an existence result which involves the fixed point index. Two examples of existence of positive solutions for two-parameters systems with additive and multiplicative Lane–Emden nonlinearities are also provided. Section 3 is devoted to the lower (and upper) solutions method and some fixed point index estimations in the case when g_1 and g_2 have superlinear behavior near the origin. The main non-existence, existence and multiplicity result for a two-parameters Lane–Emden system is stated and proved in Section 4.

2. Preliminaries

Throughout this paper, the space $C^1 := C^1[0, R]$ will be understood with the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, while the product space $C^1 \times C^1$ will be endowed with the norm $\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\} + \max\{\|u'\|_\infty, \|v'\|_\infty\}$. We consider the closed subspace

$$C_M^1 := \{(u, v) \in C^1 \times C^1 : u'(0) = u(R) = 0 = v(R) = v'(0)\}$$

and its closed, convex cone

$$K := \{(u, v) \in C_M^1 : u \geq 0 \leq v \text{ on } [0, R]\}.$$

Also we denote $\tilde{B}(\rho) := \{(u, v) \in C_M^1 : \|(u, v)\| < \rho\}$ and $B(\rho) := K \cap \tilde{B}(\rho)$.

Let us define the linear operators

$$S : C \rightarrow C, \quad Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t) dt \quad (r \in (0, R]), \quad Su(0) = 0;$$

$$P : C \rightarrow C^1, \quad Pu(r) = \int_r^R u(t) dt \quad (r \in [0, R]).$$

It is easy to see that P is bounded and S is compact. Hence, the nonlinear operator $P \circ \varphi^{-1} \circ S : C \rightarrow C^1$ is compact. Denoting by N_{g_i} the Nemytskii operator associated to g_i ($i = 1, 2$), i.e.,

$$N_{g_i} : C \times C \rightarrow C, \quad N_{g_i}(u, v) = g_i(\cdot, u(\cdot), v(\cdot)) \quad (u, v \in C),$$

we have that N_{g_i} is continuous and takes bounded sets into bounded sets. We have the following fixed point reformulation of problem (1.5).

Proposition 2.1. *A couple of functions $(u, v) \in K$ is a solution of (1.5) iff it is a fixed point of the compact nonlinear operator*

$$\mathcal{D}_g : K \rightarrow K, \quad \mathcal{D}_g = \left(P \circ \varphi^{-1} \circ S \circ N_{g_1}, P \circ \varphi^{-1} \circ S \circ N_{g_2} \right).$$

In addition, for all $(u, v) \in K$, it holds

$$\|\mathcal{D}_g(u, v)\| < R + 1. \quad (2.1)$$

Proof. The inequality in (2.1) follows from the fact that the range of φ^{-1} is $(-1, 1)$. ■

If A is a subset of K , we set

$$\mathcal{K}(A) := \{T \mid T : A \rightarrow K \text{ is a compact operator}\}.$$

Also, given a bounded open (in K) subset $\mathcal{O}$ of K , we denote by $i(T, \mathcal{O})$ the fixed point index of the operator $T \in \mathcal{K}(\overline{\mathcal{O}})$ on $\mathcal{O}$ with respect to K [23]. Here and below, as usually, the closure and the boundary of the subsets of K will be understood with respect to the topology induced on K .

Proposition 2.2. For all $d \geq R + 1$ it holds

$$i(\mathcal{D}_g, B(d)) = 1.$$

In particular, problem (1.5) always has a solution.

Proof. We consider the homotopy $h : [0, 1] \rightarrow \mathcal{K}(\overline{B}(d))$, given by

$$h(t)(u, v) = t\mathcal{D}_g(u, v) \quad (t \in [0, 1], (u, v) \in \overline{B}(d)).$$

Since $h(t)(\overline{B}(d)) \subset B(R + 1)$, it follows that, for all $t \in [0, 1]$, $h(t)$ has no fixed points in $\partial B(d)$. Then, by the invariance under homotopy of fixed point index, we get

$$i(\mathcal{D}_g, B(d)) = i((0, 0), B(d)) = 1;$$

this also ensures the existence of the solution, by virtue of Proposition 2.1. ■

Now, we fix the constants

$$b \in (0, R), \quad 0 < \alpha < R - b, \quad d \geq R + \alpha + 1. \quad (2.2)$$

Also, we introduce the continuous function $\phi : C_M^1 \rightarrow \mathbb{R}$ defined by

$$\phi(u, v) = \min\{\min_{[0, b]} u, \min_{[0, b]} v\}$$

and we put

$$\begin{aligned} D_\alpha &:= \{(u, v) \in C_M^1 : \phi(u, v) < \alpha\}, \\ U_\alpha &:= \{(u, v) \in B(d) : \phi(u, v) < \alpha\}. \end{aligned}$$

Notice that D_α is an open set in C_M^1 , hence $U_\alpha = K \cap \widetilde{B}(d) \cap D_\alpha$ is a bounded open, nonempty $((0, 0) \in U_\alpha)$ subset of K .

Proposition 2.3. *If $(u, v) \in \partial U_\alpha$, then $\|(u, v)\| = d$ or $\phi(u, v) = \alpha$.*

Proof. Since $\partial U_\alpha \subset K \cap \partial_{C_M^1} [\tilde{B}(d) \cap D_\alpha] \subset \partial_{C_M^1} [\tilde{B}(d) \cap D_\alpha]$ we get that $(u, v) \in \overline{\tilde{B}(d) \cap D_\alpha} \setminus [\tilde{B}(d) \cap D_\alpha]$ (here, the closure is understood in C_M^1). From $(u, v) \in \overline{\tilde{B}(d) \cap D_\alpha}$ it follows

$$\|(u, v)\| \leq d \text{ and } \phi(u, v) \leq \alpha,$$

while from $(u, v) \notin \tilde{B}(d) \cap D_\alpha$ we obtain

$$\|(u, v)\| \geq d \text{ or } \phi(u, v) \geq \alpha,$$

which leads to $\|(u, v)\| = d$ or $\phi(u, v) = \alpha$. ■

Proposition 2.4. *If*

$$\mathcal{D}_g(u, v) + t(\alpha, \alpha) \neq (u, v), \text{ for all } t \in [0, 1] \text{ and } (u, v) \in K \text{ with } \phi(u, v) = \alpha, \quad (2.3)$$

then

$$i(\mathcal{D}_g, U_\alpha) = 0.$$

Proof. We consider the homotopy $h : [0, 1] \rightarrow \mathcal{K}(\overline{U}_\alpha)$ defined by

$$h(t)(u, v) = \mathcal{D}_g(u, v) + t(\alpha, \alpha), \quad (t \in [0, 1], (u, v) \in \overline{U}_\alpha).$$

We claim that $(0, 0) \notin (I - h(t))(\partial U_\alpha)$. Otherwise, there exists $t \in [0, 1]$ and $(u, v) \in \partial U_\alpha$ with $(u, v) = \mathcal{D}_g(u, v) + t(\alpha, \alpha)$. From Proposition 2.3, (u, v) satisfies $\|(u, v)\| = d$ or $\phi(u, v) = \alpha$. On account of (2.1), the first case leads to

$$d = \|(u, v)\| = \|\mathcal{D}_g(u, v) + t(\alpha, \alpha)\| \leq \|\mathcal{D}_g(u, v)\| + t\alpha < R + 1 + \alpha,$$

which contradicts the choice of d , while the second one contradicts (2.3). By contradiction, suppose that $i(\mathcal{D}_g, U_\alpha) \neq 0$. Then, by the invariance under homotopy of the fixed point index we infer

$$i(h(1), U_\alpha) = i(h(0), U_\alpha) = i(\mathcal{D}_g, U_\alpha) \neq 0.$$

From the existence property of the fixed point index, there exists $(u, v) \in U_\alpha$ such that

$$\mathcal{D}_g(u, v) + (\alpha, \alpha) = (u, v).$$

As $\mathcal{D}_g(u, v) \in K$, we get the contradiction

$$\alpha > \phi(u, v) = \phi(\mathcal{D}_g(u, v) + (\alpha, \alpha)) \geq \phi(\alpha, \alpha) = \alpha$$

and the proof is complete. ■

Theorem 2.1. If (2.3) is satisfied, then (1.5) has a nontrivial solution.

Proof. Notice that, if $(0, 0) \in (I - \mathcal{D}_g)(\overline{B}(d) \setminus U_\alpha)$ then the proof is done. Otherwise, using the excision property of fixed point index, together with Propositions 2.2 and 2.4, it follows

$$1 = i(\mathcal{D}_g, B(d)) = i(\mathcal{D}_g, U_\alpha) = 0,$$

which is a contradiction and the proof is complete. ■

In the terminology of [9,24,28], a function $g = g(r, s, t) : [0, R] \times [0, \infty)^2 \rightarrow [0, \infty)$ is said to be *quasi-monotone nondecreasing* with respect to t (resp. s) if for fixed r, s (resp. r, t) one has

$$g(r, s, t_1) \leq g(r, s, t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp. } g(r, s_1, t) \leq g(r, s_2, t) \text{ as } s_1 \leq s_2).$$

Theorem 2.2. Assume that each $g_i(r, s, t)$ is quasi-monotone nondecreasing with respect to both s, t , together with

$$\varphi\left(\frac{\alpha}{R-b}\right) < \frac{1}{R^{N-1}} \int_0^b \tau^{N-1} g_i(\tau, \alpha, \alpha) d\tau \quad (i = 1, 2). \quad (2.4)$$

Then (1.5) has a nontrivial solution.

Proof. We show that (2.3) holds true. Otherwise, there exists $t \in [0, 1]$ and $(u, v) \in K$ with $\phi(u, v) = \alpha$ such that $\mathcal{D}_g(u, v) + t(\alpha, \alpha) = (u, v)$. Without loss of generality, we may assume that $\min_{[0, b]} u \leq \min_{[0, b]} v$. Let $r_0 \in [0, b]$ be such that $\phi(u, v) = \alpha = u(r_0)$; hence, $u(r) \geq u(r_0)$ for all $r \in [0, b]$. Then,

$$\begin{aligned} \alpha = u(r_0) &= P \circ \varphi^{-1} \circ S \circ N_{g_1}(r_0) + t\alpha \geq \\ &\int_{r_0}^R \varphi^{-1} \left(\frac{1}{t^{N-1}} \int_0^t \tau^{N-1} g_1(\tau, u(\tau), v(\tau)) d\tau \right) dt \geq \\ &\int_b^R \varphi^{-1} \left(\frac{1}{t^{N-1}} \int_0^b \tau^{N-1} g_1(\tau, u(\tau), v(\tau)) d\tau \right) dt \geq \\ &\int_b^R \varphi^{-1} \left(\frac{1}{R^{N-1}} \int_0^b \tau^{N-1} g_1(\tau, u(r_0), v(\tau)) d\tau \right) dt \geq \\ &\int_b^R \varphi^{-1} \left(\frac{1}{R^{N-1}} \int_0^b \tau^{N-1} g_1(\tau, \alpha, \alpha) d\tau \right) dt = \end{aligned}$$

$$(R-b)\varphi^{-1}\left(\frac{1}{R^{N-1}}\int_0^b\tau^{N-1}g_1(\tau,\alpha,\alpha)d\tau\right),$$

which contradicts (2.4) and completes the proof. ■

We shall need to invoke the following two lemmas, which are immediate from Lemmas 2.3, respectively 2.4 in [24].

Lemma 2.1. Assume that $g_1(r, s, t)$ (resp. $g_2(r, s, t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and it holds

$$(H_g^1) \text{ (i) } g_1(r, \xi, 0) > 0 < g_2(r, 0, \xi), \quad \forall \xi > 0, \forall r \in (0, R];$$

$$(ii) \quad g_1(r, 0, \xi) > 0 < g_2(r, \xi, 0), \quad \forall \xi > 0, \forall r \in (0, R].$$

If (u, v) is a nontrivial solution of problem (1.5), then (u, v) is a positive solution with both u and v strictly decreasing.

Lemma 2.2. Assume that

$$(H_g^2) \text{ (i) } g_1(r, s, t) > 0 < g_2(r, s, t), \quad \forall s, t > 0, \quad \forall r \in (0, R];$$

$$(ii) \quad g_1(r, \xi, 0) = g_2(r, 0, \xi) = 0, \quad \forall \xi > 0, \quad \forall r \in (0, R].$$

If (u, v) is a nontrivial solution of problem (1.5), then (u, v) is a positive solution with both u and v strictly decreasing.

As an immediate application of Theorem 2.2, we consider the two-parameters $(\lambda_1 > 0 < \lambda_2)$ problem

$$\begin{cases} [r^{N-1}\varphi(u')] + r^{N-1}\lambda_1 g_1(r, u, v) = 0, \\ [r^{N-1}\varphi(v')] + r^{N-1}\lambda_2 g_2(r, u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \quad (2.5)$$

under the hypothesis:

(H_g) The functions $g_{1,2}(r, s, t) : [0, R] \times [0, \infty)^2 \rightarrow [0, \infty)$ are continuous, quasi-monotone nondecreasing with respect to both s, t and satisfy

$$\int_0^b \tau^{N-1} g_i(\tau, \alpha, \alpha) d\tau > 0 \quad (i = 1, 2).$$

Theorem 2.3. Under hypothesis (H_g) , there exist $\lambda_1^* > 0 < \lambda_2^*$ such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem (2.5) has a nontrivial solution.

If in addition (H_g^1) or (H_g^2) are satisfied, then problem (2.5) has at least one positive solution.

Proof. We apply Theorem 2.2 with $\lambda_i g_i$ instead of g_i ($i = 1, 2$), by noticing that (2.4) is satisfied for all

$$\lambda_i > \frac{R^{N-1} \varphi\left(\frac{\alpha}{R-b}\right)}{\int_0^b \tau^{N-1} g_i(\tau, \alpha, \alpha) d\tau} =: \lambda_i^*, \quad (i = 1, 2).$$

The positivity of the nontrivial solution follows by Lemmas 2.1 and 2.2. ■

Corollary 2.1. *Under hypothesis (H_g) , there exist $\lambda_1^* > 0 < \lambda_2^*$ such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem*

$$\begin{cases} \mathcal{M}(u) + \lambda_1 g_1(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + \lambda_2 g_2(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)} \end{cases} \quad (2.6)$$

has a nontrivial radial solution.

If in addition (H_g^1) or (H_g^2) are satisfied, then problem (2.6) has at least one solution (u, v) with both u and v positive in $\mathcal{B}(R)$ and radially strictly decreasing.

The following examples illustrate the application of the previous result in the case of two-parameters problems involving nonlinearities of Lane–Emden type.

Example 2.1. Let p_1, q_1, p_2, q_2 be nonnegative exponents and the functions $\mu_1, \mu_2 : [0, R] \rightarrow [0, \infty)$ be continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$. Then there exist $\lambda_1^* > 0 < \lambda_2^*$, such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, the system

$$\begin{cases} \mathcal{M}(u) + \lambda_1 \mu_1(|x|)[u^{p_1} + v^{q_1}] = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + \lambda_2 \mu_2(|x|)[u^{p_2} + v^{q_2}] = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)} \end{cases} \quad (2.7)$$

has a solution (u, v) with both u and v positive in $\mathcal{B}(R)$ and radially strictly decreasing. If, in addition $q_1, p_2 > 0$, then the same holds true for the system (1.4). These follow from Corollary 2.1.

Remark 2.1. The results for the above systems appear as being interesting in the cases when, for system (2.7) it holds $\min\{p_1, q_1\} \geq 1$ or $\min\{p_2, q_2\} \geq 1$, while for system (1.4) one has $p_1 + q_1 \geq 1$ or $p_2 + q_2 \geq 1$. Indeed, it can be easily seen that if $\min\{p_1, q_1\} < 1$ and $\min\{p_2, q_2\} < 1$ in the case of system (2.7), and if $p_1 + q_1 < 1$ and $p_2 + q_2 < 1$ for (1.4), the inequality (2.4) is always satisfied for sufficiently small α . This means that for all $\lambda_1 > 0 < \lambda_2$, problems (2.7) and (1.4) have a solution (u, v) with both u and v positive and radially strictly decreasing. The conclusion for system (2.7) is in accordance with the one of Example 3.4 (ii) in [24], where $u^{p_i} + v^{q_i}$ is considered instead of $\lambda_i \mu_i(|x|)[u^{p_i} + v^{q_i}]$ ($i = 1, 2$).

3. Lower and upper solutions; fixed point index estimations

A lower solution of (1.5) is a couple of nonnegative functions $(\alpha_u, \alpha_v) \in C^1 \times C^1$, such that $\|\alpha'_u\|_\infty < 1$, $\|\alpha'_v\|_\infty < 1$, the mappings $r \mapsto r^{N-1}\varphi(\alpha'_u(r))$, $r \mapsto r^{N-1}\varphi(\alpha'_v(r))$ are of class C^1 on $[0, R]$ and satisfies

$$\begin{cases} [r^{N-1}\varphi(\alpha'_u)]' + r^{N-1}g_1(r, \alpha_u, \alpha_v) \geq 0, \\ [r^{N-1}\varphi(\alpha'_v)]' + r^{N-1}g_2(r, \alpha_u, \alpha_v) \geq 0, \\ \alpha_u(R) = 0, \quad \alpha_v(R) = 0. \end{cases} \quad (3.1)$$

An *upper solution* $(\beta_u, \beta_v) \in C^1 \times C^1$ is defined by reversing the first two inequalities in (3.1) and asking $\beta_u(R) \geq 0$, $\beta_v(R) \geq 0$ instead of $\alpha_u(R) = 0$, $\alpha_v(R) = 0$.

Lemma 3.1. Assume that (1.5) has a lower solution (α_u, α_v) and $g_1(r, s, t)$ (resp. $g_2(r, s, t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and let

$$\mathcal{A}_\alpha = \mathcal{A}_{(\alpha_u, \alpha_v)} := \{(u, v) \in K : \alpha_u \leq u, \alpha_v \leq v\}.$$

Then, the following hold true:

(i) problem (1.5) has always a solution in $\mathcal{A}_\alpha$;

(ii) if (1.5) has an unique solution (u_0, v_0) in $\mathcal{A}_\alpha$ and there exists $\rho_0 > 0$ such that $\bar{B}((u_0, v_0), \rho_0) := \{(u, v) \in K : \|(u - u_0, v - v_0)\| \leq \rho_0\} \subset \mathcal{A}_\alpha$, then

$$i(\mathcal{D}_g, B((u_0, v_0), \rho)) = 1, \quad \text{for all } 0 < \rho \leq \rho_0.$$

Proof. This makes use of some ideas from the proofs of Proposition 4.1 and Lemma 4.2 in [24].

(i) We define the continuous functions $\Gamma_1, \Gamma_2 : [0, R] \times [0, \infty)^2 \rightarrow [0, \infty)$,

$$\Gamma_1(r, s, t) = g_1(r, \gamma_1(r, s), \gamma_2(r, t)) - s + \gamma_1(r, s),$$

$$\Gamma_2(r, s, t) = g_2(r, \gamma_1(r, s), \gamma_2(r, t)) - t + \gamma_2(r, t),$$

with γ_i given by

$$\gamma_1(r, s) = \max\{\alpha_u(r), s\}, \quad \gamma_2(r, t) = \max\{\alpha_v(r), t\}$$

and we consider the modified problem

$$\begin{cases} [r^{N-1}\varphi(u')] + r^{N-1}\Gamma_1(r, u, v) = 0, \\ [r^{N-1}\varphi(v')] + r^{N-1}\Gamma_2(r, u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0). \end{cases} \quad (3.2)$$

From Proposition 2.2 problem (3.2) has at least one solution. We show now that if (u, v) is a solution of (3.2), then $(u, v) \in \mathcal{A}_\alpha$. We only prove that $\alpha_u \leq u$ on $[0, R]$ (similar reasoning for $\alpha_v \leq v$).

By contradiction, suppose that exists $r_0 \in [0, R]$ such that

$$\max_{[0, R]}(\alpha_u - u) = \alpha_u(r_0) - u(r_0) > 0. \quad (3.3)$$

If $r_0 \in (0, R)$, then $\alpha'_u(r_0) = u'(r_0)$ and there exists a sequence $\{r_k\} \subset (0, r_0)$ converging to r_0 such that $\alpha'_u(r_k) - u'(r_k) \geq 0$. Since φ is increasing, this implies

$$r_k^{N-1}\varphi(\alpha'_u(r_k)) - r_0^{N-1}\varphi(\alpha'_u(r_0)) \geq r_k^{N-1}\varphi(u'(r_k)) - r_0^{N-1}\varphi(u'(r_0)),$$

which yields

$$[r^{N-1}\varphi(\alpha'_u(r))]_{r=r_0}' \leq [r^{N-1}\varphi(u'(r))]_{r=r_0}'.$$

Hence, because (α_u, α_v) is a lower solution of (1.5) and g_1 is quasi-monotone nondecreasing with respect to t , we obtain

$$\begin{aligned} [r^{N-1}\varphi(\alpha'_u(r))]_{r=r_0}' &\leq [r^{N-1}\varphi(u'(r))]_{r=r_0}' \\ &= r_0^{N-1}[-g_1(r_0, \alpha_u(r_0), \gamma_2(r_0, v(r_0))) + u(r_0) - \alpha_u(r_0)] \\ &< r_0^{N-1}[-g_1(r_0, \alpha_u(r_0), \gamma_2(r_0, v(r_0)))] \\ &\leq r_0^{N-1}[-g_1(r_0, \alpha_u(r_0), \alpha_v(r_0))] \\ &\leq [r^{N-1}\varphi(\alpha'_u(r))]_{r=r_0}', \end{aligned}$$

a contradiction. If $r_0 = R$ then $\alpha_u(R) - u(R) > 0$, contradicting $\alpha_u(R) = 0 = u(R)$. Finally, if $r_0 = 0$, then there exists $r_1 \in (0, R]$ such that $\alpha_u(r) - u(r) > 0$ for all $r \in [0, r_1]$ and $\alpha'_u(r_1) - u'(r_1) \leq 0$. It follows that

$$r_1^{N-1}\varphi(\alpha'_u(r_1)) \leq r_1^{N-1}\varphi(u'(r_1)).$$

Integrating the first equation in problem (3.2) from 0 to r_1 and using the fact that (α_u, α_v) is a lower solution of (1.5) and that g_1 is quasi-monotone nondecreasing with respect to t , we get

$$\begin{aligned} r_1^{N-1}\varphi(u'(r_1)) &= \int_0^{r_1} r^{N-1}[-g_1(r, \alpha_u(r), \gamma_2(r, v(r))) + u(r) - \alpha_u(r)]dr \\ &< \int_0^{r_1} r^{N-1}[-g_1(r, \alpha_u(r), \gamma_2(r, v(r)))]dr \\ &\leq \int_0^{r_1} r^{N-1}[-g_1(r, \alpha_u(r), \alpha_v(r)))]dr \\ &\leq \int_0^{r_1} [r^{N-1}\varphi(\alpha'_u(r))]dr \\ &= r_1^{N-1}\varphi(\alpha'_u(r_1)), \end{aligned}$$

a contradiction. Consequently, $\alpha_u(r) \leq u(r)$ for all $r \in [0, R]$.

(ii) Let $\mathcal{D}_\Gamma : K \rightarrow K$ be the fixed point operator associated to (3.2). It is easy to check that $\mathcal{D}_\Gamma = \mathcal{D}_g$ on $\mathcal{A}_\alpha$ and, as any fixed point (u, v) of $\mathcal{D}_\Gamma$ is contained in $\mathcal{A}_\alpha$, it is a fixed point of $\mathcal{D}_g$. So, (u_0, v_0) is the unique fixed point of $\mathcal{D}_\Gamma$. Since

$$(0, 0) \notin (I - \mathcal{D}_\Gamma)(\overline{B}(d) \setminus B((u_0, v_0), \rho_0)),$$

for d sufficiently large, using again Proposition 2.2 together with the excision property of the fixed point index one gets

$$1 = i(\mathcal{D}_\Gamma, B(d)) = i(\mathcal{D}_\Gamma, B((u_0, v_0), \rho)), \text{ for all } 0 < \rho \leq \rho_0.$$

The conclusion follows from the fact that $\mathcal{D}_\Gamma = \mathcal{D}_g$ on $\mathcal{A}_\alpha \supset \overline{B}((u_0, v_0), \rho_0)$. ■

Lemma 3.2. Assume that g_1 and g_2 satisfy hypothesis (H_g^2) in Lemma 2.2. If there is some $M > 0$ such that either

$$\lim_{s \rightarrow 0_+} \frac{g_1(r, s, t)}{s} = 0 \text{ uniformly with } r \in [0, R], t \in [0, M] \quad (3.4)$$

or

$$\lim_{t \rightarrow 0_+} \frac{g_2(r, s, t)}{t} = 0 \text{ uniformly with } r \in [0, R], s \in [0, M], \quad (3.5)$$

then there exists $\rho_0 > 0$ such that

$$i(\mathcal{D}_g, B(\rho)) = 1 \text{ for all } 0 < \rho \leq \rho_0.$$

Proof. Consider the homotopy $h : [0, 1] \rightarrow \mathcal{K}(K)$, given by

$$h(t)(u, v) = t\mathcal{D}_g(u, v).$$

Using the same arguments as in the proof of Lemma 4.3 in [24] it can be shown that there exists $\rho_0 > 0$ such that

$$(u, v) \neq h(t)(u, v), \text{ for all } t \in [0, 1] \text{ and } (u, v) \in \overline{B}(\rho_0) \setminus \{(0, 0)\}.$$

Then, by the invariance under homotopy of the fixed point index we have that

$$i(\mathcal{D}_g, B(\rho)) = i((0, 0), B(\rho)) = 1, \text{ for all } \rho \in (0, \rho_0],$$

which completes the proof. ■

4. Non-existence, existence and multiplicity for a Lane–Emden system

In this section we deal with the Lane–Emden system (1.4) under the hypothesis:

(H) The functions $\mu_1, \mu_2 : [0, R] \rightarrow [0, \infty)$ are continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$, $0 < q_1, p_2 < \infty$ and $1 < p_1, q_2 < \infty$.

Notice that, in view of Remark 2.1 hypothesis (H) fits with the interesting case emphasized for system (1.4).

We have to consider the corresponding radial problem

$$\begin{cases} [r^{N-1}\varphi(u')] + r^{N-1}\lambda_1\mu_1(r)u^{p_1}v^{q_1} = 0, \\ [r^{N-1}\varphi(v')] + r^{N-1}\lambda_2\mu_2(r)u^{p_2}v^{q_2} = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0). \end{cases} \quad (4.1)$$

Setting

$$\Sigma := \{(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 > 0 \text{ and (4.1) has at least one positive solution}\},$$

it is easy to see that the set Σ is nonempty and unbounded in both directions of the axes $0\lambda_1$ and $0\lambda_2$ (see Theorem 2.3).

Lemma 4.1. Assume (H). Then, the following are true:

- (i) There exist $\Lambda_1, \Lambda_2 > 0$, such that $\Sigma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ and for all $(\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$, problem (4.1) has only the trivial solution.
- (ii) If $(\bar{\lambda}_1, \bar{\lambda}_2) \in \Sigma$, then $[\bar{\lambda}_1, +\infty) \times [\bar{\lambda}_2, +\infty) \subset \Sigma$.
- (iii) If $(\bar{\lambda}_1, \bar{\lambda}_2) \in \Sigma$, then for all $(\lambda_1, \lambda_2) \in (\bar{\lambda}_1, +\infty) \times (\bar{\lambda}_2, +\infty)$, problem (4.1) has at least two positive solutions.

Proof. This improves and simplifies some ideas from the proof of Theorem 5.1 in [24].

(i) Let $\lambda_1, \lambda_2 > 0$ and (u, v) be a positive solution of (4.1). It follows from Lemma 2.2 that u and v are both strictly decreasing. Integrating the first equation in (4.1) on $[0, r]$, one obtains

$$-r^{N-1}\varphi(u'(r)) = \lambda_1 \int_0^r t^{N-1} \mu_1(t) u^{p_1}(t) v^{q_1}(t) dt, \quad \text{for all } r \in [0, R].$$

Since u, v are strictly decreasing on $[0, R]$, we deduce

$$\begin{aligned} -r^{N-1}u'(r) &\leq -r^{N-1}\varphi(u'(r)) \\ &\leq \lambda_1 \mu_1^M(r) u^{p_1}(0) v^{q_1}(0) r^N / N, \end{aligned}$$

where $\mu_i^M := \max_{[0, R]} \mu_i$ ($i = 1, 2$). Integrating on $[0, R]$ we get

$$u(0) \leq \lambda_1 \mu_1^M u^{p_1}(0) v^{q_1}(0) R^2 / (2N). \quad (4.2)$$

Analogously, one has

$$v(0) \leq \lambda_2 \mu_2^M u^{p_2}(0) v^{q_2}(0) R^2 / (2N). \quad (4.3)$$

From $0 < u(0), v(0) < R$ and $p_1, q_2 > 1$ we obtain

$$\lambda_i > 2N/(\mu_i^M R^{p_i+q_i+1}) > 0 \quad (i = 1, 2). \quad (4.4)$$

Consider now the nonempty (Example 2.1) sets

$$\Sigma_1 := \{\lambda_1 > 0 : \exists \lambda_2 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Sigma\},$$

$$\Sigma_2 := \{\lambda_2 > 0 : \exists \lambda_1 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Sigma\}$$

and let

$$(0 <) \Lambda_i := \inf \Sigma_i (< +\infty) \quad (i = 1, 2).$$

It follows that $\Sigma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ and for all $\lambda_1, \lambda_2 \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$, problem (4.1) has only the trivial solution by virtue of Lemma 2.2.

(ii) Let $(\lambda_1^0, \lambda_2^0) \in [\bar{\lambda}_1, +\infty) \times [\bar{\lambda}_2, +\infty)$ be arbitrarily chosen and $(\bar{u}, \bar{v})$ be a positive solution for (4.1) with $\lambda_1 = \bar{\lambda}_1$ and $\lambda_2 = \bar{\lambda}_2$. Then, $(\bar{u}, \bar{v})$ is a lower solution of (4.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. From Proposition 3.1 (i) and the fact that $(\bar{u}, \bar{v})$ is positive, we obtain $(\lambda_1^0, \lambda_2^0) \in \Sigma$.

(iii) From (ii) we get that $(\bar{\lambda}_1, +\infty) \times (\bar{\lambda}_2, +\infty) \subset \Sigma$ and let $(\lambda_1^0, \lambda_2^0) \in (\bar{\lambda}_1, +\infty) \times (\bar{\lambda}_2, +\infty)$. It remains to show that problem (4.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ has a second positive solution. For this, let $(\bar{u}, \bar{v})$ be the lower solution constructed as above. We fix (u_0, v_0) a positive solution of (4.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ such that $(u_0, v_0) \in \mathcal{A} := \mathcal{A}_{(\bar{u}, \bar{v})}$.

Now, we claim that there exists $\varepsilon > 0$ such that $\bar{B}((u_0, v_0), \varepsilon) \subset \mathcal{A}$. For all $r \in [0, R/2]$, we have

$$\begin{aligned} \bar{u}(r) &= \int_r^R \varphi^{-1} \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} [\bar{\lambda}_1 \mu_1(s) \bar{u}^{p_1}(s) \bar{v}^{q_1}(s)] ds \right) dt \\ &< \int_r^R \varphi^{-1} \left(\frac{1}{t^{N-1}} \int_0^t s^{N-1} [\lambda_1^0 \mu_1^M u_0^{p_1}(s) v_0^{q_1}(s)] ds \right) dt \\ &= u_0(r). \end{aligned}$$

Analogously we obtain that $\bar{v}(r) < v_0(r)$ on $[0, R/2]$. So, we can find $\varepsilon_1 > 0$ such that if $(u, v) \in K$ then

$$\|u - u_0\|_\infty \leq \varepsilon_1 \Rightarrow \bar{u} \leq u \text{ and } \|v - v_0\|_\infty \leq \varepsilon_1 \Rightarrow \bar{v} \leq v \text{ on } [0, R/2]. \quad (4.5)$$

On the other hand, for $r \in [R/2, R]$ one obtains $u'_0(r) < \bar{u}'(r)$ and $v'_0(r) < \bar{v}'(r)$. Thus, there is some $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $(u, v) \in K$, then

$$\|u' - u'_0\|_\infty \leq \varepsilon_2 \Rightarrow \bar{u}' > u' \text{ and } \|v' - v'_0\|_\infty \leq \varepsilon_2 \Rightarrow \bar{v}' > v' \text{ on } [R/2, R].$$

From

$$u(r) = - \int_r^R u'(s) ds > - \int_r^R \bar{u}'(s) ds = \bar{u}(r)$$

we have that $u > \bar{u}$ (and, similarly $v > \bar{v}$) on $[R/2, R)$. This means that

$$\|u' - u'_0\|_\infty \leq \varepsilon_2 \Rightarrow \bar{u} \leq u \text{ and } \|v' - v'_0\|_\infty \leq \varepsilon_2 \Rightarrow \bar{v} \leq v \text{ on } [R/2, R]. \quad (4.6)$$

The claim follows from (4.5) and (4.6), by taking $\varepsilon \in (0, \varepsilon_2)$.

Next, if (4.1) has a second solution contained in $\mathcal{A}$, then it is nontrivial and the proof is complete. If not, by Lemma 3.1 we infer that

$$i(\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}, B((u_0, v_0), \rho)) = 1 \text{ for all } 0 < \rho \leq \varepsilon,$$

where $\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}$ stands for the fixed point operator associated to problem (4.1) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. Also, from Proposition 2.2 we have

$$i(\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}, B(\rho)) = 1 \text{ for all } \rho \geq R + 1,$$

and from Lemma 3.2 we get

$$i(\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}, B(\rho)) = 1 \text{ for all } \rho > 0 \text{ sufficiently small.}$$

Let $\rho_1, \rho_2 > 0$ be sufficiently small and $\rho_3 \geq R + 1$ be such that $\bar{B}((u_0, v_0), \rho_1) \cap \bar{B}(\rho_2) = \emptyset$ and $\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}(\rho_2) \subset B(\rho_3)$. From the additivity-excision property of the fixed point index it follows that

$$i(\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}, B(\rho_3) \setminus [\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}(\rho_2)]) = -1.$$

Therefore, $\mathcal{D}_{(\lambda_1^0, \lambda_2^0)}$ has a fixed point $(u, v) \in B(\rho_3) \setminus [\bar{B}((u_0, v_0), \rho_1) \cup \bar{B}(\rho_2)]$. But this means that (4.1) has a second positive solution. ■

Now, for $\theta \in (0, \pi/2)$, we denote

$$\mathcal{S}(\theta) := \{\lambda > 0 : (\lambda \cos \theta, \lambda \sin \theta) \in \Sigma\},$$

which we know that is a nonempty set, and we rewrite problem (4.1) in the form

$$\begin{cases} [r^{N-1}\varphi(u')] + r^{N-1}\lambda \cos \theta \mu_1(r)u^{p_1}v^{q_1} = 0, \\ [r^{N-1}\varphi(v')] + r^{N-1}\lambda \sin \theta \mu_2(r)u^{p_2}v^{q_2} = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \quad (4.7)$$

where $\lambda > 0$ is a real parameter.

Proposition 4.1. *There exists a continuous function $\Lambda : (0, \pi/2) \rightarrow (0, \infty)$ such that*

$$\lim_{\theta \rightarrow 0} \Lambda(\theta) \sin \theta - \Lambda_2 = 0 = \lim_{\theta \rightarrow \pi/2} \Lambda(\theta) \cos \theta - \Lambda_1 \quad (4.8)$$

and the following hold true:

(i) $\Lambda(\theta) \in \mathcal{S}(\theta)$, for every $\theta \in (0, \pi/2)$;

(ii) system (4.1) has at least two positive solutions, for all $(\lambda_1, \lambda_2) \in (\Lambda(\theta) \cos \theta, +\infty) \times (\Lambda(\theta) \sin \theta, +\infty)$ and for every $\theta \in (0, \pi/2)$.

Proof. For each $\theta \in (0, \pi/2)$, let

$$\Lambda(\theta) := \inf \mathcal{S}(\theta). \quad (4.9)$$

Note that $\Lambda(\theta)$ is $< \infty$ because $\mathcal{S}(\theta) \neq \emptyset$ and > 0 by Lemma 4.1 (i). We first prove statements (i) and (ii).

(i) Let $\{\lambda^k\} \subset \mathcal{S}(\theta)$ be a decreasing sequence converging to $\Lambda(\theta)$ and $(u_k, v_k) \in K$ with $u_k > 0 < v_k$ on $[0, R]$ be such that

$$\begin{aligned} u_k &= P \circ \varphi^{-1} \circ S \circ [\lambda^k \cos \theta \mu_1 u_k^{p_1} v_k^{q_1}], \\ v_k &= P \circ \varphi^{-1} \circ S \circ [\lambda^k \sin \theta \mu_2 u_k^{p_2} v_k^{q_2}]. \end{aligned}$$

From (2.1) and Arzela–Ascoli theorem we obtain that there exists $(u, v) \in K$ such that, passing eventually to a subsequence, $\{(u_k, v_k)\}$ converges to (u, v) in $C \times C$ – with the usual product topology. Hence, $u \geq 0 \leq v$ and

$$\begin{aligned} u &= P \circ \varphi^{-1} \circ S \circ [\Lambda(\theta) \cos \theta \mu_1 u^{p_1} v^{q_1}], \\ v &= P \circ \varphi^{-1} \circ S \circ [\Lambda(\theta) \sin \theta \mu_2 u^{p_2} v^{q_2}]. \end{aligned}$$

From (4.2) and (4.3) we have that

$$u_k(0) \leq \lambda^k \cos \theta \mu_1^M u_k^{p_1}(0) v_k^{q_1}(0) R^2 / (2N)$$

and

$$v_k(0) \leq \lambda^k \sin \theta \mu_2^M u_k^{p_2}(0) v_k^{q_2}(0) R^2 / (2N),$$

which, taking into account that $0 < u_k(0), v_k(0) < R$, imply

$$u_k^{p_1-1}(0) > \frac{2N}{\lambda^1 \mu_1^M R^{q_1} \cos \theta}$$

and

$$v_k^{q_2-1}(0) > \frac{2N}{\lambda^1 \mu_2^M R^{p_2} \sin \theta}.$$

These ensure that there is a constant $c > 0$ such that $u_k(0), v_k(0) > c$ for all k . This leads to $u(0), v(0) \geq c$, hence by Lemma 2.2 we get $u > 0 < v$ on $[0, R)$. Consequently, $\Lambda(\theta) \in \mathcal{S}(\theta)$.

(ii) This follows from statement (iii) in Lemma 4.1.

Now we show that Λ is continuous at each $\theta_0 \in (0, \pi/2)$. Otherwise, we can find some $\varepsilon \in (0, \Lambda(\theta_0))$ such that for all sufficiently large $n \in \mathbb{N}$, there exists $\theta_n \in (\theta_0 - 1/n, \theta_0 + 1/n) \subset (0, \pi/2)$ with $\Lambda(\theta_n) \notin (\Lambda(\theta_0) - \varepsilon, \Lambda(\theta_0) + \varepsilon)$. Assume that $\Lambda(\theta_n) \geq \Lambda(\theta_0) + \varepsilon$ for infinitely many $n \in \mathbb{N}$. Then, for a subsequence of $\{\theta_n\}$ – still denoted $\{\theta_n\}$, we can write

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \cos \theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right) \cos \theta_n,$$

respectively,

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \sin \theta_n \geq \left(\Lambda(\theta_0) + \frac{\varepsilon}{2}\right) \sin \theta_n.$$

Also, there exists $n_0 \in \mathbb{N}$ such that $(\Lambda(\theta_0) + \varepsilon/2) \cos \theta_n > \Lambda(\theta_0) \cos \theta_0$ and $(\Lambda(\theta_0) + \varepsilon/2) \sin \theta_n > \Lambda(\theta_0) \sin \theta_0$, for all $n \geq n_0$. Therefore, for all $n \geq n_0$ we have

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \cos \theta_n > \Lambda(\theta_0) \cos \theta_0,$$

respectively,

$$\left(\Lambda(\theta_n) - \frac{\varepsilon}{2}\right) \sin \theta_n > \Lambda(\theta_0) \sin \theta_0.$$

Hence, using that $\Lambda(\theta_0) \in \mathcal{S}(\theta_0)$ and taking into account Lemma 4.1 (ii) it follows that $((\Lambda(\theta_n) - \varepsilon/2) \cos \theta_n, (\Lambda(\theta_n) - \varepsilon/2) \sin \theta_n) \in \Sigma$, therefore $\Lambda(\theta_n) - \varepsilon/2 \in \mathcal{S}(\theta_n)$, which contradicts the definition of $\Lambda(\theta_n)$. Under the assumption $\Lambda(\theta_n) \leq \Lambda(\theta_0) - \varepsilon$ for infinitely many $n \in \mathbb{N}$, a similar reasoning leads to a contradiction of the definition of $\Lambda(\theta_0)$.

To prove (4.8), let $\{\theta_n\} \subset (0, \pi/2)$ be a sequence with $\theta_n \rightarrow \pi/2$, as $n \rightarrow \infty$. We have to show that

$$\Lambda(\theta_n) \cos \theta_n \rightarrow \Lambda_1, \text{ as } n \rightarrow \infty.$$

In this regard, it suffices to prove that any subsequence of $\{\theta_n\}$, still denoted by $\{\theta_n\}$, contains a subsequence $\{\theta_{n_k}\}$ with

$$\Lambda(\theta_{n_k}) \cos \theta_{n_k} \rightarrow \Lambda_1, \text{ as } k \rightarrow \infty.$$

From the definition of Λ_1 there exists a sequence $\{\lambda_1^k\} \subset \Sigma_1$ with $\lambda_1^k \rightarrow \Lambda_1$, as $k \rightarrow \infty$. Since $\theta_n \rightarrow \pi/2$, using Lemma 4.1 (ii) by an inductive standard reasoning we can find a sequence $\{r_k\} \subset (0, \infty)$ and a subsequence $\{\theta_{n_k}\} \subset \{\theta_n\}$ which, for all $k \in \mathbb{N}$, satisfy

$$r_k \cos \theta_{n_k} = \lambda_1^k \tag{4.10}$$

and

$$(r_k \cos \theta_{n_k}, r_k \sin \theta_{n_k}) \in \Sigma.$$

By the definition of the mapping Λ we infer $\Lambda(\theta_{n_k}) \leq r_k$, which leads to $\Lambda(\theta_{n_k}) \cos \theta_{n_k} \leq r_k \cos \theta_{n_k}$. By virtue of (4.10) and the definition of Λ_1 we get

$$\Lambda_1 \leq \Lambda(\theta_{n_k}) \cos \theta_{n_k} \leq r_k \cos \theta_{n_k} = \lambda_1^k \rightarrow \Lambda_1, \text{ as } k \rightarrow \infty,$$

which fulfills our aim. Similarly, it can be proved that $\Lambda(\theta_n) \sin \theta_n \rightarrow \Lambda_2$ when $\theta_n \rightarrow 0$ as $n \rightarrow \infty$, and the proof is complete. ■

Theorem 4.1. Assume (H). Then, there exist $\Lambda_1, \Lambda_2 > 0$ and a continuous function $\Lambda : (0, \pi/2) \rightarrow (0, +\infty)$, generating the curve

$$(\Gamma) \begin{cases} \lambda_1(\theta) = \Lambda(\theta) \cos \theta \\ \lambda_2(\theta) = \Lambda(\theta) \sin \theta \end{cases}, \quad \theta \in (0, \pi/2)$$

such that

(i) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$;

(ii) the following asymptotic behaviors hold

$$\lim_{\theta \rightarrow \pi/2} \lambda_2(\theta) = +\infty = \lim_{\theta \rightarrow 0} \lambda_1(\theta), \quad (4.11)$$

$$\lim_{\theta \rightarrow 0} \lambda_2(\theta) - \Lambda_2 = 0 = \lim_{\theta \rightarrow \pi/2} \lambda_1(\theta) - \Lambda_1; \quad (4.12)$$

(iii) Γ separates the first quadrant $(0, +\infty) \times (0, +\infty)$ in two disjoint sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (1.4) has zero, at least one or at least two radial positive solutions, according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$.

Proof. The existence of Λ_1 and Λ_2 is given in Lemma 4.1 (i) and the existence of the continuous function Λ is proved in Proposition 4.1.

(i) This follows from Proposition 4.1 (i) together with Lemma 4.1 (i).

(ii) Equalities in (4.11) are immediate from the inequalities

$$\Lambda(\theta) \geq \frac{\Lambda_1}{\cos \theta} \text{ and } \Lambda(\theta) \geq \frac{\Lambda_2}{\sin \theta},$$

while (4.12) follows from Proposition 4.1.

(iii) Using Proposition 4.1 and taking into account the definition of $\Lambda(\theta)$ (see (4.9)) we get the conclusion. ■

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