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Extremal functions of generalized critical Hardy inequalities

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Abstract

In this paper, we show the existence and non-existence of minimizers of the following minimization problems which include an open problem mentioned by Horiuchi and Kumlin [20]:

$$G_a := \inf_{u \in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N \, dx}{\left(\int_\Omega |u|^q f_{a,\beta}(x) dx\right)^{\frac{N}{q}}}, \text{ where } f_{a,\beta}(x) := |x|^{-N} \left(\log \frac{aR}{|x|}\right)^{-\beta}.$$

First, we give an answer to the open problem when $\Omega = B_R(0)$. Next, we investigate the minimization problems on general bounded domains. In this case, the results depend on the shape of the domain Ω. Finally, symmetry breaking property of the minimizers is proved for sufficiently large β.

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Keywords: Critical Hardy inequality; Optimal constant; Extremal function; Symmetry breaking

1. Introduction

Let $N \geq 2$, Ω be a bounded domain in $\mathbb{R}^N$, $0 \in \Omega$, and $1 < p < N$. The classical Hardy inequality holds for all $u \in W_0^{1,p}(\Omega)$ as follows:

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$$\left(\frac{N-p}{p}\right)^p \int_\Omega \frac{|u|^p}{|x|^p} dx \le \int_\Omega |\nabla u|^p dx, \tag{1}$$

where $W_0^{1,p}(\Omega)$ is a completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$. We refer the celebrated work by G.H. Hardy [17]. The inequality (1) has great applications to partial differential equations, for example stability, global existence, and instantaneous blow-up and so on. See e.g. [6], [3]. It is well-known that in (1) $(\frac{N-p}{p})^p$ is the optimal constant and is not attained in $W_0^{1,p}(\Omega)$.

On the other hand, in the critical case where $p = N$, the following inequality which is called the critical Hardy inequality holds for all $u \in W_0^{1,N}(\Omega)$ and all $a \ge 1$, where $R = \sup_{x\in\Omega} |x|$:

$$\left(\frac{N-1}{N}\right)^N \int_\Omega \frac{|u|^N}{|x|^N (\log \frac{aR}{|x|})^N} dx \le \int_\Omega |\nabla u|^N dx. \tag{2}$$

See e.g. [25], [24], [4], [5], [15, Corollary 9.1.2], [28], [34]. It is known that in (2) $(\frac{N-1}{N})^N$ is the optimal constant and is not attained for any bounded domain Ω with $0 \in \Omega$ (see [2], [1], [22], [7] etc.).

In this paper, we consider optimal constants and its attainability of the following inequalities (3) which are generalizations of (2):

$$G_a \left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}} \le \int_\Omega |\nabla u|^N dx \tag{3}$$

for $u \in W_0^{1,N}(\Omega)$, $q, \beta > 1$, and $a \ge 1$. We define G_a and $G_{a,\mathrm{rad}}$ as the optimal constants of the inequalities (3) as follows:

$$G_a := \inf_{u\in W_0^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N \, dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}, \quad G_{a,\mathrm{rad}} := \inf_{u\in W_{0,\mathrm{rad}}^{1,N}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N \, dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}, \tag{4}$$

where $W_{0,\mathrm{rad}}^{1,N}(\Omega) = \{u \in W_0^{1,N}(\Omega) \,|\, u \text{ is radial}\}$. When $\Omega = B_R(0)$, $\beta = \frac{N-1}{N}q + 1$, and $q > N$, the exact optimal constant and the attainability of $G_{a,\mathrm{rad}}$ are investigated by Horiuchi and Kumlin [20]. However we do not know the attainability of G_a even if $\Omega = B_R(0)$. In fact, under Theorem 2.8 in their article [20] they mention that the attainability of G_a is an open problem. See also [19]. Note that the continuous embedding $W_0^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); |x|^{-N}(\log \frac{aR}{|x|})^{-\beta} dx)$ is not compact when $\beta = \frac{N-1}{N}q + 1, q \ge N$, and $a > 1$. In addition, the rearrangement technique does not work due to the lack of monotone decreasing property of the potential function $|x|^{-N}(\log \frac{aR}{|x|})^{-\beta}$ when $1 \le a < e^{\frac{\beta}{N}}$.

In this paper, we study the existence, non-existence, and symmetry breaking property of the minimizers of G_a. First, we give an answer to the open problem except for $a = a_*$ which is a threshold number when $\Omega = B_R(0)$. More precisely, we show that $G_{a,\mathrm{rad}}$ is the concentration level of minimizing sequence of G_a and $G_a < G_{a,\mathrm{rad}}$ for $a \in (1, a_*)$. By concentration-compactness alternative, this implies that there exists a minimizer of G_a for $a \in (1, a_*)$. We also show that there is no minimizer of G_a for $a > a_*$. Next, we extend the results to general bounded domains. Furthermore we investigate the positivity and the attainability of G_1 in general bounded domains. When $a = 1$, the positivity and the attainability of G_1 depend on geometry of the boundary of the domain since the potential function has singularities on the boundary. Finally, we show that when $\Omega = B_R(0)$, any minimizers of G_a are non-radial for large β and fixed $q > N$, and any minimizers are radial for any β and any $q \le N$.

Our problem is regarded as the critical case of one of Caffarelli-Kohn-Nirenberg type inequalities, see [20]. In the weighted subcritical Sobolev spaces $W_0^{1,p}(|x|^\alpha dx)$ where $p < N + \alpha$, the existence, nonexistence, and symmetry breaking property of the minimizers of Caffarelli-Kohn-Nirenberg type inequalities are well-studied especially for $p = 2$, see [35], [26], [12], [18], [8], [9], [10], [33], [14], [16], [11] and references therein.

Our minimization problem (4) is related to the following nonlinear elliptic equation with the singular potential:

$$\begin{cases} -\mathrm{div}\,(\,|\nabla u|^{N-2}\nabla u\,) = b\frac{|u|^{q-2}u}{|x|^N(\log\frac{aR}{|x|})^\beta} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The minimizer for G_a is a ground state solution of the Euler-Lagrange equation (5) with a Lagrange multiplier b.

This paper is organized as follows: In section 2, necessary preliminary facts are presented. In section 3, we prove the (non-)attainability of G_a when $\Omega = B_R(0)$ and $a > 1$. In section 4, we extend the results to several bounded domains, and we investigate the positivity and the attainability of G_1 in several bounded domains. In section 5, we show that symmetry breaking phenomena of the minimizers of G_a occur for large β. As a corollary, we obtain a result of multiplicity of solution of the equation (5) for large β.

We fix several notations: $B_R(0)$ and $B_R^N(0)$ denote a N-dimensional ball centered 0 with radius R and ω_{N-1} denotes an area of the unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$. The Schwarz symmetrization $u^\# \colon \mathbb{R}^N \to [0, \infty]$ of u is given by

$$u^\#(x) = u^\#(|x|) = \inf\left\{\tau > 0 : |\{y \in \mathbb{R}^N : |u(y)| > \tau\}| \le |B_{|x|}(0)|\right\}.$$

2. Preliminaries

In this section, we give a necessary and sufficient condition of the positivity of G_a for $a \in [1, \infty)$. Furthermore we give the explicit value of G_a, and the minimizers when $\beta = \frac{N-1}{N}q + 1$ and $q > N$. First, we give a necessary and sufficient condition (6) of the positivity of G_a when $a > 1$.

Proposition 1. *Let $a > 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$, $R = \sup_{x\in\Omega}|x|$, $N \ge 2$ and $q, \beta > 1$. Then $G_a > 0$ if and only if β and q satisfy*

$$\textit{either } \beta > \frac{N-1}{N}q + 1 \textit{ or } \beta = \frac{N-1}{N}q + 1, q \ge N. \tag{6}$$

Essentially, Proposition 1 is proved by the following theorem in [27]. The authors in [27] show a necessary and sufficient condition of the positivity for more general inequalities in the critical Sobolev-Lorentz spaces $H^s_{p,q}(\mathbb{R}^N)$. Note that the norm of $H^1_{N,N}(\mathbb{R}^N)$ is equivalent to it of $W^{1,N}(\mathbb{R}^N)$. We can obtain Proposition 1 from Theorem A and simple calculations. We omit the proof here.

Theorem A. ([27], Theorem 1.1.) *Let $N \in \mathbb{N}$, $1 < p < \infty$, $1 < r \le \infty$ and $1 < \alpha, \beta < \infty$. Then there exists a constant $C > 0$ such that for all $u \in H^{\frac{N}{p}}_{p,r}(\mathbb{R}^N)$, the inequality*

$$\left(\int_{B_{\frac{1}{2}}(0)} \frac{|u|^\alpha}{|x|^N (\log \frac{1}{|x|})^\beta} dx \right)^{\frac{1}{\alpha}} \le C \|u\|_{H^{\frac{N}{p}}_{p,r}(\mathbb{R}^N)} \tag{7}$$

holds true if and only if one of the following conditions (i)–(iii) is fulfilled

$$\begin{cases} (i) & 1 + \alpha - \beta < 0, \\ (ii) & 1 + \alpha - \beta \ge 0 \text{ and } r < \frac{\alpha}{1+\alpha-\beta}, \\ (iii) & 1 + \alpha - \beta > 0, r = \frac{\alpha}{1+\alpha-\beta}, \text{ and } \alpha \ge \beta. \end{cases} \tag{8}$$

Next, we give a necessary and sufficient condition of the positivity of G_a when $a = 1$ and $\Omega = B_R(0)$. Essentially, the following proposition follows from results in [20].

Proposition 2. *Let $\Omega = B_R(0)$. Then $G_1 > 0$ if and only if $\beta = q = N$.*

In §4, we extend Proposition 2 to general bounded domains. Proposition 2 follows from Proposition 4 in §4. Thus we omit the proof of Proposition 2 here.

Finally, we give the explicit value of the optimal constant $G_{a,\mathrm{rad}}$ and the minimizers when $\beta = \frac{N-1}{N} q + 1$ and $q > N$.

Logarithmic transformations related to $G_{a,\mathrm{rad}}$ are founded by [20], [21], [36], [30]. Especially, in the radial setting, the authors in [30] show an unexpected relation (9) that the critical Hardy inequality in dimension $N \ge 2$ is equivalent to the one of the subcritical Hardy inequalities in higher dimension $m > N$ by using a transformation (10) as follows:

$$\begin{aligned} &\int_{\mathbb{R}^m} |\nabla u|^N \, dx - \left(\frac{m-N}{N} \right)^N \int_{\mathbb{R}^m} \frac{|u|^N}{|x|^N} dx \\ &= \frac{\omega_{m-1}}{\omega_{N-1}} \left(\frac{m-N}{N-1} \right)^{N-1} \left(\int_{B^N_R(0)} |\nabla w|^N \, dy - \left(\frac{N-1}{N} \right)^N \int_{B^N_R(0)} \frac{|w|^N}{|y|^N \left(\log \frac{R}{|y|} \right)^N} dy \right), \end{aligned} \tag{9}$$

$$\text{where } u(|x|) = w(|y|) \text{ and } \left(\log \frac{R}{|y|} \right)^{\frac{N-1}{N}} = |x|^{-\frac{m-N}{N}}. \tag{10}$$

By using the transformation (10) and direct calculations, we can observe not only an equivalence between two Hardy inequalities but also the equivalence between Hardy-Sobolev type inequalities and generalized critical Hardy inequalities in the radial setting as follows:

$$G_{1,\mathrm{rad}} = \inf_{w \in W^{1,N}_{0,\mathrm{rad}}(B^N_R(0))\setminus\{0\}} \frac{\int_{B^N_R(0)} |\nabla w|^N \, dy}{\left(\int_{B^N_R(0)} \frac{|w|^q}{|y|^N (\log \frac{R}{|y|})^\beta} dy \right)^{\frac{N}{q}}}$$

$$= \left(\frac{\omega_{N-1}}{\omega_{m-1}} \right)^{1-\frac{N}{q}} \left(\frac{N-1}{m-N} \right)^{N-1+\frac{N}{q}} \inf_{u \in W^{1,N}_{0,\mathrm{rad}}(\mathbb{R}^m)\setminus\{0\}} \frac{\int_{\mathbb{R}^m} |\nabla u|^N \, dx}{\left(\int_{\mathbb{R}^m} |x|^\alpha |u|^q dx \right)^{\frac{N}{q}}}, \tag{11}$$

where $\alpha = \frac{m-N}{N-1}(\beta - 1) - m$. The authors in [30] also give a transformation which is a modification of (10) when $a > 1$. Since the minimization problems on the right hand side of (11) are well-known (see e.g. [35], [26]), we can obtain the following proposition by using these transformations.

Proposition 3. *Let $\beta = \frac{N-1}{N} q + 1, q > N$, and $\Omega = B_R(0)$. Then the followings hold.*

(i) $G_{a,\mathrm{rad}}$ is independent of $a \geq 1$. Furthermore, the exact value of the optimal constant is as follows:

$$G_{a,\mathrm{rad}} = G_{\mathrm{rad}}$$

$$:= \omega_{N-1}^{1-\frac{N}{q}} (N-1) \left(\frac{N}{q} \right)^{1-\frac{2N}{q}} \left(1 - \frac{N}{q} \right)^{-2+\frac{2N}{q}} \left(\frac{\Gamma\left(\frac{q(N-1)}{q-N}\right) \Gamma\left(\frac{N}{q-N}\right)}{\Gamma\left(\frac{qN}{q-N}\right)} \right)^{1-\frac{N}{q}},$$

where $\Gamma(\cdot)$ is the gamma function.

(ii) $G_{a,\mathrm{rad}}$ is not attained for any $a > 1$.

(iii) $G_{1,\mathrm{rad}}$ is attained by the family of the following functions U_λ:

$$U_\lambda(y) = C \lambda^{-\frac{N-1}{N}} \left(1 + \left(\lambda \log \frac{R}{|y|} \right)^{-\frac{q-N}{N}} \right)^{-\frac{N}{q-N}}, \text{ where } C \in \mathbb{R} \setminus \{0\} \text{ and } \lambda > 0.$$

Here, we give a simple proof of Proposition 3 (ii) by using a scaling argument.

Proof of Proposition 3 (ii). Let $\beta = \frac{N-1}{N} q + 1, q > N$, and $a > 1$. Assume that $u \in W^{1,N}_{0,\mathrm{rad}}(B_R(0))$ is a radial minimizer of $G_{a,\mathrm{rad}}$. We can assume that u is nonnegative without loss of generality. We shall derive a contradiction. For $\lambda \in (0, 1)$, we consider a scaled function $u_\lambda \in W^{1,N}_{0,\mathrm{rad}}(B_R(0))$ which is given by

$$u_\lambda(x) = \begin{cases} \lambda^{-\frac{N-1}{N}} u\left(\left(\frac{|x|}{aR} \right)^{\lambda - 1} x \right) & \text{if } x \in B_{a^{(1-\lambda^{-1})}R}(0), \\ 0 & \text{if } x \in B_R(0) \setminus B_{a^{(1-\lambda^{-1})}R}(0). \end{cases}$$

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Then we have

$$\frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left(\int_{B_R(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}} = \frac{\int_{B_R(0)} |\nabla u_\lambda|^N \, dx}{\left(\int_{B_R(0)} \frac{|u_\lambda|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}$$

which yields that u_λ is also a nonnegative minimizer of $G_{a,\mathrm{rad}}$. On the other hand, we can show that $u_\lambda \in C^1(B_R(0) \setminus \{0\})$ and $u_\lambda > 0$ in $B_R(0) \setminus \{0\}$ by standard regularity argument and strong maximum principle to the Euler-Lagrange equation (5), see e.g. [13], [29]. However $u_\lambda \equiv 0$ in $B_R(0) \setminus B_{a^{(1-\lambda^{-1})} R}(0)$. This is a contradiction. Therefore $G_{a,\mathrm{rad}}$ is not attained. □

3. Existence and non-existence of the minimizers

Let $\Omega = B_R(0)$. In this section, we prove an existence and non-existence of the minimizers of G_a. First result is as follows.

Theorem 1. *Let $a > 1$ and $q, \beta > 1$ satisfy (6). Then the followings hold.*

(i) If $\beta > \frac{N-1}{N} q + 1$, then G_a is attained.
(ii) If $\beta = \frac{N-1}{N} q + 1$ and $q > N$, then there exists $a_* \in (1, e^{\frac{\beta}{N}}]$ such that $G_a < G_{a,rad}$ for $a \in (1, a_*)$ and G_a is attained for $a \in (1, a_*)$, on the other hand, $G_a = G_{a,rad}$ for $a > a_*$ and G_a is not attained for $a > a_*$.

Remark 1. If $a_* = e^{\frac{\beta}{N}}$, then we can show that G_{a_*} is not attained. In fact, if we assume that G_{a_*} is attained by u, then $u^\#$ is a radial minimizer of $G_{a,\mathrm{rad}}$ which contradicts Proposition 3 (ii), see the proof of Theorem 1 (ii). However we do not know the value of a_*.

In order to show Theorem 1, we need three lemmas. First we show the (non-)compactness of the embedding $W_0^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx)$, where $f_{a,\beta}(x) = |x|^{-N} \left(\log \frac{aR}{|x|}\right)^{-\beta}$.

Lemma 1. *Let $a > 1$ and $q, \beta > 1$ satisfy (6). Then the continuous embedding $W_0^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx)$ is*

(i) compact if $\beta > \frac{N-1}{N} q + 1$,
(ii) non-compact if $\beta = \frac{N-1}{N} q + 1$ and $q \geq N$.

Proof of Lemma 1. (i) It is proved in [31]. However we give a proof here for the convenience of readers. Let $(u_m)_{m=1}^\infty \subset W_0^{1,N}(B_R(0))$ be a bounded sequence. Then there exists a subsequence $(u_{m_k})_{k=1}^\infty$ such that

$$\begin{aligned} &u_{m_k} \rightharpoonup u \text{ in } W_0^{1,N}(B_R(0)), \\ &u_{m_k} \to u \text{ in } L^r(B_R(0)) \quad \text{for any } r \in [1, \infty). \end{aligned} \tag{12}$$

Let α satisfy $\frac{N-1}{N} q + 1 < \alpha < \beta$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that

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$$\left(\log \frac{aR}{|x|}\right)^{\alpha-\beta} < \varepsilon \quad \text{for all } x \in B_\delta(0). \tag{13}$$

From (12) and (13), we have

$$\begin{aligned}\int\limits_{B_R(0)} \frac{|u_{m_k}-u|^q}{|x|^N(\log\frac{aR}{|x|})^\beta}dx &\le \varepsilon \int\limits_{B_\delta(0)} \frac{|u_{m_k}-u|^q}{|x|^N(\log\frac{aR}{|x|})^\alpha}dx + \delta^{-N}\left(\log\frac{aR}{\delta}\right)^{-\beta}\|u_{m_k}-u\|^q_{L^q(B_R(0))} \\ &\le \varepsilon C\|\nabla(u_{m_k}-u)\|^q_{L^N(B_R(0))} + C\|u_{m_k}-u\|^q_{L^q(B_R(0))} \\ &\le C\varepsilon + C\|u_{m_k}-u\|^q_{L^q(B_R(0))} \to 0 \quad \text{as } \varepsilon\to 0, k\to\infty.\end{aligned}$$

Thus the continuous embedding $W_0^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx)$ is compact if $\beta > \frac{N-1}{N}q+1$.

(ii) We can see a non-compact sequence $(u_{\frac{1}{m}})_{m=1}^\infty$ in $W_0^{1,N}(B_R(0))$, where for $\lambda\in(0,1]$ u_λ is defined in the proof of Proposition 3 (ii). Hence the continuous embedding $W_0^{1,N}(B_R(0)) \hookrightarrow L^q(B_R(0); f_{a,\beta}(x)dx)$ is non-compact if $\beta = \frac{N-1}{N}q+1$ and $q\ge N$. □

In [20], a continuity of G_a with respect to a is proved for $a\in(1,\infty)$. However, in our argument, the continuity of G_a at $a=1$ is needed.

Lemma 2. *G_a is monotone increasing and continuous with respect to $a\in[1,\infty)$.*

Proof of Lemma 2. It is enough to show only the continuity of G_a at $a=1$. From the definition of G_1, we can take $(u_m)_{m=1}^\infty \subset C_c^\infty(B_R(0))$ and $R_m<R$ for any m such that supp $u_m \subset B_{R_m}(0)$, $R_m \nearrow R$, and

$$\frac{\int_{B_{R_m}(0)}|\nabla u_m|^N\,dx}{\left(\int_{B_{R_m}(0)}|u_m|^q f_{1,\beta}(x)\,dx\right)^{\frac{N}{q}}} = G_1+o(1) \quad \text{as } m\to\infty.$$

Set $v(y)=u_m(x)$, where $y=\frac{R}{R_m}x$. Then

$$\frac{\int_{B_{R_m}(0)}|\nabla u_m|^N\,dx}{\left(\int_{B_{R_m}(0)}|u_m|^q f_{1,\beta}(x)\,dx\right)^{\frac{N}{q}}} = \frac{\int_{B_R(0)}|\nabla v|^N\,dx}{\left(\int_{B_R(0)}|v|^q f_{a_m,\beta}(x)\,dx\right)^{\frac{N}{q}}} \ge G_{a_m},$$

where $a_m=\frac{R}{R_m}\searrow 1$ as $m\to\infty$. Therefore we have $G_{a_m}\le G_1+o(1)$. Since $f_{a_m,\beta}(x)\le f_{1,\beta}(x)$ for any $x\in B_R(0)$, we have $G_1\le G_{a_m}$. Hence we see that $\lim_{a\searrow 1} G_a = G_1$. □

Third Lemma is concerned with the concentration level of minimizing sequences of G_a.

Lemma 3. *Let $\beta=\frac{N-1}{N}q+1$, $q>N$, and $a>1$. If $G_a<G_{\mathrm{rad}}$, then G_a is attained, where G_{rad} is given by Proposition 3 (i).*

It is easy to show Theorem 1 by these three lemmas. Therefore we give a proof of Theorem 1 before showing Lemma 3.

Proof of Theorem 1. (i) This is proved by Lemma 1 (i). We omit the proof here.

(ii) Let $\beta = \frac{N-1}{N}q + 1$ and $q > N$. When $a \geq e^{\frac{\beta}{N}}$, the potential function $f_{a,\beta}$ is radially decreasing. Thus the Pólya-Szegö inequality and the Hardy-Littlewood inequality imply that

$$\frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left(\int_{B_R(0)} |u|^q f_{a,\beta}(x)\, dx\right)^{\frac{N}{q}}} \geq \frac{\int_{B_R(0)} |\nabla u^\#|^N \, dx}{\left(\int_{B_R(0)} |u^\#|^q f_{a,\beta}(x)\, dx\right)^{\frac{N}{q}}} \geq G_{a,\mathrm{rad}}$$

for any $u \in W_0^{1,N}(B_R(0))$ and $a \geq e^{\frac{\beta}{N}}$. Therefore $G_a = G_{a,\mathrm{rad}} = G_{\mathrm{rad}}$ for any $a \geq e^{\frac{\beta}{N}}$. Moreover we see $G_1 = 0$ by Proposition 2. Since G_a is continuous and monotone increasing with respect to $a \in [1, \infty)$ by Lemma 2, there exists $a_* \in (1, e^{\frac{\beta}{N}}]$ such that $G_a < G_{\mathrm{rad}}$ for $a \in [1, a_*)$ and $G_a = G_{\mathrm{rad}}$ for $a \in [a_*, \infty)$. Hence G_a is attained for $a \in (1, a_*)$ by Lemma 3. On the other hand, if we assume that there exists a nonnegative minimizer u of G_a for $a > a_*$, then we can show that $u \in C^1(B_R(0) \setminus \{0\})$ and $u > 0$ in $B_R(0) \setminus \{0\}$ by standard regularity argument and strong maximum principle to the Euler-Lagrange equation (5), see e.g. [13], [29]. Therefore we see that

$$G_{\mathrm{rad}} = G_a = \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left(\int_{B_R(0)} |u|^q f_{a,\beta}(x)\, dx\right)^{\frac{N}{q}}} > \frac{\int_{B_R(0)} |\nabla u|^N \, dx}{\left(\int_{B_R(0)} |u|^q f_{a_*,\beta}(x)\, dx\right)^{\frac{N}{q}}} \geq G_{\mathrm{rad}}.$$

This is a contradiction. Therefore G_a is not attained for $a > a_*$. $\square$

Finally, we prove Lemma 3.

Proof of Lemma 3. Take a minimizing sequence $(u_m)_{m=1}^\infty \subset W_0^{1,N}(B_R(0))$ of G_a. Without loss of generality, we can assume that

$$\int\limits_{B_R(0)} |u_m|^q f_{a,\beta}(x)\, dx = 1, \quad \int\limits_{B_R(0)} |\nabla u_m|^N \, dx = G_a + o(1) \text{ as } m \to \infty.$$

Since (u_m) is bounded in $W_0^{1,N}(B_R(0))$, passing to a subsequence if necessary, $u_m \rightharpoonup u$ in $W_0^{1,N}(B_R(0))$. Then by Brezis-Lieb lemma, we have

$$\begin{aligned} G_a &= \int\limits_{B_R(0)} |\nabla u_m|^N \, dx + o(1) \\ &= \int\limits_{B_R(0)} |\nabla (u_m - u)|^N \, dx + \int\limits_{B_R(0)} |\nabla u|^N \, dx + o(1) \end{aligned}$$

$$\ge G_a\left(\int\limits_{B_R(0)} |u_m-u|^q f_{a,\beta}(x)\,dx\right)^{\frac{N}{q}} + G_a\left(\int\limits_{B_R(0)} |u|^q f_{a,\beta}(x)\,dx\right)^{\frac{N}{q}} + o(1)$$

$$\ge G_a\left(\int\limits_{B_R(0)} \left(|u_m-u|^q+|u|^q\right) f_{a,\beta}(x)\,dx\right)^{\frac{N}{q}} + o(1)$$

$$= G_a\left(\int\limits_{B_R(0)} |u_m|^q f_{a,\beta}(x)\,dx\right)^{\frac{N}{q}} + o(1) = G_a$$

which implies that either $u\equiv 0$ or $u_m\to u\not\equiv 0$ in $L^q(B_R(0); f_{a,\beta}(x)dx)$ holds true from the equality condition of the last inequality. We shall show that $u\not\equiv 0$. Assume that $u\equiv 0$. Then we claim that

$$G_{\mathrm{rad}} \le \int\limits_{B_R(0)} |\nabla u_m|^N\,dx + o(1). \tag{14}$$

If the claim (14) is true, then we see that $G_{\mathrm{rad}}\le G_a$ which contradicts the assumption. Therefore $u\not\equiv 0$ which implies that $u_m\to u\not\equiv 0$ in $L^q(B_R(0); f_{a,\beta}(x)dx)$. Hence we have

$$1=\int\limits_{B_R(0)} |u|^q f_{a,\beta}(x)\,dx, \qquad \int\limits_{B_R(0)} |\nabla u|^N\,dx \le \liminf_{m\to\infty}\int\limits_{B_R(0)} |\nabla u_m|^N\,dx = G_a.$$

Thus we can show that u is a minimizer of G_a. We shall show the claim (14). Since $u_m\to 0$ in $L^r(B_R(0))$ for any $r\in[1,\infty)$ and the potential function $f_{a,\beta}$ is bounded away from the origin, for any small $\varepsilon>0$ we have

$$1=\int\limits_{B_R(0)} |u_m|^q f_{a,\beta}(x)\,dx = \int\limits_{B_{\frac{\varepsilon R}{2}}(0)} |u_m|^q f_{a,\beta}(x)\,dx + o(1).$$

Let ϕ_ε be a smooth cut-off function which satisfies the followings:

$$0\le\phi_\varepsilon\le 1,\ \phi_\varepsilon\equiv 1 \text{ on } B_{\frac{\varepsilon R}{2}}(0),\ \operatorname{supp}\phi_\varepsilon\subset B_{\varepsilon R}(0),\ |\nabla\phi_\varepsilon|\le C\varepsilon^{-1}.$$

Set $\tilde{u_m}(y)=u_m(x)$ and $\tilde{\phi_\varepsilon}(y)=\phi_\varepsilon(x)$, where $y=\frac{x}{\varepsilon}$. Then we have

$$1=\left(\int\limits_{B_{\frac{\varepsilon R}{2}}(0)} |u_m|^q f_{a,\beta}(x)\,dx\right)^{\frac{N}{q}} + o(1)$$

$$
\begin{aligned}
&\le \left(\int_{B_{\varepsilon R}(0)} |u_m\phi_\varepsilon|^q f_{a,\beta}(x)\,dx \right)^{\frac{N}{q}} + o(1) \\
&= \left(\int_{B_R(0)} |\tilde{u_m}\tilde{\phi_\varepsilon}|^q f_{a\varepsilon^{-1},\beta}(x)\,dx \right)^{\frac{N}{q}} + o(1) \le G_{a\varepsilon^{-1}}^{-1} \int_{B_R(0)} |\nabla(\tilde{u_m}\tilde{\phi_\varepsilon})|^N\,dx + o(1).
\end{aligned}
$$

We see that $a\varepsilon^{-1} \ge e^{\frac{\beta}{N}}$ for small ε. Since $G_{a\varepsilon^{-1}} = G_{a,\,\mathrm{rad}} = G_{\mathrm{rad}}$ from the proof of Theorem 1 (ii) and Proposition 3 (i), we have

$$
\begin{aligned}
1 &\le G_{\mathrm{rad}}^{-1} \int_{B_R(0)} |\nabla(\tilde{u_m}\tilde{\phi_\varepsilon})|^N\,dx + o(1) \\
&\le G_{\mathrm{rad}}^{-1} \left(\int_{B_{\varepsilon R}(0)} |\nabla u_m|^N\,dx + C \int_{B_{\varepsilon R}(0)} |\nabla u_m|^{N-1}|\nabla\phi_\varepsilon||u_m|\phi_\varepsilon^{N-1} + |u_m|^N|\nabla\phi_\varepsilon|^N\,dx \right) + o(1) \\
&\le G_{\mathrm{rad}}^{-1} \left(\int_{B_{\varepsilon R}(0)} |\nabla u_m|^N\,dx + NC\varepsilon^{-1}\|\nabla u_m\|_{L^N}^{N-1}\|u_m\|_{L^N} + C\varepsilon^{-N}\|u_m\|_{L^N}^N \right) + o(1) \\
&\le G_{\mathrm{rad}}^{-1} \int_{B_{\varepsilon R}(0)} |\nabla u_m|^N\,dx + o(1) \le G_{\mathrm{rad}}^{-1} \int_{B_R(0)} |\nabla u_m|^N\,dx + o(1).
\end{aligned}
$$

Therefore we obtain the claim (14). The proof of Lemma 3 is now complete. □

4. In the case of general bounded domain

We extend Theorem 1 and Proposition 2 to bounded domains. Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, $0 \in \Omega$, and β and q satisfy (6). Set $R = \sup_{x\in\Omega}|x|$.

First we extend Proposition 2 to general bounded domains. If there exists $\Gamma \subset \partial\Omega \cap \partial B_R(0)$ such that Γ is open in $\partial B_R(0)$, then we can obtain the same result as Proposition 2 as follows.

Proposition 4. *Assume that there exists $\Gamma \subset \partial\Omega \cap \partial B_R(0)$ such that Γ is open in $\partial B_R(0)$. Then $G_1 > 0$ if and only if $\beta = q = N$.*

Proof of Proposition 4. First we show that $G_1 = 0$ if $\beta > \frac{N-1}{N}q + 1$. Set $x = r\omega\,(r = |x|, \omega \in S^{N-1})$ for $x \in \mathbb{R}^N$. From the assumption, we can take $\delta > 0$ and $\tilde{\Gamma} \subset \Gamma$ such that $\tilde{\Gamma}$ is open in $\partial B_R(0)$ and

$$
\left\{ (r,\omega) \in [0,R) \times S^{N-1} \,\middle|\, R - 2\delta \le r \le R, \omega \in \frac{1}{R}\tilde{\Gamma} \right\} \subset \Omega.
$$

Let $0 \not\equiv \psi \in C_c^\infty(\frac{1}{R}\tilde{\Gamma})$ and $\phi \in C^\infty([0,\infty))$ satisfy $\phi \equiv 1$ on $[R-\delta, R]$ and $\phi \equiv 0$ on $[0, R-2\delta]$. Set $u_s(x) = \left(\log \frac{R}{r}\right)^s \psi(\omega)\phi(r)$. Then we have

$$\begin{aligned}\int_\Omega |\nabla u_s|^N dx &= \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r}\omega + \frac{1}{r}\nabla_{S^{N-1}} u_s \right|^N r^{N-1}\, dr dS_\omega \\ &\le 2^{N-1} \int_{S^{N-1}} \int_0^R \left| \frac{\partial u_s}{\partial r} \right|^N r^{N-1} + \left|\nabla_{S^{N-1}} u_s\right|^N r^{-1}\, dr dS_\omega \\ &\le s^N C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{(s-1)N} \frac{dr}{r} + C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{sN} \frac{dr}{r} + C \\ &\le s^N C \int_0^{\log \frac{R}{R-\delta}} t^{(s-1)N}\, dt + C < \infty \quad \text{if } s > \frac{N-1}{N}.\end{aligned}$$

Thus $u_s \in W_0^{1,N}(\Omega)$ for all $s > \frac{N-1}{N}$. However, direct calculation shows that

$$\int_\Omega \frac{|u_s|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \ge C \int_{R-\delta}^R \left(\log \frac{R}{r}\right)^{sq-\beta} \frac{dr}{r} = C \int_0^{\log \frac{R}{R-\delta}} t^{sq-\beta}\, dt$$

which implies that

$$\int_\Omega \frac{|u_s|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx = \infty$$

for s close to $\frac{N-1}{N}$ since $\beta > \frac{N-1}{N}q + 1$. Therefore we see that

$$G_1 = 0 \quad \text{if} \quad \beta > \frac{N-1}{N}q + 1. \tag{15}$$

Next we show that $G_1 = 0$ if $\beta > N$. Set $x_\varepsilon = (R-2\varepsilon)\frac{y}{R}$ for $y \in \partial B_R(0)$. Note that $B_\varepsilon(x_\varepsilon) \subset \Omega$ for small $\varepsilon > 0$ and some $y \in \Gamma$. Then we define u_ε as follows:

$$u_\varepsilon(x) = \begin{cases} v\left(\frac{|x-x_\varepsilon|}{\varepsilon}\right) & \text{if } x \in B_\varepsilon(x_\varepsilon), \\ 0 & \text{if } x \in \Omega \setminus B_\varepsilon(x_\varepsilon), \end{cases} \quad \text{where } v(t) = \begin{cases} 1 & \text{if } 0 \le t \le \frac{1}{2}, \\ 2(1-t) & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

Since $\log t \le t - 1$ for $t \ge 1$, we obtain

$$\int_\Omega |\nabla u_\varepsilon(x)|^N\,dx = \int_{B_1(0)} |\nabla v(|z|)|^N\,dz < \infty,$$

$$\int_\Omega \frac{|u_\varepsilon(x)|^q}{|x|^N\left(\log\frac{R}{|x|}\right)^\beta}dx \geq C\int_{B_\varepsilon(x_\varepsilon)} \frac{|u_\varepsilon(x)|^q}{(R-|x|)^\beta}dx \geq \frac{C}{(3\varepsilon)^\beta}\int_{B_{\frac{\varepsilon}{2}}(x_\varepsilon)} dx = C\,\varepsilon^{N-\beta} \to \infty$$

as $\varepsilon \to 0$ when $\beta > N$. Hence we see that

$$G_1 = 0 \quad \text{if} \quad \beta > N. \tag{16}$$

From (15), (16), and (6), we see that $G_1 > 0$ if and only if $q = \beta = N$. □

If there does not exist Γ in Proposition 4, then we can expect that the relation between q, β and the positivity of G_1 depends on geometry of the boundary $\partial\Omega$. In order to see it, we consider special cuspidal domains which satisfy the following conditions:

(Ω_1): $\partial\Omega \cap \partial B_R(0) = \{(0, \cdots, 0, -R)\}$.
(Ω_2): $\partial\Omega$ is represented by a graph $\phi : \mathbb{R}^{N-1} \to [-R, \infty)$ near the point $(0, \cdots, 0, -R)$. Namely, for small $\delta > 0$ the following holds true:

$$Q_\delta := \Omega \cap (\mathbb{R}^{N-1} \times [-R, -R+\delta]) = \{(x', x_N) \in \mathbb{R}^{N-1} \times [-R, -R+\delta] \,|\, x_N > \phi(x')\}.$$

(Ω_3): There exist $C_1, C_2 > 0$ and $\alpha \in (0, 1]$ such that

$$C_1|x'|^\alpha \leq \phi(x') + R \leq C_2|x'|^\alpha \text{ for any } x' \in \mathbb{R}^{N-1}.$$

α in (Ω_3) expresses the sharpness of the cusp at the point $(0, \cdots, 0, -R)$. Then we can obtain the following theorem concerned with the positivity and the attainability of G_1.

Theorem 2. *Assume that Ω satisfies the assumptions (Ω_1)–(Ω_3). Then there exists $\beta^* = \beta^*(\alpha, q) \in [\frac{N-1}{\alpha} + 1, \frac{N}{\alpha}]$ such that $G_1 = 0$ for $\beta > \beta_*$ and $G_1 > 0$ for $\beta < \beta^*$. Furthermore G_1 is attained for $\beta \in (\frac{N-1}{N}q + 1, \beta^*)$.*

Remark 2. When $\beta = q = N$ and $0 \in \Omega$, G_1 is not attained for any bounded domain. However, when $0 \notin \Omega$, the attainability of G_1 depends on a geometry of the boundary $\partial\Omega$. Very recently, Byeon and Takahashi investigate the attainability of G_1 on cuspidal domains in their article [7] when $\beta = q = N$.

Proof of Theorem 2. First we shall show that $G_1 = 0$ if $\beta > \frac{N}{\alpha}$. From (Ω_3), we can observe that $B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon) \subset \Omega$ for small $\varepsilon > 0$ and small $A > 0$, where $x_\varepsilon = (0, \cdots, 0, -R+2\varepsilon)$. Then we define w_ε as follows:

$$w_\varepsilon(x) = \begin{cases} v\left(\frac{|x-x_\varepsilon|}{A\varepsilon^{\frac{1}{\alpha}}}\right) & \text{if } x \in B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon), \\ 0 & \text{if } x \in \Omega \setminus B_{A\varepsilon^{\frac{1}{\alpha}}}(x_\varepsilon), \end{cases}$$

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where v is the same function in the proof of Proposition 4. In the same way as the proof of Proposition 4, we have

$$\int_{\Omega} |\nabla_x w_\varepsilon(x)|^N \, dx < \infty, \quad \int_{\Omega} \frac{|w_\varepsilon(x)|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \geq C\, \varepsilon^{\frac{N}{\alpha}-\beta} \to \infty$$

as $\varepsilon \to 0$ if $\beta > \frac{N}{\alpha}$. Therefore we have

$$G_1 = 0 \quad \text{at least for } \beta > \frac{N}{\alpha}. \tag{17}$$

Next we shall show that $G_1 > 0$ if $\beta < \frac{N-1}{\alpha} + 1$. For $u \in W_0^{1,N}(\Omega)$, we divide the domain Ω into three parts as follows:

$$\int_{\Omega} \frac{|u(x)|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx = \int_{\Omega \cap B_{\frac{R}{2}}(0)} + \int_{\Omega \setminus \left(B_{\frac{R}{2}}(0) \cup Q_\delta\right)} + \int_{Q_\delta} =: I_1 + I_2 + I_3. \tag{18}$$

From Theorem A, we obtain

$$I_1 \leq C \left(\int_{\Omega} |\nabla u|^N dx \right)^{\frac{q}{N}}. \tag{19}$$

Since the potential function $|x|^{-N}(\log \frac{R}{|x|})^{-\beta}$ does not have any singularity in $\Omega \setminus \left(B(\frac{R}{2}) \cup Q_\delta\right)$, the Sobolev inequality yields that

$$I_2 \leq C \int_{\Omega} |u|^q dx \leq C \left(\int_{\Omega} |\nabla u|^N dx \right)^{\frac{q}{N}}. \tag{20}$$

Finally, we shall derive an estimate of I_3 from above. Since $\log t \geq \frac{1}{2}(t-1)$ $(1 \leq t \leq 2)$, we obtain

$$I_3 \leq C \int_{Q_\delta} \frac{|u(x)|^q}{(R-|x|)^\beta} dx \leq C \int_{z_N=0}^{z_N=\delta} \int_{z_N \geq C_1 |z'|^\alpha} \frac{|\tilde{u}(z', z_N)|^q}{|z|^\beta} dz, \tag{21}$$

where $u(x) = \tilde{u}(z)$ $(z = x + (0, \cdots, 0, R))$. If $\beta < \frac{N-1}{\alpha} + 1$, then there exists $\varepsilon > 0$ and $p > \frac{N}{N-\varepsilon}$ such that $(\beta - \varepsilon)p < \frac{N-1}{\alpha} + 1$. By using the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\int\limits_{z_N=0}^{z_N=\delta}\int\limits_{z_N\geq C_1|z'|^\alpha}\frac{|\tilde{u}(z',z_N)|^q}{|z|^\beta}dz\\
&=\iint\frac{|\tilde{u}|^\varepsilon}{|z|^\varepsilon}|\tilde{u}|^{q-\varepsilon}|z|^{\beta-\varepsilon}dz\\
&\leq\left(\iint\frac{|\tilde{u}|^N}{|z|^N}dz\right)^{\frac{\varepsilon}{N}}\left(\iint|\tilde{u}|^{(q-\varepsilon)\frac{Np}{Np-N-p\varepsilon}}dz\right)^{\frac{Np-N-p\varepsilon}{Np}}\left(\iint|z|^{-(\beta-\varepsilon)p}dz\right)^{\frac{1}{p}}\\
&\leq C\left(\iint\frac{|\tilde{u}|^N}{|z_N|^N}dz\right)^{\frac{\varepsilon}{N}}\left(\int\limits_\Omega|\nabla\tilde{u}|^Ndz\right)^{\frac{q-\varepsilon}{N}}\left(\iint\limits_{|z'|\leq\left(\frac{z_N}{C_1}\right)^{\frac{1}{\alpha}}}z_N^{-(\beta-\varepsilon)p}dz\right)^{\frac{1}{p}}\\
&\leq C\left(\iint\frac{|\tilde{u}|^N}{|z_N|^N}dz\right)^{\frac{\varepsilon}{N}}\left(\int\limits_\Omega|\nabla\tilde{u}|^Ndz\right)^{\frac{q-\varepsilon}{N}}\left(\int\limits_{z_N=0}^{z_N=\delta}z_N^{\frac{N-1}{\alpha}-(\beta-\varepsilon)p}dz_N\right)^{\frac{1}{p}}.
\end{aligned}$$

Since $\frac{N-1}{\alpha}-(\beta-\varepsilon)p>-1$, $\int_0^\delta z_N^{\frac{N-1}{\alpha}-(\beta-\varepsilon)p}dz_N<\infty$. Furthermore, applying the Hardy inequality on the half space $\mathbb{R}^N_+:=\{(x',x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}\,|\,x_N>0\}$:

$$\left(\frac{r-1}{r}\right)^r\int\limits_{\mathbb{R}^N_+}\frac{|u|^r}{|x_N|^r}dx\leq\int\limits_{\mathbb{R}^N_+}|\nabla u|^rdx\quad(1\leq r<\infty)$$

yields that

$$\int\limits_{z_N=0}^{z_N=\delta}\int\limits_{z_N\geq C_1|z'|^\alpha}\frac{|\tilde{u}(z',z_N)|^q}{|z|^\beta}dz\leq C\left(\int\limits_{\Omega+(0,\cdots,0,R)}|\nabla\tilde{u}|^Ndz\right)^{\frac{q}{N}}.\tag{22}$$

By (21) and (22), we have

$$I_3\leq C\left(\int\limits_\Omega|\nabla u|^Ndx\right)^{\frac{q}{N}}.\tag{23}$$

Therefore, from (18), (19), (20), and (23), for all $u\in W_0^{1,N}(\Omega)$,

$$C\left(\int\limits_\Omega\frac{|u(x)|^q}{|x|^N\left(\log\frac{R}{|x|}\right)^\beta}dx\right)^{\frac{N}{q}}\leq\int\limits_\Omega|\nabla u|^Ndx.$$

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Hence

$$G_1 > 0 \quad \text{at least for} \quad \beta < \frac{N-1}{\alpha} + 1. \tag{24}$$

From (17) and (24), there exists $\beta^* \in [\frac{N-1}{\alpha} + 1, \frac{N}{\alpha}]$ such that $G_1 > 0$ for $\beta < \beta^*$ and $G_1 = 0$ for $\beta > \beta^*$.

Lastly we shall show that G_1 is attained for $\beta \in (\frac{N-1}{N}q + 1, \beta^*)$. In order to show it, we show that the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{1,\beta}(x)dx)$ is compact if $\frac{N-1}{N}q + 1 < \beta < \beta^*$. Let $(u_m)_{m=1}^\infty \subset W_0^{1,N}(\Omega)$ be a bounded sequence. Then there exists a subsequence $(u_{m_k})_{k=1}^\infty$ such that

$$\begin{aligned} &u_{m_k} \rightharpoonup u \text{ in } W_0^{1,N}(\Omega), \\ &u_{m_k} \to u \text{ in } L^r(\Omega) \quad \text{for all } 1 \le r < \infty. \end{aligned} \tag{25}$$

We divide the domain into two parts as follows:

$$\int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx = \int_{\Omega \setminus Q_\delta} + \int_{Q_\delta} =: J_1(u_{m_k} - u) + J_2(u_{m_k} - u). \tag{26}$$

Since $\log \frac{R}{|x|} \ge C \log \frac{aR}{|x|}$ for any $x \in \Omega \setminus Q_\delta$ for some $a > 1$ and $C > 0$, it holds that

$$J_1(u_{m_k} - u) \le C \int_{\Omega \setminus Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{aR}{|x|}\right)^\beta} dx \le C \int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{aR}{|x|}\right)^\beta} dx.$$

Note that the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$ is compact for $\beta > \frac{N-1}{N}q + 1$ from Lemma 1, we obtain

$$J_1(u_{m_k} - u) \to 0 \quad \text{as } k \to \infty. \tag{27}$$

On the other hand, for any $\varepsilon > 0$, we take $\gamma > 0$ which satisfies $\beta < \gamma < \beta^*$ and $(\log \frac{R}{|x|})^{\gamma - \beta} < \varepsilon$ for $x \in Q_\delta$. (If necessary, we take small $\delta > 0$ again.) Then we have

$$J_2(u_{m_k} - u) \le \varepsilon \int_{Q_\delta} \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\gamma} dx \le C\varepsilon \left(\int_\Omega |\nabla (u_{m_k} - u)|^N \, dx \right)^{\frac{q}{N}} \le C\varepsilon. \tag{28}$$

From (26), (27), and (28), we have

$$\int_\Omega \frac{|u_{m_k} - u|^q}{|x|^N \left(\log \frac{R}{|x|}\right)^\beta} dx \to 0 \quad \text{as } k \to \infty.$$

Therefore the continuous embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{1,\beta}(x)dx)$ is compact if $\frac{N-1}{N}q + 1 < \beta < \beta^*$. In conclusion, we have showed that G_1 is attained if $\frac{N-1}{N}q + 1 < \beta < \beta^*$. $\square$

Next we extend Theorem 1 to general bounded domains.

Theorem 3. *Let $a > 1$. Then the followings hold.*

(i) If $\beta > \frac{N-1}{N}q + 1$, then G_a is attained for any bounded domains Ω.
(ii) If $\beta = \frac{N-1}{N}q + 1, q > N$, and $a \geq e^{\frac{\beta}{N}}$, then $G_a = G_{\mathrm{rad}}$ and G_a is not attained for any bounded domain Ω.
(iii) If $\beta = \frac{N-1}{N}q + 1, q > N$, and Ω satisfies either (Ω_4) or (Ω_5), where
(Ω_4): $\partial\Omega$ satisfies the Lipschitz condition at some point $x_0 \in \Omega \cap B_R(0)$,
(Ω_5): Ω satisfies (Ω_1)–(Ω_3) and α in (Ω_3) is greater than $\frac{N}{\beta}$,

then there exists $a_* \in (1, e^{\frac{\beta}{N}}]$ such that G_a is attained for $a \in (1, a_*)$ and G_a is not attained for $a > a_*$.

In order to show Theorem 3 (iii), we need the continuity of G_a with respect to a at $a = 1$. Under the assumptions (Ω_4), (Ω_5), we can show the continuity of G_a at $a = 1$ as follows.

Lemma 4. *Let $\beta > N$. If Ω satisfies either (Ω_4) or (Ω_5), then $G_1 = \lim_{a \searrow 1} G_a = 0$.*

Lemma 4 follows from the following proposition.

Proposition 5. *Let $a > 1$. If Ω satisfies either (Ω_4) or (Ω_5), then there exists $C > 0$ such that for a close to 1, $G_a \leq C(a-1)^{\frac{N}{q}(\beta - \frac{N}{\alpha})}$, where α is regarded as 1 if Ω satisfies (Ω_4).*

Proof of Proposition 5. Let $x_a = R(2-a)\frac{x_0}{|x_0|}$ and $\phi \in C_c^\infty(B_1(0))$. Here x_0 is regarded as $(0, \cdots, 0, -R)$ if Ω satisfies (Ω_5). Then $B_{c(a-1)^{\frac{1}{\alpha}}}(x_a) \subset \Omega$ for a close to 1 and for sufficiently small $c > 0$. Set $\phi_a(x) = \phi\left(\frac{x - x_a}{c(a-1)^{\frac{1}{\alpha}}}\right)$. Since $\log \frac{1}{t} \leq \frac{1-t}{2}$ for t close to 1, we have the followings for a close to 1.

$$\int_\Omega |\nabla \phi_a|^N \, dx = \int_{B_1(0)} |\nabla \phi|^N \, dz < \infty,$$

$$\int_\Omega \frac{|\phi_a|^q}{|x|^N \left(\log \frac{aR}{|x|}\right)^\beta} dx \geq c^N (a-1)^{\frac{N}{\alpha}} \|\phi\|_{L^q}^q \left(\log \frac{aR}{R(2-a) - c(a-1)^{\frac{1}{\alpha}}}\right)^{-\beta}$$
$$\geq C(a-1)^{\frac{N}{\alpha} - \beta}. \qquad \square$$

Proof of Theorem 3. (i) We can check that Lemma 1 holds true for any bounded domains Ω. Therefore (i) follows from the compactness of the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega; f_{a,\beta}(x)dx)$. We omit the proof.

(ii) Note that $W^{1,N}_{0,\mathrm{rad}}(B_\varepsilon(0)) \subset W^{1,N}_0(\Omega) \subset W^{1,N}_0(B_R(0))$ for small ε by zero extension. Then we have

$$\inf_{u \in W^{1,N}_{0,\mathrm{rad}}(B_\varepsilon(0))\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}} \geq G_a \geq \inf_{u \in W^{1,N}_0(B_R(0))\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}$$
$$= \inf_{u \in W^{1,N}_{0,\mathrm{rad}}(B_R(0))\setminus\{0\}} \frac{\int_\Omega |\nabla u|^N dx}{\left(\int_\Omega \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}}, \tag{29}$$

where the last equality comes from $a \geq e^{\frac{\beta}{N}}$. From the proof of Proposition 3 (ii), we can observe that G_{rad} does not vary even if we replace $W^{1,N}_{0,\mathrm{rad}}(B_R(0))$ to $W^{1,N}_{0,\mathrm{rad}}(B_\varepsilon(0))$ for any small $\varepsilon > 0$. Thus the right hand side and the left hand side of (29) take same value, that is G_{rad}. Therefore we have $G_a = G_{\mathrm{rad}}$. Furthermore if we assume that G_a is attained by $u \in W^{1,N}_0(\Omega)$, then $u \in W^{1,N}_0(B_R(0))$ is also a minimizer on a ball. This contradicts Theorem 1 (ii) in §2. Hence G_a is not attained for any bounded domains Ω.

(iii) Note that G_a is continuous with respect to $a \in (1, \infty)$, and is monotone increasing with respect to $a \in [1, \infty)$ for any bounded domains. From Lemma 4 and Theorem 3 (ii), we can show that there exists $a_* \in (1, e^{\frac{\beta}{N}}]$ such that $G_a < G_{\mathrm{rad}}$ for $a \in (1, a_*)$ and $G_a = G_{\mathrm{rad}}$ for $a > a_*$ in the same way as the proof of Theorem 1 (ii). The remaining parts of the proof are similar to the proof of Theorem 1 (ii). □

5. Symmetry breaking

In this section, we consider radially symmetry of the minimizers of G_a when $\Omega = B_R(0)$. We can show that any minimizer of G_a has axial symmetry by using *spherical symmetric rearrangement*, see [23]. Namely, for any minimizer u_β of G_a there exists some $\xi \in \mathbb{S}^{N-1}$ such that the restriction of u_β to any sphere $\partial B_r(0)$ is symmetric decreasing with respect to the distance to $r\xi$. See also [32]. The last result is as follows.

Theorem 4. *Let $\beta > \frac{N-1}{N}q + 1$, $a > 1$, and u_β be a minimizer of G_a in Theorem 1 (i). Then the followings hold true.*

(i) For fixed $q > N$, there exists β_* such that u_β is non-radial for $\beta > \beta_*$.
(ii) u_β is radial for any β and $q \leq N$.

From the compactness of the embedding: $W^{1,N}_{0,\mathrm{rad}}(B_1) \hookrightarrow L^q(B_1; f_{a,\beta}dx)$, we can easily show that $G_{a,\mathrm{rad}}$ is attained which implies that there is a radial solution of (5) when $\beta > \frac{N-1}{N}q + 1$ and $a > 1$. Furthermore, if $q > N$ and β is large, then we also find a non-radial solution of (5) by Theorem 4 (i). Therefore we obtain a result of multiplicity of solution of (5).

Corollary 1. *Let $\beta > \frac{N-1}{N}q + 1$, $q > N$, and $a > 1$. Then the equation (5) has at least two weak solutions for large β.*

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In order to show Theorem 4 (i), we need two lemmas concerning growth orders of G_a and $G_{a,\mathrm{rad}}$ with respect to β.

Lemma 5. *For fixed $q > N$, there exists $C > 0$ such that for sufficiently large β the following estimate holds true.*

$$G_a \le C\beta^{\frac{N^2}{q}} (\log a)^{\frac{N\beta}{q}}.$$

Proof of Lemma 5. Let $u \in C_c^\infty(B_R(0))$. Following [33] we consider $u_\beta(x) := u(\beta(x - x_\beta))$ for $x \in B_{\beta^{-1}}(x_\beta)$, where $x_\beta := (R - \beta^{-1}, 0, \cdots, 0) \in B_R(0)$. Then for sufficiently large β we obtain

$$\int\limits_{B_{\beta^{-1}}(x_\beta)} |\nabla u_\beta(x)|^N dx = \int\limits_{B_R(0)} |\nabla u(y)|^N dy, \tag{30}$$

$$\int\limits_{B_{\beta^{-1}}(x_\beta)} \frac{|u_\beta(x)|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \ge \left(R - 2\beta^{-1}\right)^{-N} \left(\log \frac{aR}{R - 2\beta^{-1}}\right)^{-\beta} \beta^{-N} \int\limits_{B_R(0)} |u(y)|^q dy. \tag{31}$$

We set $f(\beta) := (R - 2\beta^{-1})^{-N} (\log \frac{aR}{R-2\beta^{-1}})^{-\beta}$. Since $\log \frac{1}{1-x} \le 2x$ for all $x \in [0, \frac{1}{2}]$, for large β we have

$$\begin{aligned} f(\beta) &\ge \frac{1}{2} \left(\log a + \log \frac{1}{1 - 2\beta^{-1}R^{-1}}\right)^{-\beta} \\ &\ge \frac{1}{2} \left(\log a + 4\beta^{-1}R^{-1}\right)^{-\beta} \\ &= \frac{1}{2} (\log a)^{-\beta} \left(1 + \frac{4}{\beta R \log a}\right)^{-\beta} \end{aligned}$$

which yields that

$$f(\beta) \ge C(\log a)^{-\beta} \quad \text{for large } \beta. \tag{32}$$

From (30), (31), and (32), we obtain

$$G_a \le \frac{\int_{B_R(0)} |\nabla u_\beta|^N dx}{\left(\int_{B_R(0)} \frac{|u_\beta|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}} \le C\beta^{\frac{N^2}{q}} (\log a)^{\frac{N\beta}{q}}. \qquad \square$$

Lemma 6. *For fixed $q > N$, there exists $C > 0$ such that for sufficiently large β the following estimate holds true.*

$$G_{a,\mathrm{rad}} \ge C\beta^{N-1+\frac{N}{q}} (\log a)^{\frac{N\beta}{q} - (N-1+\frac{N}{q})}.$$

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Proof of Lemma 6. For $u \in W^{1,N}_{0,\mathrm{rad}}(B_R(0))$ we define $v \in W^{1,N}_{0,\mathrm{rad}}(B_R(0))$ as follows:

$$v(s) = u(r), \text{ where } (\log a)^{A-1} \log \frac{aR}{s} = \left(\log \frac{aR}{r}\right)^A \text{ and } A = \frac{N(\beta-1)}{(N-1)q}.$$

Direct calculation shows that

$$\begin{aligned}
\int_{B_R(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx &= \omega_{N-1} \int_0^R |u(r)|^q \left(\log \frac{aR}{r}\right)^{-\beta} \frac{dr}{r} \\
&= \omega_{N-1} A^{-1} (\log a)^{\frac{A-1}{A}(1-\beta)} \int_0^R \frac{|v(s)|^q}{s(\log \frac{aR}{s})^{\frac{A-1+\beta}{A}}} ds \\
&= A^{-1} (\log a)^{\frac{A-1}{A}(1-\beta)} \int_{B_R(0)} \frac{|v|^q}{|y|^N (\log \frac{aR}{|y|})^{\frac{N-1}{N}q+1}} dy.
\end{aligned}$$

In the same way as above, we have

$$\begin{aligned}
\int_{B_R(0)} |\nabla u|^N dx &= A^{N-1} (\log a)^{-\frac{A-1}{A}} \int_{B_R(0)} |\nabla v|^N \left(\log \frac{aR}{|y|}\right)^{\frac{A-1}{A}} dy \\
&\geq A^{N-1} \int_{B_R(0)} |\nabla v|^N dy.
\end{aligned}$$

Therefore we have

$$\frac{\int_{B_R(0)} |\nabla u|^N dx}{\left(\int_{B_R(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx\right)^{\frac{N}{q}}} \geq A^{N-1+\frac{N}{q}} (\log a)^{\frac{N}{q}(\beta-1)\frac{A-1}{A}} \frac{\int_{B_R(0)} |\nabla v|^N dy}{\left(\int_{B_R(0)} \frac{|v|^q}{|y|^N (\log \frac{aR}{|y|})^{\frac{N-1}{N}q+1}} dy\right)^{\frac{N}{q}}}$$

which yields that

$$G_{a,\mathrm{rad}} \geq \left(\frac{N(\beta-1)}{(N-1)q}\right)^{N-1+\frac{N}{q}} (\log a)^{\frac{N\beta}{q} - \left(N-1+\frac{N}{q}\right)} \inf_v \frac{\int_{B_R(0)} |\nabla v|^N dy}{\left(\int_{B_R(0)} \frac{|v|^q}{|y|^N (\log \frac{aR}{|y|})^{\frac{N-1}{N}q+1}} dy\right)^{\frac{N}{q}}}.$$

Therefore, for sufficiently large β we have

$$G_{a,\mathrm{rad}} \geq C\beta^{N-1+\frac{N}{q}} (\log a)^{\frac{N\beta}{q} - (N-1+\frac{N}{q})}. \qquad \square$$

Finally we shall show Theorem 4.

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Proof of Theorem 4. (i) It is enough to show that $G_a < G_{a,\mathrm{rad}}$. By Lemma 5 and Lemma 6, for fixed $q > N$ there exists β_* such that for $\beta > \beta_*$

$$G_a \le C\beta^{\frac{N^2}{q}} (\log a)^{\frac{N\beta}{q}} < C\beta^{N-1+\frac{N}{q}} (\log a)^{\frac{N\beta}{q}-(N-1+\frac{N}{q})} \le G_{a,\mathrm{rad}},$$

since $\frac{N^2}{q} < N-1+\frac{N}{q}$. Therefore we see that $G_a < G_{a,\mathrm{rad}}$.

(ii) Let $x = r\omega\,(r = |x|, \omega \in S^{N-1})$ for $x \in B_R(0)$. For $u \in W_0^{1,N}(B_R(0))$ we consider the following radial function U:

$$U(r) = \left(\omega_{N-1}^{-1} \int_{S^{N-1}} |u(r\omega)|^N dS_\omega \right)^{\frac{1}{N}}.$$

Then we have

$$U'(r) \le \left(\omega_{N-1}^{-1} \int_{S^{N-1}} \left| \frac{\partial}{\partial r} u(r\omega) \right|^N dS_\omega \right)^{\frac{1}{N}}$$

which yields that

$$\int_{B_R(0)} |\nabla U|^N dx \le \int_{B_R(0)} \left| \nabla u \cdot \frac{x}{|x|} \right|^N dx \le \int_{B_R(0)} |\nabla u|^N dx. \tag{33}$$

On the other hand, we have

$$\begin{aligned} \int_{B_R(0)} \frac{|U|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx &= \omega_{N-1} \int_0^R \left(\omega_{N-1}^{-1} \int_{S^{N-1}} |u(r\omega)|^N dS_\omega \right)^{\frac{q}{N}} \frac{dr}{r(\log \frac{aR}{|x|})^\beta} \\ &\ge \int_0^R \int_{S^{N-1}} |u(r\omega)|^q dS_\omega \frac{dr}{r(\log \frac{aR}{|x|})^\beta} \\ &= \int_{B_R(0)} \frac{|u|^q}{|x|^N (\log \frac{aR}{|x|})^\beta} dx \end{aligned} \tag{34}$$

where the inequality follows from Jensen's inequality and $q \le N$. From (33) and (34), we obtain $G_{a,\mathrm{rad}} \le G_a$. Therefore $G_{a,\mathrm{rad}} = G_a$ for any $q \le N$ and β. Moreover we observe that any minimizers of G_a must be radial from the equality condition of (33). □

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