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Reaction–Diffusion in Irregular Domains

Ugur G. Abdulla[1]

Faculty of Applied Mathematics and Cybernetics, Baku State University, Baku, Azerbaijan, and School of Mathematical Sciences, University of Nottingham, Nottingham, United Kingdom
E-mail: Ugur.Abdulla@mis.mpg.de

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We consider the Cauchy–Dirichlet and Dirichlet problems for the nonlinear parabolic equation

$$u_t - a(u^m)_{xx} + bu^\beta = 0,$$

where $a > 0$, $b \in R^1$, $m > 0$, and $\beta > 0$. The problems are considered in noncylindrical domains with nonsmooth boundaries. Existence, uniqueness, and comparison results are established. Constructed solutions are continuous up to the nonsmooth boundary if at each interior point the left modulus of the lower (respectively upper) semicontinuity of the left (respectively right) boundary curve satisfies an upper (respectively lower) Hölder condition near zero with Hölder exponent $\nu > \frac{1}{2}$. The value $\frac{1}{2}$ is critical as in the classical theory of the heat equation $u_t = u_{xx}$.

Key Words: Cauchy–Dirichlet problem; Dirichlet problem; nonlinear degenerate parabolic equation; singular parabolic equation; reaction–diffusion; irregular domains; boundary regularity.

1. INTRODUCTION

In this paper we consider the nonlinear parabolic equation

$$u_t - a(u^m)_{xx} + bu^\beta = 0, \tag{1.1}$$

with $u = u(x, t)$, $a > 0$, $b \in R^1$, $m > 0$, and $\beta > 0$. Equation (1.1) is usually called a reaction–diffusion equation. It is a simple and widely used model for various physical, chemical, and biological problems involving diffusion with a source or with absorption, as in modeling filtration in porous media, transport of thermal energy in a plasma, flow of a chemically reacting fluid from a flat surface, the evolution of populations, etc.

[1] Current address: Max-Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany.

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The mathematical theory of degenerate parabolic equations begins with the paper [21], where the first existence, uniqueness, and regularity results, as well as some qualitative properties of solutions of different initial and boundary value problems for the general diffusion equation (including as a particular case an equation (1.1) with $b=0$, $m>1$), have been established. There has been a considerable amount of published work on this subject during the past four decades. For a general study we can refer the reader to the survey articles [3, 17, 22, 25] in the case of the porous medium equation and to article [17] in the case of general nonlinear parabolic equations with implicit degeneracy. However, in paper [21] and in many of the papers which followed, the boundary-value problems were investigated in cylindrical domains, and at least in the one-dimensional case, there is now a complete picture of the general theory (existence, uniqueness, regularity, and comparison results) of these problems for the reaction–diffusion equation (1.1). As for the boundary-value problems in noncylindrical domains with nonsmooth boundaries, there is a complete picture only in the case of the heat equation (1.2). To explain, consider a Dirichlet problem for the heat equation

$$u_t = u_{xx} \qquad \text{in } E=\{(x,t): \phi_1(t)<x<\phi_2(t),\ 0<t\leqslant T\}, \tag{1.2}$$

$$u(x,0)=u_0(x), \qquad \phi_1(0)\leqslant x\leqslant \phi_2(0), \tag{1.3}$$

$$u(\phi_i(t),t)=\psi_i(t), \qquad 0\leqslant t\leqslant T, \tag{1.4}$$

where $0<T\leqslant +\infty$, $\phi_i, \psi_i \in \mathrm{C}[0;T]$, $\phi_1(t)<\phi_2(t)$ for $t\in[0;T]$, $u_0\in \mathrm{C}([\phi_1(0);\phi_2(0)])$ and $u_0(\phi_i(0))=\psi_i(0)$, $i=1,2$. Gevrey [13] proved in 1913 that there exists a classical solution to the problem (1.2)–(1.4) if $\phi_i(t)$ satisfies a Hölder condition with Hölder exponent more than $\frac{1}{2}$. Petrovsky [23] in 1935 generalized this result as follows: There exists a classical solution to the problem (1.2)–(1.4) (which is continuous in $\bar{E}$) if for each $t_0>0$ there exists a function $p(h)$ such that p is defined for all negative h with sufficiently small absolute value, p is positive and monotonically convergent to 0 as $h\to -0$, for sufficiently small $|h|$,

$$\phi_1(t_0)-\phi_1(t)\leqslant 2(t_0-t)^{1/2}\,(-\log p(t-t_0))^{1/2}, \qquad t\in[t_0-|h|;t_0], \tag{1.5}$$

$$\phi_2(t_0)-\phi_2(t)\geqslant -2(t_0-t)^{1/2}\,(-\log p(t-t_0))^{1/2}, \qquad t\in[t_0-|h|;t_0], \tag{1.6}$$

and

$$\lim_{\varepsilon\to 0-}\int_c^{\varepsilon}\frac{p(h)\,|\log p(h)|^{1/2}}{h}\,dh=-\infty,$$

where c is a suitable negative constant. In [23] a necessary condition was also derived which is close to the sufficient one but still differs slightly.

Let $\phi \in C[0; T]$ and for any fixed $t_0 > 0$ define the functions

$$\omega_{t_0}^-(\phi; \delta) = \max(\phi(t_0) - \phi(t) : t_0 - \delta \leqslant t \leqslant t_0)$$
$$\omega_{t_0}^+(\phi; \delta) = \min(\phi(t_0) - \phi(t) : t_0 - \delta \leqslant t \leqslant t_0).$$

For sufficiently small $\delta > 0$ these functions are well defined and converge to zero as $\delta \to 0+$. The function $\omega_{t_0}^-(\phi; \cdot)$ will be called the left modulus of lower semicontinuity of the function ϕ at the point t_0 and accordingly the function $\omega_{t_0}^+(\phi; \cdot)$ will be called the left modulus of upper semicontinuity of the function ϕ at the point t_0.

Hence, the conditions of Petrovsky consist of the upper (respectively lower) estimation for the left modulus of lower (respectively upper) semicontinuity of the left (respectively right) boundary curve at each $t_0 > 0$. In particular, if at some point $t_0 > 0$, $\omega_{t_0}^-(\phi_1; \delta) \leqslant \kappa\delta^\alpha$ or $\omega_{t_0}^+(\phi_2; \delta) \geqslant -\kappa\delta^\alpha$ for sufficiently small $\delta > 0$ and with $\kappa > 0$, $0 < \alpha < \frac{1}{2}$, then the nonexistence of a classical solution to problem (1.2)–(1.4) is possible (see also [5]).

In [19], a necessary and sufficient condition for the regularity of a boundary point in the Dirichlet problem for the heat equation in an arbitrary spatial dimension has been announced. A necessary and sufficient condition which is a geometric characterization for a boundary point of an arbitrary bounded open subset of R^{N+1} to be regular for the heat equation has been established in [9]. A similar criterion for linear parabolic operators with smooth, variable coefficients was established in [12]. Sufficient conditions for boundary regularity in the case of general quasilinear nondegenerate parabolic equations were found in [11, 26]. As far as we know, there are two papers concerning boundary-value problems for nonlinear degenerate parabolic equations in noncylindrical domains. In [6] some existence and boundary regularity results were announced regarding the Dirichlet problem

$$u_t = u\,\Delta u - \gamma\,|\nabla u|^2 \qquad \text{in } D_T = \{(x, t) \in R^N \times (0, T),\ |x| < s(t)\}$$
$$u = 0 \text{ for } |x| = s(t),\ t \in [0, T); \qquad u = u_0 \qquad \text{for} \quad |x| < s(0),\ t = 0,$$

where $\gamma \in R^1$, $s \in \mathrm{C}[0, T] \cap \mathrm{C}^1[0, T)$ is a monotonic function such that $s(T) = 0$, and $s(t) > 0$ for $t \in [0, T)$. It is given that in the particular case $s(t) = c(T-t)_+^p$, $c > 0$, $p > 0$, the constructed limit solution is continuous up to the vertex $(0, T)$ if $p \geqslant \frac{1}{2}$, and respectively is not continuous if $0 < p < \frac{1}{2}$. It should be mentioned that in the particular case of $\gamma < 0$, the function $v = ((am)^{-1}\,u)^{1/(m-1)}$ with $m = 1 - 1/\gamma > 1$ is a solution of a similar problem for the equation $v_t = a\,\Delta v^m$. However, the complete proof of the results from [6] has not been published. In [4] similar results were proved for a class of one-dimensional nonlinear degenerate parabolic equations with mean curvature operator. This class of equations has no relationship to that of the reaction–diffusion equations (1.1).

In this paper we are interested in Cauchy–Dirichlet and Dirichlet problems for Equation (1.1). Let us formulate the problems:

I. *The Cauchy–Dirichlet Poblem* (CDP). Find a solution of Eq. (1.1) in

$$D=\{(x,t): s(t)<x<+\infty, 0<t\leqslant T\},$$

with conditions

$$u(s(t),t)=\psi(t), \qquad 0\leqslant t\leqslant T, \tag{1.7}$$

$$u(x,0)=u_0(x), \qquad s(0)\leqslant x<+\infty, \tag{1.8}$$

where $s\in C[0;T]$, $\psi\in C[0;T]$, $\psi\geqslant 0$ for $t\in[0;T]$, $\sup\psi<+\infty$, $u_0\in C([s(0);+\infty))$, $u_0\geqslant 0$ for $x\in[s(0);+\infty)$, $u_0(s(0))=\psi(0)$, $\sup u_0<+\infty$.

II. *The Dirichlet Problem* (DP). Find a solution of Eq. (1.1) in E with conditions (1.3), and (1.4), where u_0, ϕ_i, ψ_i satisfy the same conditions as in (1.3), (1.4) and also $u_0\geqslant 0$, $\psi_i\geqslant 0$ and $\sup\psi_i<+\infty$.

Obviously, in general the equation (1.1) degenerates at points (x,t) where $u=0$ and we cannot expect the considered problems to have classical solutions. We shall follow the following notions of generalized solutions:

DEFINITION 1.1. We shall say that the function $u(x,t)$ is a solution of CDP in D if

(a) u is nonnegative and continuous in $\bar{D}$, satisifying (1.7), (1.8), and $u\in L_\infty(D\cap(t\leqslant T_1))$ for any finite $T_1\in(0;T]$.

(b) For any finite t_0, t_1 such that $0\leqslant t_0<t_1\leqslant T$ and for any C^∞ functions $\mu_i(t)$, $t_0\leqslant t\leqslant t_1$, $i=1,2$, such that $s(t)<\mu_1(t)<\mu_2(t)$ for $t\in[t_0;t_1]$, the integral identity

$$I(u,f,D_1)=\int_{t_0}^{t_1}\int_{\mu_1(t)}^{\mu_2(t)}(uf_t+au^mf_{xx}-bu^\beta f)\,dx\,dt - \int_{\mu_1(t)}^{\mu_2(t)} uf\Big|_{t=t_0}^{t=t_1}dx - \int_{t_0}^{t_1} au^mf_x\Big|_{x=\mu_1(t)}^{x=\mu_2(t)}dt=0, \tag{1.9}$$

holds, where

$$D_1=\{(x,t):\mu_1(t)<x<\mu_2(t), t_0<t<t_1\}$$

and $f \in C^{2,1}_{x,t}(\bar{D}_1)$ is an arbitrary function that equals zero when $x = \mu_i(t)$, $t_0 \leqslant t \leqslant t_1$, $i = 1, 2$.

DEFINITION 1.2. We shall say that the function $u(x, t)$ is a solution of DP in E if

(a) u is nonnegative and continuous in $\bar{E}$, satisfying (1.3), (1.4);

(b) for any finite t_0, t_1 such that $0 \leqslant t_0 < t_1 \leqslant T$ and for any C^∞ functions $\mu_i(t)$, $t_0 \leqslant t \leqslant t_1$, $i = 1, 2$, such that $\phi_1(t) < \mu_1(t) < \mu_2(t) < \phi_2(t)$ for $t \in [t_0; t_1]$, the integral identity (1.9) holds, where $f \in C^{2,1}_{x,t}(\bar{D}_1)$ is an arbitrary function that equals zero when $x = \mu_i(t)$, $t_0 \leqslant t \leqslant t_1$, $i = 1, 2$.

Furthermore, for both problems we assume that $0 < T \leqslant +\infty$ if $b \geqslant 0$ or $b < 0$ and $0 < \beta \leqslant 1$, and $T \in (0; T^*)$ if $b < 0$ and $\beta > 1$, where $T^* = M^{1-\beta}/(b(1-\beta))$ and $M = \max(\sup u_0, \sup \psi) + \varepsilon$ in CDP and $M = \max(\sup u_0, \sup \psi_1, \sup \psi_2) + \varepsilon$ in DP and $\varepsilon > 0$ is an arbitrary sufficiently small number.

In Section 2 we consider a CDP. In Section 2.1 (Theorem 2.1) we prove that there exists a solution of CDP if for each $t_0 > 0$ there exists a function $F(\delta)$ such that F is defined for all positive and sufficiently small δ, F is positive and convergent to 0 as $\delta \to +0$, and

$$\omega^-_{t_0}(s; \delta) \leqslant \delta^{1/2} F(\delta). \tag{1.10}$$

Furthermore, this assumption will be called assumption (L). In particular, the assumption (L) is satisfied if at every fixed point $t_0 > 0$, s is a left-lower-Hölder continuous with Hölder exponent $\nu > \frac{1}{2}$, i.e., for each $t_0 > 0$ there exists a $\kappa > 0$ and $\nu > \frac{1}{2}$ such that $\omega^-_{t_0}(s; \delta) \leqslant \kappa \delta^\nu$ for sufficiently small positive δ. Then we prove in Section 2.2 (Theorem 2.2) the uniqueness of the solution of the CDP if $a > 0$, $m > 0$, and either $b \geqslant 0$, $\beta > 0$ or $b < 0$, $\beta \geqslant 1$, and if s satisfies the assumption (L) and for each compact subsegment $[0; T_1] \subset [0; T]$ there exists a positive constant M_0 such that

$$s(t) - s(\tau) \geqslant -M_0(t - \tau) \qquad \text{for} \quad 0 \leqslant \tau \leqslant t \leqslant T_1. \tag{1.11}$$

If the initial and boundary data have a positive infimum under the assumption (L) on the curve s, there is also uniqueness in the case when $a > 0$, $m > 0$, $b < 0$, and $0 < \beta < 1$ (see Remark 2.1 and Theorem 2.3). In Section 2.3 we prove the comparison theorem under the same conditions as in the case of uniqueness (see Theorem 2.4 and Remark 2.2).

In Section 3 we consider DP. In Section 3.1 (Theorem 3.1) we prove the existence of a solution of the DP, if ϕ_1 satisfies assumption (L) and ϕ_2

satisfies assumption (R), that is to say, for each $t_0 > 0$ there exists a function $F(\delta)$ as before, such that

$$\omega_{t_0}^{+}(\phi_2; \delta) \geqslant -\delta^{1/2} F(\delta), \tag{1.12}$$

for sufficiently small positive δ. In particular, the assumption (R) is satisfied if at every fixed point $t_0 > 0$, ϕ_2 is a left-upper-Hölder continuous with Hölder exponent $\nu > \frac{1}{2}$, i.e. for each $t_0 > 0$ there exists a $\kappa > 0$ and $\nu > \frac{1}{2}$ such that $\omega_{t_0}^{+}(\phi_2; \delta) \geqslant -\kappa\delta^{\nu}$, for all sufficiently small positive δ. Then in Section 3.2 (Theorem 3.2) we prove the uniqueness of the solution of the DP if

(a) $a > 0$, $m > 0$, and either $b \geqslant 0$, $\beta > 0$ or $b < 0$, $\beta \geqslant 1$;

(b) ϕ_1 satisfies assumption (L) and ϕ_2 satisfies assumption (R); and

(c) for each compact subsegment $[\delta; T_1] \subset (0; T]$ there exists a positive constant M_0 such that

$$\phi_1(t) - \phi_1(\tau) \geqslant -M_0(t-\tau) \qquad \text{for} \quad 0 < \delta \leqslant \tau \leqslant t \leqslant T_1, \tag{1.13a}$$

$$\phi_2(t) - \phi_2(\tau) \leqslant M_0(t-\tau) \qquad \text{for} \quad 0 < \delta \leqslant \tau \leqslant t \leqslant T_1. \tag{1.13b}$$

From these results it easily follows that under the conditions (a) and (b), the solution of the DP is unique even if there exists a finite number of points t_i, $i = 1, ..., k$, such that $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$ and for an arbitrary compact subsegment $[\delta; T_1] \subset (t_i, t_{i+1})$, $i = 0, 1, ..., k$, there exists a positive constant M_0 such that (1.13) is valid. If $T = +\infty$ then uniqueness is still the case if (1.13) violates on a numerate number of points t_k, $k = 1, 2, ...,$ with $t_k \uparrow +\infty$ as $k \to +\infty$ (see Corollary 3.1). If the initial and boundary data have a positive infimum under the assumptions (L) and (R), there is also uniqueness in the case when $a > 0$, $m > 0$, $b < 0$, and $0 < \beta < 1$ (see Theorem 3.3 and Remark 2.1). Finally, we present the comparison theorem under the same conditions as in the case of uniqueness (see Theorem 3.4 and Remark 3.1).

It should be noted that the methods we use are essentially standard, and include parabolic regularization, Holmgrem's method, and construction of barriers. These methods have been developed during the past three decades due to papers [7, 14–16, 18, 21, 24], etc. The most difficult step is the proof of continuity of the constructed limit solution to CDP or DP up to the nonsmooth boundary. This step is proved by using the classical method of barriers and a limiting process. It should also be mentioned that our assumptions made on the boundary curves and boundary data are more general than those made in [6].

A particular motivation for this work arises from the problem of the evolution of interfaces and the local behavior of solutions near the interface in problems for Eq. (1.1). In a recent paper [1], barrier techniques using standard comparison theorems in cylindrical domains have been applied to this problem. As a result, explicit formulae for the interface and for the local solution have been derived, but only in particular cases when the small-time behavior of the solution has a uniform character near the interface. In many cases, however, the latter has a nonuniform behavior. By constructing local barriers it is possible to prove similar results in these cases as well, but only if there is a comparison theorem for Eq. (1.1) in a noncylindrical domain with a boundary curve which has the same kind of behavior as the interface. In many cases this may be nonsmooth and characteristic at the origin. Using the results of this paper, a full description of the evolution of interfaces and of the local solution near the interface for all relevant values of parameters is presented in a forthcoming paper [2].

2. THE CAUCHY–DIRICHLET PROBLEM

2.1. *Existence*

In this section we shall suppose that $a>0$, $m>0$, $b\in R^1$, and $\beta>0$. Our purpose is to prove the following theorem.

THEOREM 2.1. *If s satisfies the assumption* (L) *then there exists a solution of the CDP.*

Proof. Let $\{\varepsilon_n\}$ be an arbitrary real monotonic sequence with $\varepsilon_n\to+0$ and $\{r_n\}$ an arbitrary real monotonic sequence with $r_n\to+\infty$. Let $T_n\equiv T$, $n=1, 2, \ldots$, if $T<+\infty$, and let $\{T_n\}$ be a monotonic positive sequence such that $T_n\to+\infty$ as $n\to+\infty$ if $T=+\infty$. Suppose that $\{s_n\}$ is an arbitrary sequence of functions such that $s_n\in C^\infty[0; T_n]$ and

$$\lim_{n\to+\infty}\max_{0\leqslant t\leqslant T_n}|s_n(t)-s(t)|=0.$$

For simplicity, suppose that $s(0)=0$ and let $s_n(0)=s_{0n}\geqslant 0$, $n=1, 2, \ldots$. Some restriction on the sequence of numbers $\{s_{0n}\}$ will be formulated below. Let $\gamma_b=1$ if $b<0$, and let γ_b be an arbitrary number such that $\gamma_b>\max(m^{-1}; \beta^{-1}; 1)$ if $b>0$ and $\gamma_b>\max(m^{-1}; 1)$ if $b=0$. Henceforth, we shall write γ instead of γ_b. Without loss of generality we may suppose that $\varepsilon_1^\gamma<M$. Take two functional sequences $\{\psi_n\}$ and $\{u_{0n}\}$ and a sequence of numbers $\{s_{0n}\}(s_n(0)=s_{0n})$ such that

1. $s_{0n}\geqslant 0$, $n=1, 2, \ldots$, $\lim_{n\to\infty} s_{0n}=0$,
2. $u_0(0)-\chi(\varepsilon_n)/2\leqslant u_0(s_{0n})\leqslant(u_0^m(0)+(\chi(\varepsilon_n)/2)^m)^{1/m}$, $n=1, 2, \ldots$;
3. $\varepsilon_n^\gamma\leqslant u_{0n}(x)$, $\psi_n(t)\leqslant M$ for $(x, t)\in[0; r_n]\times[0; T_n]$;

4. $u_{0n} \in C^{\infty}[0; r_n]$, $\psi_n \in C^{\infty}[0; T_n]$, $n = 1, 2, \ldots;$

5. $u_{0n}(s_{0n}) = \psi_n(0)$, $a(u_{0n}^m)''(s_{0n}) + s_n'(0)\, u_{0n}'(s_{0n}) - bu_{0n}^{\beta}(s_{0n}) + b\theta_b\, \varepsilon_n^{\beta\gamma} = \psi_n'(0)$, $\theta_b = (1$, if $b > 0$; 0, if $b \leqslant 0)$;

6. $u_{0n}(r_n) = M$, $a(u_{0n}^m)''(r_n) - b\theta_b M^{\beta} + b\theta_b\, \varepsilon_n^{\beta\gamma} = 0;$

7. $0 \leqslant u_{0n}(x) - u_0(x) \leqslant \chi(\varepsilon_n)$ for $0 \leqslant x \leqslant r_n - 1;$

8. $0 \leqslant \psi_n^m(t) - \psi^m(t) \leqslant \chi^m(\varepsilon_n)$ for $0 \leqslant t \leqslant T_n$,

where $\chi(x) = Kx^{\gamma}$ for $x \geqslant 0$ and $K > 1$ is a fixed constant. If the initial and boundary data have a positive infimum, then we may assume that $\chi(x)$, $x > 0$, is an arbitrary continuous positive monotonic function with $\lim_{x \to 0^+} \chi(x) = 0$. Obviously, it is possible to construct such sequences. Consider an auxiliary problem

$$u_t = a(u^m)_{xx} - bu^{\beta} + b\theta_b \varepsilon_n^{\beta\gamma} \qquad \text{in} \quad D_n, \tag{2.1}$$

$$u(x, 0) = u_{0n}(x), \qquad s_{0n} \leqslant x \leqslant r_n, \tag{2.2}$$

$$u(s_n(t), t) = \psi_n(t),\ u(r_n, t) = \psi^1(t), \qquad 0 \leqslant t \leqslant T_n, \tag{2.3}$$

where $D_n = \{(x, t) : s_n(t) < x < r_n, 0 < t \leqslant T_n\}$ and

$$\psi^1(t) = \begin{cases} [M^{1-\beta} - b(1-\theta_b)(1-\beta)\, t]^{1/(1-\beta)}, & \text{if} \quad \beta \neq 1, \\ M \exp(-b(1-\theta_b)\, t), & \text{if} \quad \beta = 1. \end{cases}$$

Without loss of generality, we may suppose that

$$r_n > V_n \equiv 1 + \max_{[0;\, T_n]} |s_n(t)|,\ n = 1, 2, \ldots, \qquad r_n V_n^{-1} \to +\infty \quad \text{as} \quad n \to +\infty.$$

If we introduce new variables $r_n(x - s_n(t))(r_n - s_n(t))^{-1} \to y$, $t \to t$, then (2.1)–(2.3) will be transformed into the problem

$$\begin{aligned} L_n v \equiv v_t - ar_n^2(r_n - s_n(t))^{-2}\, (v^m)_{yy} \\ -(r_n - y)(r_n - s_n(t))^{-1}\, s_n'(t)\, v_y + bv^{\beta} - b\theta_b\, \varepsilon_n^{\beta\gamma} = 0 \quad \text{in} \ D_n' \end{aligned} \tag{2.4}$$

$$v(y, 0) = u_{0n}(s_{0n} + r_n^{-1}(r_n - s_{0n})\, y), \qquad 0 \leqslant y \leqslant r_n, \tag{2.5}$$

$$v(0, t) = \psi_n(t),\ v(r_n, t) = \psi^1(t), \qquad 0 \leqslant t \leqslant T_n, \tag{2.6}$$

where $D_n' = \{(y, t) : 0 < y < r_n, 0 < t \leqslant T_n\}$. From [20, Theorem 6.1, Sect. 6, Chap. 5] it easily follows that there exists a unique classical solution $v = v_n(y, t)$ of the problem (2.4)–(2.6) such that $v_n \in C_{x,t}^{2+\mu, 1+\mu/2}(\bar{D}_n')$ with some $\mu > 0$. The maximum principle implies

$$\varepsilon_n^{\gamma} \leqslant v_n(y, t) \leqslant \psi^1(t) \qquad \text{in} \quad \bar{D}_n'. \tag{2.7}$$

Therefore, the function $u_n(x, t) = v_n(r_n(x - s_n(t))(r_n - s_n(t))^{-1}, t)$ is the classical solution from $C^{2+\mu, 1+\mu/2}_{x, t}(\bar{D}_n)$ of the problem (2.1)–(2.3) and

$$\varepsilon_n^{\gamma} \leqslant u_n(x, t) \leqslant \psi^1(t) \qquad \text{for } (x, t) \in \bar{D}_n. \tag{2.8}$$

From [10, Theorem 10, Sect. 5, Chap. 3] it follows that $(u_n)_x \in C^{2+\mu_1, 1+\mu_1/2}_{x, t}(D_n)$ for some $\mu_1 > 0$.

The next step consists in proving the uniform Hölder continuity of the sequence u_n on every compact subset G of D (obviously u_n is defined on G for $n \geqslant N$ if N is chosen large enough). Consider a sequence of compacts $D^{(k)} = \{(x, t) : s^{(k)}(t) \leqslant x \leqslant \ell_k,\ k^{-1} \leqslant t \leqslant T_k\}$, $k = 1, 2, \ldots$, where ℓ_k is a monotonic sequence such that $\ell_k \to +\infty$ as $k \to +\infty$; $T_k \equiv T$ if $T < +\infty$ and $\{T_k\}$ is a monotonic sequence such that $T_k \to +\infty$ as $k \to +\infty$ if $T = +\infty$; $\{s^{(k)}\}$ is a sequence of functions such that $s^{(k)} \in C^\infty[0; T_k]$, $s^{(k)}(t) > s^{(k+1)}(t) > s(t)$ for $0 \leqslant t \leqslant T_k$, $k = 1, 2, \ldots$, and $\lim_{k \to +\infty} \max_{0 \leqslant t \leqslant T_k} |s^{(k)}(t) - s(t)| = 0$. Hence, we have

$$D = \bigcup_{k=1}^{\infty} D^{(k)}, D^{(k)} \subset D^{(k+1)}, \qquad k = 1, 2, \ldots. \tag{2.9}$$

Obviously, for each fixed k there exists a number $n(k)$ such that $D^{(k)} \subset D_n$ for $n \geqslant n(k)$. The sequence $\{u_n\}$, $n \geqslant n(k)$, satisfies the inequality

$$|(u_n^m)_x| \leqslant M_1(k) \qquad \text{in } D^{(k)}, \tag{2.10}$$

where $M_1(k)$ is a constant which depends on k and does not depend on n. The estimation (2.10) may be proved by Bernstein's method, for example in the form given in [24]. It implies that

$$|u_n(x, t) - u_n(y, t)| \leqslant M_2(k)\, |x - y|^\alpha \qquad \text{in } D^{(k)}, \tag{2.11}$$

where $\alpha = \min(1; m^{-1})$. It is well known from (2.11) that the Hölder estimate follows with respect to the time variable as well. As a matter of fact, the following Hölder estimate may be proved exactly as it is proved in [24],

$$|u_n(x, t) - u_n(y, \tau)| \leqslant M_3(k)(|x - y|^\alpha + |t - \tau|^{\alpha/1+\alpha}) \qquad \text{in } D^{(k)}. \tag{2.12}$$

Thus $\{u_n\}$, $n \geqslant n(k)$, is uniformly bounded and equicontinuous in $D^{(k)}$. It should be pointed out that the equicontinuity of the sequence $\{u_n\}$ in $D^{(k)}$ may be established by using more general results of [8]. From (2.12), (2.9), by a diagonalisation argument and the Arzela-Ascoli theorem, we may find a subsequence n' and a limit function $\tilde{u} \in C(D)$ such that $u_{n'} \to \tilde{u}$ as $n' \to \infty$, pointwise in D, and the convergence is uniform on compact subsets of D. Obviously, $\tilde{u} \in L_\infty(D)$ if $b \geqslant 0$ or $b < 0$ and $\beta > 1$, and

$\tilde{u} \in L_\infty(D \cap (t \leqslant T_1))$ for any finite $T_1 > 0$ if $b < 0$ and $0 < \beta \leqslant 1$. Now consider a function $u(x, t)$ such that $u(x, t) = \tilde{u}(x, t)$ for $(x, t) \in D$, $u(x, 0) = u_0(x)$ for $s(0) \leqslant x < +\infty$, and $u(s(t), t) = \psi(t)$ for $0 \leqslant t \leqslant T$.

Obviously the function $u(x, t)$ satisfies the integral identity (1.9). The continuity of u at the points $(x_0, 0)$, $x_0 > s(0)$ of the line $t = 0$ may easily be established. If u_0^m is locally Lipschitz continuous, it follows from the estimations (2.11), (2.12) which may be proved up to the line $t = 0$. In general, the continuity on the line $t = 0$ may be established by constructing barriers. It remains only to prove the continuity of $u(x, t)$ at the points $(s(t), t)$, $t \geqslant 0$. For that, first consider a function $v(y, t) = u(y + s(t), t)$ in $\bar{D}'$, where $D' = \{(y, t) : 0 < y < +\infty,\ 0 < t \leqslant T\}$. Obviously $v \in C(D') \cap L^\infty(D')$ if $b \geqslant 0$ or $b < 0$ and $\beta > 1$, and $v \in C(D') \cap L^\infty(D' \cap (t \leqslant T_1))$ if $b < 0$, $0 < \beta \leqslant 1$, and T_1 is an arbitrary finite number from $(0; T]$.

The sequence $\{v_{n'}\}$ converges to v as $n' \to +\infty$ pointwise in $\bar{D}'$ and the convergence is uniform on compact subsets of D'. Continuity of the function $u(x, t)$ at the points $(s(t), t)$, $t \geqslant 0$, is equivalent to continuity of the function $v(y, t)$ at the points $(0, t)$, $t \geqslant 0$.

If $t_0 \geqslant 0$ and $\psi(t_0) > 0$, we shall prove that for arbitrary sufficiently small $\varepsilon > 0$ two inequalities

$$\liminf v(y, t) \geqslant \psi(t_0) - \varepsilon \quad \text{as} \quad (y, t) \to (0, t_0),\ (y, t) \in D', \tag{2.13}$$

$$\limsup v(y, t) \leqslant \psi(t_0) + \varepsilon \quad \text{as} \quad (y, t) \to (0, t_0),\ (y, t) \in D'. \tag{2.14}$$

are valid.

As $\varepsilon > 0$ is arbitrary, the continuity of $v(y, t)$ at the boundary point $(0, t_0)$ follows from (2.13), (2.14). If $\psi(t_0) = 0$, then it is sufficient to prove (2.14), since (2.13) (with $\varepsilon = 0$ in the right-hand side) directly follows from the fact that v is non-negative in $\bar{D}'$.

Let $\psi(t_0) > 0$. Take an arbitrary $\varepsilon \in (0; \psi(t_0))$ and prove the inequality (2.13). Consider a function $\omega_n(y, t) = f(h(\mu) - (1 - r_n^{-1} s_n(t))\, y + \mu(t - t_0) + s_n(t_0) - s_n(t))$, $\mu > 0$, $h > 0$, where $f(\zeta) = M_1(\zeta / h(\mu))^\alpha$, $M_1 = \psi(t_0) - \varepsilon$. Then if $b \leqslant 0$, we take the two cases:

(a) $\alpha > m^{-1}$, if $0 < m \leqslant 1$ and,

(b) $m^{-1} < \alpha < (m-1)^{-1}$ if $m > 1$.

If $b > 0$ we take six different cases (as shown in Fig. 1).

I.	$m^{-1} < \alpha \leqslant \min((m-1)^{-1}; (1-\beta)^{-1})$,	if $m > 1$, $0 < \beta < 1$,
II.	$m^{-1} < \alpha \leqslant (m-1)^{-1}$,	if $m > 1$, $\beta \geqslant 1$,
III.	$\alpha > m^{-1}$,	if $0 < m \leqslant 1$, $\beta \geqslant 1$,
IV.	$m^{-1} < \alpha \leqslant (1-\beta)^{-1}$,	if $0 < \beta < 1$, $1 - \beta < m \leqslant 1$,

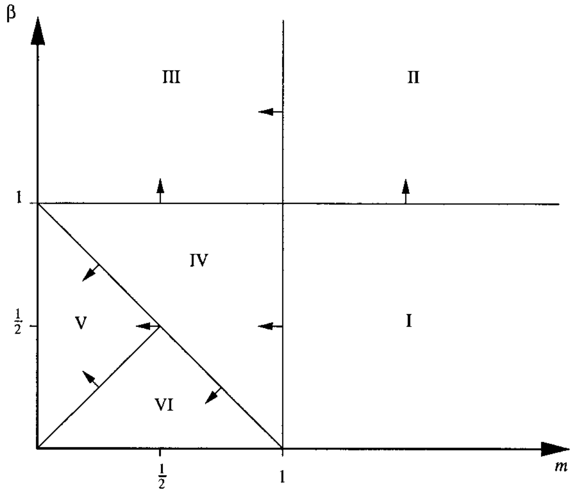

FIGURE 1

V. $\alpha > m^{-1}$, if $0 < m \leqslant \frac{1}{2}$, $m \leqslant \beta \leqslant 1 - m$,

VI. $m^{-1} < \alpha \leqslant 2/(m - \beta)$, if $0 < \beta < \frac{1}{2}$, $\beta < m \leqslant 1 - \beta$.

If $t_0 > 0$ then we choose

$$h(\mu) = M_3 \mu^{-1} F(\mu^{-2}), \qquad M_3 = ((M_2 M_1^{-1})^{1/\alpha} - 1)^{-1},$$

$$M_2 = \psi(t_0) - \varepsilon/2, \ \mu \geqslant \mu_0,$$

where $\mu_0 = \delta_0^{-1/2}$ and we assume that the curve s satisfies the condition (1.10) at the point t_0 for $\delta \in (0; \delta_0]$. If $t_0 = 0$ we choose $h(\mu) = \mu^{-2}$, $\mu \geqslant \mu_0 = 1$. We consider a function $f(\zeta)$ for $\zeta \in [0; h(\mu)(1 + \lambda)]$, where λ is a positive number such that $\lambda \geqslant (M_2 M_1^{-1})^{1/\alpha} - 1$. It may easily be checked that

$$L_n \omega_n = \mu f' - a(f^m)'' + b f^\beta - b\theta_b \, \varepsilon_n^{\beta\gamma}. \tag{2.15}$$

If either $b \leqslant 0$ or $b > 0$ and m, β belong to one of the regions I–IV (Fig. 1), then from (2.15) it follows that

$$L_n \omega_n \leqslant f^{(\alpha-1)/\alpha} \{ \alpha M_1^{1/\alpha} \mu h^{-1}(\mu) - h^{-2}(\mu) \, a M_1^{2/\alpha}$$
$$\times m\alpha(m\alpha - 1) \, M_4^{((m-1)\alpha - 1)/\alpha} + b\theta_b M_4^{\beta - 1 + 1/\alpha} \}, \tag{2.16}$$

where $M_4 = M_1(1+\lambda)^\alpha$. Since $\mu h(\mu) \to 0$ as $\mu \to \infty$, we can choose and fix $\mu_1 \geqslant \mu_0$ so large that if $\mu \geqslant \mu_1$,

$$L_n \omega_n \leqslant 0 \qquad \text{for} \quad 0 \leqslant \zeta \leqslant h(\mu)(1+\lambda). \tag{2.17}$$

If $b > 0$ and m, β belong to one of the regions V, VI (Fig. 1), then from (2.15) it follows that

$$\begin{aligned} L_n \omega_n \leqslant f^\beta (\mu h^{-1}(\mu)\, \alpha M_1^{1/\alpha} M_4^{1-\beta-1/\alpha} \\ - h^{-2}(\mu)\, a M_1^{2/\alpha} m\alpha(m\alpha - 1)\, M_4^{m-\beta-2/\alpha} + b). \end{aligned} \tag{2.18}$$

As before, from (2.18) it follows that we can choose and fix $\mu_1 \geqslant \mu_0$ so large that if $\mu \geqslant \mu_1$ then (2.17) is valid. Let $t_0 > 0$. Since $\psi(t)$ is continuous there exist the numbers $\mu_2 \geqslant \mu_1$ and δ_1 such that $\psi(t) > \psi(t_0) - \varepsilon/2$ for $t_0 - \mu_2^{-2} \leqslant t \leqslant t_0 + \delta_1$, where if $t_0 = T$ (and T is finite) we choose $\delta_1 = 0$, and if $t_0 < T$ then $\delta_1 = \delta_1(\varepsilon) > 0$ is such that $t_0 + \delta_1 < T$. If $t_0 = 0$ then we choose $\delta_1 = \delta_1(\varepsilon) > 0$ such that $\psi(t) > \psi(0) - \varepsilon/2$ for $0 \leqslant t \leqslant \delta_1$. Let us now estimate $\omega_n(0, t)$ in the neighbourhood of t_0. Since $\omega_n(0, t_0) = \psi(t_0) - \varepsilon$ and f is continuous and the sequence $\{s_n\}$ uniformly converges to a continuous function s as $n \to +\infty$, for all fixed $\mu \geqslant \mu_2$ there exists a number $\delta_2 = \delta_2(\mu, \varepsilon) \leqslant \delta_1$ which does not depend on n and a number $N_1 = N_1(\mu, \varepsilon)$ such that for arbitrary $n \geqslant N_1$,

$$\omega_n(0, t) < \psi(t_0) - \varepsilon/2 \qquad \text{for} \quad t_0 \leqslant t \leqslant t_0 + \delta_2$$

(we choose $\delta_2 = 0$ if $t_0 = T$ and $\delta_2 > 0$ if $t_0 < T$). Now suppose that $t_0 > 0$ and consider $\omega_n(0, t)$ for $t_0 - \mu^{-2} \leqslant t \leqslant t_0 + \delta_2$, $\mu \geqslant \mu_2$, and $n \geqslant N_1$. Since the sequence $\{s_n\}$ uniformly converges to s as $n \to +\infty$, we may suppose without loss of generality that $\omega_{t_0}^-(s_n; \delta)$ satisfies (1.10) for $\delta \in (0; \delta_0]$ uniformly with respect to $n \geqslant N_1$. If $t \in [t_0 - \mu^{-2}; t_0]$ then we have

$$\omega_n(0, t) \leqslant f(h(\mu) + s_n(t_0) - s_n(t)) \leqslant f((M_3^{-1} + 1)\, h(\mu)) = \psi(t_0) - \varepsilon/2.$$

If $t_0 = 0$ we choose and fix $\mu_2 \geqslant \mu_1$ and $N_2 \geqslant N_1$ so large that if $\mu \geqslant \mu_2$ and $n \geqslant N_2$ then

$$\begin{aligned} u_0((1 - s_{0n} r_n^{-1})\, y + s_{0n}) \geqslant u_0(0) - \varepsilon/2 = \psi(0) - \varepsilon/2 \\ \text{for} \quad 0 \leqslant y \leqslant r_n (r_n - s_{0n})^{-1}\, h(\mu). \end{aligned}$$

Now let $N_3 \geqslant N_2$ be chosen so large that $\varepsilon_n^\gamma < \psi(t_0) - \varepsilon$ for $n \geqslant N_3$. Let $\eta_n = [M_1^{-1}\, \varepsilon_n^\gamma]^{1/\alpha}\, h(\mu)$, $n \geqslant N_3$. Obviously, $\eta_n \to 0$ as $n \to \infty$ uniformly with respect to $\mu \geqslant \mu_2$. Then we set

$$\Omega_n = \{(y, t) : t_0 - d_{t_0}(\mu) < t \leqslant t_0 + \delta_2, 0 < y < \xi_n(t)\},$$
$$\Lambda_n = \{(y, t) : t_0 - d_{t_0}(\mu) < t \leqslant t_0 + \delta_2, y = \xi_n(t)\},$$
$$\xi_n(t) = r_n(r_n - s_n(t))^{-1} (h(\mu) + \mu(t - t_0) + s_n(t_0) - s_n(t) - \eta_n),$$

and

$$d_{t_0}(\mu) = (0 \text{ if } t_0 = 0;\ \mu^{-2}, \text{ if } t_0 > 0).$$

If $t_0 > 0$ then, since

$$\xi_n(t_0 - \mu^{-2}) \leqslant r_n(r_n - s_n(t_0 - \mu^{-2}))^{-1} ((1 + M_3^{-1})\, h(\mu) - \mu^{-1}), \quad (2.19)$$

we may choose and fix $\mu_3 \geqslant \mu_2$ so large that for arbitrary $\mu \geqslant \mu_3$,

$$\xi_n(t_0 - \mu^{-2}) < 0 \qquad \text{for} \quad n \geqslant N_3. \quad (2.20)$$

Without loss of generality we may suppose that if $T = +\infty$ then for arbitrary fixed $\mu \geqslant \mu_3$

$$t_0 + \delta_2 \leqslant T_n \qquad \text{for} \quad n \geqslant N_3. \quad (2.21)$$

Let us now compare $\omega_n(y, t)$ and $v_n(y, t)$ in Ω_n for fixed $\mu \geqslant \mu_3$ and for $n \geqslant N_3(\mu, \varepsilon)$:

$$\omega_n = f(\eta_n) = \varepsilon_n^\gamma \leqslant v_n \qquad \text{for} \quad (y, t) \in \Lambda_n$$
$$\omega_n(0, t) \leqslant \psi(t_0) - \varepsilon/2 < \psi(t) \leqslant \psi_n(t) = v_n(0, t)$$
$$\text{for} \quad t_0 - d_{t_0}(\mu) \leqslant t \leqslant t_0 + \delta_2.$$

If $t_0 = 0$ we also have

$$\omega_n(y, 0) \leqslant f(h(\mu)) = u_0(0) - \varepsilon \leqslant u_0((1 - r_n^{-1} s_{0n})\, y + s_{0n})$$
$$\leqslant v_n(y, 0) \qquad \text{for} \quad 0 \leqslant y \leqslant r_n(r_n - s_{0n})^{-1} (h(\mu) - \eta_n).$$

We can now apply the maximum principle. Obviously, ω_n is a smooth function in $\bar{\Omega}_n$. Moreover, ω_n is bounded away from zero in $\bar{\Omega}_n$ by ε_n^γ. Consider a function $z(y, t) = v_n(y, t) - \omega_n(y, t)$. Since $z \geqslant 0$ on the parabolic boundary of Ω_n, by applying the maximum principle it follows that $z \geqslant 0$ in $\bar{\Omega}_n$. Let $P = \{(y, t) : 0 < y < y_0, 0 < t \leqslant t_0 + \delta_2\}$, where $0 < y_0 < r_n$ and $\Omega_n \subset P \subset D'_n$ for $\mu \geqslant \mu_3$ and $n \geqslant N_3$. Let $\bar{\omega}_n(y, t) = \{\omega_n(y, t) \text{ in } \bar{\Omega}_n;\ \varepsilon_n^\gamma \text{ in } \bar{P} \setminus \bar{\Omega}_n\}$. Since $v_n \geqslant \varepsilon_n^\gamma$ in $\bar{P}$, we have $\bar{\omega}_n(y, t) \leqslant v_n(y, t)$ in $\bar{P}$. In the limit $n' \to +\infty$ we have

$$\omega(y, t) \leqslant v(y, t) \qquad \text{in} \quad \bar{P} \quad (2.22)$$

where

$$\omega(y,t)=\begin{cases} f(h(\mu)-y+\mu(t-t_0)+s(t_0)-s(t)), & (y,t)\in\bar{\Omega} \\ 0, & (y,t)\in\bar{P}\setminus\bar{\Omega} \end{cases}$$

and

$$\Omega=\{(y,t): t_0-d_{t_0}(\mu)<t\leqslant t_0+\delta_2,\ 0<y<h(\mu)+\mu(t-t_0)+s(t_0)-s(t)\}.$$

Obviously, we have

$$\lim_{\substack{(y,t)\in\bar{P}\\(y,t)\to(0,t_0)}}\omega(y,t)=\lim_{\substack{(y,t)\in\bar{\Omega}\\(y,t)\to(0,t_0)}}\omega(y,t)=\psi(t_0)-\varepsilon.$$

Hence, from (2.22), (2.13) follows.

Let us now prove (2.14). Let $\bar{M}=\psi^1(t_0+\bar{\delta})$, where $\bar{\delta}>0$, be so small that the function ψ^1 is defined and continuous at the point $t_0+\bar{\delta}$. Take an arbitrary $\varepsilon>0$ such that $\psi(t_0)+\varepsilon<\bar{M}$. As before, consider a function

$$\omega_n(y,t)=f_1(h_1(\mu)-(1-r_n^{-1}s_n(t))\,y+\mu(t-t_0)$$
$$+s_n(t_0)-s_n(t)),\qquad \mu>0,\ h_1>0,$$

where

$$f_1(\zeta)=[\bar{M}^{1/\alpha}+\zeta h_1^{-1}(\mu)(M_5^{1/\alpha}-\bar{M}^{1/\alpha})]^{\alpha},\qquad M_5=\psi(t_0)+\varepsilon,$$

and α is an arbitrary number such that $0<\alpha<m^{-1}$. If $t_0>0$ then we choose

$$h_1(\mu)=M_7\mu^{-1}F(\mu^{-2}),\qquad M_7=(\bar{M}^{1/\alpha}-M_5^{1/\alpha})(M_5^{1/\alpha}-M_6^{1/\alpha})^{-1},$$
$$M_6=\psi(t_0)+\varepsilon/2,\ \mu\geqslant\mu_0,$$

where $\mu_0=\delta_0^{-1/2}$, and as before we assume that the curve s satisfies the condition (1.10) at the point t_0 for $\delta\in(0;\delta_0]$. If $t_0=0$ we choose $h_1(\mu)=\mu^{-2}$, $\mu\geqslant\mu_0=1$. We consider a function $f_1(\zeta)$ for $\zeta\in[0;h_1(\mu)(1+\lambda_1)]$, where λ_1 is a positive number such that

$$(M_5^{1/\alpha}-M_6^{1/\alpha})(\bar{M}^{1/\alpha}-M_5^{1/\alpha})^{-1}\leqslant\lambda_1<M_5^{1/\alpha}(\bar{M}^{1/\alpha}-M_5^{1/\alpha})^{-1}.$$

Let us transform $L_n\omega_n$,

$$L_n\omega_n=\mu f_1'-a(f_1^m)''+bf_1^\beta-b\theta_b\varepsilon_n^{\beta\gamma}\geqslant\mu h_1^{-1}(\mu)\,\alpha(M_5^{1/\alpha}-\bar{M}^{1/\alpha})\,M_{10}$$
$$+am\alpha(1-m\alpha)\,h_1^{-2}(\mu)(\bar{M}^{1/\alpha}-M_5^{1/\alpha})^2\,M_9+b(1-\theta_b)\,\bar{M}^\beta-b\theta_b\varepsilon_n^{\beta\gamma},$$

where $M_{10}=\bar{M}^{(\alpha-1)/\alpha}$ if $\alpha\geqslant 1$, $M_{10}=M_8^{(\alpha-1)/\alpha}$ if $\alpha<1$, and

$$M_8=[M_5^{1/\alpha}-\lambda_1(\bar{M}^{1/\alpha}-M_5^{1/\alpha})]^\alpha>0,\qquad M_9=\bar{M}^{(m\alpha-2)/\alpha}.$$

Since $\mu h_1(\mu)\to 0$ as $\mu\to\infty$, we can choose and fix $\mu_1\geqslant\mu_0$ so large that if $\mu\geqslant\mu_1$ then $L_n\omega_n>0$ for $0\leqslant\zeta\leqslant h_1(\mu)(1+\lambda_1)$. Let $t_0>0$. Since $\psi(t)$ is continuous, there exist the numbers $\mu_2\geqslant\mu_1$ and δ_1 such that $\psi(t)<\psi(t_0)+\varepsilon/2$ for $t_0-\mu_2^{-2}\leqslant t\leqslant t_0+\delta_1$, where, if $t_0=T$ (and T is finite), we choose $\delta_1=0$ and if $t_0<T$ then $\delta_1=\delta_1(\varepsilon)\in(0;\bar{\delta}]$ is some number such that $t_0+\delta_1<T$. If $t_0=0$ then we choose $\delta_1=\delta_1(\varepsilon)>0$ such that $\psi(t)<\psi(0)+\varepsilon/2$ for $0\leqslant t\leqslant\delta_1$. Let us now estimate $\omega_n(0,t)$ in the neighbourhood of t_0. Since $\omega_n(0,t_0)=\psi(t_0)+\varepsilon$ and f is continuous and the sequence $\{s_n\}$ uniformly converges to a continuous function s as $n\to+\infty$, there exists a number $0\leqslant\delta_2=\delta_2(\mu,\varepsilon)\leqslant\delta_1$ which does not depend on n and a number $N_1=N_1(\mu,\varepsilon)$ such that for arbitrary $n\geqslant N_1$,

$$\omega_n(0,t)>\psi(t_0)+\varepsilon/2\qquad\text{for}\quad t_0\leqslant t\leqslant t_0+\delta_2$$

(we choose $\delta_2=0$ if $t_0=T$ and $\delta_2>0$ if $t_0<T$). Now suppose that $t_0>0$ and consider $\omega_n(0,t)$ for $t_0-\mu^{-2}\leqslant t\leqslant t_0+\delta_2$, $\mu\geqslant\mu_2$, and $n\geqslant N_1$. As before, we may suppose that $\omega_{t_0}^-(s_n;\delta)$ satisfies (1.10) for $\delta\in(0;\delta_0]$ uniformly with respect to $n\geqslant N_1$. If $t_0\in[t_0-\mu^{-2};t_0]$ then we have

$$\omega_n(0,t)\geqslant f_1(h_1(\mu)+s_n(t_0)-s_n(t))\geqslant f_1((M_7^{-1}+1)\,h_1(\mu))=\psi(t_0)+\varepsilon/2.$$

Now we can choose $N_2=N_2(\mu,\varepsilon)\geqslant N_1$ so large that for $n\geqslant N_2$,

$$\psi(t)\leqslant\psi_n(t)<\psi(t_0)+\varepsilon/2\text{ for }t_0-\mu^{-2}\leqslant t\leqslant t_0+\delta_2.$$

If $t_0=0$ we choose $\mu_2\geqslant\mu_1$ and $N_2\geqslant N_1$ so large that if $\mu\geqslant\mu_2$ and $n\geqslant N_2$, then

$$\psi(t)\leqslant\psi_n(t)<\psi(0)+\varepsilon/2\qquad\text{for}\quad 0\leqslant t\leqslant\delta_2$$

$$u_{0n}((1-s_{0n}r_n^{-1})\,y+s_{0n})\leqslant u_0((1-s_{0n}r_n^{-1})\,y+s_{0n})+K\varepsilon_n^\gamma<u_0(0)+\varepsilon$$
$$\text{for}\quad 0\leqslant y\leqslant r_n(r_n-s_{0n})^{-1}\,h_1(\mu).$$

Then we set Ω_n,Λ_n,ξ_n as before, by replacing h and η_n with h_1 and 0 respectively. We then derive (2.19)–(2.21), replacing M_3 and N_3 with M_7 and N_2 respectively.

Let us now compare $\omega_n(y,t)$ and $v_n(y,t)$ in Ω_n for fixed $\mu\geqslant\mu_3$ and for $n\geqslant N_2$. We have

$$\omega_n(0,t)>v_n(0,t)\qquad\text{for}\quad t_0-d_{t_0}(\mu)\leqslant t\leqslant t_0+\delta_2$$
$$\omega_n=\bar{M}=\psi^1(t_0+\bar{\delta})\geqslant\psi^1(t_0+\delta_2)\geqslant v_n\qquad\text{for}\quad(x,t)\in\bar{\Lambda}_n.$$

If $t_0=0$ then

$$\omega_n(y,0)\geqslant f_1(\mu^{-2})=u_0(0)+\varepsilon>u_{0n}((1-s_{0n}r_n^{-1})y+s_{0n})=v_n(y,0)$$
$$\text{for}\quad 0\leqslant y\leqslant r_n(r_n-s_{0n})^{-1}\mu^{-2}.$$

Consider a function $z(y,t)=v_n(y,t)-\omega_n(y,t)$. Since $z\leqslant 0$ on the parabolic boundary of Ω_n, by applying the maximum principle it follows that $z\leqslant 0$ in $\bar{\Omega}_n$. As before, consider a rectangular P, where $0<y_0<r_n$ and $\Omega_n\subset P\subset D'_n$ for $\mu\geqslant\mu_3$ and $n\geqslant N_2$. Let

$$\bar{\omega}_n(y,t)=\{\omega_n(y,t)\text{ in }\bar{\Omega}_n;\ \bar{M}\text{ in }\bar{P}\backslash\bar{\Omega}_n\}.$$

Since $v_n(y,t)\leqslant\bar{M}$ in $\bar{P}$, we have $\bar{\omega}_n\geqslant v_n$ in $\bar{P}$. In the limit as $n'\to\infty$, we have

$$\omega(y,t)\geqslant v(y,t)\qquad\text{in }\bar{P},\tag{2.23}$$

where $\omega(y,t)=\{f_1(h_1(\mu)+\mu(t-t_0)+s(t_0)-s(t)-y)$ in $\bar{\Omega}$; $\bar{M}$ in $\bar{P}\backslash\bar{\Omega}\}$ and Ω is defined as before with h being replaced by h_1. Obviously

$$\lim_{\substack{(y,t)\in\bar{P}\\(y,t)\to(0,t_0)}}\omega(y,t)=\lim_{\substack{(y,t)\in\bar{\Omega}\\(y,t)\to(0,t_0)}}\omega(y,t)=\psi(t_0)+\varepsilon.$$

Hence, from (2.23), (2.14) follows and we have completed the proof of the continuity of $v(y,t)$ on the line $y=0$, that is to say, the continuity of $u(x,t)$ on the curve $x=s(t)$, $t\geqslant 0$. The theorem is proved.

Remark 2.1. It should be noted that since we construct the solution as a limit of a sequence of classical solutions to nondegenerate parabolic problems, by using a generalization of the Nash theorem [20, Theorem 10.1, Chap. III] and Friedman's a priori interior estimations [10, Theorem 10, Chap. III] one may show by standard methods that the generalized solution is a classical solution in a neighbourhood of any interior point (x_0,t_0), where $u(x_0,t_0)>0$. If, in particular, a constructed solution has a positive infimum, then it is a classical solution and the uniqueness of this solution immediately follows from the existence theorem (which includes continuity up to the boundaries) and from the classical maximum principle. This observation is of a general nature and it relates to both problems considered in this paper (see Theorems 2.3 and 3.3 below).

2.2. *Uniqueness*

Throughout this section we shall suppose that the boundary curve s satisfies the assumption (L).

THEOREM 2.2. *Let $a>0$, $m>0$, and either $b\geqslant 0$, $\beta>0$ or $b<0$, $\beta\geqslant 1$. If s satisfies* (1.11) *then the solution of the CDP is unique.*

Proof. Suppose that g_1 and g_2 are two solutions of the CDP. Let $\bar{t}\in(0;T]$ be an arbitrary finite number. We shall prove uniqueness by proving that for some limit solution $u=\lim u_n$ the inequalities

$$\int_{s(t)}^{+\infty}(u(x,t)-g_i(x,t))\,\omega(x)\,dx\leqslant 0,\quad i=1,2, \tag{2.24}$$

for every $t\in(0;\bar{t}]$ and for every $\omega\in C_0^\infty((s(t);+\infty))$ such that $|\omega|\leqslant 1$, are valid. Obviously, from (2.24) it follows that $g_1=u=g_2$ for $s(t)\leqslant x<+\infty$, $0\leqslant t\leqslant\bar{t}$, which implies uniqueness in view of the arbitrariness of $\bar{t}$.

Let $t\in(0;\bar{t}]$ be fixed and let $\omega\in C_0^\infty((s(t);+\infty))$ be an arbitrary function such that $|\omega|\leqslant 1$. Assume that

$$\operatorname{supp}\omega=(p;q),\qquad s(t)<p<q<+\infty. \tag{2.25}$$

Let $\chi(x)=Kx^\gamma$ for $x\geqslant 0$ (see the proof of Theorem 2.1). Suppose also that the sequences $\{r_n\}$, $\{s_n\}$ satisfy, in addition to the conditions from Theorem 2.1,

$$s'_n(\tau)\geqslant -M_0\qquad\text{for}\quad 0\leqslant\tau\leqslant\bar{t},\qquad n=1,2,\ldots,$$

(the possibility of which follows from (1.11)),

$$\begin{aligned}
&s_n(\tau)>s_{n+1}(\tau)>s(\tau)\text{ for }0\leqslant\tau\leqslant\bar{t}, && n=1,2,\ldots,\\
&\max_{0\leqslant\tau\leqslant\bar{t}}|g_i^m(s_n(\tau),\tau)-g_i^m(s(\tau),\tau)|\leqslant\chi^m(\varepsilon_n), && n=1,2,\ldots,\ i=1,2,\\
&r_n\geqslant\max(nc_n^{-1};\alpha_n+d_n\bar{t}+\varepsilon_n^{-1}), && n=1,2,\ldots,
\end{aligned}$$

where

$$\begin{aligned}
&c_n=A_0^{-1/2}(\varepsilon_n^{\gamma-1}+b\varepsilon_n^{\beta\gamma-1})^{1/2}, && d_n=A_0\varepsilon_n^{1-\gamma} && \text{if } b\geqslant 0,\\
&c_n=a^{-1/2}\varepsilon_n^{(1-m)/2}, && d_n=a\varepsilon_n^{m-1} && \text{if } b<0,\ 0<m<1,\\
&c_n=A_0^{-1/2}, && d_n=A_0 && \text{if } b<0,\ m\geqslant 1,\\
&\alpha_n=\max(\ln 2;\ B_0^{1/2}a^{-1/2}\varepsilon_n^{(1-m\gamma)/2}) && && \text{if } b\geqslant 0,\text{ or } b<0\text{ and } m\geqslant 1,\\
&\alpha_n=\max(\ln 2;\ (B_0\Delta_0^{-1})^{1/2}) && && \text{if } b<0,\ 0<m<1,
\end{aligned}$$

and A_0, B_0, Δ_0 are positive constants (defined below). Without loss of generality we may assume that $s_n(t) < p$, $r_n \geqslant q+1$, $n = 1, 2, \ldots$. Since the proof of (2.24) is similar for each i, we shall henceforth let $g = g_i$. Let $R_n = \{(x, \tau) : s_n(\tau) < x < r_n,\ 0 \leqslant \tau < t\}$. Take any function $f \in C^{2,1}_{x,t}(\bar{R}_n)$ such that $f = 0$ for $x = s_n(\tau)$ and $x = r_n$, $0 \leqslant \tau \leqslant t$. Let $u = \lim u_n$ be a limit solution of CDP constructed in the proof of Theorem 2.1. We have

$$I(u_n, f, R_n) + b\theta_b \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f(x, \tau)\, dx\, d\tau - I(g, f, R_n) = 0, \tag{2.26}$$

If $b \geqslant 0$ then we transform (2.26) as

$$\begin{aligned}
\int_{s_n(t)}^{r_n} (u_n(x, t) - g(x, t))\, f(x, t)\, dx &= \int_{s_{0n}}^{r_n} (u_{0n}(x) - u_0(x))\, f(x, 0)\, dx \\
&+ a \int_0^t (\psi_n^m(\tau) - g^m(s_n(\tau), \tau))\, f_x(s_n(\tau), \tau)\, d\tau \\
&- a \int_0^t u_n^m(r_n, \tau) - g^m(r_n, \tau)\, f_x(r_n, \tau)\, d\tau \\
&+ \int_0^t \int_{s_n(\tau)}^{r_n} (C_n^k f_\tau + A_n^k f_{xx} - B_n^k f)(u_n^{1/\gamma} - g^{1/\gamma})\, dx\, d\tau + b\, \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f\, dx\, d\tau \\
&+ \int_0^t \int_{s_n(\tau)}^{r_n} ((C_n - C_n^k)\, f_\tau + (A_n - A_n^k)\, f_{xx} \\
&- (B_n - B_n^k)\, f)(u_n^{1/\gamma} - g^{1/\gamma})\, dx\, d\tau,
\end{aligned} \tag{2.27}$$

where

$$A_n(x, t) = am\gamma \int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)\, g^{1/\gamma})^{m\gamma - 1}\, d\theta,$$

$$B_n(x, t) = b\beta\gamma \int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)\, g^{1/\gamma})^{\beta\gamma - 1}\, d\theta,$$

$$C_n(x, t) = \gamma \int_0^1 (\theta u_n^{1/\gamma} + (1-\theta)\, g^{1/\gamma})^{\gamma - 1}\, d\theta,$$

and A_n^k, B_n^k, C_n^k, $k = 1, 2, \ldots$ are C^∞ approximations of A_n, B_n, C_n, respectively, in $\bar{R}_n$. We assume that

$$\max L_n^k \leqslant \max L_n, \qquad \min L_n^k \geqslant \min L_n \qquad \text{in } \bar{R}_n, \tag{2.28}$$

where L stands for A to C, respectively. If $b < 0$ then we transform (2.26) as

$$\int_{s_n(t)}^{r_n} (u_n(x,t)-g(x,t))\, f(x,t)\, dx = \int_{s_{0n}}^{r_n} (u_{0n}(x)-u_0(x))\, f(x,0)\, dx$$

$$+a\int_0^t (\psi_n^m(\tau)-g^m(s_n(\tau),\tau))\, f_x(s_n(\tau),\tau)\, d\tau$$

$$-a\int_0^t (u_n^m(r_n,\tau)-g^m(r_n,\tau))\, f_x(r_n,\tau)\, d\tau$$

$$+\int_0^t \int_{s_n(\tau)}^{r_n} (f_\tau + A_n^k f_{xx} - B_n^k f)(u_n - g)\, dx\, d\tau$$

$$+\int_0^t \int_{s_n(\tau)}^{r_n} ((A_n - A_n^k)\, f_{xx} - (B_n - B_n^k)\, f)(u_n - g)\, dx\, d\tau, \tag{2.29}$$

where A_n, B_n, A_n^k, B_n^k are the same as before (it should be stressed that $\gamma = 1$ if $b<0$).

Since $\varepsilon_n^\gamma \leqslant u_n(x,\tau) \leqslant \psi^1(t)$, in $\bar{R}_n$, we have

$$\begin{aligned}
&a\,\varepsilon_n^{m\gamma-1} \leqslant A_n,\, A_n^k \leqslant \bar{A} && \text{in } \bar{R}_n \text{ if } b\geqslant 0 \text{ or } b<0 \text{ and } m\geqslant 1,\\
&0<\Delta \leqslant A_n,\, A_n^k \leqslant a\,\varepsilon_n^{m-1} && \text{in } \bar{R}_n, \text{ if } b<0 \text{ and } 0<m<1,\\
&|b|\,\varepsilon_n^{\beta\gamma-1} \leqslant |B_n|,\, |B_n^k| \leqslant \bar{B} && \text{in } \bar{R}_n,\\
&\varepsilon_n^{\gamma-1} \leqslant C_n,\, C_n^k \leqslant \bar{C}, && \text{in } \bar{R}_n, \text{ if } b\geqslant 0,
\end{aligned} \tag{2.30}$$

where $\bar{A}$, $\bar{B}$, $\bar{C}$, Δ are some positive constants which do not depend on n, k. Furthermore, we shall suppose that $A_0=\bar{A}$, $C_0=\bar{C}$, $\Delta_0=\Delta$, and $B_0>\bar{B}$. Then consider the problem

$$\mathscr{L}_1 f = D_n^k f_\tau + A_n^k f_{xx} - B_n^k f = 0 \qquad \text{in } R_n, \tag{2.31a}$$

$$f(x,t)=\omega(x), \qquad s_n(t)\leqslant x\leqslant r_n, \tag{2.31b}$$

$$f(s_n(\tau),\tau)=f(r_n,\tau)=0. \qquad 0\leqslant\tau\leqslant t, \tag{2.31c}$$

where $D_n^k = C_n^k$ if $b\geqslant 0$ and $D_n^k\equiv 1$ if $b<0$. The existence and uniqueness of the classical solution to (2.31) follows from [10]. The solution $f=f(x,\tau)$ has the properties

I. $|f|\leqslant \exp(\sigma_b B_0(t-\tau))$, $\quad (x,\tau)\in\bar{R}_n$, $\sigma_b=(1 \text{ if } b<0; 0 \text{ if } b\geqslant 0)$,

II. $|f|\leqslant \exp[c_n(q-x)+(1+\sigma_b B_0)(t-\tau)]$, $\quad (x,\tau)\in\bar{R}_n$,

$$\text{III.} \qquad |f| \leqslant \exp\left[q - x + (d_n + \sigma_b B_0)(t-\tau)\right], \qquad (x,\tau) \in \bar{R}_n,$$

$$\text{IV.} \quad |f_x(r_n,\tau)| = O(\alpha_n \exp(-\varepsilon_n^{-1})) \qquad \text{as } n \to +\infty \text{ for } \tau \in [0;t],$$

$$\text{V.} \qquad |f_x(s_n(\tau),\tau)| = O(\varepsilon_n^s) \qquad \text{as } n \to +\infty \text{ for } \tau \in [0;t],$$

where $s = (1 - m\gamma$, if $b \geqslant 0$; $1-m$, if $b<0$ and $m \geqslant 1$; 0, if $b<0$ and $0<m<1)$.

$$\text{VI.} \qquad \|f\|_{W_q^{2,1}(R_n)} \leqslant M_*(n), \qquad q>1,$$

where the constant M_* does not depend on k.

The proof of I–V is standard and based on the maximum principle (see e.g. [7, 16, 18, 24]). Let us prove Property V. Assume that

$$\begin{aligned}
&e_n = a(2C_0M_0)^{-1}\,\varepsilon_n^{m\gamma-1} && \text{if} \quad b \geqslant 0,\\
&e_n = \bar{e}\varepsilon_n^{m-1},\ \bar{e} = \min(2\tilde{\delta};\, a(2M_0)^{-1};\, (a/B_0)^{1/2}), && \text{if} \quad b<0,\ m \geqslant 1,\\
&e_n = \bar{e} = \min(2\tilde{\delta};\, B_0^{-1}((M_0^2 + 2B_0\Delta)^{1/2} - M_0)), && \text{if} \quad b<0,\ 0<m<1,
\end{aligned}$$

where $\tilde{\delta}>0$ is chosen such that $s_n(t) + \tilde{\delta} < p$, $n = 1, 2, \ldots$.

Let $R_{2n} = \{(x,\tau) : s_n(\tau) < x < s_n(\tau) + e_n/2,\ 0 \leqslant \tau < t\}$. Obviously, $s_n(t) + e_n/2 < p$ and $s_n(\tau) + e_n/2 < r_n$ for $0 \leqslant \tau \leqslant t$, if n is chosen large enough. If $b>0$ consider a function $f_1(x,\tau) = 4e_n^{-2}(s_n(\tau) - x + e_n)^2 \pm f(x,\tau)$. Obviously $f_1 \geqslant 0$ in $\bar{R}_{2n} \backslash R_{2n}$. Moreover, we have $\mathscr{L}_1 f_1 \geqslant -8C_0M_0e_n^{-1} + 8a\varepsilon_n^{m\gamma-1}e_n^{-2} - 4B_0 \geqslant 0$ in R_{2n}, if n is chosen large enough. It follows that f_1 cannot attain its maximum in $\bar{R}_{2n}$ at some point of R_{2n}. Since

$$\begin{aligned}
&f_1(s_n(\tau) + e_n/2, \tau) \leqslant f_1(s_n(\tau), \tau) && \text{for } 0 \leqslant \tau \leqslant t\\
&f_1(x,t) \leqslant f_1(s_n(t), t) && \text{for } s_n(t) \leqslant x \leqslant s_n(t) + e_n/2,
\end{aligned} \tag{2.32}$$

the function f_1 attains its maximum in $\bar{R}_{2n}$ on the whole curve $x = s_n(\tau)$, $0 \leqslant \tau \leqslant t$. That is to say, $f_{1x}(s_n(\tau),\tau) \leqslant 0$ for $0 \leqslant \tau \leqslant t$ or $|f_x(s_n(\tau),\tau)| \leqslant 8e_n^{-1}$ for $0 \leqslant \tau \leqslant t$, which implies Property V. If $b=0$, the same arguments may be applied to the function $f_1(x,\tau) = \exp(\tau)[4e_n^{-2}(s_n(\tau) - x + e_n)^2 \pm f(x,\tau)]$.

If $b<0$ then we first consider a function $f_1 = \exp(B_0(\tau - t))\, f$. Then we apply the same arguments to the function $f_2(x,\tau) = 4e_n^{-2}(s_n(\tau) - x + e_n)^2 \pm f_1(x,\tau)$. Then Property VI follows from [20, Theorem 9.1, Sect. 9, Chap. IV].

Now consider (2.27) (resp. (2.29)) with $f = f(x,\tau)$, which is a solution of the problem (2.31). Then we have

$$\int_{s_n(t)}^{r_n} (u_n(x,t)-g(x,t))\,\omega(x)\,dx = \int_{s_{0n}}^{r_n} (u_{0n}(x)-u_0(x))\,f(x,0)\,dx$$
$$+a\int_0^t (\psi_n^m(\tau)-g^m(s_n(\tau),\tau))\,f_x(s_n(\tau),\tau)\,d\tau$$
$$-a\int_0^t (u_n^m(r_n,\tau)-g^m(r_n,\tau))\,f_x(r_n,\tau)\,d\tau$$
$$+\int_0^t \int_{s_n(\tau)}^{r_n} ((1-\sigma_b)(C_n-C_n^k)\,f_\tau+(A_n-A_n^k)\,f_{xx}$$
$$-(B_n-B_n^k)\,f)(u_n^{1/\gamma}-g^{1/\gamma})\,dx\,d\tau$$
$$+b\theta_b\varepsilon_n^{\beta\gamma}\int_0^t \int_{s_n(\tau)}^{r_n} f\,dx\,d\tau=\sum_{i=1}^5 J_i. \tag{2.33}$$

By using Properties I–VI we estimate the right-hand side of (2.33) as:

$$|J_1|\leqslant\int_{s_{0n}}^{r_n-1} |u_{0n}(x)-u_0(x)|\,|f(x,0)|\,dx+\int_{r_n-1}^{r_n} |u_{0n}(x)-u_0(x)|\,|f(x,0)|\,dx$$
$$\leqslant K\exp\left[c_nq+(1+\sigma_bB_0)\,t\right]\varepsilon_n^\gamma c_n^{-1}$$
$$+2M\exp\left(c_n(q+1)+(1+\sigma_bB_0)\,t-n\right)=o(1),\qquad n\to+\infty,$$

$$|J_2|\leqslant a\int_0^t (|\psi_n^m(\tau)-\psi^m(\tau)|+|g^m(s_n(\tau),\tau)-g^m(s(\tau),\tau)|)\,|f_x(s_n(\tau),\tau)|\,d\tau$$
$$=O(\varepsilon_n^{m\gamma+s})\qquad \text{as } n\to+\infty,$$

$$|J_3|\leqslant a\int_0^t |u_n^m(r_n,\tau)-g^m(r_n,\tau)|\,|f_x(r_n,\tau)|\,d\tau$$
$$=O\Big(\alpha_n\exp(-\varepsilon_n^{-1})\Big),\ \text{as } n\to+\infty.$$

In view of Property VI we have $\lim_{k\to+\infty} J_4=0$. If $b\leqslant 0$, then $J_5=0$, but if $b>0$ then we have

$$|J_5|\leqslant b\varepsilon_n^{\beta\gamma}\int_0^t \int_{s_n(\tau)}^{r_n} \exp\left[c_n(q-x)+t-\tau\right]dx\,d\tau$$
$$\leqslant b[\exp(t)-1]\exp\left(c_n(q-\tilde{s})\right)\varepsilon_n^{\beta\gamma}\,c_n^{-1}=o(1)\qquad\text{as}\quad n\to+\infty,$$

where $\tilde{s}$ is an arbitrary number such that $\tilde{s}\leqslant\min_{0\leqslant\tau\leqslant t} s_n(\tau)$, $n=1,2,\ldots$. By using these estimates in (2.33) and passing to the limit first with respect to $k\to+\infty$ and then with respect to $n\to+\infty$ from (2.33), (2.24) follows. The theorem is proved.

THEOREM 2.3. *Let* $a>0$, $b<0$, $m>0$, and $0<\beta<1$. *Then if*

$$u_0(x),\ \psi(t) \geqslant \delta > 0 \qquad \text{for} \quad (x, t) \in [0; +\infty) \times [0; T], \tag{2.34}$$

then the CDP *has a unique solution.*

From (2.34) and the proof of Theorem 2.1 it easily follows that the constructed solution of the CDP satisfies

$$u(x, t) \geqslant \delta > 0 \qquad \text{for} \quad (x, t) \in \bar{D}.$$

Hence u is a classical solution (see Remark 2.1) and the uniqueness of the solution immediately follows from the maximum principle.

2.3. *Comparison Theorem*

In this section we shall prove the comparison theorem for the solution of the CDP.

DEFINITION 2.1. We shall say that the function $g(x, t)$ is a supersolution (respectively sub-solution) of Eq. (1.1) in D if

(a) g is non-negative and continuous in $\bar{D}$ and $g \in L_\infty(D \cap (t \leqslant T_1))$ for any finite $T_1 \in (0; T]$,

(b) for any finite t_0, t_1 such that $0 \leqslant t_0 < t_1 \leqslant T$ and for any C^∞ functions $\mu_i(t)$, $t_0 \leqslant t \leqslant t_1$, $i=1, 2$ such that $s(t) < \mu_1(t) < \mu_2(t)$ for $t \in [t_0; t_1]$ (see Definition 1.1), the integral inequality

$$I(g, f, D_1) \leqslant 0 \qquad (\geqslant 0) \tag{2.35}$$

holds where $f \in C^{2,1}_{x,t}(\bar{D}_1)$ is an arbitrary nonnegative function such that $f(\mu_i(t), t) = 0$ for $t_0 \leqslant t \leqslant t_1$, $i=1, 2$.

The next lemma gives a sufficient condition for super- or subsolutions.

LEMMA 2.1. *Let g be a nonnegative and continuous function in $\bar{D}$ belonging to $C^{2,1}_{x,t}$ in D outside a finite number of curves $x=\eta_i(t)$, which divide D into a finite number of subdomains D^j, where $\eta_i \in C[0; T]$; for arbitrary $\delta>0$ and finite $\Delta \in (\delta; T]$ the function η_i is absolutely continuous in $[\delta; \Delta]$. Let g satisfy the inequality*

$$Lg = g_t - a(g^m)_{xx} + bg^\beta \geqslant 0 \qquad (\leqslant 0)$$

at the points of D, where $g \in C^{2,1}_{x,t}$. Assume also that the function $(g^m)_x$ is continuous in D and $g \in L_\infty(D \cap (t \leqslant T_1))$ for any finite $T_1 \in (0; T]$. Then g is a supersolution (subsolution) of Eq. (1.1) in D.

Proof. Let D_1 be given and take non-negative $f \in C^{2,1}_{x,t}(\bar{D}_1)$ such that $f(\mu_i(t), t) = 0$ for $t_0 \leqslant t \leqslant t_1$, $i = 1, 2$. Denote $D^j_1 = D_1 \cap D^j$. Let $\delta_n \equiv t_0$, $n = 1, 2, ...$, if $t_0 > 0$, whilst if $t_0 = 0$, then δ_n is a positive monotone sequence such that $\lim \delta_n = 0$, $0 < \delta_{n+1} < \delta_n < t_1$, $n = 1, 2, ...$ Integrating by parts the expression $(-Lg) f$ in all regions $D^j_1 \cap ((x, t) \colon t \geqslant \delta_n)$, and then summing and taking the limit as $n \to +\infty$ yields (2.35). The lemma is proved.

THEOREM 2.4 (Comparison). *Let the conditions of Theorem 2.2 be satisfied. Let u be a solution of the CDP, g be a supersolution (respectively, subsolution) of Eq. (1.1) in D, and*

$$u_0(x) \leqslant (\geqslant)\, g(x, 0) \qquad \textit{for} \quad s(0) \leqslant x < +\infty, \tag{2.36a}$$

$$\psi(t) \leqslant (\geqslant)\, g(s(t), t) \qquad \textit{for} \quad 0 \leqslant t \leqslant T. \tag{2.36b}$$

Then

$$u(x, t) \leqslant (\geqslant)\, g(x, t) \qquad \textit{in} \quad \bar{D}.$$

Proof. First, we prove the theorem for supersolutions. The proof is similar to the proof of uniqueness. Suppose on the contrary that $g(x_*, t) < u(x_*, t)$ for some $(x_*, t) \in D$. The continuity of g and u implies that $g(x, t) < u(x, t)$ for $x \in [x_* - \mu; x_* + \mu]$, where $\mu > 0$ is chosen such that $s(t) < x_* - \mu$. Then we take an arbitrary function $\omega \in C^\infty_0((s(t); +\infty))$ such that

$$0 \leqslant \omega \leqslant 1 \qquad \text{for} \quad s(t) \leqslant x < +\infty,$$

$$\omega > 0 \qquad \text{for} \quad |x - x_*| < \mu; \ \omega = 0 \qquad \text{for} \quad |x - x_*| \geqslant \mu.$$

Our goal will be achieved if we prove the inequality

$$\int_{s(t)}^{+\infty} (u(x, t) - g(x, t))\, \omega(x)\, dx \leqslant 0, \tag{2.37}$$

which is a contradiction of our assumption. To prove (2.37), first we construct a sequence $\{u_n\}$ as in Theorems 2.1 and 2.2. Since u is a unique solution of CDP we have $u = \lim u_n$. Since g is a supersolution of Eq. (1.1) in R_n and u_n is a solution of Eq. (2.1) in R_n, we have instead of (2.26)

$$I(u_n, f, R_n) + b\theta_b\, \varepsilon_n^{\beta\gamma} \int_0^t \int_{s_n(\tau)}^{r_n} f(x, \tau)\, dx\, d\tau - I(g, f, R_n) \geqslant 0. \tag{2.38}$$

Then instead of f in (2.38) we take the classical solution of the problem (2.31). Since ω is a non-negative function, from the maximum principle it follows that $f \geqslant 0$ for $(x, \tau) \in \bar{R}_n$ and hence $f_x(s_n(\tau), \tau) \geqslant 0$ for $0 \leqslant \tau \leqslant t$. By using this, from (2.36) and (2.38), (2.37) follows. The proof coincides with the proof given in the uniqueness theorem, Theorem 2.2. The proof for the subsolution is similar. The theorem is proved.

Remark 2.2. If the conditions of Theorem 2.3 are satisfied, then the comparison theorem, Theorem 2.4, is valid if we require g to be a classical smooth supersolution (respectively, subsolution) of Eq. (1.1) in D satisfying (2.36) (see Remark 2.1 and the proof of Theorem 2.3). If in addition to the conditions of Theorem 2.3, the boundary curve s also satisfies (1.11), then the assertion of Theorem 2.4 is valid. The proof is similar to that of Theorem 2.4.

Remark 2.3. It should be noted that the definition of super- and subsolutions and the sufficient condition for super- and subsolutions in the case of DP coincide with the definition and Lemma 2.1 given in this section. The only difference is that the domain D should be replaced by the domain E.

3. THE DIRICHLET PROBLEM

3.1. *Existence*

In this section we shall suppose that $a>0$, $m>0$, $b\in R^1$, and $\beta>0$.

THEOREM 3.1. *If* ϕ_1 *satisfies the assumption* (L) *and* ϕ_2 *satisfies the assumption* (R) *then there exists a solution of the DP.*

Proof. Let the sequences $\{\varepsilon_n\}$, $\{T_n\}$ be chosen as in the proof of Theorem 2.1. Suppose that $\{\phi_{in}\}$, $i=1,2$, are arbitrary sequences of functions such that

$$\phi_{in}\in C^\infty[0;T_n], \qquad \phi_{1n}(t)<\phi_{2n}(t) \text{ for } t\in[0;T_n]$$
$$\lim_{n\to+\infty}\max_{0\leqslant t\leqslant T_n}|\phi_{in}(t)-\phi_i(t)|=0.$$

For simplicity, suppose that $\phi_1(0)=0$, $\phi_2(0)=H>0$, and let $\phi_{1n}(0)=\phi^0_{1n}$, $\phi_{2n}(0)=\phi^0_{2n}$, $n=1,2,\ldots$. Some additional restrictions on the sequence of numbers $\{\phi^0_{in}\}$, $i=1,2$, will be formulated below. Let γ_b be defined as in the proof of Theorem 2.1 and as before we will write γ instead of γ_b. Without loss of generality we may suppose that $\varepsilon_1^\gamma<M$.

Now take the functional sequences $\{u_{0n}\}$, $\{\psi_{in}\}$, $i=1,2$, and a sequence of numbers $\{\phi^0_{in}\}$, $i=1,2$ such that

1. $\phi^0_{1n}\in[0;H/4]$, $\phi^0_{2n}\in[(3/4)H;H]$, $n=1,2,\ldots$, $\lim_{n\to\infty}\phi^0_{1n}=0$, $\lim_{n\to\infty}\phi^0_{2n}=H$,

2. $u_0(0) - \chi(\varepsilon_n)/2 \leqslant u_0(\phi^0_{1n}) \leqslant (u_0^m(0) + (\chi(\varepsilon_n)/2)^m)^{1/m}$, $n = 1, 2, \ldots$,
3. $u_0(H) - \chi(\varepsilon_n)/2 \leqslant u_0(\phi^0_{2n}) \leqslant (u_0^m(H) + (\chi(\varepsilon_n)/2)^m)^{1/m}$, $n = 1, 2, \ldots$,
4. $\varepsilon_n^\gamma \leqslant u_{0n}(x), \psi_{in}(t) \leqslant M$ for $(x, t) \in [0; H] \times [0; T_n]$,
5. $u_{0n} \in C^\infty[0; H]$, $\psi_{in} \in C^\infty[0; T_n]$, $i = 1, 2$, $n = 1, 2, \ldots$,
6. $u_{0n}(\phi^0_{1n}) = \psi_{1n}(0)$, $a(u^m_{0n})''(\phi^0_{1n}) + \phi'_{1n}(0)\, u'_{0n}(\phi^0_{1n}) - bu^\beta_{0n}(\phi^0_{1n}) + b\theta_b \varepsilon_n^{\beta\gamma} = \psi'_{1n}(0)$,
7. $u_{0n}(\phi^0_{2n}) = \psi_{2n}(0)$, $a(u^m_{0n})''(\phi^0_{2n}) + \phi'_{2n}(0)\, u'_{0n}(\phi^0_{2n}) - bu^\beta_{0n}(\phi^0_{2n}) + b\theta_b \varepsilon_n^{\beta\gamma} = \psi'_{2n}(0)$,
8. $0 \leqslant u_{0n}(x) - u_0(x) \leqslant \chi(\varepsilon_n)$ for $0 \leqslant x \leqslant H$,
9. $0 \leqslant \psi^m_{in}(t) - \psi^m_i(t) \leqslant \chi^m(\varepsilon_n)$ for $0 \leqslant t \leqslant T_n$, $i = 1, 2$,

where the constant θ_b and the function χ are the same as in the proof of Theorem 2.1.

Consider the auxiliary problem

$$u_t = a(u^m)_{xx} - bu^\beta + b\theta_b\, \varepsilon_n^{\beta\gamma} \qquad \text{in } E_n, \tag{3.1}$$

$$u(x, 0) = u_{0n}(x), \qquad \phi^0_{1n} \leqslant x \leqslant \phi^0_{2n}, \tag{3.2}$$

$$u(\phi_{in}(t), t) = \psi_{in}(t), \qquad 0 \leqslant t \leqslant T_n,\ i = 1, 2, \tag{3.3}$$

where $E_n = \{(x, t) : \phi_{1n}(t) < x < \phi_{2n}(t), 0 < t \leqslant T_n\}$. If we introduce new variables $H(x - \phi_{1n}(t))(\phi_{2n}(t) - \phi_{1n}(t))^{-1} \to y$, $t \to t$, then (3.1)–(3.3) will be transformed into the problem

$$\begin{aligned} v_t = {} & aH^2(\phi_{2n}(t) - \phi_{1n}(t))^{-2}\, (v^m)_{yy} \\ & + [H\phi'_{1n}(t) + (\phi'_{2n}(t) - \phi'_{1n}(t))\, y] \\ & \times (\phi_{2n}(t) - \phi_{1n}(t))^{-1}\, v_y - bv^\beta + b\theta_b\, \varepsilon_n^{\beta\gamma} \qquad \text{in } E'_n, \end{aligned} \tag{3.4}$$

$$v(y, 0) = u_{0n}(\phi^0_{1n} + H^{-1}(\phi^0_{2n} - \phi^0_{1n})\, y) \qquad \text{for } 0 \leqslant y \leqslant H, \tag{3.5}$$

$$v(0, t) = \psi_{1n}(t),\ v(H, T) = \psi_{2n}(t) \qquad \text{for } 0 \leqslant t \leqslant T_n, \tag{3.6}$$

where $E'_n = \{(y, t) : 0 < y < H,\ 0 < t \leqslant T_n\}$. From [20, Theorem 6.1, Sect. 6, Chap. 5] it follows that there exists a unique classical solution $v = v_n(y, t)$ of the problem (3.4)–(3.6) such that $v_n \in C^{2+\mu, 1+\mu/2}_{x,t}(\bar{E}'_n)$ for some $\mu > 0$. The maximum principle implies (2.7) in E'_n. Therefore, the function $u_n(x, t) = v_n(H(x - \phi_{1n}(t))\,(\phi_{2n}(t) - \phi_{1n}(t))^{-1}, t)$ is the classical solution from $C^{2+\mu, 1+\mu/2}_{x,t}(\bar{E}_n)$ of the problem (3.1)–(3.3) and (2.8) is valid in $\bar{E}_n$.

The sequence $\{u_n\}$ is uniformly bounded and equicontinuous on every compact subset of E. The proof completely coincides with that given in the proof of Theorem 2.1. As before, by a diagonalization argument and the

Arzela–Ascoli theorem we may find a subsequence n' and a limit function $\tilde{u} \in C(E)$ such that $u_{n'} \to \tilde{u}$ as $n' \to +\infty$, pointwise in E, and the convergence is uniform on compact subsets of E. Obviously, $\tilde{u} \in L_\infty(E)$ if $b \geqslant 0$ or $b<0$ and $\beta>1$, and $\tilde{u} \in L_\infty(E \bigcap (t \leqslant T_1))$ for any finite $T_1>0$, if $b<0$ and $0<\beta \leqslant 1$.

Now consider a function $u(x, t)$ such that $u=\tilde{u}$ for $(x, t) \in E$, $u(x, 0)=u_0(x)$ for $0 \leqslant x \leqslant H$, and $u(\phi_i(t), t)=\psi_i(t)$ for $0 \leqslant t \leqslant T$, $i=1, 2$. The function $u(x, t)$ satisfies the integral identity (1.9) in the sense of Definition 1.2. The continuity of u at points $(x_0, 0)$, $0<x_0<H$, of the line $t=0$ may be established as is mentioned in the proof of Theorem 2.1.

It remains only to prove the continuity of $u(x, t)$ at the points $(\phi_i(t), t)$, $t \geqslant 0$. For that, first consider a function $v(y, t)=u(H^{-1}(\phi_2(t)-\phi_1(t))\, y+\phi_1(t), t)$, $(y, t) \in \bar{E}'$, where $E'=\{(y, t): 0<y<H,\ 0<t \leqslant T\}$. Obviously $v \in \mathrm{C}(E') \cap \mathrm{L}_\infty(E')$ if $b \geqslant 0$ or $b<0$ and $\beta>1$, and $v \in \mathrm{C}(E') \cap \mathrm{L}_\infty (E' \cap (t \leqslant T_1))$ if $b<0$, $0<\beta \leqslant 1$, and T_1 is an arbitrary finite number from $(0; T]$. The sequence $\{v_{n'}\}$ converges to v as $n' \to +\infty$ pointwise in $\bar{E}'$ and convergence is uniform on compact subsets of E'. Continuity of the function $u(x, t)$ at the points $(\phi_i(t), t)$ $t \geqslant 0$, $i=1, 2$, is equivalent to continuity of the function $v(y, t)$ at the points $(0, t)$, (H, t), $t \geqslant 0$. First, we prove the continuity at the points (H, t), $t \geqslant 0$.

If $t_0 \geqslant 0$ and $\psi_2(t_0)>0$, it is enough to show that for arbitrary sufficiently small $\varepsilon>0$ the two inequalities

$$\liminf v(y, t) \geqslant \psi_2(t_0)-\varepsilon \qquad \text{as} \quad (y, t) \to (H, t_0), \tag{3.7}$$

$$\limsup v(y, t) \leqslant \psi_2(t_0)+\varepsilon \qquad \text{as} \quad (y, t) \to (H, t_0), \tag{3.8}$$

are valid.

Because $\varepsilon>0$ is arbitrary, from (3.7) and (3.8), the continuity of v at the boundary points (H, t_0) follows. If $\psi_2(t_0)=0$, however, then it is sufficient to prove (3.8), since (3.7) (with $\varepsilon=0$ in the right-hand side) directly follows from the fact that v is nonnegative in $\bar{E}'$.

Let $\psi_2(t_0)>0$. Take an arbitrary $\varepsilon \in (0; \psi_2(t_0))$ and prove the inequality (3.7). The proof is similar to that of (2.13). Consider a function

$$\begin{aligned}\omega_n(y, t)=f(h(\mu)&+H^{-1}(\phi_{2n}(t)-\phi_{1n}(t))\, y\\ &+\mu(t-t_0)-\phi_{2n}(t_0)+\phi_{1n}(t)), \qquad \mu>0,\ h>0,\end{aligned}$$

where

$$f(\xi)=M_1(\xi/h(\mu))^\alpha, \qquad M_1=\psi_2(t_0)-\varepsilon,$$

and by choosing the value of α appropriately we divide the analysis into different cases, as in the proof of Theorem 2.1 (see also Fig. 1 if $b>0$). We then choose h, M_i, $i=\overline{1,3}$, as in the proof of (2.13) (replacing ψ by ψ_2), and similar analysis leads to the estimation

$$\bar{\omega}_n(y,t)\leqslant v_n(y,t) \quad \text{in } \bar{E}'_n$$

where

$$\bar{\omega}_n=\{\omega_n \text{ in } \bar{\Omega}_n;\ \varepsilon_n^{\gamma} \text{ in } \bar{E}'_n\backslash\bar{\Omega}_n\},$$

$$\Omega_n=\{(y,t): t_0-d_{t_0}(\mu)<t\leqslant t_0+\delta_1, \zeta_n(t)<y<H\},$$

$$\zeta_n(t)=H(\phi_{2n}(t)-\phi_{1n}(t))^{-1}[-h(\mu)-\mu(t-t_0)+\phi_{2n}(t_0)-\phi_{1n}(t)+\eta_n],$$

and $d_{t_0}(\mu)$, η_n are defined as before. Since ϕ_2 satisfies (1.12), for arbitrary μ fixed and large enough there exists $N=N(\mu)$ such that $\zeta_n(t_0-\mu^{-2})>H$ for $n\geqslant N$. In the final limit as $n'\to+\infty$, we have

$$\omega(y,t)\leqslant v(y,t) \qquad \text{in} \quad \bar{E}, \tag{3.9}$$

where

$$\omega(y,t)=\begin{cases} f(h(\mu)+H^{-1}(\phi_2(t)-\phi_1(t))\,y+\mu(t-t_0) & (y,t)\in\bar{\Omega}, \\ \qquad -\phi_2(t_0)+\phi_1(t)), & \\ 0 & (y,t)\in\bar{E}'\backslash\bar{\Omega} \end{cases}$$

and

$$\Omega=\{(y,t): t_0-d_{t_0}(\mu)<t\leqslant t_0+\delta_1, \zeta(t)<y<H\},$$

$$\zeta(t)=H(\phi_2(t)-\phi_1(t))^{-1}[-h(\mu)-\mu(t-t_0)+\phi_2(t_0)-\phi_1(t)].$$

Obviously, we have

$$\lim_{\substack{(y,t)\in\bar{E}'\\(y,t)\to(H,t_0);}}\omega(y,t)=\lim_{\substack{(y,t)\in\bar{\Omega}\\(y,t)\to(H,t_0);}}\omega(y,t)=\psi_2(t_0)-\varepsilon.$$

Hence, from (3.9), (3.7) follows. The proof of (3.8) is similar to the given proof of (3.7) and to that of (2.14), therefore we omit it. Thus we have proved the continuity of the limit solution at the boundary points $(\phi_2(t),t)$, $t\geqslant 0$. The proof of continuity of the limit solution at the points $(\phi_1(t),t)$, $t\geqslant 0$, is similar to the given proof and to the proof of continuity of the limit solution to CDP on the boundary curve $(s(t),t)$, $t\geqslant 0$. The theorem is proved.

3.2. *Uniqueness and Comparison Results*

In this section we shall suppose that the boundary curve ϕ_1 (respectively ϕ_2) satisfies the assumption (L) (respectively assumption (R)).

THEOREM 3.2. *Let $a>0$, $m>0$ and either $b\geqslant 0$, $\beta>0$ or $b<0$, $\beta\geqslant 1$. Suppose that for each compact subsegment $[\delta; T_1]\subset(0; T]$ there exists a positive constant M_0 such that the conditions* (1.13) *are satisfied. Then the solution of the DP is unique.*

Proof. The proof is similar to the proof given in the case of the CDP (Theorem 2.2). Suppose that g_1 and g_2 are two solutions of the DP. Let $\bar{t}\in(0; T]$ be an arbitrary finite number. As before, uniqueness will be proved by confirming that for some limit solution $u=\lim u_n$ the inequalities

$$\int_{\phi_1(t)}^{\phi_2(t)} (u(x,t)-g_i(x,t))\,\omega(x)\,dx\leqslant 0, \qquad i=1,2, \tag{3.10}$$

for every $t\in(0;\bar{t}\,]$ and for an arbitrary function $\omega\in C_0^\infty((\phi_1(t);\phi_2(t))$ such that $|\omega|\leqslant 1$ are valid. Let (2.25) be valid with $\phi_1(t)<p<q<\phi_2(t)$. Suppose that $\chi(x)=Kx^\gamma$ for $x\geqslant 0$ (see the proof of Theorem 3.1). Take an arbitrary sequence of real numbers $\{\delta_\ell\}$ such that

$$0<\delta_{\ell+1}<\delta_\ell<\bar{t},\ \ell=1,2,...;\qquad \delta_\ell\to 0+\quad \text{as}\quad \ell\to+\infty.$$

Suppose also that the sequences $\{\phi_{jn}\}$, $j=1,2$, in addition to conditions from the proof of Theorem 3.1, satisfy the properties

$$\phi'_{1n}(\tau)\geqslant -M_0(\ell),\ \phi'_{2n}(\tau)\leqslant M_0(\ell),\qquad \text{for}\quad \delta_\ell\leqslant\tau\leqslant\bar{t},\ n=1,2,...,$$

(the possibility of which follows from (1.13)),

$$\phi_1(\tau)<\phi_{1,n+1}(\tau)<\phi_{1n}(\tau)<\phi_{2n}(\tau)<\phi_{2,n+1}(\tau)<\phi_2(\tau)$$
$$\text{for}\quad 0\leqslant\tau\leqslant\bar{t},\ n=1,2,...,$$

and

$$\max_{0\leqslant\tau\leqslant\bar{t}}|g_i^m(\phi_{jn}(\tau),\tau)-g_i^m(\phi_j(\tau),\tau)|\leqslant\chi^m(\varepsilon_n),\qquad n=1,2,...,\ j=1,2.$$

Without loss of generality we may assume that $\phi_{1n}(t)<p<q<\phi_{2n}(t)$, $n=1,2,...$, $\delta_\ell<t$, $\ell=1,2,....$ Since the proof of (3.10) is similar for each i, we shall henceforth write $g=g_i$. Let $E^\ell_{1n}=\{(x,\tau):\phi_{1n}(\tau)<x<\phi_{2n}(\tau),\ \delta_\ell\leqslant\tau<t\}$. Take any function $f\in C^{2,1}_{x,t}(\bar{E}^\ell_{1n})$ such that $f=0$ for $x=\phi_{in}(\tau)$, $\delta_\ell\leqslant\tau\leqslant t$, $i=1,2$. Let $u=\lim u_n$ be the limit solution of the DP constructed in the proof of Theorem 3.1. We have

$$I(u_n,f,E^\ell_{1n})+b\theta_b\,\varepsilon_n^{\beta\gamma}\int_{\delta_\ell}^t\int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f(x,\tau)\,dx\,d\tau-I(g,f,E^\ell_{1n})=0. \tag{3.11}$$

If $b \geqslant 0$ we transform (3.11) as

$$\begin{aligned}
&\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} (u_n(x,t) - g(x,t))\, f(x,t)\, dx \\
&\quad = \int_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} (u_n(x,\delta_\ell) - g(x,\delta_\ell))\, f(x,\delta_\ell)\, dx \\
&\quad + a \int_{\delta_\ell}^{t} (\psi_{1n}^m(\tau) - g^m(\phi_{1n}(\tau),\tau))\, f_x(\phi_{1n}(\tau),\tau)\, d\tau \\
&\quad - a \int_{\delta_\ell}^{t} (\psi_{2n}^m(\tau) - g^m(\phi_{2n}(\tau),\tau))\, f_x(\phi_{2n}(\tau),\tau)\, d\tau \\
&\quad + \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} (C_n^k f_\tau + A_n^k f_{xx} - B_n^k f)(u_n^{1/\gamma} - g^{1/\gamma})\, dx\, d\tau \\
&\quad + b\varepsilon_n^{\beta\gamma} \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f\, dx\, d\tau \\
&\quad + \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} ((C_n - C_n^k)\, f_\tau + (A_n - A_n^k)\, f_{xx} \\
&\quad - (B_n - B_n^k)\, f)(u_n^{1/\gamma} - g^{1/\gamma})\, dx\, d\tau, \qquad (3.12a)
\end{aligned}$$

where A_n, B_n, C_n are the same as in the proof of Theorem 2.2. As before, assume that A_n^k, B_n^k, C_n^k, $k = 1, 2, \ldots$, are C^∞ approximations of A_n, B_n, C_n, respectively, in $\bar{E}_{1n}^{\ell}$ and that they satisfy (2.28). If $b < 0$ then we transform (3.11) as

$$\begin{aligned}
&\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} (u_n(x,t) - g(x,t))\, f(x,t)\, dx \\
&\quad = \int_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} (u_n(x,\delta_\ell) - g(x,\delta_\ell))\, f(x,\delta_\ell)\, dx \\
&\quad + a \int_{\delta_\ell}^{t} (\psi_{1n}^m(\tau) - g^m(\phi_{1n}(\tau),\tau))\, f_x(\phi_{1n}(\tau),\tau)\, d\tau \\
&\quad - a \int_{\delta_\ell}^{t} (\psi_{2n}^m(\tau) - g^m(\phi_{2n}(\tau),\tau))\, f_x(\phi_{2n}(\tau),\tau)\, d\tau \\
&\quad + \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} (f_\tau + A_n^k f_{xx} - B_n^k f)(u_n - g)\, dx\, d\tau \\
&\quad + \int_{\delta_\ell}^{t} \int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} ((A_n - A_n^k)\, f_{xx} - (B_n - B_n^k)\, f)(u_n - g)\, dx\, d\tau, \qquad (3.12b)
\end{aligned}$$

where A_n, B_n, A_n^k, B_n^k are the same as before. Since u_n satisfies (2.8), the estimations (2.30) are valid in this case as well (note that $\gamma=1$ if $b<0$). Then consider a problem

$$\mathscr{L}_1 f = D_n^k f_\tau + A_n^k f_{xx} - B_n^k f = 0 \qquad \text{in } E_{1n}^\ell, \tag{3.13a}$$

$$f(x,t)=\omega(x), \qquad \phi_{1n}(t)\leqslant x\leqslant \phi_{2n}(t), \tag{3.13b}$$

$$f(\phi_{1n}(\tau),\tau)=f(\phi_{2n}(\tau),\tau)=0, \qquad \delta_\ell\leqslant\tau\leqslant t, \tag{3.13c}$$

where D_n^k is defined as in (2.31). There exists a unique classical solution to problem (3.13) [10]. The solution $f=f(x,\tau)$ has the properties:

I. $$|f|\leqslant \exp(\sigma_b B_0(t-\tau)), \qquad (x,\tau)\in \bar{E}_{1n}^\ell,$$

II. $|f_x(\phi_{in}(\tau),\tau)| = O(\varepsilon_n^s)$ as $n\to+\infty$ for $\tau\in[\delta_\ell; t]$, $i=1, 2$,

where $s=(1-m\gamma$ if $b\geqslant 0$; $1-m$ if $b<0$ and $m\geqslant 1$; 0, if $b<0$ and $0<m<1)$,

III. $$\|f\|_{W_q^{2,1}(E_{1n}^\ell)}\leqslant M_*(n), \qquad q>1,$$

where the constant $M_*(n)$ does not depend on k and ℓ.

The proof is similar to that given in the case of problem (2.31). Let us consider (3.12) with $f=f(x,\tau)$, which is a solution of the problem (3.13). Then we have

$$\begin{aligned}\int_{\phi_{1n}(t)}^{\phi_{2n}(t)} (u_n(x,t)-g(x,t))\,\omega(x)\,dx &= \int_{\phi_{1n}(\delta_\ell)}^{\phi_{2n}(\delta_\ell)} (u_n(x,\delta_\ell)-g(x,\delta_\ell))\,f(x,\delta_\ell)\,dx\\ &+a\int_{\delta_\ell}^{t} (\psi_{1n}^m(\tau)-g^m(\phi_{1n}(\tau),\tau))\,f_x(\phi_{1n}(\tau),\tau)\,d\tau\\ &-a\int_{\delta_\ell}^{t} (\psi_{2n}^m(\tau)-g^m(\phi_{2n}(\tau),\tau))\,f_x(\phi_{2n}(\tau),\tau)\,d\tau\\ &+b\theta_b\varepsilon_n^{\beta\gamma}\int_{\delta_\ell}^{t}\int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)} f\,dx\,d\tau+\int_{\delta_\ell}^{t}\int_{\phi_{1n}(\tau)}^{\phi_{2n}(\tau)}[(1-\sigma_b)(C_n-C_n^k)\,f_\tau\\ &+(A_n-A_n^k)\,f_{xx}-(B_n-B_n^k)\,f](u_n^{1/\gamma}-g^{1/\gamma})\,dx\,d\tau=\sum_{i=1}^{5} J_i.\end{aligned} \tag{3.14}$$

To estimate the right-hand side of (3.14) we can now use Properties I–III. Obviously, from Properties I and III it follows that $\lim_{n\to+\infty} J_4=0$ and $\lim_{k\to+\infty} J_5=0$. In view of Property II we have

$$|J_{i+1}|\leqslant a\int_{\delta_\ell}^{t}(|\psi_{in}^m(\tau)-\psi_i^m(\tau)|+|g^m(\phi_{in}(\tau),\tau)-g^m(\phi_i(\tau),\tau)|)$$
$$\times|f_x(\phi_{in}(\tau),\tau)|\,d\tau=O(\varepsilon_n^{m\gamma+s})\qquad\text{as}\quad n\to+\infty,\ i=1,2.$$

To estimate J_1, we first introduce a function

$$u_n^\ell(x)=\begin{cases}u_n(x,\delta_\ell), & \text{if}\quad \phi_{1n}(\delta_\ell)\leqslant x\leqslant\phi_{2n}(\delta_\ell),\\ \psi_{1n}(\delta_\ell), & \text{if}\quad x<\phi_{1n}(\delta_\ell),\\ \psi_{2n}(\delta_\ell), & \text{if}\quad x>\phi_{2n}(\delta_\ell).\end{cases}$$

Obviously u_n^ℓ, $x\in R^1$, is bounded uniformly with respect to n,ℓ. By using Property I we have

$$|J_1|\leqslant\exp(\sigma_b B_0 t)\,J_1^1,\qquad J_1^1=\int_{\phi_1(\delta_\ell)}^{\phi_2(\delta_\ell)}|u_n^\ell(x)-g(x,\delta_\ell)|\,dx.$$

From Lebesgue's theorem it follows that

$$\lim_{n\to+\infty} J_1^1=J_1^2,\qquad J_1^2=\int_{\phi_1(\delta_\ell)}^{\phi_2(\delta_\ell)}|u(x,\delta_\ell)-g(x,\delta_\ell)|\,dx.$$

By using these estimations in (3.14) and passing to the limit first with respect to $k\to+\infty$, and then with respect to $n\to+\infty$, from (3.14) it follows that

$$\int_{\phi_1(t)}^{\phi_2(t)}(u(x,t)-g(x,t))\,\omega(x)\,dx\leqslant\exp(\sigma_b\,B_0\,t)\,J_1^2.\tag{3.15}$$

Let

$$u_\ell(x)=\begin{cases}u(x,\delta_\ell)-g(x,\delta_\ell), & \text{if}\quad \phi_1(\delta_\ell)\leqslant x\leqslant\phi_2(\delta_\ell),\\ 0, & \text{if}\quad x\notin[\phi_1(\delta_\ell);\phi_2(\delta_\ell)].\end{cases}$$

Obviously, u_ℓ, $x\in R^1$, is bounded uniformly with respect to ℓ. Hence, we have (noting that $\phi_1(0)=0$, $\phi_2(0)=H$)

$$J_1^2\leqslant C(|\phi_1(\delta_\ell)|+|H-\phi_2(\delta_\ell)|)+J_1^3,\qquad J_1^3=\int_0^H|u_\ell|\,dx,$$

where the constant C does not depend on ℓ. From Lebesgue's theorem it follows that $\lim_{\ell \to \infty} J_1^3 = 0$. By using these estimations in (3.15) and passing to the limit $\ell \to +\infty$, from (3.15), (3.10) follows. The theorem is proved.

From Theorems 3.1 and 3.2 the following corollary easily follows.

COROLLARY 3.1. *Let $a > 0$, $m > 0$, and either $b \geqslant 0$, $\beta > 0$ or $b < 0$, $\beta \geqslant 1$. The solution of the DP is unique if there exists a finite number of points t_i, $i = 1, ..., k$, such that $t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = T$ and for the arbitrary compact subsegment $[\delta_1; \delta_2] \subset (t_i; t_{i+1})$, $i = 0, 1, ..., k$, there exists a positive constant M_0 such that* (1.13) *is satisfied for $0 < \delta_1 \leqslant \tau \leqslant t \leqslant \delta_2$. If $T = +\infty$ the uniqueness is still the case even if $k = +\infty$ and $t_i \uparrow +\infty$ as $i \to +\infty$.*

THEOREM 3.3. *Let $a > 0$, $m > 0$, $b < 0$,* and *$0 < \beta < 1$. Then, if $u_0(x)$, $\psi_1(t)$, $\psi_2(t) \geqslant \bar{\delta} > 0$ for $(x, t) \in [0; H] \times [0; T]$, the DP has a unique solution.*

The proof is similar to that of Theorem 2.3 (see also Remark 2.1).

Finally, we present the following comparison result (see Remark 2.3 in Section 2.3).

THEOREM 3.4. *Let the conditions of Theorem* 3.2 (*or Corollary* 3.1) *be satisfied. Let u be a solution of the DP and g be a supersolution* (*respectively subsolution*) *of Eq.* (1.1) *in E and $u \leqslant (\geqslant) g$ in $\bar{E} \setminus E$. Then $u \leqslant (\geqslant) g$ in $\bar{E}$.*

As in the case of Theorem 2.4, the proof is similar to that of the uniqueness theorem, Theorem 3.2.

Remark 3.1. If the conditions of Theorem 3.3 are satisfied, then Theorem 3.4 (comparison) is valid if we require g to be a classical smooth supersolution (respectively subsolution) of Eq. (1.1) in E (see Remark 2.1). Suppose that in addition to the conditions of Theorem 3.3, for each compact subsegment $[0; T_1] \subset [0; T]$ there exists a positive constant M_0 such that (1.13) is satisfied with $\delta = 0$. Then the assertion of Theorem 3.4 is valid.

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