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Necessary and sufficient conditions of existence for a system involving the p-Laplacian ($1<p<N$)

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Abstract

In this work, we study the existence of positive solutions for the system

$$(\mathscr{P})\begin{cases} -\Delta_p u = \mu f(x,u,v) \text{ in } \mathbb{R}^N, \\ -\Delta_p v = \mu g(x,u,v) \text{ in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0 = \lim\limits_{|x|\to\infty} v(x), \end{cases}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian $(1<p<N)$. We prove that for $0<\mu\leqslant\mu_*$ $(\mathscr{P})$ admits a positive solution and that $(\mathscr{P})$ has no positive solution for $\mu>\mu^*$. In some case we can have $\mu_* = \mu^*$ because these parameters depend on p, N and the growth of f and g. The existence result is related to a sub,super-solution methods and the nonexistence result is a consequence of Picone's identity.

Keywords: p-Laplacian; Existence; System; Unbounded domain; Sub, super-solution; Picone's identity

In this paper, we assume that $1<p<N$ and we study the following nonlinear and nonvariational elliptic system:

$$(\mathscr{P})\begin{cases} -\Delta_p u = \mu f(x,u,v) \text{ in } \mathbb{R}^N, \\ -\Delta_p v = \mu g(x,u,v) \text{ in } \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} u(x) = 0 = \lim\limits_{|x|\to\infty} v(x), \end{cases}$$

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where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. We make some suitable hypotheses on the regularity and the growth of the functions f and g which will be detailed later.

Here, we present necessary and sufficient conditions of existence for positive solutions of Problem $(\mathscr{P})$. More precisely:

- There exists some $\mu^* \in \mathbb{R}_*^+$ such that for $\mu \in (0, \mu^*]$ $(\mathscr{P})$ admits a positive solution.
- Conversely, there exists some $\mu_* \in \mathbb{R}_*^+$ such that $(\mathscr{P})$ has no positive solution for $\mu > \mu_*$.

In fact, these real parameters μ^* and μ_* depend on some weight functions appearing in the problem. So if we choose correctly these weight functions, we obtain a more precise result.

There exists $\mu_0 \in \mathbb{R}_*^+$ such that $(\mathscr{P})$ admits a positive solution if and only if $0 < \mu \leqslant \mu_0$. To treat this subject we employ two methods very different from each other.

In the proof of existence for a positive solution, we use the sub,super-solution method (see for example [4,8]). By the Mountain Pass Lemma, we prove the existence of a nonnegative super-solution. Thanks to Serrin's estimates [13], we can apply the Vázquez's strong maximum principle [14] to establish the positivity of this solution. Moreover, we take $(0,0)$ as a sub-solution.

Concerning the necessary condition, our work follows Maya and Shivaji's results [11] for the semi-linear case $p=2$. In their proof, the self-adjointness of the usual Laplacian is essential. Since the p-Laplacian does not have this property, we apply here the Díaz–Saa's inequality. We can find the proof of this inequality in Díaz and Saa's article [5] for bounded domains and in Chaïb's paper [3] for unbounded domains; in fact it appears as a consequence of Picone's identity.

In this problem, the functions f and g satisfy the following hypotheses:

$(\mathscr{H}1)$ f and g are Carathéodory functions, i.e.:

- $(r,s) \mapsto f(x,r,s)$ and $(r,s) \mapsto g(x,r,s)$ are continuous on $[0,+\infty)^2$ for almost every $x \in \mathbb{R}^N$,
- $x \mapsto f(x,r,s)$ and $x \mapsto g(x,r,s)$ are measurable.

$(\mathscr{H}2)$ For all $u, v \geqslant 0$ there exist two smooth positive functions $A, B \in L^{\frac{N}{p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$f(x,u,v) \geqslant A(x)v^{p-1} \quad \text{a.e. in } \mathbb{R}^N,$$

$$g(x,u,v) \geqslant B(x)u^{p-1} \quad \text{a.e. in } \mathbb{R}^N.$$

$(\mathscr{H}3)$ For all $u, v \geqslant 0$ there exist four smooth functions $a, c \in L^{\delta}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ and $b, d \in L^{\delta'}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ such that

$$0 \leqslant f(x, u, v) \leqslant \tfrac{1}{2} a(x)(u^{q-1} + v^{q-1}) + b(x) \quad \text{a.e. in } \mathbb{R}^N,$$

$$0 \leqslant g(x, u, v) \leqslant \tfrac{1}{2} c(x)(u^{q-1} + v^{q-1}) + d(x) \quad \text{a.e. in } \mathbb{R}^N,$$

where

$$\delta = Np(Np - q(N-p))^{-1}, \quad \delta' = Np(Np - N + p)^{-1},$$

$$p < q < p^* \quad \text{and} \quad 1 < p < N,$$

$$a, b, c \text{ and } d \text{ are nonnegative functions non identically zero,}$$

$$|\mathscr{A}^+| := |\{x \in \mathbb{R}^N \text{ such that } a > 0\}| > 0,$$

$$|\mathscr{C}^+| := |\{x \in \mathbb{R}^N \text{ such that } c > 0\}| > 0.$$

$(\mathscr{H}4)$ The function $x \mapsto f(x, 0, 0) + g(x, 0, 0)$ is not identically zero.

1. Sufficient condition of existence

In the following, we can assume without loss of generality that $a \equiv c$ and $b \equiv d$; we also define the space $D^{1,p}(\mathbb{R}^N)$ as the closure of the set $C_c^{\infty}(\mathbb{R}^N)$ for the norm

$$||u||_{D^{1,p}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

We introduce the nonhomogeneous problem

$$(\mathscr{V}_{p,q}) \; -\Delta_p \phi = \mu_q a(x) |\phi|^{q-2} \text{ in } \mathbb{R}^N.$$

Resolving this problem, we obtain

$$\mu_q := \inf_{\phi \in \Gamma_q} \left\{ \frac{1}{p} \int_{\mathbb{R}^N} |\nabla \phi|^p \, dx \right\},$$

where $\Gamma_q := \{\phi \in D^{1,p}(\mathbb{R}^N) \text{ such that } \phi \not\equiv 0 \text{ and } \frac{1}{q} \int_{\mathbb{R}^N} a(x) |\phi|^q \, dx = 1\}$.

We can see more details on the properties of μ_q in [9]; in particular, there exists a unique minimizer of $I : u \mapsto \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx$ on the set Γ_q.

Theorem 1. *Under Hypotheses* $(\mathscr{H}1), (\mathscr{H}3)$ *and* $(\mathscr{H}4)$, *if* μ *satisfies*

$$0 < \mu \leqslant \min\left(\frac{p}{q}\mu_q, \left(2\frac{p}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)} + 2pC(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)}\right)^{-1}\right),$$

where $C(N,p)$ *is the Sobolev's constant, then Problem* $(\mathscr{P})$ *admits a nontrivial nonnegative solution.*

Moreover, by Hypothesis $(\mathscr{H}3)$, *if* (u,v) *is a nonnegative solution of* $(\mathscr{P})$, *then for all* $r > 0$, *there exist* $\alpha, \gamma \in (0,1)$ *such that* $(u,v) \in C^{1,\alpha}(B_r) \times C^{1,\gamma}(B_r)$ *and* $u > 0, v > 0$ *on* $\mathbb{R}^N$.

To prove the existence of a solution for Problem $(\mathscr{P})$, we have to prove the existence of sub, super-solution pair.

Definition 1 (Díaz and Hernández [4]). $(u_0, v_0) - (u^0, v^0)$ is a sub,super-solution of $(\mathscr{P})$ if these functions satisfy:

(S1) $(u_0, v_0) \in (D^{1,p}(\mathbb{R}^N))^2$ and $(u^0, v^0) \in (D^{1,p}(\mathbb{R}^N))^2$;
(S2) $u_0 \leqslant u^0$ and $v_0 \leqslant v^0$;
(S3) $-\Delta_p u_0 - \mu f(x, u_0, v) \leqslant 0 \leqslant -\Delta_p u^0 - \mu f(x, u^0, v)$ for all v in $[v_0, v^0]$;
(S4) $-\Delta_p v_0 - \mu g(x, u, v_0) \leqslant 0 \leqslant -\Delta_p v^0 - \mu g(x, u, v^0)$ for all u in $[u_0, u^0]$.

Proof. First, Hypothesis $(\mathscr{H}3)$ implies obviously that $(0,0)$ is a sub-solution of $(\mathscr{P})$. But $(0,0)$ is not a solution of this problem by Hypothesis $(\mathscr{H}4)$. So, it only remains to prove the existence of a nonnegative super-solution of $(\mathscr{P})$. By $(\mathscr{H}3)$, we have to look for a nonnegative solution of the system:

$$\begin{cases} -\Delta_p u = \mu(\frac{1}{2}a(x)(|u|^{q-2}u + |v|^{q-2}v) + b(x)) \text{ in } \mathbb{R}^N, \\ -\Delta_p v = \mu(\frac{1}{2}a(x)(|u|^{q-2}u + |v|^{q-2}v) + b(x)) \text{ in } \mathbb{R}^N. \end{cases}$$

The above system being symmetric, we can seek a solution of the form $u \equiv v \equiv w$ where w is a solution of

$$-\Delta_p w = \mu(a(x)|w|^{q-2}w + b(x)) \quad \text{in } \mathbb{R}^N. \qquad (\mathscr{P}^0)$$

To prove the existence of solution of $(\mathscr{P}^0)$, we will apply the Mountain Pass Lemma (see for example [12]) to the energy functional

$$J(w) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla w|^p \, dx - \frac{\mu}{q}\int_{\mathbb{R}^N} a(x)|w|^q \, dx - \mu\int_{\mathbb{R}^N} b(x)w \, dx.$$

The facts that $D^{1,p}(\mathbb{R}^N)$ is a reflexive Banach space and that J is a continuous function in $C^1(D^{1,p}(\mathbb{R}^N), \mathbb{R})$ satisfying the Palais–Smale condition are basic results (see [10,12]). It remains to verify the two following claims to prove that the functional J has a mountain pass geometry:

(C1) There exist $R>0$ and $a>0$ such that $||u||_{D^{1,p}(\mathbb{R}^N)}=R$ implies $J(u)\geqslant a$;

(C2) There exists $u_0\in D^{1,p}(\mathbb{R}^N)$ such that $||u_0||_{D^{1,p}(\mathbb{R}^N)}>R$ and $J(u_0)<a$.

(C1) Let $\int_{\mathbb{R}^N}|\nabla w|^p=1$

$$J(w)=\frac{1}{p}-\frac{\mu}{q}\int_{\mathbb{R}^N}a(x)|w|^q\,dx-\mu\int_{\mathbb{R}^N}b(x)w\,dx.$$

By Hölder inequality and Sobolev embeddings, we get

$$J(w)\geqslant\frac{1}{p}-\frac{\mu}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)}-\mu C(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)},$$

where $C(N,p)$ is the Sobolev constant.

So, when $\mu\leqslant(2\frac{p}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)}+2pC(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)})^{-1}$, if we choose the constants $R=1$ and $a=\frac{1}{p}-\frac{\mu}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)}-\mu C(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)}$, the functional J satisfies the first condition.

(C2) Let $w\in C_0^\infty(\mathbb{R}^N)$ be fixed such that $w>0$ on $\mathscr{A}^+$ and $w\geqslant0$ on $\mathbb{R}^N$. For all $k\geqslant0$ we have

$$J(kw)=\frac{k^p}{p}\int_{\mathbb{R}^N}|\nabla w|^p\,dx-\frac{\mu}{q}k^q\int_{\mathbb{R}^N}a(x)|w|^q\,dx-\mu k\int_{\mathbb{R}^N}b(x)w\,dx.$$

As $a>0$ on $\mathscr{A}^+$ and $q>p>1$, we obtain

$$J(kw)\text{ tends to }-\infty\text{ as }k\text{ tends to }+\infty.$$

So putting $w^0=kw$, there exists some k great enough so that $||w^0||_{D^{1,p}}>R$ and $J(w^0)<a$ which is exactly condition (C2).

By Mountain Pass Lemma, we have obtained the existence of a solution for Problem $(\mathscr{P}^0)$. But before affirming that this solution is a super-solution of $(\mathscr{P})$, it remains to prove that it is a nonnegative solution, i.e. greater than the sub-solution.

Let $w^-=\max(0,-w)$ where $w\in D^{1,p}(\mathbb{R}^N)$ is a solution of $(\mathscr{P}^0)$

$$-\Delta_p w=\mu(a(x)|w|^{q-2}w+b(x))\quad\text{in }\mathbb{R}^N.$$

Assuming that $w^-\not\equiv0$, we multiply Eq. $(\mathscr{P}^0)$ by w^- and integrate on $\mathbb{R}^N$.

$$\int_{\mathbb{R}^N}|\nabla w|^{p-2}\nabla w\nabla w^-\,dx=\mu\int_{\mathbb{R}^N}a(x)|w|^{q-2}ww^-\,dx+\mu\int_{\mathbb{R}^N}b(x)w^-\,dx$$

$$\int_{\mathbb{R}^N}|\nabla w^-|^p\,dx=\mu\int_{\mathbb{R}^N}a(x)|w^-|^q\,dx-\mu\int_{\mathbb{R}^N}b(x)w^-\,dx.\qquad(1)$$

Since $b \geqslant 0$ we obtain

$$\int_{\mathbb{R}^N} |\nabla w^-|^p \, dx \leqslant \mu \int_{\mathbb{R}^N} a(x)|w^-|^q \, dx.$$

We can assume that $\frac{1}{q}\int_{\mathbb{R}^N} a(x)|w^-|^q \, dx = 1$ because $w^- \not\equiv 0$ and $a \not\equiv 0$. Now we choose μ small enough so that $\mu \leqslant \frac{p}{q}\mu_q$ and then

$$\int_{\mathbb{R}^N} |\nabla w^-|^p \, dx \leqslant p\mu_q. \tag{2}$$

But, by the definition of μ_q, for all ϕ in Γ_q we have $\mu_q \leqslant \frac{1}{p}\int_{\mathbb{R}^N} |\nabla \phi|^p \, dx$.

Since there exists a unique minimizer of I on Γ_q(see [9]), we have $w^- = \phi$ and by Eq. (1) it implies that $b \equiv 0$ which contradicts $(\mathscr{H}3)$; so $w \geqslant 0$.

Finally, after this study, we can apply the results of the sub,super-solution method.

Indeed, $(0,0)-(w,w)$ is a sub,super-solution pair for the problem $(\mathscr{P})$:

(S1) $w \in D^{1,p}(\mathbb{R}^N)$;
(S2) $0 \leqslant w$;
(S3) By $(\mathscr{H}3)$,

$$-\mu f(x,0,v) \leqslant 0 \quad \text{for all } v \text{ in } [0,w].$$

Moreover, for all v in $[0,w]$ we have

$$\begin{aligned} -\Delta_p w - \mu f(x,w,v) &= \mu(a(x)w^{q-1} + b(x) - f(x,w,v)) \\ &\geqslant \mu(\tfrac{1}{2}a(x)(w^{q-1} + v^{q-1}) + b(x) - f(x,w,v)) \\ &\geqslant 0. \end{aligned}$$

(S4) In the same way, for all u in $[0,w]$ we obtain

$$-\mu g(x,u,0) \leqslant 0$$

and

$$-\Delta_p w - \mu g(x,u,w) \geqslant 0.$$

Now, we will use the Leray–Schauder's fixed point theorem to prove the existence and the uniqueness of the positive solution of $(\mathscr{P})$.

We denote $E = (L^{p*}(\mathbb{R}^N))^2$ with $p^* = \frac{Np}{N-p}$ and $K = [u_0, u^0] \times [v_0, v^0] = K_u \times K_v$ and we define the operator $T : K \to E$ by

for all $(\bar{u}, \bar{v})$ in K,

$(w,z)=T(\bar{u},\bar{v})$ is the unique solution of the decoupled system $(\mathscr{S}_d)$.

$$(\mathscr{S}_d)\begin{cases} -\Delta_p w + Mk(x)|w|^{q-2}w = \mu f(x,\bar{u},\bar{v}) + Mk(x)\bar{u}^{q-1} \text{ in } \mathbb{R}^N, \\ -\Delta_p z + Mk(x)|z|^{q-2}z = \mu g(x,\bar{u},\bar{v}) + Mk(x)\bar{v}^{q-1} \text{ in } \mathbb{R}^N \end{cases}$$

with $k\in L^\delta(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and for all $R>0$ there exists ε_R such that $\inf_{B_R} k(x)>\varepsilon_R$.

The operator T has a fixed point in K if the four following assertions are satisfied:

(LS1) K is a convex, bounded and closed set in E.
(LS2) T is well defined on K.
(LS3) K is stable by T, i.e. $T(K)\subset K$.
(LS4) T is a compact operator.

(LS1) It is easy to verify the first point because u_0, u^0, v_0 and v^0 are in $D^{1,p}(\mathbb{R}^N)$ which is a reflexive Banach space.

(LS2) To prove that T is well defined is equivalent to prove that problem $(\mathscr{S}_d)$ admits a unique solution for all $(\bar{u},\bar{v})$ in K.

As we are considering a decoupled system, we only have to study the existence and uniqueness of the solution for every equation separately. We state

$$J_f(w)=\int_{\mathbb{R}^N}|\nabla w|^p\,dx + M\int_{\mathbb{R}^N}k(x)|w|^q\,dx - \int_{\mathbb{R}^N}\tilde{f}(x)w\,dx$$

and

$$J_g(z)=\int_{\mathbb{R}^N}|\nabla z|^p\,dx + M\int_{\mathbb{R}^N}k(x)|z|^q\,dx - \int_{\mathbb{R}^N}\tilde{g}(x)z\,dx,$$

where $\tilde{f}(x)=\mu f(x,\bar{u},\bar{v})+Mk(x)\bar{u}^{q-1}$, $\tilde{g}(x)=\mu g(x,\bar{u},\bar{v})+Mk(x)\bar{v}^{q-1}$.

J_f and J_g are $C^1(D^{1,p}(\mathbb{R}^N),\mathbb{R})$, lower weakly semi-continuous, strictly convex and coercive on $D^{1,p}(\mathbb{R}^N)$. These properties imply the existence of a unique couple solution of $(\mathscr{S}_d)$.

Since $\tilde{f},\tilde{g}\geqslant 0$, we apply the Vázquez's strong maximum principle and we obtain that these solutions are positive (see for example [2]).

(LS3) For the stability of K by T, we are going to prove that $w\geqslant u_0$ because the other inequalities can be proved by the same way.

The pair (u_0,v_0) is a sub-solution of problem $(\mathscr{P})$ and satisfies

$$\begin{cases} -\Delta_p u_0 - \mu f(x,u_0,v)\leqslant 0 & \forall v\in[v_0,v^0], \\ -\Delta_p v_0 - \mu g(x,u,v_0)\leqslant 0 & \forall u\in[u_0,u^0]. \end{cases} \tag{3}$$

The pair (w,z) is defined by $(w,z) = T(\bar{u},\bar{v})$ where $(\bar{u},\bar{v}) \in K$ and satisfies

$$\begin{cases} -\Delta_p w + Mk(x)w^{q-1} = \mu f(x,\bar{u},\bar{v}) + Mk(x)\bar{u}^{q-1}, \\ -\Delta_p z + Mk(x)z^{q-1} = \mu g(x,\bar{u},\bar{v}) + Mk(x)\bar{v}^{q-1}. \end{cases} \tag{4}$$

Since $(\bar{u},\bar{v})$ is in K we have

$$\begin{cases} -\Delta_p w + Mk(x)w^{q-1} \geqslant \mu f(x,\bar{u},\bar{v}) + Mk(x)u_0^{q-1}, \\ -\Delta_p z + Mk(x)z^{q-1} \geqslant \mu g(x,\bar{u},\bar{v}) + Mk(x)v_0^{q-1}. \end{cases} \tag{5}$$

We multiply the first equations of (3) and (5) by $(u_0 - w)^+ = \max(0, u_0 - w)$, we integrate on $\mathbb{R}^N$ and subtract the inequalities obtained. We assume that $(u_0 - w)^+ \not\equiv 0$,

$$0 \geqslant \int_{\mathbb{R}^N} (-\Delta_p u_0 + \Delta_p w)(u_0 - w)^+ \, dx + \int_{\mathbb{R}^N} [\mu(f(x,\bar{u},\bar{v}) - f(x,u_0,\bar{v})) \\ + \ Mk(x)(u_0^{q-1} - w^{q-1})](u_0 - w)^+ \, dx.$$

Since $k(x)(u_0^{q-1} - w^{q-1})(u_0 - w)^+ \not\equiv 0$, we can choose M such that the second integral is positive. So

$$\int_{\mathbb{R}^N} (-\Delta_p u_0 + \Delta_p w)(u_0 - w)^+ \, dx < 0,$$

which contradicts the monotonicity of the operator $-\Delta_p$ (see [6, Chapter 1]); consequently, $(u_0 - w)^+ = 0$, i.e., $u_0 \leqslant w$. We have proved that K is stable by T.

(LS4) Now, we have to prove that T is continuous from K to K and T maps bounded subsets of K into conditionally compact subsets of K. We will see in details the second point; concerning the continuity, the proof uses the same ingredients.

Let $(\bar{u}_n, \bar{v}_n)_{n\in\mathbb{N}}$ be a sequence in K converging weakly to $(\bar{u},\bar{v})$ for the norm of $(L^{p*}(\mathbb{R}^N))^2$. By the preceding point, we can say that $(T(\bar{u}_n,\bar{v}_n))_{n\in\mathbb{N}} = (w_n, z_n)_{n\in\mathbb{N}} \subset K$. For example, we will prove that there exists a sub-sequence $(w_n)_{n\in\mathbb{N}}$ which converges strongly in K_u for the norm of $L^{p*}(\mathbb{R}^N)$; for the convergence in K_v of the other sequence $(z_n)_{n\in\mathbb{N}}$ the method is the same.

First, we prove that the sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in K_u for the norm of $D^{1,p}(\mathbb{R}^N)$ by estimating

$$I_n := \int_{\mathbb{R}^N} |\nabla w_n|^p \, dx + M \int_{\mathbb{R}^N} k(x) w_n^q \, dx.$$

$$\begin{aligned}
I_n &= \mu \int_{\mathbb{R}^N} f(x, \bar{u}_n, \bar{v}_n) w_n \, dx + M \int_{\mathbb{R}^N} k(x) \bar{u}_n^{q-1} w_n \, dx \\
&\leqslant \mu \int_{\mathbb{R}^N} f(x, \bar{u}_n, \bar{v}_n) u^0 \, dx + M \int_{\mathbb{R}^N} k(x) u^{0q} \, dx \\
&\leqslant \frac{\mu}{2} \int_{\mathbb{R}^N} a(x) \ (u^{0q} + v^{0q-1} u^0) \, dx \\
&\quad + \mu \int_{\mathbb{R}^N} b(x) u^0 \, dx + M \int_{\mathbb{R}^N} k(x) u^{0q} \, dx \\
&\leqslant C(\mu, M)[||a||_{L^{\delta}(\mathbb{R}^N)} (||u^0||^q_{L^{p*}(\mathbb{R}^N)} + ||v^0||^q_{L^{p*}(\mathbb{R}^N)}) \\
&\quad + ||b||_{L^{\delta'}(\mathbb{R}^N)} ||u^0||_{L^{p*}(\mathbb{R}^N)} + ||k||_{L^{\delta}(\mathbb{R}^N)} ||u^0||^q_{L^{p*}(\mathbb{R}^N)}].
\end{aligned}$$

Because $a \in L^{\delta}(\mathbb{R}^N)$, $b \in L^{\delta'}(\mathbb{R}^N)$, $k \in L^{\delta}(\mathbb{R}^N)$, $u^0, v^0 \in L^{p*}(\mathbb{R}^N)$, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p \, dx < C.$$

We can extract a sub-sequence also denoted by $(w_n)_{n \in \mathbb{N}}$ converging weakly in $D^{1,p}(\mathbb{R}^N)$. Then $(w_n)_{n \in \mathbb{N}}$ converges strongly in $L^s(B_R)$ for all $R > 0$ and all $s \leqslant p^*$ so it converges almost everywhere in $\mathbb{R}^N$ and is dominated by the super-solution u^0. Consequently, it converges in K_u for the norm of $L^{p^*}(\mathbb{R}^N)$.

We do the same work for $(z_n)_{n \in \mathbb{N}}$, so we prove that the operator $T : K \to K \subset E = (L^{p*}(\mathbb{R}^N))^2$ is compact.

After these proofs, we apply the Leray–Schauder's fixed point theorem to this operator T. So, there exists (u, v) in K such that $(u, v) = T(u, v)$. It signifies that (u, v) is a nonnegative solution of $(\mathscr{P})$.

Moreover, by maximum principle and by some regularity results, we prove that the nonnegative solution (u, v) is positive. We only have to use the Vázquez strong maximum principle. Indeed since $\mu > 0$ and $(\mathscr{H}3)$ is satisfied, we have $-\Delta_p u \geqslant 0$ and $-\Delta_p v \geqslant 0$. Vázquez's strong maximum principle [14] says that if the problem admits a nontrivial smooth solution ($C^{1,\gamma}_{\mathrm{loc}}(\mathbb{R}^N)$) satisfying $u \geqslant 0$, $v \geqslant 0$ on $\mathbb{R}^N$ then this solution is positive: $u > 0$, $v > 0$ on $\mathbb{R}^N$. $\square$

2. Necessary condition of existence

Recall the problem we study in this paper

$$(\mathscr{P}) \begin{cases} -\Delta_p u = \mu f(x, u, v) \ \text{ in } \ \mathbb{R}^N, \\ -\Delta_p v = \mu g(x, u, v) \ \text{ in } \ \mathbb{R}^N, \\ \lim\limits_{|x| \to \infty} u(x) = 0 = \lim\limits_{|x| \to \infty} v(x). \end{cases}$$

We also consider the eigenvalue problem

$$(\mathscr{V}_p) - \Delta_p \phi = \lambda_1 C(x)\phi^{p-1} \quad \text{in } \mathbb{R}^N$$

with $C \in L^{\frac{N}{p}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $C \not\equiv 0$ and $A(x) \geqslant C(x) \geqslant 0$ a.e. in $\mathbb{R}^N$.

The properties of the principal eigenvalue of Problem $(\mathscr{V}_p)$ are studied by Fleckinger et al. [7]. In particular, there exists a positive eigenfunction in $D^{1,p}(\mathbb{R}^N)$ corresponding to the first eigenvalue λ_1. In this section, we can assume without loss of generality that $A \equiv C$.

Theorem 2. *We assume that Hypotheses $(\mathscr{H}1)$ and $(\mathscr{H}2)$ are satisfied. If (u, v) is a positive solution in $(D^{1,p}(\mathbb{R}^N) \cap C^{1,\gamma}_{\text{loc}}(\mathbb{R}^N))^2$ $(0<\gamma<1)$ of $(\mathscr{P})$ then $\mu \leqslant \lambda_1$, where λ_1 is the first eigenvalue of $(\mathscr{V}_p)$.*

When we study $(\mathscr{P})$ on a bounded domain Ω and only in this case, we can apply Díaz–Saa's inequality to positive solutions in $D^{1,p}(\Omega) \cap C^{1,\gamma}_{\text{loc}}(\Omega)$ $(\gamma \in (0,1))$. But, if $\Omega = \mathbb{R}^N$ we can use a method introduced in [7] by Fleckinger et al. or in [2] by Bechah et al. We simplify this proof by using Picone's identity which is presented by Allegretto and Huang [1] with some applications.

Lemma 1 (Picone identity). *Let $v>0$, $u \geqslant 0$ differentiable. Let*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u \cdot \nabla v |\nabla v|^{p-2},$$

$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v.$$

Then $L(u,v) = R(u,v) \geqslant 0$.

Moreover, $L(u,v) = 0$ a.e. on Ω if and only if $\nabla(\frac{u}{v}) = 0$ a.e. on Ω.

The reader can see the proof of this lemma in Allegretto and Huang's paper [1] or in Chaïb's one [3].

Proof of Theorem 2. We consider a sequence of nonnegative functions $(\phi_n)_{n \in \mathbb{N}}$ in $C_0^\infty(\mathbb{R}^N)$ converging to ϕ for the norm of $D^{1,p}(\mathbb{R}^N)$; where ϕ is a positive eigenfunction associated to λ_1 for Problem $(\mathscr{V}_p)$. Such a sequence exists because $C_0^\infty(\mathbb{R}^N)$ is dense in $D^{1,p}(\mathbb{R}^N)$. Since (u, v) is a positive solution of $(\mathscr{P})$, we apply Picone's identity to the functions ϕ_n, u and ϕ_n, v, next we add the inequalities obtained

$$0 \leqslant \int_{\mathbb{R}^N} L(\phi_n, u)\, dx + \int_{\mathbb{R}^N} L(\phi_n, v)\, dx$$

$$0 \leqslant \int_{\mathbb{R}^N} R(\phi_n, u)\, dx + \int_{\mathbb{R}^N} R(\phi_n, v)\, dx$$

$$0 \leqslant \int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx - \int_{\mathbb{R}^N} \nabla\left(\frac{\phi_n^p}{u^{p-1}}\right) |\nabla u|^{p-2} \nabla u\, dx$$

$$+ \int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx - \int_{\mathbb{R}^N} \nabla\left(\frac{\phi_n^p}{v^{p-1}}\right) |\nabla v|^{p-2} \nabla v\, dx.$$

We can take $\frac{\phi_n^p}{u^{p-1}}$ and $\frac{\phi_n^p}{v^{p-1}}$ as test functions because $\phi_n \in C_0^\infty(\mathbb{R}^N)$, $u > 0$ and $v > 0$. So we apply the divergence theorem and we obtain

$$0 \leqslant 2 \int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx + \int_{\mathbb{R}^N} \frac{\phi_n^p}{u^{p-1}} \Delta_p u\, dx + \int_{\mathbb{R}^N} \frac{\phi_n^p}{v^{p-1}} \Delta_p v\, dx,$$

$$0 \leqslant 2 \int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx - \mu \int_{\mathbb{R}^N} \frac{\phi_n^p}{u^{p-1}} f(x, u, v)\, dx - \mu \int_{\mathbb{R}^N} \frac{\phi_n^p}{v^{p-1}} g(x, u, v)\, dx.$$

Hypothesis $(\mathscr{H}2)$, $f(x, u, v) \geqslant A(x) v^{p-1}$ and $g(x, u, v) \geqslant A(x) u^{p-1}$ give

$$0 \leqslant 2 \int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx - \mu \int_{\mathbb{R}^N} A(x) \frac{\phi_n^p}{u^{p-1}} v^{p-1}\, dx - \mu \int_{\mathbb{R}^N} A(x) \frac{\phi_n^p}{v^{p-1}} u^{p-1}\, dx.$$

Since $\frac{u^{p-1}}{v^{p-1}} + \frac{v^{p-1}}{u^{p-1}} \geqslant 2$, we have

$$0 \leqslant 2\left[\int_{\mathbb{R}^N} |\nabla \phi_n|^p\, dx - \mu \int_{\mathbb{R}^N} A(x) \phi_n^p\, dx\right].$$

Taking $n \to \infty$ in the last inequality, $(\phi_n)_{n \in \mathbb{N}}$ converges to ϕ in $D^{1,p}(\mathbb{R}^N)$, so $(\phi_n)_{n \in \mathbb{N}}$ converges to ϕ in $L^{p^*}(\mathbb{R}^N)$ and $(\phi_n^p)_{n \in \mathbb{N}}$ converges to ϕ^p in $L^{\frac{p^*}{p}}(\mathbb{R}^N)$. We therefore obtain

$$\int_{\mathbb{R}^N} A(x)(\phi_n^p - \phi^p)\, dx \leqslant ||A||_{L^{\frac{N}{p}}(\mathbb{R}^N)} ||\phi_n^p - \phi^p||_{L^{\frac{p^*}{p}}(\mathbb{R}^N)} \quad \text{tends to 0 as } n \to \infty.$$

And finally

$$\int_{\mathbb{R}^N} |\nabla \phi|^p\, dx \geqslant \mu \int_{\mathbb{R}^N} A(x) \phi^p\, dx.$$

Now, by using the fact that ϕ is a positive eigenfunction associated to the eigenvalue λ_1, we arrive to the expected result

$$\mu \leqslant \lambda_1. \qquad \square$$

Remark. In fact, in this paper we have shown:

- If $\mu \leqslant \min(\frac{p}{q}\mu_q, (2\frac{p}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)} + 2pC(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)})^{-1})$ then $(\mathscr{P})$ admits a positive solution.
- If $(\mathscr{P})$ admits a positive solution then $\mu \leqslant \lambda_1$.

If we choose the functions a and b such that $||a||_{L^{\delta'}(\mathbb{R}^N)}$ and $||b||_{L^{\delta'}(\mathbb{R}^N)}$ be small enough we have

$$\min\left(\frac{p}{q}\mu_q, \left(2\frac{p}{q}C(N,p)^q||a||_{L^\delta(\mathbb{R}^N)} + 2pC(N,p)||b||_{L^{\delta'}(\mathbb{R}^N)}\right)^{-1}\right) = \frac{p}{q}\mu_q.$$

Moreover, denoting by ϕ_q the solution of $(\mathscr{V}_{p,q})$ associated to μ_q we have

$$\int_{\mathbb{R}^N} |\nabla \phi_q|^p \, dx = \frac{p}{q}\mu_q \int_{\mathbb{R}^N} a(x)|\phi_q|^q \, dx.$$

Since λ_1 is the first eigenvalue of $(\mathscr{V}_p)$, we have

$$\int_{\mathbb{R}^N} |\nabla \phi_q|^p \, dx \geqslant \lambda_1 \int_{\mathbb{R}^N} C(x)|\phi_q|^p \, dx.$$

If we take a such that $\int_{\mathbb{R}^N} a(x)|\phi_q|^q \, dx \leqslant \int_{\mathbb{R}^N} C(x)|\phi_q|^p \, dx$ we obtain $\lambda_1 \leqslant \frac{p}{q}\mu_q$. Finally, for $\mu_0 = \lambda_1$, we obtain the following:

$$(\mathscr{P}) \text{ admits a positive solution if and only if } 0 < \mu \leqslant \mu_0.$$

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