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On potential wells and vacuum isolating of solutions for semilinear wave equations☆

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Abstract

In this paper, we study the initial boundary value problem of semilinear wave equations:

$$u_{tt} - \Delta u = |u|^{p-1}u, \quad x \in \Omega, \ \ t > 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \ \ x \in \Omega,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geqslant 0,$$

where $\Omega \subset \mathrm{R}^N$ is a bounded domain, $1 < p < \infty$ for $N = 1, 2$; $1 < p \leqslant \frac{N+2}{N-2}$ for $N \geqslant 3$. First, by using a new method, we introduce a family of potential wells which include the known potential well as a special case. Then by using it, we obtain some new existence theorems of global solutions, and prove that for any $e \in (0, d)$ (d is the depth of the known potential well) all solutions with initial energy $E(0)$ satisfying $0 < E(0) \leqslant e$ can only lie either inside of some smaller ball or outside of some bigger ball of space $H_0^1(\Omega)$.

Keywords: Potential wells; Semilinear wave equations; Global solutions; Existence; Vacuum isolating

1. Introduction

The potential well was introduced by Sattinger [8] in order to prove the global existence of solutions for nonlinear hyperbolic equations which have not necessary

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positive definite energy. After that many authors [3,6,7,9,10] studied the global existence and nonexistence of solutions of the initial boundary value problem for various nonlinear evolution equations by using the potential well method, but the potential wells used in these work were defined by the same method as Sattinger [8], and the results obtained in these work were similar each other.

In this paper, we study the initial boundary value problem of semilinear wave equations:

$$u_{tt} - \Delta u = |u|^{p-1}u, \quad x \in \Omega, \ \ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \ \ t \geqslant 0, \tag{1.3}$$

where $\Omega \subset \mathrm{R}^N$ is a smooth bounded domain, p satisfies

$$(\mathrm{H}_0) \quad 1 < p < \infty \quad \text{for } N = 1,2, \quad 1 < p \leqslant \frac{N+2}{N-2} \quad \text{for } N \geqslant 3.$$

On the global existence, nonexistence and blow-up of solutions for problem (1.1)–(1.3), there have been many results [1,2,4,5,7,9]. First by using the new method, we introduce a family of potential wells which include the known potential well as a special case. And give some results on the properties of this family of potential wells. Then by using it we obtain some new existence and nonexistence theorems of global solutions for problem (1.1)–(1.3) by which the known results are improved very much. Finally, we prove that for any given $e \in (0,d)$ (d is the depth of the known potential well), all solutions of problem (1.1)–(1.3) with initial energy $E(0)$ satisfying $0 < E(0) \leqslant e$ can only lie either inside of some smaller ball or outside of some bigger ball of space $H_0^1(\Omega)$.

The method used in this paper can be applied for some class of other nonlinear evolution equations. So the phenomena of vacuum isolating of solutions are discovered for some class of nonlinear evolution equations.

In this paper, we denote $||\cdot||_{L^p(\Omega)}$ by $||\cdot||_p$, $||\cdot|| = ||\cdot||_2$ and $(u,v) = \int_\Omega uv\,\mathrm{d}x$.

2. Introducing of potential wells

For problem (1.1)–(1.3) we also define

$$E(t) = \frac{1}{2}||u_t||^2 + \frac{1}{2}||\nabla u||^2 - \frac{1}{p+1}||u||_{p+1}^{p+1},$$

$$J(u) = \frac{1}{2}||\nabla u||^2 - \frac{1}{p+1}||u||_{p+1}^{p+1},$$

$$I(u) = ||\nabla u||^2 - ||u||_{p+1}^{p+1},$$

and the potential well

$$W = \{u \in H_0^1(\Omega) \mid I(u) > 0, J(u) < d\} \cup \{0\},$$

where

$$d = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \left(\sup_{\lambda \geqslant 0} J(\lambda u) \right)$$

or equivalently [7]

$$d = \inf J(u)$$

subject to $u \in H_0^1(\Omega)$, $||\nabla u|| \neq 0$, $I(u) = 0$.

On the value of d we have [7]

$$d = \frac{1}{\alpha C_*^{\alpha}}, \quad \alpha = 2\frac{p+1}{p-1}, \tag{2.1}$$

where C_* is the imbedding constant from $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$:

$$||u||_{p+1} \leqslant C_* ||\nabla u||$$

or

$$C_* = \sup \frac{||u||_{p+1}}{||\nabla u||}.$$

In the following, d always is defined by (2.1).

Furthermore, for problem (1.1)–(1.3) and $\delta \in (0, 1)$ we define

$$J_\delta(u) = \frac{\delta}{2} ||\nabla u||^2 - \frac{1}{p+1} ||u||_{p+1}^{p+1},$$

$$d(\delta) = \frac{1-\delta}{2} \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{2}{p-1}}.$$

In the following Lemmas 2.1–2.10 we always assume that p satisfies (H_0), $0 < \delta < 1$ and $u \in H_0^1(\Omega)$.

Lemma 2.1. *If $J(u) \leqslant d(\delta)$, then $J_\delta(u) > 0$ if and only if*

$$0 < ||\nabla u|| < \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{1}{p-1}}. \tag{2.2}$$

Proof. If (2.2) holds, then we have

$$||u||_{p+1}^{p+1} \leqslant C_*^{p+1}||\nabla u||^{p+1} = C_*^{p+1}||\nabla u||^{p-1}||\nabla u||^2 < \frac{p+1}{2}\delta||\nabla u||^2,$$

hence $J_\delta(u) > 0$. On the other hand, from $J_\delta(u) > 0$ we have $||\nabla u|| > 0$, and by

$$J(u) = \frac{1-\delta}{2}||\nabla u||^2 + J_\delta(u) \leqslant d(\delta) \tag{2.3}$$

we can obtain (2.2). □

Lemma 2.2. *If $J(u) \leqslant d(\delta)$, then $J_\delta(u) < 0$ if and only if*

$$||\nabla u|| > \left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}}. \tag{2.4}$$

Proof. If $J_\delta(u) < 0$, then from

$$\frac{p+1}{2}\delta||\nabla u||^2 < ||u||_{p+1}^{p+1} \leqslant C_*^{p+1}||\nabla u||^{p-1}||\nabla u||^2$$

we can get (2.4). On the other hand, from (2.4) and (2.3) we obtain $J_\delta(u) < 0$. □

Lemma 2.3. *If $J(u) = d(\delta)$, then $J_\delta(u) = 0$ if and only if*

$$||\nabla u|| = \left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}}.$$

Lemma 2.4. *As function of δ, $d(\delta)$ possesses the following properties on the interval $0 \leqslant \delta \leqslant 1$:*

(i) $d(0) = d(1) = 0$;

(ii) *$d(\delta)$ takes the maximum, $d(\delta_0) = \frac{1}{\alpha C_*^\alpha} = d$ at $\delta_0 = \frac{2}{p+1}$, where α and d are as defined by* (2.1);

(iii) *$d(\delta)$ is increasing on $[0, \delta_0]$ and decreasing on $[\delta_0, 1]$;*

(iv) *For any given $e \in (0, d)$, the equation $d(\delta) = e$ has exactly two roots $\delta_1 \in (0, \delta_0)$ and $\delta_2 \in (\delta_0, 1)$.*

Proof. Clearly it is enough to prove (ii) and (iii), and they can be obtained by

$$d'(\delta) = A\delta^{\frac{2}{p-1}}\left(\frac{2}{p-1}\frac{1-\delta}{\delta} - 1\right) = \frac{2}{p-1}A\delta^{\frac{2}{p-1}}\left(\frac{1}{\delta} - \frac{p+1}{2}\right),$$

where

$$A = \frac{1}{2}\left(\frac{p+1}{2C_x^{p+1}}\right)^{\frac{2}{p-1}}. \qquad \square$$

Theorem 2.5.

$$d(\delta) = \inf J(u) \tag{2.5}$$

subject to $u \in H_0^1(\Omega), ||\nabla u|| \neq 0, J_\delta(u) = 0.$

Proof. If $J_\delta(u) = 0$ and $||\nabla u|| \neq 0$, then from

$$J(u) = \frac{1-\delta}{2}||\nabla u||^2 + J_\delta(u) = \frac{1-\delta}{2}||\nabla u||^2$$

and

$$\frac{p+1}{2}\delta||\nabla u||^2 = ||u||_{p+1}^{p+1} \leqslant C_*^{p+1}||\nabla u||^{p-1}||\nabla u||^2,$$

it follows that $J(u) \geqslant d(\delta)$. From this and the definition of C_* we get (2.5). $\square$

Corollary 2.6.

$$d = d(\delta_0) = \inf J(u)$$

subject to $u \in H_0^1(\Omega)$, $||\nabla u|| \neq 0$, $I(u) = 0$.

Proof. This corollary can be obtained by Theorem 2.5 and the fact that $J_{\delta_0}(u) = 0$ is equivalent to $I(u) = 0$. $\square$

Remark 2.1. The value of the depth d of the potential well W given by Lemma 2.4 and Corollary 2.6 is exactly the same as that obtained by Payne and Sattinger [7], but it is obtained by a different method in this paper.

According to Theorem 2.5 and Corollary 2.6, now we can define a family of potential wells as follows:

$$W_\delta = \{u \in H_0^1(\Omega) \mid J_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\}, \quad 0 < \delta < 1,$$

$$\bar{W}_\delta = W_\delta \cup \partial W_\delta = \{u \in H_0^1(\Omega) \mid J_\delta(u) \geqslant 0, J(u) \leqslant d(\delta)\} \cup \{0\}.$$

Clearly, we have

$$W_{\delta_0} = W.$$

Remark 2.2. From

$$J(u) = \frac{1-\delta}{2} ||\nabla u||^2 + J_\delta(u)$$

we see that $J_\delta(u) > 0$ implies $J(u) > 0$.

In addition, we define

$$V_\delta = \{u \in H_0^1(\Omega) \mid J_\delta(u) < 0, J(u) < d(\delta)\}, \quad 0 < \delta < 1,$$

$$\bar{V}_\delta = V_\delta \cup \partial V_\delta = \{u \in H_0^1(\Omega) \mid J_\delta(u) \leqslant 0, J(u) \leqslant d(\delta)\},$$

$$V = \{u \in H_0^1(\Omega) \mid I(u) < 0, J(u) < d\},$$

$$B_\delta = \left\{ u \in H_0^1(\Omega) \;\middle|\; ||\nabla u|| < \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{1}{p-1}} \right\},$$

$$\bar{B}_\delta = B_\delta \cup \partial B_\delta = \left\{ u \in H_0^1(\Omega) \;\middle|\; ||\nabla u|| \leqslant \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{1}{p-1}} \right\},$$

$$B_\delta^c = \left\{ u \in H_0^1(\Omega) \;\middle|\; ||\nabla u|| > \left(\frac{p+1}{2c_*^{p+1}} \delta \right)^{\frac{1}{p-1}} \right\}.$$

Clearly, we have $V_{\delta_0} = V$.

Note that $J(u) \leqslant \frac{1}{2} ||\nabla u||^2$, hence for any given $\delta \in (0,1)$, when

$$0 < ||\nabla u|| < (1-\delta)^{\frac{1}{2}} \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{1}{p-1}},$$

we have $J(u) < d(\delta)$ and $J_\delta(u) > 0$. This implies

$$B_{\bar{\delta}} \subset W_\delta, \quad \bar{\delta} \text{ satisfies } \left(\frac{p+1}{2C_*^{p+1}} \bar{\delta} \right)^{\frac{1}{p-1}} = (1-\delta)^{\frac{1}{2}} \left(\frac{p+1}{2C_*^{p+1}} \delta \right)^{\frac{1}{p-1}}.$$

From this, Lemmas 2.1 and 2.2 we have

Theorem 2.7. *Let W_δ, V_δ, B_δ, B_δ^c and $\bar{\delta}$ be as defined above. Then*

$$B_{\bar{\delta}} \subset W_\delta \subset B_\delta, \; V_\delta \subset B_\delta^c.$$

Corollary 2.8.

$$B_{\bar{\delta}_0} \subset W \subset B_{\delta_0}, \quad V \subset B^c_{\delta_0},$$

where

$$B_{\delta_0} = \{u \in H^1_0(\Omega) \mid ||\nabla u|| < C_*^{-\frac{p+1}{p-1}}\},$$

$$\left(\frac{p+1}{2C_*^{p+1}}\bar{\delta}_0\right)^{\frac{1}{p-1}} = \left(\frac{p-1}{p+1}\right)^{\frac{1}{2}} C_*^{-\frac{p+1}{p-1}}.$$

Lemma 2.9. (i) *If* $0 < \delta' < \delta'' \leqslant \delta_0$, *then* $W_{\delta'} \subset W_{\delta''}$.
(ii) *If* $\delta_0 \leqslant \delta' < \delta'' < 1$, *then* $V_{\delta''} \subset V_{\delta'}$.

Proof. This lemma can be obtained from the definition of W_δ and V_δ and Lemma 2.4. □

Lemma 2.10. *Assume that* $0 < J(u) < d$ *for some given* $u \in H^1_0(\Omega)$, $\delta_1 < \delta_2$ *are the two roots of equation* $d(\delta) = J(u)$. *Then the sign of* $J_\delta(u)$ *is not changed for* $\delta \in (\delta_1, \delta_2)$.

Proof. First $J(u) > 0$ implies $||\nabla u|| \neq 0$. If the sign of $J_\delta(u)$ is changed, then there must exist a $\delta^* \in (\delta_1, \delta_2)$ such that $J_{\delta^*}(u) = 0$. So by Theorem 2.5 and Lemma 2.4 we have

$$J(u) \geqslant d(\delta^*) > d(\delta_1) = d(\delta_2),$$

which contradicts $J(u) = d(\delta_1) = d(\delta_2)$. □

3. Existence and nonexistence of global solutions

In [9], Tsutsumi studied the initial boundary value problem of semilinear wave equations in the following form:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + Au + f(u) = 0,$$
$$u(0) = u_0, \quad u_t(0) = u_1$$

in a Hilbert space $H \subset V$. Taking $A = -\Delta, f(u) = -|u|^{p-1}u$, $H = L^2(\Omega)$, $V = H^1_0(\Omega)$ in [9, Theorem 6], we can only obtain the following:

Theorem 3.1. *Let p satisfy*

$$(\mathrm{H}_1) \quad 2<p<\infty \quad \text{for } N=1,2; \quad 2<p\leqslant \frac{N}{N-2} \quad \text{for } N\geqslant 3.$$

Assume that $u_0(x)\in W\cap H^2(\Omega)$, $u_1(x)\in H_0^1(\Omega)$ *and* $E(0)<d$. *Then problem* (1.1)–(1.3) *has a unique global solution* $u(t)\in C([0,T];\ H^2(\Omega)\cap H_0^1(\Omega))\cap C^1([0,T];\ H_0^1(\Omega))\cap C^2([0,T]lL^2(\Omega))$, *and* $u(t)\in W$ *for* $t\in[0,T]$, *for any* $T>0$.

Remark 3.1. Note that $u_0\in W$ implies that $E(0)\geqslant J(u_0)>J_{\delta_0}(u_0)>0$ if $||\nabla u_0||\neq 0$ or that $E(0)\geqslant J(u_0)=0$ if $||\nabla u_0||=0$.

From (H_1) we see that Theorem 3.1 is only applicable for $N\leqslant 3$. On the other hand from Theorem 3.1 we cannot conclude whether there exists a global weak solution of problem (1.1)–(1.3) provided $u_0(x)\in W$, $u_1(x)\in L^2(\Omega)$.

Next we give a new existence theorem of global weak solutions for problem (1.1)–(1.3).

Theorem 3.2. *Let p satisfy* (H_0), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. *Suppose that* $0<E(0)<d$, $\delta_1<\delta_2$ *are the two roots of equation* $d(\delta)=E(0)$ *and* $J_{\delta_2}(u_0)>0$ *or* $||\nabla u_0||=0$. *Then problem* (1.1)–(1.3) *admits a global weak solution* $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ *with* $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ *and* $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$ *and* $0\leqslant t<\infty$.

Proof. Let $\{w_j(x)\}$ be a system of base functions of $H_0^1(\Omega)$. Construct approximate solutions of problem (1.1)–(1.3)

$$u_m(x,t)=\sum_{j=1}^m g_{jm}(t)w_j(x), \quad m=1,2,\dots,$$

satisfying

$$(u_{mtt},w_s)+(\nabla u_m\,\nabla w_s)=(|u_m|^{p-1}u_m,w_s), \quad s=1,2,\dots,m, \tag{3.1}$$

$$u_m(0)=\sum_{j=1}^m a_{jm}w_j(x)\to u_0(x) \quad \text{in } H_0^1(\Omega), \tag{3.2}$$

$$u_{mt}(0)=\sum_{j=1}^m b_{jm}w_j(x)\to u_1(x) \quad \text{in } L^2(\Omega). \tag{3.3}$$

Multiplying (3.1) by $g'_{sm}(t)$, summing for s and integrating with respect to t, we obtain

$$\begin{aligned} E_m(t)&=\frac{1}{2}||u_{mt}||^2+\frac{1}{2}||\nabla u_m||^2-\frac{1}{p+1}||u_m||_{p+1}^{p+1}\\ &=\frac{1}{2}||u_{mt}(0)||^2+\frac{1}{2}||\nabla u_m(0)||^2-\frac{1}{p+1}||u_m(0)||_{p+1}^{p+1}=E_m(0). \end{aligned} \tag{3.4}$$

Note that $J_{\delta_2}(u_0)>0$ implies $||\nabla u_0||\neq 0$. So by an argument similar to that of Lemma 2.10, we can prove $J_\delta(u_0)>0$ for $\delta\in(\delta_1,\delta_2)$. From this and $J(u_0)\leqslant E(0)=d(\delta_1)$ we obtain $u_0(x)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$. If $||\nabla u_0||=0$, then $u_0(x)\in W_\delta$ for $\delta\in(0,1)$. For any fixed $\delta\in(\delta_1,\delta_2)$, we have $J_\delta(u_m(0))>0$ and $E_m(0)<d(\delta)$ (if $J_{\delta_2}(u_0)>0$) or $u_m(0)\in B_{\bar\delta}$ (if $||\nabla u_0||=0$, $\bar\delta$ is defined in Theorem 2.7) thereby $u_m(0)\in W_\delta$ for sufficiently large m.

Next, we prove that $u_m(t)\in W_\delta$ for sufficiently large m and $t>0$. If it is false, then we must have a $t_0>0$ such that $u_m(t_0)\in\partial W_\delta$, i.e. $J_\delta(u_m(t_0))=0$ and $||\nabla u_m(t_0)||\neq 0$ or $J(u_m(t_0))=d(\delta)$. From (3.4) we have

$$J(u_m(t))\leqslant E_m(0)<d(\delta),\quad t>0,$$

hence $J(u_m(t_0))=d(\delta)$ is impossible. If $J_\delta(u_m(t_0))=0$ and $||\nabla u_m(t_0)||\neq 0$, then by Theorem 2.5 we have $J(u_m(t_0))\geqslant d(\delta)$, which is also impossible. Thus from (3.4) and

$$J(u_m)=\frac{1-\delta}{2}||\nabla u_m||^2+J_\delta(u_m),$$

we obtain

$$||\nabla u_m(t)||<\left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}},$$

$$||u_{mt}(t)||_{p+1}\leqslant C_*||\nabla u_m(t)||\leqslant C_*\left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}}$$

and

$$||u_{mt}(t)||^2<2d(\delta)$$

for $t>0$ and sufficiently large m. From these and the compactness method, we can prove that problem (1.1)–(1.3) admits a global weak solution $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ with $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ and $u(t)\in W_\delta$ for $0\leqslant t<\infty$. From the arbitrariness of δ, we obtain $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$ and $0\leqslant t<\infty$. Theorem 3.2 is proved. □

Corollary 3.3. *If in Theorem* 3.2 *the assumption* $J_{\delta_2}(u_0)>0$ *or* $||\nabla u_0||=0$ *is replaced by* $I(u_0)>0$ *or* $||\nabla u_0||=0$, *i.e.* $u_0(x)\in W$, *then the conclusion of Theorem* 3.2 *also holds.*

Proof. This corollary can be obtained from the fact that $I(u_0)>0$ implies $J_{\delta_0}(u_0)>0$ and $J_{\delta_2}(u_0)\geqslant J_{\delta_0}(u_0)$. □

From Corollary 4.4 of this paper, we can obtain the following:

Corollary 3.4. *Under the conditions of Theorem* 3.2 *we have* $u(t)\in\bar W_{\delta_1}$ *for* $0\leqslant t<\infty$.

From Lemma 2.1, Corollary 3.4 and proof of Theorem 3.2, we can obtain the following:

Theorem 3.5. *Suppose that* p, $u_i(x)$ $(i=0,1)$ *and* $E(0)$ *satisfy the conditions of Theorem* 3.2, $\delta_1<\delta_2$ *are the two roots of equation* $d(\delta)=E(0)$, *and* $u_0(x)\in B_{\delta_2}$. *Then problem* (1.1)–(1.3) *admits a global weak solution* $u(t)\in L^\infty(0,\infty;H_0^1(\Omega))$ *with* $u_t(t)\in L^\infty(0,\infty;L^2(\Omega))$ *and* $u(t)\in \bar{B}_{\delta_1}$ *with* $||u_t(t)||\leqslant\sqrt{2d(\delta_1)}$ *for* $0\leqslant t<\infty$.

Theorem 3.6. *Suppose that* p *and* $u_i(x)$ $(i=0,1)$ *satisfy the conditions of Theorem* 3.2, *and* $E(0)=0$, $||\nabla u_0||=0$. *Then problem* (1.1)–(1.3) *admits a unique global solution* $u(t)\equiv 0$.

Proof. Since $||\nabla u_0||=0$ implies $J(u_0)=0$, and by $E(0)=0$ we have $u_1=0$. From this we see that $u(t)\equiv 0$ is a global solution of problem (1.1)–(1.3). On the other hand, the uniqueness of solution can be obtained from the following Theorem 4.7 of this paper. □

Theorem 3.7. *Let* p satisfy (H_0), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. *Assume that* $E(0)<0$ *or* $E(0)=0$, $(u_0,u_1)>0$ *or* $0<E(0)<d$, $I(u_0)<0$, $(u_0,u_1)>0$. *Then the existence time* T *of solutions for problem* (1.1)–(1.3) *are finite*. *Furthermore*, *if we make a further assumption* $1<p\leqslant\frac{N}{N-2}$ *for* $N\geqslant 3$ *or* $(u_0,u_1)>0$ *for the case* $E(0)<0$, *then for all cases solutions* $u(t)$ *of problem* (1.1)–(1.3) *must blow-up in a finite time*, *i.e.*

$$\lim_{t\to T}||\nabla u(t)||=\infty$$

and

$$\lim_{t\to 1}||u(t)||_q=\infty \quad \textit{for some } q\geqslant 2.$$

Proof. (1) $E(0)<0$, or $E(0)=0$, $(u_0,u_1)>0$.

From the last part of the proof of Theorem 4.2 in [1] we know that the existence time T of solutions of problem (1.1)–(1.3) are finite

(i) If $E(0)<0$ and $1<p\leqslant\frac{N}{N-2}$ for $N\geqslant 3$, then in [1, Theorem 4.2], we have

$$\lim_{t\to T}\left(||u_t||^2+||\nabla u||^2\right)=\infty. \tag{3.5}$$

From (3.5) and energy equality

$$E(t)=\frac{1}{2}||u_t||^2+\frac{1}{2}||\nabla u||^2-\frac{1}{p+1}||u||_{p+1}^{p+1}=E(0)$$

we have

$$\lim_{t \to T} ||u||_{p+1} = \infty. \tag{3.6}$$

From (3.6) and

$$||u||_{p+1} \leqslant C_* ||\nabla u||$$

we obtain

$$\lim_{t \to T} ||\nabla u|| = \infty.$$

(ii) If $E(0) < 0$, p satisfies (H_0) and $(u_0, u_1) > 0$, then by

$$\frac{1}{2}||u_1||^2 + \frac{1-\delta_0}{2}||\nabla u_0||^2 + J_{\delta_0}(u_0) = E(0) \tag{3.7}$$

it follows that $J_{\delta_0}(u_0) < 0$ and $I(u_0) < 0$. The remainder of proof is the same as that of the following:

(iii) If $E(0) = 0$, $(u_0, u_1) > 0$, then from

$$0 < (u_0, u_1) \leqslant ||u_0|| \, ||u_1|| \leqslant \lambda_0 ||\nabla u_0|| \, ||u_1||$$

we have $||\nabla u_0|| > 0$, $||u_1|| > 0$. From this and (3.7) we get $J_{\delta_0}(u_0) < 0$ and $I(u_0) < 0$. The remainder of proof is the same as that of (2).

(2) $0 < E(0) < d$, $I(u_0) < 0$ and $(u_0, u_1) > 0$. We also define $M(t) = ||u||^2$. Then by the proof of Theorem 4.3 in [7] we can obtain $\ddot{M}(t) \geqslant 0$. And by $\dot{M}(0) = 2(u_0, u_1) > 0$ we get $\dot{M}(t) > 0$ and $M(t)$ is increasing for $t > 0$. Again by the proof of Theorem 4.3 in [7] we can obtain finally

$$\lim_{t \to T} ||u||_q = \infty, \quad q \geqslant 2. \tag{3.8}$$

Again by $||u|| \leqslant \lambda_0 ||\nabla u||$ and (3.8) we get

$$\lim_{t \to T} ||\nabla u|| = \infty. \qquad \square$$

Corollary 3.8. *If in the case* $0 < E(0) < d$ *of Theorem* 3.7, *assumption* $I(u_0) < 0$ *is replaced by* $J_{\delta_1}(u_0) < 0$, δ_1 *is the smaller root of equation* $d(\delta) = E(0)$, *then the conclusion of Theorem* 3.6 *also holds.*

4. Vacuum isolating of solutions

In this section we shall prove the main result, i.e. the behaviour of vacuum isolating of solutions for problem (1.1)–(1.3).

Theorem 4.1. *Let p satisfy* (H_0), $u_0(x)\in H_0^1(\Omega)$, $u_1(x)\in L^2(\Omega)$. *Assume* $0<e<d$, $\delta_1<\delta_2$ *are the two roots of equation* $d(\delta)=e$. *Then*

(i) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)=e$ *belong to* W_δ for $\delta\in(\delta_1,\delta_2)$, *provided* $I(u_0)>0$ *or* $\|\nabla u_0\|=0$.

(ii) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)=e$ *belong to* V_δ *for* $\delta\in(\delta_1,\delta_2)$, *provided* $I(u_0)<0$.

Proof. (i) Let $u(t)$ be any solution of problem (1.1)–(1.3) with initial energy $E(0)=e$ and $I(u_0)>0$ or $\|\nabla u_0\|=0$, T be the existence time of $u(t)$. First by the proof of Corollary 3.3 and Theorem 3.2 we have $u_0(x)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$. Next we prove $u(t)\in W_\delta$ for $\delta\in(\delta_1,\delta_2)$, $0<t<T$. In fact if it is false, then we must have a $t_0\in(0,T)$ such that $u(t_0)\in\partial W_\delta$ for some $\delta\in(\delta_1,\delta_2)$, i.e. $J_\delta(u(t_0))=0$, $\|\nabla u(t_0)\|\neq 0$ or $J(u(t_0))=d(\delta)$. From energy equality

$$E(t)=\tfrac{1}{2}\|u_t\|^2+J(u)=E(0)<d(\delta),\quad 0<t<T \tag{4.1}$$

we see that $J(u(t_0))=d(\delta)$ is impossible. On the other hand, if $J_\delta(u(t_0))=0$ and $\|\nabla u(t_0)\|\neq 0$, then by Theorem 2.5 we have $J(u(t_0))\geqslant d(\delta)$, which contradicts (4.1).

(ii) Let $u(t)$ be any solution of problem (1.1)–(1.3) with initial energy $E(0)=e$ and $I(u_0)<0$, T be the existence time of $u(t)$. Since the sign of $J_\delta(u_0)$ is not changed for $\delta\in(\delta_1,\delta_2)$, we have $J_\delta(u_0)<0$ for $\delta\in(\delta_1,\delta_2)$. From this and $J(u_0)\leqslant E(0)<d(\delta)$ for $\delta\in(\delta_1,\delta_2)$ we obtain $u_0(x)\in V_\delta$ for $\delta\in(\delta_1,\delta_2)$. Next we prove $u(t)\in V_\delta$ for $\delta\in(\delta_1,\delta_2)$, $0<t<T$. If it is false, then we must have a $t_0\in(0,T)$ such that $u(t_0)\in\partial V_\delta$ for some $\delta\in(\delta_1,\delta_2)$, i.e. $J_\delta(u(t_0))=0$ or $J(u(t_0))=d(\delta)$. From (4.1) we see that $J(u(t_0))=d(\delta)$ is impossible. On the other hand, let t_0 be the first time such that $J_\delta(u(t_0))=0$, then $J_\delta(u(t))<0$ for $0\leqslant t<t_0$. From (4.1) and Lemma 2.2 it follows that $\|\nabla u(t)\|>\left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}}$ for $0\leqslant t<t_0$. Hence we have $\|\nabla u(t_0)\|\geqslant\left(\frac{p+1}{2C_*^{p+1}}\delta\right)^{\frac{1}{p-1}}$. Thus by Theorem 2.5 we get $J(u(t_0))\geqslant d(\delta)$, which contradicts (4.1). $\square$

From Theorem 4.1 and Lemma 2.4 we can obtain the following:

Theorem 4.2. *Let* p, $u_i(x)$ $(i=0,1)$, e *and* δ_i $(i=1,2)$ *be the same as those in Theorem* 4.1. *Then*

(i) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *belong to* W_δ *for* $\delta\in(\delta_1,\delta_2)$, *provided* $I(u_0)>0$ *or* $\|\nabla u_0\|=0$.

(ii) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *belong to* V_δ *for* $\delta\in(\delta_1,\delta_2)$, *provided* $I(u_0)<0$.

From Theorem 4.2 and the invariance of the sign of $J_\delta(u_0)$ for $\delta\in(\delta_1,\delta_2)$ we can get the following:

Corollary 4.3. *Let* p, $u_i(x)$ $(i=0,1)$, e *and* δ_i $(i=1,2)$ *be the same as those in Theorem* 4.1. *Then for any one* $\delta\in(\delta_1,\delta_2)$, *both* W_δ *and* V_δ *are invariant under the flow of* (1.1)–(1.3), *respectively*, *provided* $0<E(0)\leqslant e$.

Corollary 4.4. *Let* p, $u_i(x)$ $(i=0,1)$, e *and* δ_i $(i=1,2)$ *be the same as those in Theorem* 4.1. *Then*

(i) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *belong to* $\bar{W}_{\delta_1}$, *provided* $I(u_0)>0$ *or* $\|\nabla u_0\|=0$.

(ii) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *belong to* $\bar{V}_{\delta_2}$, *provided* $I(u_0)<0$.

Proof. Let $u(t)$ be any solution of problem (1.1)–(1.3) with initial energy $E(0)$ satisfying $0<E(0)\leqslant e$, T be the existence time of $u(t)$. First from energy equality

$$\tfrac{1}{2}\|u_t\|^2+J(u)=E(0)\leqslant d(\delta_1)(d(\delta_2)),$$

we have $J(u)\leqslant d(\delta_1)(=d(\delta_2))$ for $0\leqslant t<T$.

For fixed $t\in[0,T)$ letting $\delta\to\delta_1$ $(\delta\to\delta_2)$ in $J_\delta(u)>0$ $(J_\delta(u)<0)$ for the case (i) (case(ii)) we obtain $J_{\delta_1}(u)\geqslant 0$ $(J_{\delta_2}(u)\leqslant 0)$ for $0\leqslant t<T$.

From Corollary 4.4, Lemmas 2.1 and 2.2 we can obtain Theorem 4.5.

Theorem 4.5. *Let* p, $u_i(x)$ $(i=0,1)$, e *and* δ_i $(i=1,2)$ *be the same as those in Theorem* 4.1. *Then*,

(i) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *lie inside of the ball* $\bar{B}_{\delta_1}$ (*may be in* ∂B_{δ_1}), *provided* $u_0(x)\in B_{\delta_0}$.

(ii) *All solutions of problem* (1.1)–(1.3) *with initial energy* $E(0)$ *satisfying* $0<E(0)\leqslant e$ *lie outside of the ball* B_{δ_2} (*may be in* ∂B_{δ_2}), *provided* $u_0(x)\in B^c_{\delta_0}$.

Corollary 4.6. *Let* p, $u_i(x)$ $(i=0,1)$, e *and* δ_i $(i=1,2)$ *be the same as those in Theorem* 4.1. *Then for any one* $\delta\in(\delta_1,\delta_2)$, *both* B_δ *and* B^c_δ *are invariant under the flow of* (1.1)–(1.3), *respectively*, *provided* $0<E(0)\leqslant e$.

The result of Theorem 4.5 shows that for any given $e\in(0,d)$, there is a corresponding vacuum region of solutions

$$U_e=\left\{u\in H^1_0(\Omega)\;\middle|\;\left(\frac{p+1}{2C_*^{p+1}}\delta_1\right)^{\frac{1}{p-1}}<\|\nabla u\|<\left(\frac{p+1}{2C_*^{p+1}}\delta_2\right)^{\frac{1}{p-1}}\right\}$$

for the set of all solutions of problem (1.1)–(1.3) with initial energy $E(0)$ satisfying $0<E(0)\leqslant e$, there is no solution in U_e and all solutions are isolated by U_e. This phenomenon which is first discovered in this paper can be called the phenomenon of vacuum isolating of solutions.

The vacuum region U_e of solutions become bigger and bigger with decreasing of e. As the limit case $e=0$, we obtain the biggest vacuum region of solutions (for $E(0)\geqslant 0$)

$$U_0=\left\{u\in H^1_0(\Omega)\;\middle|\;0<\|\nabla u\|<\left(\frac{p+1}{2C_*^{p+1}}\right)^{\frac{1}{p-1}}\right\}.$$

In fact we have the following:

Theorem 4.7. *Let p, $u_i(x)$ $(i=0,1)$ be the same as those in Theorem* 4.1. *Then all nontrivial solutions of problem* (1.1)–(1.3) *with initial energy $E(0)=0$ lie outside of the ball B_1 (may be in ∂B_1).*

Proof. Let $u(t)$ be any solution of problem (1.1)–(1.3) with initial energy $E(0)=0$, T be the existence time of $u(t)$. From energy equality

$$E(t)=\tfrac{1}{2}||u_t||^2+J(u)=E(0)=0$$

we have $J(u(t))\leqslant 0$ for $0\leqslant t<T$. So by

$$\frac{p+1}{2}||\nabla u||^2\leqslant ||u||_{p+1}^{p+1}\leqslant C_*^{p+1}||\nabla u||^{p-1}||\nabla u||^2,$$

we obtain that any $t\in[0,T)$ must have either $||\nabla u(t)||=0$ or $||\nabla u(t)||\geqslant(\frac{p+1}{2C_*^{p+1}})^{\frac{1}{p-1}}$. Assume $||\nabla u_0||=0$. We prove $||\nabla u(t)||\equiv 0$ for $0\leqslant t<T$. If it is false, then we must have $t\in(0,T)$ such that $0<||\nabla u(t)||<(\frac{p+1}{2C_*^{p+1}})^{\frac{1}{p-1}}$, which contradicts the above conclusion on $||\nabla u(t)||$. By a similar argument we can prove that if $||\nabla u_0||\geqslant(\frac{p+1}{2C_*^{p+1}})^{\frac{1}{p-1}}$, then we must have $||\nabla u(t)||\geqslant(\frac{p+1}{2C_*^{p+1}})^{\frac{1}{p-1}}$ for $0\leqslant t<T$. Theorem 4.7 is proved. $\square$

Theorem 4.8. *Let p, $u_i(x)$ $(i=0,1)$ be the same as those in Theorem* 4.1. *Then all nontrivial solutions of problem* (1.1)–(1.3) *with initial energy $E(0)<0$ satisfy*

$$||\nabla u||>\left(\frac{p+1}{2C_*^{p+1}}\right)^{\frac{1}{p-1}} \tag{4.2}$$

and

$$||\nabla u||\geqslant\left(\frac{(p+1)\sqrt{-2E(0)}}{C_*^{p+1}}\right)^{\frac{1}{p}}. \tag{4.3}$$

Proof. First, from the proof of Theorem 4.7 we can obtain (4.2). On the other hand, by energy equality we have

$$\frac{1}{p+1}||u||_{p+1}^{p+1}\geqslant\frac{1}{2}||\nabla u||^2-E(0)\geqslant 2\frac{1}{\sqrt{2}}\sqrt{-E(0)}||\nabla u||.$$

From this and

$$||u||_{p+1}^{p+1} \leqslant C_*^{p+1} ||\nabla u||^p ||\nabla u||$$

we get (4.3). $\square$

Remarks. Clearly the method and results of this paper can be generalized to more general semilinear wave equations

$$u_{tt} - \Delta u = a|u|^{p-1}u$$

and

$$u_{tt} - \Delta u = f(u).$$

Furthermore, the method of this paper can be applied to some other class of nonlinear evolution equations.

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