

Well-posedness of higher-order Camassa–Holm equations [☆]

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Abstract

We consider higher-order Camassa–Holm equations describing exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane. We establish the existence of global weak solutions. We also present some invariant spaces under the action of the equation. Moreover, we prove a “weak equals strong” uniqueness result.

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1. Introduction

Consider the unit circle S^1 in the plane and the manifold \mathcal{D} of the smooth orientation-preserving diffeomorphisms of S^1 . Following [13] we study the equation for the exponential

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curves on \mathcal{D} using the Riemannian structure induced by the Sobolev inner product $(\cdot, \cdot)_{H^k(\mathbb{R})}$, $k \in \mathbb{N}$ (where we identify $H^0(\mathbb{R})$ and $L^2(\mathbb{R})$). Let $k \in \mathbb{N}$ and

$$\Gamma : t \geq 0 \mapsto u(t, \cdot) \in \mathcal{D}$$

be a curve. It is an exponential curve if it satisfies the following equation [13, (3.7)]

$$\partial_t u = B_k(u, u), \quad t > 0, x \in \mathbb{R}, \quad (1.1)$$

where (see [13, (3.2), (3.3), and Proof of Theorem 2])

$$\begin{aligned} B_k(u, u) &:= A_k^{-1} C_k(u) - u \partial_x u, \\ A_k(u) &:= \sum_{j=0}^k (-1)^j \partial_x^{2j} u, \\ C_k(u) &:= -u A_k(\partial_x u) + A_k(u \partial_x u) - 2 \partial_x u A_k(u). \end{aligned}$$

In the cases $k = 0$ and $k = 1$, (1.1) becomes the inviscid Burgers equation [23]

$$\partial_t u + 3u \partial_x u = 0 \quad (1.2)$$

and the Camassa–Holm equation [2,8]

$$\partial_t u - \partial_t \partial_x^2 u + 3u \partial_x u = 2 \partial_x u \partial_x^2 u + u \partial_x^3 u, \quad (1.3)$$

respectively (see [13, Examples 1 and 2]). This infinite sequence of higher-order Camassa–Holm equations is distinct from what is normally called the Camassa–Holm hierarchy, where the equations beyond the Camassa–Holm equation itself are nonlocal and all equations are completely integrable in the sense that one can find a zero-curvature formulation for each equation in the hierarchy. Indeed, that is the main mechanism behind their construction. For details about the Camassa–Holm hierarchy, see [19,20] and references therein.

In this paper we study the well-posedness of Eq. (1.1). In particular, we show that it possesses a globally defined weak solution u in $C([0, \infty); C^{k-1}(\mathbb{R})) \cap L^\infty([0, \infty); H^k(\mathbb{R}))$ when the initial data $u_0 \in H^k(\mathbb{R})$, $\partial_x^k u_0 \in L^p(\mathbb{R})$, for some $2 < p < \infty$, see Definition 2.3 and Theorem 2.4. Moreover we show the uniqueness of the solution of the Cauchy problem (2.8) within the class of the maps with bounded second spatial derivative.

Similar results hold for the Camassa–Holm equation (1.3) in the case $k = 1$ [4,29]. This equation models the propagation of unidirectional shallow water waves on a flat bottom, and $u(t, x)$ represents the fluid velocity at time t in the horizontal direction x [2,22]. The Camassa–Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [2,18] and is completely integrable [1,2,7,11]. Moreover, it has an infinite number of solitary wave solutions, called *peakons* due to the discontinuity of their first derivatives at the wave peak, interacting like solitons: $u(t, x) = ce^{-|x-ct|}$, $c \in \mathbb{R}$. From a mathematical point of view the Camassa–Holm equation is well studied. Local well-posedness results are proved in [9,21,24,26]. It is also known that there exist global solutions for a particular class of initial data and also solutions that blow up in finite time for a large class of initial data [6,8,9]. Here blow up means that the slope of the solution becomes unbounded while the solution itself stays bounded. More

relevant for the present paper, we recall that existence and uniqueness results for global weak solutions of (1.3) are proved in [10,12,14,15,28,29], see also [4].

On the other hand we recall that the solutions of the Burgers equation (1.2) in the case $k = 0$ experience shock formation and indeed it is well-posed in the space $L^\infty([0, \infty); BV(\mathbb{R}))$. Let us mention the Degasperis–Procesi equation [16,17]

$$\partial_t u - \partial_t \partial_x^2 u + 4u \partial_x u = 3 \partial_x u \partial_x^2 u + u \partial_x^3 u. \quad (1.4)$$

It appears to be similar to the Camassa–Holm equation (1.3), but its solutions are in general discontinuous, see [5] and the references cited therein.

To keep the presentation short, details are presented for the case $k = 2$ only. In Appendices A and B we show how to extend the theory to general $k > 2$.

The paper is organized as follows. In Section 2 we introduce the equations and state the main result. The existence result is obtained as a singular limit of a viscous regularization. The key a priori estimates are treated in Section 3. The necessary compactness arguments as well as regularity of the solution are obtained in Sections 4 and 5. In Section 6 we prove a “weak equals strong” uniqueness result. Appendices A and B deal with the general case $k > 2$.

2. The governing equations and the main theorem

We construct a family of higher-order Camassa–Holm equations as follows. Let $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Consider the equation

$$\partial_t u = B_k(u, u), \quad (2.1)$$

where $u = u(t, x) : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is the unknown function and

$$\begin{aligned} B_k(u, u) &:= A_k^{-1} C_k(u) - u \partial_x u, \\ A_k(u) &:= \sum_{j=0}^k (-1)^j \partial_x^{2j} u, \\ C_k(u) &:= -u A_k(\partial_x u) + A_k(u \partial_x u) - 2 \partial_x u A_k(u). \end{aligned} \quad (2.2)$$

It turns out that the operator $C_k(u)$ is a total derivative, that is, there exists a differential polynomial in u denoted by \mathcal{F}_k such that

$$C_k(u) = -\partial_x \mathcal{F}_k(u).$$

One can see this as follows

$$\begin{aligned} -\mathcal{F}_k(u) &= \int_{-\infty}^x C_k(u(\xi)) d\xi \\ &= \int_{-\infty}^x (-u A_k(\partial_x u) + A_k(u \partial_x u) - 2 \partial_x u A_k(u)) d\xi \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^k (-1)^j \int_{-\infty}^x \left(-u \partial_x^{2j} \partial_x u + \partial_x^{2j} (u \partial_x u) - 2 \partial_x u \partial_x^{2j} u \right) d\xi \\
&= \frac{1}{2} \sum_{j=0}^k (-1)^j \partial_x^{2j} (u^2) - \sum_{j=0}^k (-1)^j \int_{-\infty}^x \left(u \partial_x^{2j} \partial_x u + 2 \partial_x u \partial_x^{2j} u \right) d\xi.
\end{aligned}$$

Lemma A.2 shows that indeed the integrand in each term is a total derivative, making $\mathcal{F}_k(u)$ a differential polynomial in u .

Remark 2.1. The operator A_k^{-1} has a convolution structure, more precisely

$$A_k^{-1}(f)(x) = \int_{\mathbb{R}} G_k(x-y) f(y) dy, \quad x \in \mathbb{R}, \quad (2.3)$$

where G_k has Fourier transform \widehat{G}_k given by

$$\widehat{G}_k(\zeta) = \frac{1}{1 + \zeta^2 + \dots + \zeta^{2k}}, \quad \zeta \in \mathbb{R}.$$

We also have

$$G_k \geq 0, \quad \|G_k\|_{W^{2k-1,1}(\mathbb{R})}, \|G_k\|_{W^{2k-1,\infty}(\mathbb{R})} \leq C_0, \quad (2.4)$$

for some constant $C_0 > 0$. In the special case $k = 1$ we find

$$G_1(x) = \frac{1}{2} e^{-|x|}.$$

We will repeatedly use that

$$\partial_x^j A_k(u) = A_k(\partial_x^j u)$$

as well as

$$\int_{\mathbb{R}} v A_k(w) dx = \int_{\mathbb{R}} A_k(v) w dx.$$

Example 2.2. (2.1) reads in the cases $k = 0, 1, 2, 3$ as follows [13].

(i) For $k = 0$ we find the following:

$$\partial_t u + u \partial_x u = -\partial_x (u^2) \quad \text{or} \quad \partial_t u + 3u \partial_x u = 0,$$

which constitutes the inviscid Burgers equation, and

$$A_0(u) = u, \quad C_0(u) = -2u \partial_x u, \quad \mathcal{F}_0(u) = u^2.$$

(ii) For $k = 1$ we obtain the following equation:

$$\partial_t u + u \partial_x u = -\partial_x A_1^{-1} \left(u^2 + \frac{1}{2} (\partial_x u^2) \right)$$

or

$$\partial_t u - \partial_t \partial_x^2 u + 3u \partial_x u = 2\partial_x u \partial_x^2 u + u \partial_x^3 u,$$

which is the Camassa–Holm equation. Furthermore,

$$A_1(u) = u - \partial_x^2 u, \quad C_1(u) = -2u \partial_x u - \partial_x u \partial_x^2 u, \quad \mathcal{F}_1(u) = u^2 + \frac{1}{2} (\partial_x u)^2.$$

(iii) For $k = 2$, Eq. (2.1) becomes

$$\partial_t u + u \partial_x u = -A_2^{-1} \partial_x \left[u^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{2} (\partial_x^2 u)^2 - 3\partial_x (u \partial_x^2 u) \right], \quad (2.5)$$

or equivalently

$$\partial_t u - \partial_t \partial_x^2 u + \partial_t \partial_x^4 u + 3u \partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u + 2\partial_x u \partial_x^4 u + u \partial_x^5 u = 0. \quad (2.6)$$

In particular,

$$\begin{aligned} A_2(u) &= \partial_x^4 u - \partial_x^2 u + u, \\ C_2(u) &= -u A_2(\partial_x u) + A_2(u \partial_x u) - 2\partial_x u A_2(u) \\ &= -\partial_x u \partial_x^2 u + 10\partial_x^2 u \partial_x^3 u + 3\partial_x u \partial_x^4 u - 2u \partial_x u, \\ \mathcal{F}_2(u) &= - \int_{-\infty}^x C_2(u) dx \\ &= u^2 + \frac{1}{2} (\partial_x u)^2 - \frac{7}{2} (\partial_x^2 u)^2 - 3\partial_x u \partial_x^3 u \\ &= u^2 + \frac{1}{2} (\partial_x u)^2 - \frac{1}{2} (\partial_x^2 u)^2 - 3\partial_x (u \partial_x^2 u). \end{aligned}$$

(iv) For $k = 3$, Eq. (2.1) becomes

$$\begin{aligned} \partial_t u + u \partial_x u &= -A_2^{-1} \partial_x \left[u^2 + \frac{1}{2} (\partial_x u)^2 - \frac{7}{2} (\partial_x^2 u)^2 - 3\partial_x u \partial_x^3 u \right. \\ &\quad \left. + 5\partial_x u \partial_x^5 u + 16\partial_x^2 u \partial_x^4 u + \frac{19}{2} (\partial_x^3 u)^2 \right], \end{aligned} \quad (2.7)$$

or equivalently

$$\begin{aligned} \partial_t u - \partial_t \partial_x^2 u + \partial_t \partial_x^4 u - \partial_t \partial_x^6 u + 3u \partial_x u - 2\partial_x u \partial_x^2 u - u \partial_x^3 u \\ + 2\partial_x u \partial_x^4 u + u \partial_x^5 u - 2\partial_x u \partial_x^6 u - u \partial_x^7 u = 0. \end{aligned}$$

In particular,

$$\begin{aligned} A_3(u) &= -\partial_x^6 u + \partial_x^4 u - \partial_x^2 u + u = A_2(u) - \partial_x^6 u, \\ C_3(u) &= -u A_3(\partial_x u) + A_3(u \partial_x u) - 2\partial_x u A_3(u) \\ &= C_2(u) - 35\partial_x^3 u \partial_x^4 u - 21\partial_x^2 u \partial_x^5 u - 5\partial_x u \partial_x^6 u, \\ \mathcal{F}_3(u) &= - \int_{-\infty}^x C_3(u) dx \\ &= u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{7}{2}(\partial_x^2 u)^2 - 3\partial_x u \partial_x^3 u \\ &\quad + 5\partial_x u \partial_x^5 u + 16\partial_x^2 u \partial_x^4 u + \frac{19}{2}(\partial_x^3 u)^2. \end{aligned}$$

We are interested in the Cauchy problem

$$\begin{cases} \partial_t u = B_k(u, u), & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (2.8)$$

in the case $k \geq 2$. We will assume

$$u_0 \in H^k(\mathbb{R}), \quad \partial_x^k u_0 \in L^p(\mathbb{R}) \quad \text{for some } 2 < p < \infty. \quad (2.9)$$

For the definition of weak solutions of (2.8) we reformulate the equation as a system of a hyperbolic equation and a higher-order elliptic one, namely

$$\begin{cases} \partial_t u + u \partial_x u + \partial_x P = 0, \\ A_k(P) = \mathcal{F}_k(u). \end{cases} \quad (2.10)$$

This formulation is formally equivalent to (2.8).

Definition 2.3. We call a function $u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ a weak solution of (2.8) if

- (i) $u \in C([0, \infty); C^{k-1}(\mathbb{R})) \cap L^\infty([0, \infty); H^k(\mathbb{R}))$;
- (ii) u satisfies (2.10) in the sense of distributions;
- (iii) $u(0, x) = u_0(x)$ for every $x \in \mathbb{R}$;
- (iv) $\|u(t, \cdot)\|_{H^k(\mathbb{R})} \leq \|u_0\|_{H^k(\mathbb{R})}$, for each $t > 0$.

Our main result is the following.

Theorem 2.4. Let $2 < p < \infty$. For any $u_0 \in \mathcal{H}_{k,p}$, the Cauchy problem (2.8) admits a weak solution $u = u(t, x)$ in the sense of Definition 2.3, where

$$\mathcal{H}_{k,p} := \{f \in H^k(\mathbb{R}) \mid \partial_x^k f \in L^p(\mathbb{R})\}.$$

Moreover, the spaces $H^{k+1}(\mathbb{R})$ and $\mathcal{H}_{k,r}$, $2 \leq r < \infty$, are invariant under the action of equation, i.e.,

$$u_0 \in H^{k+1}(\mathbb{R}) \implies u \in L^\infty([0, T]; H^{k+1}(\mathbb{R})), \quad (2.11)$$

$$u_0 \in \mathcal{H}_{k,r} \implies u \in L^\infty([0, T]; \mathcal{H}_{k,r}), \quad 2 \leq r < \infty, \quad (2.12)$$

for each $T > 0$.

In addition we prove the following “weak equals strong” uniqueness principle.

Theorem 2.5. Assume $k = 2$. Let u be a weak solution of the Cauchy problem (2.8) in the sense of Definition 2.3. If there exists a map $b \in L^1([0, T])$, $T > 0$, such that

$$\|\partial_x^2 u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq b(t), \quad t \geq 0,$$

then u is unique within the class of the maps satisfying such a condition.

In particular, here we assume $b \in L^1([0, T])$, $T > 0$, and in [29] when $k = 1$, the authors assumed $b \in L^2([0, T])$, $T > 0$.

One should observe that the behavior of the Camassa–Holm equation ($k = 1$) is quite different from the behavior of (2.8). Indeed the equation for $q = \partial_x^2 u$, which is a relevant quantity for (2.8), is

$$\partial_t q + u \partial_x q + \tilde{P} = 0,$$

where \tilde{P} is a given function that will be defined later on. On the other hand, if u solves the Camassa–Holm equation (1.3), then $q = \partial_x u$, which is the corresponding relevant quantity, satisfies (P is another given function)

$$\partial_t q + u \partial_x q + \frac{1}{2} q^2 - u^2 + P = 0,$$

which now contains the nonlinear term q^2 .

We apply the following singular perturbation approach. Let $\varepsilon > 0$, and consider the system

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ A_k(P_\varepsilon) = \mathcal{F}_k(u_\varepsilon), & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases} \quad (2.13)$$

We call the solution $u_\varepsilon = u_\varepsilon(t, x)$ of (2.13) a *viscous approximant* to the solution $u = u(t, x)$ of (2.8). Furthermore, we shall assume

$$u_{0,\varepsilon} \in H^{k+1}(\mathbb{R}), \quad \|u_{0,\varepsilon}\|_{H^k(\mathbb{R})} \leq \|u_0\|_{H^k(\mathbb{R})}, \quad u_{0,\varepsilon} \rightarrow u_0 \quad \text{in } H^k(\mathbb{R}). \quad (2.14)$$

Example 2.6. Eqs. (2.10) and (2.13) read in the special cases $k = 0, 1, 2, 3$ as follows.

(i) For $k = 0$ we find the following:

$$\partial_t u + u \partial_x u + \partial_x P = 0, \quad P = u^2,$$

and

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \quad P_\varepsilon = u_\varepsilon^2.$$

(ii) For $k = 1$ we obtain the following equations

$$\partial_t u + u \partial_x u + \partial_x P = 0, \quad P - \partial_x^2 P = u^2 + \frac{1}{2}(\partial_x u)^2,$$

and

$$\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon, \quad P_\varepsilon - \partial_x^2 P_\varepsilon = u_\varepsilon^2 + \frac{1}{2}(\partial_x u_\varepsilon)^2.$$

(iii) For $k = 2$ we find

$$\begin{aligned} \partial_t u + u \partial_x u + \partial_x P &= 0, \\ \partial_x^4 P - \partial_x^2 P + P &= u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}(\partial_x^2 u)^2 - 3\partial_x(\partial_x u \partial_x^2 u), \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon &= \varepsilon \partial_x^2 u_\varepsilon, \\ \partial_x^4 P_\varepsilon - \partial_x^2 P_\varepsilon + P_\varepsilon &= u_\varepsilon^2 + \frac{1}{2}(\partial_x u_\varepsilon)^2 - \frac{7}{2}(\partial_x^2 u_\varepsilon)^2 - 3\partial_x u_\varepsilon \partial_x^3 u_\varepsilon, \end{aligned} \quad (2.16)$$

or equivalently

$$\begin{aligned} \partial_t u_\varepsilon - \partial_t \partial_x^2 u_\varepsilon + \partial_t \partial_x^4 u_\varepsilon + 3u_\varepsilon \partial_x u_\varepsilon - 2\partial_x u_\varepsilon \partial_x^2 u_\varepsilon - u \partial_x^3 u_\varepsilon + 2\partial_x u_\varepsilon \partial_x^4 u_\varepsilon + u \partial_x^5 u_\varepsilon \\ = \varepsilon \partial_x^2 u_\varepsilon - \varepsilon \partial_x^4 u_\varepsilon + \varepsilon \partial_x^6 u_\varepsilon. \end{aligned} \quad (2.17)$$

(iv) For $k = 3$ we find

$$\begin{aligned} \partial_t u + u \partial_x u + \partial_x P &= 0, \\ -\partial_x^6 P + \partial_x^4 P - \partial_x^2 P + P &= u^2 + \frac{1}{2}(\partial_x u)^2 - \frac{7}{2}(\partial_x^2 u)^2 - 3\partial_x u \partial_x^3 u \\ &\quad + 5\partial_x^2(\partial_x u \partial_x^3 u) + 6\partial_x(\partial_x^2 u \partial_x^3 u) - \frac{3}{2}(\partial_x^3 u)^2. \end{aligned} \quad (2.18)$$

Remark 2.7. Introducing the quantities

$$m := A_k(u), \quad m_\varepsilon := A_k(u_\varepsilon),$$

we have, see [13], that Eqs. (2.10) and (2.13) equal

$$\partial_t m + u \partial_x m + 2m \partial_x u = 0 \quad (2.19)$$

and

$$\partial_t m_\varepsilon + u_\varepsilon \partial_x m_\varepsilon + 2m_\varepsilon \partial_x u_\varepsilon = \varepsilon \partial_x^2 m_\varepsilon, \quad (2.20)$$

respectively.

3. Viscous approximants: Global existence and energy estimate

We begin with the existence of the viscous approximants to (2.8).

Lemma 3.1. Assume (2.9) and (2.14). Let $\varepsilon > 0$. Then there exists a unique global smooth solution $u_\varepsilon = u_\varepsilon(t, x)$ of the Cauchy problem (2.13) belonging to $C([0, \infty); H^{k+1}(\mathbb{R}))$.

Proof. The proof of this statement is similar to the one of [3, Theorem 2.3], and is therefore omitted. \square

Lemma 3.2 (Energy estimate). Assume (2.9) and (2.14). The identity

$$\|u_\varepsilon(t, \cdot)\|_{H^k(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\tau, \cdot)\|_{H^k(\mathbb{R})}^2 d\tau = \|u_{0,\varepsilon}\|_{H^k(\mathbb{R})}^2 \quad (3.1)$$

holds for each $t \geq 0$ and $\varepsilon > 0$. In addition,

$$\|u_\varepsilon\|_{L^\infty([0,\infty)\times\mathbb{R})}, \dots, \|\partial_x^{k-1} u_\varepsilon\|_{L^\infty([0,\infty)\times\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^k(\mathbb{R})}, \quad (3.2)$$

for each $\varepsilon > 0$.

Proof. Fix $t > 0$. Multiplying the first equation of (2.13) by $A_k(u_\varepsilon)$ and integrating over \mathbb{R} , we get

$$\begin{aligned} & \int_{\mathbb{R}} \partial_t u_\varepsilon A_k(u_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon A_k(u_\varepsilon) dx \\ &= - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon A_k(u_\varepsilon) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon A_k(u_\varepsilon) dx. \end{aligned} \quad (3.3)$$

Integrating by parts we have for the left-hand side

$$\begin{aligned} & \int_{\mathbb{R}} \partial_t u_\varepsilon A_k(u_\varepsilon) dx - \varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon A_k(u_\varepsilon) dx \\ &= \frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{H^k(\mathbb{R})}^2 + \varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{H^k(\mathbb{R})}^2, \end{aligned} \quad (3.4)$$

and, using the second equation of (2.13), we have for the right-hand side

$$\begin{aligned}
& - \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} A_k(u_{\varepsilon}) dx - \int_{\mathbb{R}} \partial_x P_{\varepsilon} A_k(u_{\varepsilon}) dx \\
& = - \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} A_k(u_{\varepsilon}) dx - \int_{\mathbb{R}} \partial_x (A_k(P_{\varepsilon})) u_{\varepsilon} dx \\
& = - \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} A_k(u_{\varepsilon}) dx + \int_{\mathbb{R}} C_k(u_{\varepsilon}) u_{\varepsilon} dx \\
& = -3 \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} A_k(u_{\varepsilon}) dx - \int_{\mathbb{R}} u_{\varepsilon}^2 A_k(\partial_x u_{\varepsilon}) + \int_{\mathbb{R}} u_{\varepsilon} A_k(u_{\varepsilon} \partial_x u_{\varepsilon}) dx \\
& = -3 \int_{\mathbb{R}} u_{\varepsilon} \partial_x u_{\varepsilon} A_k(u_{\varepsilon}) dx + \int_{\mathbb{R}} \partial_x (u_{\varepsilon}^2) A_k(u_{\varepsilon}) + \int_{\mathbb{R}} A_k(u_{\varepsilon}) u_{\varepsilon} \partial_x u_{\varepsilon} dx = 0. \quad (3.5)
\end{aligned}$$

Substituting (3.4) and (3.5) in (3.3),

$$\frac{d}{dt} \|u_{\varepsilon}(t, \cdot)\|_{H^k(\mathbb{R})}^2 + 2\varepsilon \|\partial_x u_{\varepsilon}(t, \cdot)\|_{H^k(\mathbb{R})}^2 = 0.$$

Integrating over $[0, t]$, we get (3.1). Finally, (3.2) is direct consequence of [25, Theorem 8.5], Eqs. (2.14) and (3.1). \square

4. Bounds on the source term P_{ε} and invariance properties with $k = 2$

From now on we assume $k = 2$. We show in Appendix B how to extend the proofs to the general case $k > 2$.

Using Remark 2.1, we may write

$$P_{\varepsilon} = P_{1,\varepsilon} + P_{2,\varepsilon}, \quad (4.1)$$

where

$$\begin{aligned}
P_{1,\varepsilon}(t, x) &:= \int_{\mathbb{R}} G_2(x - y) \left[u_{\varepsilon}^2(t, y) + \frac{1}{2} (\partial_x u_{\varepsilon}(t, y))^2 - \frac{1}{2} (\partial_x^2 u_{\varepsilon}(t, y))^2 \right] dy, \\
P_{2,\varepsilon}(t, x) &:= -3 \int_{\mathbb{R}} G_2(x - y) \left[(\partial_x^2 u_{\varepsilon}(t, y))^2 + \partial_x u_{\varepsilon}(t, y) \partial_x^3 u_{\varepsilon}(t, y) \right] dy.
\end{aligned}$$

Moreover, since G_2 is the Green's function of the operator A_2 , we have

$$\begin{aligned}
\partial_x^3 P_{2,\varepsilon}(t, x) &= -3 \int_{\mathbb{R}} G_2'''(x - y) \left[(\partial_x^2 u_{\varepsilon}(t, y))^2 + \partial_x u_{\varepsilon}(t, y) \partial_x^3 u_{\varepsilon}(t, y) \right] dy \\
&= -3 \partial_x u_{\varepsilon}(t, x) \partial_x^2 u_{\varepsilon}(t, x) \\
&\quad - 3 \int_{\mathbb{R}} (G_2''(x - y) - G_2(x - y)) \partial_x u_{\varepsilon}(t, y) \partial_x^2 u_{\varepsilon}(t, y) dy.
\end{aligned}$$

Hence

$$\partial_x^3 P_\varepsilon = -3\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon}, \quad (4.2)$$

where

$$P_{3,\varepsilon}(t, x) := -3 \int_{\mathbb{R}} (G_2''(x - y) - G_2(x - y)) \partial_x u_\varepsilon(t, y) \partial_x^2 u_\varepsilon(t, y) dy,$$

for each $\varepsilon > 0$, $t \geq 0$, $x \in \mathbb{R}$.

Lemma 4.1. Assume $k = 2$, (2.9) and (2.14). The following inequalities hold

$$\|P_\varepsilon(t, \cdot)\|_{W^{2,1}(\mathbb{R})}, \|P_\varepsilon(t, \cdot)\|_{W^{2,\infty}(\mathbb{R})} \leq 4C_0 \|u_0\|_{H^2(\mathbb{R})}^2, \quad (4.3)$$

$$\|P_{1,\varepsilon}(t, \cdot)\|_{W^{4,1}(\mathbb{R})}, \|P_{1,\varepsilon}(t, \cdot)\|_{W^{4,\infty}(\mathbb{R})} \leq (6C_0 + 1) \|u_0\|_{H^2(\mathbb{R})}^2, \quad (4.4)$$

$$\|P_{2,\varepsilon}(t, \cdot)\|_{W^{2,1}(\mathbb{R})}, \|P_{2,\varepsilon}(t, \cdot)\|_{W^{2,\infty}(\mathbb{R})} \leq 2C_0 \|u_0\|_{H^2(\mathbb{R})}^2, \quad (4.5)$$

$$\|\partial_x^3 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq (7C_0 + 3) \|u_0\|_{H^2(\mathbb{R})}^2, \quad (4.6)$$

$$\|P_{3,\varepsilon}(t, \cdot)\|_{W^{1,1}(\mathbb{R})}, \|P_{3,\varepsilon}(t, \cdot)\|_{W^{1,\infty}(\mathbb{R})} \leq 12C_0 \|u_0\|_{H^2(\mathbb{R})}^2, \quad (4.7)$$

for each $t \geq 0$ and $\varepsilon > 0$.

Proof. Fix $t > 0$. We begin by proving (4.4). Observing that

$$\partial_x^i P_{1,\varepsilon}(t, x) = \int_{\mathbb{R}} \frac{d^i G_2}{dx^i}(x - y) \left[u_\varepsilon^2 + \frac{1}{2} (\partial_x u_\varepsilon(t, y))^2 - \frac{1}{2} (\partial_x^2 u_\varepsilon(t, y))^2 \right] dy,$$

from (2.4) and (3.2),

$$\begin{aligned} \|\partial_x^i P_{1,\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \left\| \frac{d^i G_2}{dx^i} \right\|_{L^p(\mathbb{R})} \int_{\mathbb{R}} \left[u_\varepsilon^2 + \frac{1}{2} (\partial_x u_\varepsilon)^2 + \frac{1}{2} (\partial_x^2 u_\varepsilon)^2 \right] dy \\ &\leq C_0 \|u(t, \cdot)\|_{H^2(\mathbb{R})}^2 \leq C_0 \|u_0\|_{H^2(\mathbb{R})}^2, \end{aligned} \quad (4.8)$$

for each $p \in \{1, \infty\}$, $i \in \{0, 1, 2, 3\}$. Recalling that G_2 is the Green's function of the operator A_2 (see Remark 2.1), we find

$$\partial_x^4 P_{1,\varepsilon} = \partial_x^2 P_{1,\varepsilon} - P_{1,\varepsilon} + u_\varepsilon^2 + \frac{1}{2} (\partial_x u_\varepsilon)^2 - \frac{1}{2} (\partial_x^2 u_\varepsilon)^2, \quad (4.9)$$

hence, (4.4) is a direct consequence of (3.1), (4.8), and (4.9).

We continue by proving (4.5). Observing that

$$\begin{aligned}\partial_x^j P_{2,\varepsilon}(t, x) &= -3 \int_{\mathbb{R}} \frac{d^j G_2}{dx^j} (x - y) \left[(\partial_x^2 u_\varepsilon(t, y))^2 + \partial_x u_\varepsilon(t, y) \partial_x^3 u_\varepsilon(t, y) \right] dy \\ &= -3 \int_{\mathbb{R}} \frac{d^{j+1} G_2}{dx^{j+1}} (x - y) \partial_x u_\varepsilon(t, y) \partial_x^2 u_\varepsilon(t, y) dy,\end{aligned}$$

we conclude, using the Hölder inequality, (2.4) and (3.2), that

$$\begin{aligned}\|\partial_x^j P_{2,\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \left\| \frac{d^{j+1} G_2}{dx^{j+1}} \right\|_{L^p(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dy \\ &\leq C_0 \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq C_0 \|u(t, \cdot)\|_{H^2(\mathbb{R})}^2 \leq C_0 \|u_0\|_{H^2(\mathbb{R})}^2,\end{aligned}$$

for each $p \in \{1, \infty\}$, $j \in \{0, 1, 2\}$. This proves (4.5). Clearly, estimates (4.4) and (4.5) imply (4.3).

Finally, using the Hölder inequality, (2.4) and (3.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dx &\leq \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq \|u_\varepsilon(t, \cdot)\|_{H^2(\mathbb{R})}^2 \leq \|u_0\|_{H^2(\mathbb{R})}^2,\end{aligned}\tag{4.10}$$

$$\begin{aligned}\|\partial_x^i P_{3,\varepsilon}(t, \cdot)\|_{L^p(\mathbb{R})} &\leq 3 \left(\left\| \frac{d^{2+i} G_2}{dx^{2+i}} \right\|_{L^p(\mathbb{R})} + \left\| \frac{d^i G_2}{dx^i} \right\|_{L^p(\mathbb{R})} \right) \int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dx \\ &\leq 6C_0 \|u_0\|_{H^2(\mathbb{R})}^2,\end{aligned}\tag{4.11}$$

for $p \in \{1, \infty\}$ and $i \in \{0, 1\}$. The estimates (4.4), (4.10), and (4.11) imply (4.6) and (4.7). \square

Next we turn to estimates of time derivatives. Introduce the notation

$$\Pi_T := [0, T] \times \mathbb{R},$$

for T positive.

Lemma 4.2. Assume $k = 2$, (2.9) and (2.14). The following inequalities hold

$$\|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})}^2 + 4C_0 \|u_0\|_{H^2(\mathbb{R})}^2 + \|u_0\|_{H^2(\mathbb{R})},\tag{4.12}$$

$$\|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} \leq \sqrt{2T} \|u_0\|_{H^2(\mathbb{R})}^2 + 4C_0 \|u_0\|_{H^2(\mathbb{R})}^2 \sqrt{T} + \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})},\tag{4.13}$$

for each $T, t > 0$ and $0 < \varepsilon < 1$.

Proof. Let $T, t > 0$ and $0 < \varepsilon < 1$. From (2.16) and Lemmas 3.2 and 4.1,

$$\begin{aligned} \|\partial_t u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} &\leq \|u_\varepsilon(t, \cdot) \partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} + \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\quad + \varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq \|u_\varepsilon\|_{L^\infty([0, \infty) \times \mathbb{R})} \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} + \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\quad + \varepsilon \|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})}^2 + 4C_0 \|u_0\|_{H^2(\mathbb{R})}^2 + \varepsilon \|u_0\|_{H^2(\mathbb{R})}, \end{aligned}$$

this proves (4.12).

Moreover, differentiating (2.16) with respect to x , we get

$$\partial_t \partial_x u_\varepsilon + (\partial_x u_\varepsilon)^2 + u_\varepsilon \partial_x^2 u_\varepsilon + \partial_x^2 P_\varepsilon = \varepsilon \partial_x^3 u_\varepsilon, \quad (4.14)$$

then, from Lemmas 3.2 and 4.1 we find that

$$\begin{aligned} \|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} &\leq \|\partial_x u_\varepsilon\|_{L^4(\Pi_T)}^2 + \|u_\varepsilon \partial_x^2 u_\varepsilon\|_{L^2(\Pi_T)} + \|\partial_x^2 P_\varepsilon\|_{L^2(\Pi_T)} + \varepsilon \|\partial_x^3 u_\varepsilon\|_{L^2(\Pi_T)} \\ &\leq \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})}^2 \sqrt{T} + \|u_\varepsilon\|_{L^\infty} \|\partial_x^2 u_\varepsilon\|_{L^2(\Pi_T)} + \|\partial_x^2 P_\varepsilon\|_{L^2(\Pi_T)} + \varepsilon \|\partial_x^3 u_\varepsilon\|_{L^2(\Pi_T)} \\ &\leq \sqrt{2T} \|u_0\|_{H^2(\mathbb{R})}^2 + 4C_0 \|u_0\|_{H^2(\mathbb{R})}^2 \sqrt{T} + \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})}, \end{aligned}$$

which proves (4.13). \square

Lemma 4.3. Assume $k = 2$, (2.9) and (2.14). Let $T > 0$. There exist two positive constants $K_{1,T}$, $K_{2,T}$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$ and T and independent of ε , such that

$$\|\partial_t \partial_x^3 P_{1,\varepsilon}\|_{L^1(\Pi_T)}, \|\partial_t \partial_x^3 P_{1,\varepsilon}\|_{L^\infty(\Pi_T)} \leq K_{1,T}, \quad (4.15)$$

$$\|\partial_t P_{3,\varepsilon}\|_{L^1(\Pi_T)}, \|\partial_t P_{3,\varepsilon}\|_{L^1((0,T);L^\infty(\mathbb{R}))} \leq K_{2,T}, \quad (4.16)$$

for each $0 < \varepsilon < 1$.

Proof. Fix $0 < \varepsilon < 1$ and $T > 0$. We begin by proving (4.15). Observe that

$$\partial_t \partial_x^2 u_\varepsilon + 3\partial_x^2 u_\varepsilon \partial_x u_\varepsilon + u_\varepsilon \partial_x^3 u_\varepsilon + \partial_x^3 P_\varepsilon = \varepsilon \partial_x^4 u_\varepsilon, \quad (4.17)$$

and, from (4.2),

$$\partial_t \partial_x^2 u_\varepsilon + u_\varepsilon \partial_x^3 u_\varepsilon + \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon} = \varepsilon \partial_x^4 u_\varepsilon. \quad (4.18)$$

Hence, since G_2 is the Green's function of the operator A_2 (see Remark 2.1), we find from the definition of $P_{1,\varepsilon}$ and (4.18) that

$$\begin{aligned}
\partial_t \partial_x^3 P_{1,\varepsilon}(t, x) &= \int_{\mathbb{R}} G_2'''(x-y) [\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon + 2u_\varepsilon \partial_t u_\varepsilon - \partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon] dy \\
&= \int_{\mathbb{R}} G_2'''(x-y) [\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon + 2u_\varepsilon \partial_t u_\varepsilon] dy \\
&\quad + \int_{\mathbb{R}} G_2'''(x-y) [\partial_x^2 u_\varepsilon \partial_x^3 P_{1,\varepsilon} + \partial_x^2 u_\varepsilon P_{3,\varepsilon}] dy \\
&\quad + \int_{\mathbb{R}} G_2'''(x-y) [u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon - \varepsilon \partial_x^4 u_\varepsilon \partial_x^2 u_\varepsilon] dy \\
&= \int_{\mathbb{R}} G_2'''(x-y) [\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon + 2u_\varepsilon \partial_t u_\varepsilon] dy \\
&\quad + \int_{\mathbb{R}} G_2'''(x-y) [\partial_x^2 u_\varepsilon \partial_x^3 P_{1,\varepsilon} + \partial_x^2 u_\varepsilon P_{3,\varepsilon}] dy + \frac{1}{2} u_\varepsilon (\partial_x^2 u_\varepsilon)^2 - \varepsilon \partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon \\
&\quad + \int_{\mathbb{R}} (G_2'' - G_2)(x-y) \left[\frac{1}{2} u_\varepsilon (\partial_x^2 u_\varepsilon)^2 - \varepsilon \partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon \right] dy \\
&\quad - \int_{\mathbb{R}} G_2'''(x-y) \left[\frac{1}{2} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 - \varepsilon (\partial_x^3 u_\varepsilon)^2 \right] dy. \tag{4.19}
\end{aligned}$$

Using the Hölder inequality,

$$\begin{aligned}
\|\partial_t \partial_x^3 P_{1,\varepsilon}\|_{L^1(\Pi_T)} &\leq C_0 \int_{\Pi_T} \left[|\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon| + 2|u_\varepsilon \partial_t u_\varepsilon| \right. \\
&\quad + |\partial_x^2 u_\varepsilon \partial_x^3 P_{1,\varepsilon}| + |\partial_x^2 u_\varepsilon P_{3,\varepsilon}| + |u_\varepsilon| (\partial_x^2 u_\varepsilon)^2 \\
&\quad + 2\varepsilon |\partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon| + \frac{1}{2} |\partial_x u_\varepsilon| (\partial_x^2 u_\varepsilon)^2 + \varepsilon (\partial_x^3 u_\varepsilon)^2 \Big] dt dx \\
&\quad + \int_{\Pi_T} \left[\frac{1}{2} |u_\varepsilon| (\partial_x^2 u_\varepsilon)^2 + \varepsilon |\partial_x^3 u_\varepsilon \partial_x^2 u_\varepsilon| \right] dt dx \\
&\leq C_0 \left[\|\partial_x u_\varepsilon\|_{L^2} \|\partial_t \partial_x u_\varepsilon\|_{L^2} + 2\|u_\varepsilon\|_{L^2} \|\partial_t u_\varepsilon\|_{L^2} \right. \\
&\quad + \|\partial_x^2 u_\varepsilon\|_{L^2} \|\partial_x^3 P_{1,\varepsilon}\|_{L^2} + \|\partial_x^2 u_\varepsilon\|_{L^2} \|P_{3,\varepsilon}\|_{L^2} \\
&\quad + \|u_\varepsilon\|_{L^\infty} \|\partial_x^2 u_\varepsilon\|_{L^2}^2 + 2\varepsilon \|\partial_x^3 u_\varepsilon\|_{L^2} \|\partial_x^2 u_\varepsilon\|_{L^2} \\
&\quad + \frac{1}{2} \|\partial_x u_\varepsilon\|_{L^\infty} \|\partial_x^2 u_\varepsilon\|_{L^2}^2 + \varepsilon \|\partial_x^3 u_\varepsilon\|_{L^2}^2 \Big] \\
&\quad + \frac{1}{2} \|u_\varepsilon\|_{L^\infty} \|\partial_x^2 u_\varepsilon\|_{L^2}^2 + \varepsilon \|\partial_x^3 u_\varepsilon\|_{L^2} \|\partial_x^2 u_\varepsilon\|_{L^2}. \tag{4.20}
\end{aligned}$$

Then, the estimate (4.15) is a consequence of (2.4), (3.1), (3.2), (4.4), (4.7), (4.12), and (4.13).

We continue by proving (4.16). Observing that

$$P_{3,\varepsilon}(t, x) = -\frac{3}{2} \int_{\mathbb{R}} (G_2'''(x-y) - G_2'(x-y)) (\partial_x u_\varepsilon(t, y))^2 dy,$$

$$\partial_t P_{3,\varepsilon}(t, x) = -3 \int_{\mathbb{R}} (G_2'''(x-y) - G_2'(x-y)) \partial_x u_\varepsilon(t, y) \partial_t \partial_x u_\varepsilon(t, y) dy,$$

we have

$$\begin{aligned} \|\partial_t P_{3,\varepsilon}\|_{L^1((0,T);L^p(\mathbb{R}))} &\leq 3(\|G_2'''\|_{L^p(\mathbb{R})} + \|G_2'\|_{L^p(\mathbb{R})}) \int_{\Pi_T} |\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon| dx dt \\ &\leq 3(\|G_2'''\|_{L^p(\mathbb{R})} + \|G_2'\|_{L^p(\mathbb{R})}) \|\partial_x u_\varepsilon\|_{L^2(\Pi_T)} \|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)}, \end{aligned} \quad (4.21)$$

for $p \in \{1, \infty\}$. Hence, the estimate (4.16) follows from (2.4), (3.1) and (4.13). \square

Now we look for invariance properties of the problem (2.16).

Lemma 4.4. Assume $k = 2$, (2.9) and (2.14). The following estimate holds

$$\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leq \|\partial_x^2 u_{0,\varepsilon}\|_{L^p(\mathbb{R})} e^{K_1 t} + K_2 \frac{e^{K_1 t} - 1}{K_1}, \quad (4.22)$$

for each $t \geq 0$, $2 \leq p < \infty$ and $\varepsilon > 0$, where

$$K_1 := \frac{1}{p\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})}, \quad K_2 := (18C_0 + 1)^2 \|u_0\|_{H^2(\mathbb{R})}^6.$$

Proof. Let $2 \leq p < \infty$. Denote

$$q_\varepsilon := \partial_x^2 u_\varepsilon,$$

there results

$$\partial_t q_\varepsilon + 3q_\varepsilon \partial_x u_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + \partial_x^3 P_\varepsilon = \varepsilon \partial_x^2 q_\varepsilon, \quad (4.23)$$

and, from (4.2),

$$\partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + \tilde{P}_\varepsilon = \varepsilon \partial_x^2 q_\varepsilon, \quad (4.24)$$

where

$$\tilde{P}_\varepsilon := \partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon}. \quad (4.25)$$

Multiplying (4.24) by $p q_\varepsilon |q_\varepsilon|^{p-2}$ there results

$$\begin{aligned} \partial_t (|q_\varepsilon|^p) + u_\varepsilon \partial_x (|q_\varepsilon|^p) + p \tilde{P}_\varepsilon q_\varepsilon |q_\varepsilon|^{p-2} &= p \varepsilon q_\varepsilon |q_\varepsilon|^{p-2} \partial_x^2 q_\varepsilon \\ &= \varepsilon \partial_x^2 (|q_\varepsilon|^p) - \varepsilon p(p-1) |q_\varepsilon|^{p-2} (\partial_x q_\varepsilon)^2. \end{aligned} \quad (4.26)$$

By (3.2), (4.4), and (4.7),

$$\begin{aligned} p \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1} \frac{d}{dt} \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &= \frac{d}{dt} \int_{\mathbb{R}} |q_\varepsilon|^p dx \\ &\leq \int_{\mathbb{R}} \partial_x u_\varepsilon |q_\varepsilon|^p dx + p \int_{\mathbb{R}} |\tilde{P}_\varepsilon| |q_\varepsilon|^{p-1} dx \\ &\leq K_1 \int_{\mathbb{R}} |q_\varepsilon|^p dx + p \|\tilde{P}_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1} \\ &\leq K_1 \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p + p K_2 \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1}, \end{aligned}$$

hence

$$\frac{d}{dt} \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leq \frac{K_1}{p} \|q_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} + K_2.$$

The claim is a direct consequence of the Gronwall inequality. \square

Lemma 4.5. Assume $k = 2$, (2.9) and (2.14). The following estimate holds

$$\begin{aligned} \|\partial_x^3 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t e^{K_3(t-\tau)} \|\partial_x^4 u_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 d\tau \\ \leq \|\partial_x^3 u_{0,\varepsilon}\|_{L^2(\mathbb{R})}^2 e^{K_3 t} + K_4 \frac{e^{K_3 t} - 1}{K_3}, \end{aligned} \quad (4.27)$$

for each $t \geq 0$ and $\varepsilon > 0$, where

$$K_3 := \frac{1}{\sqrt{2}} \|u_0\|_{H^2(\mathbb{R})} + \frac{7}{2}, \quad K_4 := \left(\frac{3}{4} + 16C_0^2\right) \|u_0\|_{H^2(\mathbb{R})}^4.$$

Proof. Using the notation from the proof of Lemma 4.4, we have

$$\partial_t \partial_x q_\varepsilon + \partial_x u_\varepsilon \partial_x q_\varepsilon - \frac{1}{2} q_\varepsilon^2 + u_\varepsilon \partial_x^2 q_\varepsilon + \frac{1}{2} (\partial_x u_\varepsilon)^2 + u_\varepsilon^2 + \partial_x^2 P_\varepsilon - P_\varepsilon = \varepsilon \partial_x^3 q_\varepsilon. \quad (4.28)$$

By (4.28), (3.2), and (4.3),

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x q_\varepsilon)^2 dx &= \int_{\mathbb{R}} \partial_x q_\varepsilon \partial_t \partial_x q_\varepsilon dx \\
&= \varepsilon \int_{\mathbb{R}} \partial_x^3 q_\varepsilon \partial_x q_\varepsilon dx - \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x q_\varepsilon)^2 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} q_\varepsilon^2 \partial_x q_\varepsilon dx - \int_{\mathbb{R}} u_\varepsilon \partial_x^2 q_\varepsilon \partial_x q_\varepsilon dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x q_\varepsilon dx - \int_{\mathbb{R}} u_\varepsilon^2 \partial_x q_\varepsilon dx \\
&\quad - \int_{\mathbb{R}} \partial_x^2 P_\varepsilon \partial_x q_\varepsilon dx + \int_{\mathbb{R}} P_\varepsilon \partial_x q_\varepsilon dx \\
&= -\varepsilon \int_{\mathbb{R}} (\partial_x^2 q_\varepsilon)^2 dx - \frac{1}{2} \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x q_\varepsilon)^2 dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}} \partial_x q_\varepsilon (\partial_x u_\varepsilon)^2 dx - \int_{\mathbb{R}} u_\varepsilon^2 \partial_x q_\varepsilon dx \\
&\quad - \int_{\mathbb{R}} \partial_x^2 P_\varepsilon \partial_x q_\varepsilon dx + \int_{\mathbb{R}} P_\varepsilon \partial_x q_\varepsilon dx \\
&\leq -\varepsilon \int_{\mathbb{R}} (\partial_x^2 q_\varepsilon)^2 dx + \left(\frac{1}{2} \|\partial_x u_\varepsilon\|_{L^\infty([0,\infty) \times \mathbb{R})} + \frac{7}{4} \right) \int_{\mathbb{R}} (\partial_x q_\varepsilon)^2 dx \\
&\quad + \frac{1}{4} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 dx + \frac{1}{2} \int_{\mathbb{R}} u_\varepsilon^4 dx + \frac{1}{2} \int_{\mathbb{R}} (\partial_x^2 P_\varepsilon)^2 dx + \frac{1}{2} \int_{\mathbb{R}} P_\varepsilon^2 dx \\
&\leq -\varepsilon \int_{\mathbb{R}} (\partial_x^2 q_\varepsilon)^2 dx + \frac{K_3}{2} \int_{\mathbb{R}} (\partial_x q_\varepsilon)^2 dx + \frac{K_4}{2},
\end{aligned}$$

hence, using the Gronwall inequality, we get (4.27). \square

Remark 4.6. Assume $k = 2$, (2.9) and (2.14). From (4.24), (4.27) and Lemmas 3.2 and 4.1, we get

$$\|\partial_t \partial_x^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq K_5 e^{K_5 t} (1+t) \|u_{0,\varepsilon}\|_{H^3(\mathbb{R})}, \quad (4.29)$$

$$\|\partial_x^2 u_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq M_\varepsilon(t), \quad (4.30)$$

for each $t \geq 0$ and $\varepsilon > 0$, where $K_5 > 0$ is a constant depending only on $\|u_0\|_{H^2(\mathbb{R})}$ but independent of ε , and

$$M_\varepsilon(t)^2 := \frac{1}{2} \|u_0\|_{H^2(\mathbb{R})}^2 + \frac{1}{2} \|\partial_x^3 u_{0,\varepsilon}\|_{L^2(\mathbb{R})}^2 e^{K_3 t} + K_4 \frac{e^{K_3 t} - 1}{2K_3}.$$

Lemma 4.7. Assume $k = 2$, (2.9) and (2.14). There exists a constant $K_6 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$ but independent of ε , such that

$$\|\partial_t \partial_x^i P_\varepsilon\|_{L^1(\Pi_T)} \leq K_6 e^{K_6 T} (1 + T) \|u_{0,\varepsilon}\|_{H^3(\mathbb{R})}, \quad (4.31)$$

for each $i \in \{0, 1, 2\}$, $T \geq 0$ and $\varepsilon > 0$.

Proof. Using Remark 2.1, we know

$$P_\varepsilon = P_{4,\varepsilon} + P_{5,\varepsilon}, \quad (4.32)$$

where

$$P_{4,\varepsilon}(t, x) := \int_{\mathbb{R}} G_2(x - y) \left[\frac{1}{2} (\partial_x u_\varepsilon(t, y))^2 + u_\varepsilon^2(t, y) \right] dy,$$

$$P_{5,\varepsilon}(t, x) := - \int_{\mathbb{R}} G_2(x - y) \left[\frac{7}{2} (\partial_x^2 u_\varepsilon(t, y))^2 + 3 \partial_x u_\varepsilon(t, y) \partial_x^3 u_\varepsilon(t, y) \right] dy.$$

Observe that, for each $i \in \{0, 1, 2\}$,

$$\partial_t \partial_x^i P_{4,\varepsilon}(t, x) = \int_{\mathbb{R}} \frac{d^i G_2}{dx^i}(x - y) [\partial_x u_\varepsilon(t, y) \partial_t \partial_x u_\varepsilon(t, y) + 2u_\varepsilon(t, y) \partial_t u_\varepsilon(t, y)] dy,$$

then, by (2.4), (3.2), and Remark 4.2,

$$\begin{aligned} & \|\partial_t \partial_x^i P_{4,\varepsilon}\|_{L^1(\Pi_T)} \\ & \leq \int_{\Pi_T \times \mathbb{R}} \left| \frac{d^i G_2}{dx^i}(x - y) \right| |\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon + 2u_\varepsilon \partial_t u_\varepsilon| dt dx dy \\ & \leq C_0 \int_{\Pi_T} (|\partial_x u_\varepsilon| |\partial_t \partial_x u_\varepsilon| + 2|u_\varepsilon| |\partial_t u_\varepsilon|) dt dx \\ & \leq C_0 (\|\partial_x u_\varepsilon\|_{L^2(\Pi_T)} \|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} + 2\|u_\varepsilon\|_{L^2(\Pi_T)} \|\partial_t u_\varepsilon\|_{L^2(\Pi_T)}) \\ & \leq c_1 (1 + T), \end{aligned} \quad (4.33)$$

for some constant $c_1 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$.

Moreover,

$$\begin{aligned} \partial_t P_{5,\varepsilon}(t, x) &= - \int_{\mathbb{R}} G_2(x - y) [7 \partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon + 3 \partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon + 3 \partial_x u_\varepsilon \partial_t \partial_x^3 u_\varepsilon] dy \\ &= 4 \int_{\mathbb{R}} G_2(x - y) \partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dy - 7 \int_{\mathbb{R}} G'_2(x - y) \partial_t \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \end{aligned}$$

$$\begin{aligned}
& -3 \int_{\mathbb{R}} G'_2(x-y) \partial_t \partial_x^2 u_\varepsilon \partial_x u_\varepsilon dy + 3 \int_{\mathbb{R}} G_2(x-y) \partial_t \partial_x^2 u_\varepsilon \partial_x^2 u_\varepsilon dy \\
& = \int_{\mathbb{R}} G_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dy - \int_{\mathbb{R}} G'_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \\
& \quad - 3 \int_{\mathbb{R}} G''_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon dy,
\end{aligned} \tag{4.34}$$

then, by (2.4), (3.2), (4.27), and Remark 4.2, using the same argument as for (4.33), for $i \in \{0, 1\}$ we infer

$$\begin{aligned}
\|\partial_t \partial_x^{i+1} P_{5,\varepsilon}\|_{L^1(\Pi_T)} & \leq \int_{\Pi_T \times \mathbb{R}} \left| \frac{d^i G_2}{dx^i}(x-y) \right| |\partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon| dt dx dy \\
& \quad + \int_{\Pi_T \times \mathbb{R}} \left| \frac{d^{i+1} G_2}{dx^{i+1}}(x-y) \right| |\partial_t \partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dt dx dy \\
& \quad + 3 \int_{\Pi_T \times \mathbb{R}} \left| \frac{d^{i+2} G_2}{dx^{i+2}}(x-y) \right| |\partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon| dt dx dy \\
& \leq 3C_0 \|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} \|u_\varepsilon\|_{H^3(\Pi_T)} \\
& \leq c_2 e^{c_2 T} (1+T) \|u_{0,\varepsilon}\|_{H^3(\mathbb{R})},
\end{aligned} \tag{4.35}$$

for some constant $c_2 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$.

Since G_2 is the Green's function of A_2 , we have

$$\begin{aligned}
\partial_t \partial_x^2 P_{5,\varepsilon}(t, x) & = \int_{\mathbb{R}} G''_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dy - \int_{\mathbb{R}} G'''_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \\
& \quad - 3 \partial_t \partial_x u_\varepsilon(t, x) \partial_x u_\varepsilon(t, x) - 3 \int_{\mathbb{R}} G''_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon dy \\
& \quad + 3 \int_{\mathbb{R}} G_2(x-y) \partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon dy,
\end{aligned} \tag{4.36}$$

then, by (2.4), (3.2), (4.27), and Remark 4.2, using the same argument of (4.33),

$$\begin{aligned}
\|\partial_t \partial_x^2 P_{5,\varepsilon}\|_{L^1(\Pi_T)} & \leq \int_{\Pi_T \times \mathbb{R}} |G''_2(x-y)| |\partial_t \partial_x u_\varepsilon \partial_x^3 u_\varepsilon| dt dx dy \\
& \quad + \int_{\Pi_T \times \mathbb{R}} |G'''_2(x-y)| |\partial_t \partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dt dx dy
\end{aligned}$$

$$\begin{aligned}
& + 3 \int_{\Pi_T} |\partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon| dt dx \\
& + 3 \int_{\Pi_T \times \mathbb{R}} |G_2''(x-y)| |\partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon| dt dx dy \\
& + 3 \int_{\Pi_T \times \mathbb{R}} G_2(x-y) |\partial_t \partial_x u_\varepsilon \partial_x u_\varepsilon| dt dx dy \\
& \leq 6C_0 \|\partial_t \partial_x u_\varepsilon\|_{L^2(\Pi_T)} \|u_\varepsilon\|_{H^3(\Pi_T)} \\
& \leq c_3 e^{c_3 T} (1+T) \|u_{0,\varepsilon}\|_{H^3(\mathbb{R})},
\end{aligned} \tag{4.37}$$

for some constant $c_3 > 0$ depending only on $\|u_0\|_{H^2(\mathbb{R})}$. \square

5. Existence of solutions

In this section we prove Theorem 2.4 for $k = 2$. We show that the family of viscous approximants is compact in the space $L_{\text{loc}}^\infty([0, \infty); \mathcal{H}_{2,p})$, and the converging subsequence tends to a weak solution of (2.8).

Lemma 5.1. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. Then the family $\{u_\varepsilon\}_{\varepsilon>0}$ that solves (2.13) for $k = 2$ is compact in $L_{\text{loc}}^\infty([0, \infty); H^2(\mathbb{R}))$. Thus there exist a positive sequence $\{\varepsilon_h\}_{h \in \mathbb{N}}$ decreasing to 0 and a function $u \in L^\infty([0, \infty); H^2(\mathbb{R})) \cap H^1([0, T]; H^1(\mathbb{R}))$, for each $T > 0$, such that*

- (i) $u_{\varepsilon_h} \rightarrow u$ in $L^\infty([0, T]; H^2(\mathbb{R}))$, for each $T > 0$;
- (ii) u is a weak solution of (2.8) for $k = 2$.

Before we prove this lemma, we need to establish some further properties. We begin with the following result on basic compactness.

Lemma 5.2. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. Let u_ε , P_ε and \tilde{P}_ε be given by Lemma 3.1, Eqs. (4.1) and (4.25), respectively. There exist a positive sequence $\{\varepsilon_h\}_{h \in \mathbb{N}}$ decreasing to zero and three functions $u \in L^\infty([0, \infty); H^2(\mathbb{R})) \cap H^1([0, T]; H^1(\mathbb{R})) \subseteq C([0, \infty); C^1(\mathbb{R}))$ for each $T > 0$, $P \in L^\infty([0, \infty); W^{2,\infty}(\mathbb{R}))$ and $\tilde{P} \in L^\infty([0, \infty); W^{1,\infty}(\mathbb{R}))$ such that*

$$u_{\varepsilon_h} \rightharpoonup u \quad \text{weakly in } H^1([0, T]; H^1(\mathbb{R})), \text{ for each } T \geq 0; \tag{5.1}$$

$$u_{\varepsilon_h} \rightarrow u \quad \text{strongly in } L_{\text{loc}}^\infty([0, \infty); H_{\text{loc}}^1(\mathbb{R})); \tag{5.2}$$

$$P_{\varepsilon_h} \rightarrow P \quad \text{strongly in } L_{\text{loc}}^p([0, \infty); W_{\text{loc}}^{1,p}(\mathbb{R})), \text{ for each } 1 \leq p < \infty; \tag{5.3}$$

$$\tilde{P}_{\varepsilon_h} \rightarrow \tilde{P} \quad \text{strongly in } L_{\text{loc}}^p([0, \infty) \times \mathbb{R}), \text{ for each } 1 \leq p < \infty. \tag{5.4}$$

Proof. Due to Lemmas 3.2 and 4.2, we have that

$$\begin{aligned} \{u_\varepsilon\}_\varepsilon &\text{ is uniformly bounded in } L^\infty((0, \infty); H^2(\mathbb{R})), \\ \{\partial_t u_\varepsilon\}_\varepsilon &\text{ is uniformly bounded in } L^2((0, T); H^1(\mathbb{R})), \quad T > 0. \end{aligned} \quad (5.5)$$

In particular $\{u_\varepsilon\}_\varepsilon$ is uniformly bounded in $H^1((0, T); H^1(\mathbb{R}))$ and then we have (5.1). Moreover, by observing that $W^{2,\infty}(\mathbb{R}) \subseteq W_{\text{loc}}^{1,p}(\mathbb{R}) \subset L_{\text{loc}}^2(\mathbb{R})$, $1 \leq p < \infty$, (5.6) and [27, Lemmas 5.2 and 5.3] guarantee (5.4).

Due to Lemmas 4.1 and 4.7,

$$\begin{aligned} \{P_\varepsilon\}_\varepsilon &\text{ is uniformly bounded in } L^\infty((0, \infty); W^{1,\infty}(\mathbb{R})), \\ \{\partial_t P_\varepsilon\}_\varepsilon &\text{ is uniformly bounded in } L^1((0, T); W^{2,1}(\mathbb{R})), \quad T > 0. \end{aligned} \quad (5.6)$$

Since $H^2(\mathbb{R}) \subseteq H_{\text{loc}}^1(\mathbb{R}) \subset L_{\text{loc}}^2(\mathbb{R})$, (5.6) and [27, Lemmas 5.2 and 5.3] give (5.2).

Due to Lemmas 4.1 and 4.3,

$$\begin{aligned} \{\partial_x^3 P_{1,\varepsilon} + P_{3,\varepsilon}\}_\varepsilon &\text{ is uniformly bounded in } L^\infty((0, \infty); W^{1,\infty}(\mathbb{R})), \\ \{\partial_t \partial_x^3 P_{1,\varepsilon} + \partial_t P_{3,\varepsilon}\}_\varepsilon &\text{ is uniformly bounded in } L^\infty((0, T); L^\infty(\mathbb{R})), \quad T > 0. \end{aligned} \quad (5.7)$$

Since $W^{1,\infty}(\mathbb{R}) \subseteq L_{\text{loc}}^p(\mathbb{R}) \subset L_{\text{loc}}^1(\mathbb{R})$, $1 \leq p < \infty$, (5.7) and [27, Lemmas 5.2 and 5.3] yield (5.3). \square

Denoting

$$q := \partial_x^2 u, \quad \text{in the weak sense,}$$

we infer from (4.24), (5.1), (5.2), and (5.4) that

$$\partial_t q + u \partial_x q + \tilde{P} = 0 \quad (5.8)$$

holds in the sense of distributions in $[0, \infty) \times \mathbb{R}$.

Since in \tilde{P} we have the nonlinear term $(\partial_x^2 u)^2 = q^2$, we need to show that q_ε converges to q (strongly) in L^2 . This convergence is needed if we want to send $\varepsilon \rightarrow 0$ in the viscous problem and recover the original problem.

Lemma 5.3. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. Then there exists a map $\overline{q^2} \in L^\infty([0, \infty); L^r(\mathbb{R}))$, $1 \leq r \leq \frac{p}{2}$, such that for a subsequence we have*

$$q_{\varepsilon_h}^2 \rightharpoonup \overline{q^2} \quad \text{weakly in } L^\rho([0, T]; L^r(\mathbb{R})), \quad (5.9)$$

for each $T \geq 0$ and $1 < \rho < \infty$, $1 < r \leq \frac{p}{2}$. Moreover,

$$q^2 \leq \overline{q^2} \quad \text{a.e. in } [0, \infty) \times \mathbb{R}, \quad (5.10)$$

and the following inequality holds

$$\partial_t \overline{q^2} - \partial_x u \overline{q^2} + \partial_x (u \overline{q^2}) + 2\tilde{P}q \leq 0, \quad (5.11)$$

in the sense of distributions on $[0, \infty) \times \mathbb{R}$.

Proof. (5.9) follows from (2.9), (3.1) and Lemma 4.4.

The inequality (5.10) is a well-known consequence of Jensen's inequality.

Finally, we prove (5.11). Multiplying (4.24) by $2q_\varepsilon$ we get

$$\partial_t (q_\varepsilon^2) - \partial_x u_\varepsilon q_\varepsilon^2 + \partial_x (u_\varepsilon q_\varepsilon^2) + 2\tilde{P}_\varepsilon q_\varepsilon = \varepsilon \partial_x^2 (q_\varepsilon^2) - 2\varepsilon q_\varepsilon^2 \leq \varepsilon \partial_x^2 (q_\varepsilon^2),$$

hence (5.11) is a consequence of (5.1), (5.2), (5.4) and (5.9). \square

Arguing as in [4, Lemma 5.8], [28, Proposition 4.3], we get the following result.

Lemma 5.4. *Let $2 < p < \infty$. Assume that $u_0 \in \mathcal{H}_{2,p}$. The following identity holds*

$$\partial_t (q^2) - \partial_x u q^2 + \partial_x (u q^2) + 2\tilde{P}q = 0, \quad (5.12)$$

in the sense of distributions on $[0, \infty) \times \mathbb{R}$.

Proof of Lemma 5.1. We claim that

$$q_{\varepsilon_h} \rightarrow q \quad \text{strongly in } L^\infty([0, T]; L^2(\mathbb{R})), \text{ for each } T \geq 0. \quad (5.13)$$

Subtract (5.11) and (5.12)

$$\partial_t [\overline{q^2} - q^2] - \partial_x u [\overline{q^2} - q^2] + \partial_x [u(\overline{q^2} - q^2)] \leq 0, \quad (5.14)$$

in the sense of distributions on $[0, \infty) \times \mathbb{R}$.

Since (see, e.g., [4, Lemma 6.1])

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} q^2 dx = \lim_{t \rightarrow 0} \int_{\mathbb{R}} \overline{q^2} dx = \int_{\mathbb{R}} (\partial_x^2 u_0)^2 dx,$$

the claim is direct consequence of (5.14). \square

Proof of Theorem 2.4. The existence of a solution for (2.8) is stated in Lemma 5.1. Moreover, the invariance properties (2.11) and (2.12) are consequences of Lemmas 4.4, 4.5, and 5.1. \square

6. Weak equals strong uniqueness

In this section we prove Theorem 2.5.

The following lemma is needed.

Lemma 6.1. *Assume $k = 2$. Let u_1, u_2 be two weak solutions of the system (2.10) in the sense of Definition 2.3. If there exists a map $b \in L^1([0, T])$, $T > 0$, such that*

$$\|\partial_x^2 u_1(t, \cdot)\|_{L^\infty(\mathbb{R})}, \|\partial_x^2 u_2(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq b(t), \quad t \geq 0, \quad (6.1)$$

then,

$$\mathcal{L}(t) \leq \mathcal{L}(0) + c \int_0^t (1 + b(s)) \mathcal{L}(s) ds, \quad (6.2)$$

for each $t \geq 0$ and some constant $c > 0$, where

$$\begin{aligned} \mathcal{L}(t) = & \|w(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(t, \cdot)\|_{L^\infty(\mathbb{R})} \\ & + \|(\partial_x^3 - \partial_x) A_2^{-1} e(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-2} e(t, \cdot)\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

Proof. Introduce the notation

$$w = u_1 - u_2, \quad v = u_1 + u_2, \quad e_i = u_i^2 + (\partial_x u_i)^2 + (\partial_x^2 u_i)^2, \quad e = \frac{1}{2}(e_1 - e_2).$$

We split the proof in six steps.

Step 1. We begin with some manipulation of the equations. By fixed $i \in \{1, 2\}$, since

$$\begin{aligned} P_i = & -\frac{1}{2} A_2^{-1} e_i + A_2^{-1} \left(\frac{3}{2} u_i^2 + (\partial_x u_i)^2 \right) - \frac{3}{2} \partial_x^2 A_2^{-1} ((\partial_x u_i)^2), \\ & \partial_x^4 A_2^{-1} = \partial_x^2 A_2^{-1} - A_2^{-1} + 1, \end{aligned}$$

we have that

$$\begin{aligned} \partial_x P_i = & -\frac{1}{2} \partial_x A_2^{-1} e_i + F_i, \\ \partial_x^2 P_i = & -\frac{1}{2} \partial_x^2 A_2^{-1} e_i - \frac{3}{2} (\partial_x u_i)^2 + G_i, \\ \partial_x^3 P_i = & -\frac{1}{2} \partial_x^3 A_2^{-1} e_i - 3 \partial_x u_i \partial_x^2 u_i + \partial_x G_i, \end{aligned}$$

where

$$\begin{aligned} F_i = & \partial_x A_2^{-1} \left(\frac{3}{2} u_i^2 + (\partial_x u_i)^2 \right) - \frac{3}{2} \partial_x^3 A_2^{-1} ((\partial_x u_i)^2), \\ G_i = & \partial_x^2 A_2^{-1} \left(\frac{3}{2} u_i^2 - \frac{1}{2} (\partial_x u_i)^2 \right) + \frac{3}{2} A_2^{-1} ((\partial_x u_i)^2). \end{aligned}$$

Hence

$$\partial_t u_i + u_i \partial_x u_i - \frac{1}{2} \partial_x A_2^{-1} e_i + F_i = 0, \quad (6.3)$$

$$\partial_t \partial_x u_i + u_i \partial_x^2 u_i - \frac{1}{2} (\partial_x u_i)^2 - \frac{1}{2} \partial_x^2 A_2^{-1} e_i + G_i = 0, \quad (6.4)$$

$$\partial_t \partial_x^2 u_i + u_i \partial_x^3 u_i - \frac{1}{2} \partial_x^3 A_2^{-1} e_i + \partial_x G_i = 0. \quad (6.5)$$

In particular, from (6.3) and (6.4), we get the following equations for w , $\partial_x w$

$$\partial_t w + u_1 \partial_x w + w \partial_x u_2 - \partial_x A_2^{-1} e + F_1 - F_2 = 0, \quad (6.6)$$

$$\partial_t \partial_x w + u_1 \partial_x^2 w + w \partial_x^2 u_2 - \frac{1}{2} \partial_x w \partial_x v - \partial_x^2 A_2^{-1} e + G_1 - G_2 = 0. \quad (6.7)$$

Multiplying (6.3) by u_i , (6.4) by $\partial_x u_i$, (6.5) by $\partial_x^2 u_i$, adding the three equations, and observing

$$\begin{aligned} u_i \partial_x A_2^{-1} e_i + \partial_x u_i \partial_x^2 A_2^{-1} e_i + \partial_x^2 u_i \partial_x^3 A_2^{-1} e_i \\ = -e_i \partial_x u_i + \partial_x (u_i A_2^{-1} e_i + \partial_x u_i \partial_x^3 A_2^{-1} e_i), \end{aligned}$$

we get

$$\frac{1}{2} \partial_t e_i + \frac{1}{2} \partial_x (u_i e_i) - \frac{1}{2} (\partial_x u_i)^3 - \frac{1}{2} \partial_x (u_i A_2^{-1} e_i + \partial_x u_i \partial_x^3 A_2^{-1} e_i) + H_i = 0, \quad (6.8)$$

where

$$H_i = u_i F_i + \partial_x u_i G_i + \partial_x^2 u_i \partial_x G_i.$$

Finally, from (6.8) we get the following equation for e

$$\begin{aligned} \partial_t e + \partial_x (u_1 e) + \frac{1}{2} \partial_x (w e_2) - \frac{1}{2} ((\partial_x u_1)^3 - (\partial_x u_2)^3) + H_1 - H_2 \\ - \partial_x \left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \right) = 0. \end{aligned} \quad (6.9)$$

Step 2. We estimate the L^∞ -norm of w . Since $\partial_x u_1 \in L^\infty([0, \infty) \times \mathbb{R})$ applying [29, Lemma 2] to (6.6) we get

$$\begin{aligned} \|w(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|w(0, \cdot)\|_{L^\infty(\mathbb{R})} \\ &+ \int_0^t \|(w \partial_x u_2 - \partial_x A_2^{-1} e + F_1 - F_2)(s, \cdot)\|_{L^\infty(\mathbb{R})} ds, \end{aligned} \quad (6.10)$$

for each $t > 0$. Using Definition 2.3(iv) and the Sobolev embedding theorem [25, Theorem 8.5]

$$\|\partial_x u_i(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_i(s, \cdot)\|_{H^2(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|u_i(0, \cdot)\|_{H^2(\mathbb{R})}, \quad (6.11)$$

for any $s \geq 0$, $i = 1, 2$.

Moreover,

$$F_1 - F_2 = \partial_x A_2^{-1} \left(\frac{3}{2} w v + \partial_x w \partial_x v \right) - \frac{3}{2} \partial_x^3 A_2^{-1} (\partial_x w \partial_x v),$$

hence using the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3(iv) and the Sobolev embedding theorem [25, Theorem 8.5]

$$\|(F_1 - F_2)(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq c_1(\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})}), \quad (6.12)$$

for each $s \geq 0$ and some constant $c_1 > 0$.

Therefore, by (6.10)–(6.12),

$$\begin{aligned} \|w(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|w(0, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + c_2 \int_0^t (\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x A_2^{-1} e(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds, \end{aligned} \quad (6.13)$$

for each $t > 0$ and some constant $c_2 > 0$.

Step 3. We use the same argument for the estimate of the L^∞ -norm of $\partial_x w$. The boundedness of $\partial_x u_1$, [29, Lemma 2] and (6.6) give

$$\begin{aligned} \|\partial_x w(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x w(0, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + \int_0^t \left\| \left(w \partial_x^2 u_2 - \frac{1}{2} \partial_x w \partial_x v - \partial_x^2 A_2^{-1} e + G_1 - G_2 \right)(s, \cdot) \right\|_{L^\infty(\mathbb{R})} ds, \end{aligned} \quad (6.14)$$

for each $t > 0$. (6.1) and (6.11) yield

$$\|(w \partial_x^2 u_2)(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq 2b(s) \|w(s, \cdot)\|_{L^\infty(\mathbb{R})}, \quad (6.15)$$

$$\|(\partial_x w \partial_x v)(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} \|v(0, \cdot)\|_{H^2(\mathbb{R})} \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})}, \quad (6.16)$$

for each $s \geq 0$. Finally

$$G_1 - G_2 = \partial_x^2 A_2^{-1} \left(\frac{3}{2} w v + \frac{5}{2} \partial_x w \partial_x v \right) - \frac{3}{2} \partial_x^2 A_2^{-1} (\partial_x w \partial_x v),$$

hence using the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3(iv) and the Sobolev embedding theorem [25, Theorem 8.5]

$$\|(G_1 - G_2)(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq c_3(\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})}), \quad (6.17)$$

for each $s \geq 0$ and some constant $c_3 > 0$.

Therefore, by (6.14)–(6.17)

$$\begin{aligned} \|\partial_x w(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|\partial_x w(0, \cdot)\|_{L^\infty(\mathbb{R})} + c_4 \int_0^t (1 + b(s)) \\ &\quad \times (\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 A_2^{-1} e(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds, \end{aligned} \quad (6.18)$$

for each $t > 0$ and some constant $c_4 > 0$.

Step 4. Introduce the operator

$$\Lambda = (\partial_x^3 - \partial_x) A_2^{-1}$$

and observe that

$$\partial_x \Lambda = (\partial_x^4 - \partial_x^2) A_2^{-1} = 1 - A_2^{-1}. \quad (6.19)$$

Applying Λ to (6.9)

$$\begin{aligned} \partial_t \Lambda e + \partial_x \Lambda(u_1 e) + \frac{1}{2} \partial_x \Lambda(w e_2) - \frac{1}{2} \Lambda((\partial_x u_1)^3 - (\partial_x u_2)^3) + \Lambda(H_1 - H_2) \\ - \partial_x \Lambda \left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \right) = 0. \end{aligned} \quad (6.20)$$

Due to (6.19)

$$\partial_x \Lambda(u_1 e) = u_1 e - A_2^{-1}(u_1 e) = u_1 \partial_x \Lambda e + u_1 A_2^{-1} e - A_2^{-1}(u_1 e), \quad (6.21)$$

$$\frac{1}{2} \partial_x \Lambda(w e_2) = \frac{1}{2} w e_2 - \frac{1}{2} A_2^{-1}(w e_2), \quad (6.22)$$

$$\begin{aligned} -\partial_x \Lambda \left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \partial_x^3 A_2^{-1} e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \right) \\ = -u_1 A_2^{-1} e - \frac{w}{2} A_2^{-1} e_2 - \partial_x u_1 \Lambda e - \partial_x u_1 \partial_x A_2^{-1} e - \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \\ + A_2^{-1} \left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \Lambda e + \partial_x u_1 \partial_x A_2^{-1} e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \right). \end{aligned} \quad (6.23)$$

Using (6.21)–(6.23) in (6.20),

$$\partial_t \Lambda e + u_1 \partial_x \Lambda e + h = 0, \quad (6.24)$$

where

$$\begin{aligned} h = u_1 A_2^{-1} e - A_2^{-1}(u_1 e) + \frac{1}{2} w e_2 - \frac{1}{2} A_2^{-1}(w e_2) + \Lambda(H_1 - H_2) \\ - u_1 A_2^{-1} e - \frac{w}{2} A_2^{-1} e_2 - \partial_x u_1 \Lambda e - \partial_x u_1 \partial_x A_2^{-1} e - \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \\ + A_2^{-1} \left(u_1 A_2^{-1} e + \frac{w}{2} A_2^{-1} e_2 + \partial_x u_1 \Lambda e + \partial_x u_1 \partial_x A_2^{-1} e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1} e_2 \right). \end{aligned}$$

Due to the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3(iv) and the Sobolev embedding theorem [25, Theorem 8.5]

$$\begin{aligned} \|h(s, \cdot)\|_{L^\infty(\mathbb{R})} &\leq c_5(\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + \|Ae(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}), \end{aligned}$$

for each $s \geq 0$ and some constant $c_5 > 0$. Hence, [29, Lemma 2] and (6.24) give

$$\begin{aligned} \|Ae(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|Ae(0, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + c_5 \int_0^t (\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|Ae(s, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + \|\partial_x A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds, \end{aligned} \quad (6.25)$$

for each $s \geq 0$.

Step 5. Applying A_2^{-2} to (6.9)

$$\partial_t A_2^{-2}e + \partial_x A_2^{-2}(u_1 e) + k = 0,$$

where

$$\begin{aligned} k &= \frac{1}{2} \partial_x A_2^{-2}(w e_2) - \frac{1}{2} A_2^{-2}((\partial_x u_1)^3 - (\partial_x u_2)^3) + A_2^{-2}(H_1 - H_2) \\ &\quad - \partial_x A_2^{-2} \left(u_1 A_2^{-1}e + \frac{w}{2} A_2^{-1}e_2 + \partial_x u_1 \partial_x^3 A_2^{-1}e + \frac{\partial_x w}{2} \partial_x^3 A_2^{-1}e_2 \right). \end{aligned}$$

Due to the boundedness of the derivatives of the Green's function of A_2 and again Definition 2.3(iv) and the Sobolev embedding theorem [25, Theorem 8.5]

$$\begin{aligned} \|k(s, \cdot)\|_{L^\infty(\mathbb{R})} &\leq c_6(\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}), \end{aligned}$$

for each $s \geq 0$ and some constant $c_6 > 0$. Therefore integrating (6.24) on $(0, t)$ we have that

$$\begin{aligned} \|A_2^{-2}e(t, \cdot)\|_{L^\infty(\mathbb{R})} &\leq \|A_2^{-2}e(0, \cdot)\|_{L^\infty(\mathbb{R})} + \int_0^t (\|\partial_x A_2^{-2}(u_1 e)(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|k(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds \\ &\leq \|A_2^{-2}e(0, \cdot)\|_{L^\infty(\mathbb{R})} + c_7 \int_0^t (\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} \\ &\quad + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds, \end{aligned} \quad (6.26)$$

for each $s \geq 0$ and some constant $c_7 > 0$.

Step 6. Adding together (6.13), (6.18), (6.25), (6.26)

$$\begin{aligned} & \|w(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(t, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-2}e(t, \cdot)\|_{L^\infty(\mathbb{R})} \\ & \leq \|w(0, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(0, \cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(0, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-2}e(0, \cdot)\|_{L^\infty(\mathbb{R})} \\ & \quad + c_8 \int_0^t (1 + b(s)) (\|w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x w(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\Lambda e(s, \cdot)\|_{L^\infty(\mathbb{R})} \\ & \quad + \|\partial_x A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|\partial_x^2 A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}) ds, \quad (6.27) \end{aligned}$$

for each $s \geq 0$ and some constant $c_8 > 0$.

From (6.19)

$$A_2^{-1}e = \partial_x \Lambda A_2^{-1}e + A_2^{-2}e = \partial_x A_2^{-1}(\Lambda e) + A_2^{-2}e,$$

so

$$\|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} \leq c_9 (\|\Lambda e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-2}e(s, \cdot)\|_{L^\infty(\mathbb{R})}), \quad (6.28)$$

for each $s \geq 0$ and some constant $c_9 > 0$. By fixed $j = 1, 2$ again by (6.19)

$$\partial_x^j A_2^{-1}e = \partial_x^{j+1} \Lambda A_2^{-1}e + \partial_x^j A_2^{-2}e = \partial_x^{j+1} A_2^{-1}(\Lambda e) + \partial_x^j A_2^{-1}(A_2^{-1}e),$$

using now (6.27) we have that

$$\begin{aligned} \|\partial_x^j A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})} & \leq c_{10} (\|\Lambda e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-1}e(s, \cdot)\|_{L^\infty(\mathbb{R})}) \\ & \leq c_{11} (\|\Lambda e(s, \cdot)\|_{L^\infty(\mathbb{R})} + \|A_2^{-2}e(s, \cdot)\|_{L^\infty(\mathbb{R})}), \quad (6.29) \end{aligned}$$

for each $s \geq 0$ and some constants $c_{10}, c_{11} > 0$.

Finally, (6.27)–(6.29) imply (6.2). \square

Proof of Theorem 2.5. Assume that there exist two weak solutions u_1, u_2 of the Cauchy problem (2.8) satisfying (6.1). The Gronwall lemma and (6.2) imply $\mathcal{L} = 0$ that means $u_1 = u_2$. \square

Appendix A. Consistency of the weak formulation

In this section we prove the consistency of Definition 2.3 for a general k .

Lemma A.1. Let $f \in C^\infty([0, \infty) \times \mathbb{R})$. Then

$$f \in L^\infty([0, \infty); H^k(\mathbb{R})) \quad \text{implies} \quad \int_{[0, \infty) \times \mathbb{R}} \mathcal{F}_k(f) \varphi \, dt \, dx < \infty, \quad (A.1)$$

for each $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R})$ and $k \geq 2$.

The following lemma is needed.

Lemma A.2. Let $f \in C_c^\infty(\mathbb{R})$. The following identity holds

$$\int_{-\infty}^x (2\partial_x f \partial_x^{2j} f + f \partial_x^{2j+1} f) d\xi = f \partial_x^{2j} f + \sum_{i=1}^{j-1} (-1)^{i+1} \partial_x^i f \partial_x^{2j-i} f - \frac{(-1)^j}{2} (\partial_x^j f)^2, \quad (\text{A.2})$$

for each $x \in \mathbb{R}$ and $j \geq 2$.

Proof. Fix $f \in C_c^\infty(\mathbb{R})$, $x \in \mathbb{R}$ and $j \geq 2$. Integrating by parts we get

$$\begin{aligned} & \int_{-\infty}^x (2\partial_x f \partial_x^{2j} f + f \partial_x^{2j+1} f) d\xi \\ &= f \partial_x^{2j} f + \int_{-\infty}^x \partial_x f \partial_x^{2j} f d\xi \\ &= f \partial_x^{2j} f + \partial_x f \partial_x^{2j-1} f - \int_{-\infty}^x \partial_x^2 f \partial_x^{2j-1} f d\xi \\ &= f \partial_x^{2j} f + \partial_x f \partial_x^{2j-1} f - \partial_x^2 f \partial_x^{2j-2} f + \int_{-\infty}^x \partial_x^3 f \partial_x^{2j-2} f d\xi \\ &= f \partial_x^{2j} f + \sum_{i=1}^{j-1} (-1)^{i+1} \partial_x^i f \partial_x^{2j-i} f - (-1)^j \int_{-\infty}^x \partial_x^j f \partial_x^{j+1} f d\xi \\ &= f \partial_x^{2j} f + \sum_{i=1}^{j-1} (-1)^{i+1} \partial_x^i f \partial_x^{2j-i} f - \frac{(-1)^j}{2} (\partial_x^j f)^2. \quad \square \end{aligned}$$

Proof of Lemma A.1. Let $f \in L^\infty([0, \infty); H^k(\mathbb{R}))$, $\varphi \in C_c^\infty([0, \infty) \times \mathbb{R})$ and $k \geq 2$. From (A.2) we find, integrating by parts, that

$$\begin{aligned} & \int_{[0, \infty) \times \mathbb{R}} \mathcal{F}_k(f) \varphi dt dx \\ &= - \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x C_k(f)(t, y) \varphi(t, x) dt dx dy \\ &= \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x [A_k(f \partial_x f) - f A_k(\partial_x f) - 2\partial_x f A_k(f)](t, y) \varphi(t, x) dt dx dy \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^k (-1)^j \int_{[0, \infty) \times \mathbb{R}} \partial_x^{2j-1} (f \partial_x f) \varphi \, dt \, dx + \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x [f \partial_x f](t, y) \varphi(t, x) \, dt \, dx \, dy \\
&\quad - \sum_{j=0}^k (-1)^j \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x [f \partial_x^{2j} (\partial_x f) + 2 \partial_x f \partial_x^{2j} (f)](t, y) \varphi(t, x) \, dt \, dx \, dy \\
&= J_1 + J_2 + J_3 + J_4 + J_5,
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
J_1 &:= \sum_{j=1}^k (-1)^j \int_{[0, \infty) \times \mathbb{R}} \partial_x^{2j-1} (f \partial_x f) \varphi \, dt \, dx, \\
J_2 &:= - \sum_{j=2}^k (-1)^j \int_{[0, \infty) \times \mathbb{R}} f \partial_x^{2j} f \varphi \, dt \, dx, \\
J_3 &:= - \sum_{j=2}^k \sum_{i=1}^{j-1} (-1)^{j+i+1} \int_{[0, \infty) \times \mathbb{R}} \partial_x^i f \partial_x^{2j-i} f \varphi \, dt \, dx, \\
J_4 &:= \frac{1}{2} \sum_{j=2}^k \int_{[0, \infty) \times \mathbb{R}} (\partial_x^j f)^2 \varphi \, dt \, dx, \\
J_5 &:= \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x [f \partial_x f](t, y) \varphi(t, x) \, dt \, dx \, dy \\
&\quad - \sum_{j=0}^1 (-1)^j \int_{[0, \infty) \times \mathbb{R}} \int_{-\infty}^x [f \partial_x^{2j} (\partial_x f) + 2 \partial_x f \partial_x^{2j} (f)](t, y) \varphi(t, x) \, dt \, dx \, dy.
\end{aligned}$$

Observe that, employing integration by parts,

$$\begin{aligned}
J_1 &= - \sum_{j=1}^k \int_{[0, \infty) \times \mathbb{R}} f \partial_x f \partial_x^{2j-1} \varphi \, dt \, dx \\
&= \sum_{j=1}^k \int_{[0, \infty) \times \mathbb{R}} \frac{f^2}{2} \partial_x^{2j} \varphi \, dt \, dx \\
&\leq \frac{k}{2} \|f\|_{L^\infty([0, \infty); L^2(\mathbb{R}))}^2 \|\varphi\|_{L^1([0, \infty); W^{2k, \infty}(\mathbb{R}))},
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
 J_2 &= \sum_{j=2}^k \int_{[0,\infty) \times \mathbb{R}} \partial_x^j f \partial_x^j (f\varphi) dt dx \\
 &\leq c_1 \|f\|_{L^\infty([0,\infty); H^k(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); W^{k,\infty}(\mathbb{R}))},
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 J_3 &= - \sum_{j=2}^k \sum_{i=1}^{j-1} \int_{[0,\infty) \times \mathbb{R}} \partial_x^i f \partial_x^{j-i} (\varphi \partial_x^i f) dt dx \\
 &\leq c_2 \sum_{j=2}^k \|f\|_{L^\infty([0,\infty); H^j(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); W^{j,\infty}(\mathbb{R}))} \\
 &\leq c_3 \|f\|_{L^\infty([0,\infty); H^k(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); W^{k,\infty}(\mathbb{R}))},
 \end{aligned} \tag{A.6}$$

$$\begin{aligned}
 J_4 &\leq \frac{1}{2} \sum_{j=2}^k (-1)^j \|f\|_{L^\infty([0,\infty); H^j(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); L^\infty(\mathbb{R}))} \\
 &\leq c_4 \|f\|_{L^\infty([0,\infty); H^k(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); L^\infty(\mathbb{R}))},
 \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 J_5 &= - \int_{[0,\infty) \times \mathbb{R}} \left[\frac{(\partial_x f)^2}{2} + f \partial_x^2 f - \frac{f^2}{2} \right] \varphi dt dx \\
 &\leq c_5 \|f\|_{L^\infty([0,\infty); H^2(\mathbb{R}))}^2 \|\varphi\|_{L^1([0,\infty); L^\infty(\mathbb{R}))},
 \end{aligned} \tag{A.8}$$

for some constants $c_1, c_2, c_3, c_4, c_5 > 0$ depending only on k . Clearly, (A.3)–(A.8) imply (A.2). \square

Appendix B. The general case $k > 2$

In this appendix we show that the ideas used in the previous sections can be applied also in the general case. More precisely, we assume

$$k > 2.$$

Due to the boundedness of the family $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty([0,\infty); H^k(\mathbb{R}))$ (see Lemma 3.2) as in Lemma 5.1, we have to prove compactness of the family $\{\partial_x^k u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty([0,\infty); L^2(\mathbb{R}))$. To this end we derive an equation for $\partial_x^k u_\varepsilon$. From (2.13) we infer

$$\partial_t \partial_x^k u_\varepsilon + \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) + \partial_x^{k+1} P_\varepsilon = \varepsilon \partial_x^{k+2} u_\varepsilon. \tag{B.1}$$

Observe that

$$\begin{aligned}
 \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) &= \sum_{j=0}^k \binom{k}{j} \partial_x^{j+1} u_\varepsilon \partial_x^{k-j} u_\varepsilon \\
 &= u_\varepsilon \partial_x^{k+1} u_\varepsilon + (k+1) \partial_x u_\varepsilon \partial_x^k u_\varepsilon + U_\varepsilon,
 \end{aligned} \tag{B.2}$$

where

$$U_\varepsilon := \sum_{j=1}^{k-2} \binom{k}{j} \partial_x^{j+1} u_\varepsilon \partial_x^{k-j} u_\varepsilon.$$

Since we have only derivatives in U_ε of order less than $k-1$, due to (3.1), it is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

In the following we analyze the nonlocal term $\partial_x^{k+1} P_\varepsilon$, employing Remark 2.1, and find

$$\begin{aligned} \partial_x^{k+1} P_\varepsilon(t, x) &= \int_{\mathbb{R}} \frac{d^{k+1} G_k}{dx^{k+1}}(x-y) \mathcal{F}_k(t, y) dy \\ &= \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) C_k(t, y) dy = (P_{1,\varepsilon} + P_{2,\varepsilon} + P_{3,\varepsilon})(t, x), \end{aligned} \quad (\text{B.3})$$

where

$$\begin{aligned} P_{1,\varepsilon}(t, x) &:= - \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) [u_\varepsilon A_k(\partial_x u_\varepsilon)](t, y) dy, \\ P_{2,\varepsilon}(t, x) &:= \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) A_k(u_\varepsilon \partial_x u_\varepsilon)(t, y) dy, \\ P_{3,\varepsilon}(t, x) &:= -2 \int_{\mathbb{R}} \frac{d^k G_k}{dx^k}(x-y) [\partial_x u_\varepsilon A_k(u_\varepsilon)](t, y) dy. \end{aligned}$$

Observe that, integrating by parts and using the fact that G_k is the Green's function of A_k , we get

$$\begin{aligned} P_{1,\varepsilon}(t, x) &= \sum_{j=0}^k (-1)^{j+1} \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} u_\varepsilon \partial_x^{2j+1} u_\varepsilon dy \\ &= - \sum_{j=0}^k \int_{\mathbb{R}} \partial_x^j \left(\frac{d^k G_k}{dx^k} u_\varepsilon \right) \partial_x^{j+1} u_\varepsilon dy \\ &= - \sum_{j=0}^{k-1} \int_{\mathbb{R}} \partial_x^j \left(\frac{d^k G_k}{dx^k} u_\varepsilon \right) \partial_x^{j+1} u_\varepsilon dy \\ &\quad - (-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} u_\varepsilon \partial_x^{k+1} u_\varepsilon dy \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=0}^{k-1} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^{k-j} u_\varepsilon \partial_x^{k+1} u_\varepsilon dy \\
& = - \sum_{j=0}^{k-1} \int_{\mathbb{R}} \partial_x^j \left(\frac{d^k G_k}{dx^k} u_\varepsilon \right) \partial_x^{j+1} u_\varepsilon dy \\
& \quad - (-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon - \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \partial_x \left(\frac{d^{2j} G_k}{dx^{2j}} u_\varepsilon \right) \partial_x^k u_\varepsilon dy \\
& \quad - \sum_{j=0}^{k-2} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j+1} G_k}{dx^{k+j+1}} \partial_x^{k-j} u_\varepsilon \partial_x^k u_\varepsilon dy \\
& \quad - \sum_{j=2}^{k-1} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^{k-j+1} u_\varepsilon \partial_x^k u_\varepsilon dy + \frac{1}{2} \int_{\mathbb{R}} \frac{d^{k+2} G_k}{dx^{k+2}} (\partial_x^k u_\varepsilon)^2 dy \\
& \quad + (-1)^k k \partial_x u_\varepsilon \partial_x^k u_\varepsilon - \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} \partial_x u_\varepsilon \partial_x^k u_\varepsilon dy \\
& \quad - \int_{\mathbb{R}} \frac{d^{2k-1} G_k}{dx^{2k-1}} \partial_x^{k+1} u_\varepsilon \partial_x^k u_\varepsilon dy \\
& = -(-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + (-1)^k k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \tilde{P}_{1,\varepsilon},
\end{aligned} \tag{B.4}$$

where $\tilde{P}_{1,\varepsilon}$ is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Concerning the second term we have

$$\begin{aligned}
P_{2,\varepsilon}(t, x) & = \sum_{j=0}^k (-1)^j \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} \partial_x^{2j} (u_\varepsilon \partial_x u_\varepsilon) dy \\
& = \sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^j (u_\varepsilon \partial_x u_\varepsilon) dy \\
& \quad + (-1)^k \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) - \sum_{j=0}^{k-1} (-1)^{j+k} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) dy \\
& = \sum_{j=0}^{k-1} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^j (u_\varepsilon \partial_x u_\varepsilon) dy \\
& \quad + (-1)^k \partial_x^k (u_\varepsilon \partial_x u_\varepsilon) + \sum_{j=0}^{k-1} (-1)^{j+k} \int_{\mathbb{R}} \frac{d^{2j+1} G_k}{dx^{2j+1}} \partial_x^{k-1} (u_\varepsilon \partial_x u_\varepsilon) dy \\
& = (-1)^k u_\varepsilon \partial_x^{k+1} u_\varepsilon + (-1)^k (k+1) \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \tilde{P}_{2,\varepsilon},
\end{aligned} \tag{B.5}$$

where $\tilde{P}_{2,\varepsilon}$ is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

For the third term we have

$$\begin{aligned}
 P_{3,\varepsilon}(t, x) &= -2 \sum_{j=0}^k (-1)^j \int_{\mathbb{R}} \frac{d^k G_k}{dx^k} \partial_x u_\varepsilon \partial_x^{2j} u_\varepsilon dy \\
 &= -2 \sum_{j=0}^{k-1} \int_{\mathbb{R}} \partial_x^j \left(\frac{d^k G_k}{dx^k} \partial_x u_\varepsilon \right) \partial_x^j u_\varepsilon dy \\
 &\quad - 2(-1)^k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + 2 \sum_{j=0}^{k-1} (-1)^{k+j} \int_{\mathbb{R}} \frac{d^{2j} G_k}{dx^{2j}} \partial_x u_\varepsilon \partial_x^k u_\varepsilon dy \\
 &\quad - 2 \sum_{j=1}^{k-1} \binom{k}{j} \int_{\mathbb{R}} \frac{d^{k+j} G_k}{dx^{k+j}} \partial_x^{k+1-j} u_\varepsilon \partial_x^k u_\varepsilon dy + \int_{\mathbb{R}} \frac{d^{k+1} G_k}{dx^{k+1}} (\partial_x^k u_\varepsilon)^2 dy \\
 &= -2(-1)^k \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \tilde{P}_{3,\varepsilon},
 \end{aligned} \tag{B.6}$$

where $\tilde{P}_{3,\varepsilon}$ is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Hence, from (B.3)–(B.6) (as in (4.2)),

$$\partial_x^{k+1} P_\varepsilon = (-1)^k (2k-1) \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \tilde{P}_\varepsilon, \tag{B.7}$$

where \tilde{P}_ε is uniformly bounded in $L^\infty([0, \infty); L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

Moreover, denoting

$$q_\varepsilon := \partial_x^k u_\varepsilon$$

from (B.1), (B.2), and (B.7), we get (as in (4.24))

$$\partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + \lambda_\varepsilon \partial_x u_\varepsilon \partial_x^k u_\varepsilon + \Lambda_\varepsilon = 0, \tag{B.8}$$

where

$$\lambda_\varepsilon := k+1 + (-1)^k (2k-1), \quad \Lambda_\varepsilon := U_\varepsilon + \tilde{P}_\varepsilon.$$

Clearly for this equation we can apply the same argument used for the previous proofs.

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