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Average value problems in ordinary differential equations

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ABSTRACT

Using Schauder's fixed point theorem, with the help of an integral representation in 'Sharp conditions for weighted 1-dimensional Poincaré inequalities', Indiana Univ. Math. J., 49 (2000) 143–175, by Chua and Wheeden, we obtain existence and uniqueness theorems and 'continuous dependence of average condition' for average value problem:

$$y' = F(x, y),$$

$$\int_a^b y(x)\,dv = y_0 \quad \text{where } v \text{ is any probability measure on } [a, b]$$

under the usual conditions for initial value problem. We also extend our existence and uniqueness theorems in the case where v is just a signed measure with $v[a, b] \neq 0$ and

$$F : \mathcal{F} \subset C[a, b] \to L^1[a, b] \text{ is a continuous operator.}$$

1. Introduction

There are abundant studies on initial value problems:

$$y' = F(x, y), \qquad y(a) = y_0, \quad \text{where } a \in \mathbb{R},\ y_0 \in \mathbb{R}^n. \tag{1.1}$$

For example, see [1,6,14,11]. However, not much has been studied for average value problems

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$$y' = F(x, y), \qquad \int_a^b y(x)\,dv = y_0 \quad \text{where } v = (v_1, \dots, v_n), \tag{1.2}$$

$-\infty < a < b < \infty$, v_i's are probability measures on $[a, b]$. Average value problems may seem to be less natural compare to initial value problems. However, in real life, a given initial value is likely to be indeed just an average value; for example, measurement of temperature or speed is indeed average temperature or average speed over a short time interval. Note that initial value problems are just special cases of average value problems (1.2) where v_i's are just Delta measure at a. Moreover, let us note that many boundary value problems for higher order ordinary differential equation are also average value problems. Nevertheless, as far as we know, there is only one study of such problem [4]. After checking through the literature more carefully, we finally realized that indeed functional boundary value problems,

$$y' = F(x, y), \qquad l(y) = y_0, \tag{1.3}$$

where l is a continuous function (vector value) from $C([a, b]; \mathbb{R}^n)$ to $\mathbb{R}^n$, have been discussed in [9,3,11,12]. Here $C([a, b]; \mathbb{R}^n)$ is the space of continuous ($\mathbb{R}^n$ value) functions on $[a, b]$. Moreover, general boundary value problems were probably first introduced by Whyburn [13]. Clearly, average value problems are just special cases of functional boundary value problems. However, none of those studies give parallel result to the case of initial value problems. Nevertheless, indeed, there is no way to obtain such result. Let us look at the following simple example. The (1-dimensional) functional boundary value problem:

$$u' = 0, \qquad l(u) = 1$$

may not have solution when l is a bounded linear functional on $C[0, 1]$ unless $l(u) \neq 0$ if u is a nonzero constant function. Thus, in order to obtain parallel results as in the case of initial value problems, it is necessary to restrict the choice of the bounded linear functional. Note that in the above example, in the case where $l(1) \neq 0$, it is easy to see that the above mentioned functional boundary value problem has a unique solution. Let us note that bounded linear functionals on $C[0, 1]$ with $l(1) \neq 0$ are just the family of signed measures v on $[a, b]$ with $v[a, b] \neq 0$. In this paper we will study such functional value problems which we call them average value problems where we are able to obtain existence and uniqueness results under the same (standard) assumptions for initial value problems (when the linear functional is arising from a probability measure).

We will consider mostly the case $n = 1$ in this paper. Note that this is the case where the intermediate value theorem holds. However, results in this paper can be easily generalized to $n \geqslant 2$ (when the intermediate value theorem is not required). By a weight w, we always mean a nonnegative measurable function that is finite almost everywhere. We will also denote the measure arising from w by w, and sometimes we write dw instead of $w(x)\,dx$. For any interval $[c, d]$, we set $w[c, d] = \int_c^d w(x)\,dx$. We use the convention that $\frac{1}{p} + \frac{1}{p'} = 1$ when $1 \leqslant p < \infty$. We say y is an absolutely continuous solution of (1.1) or (1.2) if $y' = F(x, y)$ almost everywhere on an interval $[a, b]$ besides satisfying the initial value condition or the average value condition respectively. This definition of solution is indeed just the 'Carathéodory solution' [14,2]. Let us begin with stating the following two classical theorems on initial value problems.

Theorem 1.1. *Let $x_0 \in [a, b] \subset \mathbb{R}$, $K > 0$, $y_0 \in J$ where J is an interval in $\mathbb{R}$. Let $F : [a, b] \times J \to \mathbb{R}$ be such that $F(x, \cdot)$ is continuous for each fixed $x \in [a, b]$ and $F(\cdot, y)$ is (Lebesgue) integrable (on $[a, b]$) for each fixed $y \in J$. (This condition is sometimes being called Carathéodory condition* [14, p. 121]. *Note that it can be further relaxed to generalized Carathéodory condition* [12].) *Let $\|F(x, \cdot)\|_{L^\infty(J)} = f(x)$ and $f \in L^1[a, b]$. If $\int_a^b f \leqslant K$ and $[y_0 - K, y_0 + K] \subset J$, then the initial value problem:*

$$u'(x) = F\big(x, u(x)\big) \quad \text{on } [a,b] \quad \text{and} \quad u(x_0) = y_0 \tag{1.4}$$

has at least one absolutely continuous solution (*this is also known as Carathéodory solution*; *see* [14] *or* [2]) *taking values in* $[y_0 - K, y_0 + K]$.

Theorem 1.2. *Let* $x_0 \in [a,b] \subset \mathbb{R}$, *and let* J *be an interval in* $\mathbb{R}$. *Let* $F : [a,b] \times J \to \mathbb{R}$ *be such that* $F(\cdot, y)$ *is integrable* (*on* $[a,b]$) *for each fixed* $y \in J$ *and satisfies the following generalized Lipschitz condition*: *let* $w[a,b] = \int_a^b w\,dx < \infty$ *and*

$$\big|F(x, y_1) - F(x, y_2)\big| \leqslant w(x)|y_1 - y_2|, \quad y_1, y_2 \in J, \text{ for almost every } x \in [a,b]. \tag{1.5}$$

If y *and* z *are absolutely continuous solutions of the differential equation* $u'(x) = F(x, u(x))$ *on* $[a,b]$ *such that* $z(x), y(x) \in J$ *for all* $x \in [a,b]$, *then*

$$\|y - z\|_{L^\infty[a,b]} \leqslant \big|y(x_0) - z(x_0)\big| \exp\big(w[a,b]\big). \tag{1.6}$$

In particular, if $y(x_0) = z(x_0)$, *then we know the solution of the initial value problem* (1.4) *is unique.*

Remark 1.3.

(1) The condition on F will imply that $F(x, u(x))$ is measurable if u is continuous.
(2) The two theorems above are usually stated with F being continuous (instead of just measurable) and with $w(x)$ equals to a constant L. Note that when F is continuous, it is easy to check that Carathéodory solutions are just classical solutions, that is, $u'(x) = F(x, u(x))$ everywhere instead of almost everywhere since it is now clear that u is continuously differentiable.
(3) In the case where F is given to be continuous on $[a,b] \times \mathbb{R}$, then Peano's theorem asserts that the initial value problem has at least a solution [14, p. 73].
(4) The two theorems above are sometimes being combined so as to give an existence and uniqueness theorem for initial value problem.

In 2000, Chua and Wheeden obtained the following existence and uniqueness result on average value problems.

Theorem 1.4. *(See [4, Theorem 4.2].) Let* $1 < p < \infty$, $M, K > 0$, $-\infty < a < b < \infty$ *and* $|b - a| \leqslant K/M$. *Let* σ *be a nonnegative weight and* v *be a measure on* $[a,b]$ *such that* $v[a,b] > 0$. *Suppose* F *is a measurable function on* $[a,b] \times [y_0 - K, y_0 + k]$ *such that*

$$\text{(A)} \qquad \begin{cases} \big|F(x,y)\big| \leqslant M & \text{for } x \in [a,b] \text{ and } |y - y_0| \leqslant K, \text{ and} \\ \big|F(x,y_1) - F(x,y_2)\big| \leqslant w(x)|y_1 - y_2| & \text{for } x \in [a,b],\ y_1, y_2 \in [y_0 - K, y_0 + K]. \end{cases}$$

$$\text{(B)} \qquad \lambda_\sigma = \frac{1}{v[a,b]} \big\| \big(\mu[\cdot, b] v[a, \cdot]^{p'} + \mu[a, \cdot] v[\cdot, b]^{p'}\big)^{1/p'} \big\|_{L^\infty[a,b]} \|w\|_{L^p_\sigma[a,b]} < 1, \quad \mu = \sigma^{1-p'}.$$

Then the ordinary differential equation $u'(x) = F(x, u(x))$, $x \in [a,b]$, *has a unique absolutely continuous solution* $u : [a,b] \to [y_0 - K, y_0 + K]$ *such that* $\int_a^b u\,dv / v[a,b] = y_0$.

The proof of Theorem 1.1 in [4] is basically just a modification of Picard's iterations using an integral representation [4, (2.1)] which is slightly different from the initial value problem. Note that the representation is indeed just a generalization of an integral representation obtained in [8, p. 147].

In case ν is just the Delta measure at a (and hence the average value problem is just an initial value problem), $w(x) = L$, then

$$\lambda_\sigma = \mu[a,b]^{1/p'} L\sigma[a,b]^{1/p} < 1$$

will imply $|b-a| < 1/L$ (by the Hölder inequality) which is certainly not required in standard theorem on initial value problems. Thus condition (B) is too restrictive. In this paper, we will show that one can indeed extend Theorems 1.1 and 1.2 to average value problems, that is, existence and uniqueness theorems for average value problems still holds under the same assumption for initial value problems. Surprisingly, our methods are indeed quite elementary that involved mostly Schauder's fixed point theorem, intermediate value theorem and a simple integral representation [4, (2.1)].

Theorem 1.5. *Let ν be a probability measure on $[a,b]$. Under the assumption of Theorem 1.1, the average value problem:*

$$y'(x) = F\big(x, y(x)\big) \quad \text{on } [a,b] \quad \text{and} \quad y_0 = \int_a^b y\,d\nu \tag{1.7}$$

has at least an absolutely continuous solution taking values in $[y_0 - K, y_0 + K]$.

Similarly, we also have the uniqueness and continuous dependence of average value as follows:

Theorem 1.6. *Let ν be a probability measure on $[a,b]$. Under the assumptions of Theorem 1.2, we have*

$$\|y - z\|_{L^\infty[a,b]} \leqslant \left|\int_a^b (y-z)\,d\nu\right| \exp\big(w[a,b]\big). \tag{1.8}$$

Remark 1.7.

(1) Note that $w[a,b] < \infty$ whenever $\lambda_\sigma < \infty$ (see condition (B) of Theorem 1.4) since there exists $\alpha \in [a,b]$ such that $\nu[a,\alpha], \nu[\alpha,b] > 0$ and by Hölder's inequality we have

$$w[\alpha,b] \leqslant \|w\|_{L^p_\sigma[\alpha,b]} \big(\sigma^{1-p'}[\alpha,b]\big)^{1/p'} < \infty$$

and $w[a,\alpha] < \infty$ similarly.

(2) It is essential that $w[a,b] < \infty$. For example, in the case where $w \notin L^1[0,1]$, the equation $y' = w(x)y$ will not have an absolutely continuous solution on $[0,1]$ with $y(0) \neq 0$.

(3) In case ν is just the Delta measure at a, Theorems 1.5 and 1.6 are just the classical theorems: Theorems 1.1 and 1.2.

Combining the above two theorems, we have

Corollary 1.8. *Let $x_0 \in [a,b] \subset \mathbb{R}$, $K > 0$, $y_0 \in J$ where J is an interval in $\mathbb{R}$. Let ν be a probability measure on $[a,b]$. Let $F : [a,b] \times J \to \mathbb{R}$ be such that $F(\cdot, y)$ is integrable (on $[a,b]$) for each fixed $y \in J$ and satisfies the generalized Lipschitz condition* (1.5)*. Let*

$$\big\|F(x,\cdot)\big\|_{L^\infty(J)} = f(x)$$

and $f \in L^1[a,b]$. If $\int_a^b f \leqslant K$ and $[y_0 - K, y_0 + K] \subset J$, then (1.7) *has a unique absolutely continuous solution taking values in $[y_0 - K, y_0 + K]$.*

In the case where the generalized Lipschitz condition (1.5) holds for $J = \mathbb{R}$, we also have:

Theorem 1.9. *Let $-\infty < a < b < \infty$ and let ν be a probability measure on $[a,b]$. Let $F : [a,b] \times \mathbb{R} \to \mathbb{R}$ be such that $F(\cdot, y)$ is integrable* (*on $[a,b]$*) *for each fixed $y \in \mathbb{R}$ and satisfies the generalized Lipschitz condition* (1.5) *with $J = \mathbb{R}$. Then for any $y_0 \in \mathbb{R}$, the ordinary differential equation $u'(x) = F(x, u(x))$, has a unique absolutely continuous solution y on $[a,b]$ such that $\int_a^b y\,d\nu = y_0$. Moreover, if $z'(x) = F(x, z(x))$ on $[a,b]$ such that $\int_a^b z\,d\nu = z_0$, then*

$$\|y - z\|_{L^\infty[a,b]} \leqslant |y_0 - z_0| \exp\bigl(w[a,b]\bigr). \tag{1.9}$$

Remark 1.10.

(1) In Theorem 1.9, we did not assume $\int_a^b \|F(x,\cdot)\|_{L^\infty(\mathbb{R})}\,dx < \infty$, as in Theorems 1.1 or 1.5. Thus the existence of the solution is not a consequence of Theorem 1.5.
(2) Note also that the theorem is strictly stronger than a result obtained in [4, Theorem 4.3] where it assumed a strictly stronger condition.

Moreover, we will prove the following existence and uniqueness theorem in the case where ν is just a signed measure. Instead of assuming F being a function that satisfies the Carathéodory condition (see Theorem 1.1), we will assume that $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$.

Theorem 1.11. *Let $x_0 \in [a', b'] \subset [a,b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let ν be a signed measure on $[a', b']$ with $\nu[a', b'] \neq 0$. Let $\nu_0 > 0$ be such that $|\nu[a', t]|, |\nu[t, b']| \leqslant \nu_0 |\nu[a', b']|$ on $[a', b']$. Let $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$ such that $F[y] = F[z]$ on $[a', b']$ if $y = z$ on $[a', b']$. Suppose F satisfies the following Lipschitz type condition*:

$$\bigl|F[y](x) - F[z](x)\bigr| \leqslant w(x)\|y - z\|_{L^\infty[x_0,x]} \quad \textit{for almost all } x \in (x_0, b]$$

and

$$\bigl|F[y](x) - F[z](x)\bigr| \leqslant w(x)\|y - z\|_{L^\infty[x,x_0]} \quad \textit{for almost all } x \in [a, x_0) \tag{1.10}$$

for all $y, z \in \mathcal{F}$. Let $w[a,b] < \infty$ and $w[a', b']\nu_0 < 1$. Suppose either (1) *$\mathcal{F} = C[a,b]$, or* (2) *$\mathcal{F}_0 = \{y \in C[a,b]\colon \|y - y_0\|_{L^\infty[a,b]} \leqslant \nu_0 K\} \subset \mathcal{F}$ and $\int_a^b |F[y]|\,dx \leqslant K$ for all $y \in \mathcal{F}_0$. Then the average value problem*:

$$y'(x) = F[y](x) \quad \text{on } [a,b] \quad \text{and} \quad y_0 \nu\bigl[a', b'\bigr] = \int_{a'}^{b'} y\,d\nu \tag{1.11}$$

has a unique absolutely continuous solution on $[a,b]$. Moreover, in case (2), *the solution will take values in $[y_0 - \nu_0 K, y_0 + \nu_0 K]$.*

Remark 1.12.

(1) The differential equation in (1.11) will certainly include differential equations such as

$$y'(x) = f\left(x, y\left(\frac{x + x_0}{2}\right)\right), \qquad y(x_0) = y_0.$$

Our idea can also be modified to include delay-differential equations; see [14, p. 82] for the definition.

(2) One can also extend the above theorem to boundary value problems in ordinary differential equations such as Necoletti boundary value problem [3]. See also [10,7] for some other types of existence-uniqueness results.

Finally, we would like to discuss an application/extension of our results to symmetric (rotational) solution of the Laplace equation:

$$\Delta u = F\big(|x|, u\big), \qquad \int_0^a u(r)\, d\nu(r) = u_0 \quad \big(r = |x|\big).$$

Theorem 1.13. *Let $b \geqslant b' > 0$ and $y_0 \in \mathbb{R}$. Let ν be a signed measure on $[0, b']$ such that $\nu[0, b'] \neq 0$. Let $\nu_0 > 0$ be such that $|\nu[0, t]|, |\nu[t, b']| \leqslant \nu_0 |\nu[0, b']|$ for almost all $t \in [0, b']$. Let $F : \mathcal{F} \subset C[0, b] \to L^1[0, b]$ such that $F[y_1] = F[y_2]$ on $[0, b']$ whenever $y_1 = y_2$ on $[0, b']$ and satisfies a Lipschitz-type condition*

$$\int_0^x \big|F[y_1](t) - F[y_2](t)\big|\, dt \leqslant w(x)\|y_1 - y_2\|_{L^\infty[0,x]}, \quad y_1, y_2 \in \mathcal{F}, \text{ for almost all } x \in [0, b] \tag{1.12}$$

with $w[0, b] < \infty$. Suppose either (1) $\mathcal{F} = C[a, b]$ or (2) there exists $K > 0$ such that

$$\mathcal{F}_0 = \big\{y \in C[0, b]:\ \|y - y_0\|_{L^\infty[0,b]} \leqslant \nu_0 K\big\} \subset \mathcal{F}$$

and $\int_0^b |F[y]|\, dx \leqslant K/b$ for all $y \in \mathcal{F}_0$. If $\nu_0 w[0, b'] < 1$, then the following partial differential equation

$$\Delta u = F\big(|x|, u\big), \quad \text{for all } x \in \mathbb{R}^n,\ |x| \leqslant b$$

has a unique (rotational) symmetric solution $u(x) = y(r)$, $r = |x|$, such that $y \in C^1[0, b]$ and y' is absolutely continuous, with $\int_0^{b'} y\, d\nu = y_0 \nu[0, b']$.

2. Preliminaries

We shall collect a few useful results here. First, let us state a theorem on compact operators: Schauder's fixed point theorem.

Theorem 2.1. *(See [5, Corollary 11.2].) Let X be a Banach space. If $\mathcal{D} \subset X$ is closed and convex such that $T : \mathcal{D} \to \mathcal{D}$ is continuous and $T(\mathcal{D})$ is precompact, then there exists $x_0 \in \mathcal{D}$ such that $T(x_0) = x_0$.*

We will now use the above fixed point theorem to prove existence of fixed points of some integral operators.

Lemma 2.2. *Let $[a, b] \subset \mathbb{R}$, $M, K > 0$, $y_0 \in \mathcal{F} \subset C[a, b]$. Let $F : \mathcal{F} \to L^1[a, b]$ be such that F is continuous, i.e., $F[y_n] \to F[y]$ in $L^1[a, b]$ whenever $\|y_n - y\|_{L^\infty[a,b]} \to 0$. Let $\Gamma_1, \Gamma_2 : [a, b] \to \mathbb{R}$ be two bounded measurable* *functions such that* $|\Gamma_1|, |\Gamma_2| \leqslant M$ *a.e.. Let*

$$Tu(x) = y_0(x) + \int_a^x \Gamma_1(t)F[u](t)\,dt - \int_x^b \Gamma_2(t)F[u](t)\,dt \quad \textit{for } x \in [a,b].$$

Suppose $\mathcal{F}_0 = \{y \in C[a,b]\colon \|y - y_0\|_{L^\infty[a,b]} \leqslant MK\} \subset \mathcal{F}$ *and* $\int_a^b |F[y](x)|\,dx \leqslant K$ *for all* $y \in \mathcal{F}_0$*. If there exists* $f \in L^1[a,b]$ *such that* $|F[y]| \leqslant f$ *a.e. for all* $y \in \mathcal{F}_0$*, then the operator* T *has a fixed point in* $\{y \in C[a,b]\colon \|y - y_0\|_{L^\infty[a,b]} \leqslant MK\}$.

Proof. First, since

$$\big|Tu(x) - Tu(z)\big| \leqslant \left|\int_x^z 2M\big|f(t)\big|\,dt\right| + \big|y_0(x) - y_0(z)\big| \tag{2.1}$$

it is clear that Tu is continuous for $u \in \mathcal{F}_0$ since $f \in L^1[a,b]$.

Next, it is easy to see that $\|Tu - y_0\|_{L^\infty[a,b]} \leqslant MK$ and hence $T(\mathcal{F}_0) \subset \mathcal{F}_0$. Moreover $\mathcal{F}_0$ is clearly closed and convex. Now, let us check that T is continuous on $\mathcal{F}_0$. Suppose $u_n \to u$ in $C[a,b]$ such that $u_n, u \in \mathcal{F}_0$. We have

$$\lim_{n\to\infty} \|Tu_n - T_u\|_{L^\infty[a,b]} \leqslant \lim_{n\to\infty} \int_a^b M\big|F[u_n](x) - F[u](x)\big|\,dx = 0$$

since F is a continuous operator. It is then clear that T is continuous.

Moreover, note that by (2.1), the family of functions $\{T(\mathcal{F}_0)\}$ is equi-continuous. Also, clearly the family of functions $\{T(\mathcal{F}_0)\}$ is uniformly bounded. Thus by the Ascoli Arzela theorem, we know $\{T(\mathcal{F}_0)\}$ is precompact. Hence, by Schauder's fixed point theorem, the operator T must have at least one fixed point. $\square$

Remark 2.3. Suppose F satisfies the Carathéodory condition as in Theorem 1.1, and $\|F(x,\cdot)\|_{L^\infty[a,b]} \in L^1[a,b]$. If we define

$$\tilde{F}[y](x) = F\big(x, y(x)\big) \quad \text{for } x \in [a,b],\ y \in C[a,b],$$

then $\tilde{F}$ is clearly a continuous operator from $C[a,b]$ to $L^1[a,b]$.

Next, let us state a simple fact on Laplace operator from [14].

Lemma 2.4. *(See [14, p. 72].) Let* $a > 0$ *and* $B_a(0) = \{x \in \mathbb{R}^n\colon |x| \leqslant a\}$*. If* $u \in C^2(B_a(0))$ *is (rotational) symmetric function, i.e.,* $u(x) = y(r)$, $r = |x|$*, then*

$$\Delta u = y'' + \frac{n-1}{r} y', \quad \textit{and} \quad y'(0) = 0.$$

Let us note that indeed, the following is also true:

Proposition 2.5. *Let $a > 0$ and $B_a(0) = \{x \in \mathbb{R}^n\colon |x| \leqslant a\}$. If $u \in C^1(B_a(0))$ is (rotational) symmetric function, i.e., $u(x) = y(r)$, $r = |x|$, and y' is absolutely continuous on $[0, b]$ then*

$$\Delta u(x) = y''(r) + \frac{n-1}{r} y'(r), \quad \text{for almost all } x \quad \text{and} \quad y'(0) = 0.$$

3. Average value problems with signed measure

In this section, we will study existence and uniqueness theorems for average value problem in the case where the probability measure is being replaced by just a signed measure. Note that of course, we can only obtain a weaker result under this assumption. However, on the other hand, our assumption on F can be relaxed.

First of all, let us note that the space of bounded linear functionals on $C[a, b]$ is just the collection of all signed measures on $[a, b]$. Here are our main theorems in this section.

Theorem 3.1. *Let $[a, b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let v be a signed measure on $[a, b]$ with $v[a, b] \neq 0$. Let $v_0 > 0$ be such that $|v[a, t]|, |v[t, b]| \leqslant v_0 |v[a, b]|$ for almost all $t \in [a, b]$. Let $F : \mathcal{F} \subset C[a, b] \to L^1[a, b]$ be such that F is continuous, i.e., $F[y_n] \to F[y]$ in $L^1[a, b]$ whenever $\|y_n - y\|_{L^\infty[a,b]} \to 0$. Let $\mathcal{F}_0 = \{y \in C[a, b]\colon \|y - y_0\|_{L^\infty[a,b]} \leqslant v_0 K\}$ and $\int_a^b |F[y]|\,dx \leqslant K$ for all $y \in \mathcal{F}_0$. If $\mathcal{F}_0 \subset \mathcal{F}$ and there exists $f \in L^1[a, b]$ such that $|F[y]| \leqslant f$ a.e. for all $y \in \mathcal{F}_0$, then the average value problem:*

$$y'(x) = F[y](x) \quad \text{on } [a, b] \quad \text{and} \quad y_0 v[a, b] = \int_a^b y\,dv \tag{3.1}$$

has at least an absolutely continuous solution taking values in $[y_0 - v_0 K, y_0 + v_0 K]$.

Theorem 3.2. *Let v be a signed measure on $[a, b]$ with $v[a, b] \neq 0$. Let $v_0, w_0 > 0$ and $|v[a, t]|, |v[t, b]| \leqslant v_0 |v[a, b]|$ for almost all $t \in [a, b]$. Let $F : \mathcal{F} \subset C[a, b] \to L^1[a, b]$ such that*

$$\big\| F[y_1] - F[y_2] \big\|_{L^1[a,b]} \leqslant w_0 \|y_1 - y_2\|_{L^\infty[a,b]} \tag{3.2}$$

for all $y_1, y_2 \in \mathcal{F}$. If $w_0 v_0 < 1$ and $y, z \in \mathcal{F}$ are both solutions of the differential equation in (3.1), then

$$\|y - z\|_{L^\infty[a,b]} \leqslant \frac{|\int_a^b (y - z)\,dv|}{|v[a, b]|(1 - w_0 v_0)}. \tag{3.3}$$

Before we prove these two theorems, let us use Theorem 3.1 to provide a quick proof to an interesting consequence of [12, Theorem 2].

Corollary 3.3. *Let v be a signed measure on $[a, b]$. If the functional boundary value problem:*

$$u' = 0, \qquad \int_a^b u\,dv = y_0 v[a, b]$$

has a unique solution $u \in C[a, b]$ for every $y_0 \in \mathbb{R}$, then for every bounded continuous function $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ and every $y_0 \in \mathbb{R}$, the functional boundary value problem:

$$u'(x) = g\big(x, u(x)\big), \qquad \int_a^b u\,dv = y_0 v[a,b]$$

has at least one solution.

Proof. First note that the condition will imply that $v[a,b] \neq 0$. Next, if g is a bounded continuous function on $[a,b] \times \mathbb{R}$, then there exists $M > 0$ such that $|g(x,y)| \leqslant M$ on $[a,b] \times \mathbb{R}$ and hence $\int_a^b |g(x, y(x))|\,dx \leqslant M|b-a|$. Also, since v is a signed measure and $v[a,b] \neq 0$, there exists $v_0 > 0$ such that

$$\big|v[a,t]\big|, \big|v[t,b]\big| \leqslant v_0\big|v[a,b]\big|.$$

It is now clear Corollary 3.3 follows from Theorem 3.1. □

Proof of Theorem 3.1. The key idea of the proof is to deduce the integral representation similar to [4, p. 153].

First let us show that $y \in \mathcal{F}$ is an absolutely continuous solution of the average value problem (3.1) if and only if

$$y(x) - y_0 = \frac{1}{v[a,b]}\left[\int_a^x v[a,t]F[y](t)\,dt - \int_x^b v[t,b]F[y](t)\,dt\right]. \tag{3.4}$$

Clearly, (3.4) will imply $y'(x) = F[y](x)$ for almost all $x \in [a,b]$ and y is absolutely continuous since $F[y] \in L^1[a,b]$. Next, observe that if y is absolutely continuous on $[a,b]$, following [4, p. 153], we have

$$\begin{aligned}
y(x) - \frac{1}{v[a,b]}\int_a^b y(z)\,dv(z) &= \frac{1}{v[a,b]}\int_a^b \big(y(x) - y(z)\big)\,dv(z) \\
&= \frac{1}{v[a,b]}\int_a^b\int_z^x y'(t)\,dt\,dv(z) \\
&= \frac{1}{v[a,b]}\left[\int_a^x\int_z^x y'(t)\,dt\,dv(z) - \int_x^b\int_x^z y'(t)\,dt\,dv(z)\right] \\
&= \frac{1}{v[a,b]}\left[\int_a^x v[a,t]y'(t)\,dt - \int_x^b v[t,b]y'(t)\,dt\right]
\end{aligned} \tag{3.5}$$

by Fubini's theorem. It is now easy to see that y satisfies (3.4) will imply that y is a solution of the average value problem (3.1) and conversely, y satisfies (3.4) if it is an absolutely continuous solution of (3.1).

Thus, we need only to find a fixed point of the following operator

$$Ty(x) = y_0 + \frac{1}{v[a,b]}\left[\int_a^x v[a,t]F[y](t)\,dt - \int_x^b v[t,b]F[y](t)\,dt\right].$$

It then follows from Lemma 2.2 that the operator T has a fixed point $u \in C[a,b]$ taking value in $[y_0 - v_0 K, y_0 + v_0 K]$. Moreover, it is clear that Tu is absolutely continuous and so is u. This completes the proof of Theorem 3.1. $\square$

We will now prove Theorem 3.2.

Proof of Theorem 3.2. First it is clear that we have

$$y(x) - z(x) = \frac{1}{v[a,b]} \int_a^b (y-z)\,dv + \frac{1}{v[a,b]} \left[\int_a^x v[a,t]\bigl\{F[y](t) - F[z](t)\bigr\}\,dt - \int_x^b v[t,b]\bigl\{F[y](t) - F[z](t)\bigr\}\,dt \right]$$

and hence by the triangle inequality and (3.2),

$$\begin{aligned} \bigl|y(x) - z(x)\bigr| &\leqslant \left| \frac{1}{v[a,b]} \int_a^b y - z\,dv \right| + v_0 \int_a^b \bigl|F[y](t) - F[z](t)\bigr|\,dt \\ &\leqslant \left| \frac{1}{v[a,b]} \int_a^b y - z\,dv \right| + v_0 w_0 \|y - z\|_{L^\infty[a,b]}. \end{aligned}$$

Theorem 3.2 is now clear. $\square$

Remark 3.4. In the case where conditions in both Theorems 3.1 and 3.2 hold, Picard's iteration can also be used to obtain the solution of the average value problem. Just let $u_0(x) = y_0$ and

$$u_n(x) = y_0 + \frac{1}{v[a,b]} \left[\int_a^x v[a,t] F[u_{n-1}](t)\,dt - \int_x^b v[t,b] F[u_{n-1}](t)\,dt \right]$$

for $n = 1, 2, 3, \ldots$.

4. Proof of main theorems

First, note that Theorems 1.5 and 1.1 are clearly special cases of Theorem 3.1.

Next, let us prove a stronger result instead of Theorem 1.2.

Proposition 4.1. *Let $x_0 \in [a,b] \subset \mathbb{R}$, $y_0 \in \mathbb{R}$. Let $F : \mathcal{F} \subset C[a,b] \to L^1[a,b]$ such that F satisfies the following Lipschitz type condition*:

$$\bigl|F[u](x) - F[v](x)\bigr| \leqslant w(x)\|u - v\|_{L^\infty[x_0,x]} \quad \textit{for almost all } x \in (x_0, b]$$

and

$$\bigl|F[u](x) - F[v](x)\bigr| \leqslant w(x)\|u - v\|_{L^\infty[x,x_0]} \quad \textit{for almost all } x \in [a, x_0), \tag{4.1}$$

for all $u, v \in \mathcal{F}$ with $w[a,b] < \infty$. If $y, z \in \mathcal{F}$ are solutions of the equation $u' = F[u]$, then

$$\|y - z\|_{L^\infty[a,b]} \leqslant \big|y(x_0) - z(x_0)\big|\big(\max\big\{\exp w[x_0, b], \exp w[a, x_0]\big\}\big). \tag{4.2}$$

Moreover, if $\mathcal{F} = C[a,b]$, then for each $\alpha \in \mathbb{R}$, there exists a unique absolutely continuous solution of the initial value problem:

$$u' = F[u], \qquad u(x_0) = \alpha.$$

Proof. Since $y, z \in \mathcal{F}$ are solutions to the differential equation, we have

$$y(x) = y(x_0) + \int_{x_0}^{x} F[y](t)\,dt$$

and

$$z(x) = z(x_0) + \int_{x_0}^{x} F[z](t)\,dt.$$

To simplify the computation, let us assume $x_0 = a$. Then,

$$\begin{aligned}\big|y(x) - z(x)\big| &\leqslant \big|y(x_0) - z(x_0)\big| + \int_a^x \big|F[y](t) - F[z](t)\big|\,dt \\ &\leqslant \big|y(x_0) - z(x_0)\big| + \int_a^x w(t)\|y - z\|_{L^\infty[a,t]}\,dt \\ &\leqslant \big|y(x_0) - z(x_0)\big| + w[a,x]\|y - z\|_{L^\infty[a,x]}. \end{aligned} \tag{4.3}$$

(4.2) is then just a consequence of Gronwall's inequality. Alternatively, note that

$$\|y - z\|_{L^\infty[a,x]} \leqslant \big|y(x_0) - z(x_0)\big| + w[a,x]\|y - z\|_{L^\infty[a,x]}, \tag{4.4}$$

we can repeat our previous argument. Let $\varepsilon = |y(x_0) - z(x_0)|$. Then

$$\begin{aligned}\big|y(x) - z(x)\big| &\leqslant \int_a^x w(t)\big(w[a,t]\|y - z\|_{L^\infty[a,t]}\,dt + \varepsilon\big) + \varepsilon \quad \big(\text{by (4.3) and (4.4)}\big) \\ &\leqslant \int_a^x w(t)\big(w[a,t]\|y - z\|_{L^\infty[a,x]} + \varepsilon\big)\,dt + \varepsilon \\ &= \|y - z\|_{L^\infty[a,x]} w[a,x]^2/2 + \varepsilon w[a,x] + \varepsilon. \end{aligned}$$

Repeat it n times, we have

$$\|y-z\|_{L^\infty[a,x]} \leqslant \big(w[a,x]^n/n!\big)\|y-z\|_{L^\infty[a,x]} + \varepsilon\exp\big(w[a,x]\big)$$

and hence,

$$\|y-z\|_{L^\infty[a,x]} \leqslant \varepsilon\exp\big(w[a,x]\big).$$

This completes the proof of the first part of the proposition and Theorem 1.2 is now clear. The second part of the proof is indeed again just a simple modification of the classical argument, we put them here simply because we are unable to find a reference. Again, we will just prove the case $x_0 = a$. Define

$$u_1 = y_0, \qquad u_{n+1}(x) = y_0 + \int_a^x F[u_n](t)\,dt \quad \text{for } n = 1, 2, \ldots.$$

First, we have

$$\begin{aligned}\big|u_3(x) - u_2(x)\big| &\leqslant \int_a^x w(t)\|u_2 - u_1\|_{L^\infty[a,t]}\,dt \\ &\leqslant \|u_2 - u_1\|_{L^\infty[a,x]} w[a,x].\end{aligned}$$

Hence

$$\|u_3 - u_2\|_{L^\infty[a,x]} \leqslant \|u_2 - u_1\|_{L^\infty[a,x]} w[a,x].$$

Repeat the above estimate, we then have

$$\|u_{n+2} - u_{n+1}\|_{L^\infty[a,x]} \leqslant \|u_2 - u_1\|_{L^\infty[a,x]} w[a,x]^n/n!. \tag{4.5}$$

We can then argue by standard argument that the initial value problem has an absolutely continuous solution. This completes the proof of Proposition 4.1. $\square$

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Note that by the fact that $y - z$ is continuous, we can find $x_0 \in [a,b]$ such that

$$\int_a^b (y-z)\,dv = y(x_0) - z(x_0).$$

Since y and z are absolutely continuous solutions of the differential equation $u' = F(x,u)$, by (1.6) in Theorem 1.2, we conclude that

$$\|y-z\|_{L^\infty[a,b]} \leqslant \big|y(x_0) - z(x_0)\big| \exp\big(w[a,b]\big) = \left|\int_a^b (y-z)\,dv\right| \exp\big(w[a,b]\big).$$

This completes the proof of our main theorem. $\square$

Proof of Theorem 1.9. We will provide an elementary proof instead of using Schauder's fixed point theorem. We will prove the theorem by 2 steps. First we show that the theorem holds if $w[a,b] < 1$. We then make use of Proposition 4.1 and Step 1 to conclude the proof.

Step 1. We will first prove that the theorem holds when $w[a,b] < 1$.

The proof of this part is just a simple modification of [4, Theorem 4.2]. Let $u_1(x) = y_0$ on $[a,b]$ and for each $n \in \mathbb{N}$, define

$$u_{n+1}(x) = y_0 + \left[\int_a^x v[a,t]F\big(t, u_n(t)\big)\,dt - \int_x^b v[t,b]F\big(t, u_n(t)\big)\,dt \right] \quad \text{for } x \in [a,b].$$

Observe that $u'_{n+1}(x) = F(x, u_n(x))$ almost everywhere, and hence by [4, (2.1)] or (3.5), we get $\int_a^b u_{n+1}\,dv = y_0$ and

$$\begin{aligned} u_{n+1}(x) - u_n(x) &= \int_a^x v[a,t]\big\{F\big(t, u_n(t)\big) - F\big(t, u_{n-1}(t)\big)\big\}\,dt \\ &\quad - \int_x^b v[t,b]\big\{F\big(t, u_n(t)\big) - F\big(t, u_{n-1}(t)\big)\big\}\,dt \\ &= \int_a^b \big\{F\big(t, u_n(t)\big) - F\big(t, u_{n-1}(t)\big)\big\}\big\{v[a,t]\chi_{[a,x]}(t) - v[t,b]\chi_{[x,b]}(t)\big\}\,dt. \end{aligned}$$

Hence by the generalized Lipschitz condition (1.5),

$$\begin{aligned} \|u_{n+1} - u_n\|_{L^\infty[a,b]} &\leqslant \int_a^b \big|F\big(t, u_n(t)\big) - F\big(t, u_{n-1}(t)\big)\big|\,dt \\ &\leqslant \int_a^b w(t)\big|u_n(t) - u_{n-1}(t)\big|\,dt \\ &\leqslant \|u_n - u_{n-1}\|_{L^\infty[a,b]} \int_a^b w(t)\,dt \\ &= w[a,b]\|u_n - u_{n-1}\|_{L^\infty[a,b]}. \end{aligned}$$

Moreover, note that

$$\|u_2 - u_1\|_{L^\infty} = \|u_2 - y_0\|_{L^\infty[a,b]} \leqslant \int_a^b \big|F(t, y_0)\big|\,dt < \infty,$$

since $F(\cdot, y_0)$ is integrable. Hence $\{u_n\}$ is a Cauchy sequence in $C[a,b]$ and has a limit $u \in C[a,b]$. Thus,

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= y_0 + \lim_{n\to\infty}\left[\int_a^x v[a,t]F\big(t,u_{n-1}(t)\big)\,dt - \int_x^b v[t,b]F\big(t,u_{n-1}(t)\big)\,dt\right] \\ &= y_0 + \left[\int_a^x v[a,t]F\big(t,u(t)\big)\,dt - \int_x^b v[t,b]F\big(t,u(t)\big)\,dt\right] \end{aligned}$$

by dominated convergence theorem since

$$\begin{aligned} &\big|F\big(t,u_{n-1}(t)\big)\big[v[a,t]\chi_{[a,x]}(t) - v[t,b]\chi_{[x,b]}(t)\big]\big| \\ &\quad\leqslant \big|F\big(t,u_{n-1}(t)\big)\big| \\ &\quad\leqslant \big|F\big(t,u_{n-1}(t)\big) - F(t,y_0)\big| + \big|F(t,y_0)\big| \quad \text{(by the triangle inequality)} \\ &\quad\leqslant w(t)\big|u_{n-1}(t) - y_0\big| + \big|F(t,y_0)\big| \quad \big(\text{by (1.5)}\big) \\ &\quad\leqslant w(t)\big|u_{n-1}(t)\big| + w(t)|y_0| + \big|F(t,y_0)\big| \\ &\quad\leqslant \big(C + |y_0|\big)w(t) + \big|F(t,y_0)\big| \end{aligned}$$

using the fact that $\{u_n\}$ is a Cauchy sequence in $C[a,b]$.

Also, $\int_a^b u\,dv = y_0$ and $u'(x) = F(x,u(x))$ almost everywhere. Finally, note that we can also deduce uniqueness (but we do not need it here) by the fact that $w[a,b] < 1$. This completes the first step of the proof.

Step 2. We can now complete the proof of Theorem 1.9. Since $w \in L^1[a,b]$, it is easy to see that there exists $\delta > 0$ such that if $[a',b'] \subset [a,b]$ such that $b' - a' < \delta$, then $w[a',b'] < 1$ and hence (1.7) has a unique solution y on $[a',b']$ such that $\int_{a'}^{b'} y(x)\,dv/v[a',b'] = y_0$ provided $v[a',b'] > 0$ by Step 1 as $w[a',b'] < 1$.

We will partition $[a,b]$ into closed subintervals so that the average value problem (1.7) has a solution on each subintervals with average y_0 (provided the v-measure of that subinterval is > 0). All we need to do next is to find a way to "combine" solutions on any two neighboring subintervals. We will show that this is always possible and then our proof will be complete.

First, for any $[\alpha,\gamma] \subset [a,b]$, such that there exist absolutely continuous functions y_1 on $[\alpha,\beta]$, $\alpha < \beta < \gamma$, and y_2 on $[\beta,\gamma]$, $v[\alpha,\beta], v[\beta,\gamma] > 0$, $v[\alpha,\gamma] = v[\alpha,\beta] + v[\beta,\gamma]$ such that $y_1'(x) = F(x,y_1(x))$ almost everywhere on $[\alpha,\beta]$, $y_2'(x) = F(x,y_2(x))$ almost everywhere on $[\beta,\gamma]$, and

$$y_0 = \int_\alpha^\beta y_1(x)\,dv(x)/v[\alpha,\beta] = \int_\beta^\gamma y_2(x)\,dv(x)/v[\beta,\gamma],$$

we will show that there exists y_3 on $[\alpha,\gamma]$ such that $y_3'(x) = F(x,y_3(x))$ almost everywhere on $[\alpha,\gamma]$ with

$$y_0 = \int_\alpha^\gamma y_3(x)\,dv(x)/v[\alpha,\gamma].$$

Applying Proposition 4.1 (second part) to intervals $[\alpha,\beta]$ and $[\beta,\gamma]$ with $x_0=\beta$, we know that for any z, there exists a unique absolutely continuous function y_z, $y_z'=F(\cdot,y_z)$ on $[\alpha,\gamma]$ and $y_z(\beta)=z$. Define

$$A(z)=\int_\alpha^\gamma y_z(x)\,dv(x)/v[\alpha,\gamma].$$

Note that A is a continuous function by (1.6). Without lost of generality, we may assume that $y_1(\beta)>y_2(\beta)$. Again, by uniqueness from Theorem 1.2, we know $y_1(x)=y_{y_1(\beta)}(x)>y_{y_2(\beta)}(x)=y_2(x)$ on $[\alpha,\gamma]$. Note that we can extend y_1,y_2 to $[\alpha,\gamma]$ by Proposition 4.1. It is then easy to see that

$$\frac{1}{v[\alpha,\gamma]}\int_\alpha^\gamma y_1(x)\,dv(x)\geqslant y_0\geqslant\frac{1}{v[\alpha,\gamma]}\int_\alpha^\gamma y_2(x)\,dv(x).$$

It now follows from the intermediate value theorem that there exists z_0 between $y_1(\beta)$ and $y_2(\beta)$ such that $A(z_0)=y_0$. Hence $y_3=y_{z_0}$ is a solution.

Note that in general, we may assume that at least one of $v[\alpha,\beta]$, $v[\beta,\gamma]$ is nonzero. When $v[\alpha,\beta]>0$, but $v[\beta,\gamma]=0$, we can just take $y_3=y_1$. Similarly, when $v[\beta,\gamma]>0$ but $v[\alpha,\beta]=0$, we can just take $y_3=y_2$.

Finally, (1.9) is just a consequence of (1.8). This completes the proof of our main theorem. □

Proof of Theorem 1.11. First, recall that

$$F[y_1]=F[y_2]\quad\text{on }[a',b']\quad\text{whenever }y_1=y_2\text{ on }\big[a',b'\big].$$

Next, by Picard's iteration as in Step 1 of the proof of Theorem 1.9, we see that (1.11) has an absolutely continuous solution y on $[a',b']$ such that $\int_{a'}^{b'}y\,dv=y_0v[a',b']$. (For case (2), one can also use Theorem 3.1 to get such an absolutely continuous solution.) Next, since for any $y_1,y_2\in C[a',b']$,

$$\big\|F[y_1]-F[y_2]\big\|_{L^1[a',b']}\leqslant w\big[a',b'\big]\|y_1-y_2\|_{L^\infty[a',b']},$$

and $w[a',b']v_0<1$, by Theorem 3.2, the solution y is unique on $[a',b']$.

We will now use Proposition 4.1 for case (1) and Theorem 3.1 for case (2) to find a solution u of the initial value problem on $[a,b]$:

$$u'=F[u]\quad\text{on }[a,b]\quad\text{and}\quad u(x_0)=y(x_0).$$

Note that this solution is unique by Proposition 4.1. Moreover, since the restriction of u on $[a',b']$ and the function y are both solutions of the differential equation on $[a',b']$, we conclude that $u=y$ on $[a',b']$ since $u(x_0)=y(x_0)$ and $x_0\in[a',b']$. It is now clear that

$$u'=F[u],\quad\text{and}\quad\int_{a'}^{b'}u\,dv=v\big[a',b'\big]y_0.$$

This concludes the proof of Theorem 1.11. □

Next, instead of proving Theorem 1.13, we will prove a slightly more general proposition.

Proposition 4.2. *Let $b \geqslant b' > 0$ and $y_0 \in \mathbb{R}$. Let $\nu_0 \geqslant 1$. Let ν be a signed measure on $[0, a]$ such that $\nu[0, b'] \neq 0$ and $|\nu[0, t]|, |\nu[t, b']| \leqslant \nu_0 |\nu[0, b']|$ for almost all $t \in [0, b']$. Let $F : \mathcal{F} \subset C[0, b] \to L^1[0, b]$ such that $F[y_1] = F[y_2]$ on $[0, b']$ whenever $y_1 = y_2$ on $[0, b']$ and let*

$$G[y](t) = t^{-\alpha} \int_0^t s^\alpha F[y](s)\, ds$$

satisfy the following Lipschitz type condition

$$\bigl|G[y_1](x) - G[y_2](x)\bigr| \leqslant w(x) \|y_1 - y_2\|_{L^\infty[0,x]} \quad \text{for } x \in [0, b],\ y_1, y_2 \in \mathcal{F}. \tag{4.6}$$

Suppose either (1) $\mathcal{F} = C[0, b]$ or (2) there exists $K > 0$ such that

$$\mathcal{F}_0 = \bigl\{ y \in C[0, b]\colon\ \|y - y_0\|_{L^\infty[0,b]} \leqslant \nu_0 K \bigr\} \subset \mathcal{F}$$

and $\int_0^b |G[y]|\, dx \leqslant K$ for all $y \in \mathcal{F}_0$. If $\nu_0 w[0, b'] < 1$, $\alpha \geqslant 0$, then the following second order linear ordinary differential equation

$$y'' + \frac{\alpha}{x} y' = F[y] \tag{4.7}$$

has a unique solution y such that y' is absolutely continuous, $y'(0) = 0$ and $\int_0^{b'} y\, d\nu = y_0$.

Proof. First note that $G[y]$ is absolutely continuous for all $y \in \mathcal{F}$ and $G[y_1] = G[y_2]$ whenever $y_1 = y_2$ on $[0, b']$. Next if $y \in \mathcal{F}$ is a solution of the differential equation (4.7), then $y' = G[y]$ and y' is absolutely continuous. Conversely, if $y' = G[y]$, then (4.7) holds. Hence, the proposition will follow from Theorem 1.11 with $a' = x_0 = 0$. $\square$

Theorem 1.13 is now easy to prove.

Proof of Theorem 1.13. First, note that by (1.12)

$$\begin{aligned} \bigl|G[y](t) - G[z](t)\bigr| &\leqslant \int_0^t \bigl|F[y](s) - F[z](s)\bigr|\, ds \\ &\leqslant w(t) \|y - z\|_{L^\infty[0,t]} \quad \text{for } y, z \in \mathcal{F} \end{aligned} \tag{4.8}$$

and

$$\bigl|G[y](t)\bigr| \leqslant \int_0^t \bigl|F[y](s)\bigr| \leqslant \int_0^b \bigl|F[y](s)\bigr|\, ds \leqslant K/b.$$

Theorem 1.13 is now clearly a corollary of Proposition 4.2 by Proposition 2.5. $\square$

Remark 4.3. Indeed, the condition $\int_0^b |F[y](s)|\,ds \leqslant K/b$ is clearly too restrictive. For example, in the case where $n > 2$, we only need

$$\int_0^b \frac{s^{2-n} - b^{2-n}}{n-2} s^{n-1} \big|F[y](s)\big|\,ds \leqslant K$$

in order to conclude that $\int_0^b |G[y](s)|\,ds \leqslant K$ and hence the conclusion of Theorem 1.13 will still hold. Indeed, by Fubini's theorem, we have

$$\begin{aligned}
\int_0^b \big|G[y](t)\big|\,dt &\leqslant \int_0^b t^{1-n} \int_0^t s^{n-1} \big|F[y](s)\big|\,ds\,dt \\
&\leqslant \int_0^b \int_s^b t^{1-n} s^{n-1} \big|F[y](s)\big|\,dt\,ds \\
&= \int_0^b \frac{s^{2-n} - b^{2-n}}{n-2} s^{n-1} \big|F[y](s)\big|\,ds
\end{aligned}$$

since $n > 2$.

Finally, let us prove an extension of a classical Peano's uniqueness theorem [1, Theorem 9.2]:

Theorem 4.4. *Let $F(x, y)$ be continuous in $[a, b] \times [y_0 - K, y_0 + K]$ and nonincreasing in y for each fixed x. If ν is a probability measure on $[a, b]$, $|F(x, y)| \leqslant M$ and $M(b - a) \leqslant K$, then the average value problem* (1.7) *has at most one absolutely continuous solution $u : [a, b] \to [y_0 - K, y_0 + K]$ with $\int_a^b u\,d\nu = y_0$.*

Proof. Let y_1, y_2 be two solutions. Clearly, $y_1(x), y_2(x) \in [y_0 - K, y_0 + K]$ for all $x \in [a, b]$. It suffices to see that there exists $z_0 \in [a, b]$ such that $y_1(z_0) = y_2(z_0)$ and we may assume z_0 is the minimum $z \in [a, b]$ such that $y_1(z) = y_2(z)$. The proof then follows from the classical argument that $y_1(x) = y_2(x)$ for all $x \in [z_0, b]$; see for example the proof of [1, Theorem 9.2]. By modifying that simple argument, one could also show that $y_1 = y_2$ on $[a, z_0]$. $\square$

Final remark. Consider the following boundary value problem:

$$u'' = F(x, u) \quad \text{on } [0, 1], \qquad u(0) = u(1) = 0. \tag{4.9}$$

Note that this is indeed a special case of our average value problem (in $\mathbb{R}^2$). It is known that if F is continuous and

$$\big|F(x, y_1) - F(x, y_2)\big| \leqslant L|y_1 - y_2| \quad \text{for } x \in [0, 1],\ y_1, y_2 \in \mathbb{R},$$

then this boundary value problem has a unique solution if $L < \pi^2$ [14, p. 254]. However, there are counter examples when $L = \pi^2$ (again see [14, p. 254]).

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