



# Homogenization of a parabolic model of ferromagnetism

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## ABSTRACT

This work deals with the homogenization of hysteresis-free processes in ferromagnetic composites. A degenerate, quasilinear, parabolic equation is derived by coupling the Maxwell–Ohm system without displacement current with a nonlinear constitutive law:

$$\frac{\partial \vec{B}}{\partial t} + \operatorname{curl} \left\{ A \left( \frac{x}{\varepsilon} \right) \cdot \operatorname{curl} \vec{H} \right\} = \operatorname{curl} \vec{E}_a, \quad \vec{B} \in \vec{\alpha} \left( \vec{H}, \frac{x}{\varepsilon} \right).$$

Here  $A$  is a periodic positive-definite matrix,  $\vec{\alpha}(\cdot, y)$  is maximal monotone and periodic in  $y$ ,  $\vec{E}_a$  is an applied field, and  $\varepsilon > 0$ . An associated initial- and boundary-value problem is represented by a minimization principle via an idea of Fitzpatrick. As  $\varepsilon \rightarrow 0$  a two-scale problem is obtained via two-scale convergence, and an equivalent coarse-scale formulation is derived. This homogenization result is then retrieved via  $\Gamma$ -convergence, and the continuity of the solution with respect to the operator  $\vec{\alpha}$  and the matrix  $A$  is also proved. This is then extended to some relaxation dynamics.

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## 1. Introduction

This paper deals with the homogenization of hysteresis-free processes in ferromagnetic composites. The main aim is to illustrate a method that rests upon a variational representation of maximal monotonicity, a two-scale formulation, and scale-transformations.

**Two-scale approach to homogenization.** We proceed along the following path, which may be applied to the homogenization of a large class of nonlinear partial differential equations with monotonicity.

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(i) First we represent processes in a composite by an initial- and boundary-value problem with data that periodically oscillate in space on a short length-scale, provide a formulation in the framework of Sobolev spaces, and prove existence of a solution.

(ii) We then derive a two-scale model via Nguetseng's two-scale convergence.

(iii) By integrating this problem with respect to the fine-scale variable (*upscaling*), we derive a purely coarse-scale model.

(iv) We prove that, conversely, any solution of the latter problem may be retrieved as an average of solutions of the fine-scale model (*downscaling*). This guarantees that no spurious solution was introduced by upscaling, so that the coarse-scale problem may actually be regarded as an *effective* (or *homogenized*) model – namely, a macroscopic model capable of representing the behavior of the composite.

(v) Finally, we retrieve and interpret this homogenization result via De Giorgi's notion of  $\Gamma$ -convergence. This also allows us to prove the stability with respect to variations of the nonlinear operator.

The possibility of applying variational techniques stems from a variational formulation of (either stationary or evolutionary) maximal monotone operators that is due to Fitzpatrick.

**A quasilinear parabolic model of ferromagnetic processes.** Let us briefly outline the model that we shall deal with. By coupling the Maxwell system without displacement current (so-called *eddy-current approximation*) with the Ohm law in a domain  $\Omega$ , we get an equation of the form

$$\vec{B}_t + \nabla \times (A(x) \cdot \nabla \times \vec{H}) = \nabla \times \vec{E}_a \quad (\vec{B}_t := \partial \vec{B} / \partial t, \nabla \times := \text{curl}), \quad (1.1)$$

where by  $A(x)$  we denote the resistivity matrix, and by  $\vec{E}_a$  an applied electric field. We couple this equation with the constitutive relation

$$\vec{B} \in \vec{\alpha}(\vec{H}, x). \quad (1.2)$$

Here  $\vec{\alpha}(\cdot, x)$  is a (possibly multivalued) maximal monotone mapping, which we do not assume to be cyclically monotone. We neglect hysteresis, under the assumption that the hysteresis loop is so thin that it may be replaced by a curve; this behavior is exhibited e.g. by soft iron, mild steel and other materials of engineering interest.

It may be noticed that the system (1.1) and (1.2) is degenerate, quasilinear and parabolic; the degeneration is due to the occurrence of the curl operator in the second-order operator. (1.1) entails that the field  $\vec{B}$  is divergence-free, provided it is so at the initial instant.

By a theory pioneered by Fitzpatrick [34], the monotone structure of this problem allows for a variational formulation. The system (1.1), (1.2) is actually equivalent to a minimization principle:

$$\text{find } (\vec{B}, \vec{H}) \in X \text{ such that } \mathcal{J}(\vec{B}, \vec{H}) = \inf \mathcal{J} (= 0), \quad (1.3)$$

for a suitable definition of the space  $X$  and of the functional  $\mathcal{J}: X \rightarrow \mathbf{R}^+$ , see (2.6).

If the mapping  $\vec{\alpha}(\cdot, x)$  is multivalued, the system (1.1), (1.2) may be interpreted as the weak formulation of a *free boundary problem*, see e.g. [67, Sect. IV.8] and references therein. This setting may be compared with the weak formulation of the classical Stefan model: both problems are quasilinear parabolic, include a maximal monotone discontinuous operator, and are intended to represent a free boundary problem. In the present model, however, the fields are vector-valued, and the second-order operator is degenerate.

For the Stefan problem the actual existence of a free boundary has extensively been studied, and nowadays the appropriate conditions for this to occur are fairly understood. This author does not know of investigations in that direction for electromagnetic models like this. In the present article, however, we are just concerned with the weak formulation of this vector problem.

Besides the quasi-stationary constitutive relation (1.2), we shall also consider some dynamics of relaxation, for instance

$$b\vec{H}_t + \vec{\alpha}(\vec{H}, x) \ni \vec{B} \quad (b \text{ being a positive constant}). \quad (1.4)$$

The corresponding Cauchy problem represents a maximal monotone relation between the fields  $\vec{B}$  and  $\vec{H}$  in a suitable space of time-dependent functions. The alternative dynamics

$$b\vec{B}_t + \vec{B} \in \vec{\alpha}(\vec{H}, x) \quad (1.5)$$

is also briefly discussed. In either case, as  $b \rightarrow 0$  the original problem is retrieved. As it is illustrated in Section 9 of [70], the latter dynamics account for eddy-current dissipation; see also Chap. 12 of [8]. On the other hand, this author is not aware of any physical motivation for (1.4) in ferromagnetism, which here is only considered in order to complete the analysis.

**Monotonicity and variational formulation.** By a classical transformation that was introduced for other free boundary problems by Baiocchi [4] and Duvaut [31], we shall integrate Eq. (1.1) in time. This yields

$$\vec{B} + \nabla \times \left( A(x) \cdot \nabla \times \int_0^t \vec{H}(\tau) d\tau \right) = \vec{B}^0 + \nabla \times \vec{E}_{\text{app}} * 1 =: \vec{G} \quad \text{in } \Omega_T, \quad (1.6)$$

so that eliminating the field  $\vec{B}$  by (1.2) we get

$$\vec{\gamma}(\vec{H}) := \vec{\alpha}(\vec{H}) + \nabla \times \left( A(x) \cdot \nabla \times \int_0^t \vec{H}(\tau) d\tau \right) \ni \vec{G} \quad \text{in } \Omega_T. \quad (1.7)$$

Under natural assumptions on the growth of  $\vec{\alpha}$ , the operator  $\vec{\gamma}$  is maximal monotone in a suitable space of functions of  $(x, t)$ . By the theory pioneered by Fitzpatrick [34], this equation may be represented as a minimization problem, see e.g. [69].

Here we are especially concerned with the derivation of an effective model, in the case of a fine mixture of materials, which we represent by assuming that the data  $\vec{\alpha}$ ,  $A$ ,  $\vec{B}^0$  and  $\vec{E}_a$  periodically oscillate in space on a short length-scale. We also address the stability with respect to the data, including the monotone mapping  $\vec{\alpha}$  and the matrix  $A$ .

Although physically it would be perfectly sound to assume the cyclical monotonicity of  $\vec{\alpha}$ , here we do not require it, in view of certain extensions that however we do not develop here. This however rules out an estimate procedure and the corresponding regularity: loosely speaking, it is not possible to multiply the equation by the field  $\vec{H}$ . If for instance a cyclically monotone mapping  $\vec{\alpha}$  depended explicitly on time in a nondifferentiable way, that estimate procedure would also be precluded; the present procedure would then be of interest.

**Outline.** This article is organized as follows. In Section 2 we outline the variational formulation of maximal monotone operators due to Fitzpatrick, and some results of [71] on scale-transformations. In Section 3 we illustrate the ferromagnetic model, and introduce a weak formulation of an initial- and boundary-value problem for the system (1.1), (1.2). In Section 4 we prove existence and uniqueness of the solution via a fairly standard argument,<sup>1</sup> and reformulate the problem as a minimization principle via Fitzpatrick's theory. In Section 5 we derive a two-scale model by passing to the two-scale limit (in the sense of Nguetseng [48]) in the problem with oscillating data. In Section 6 we formulate a coarse-scale homogenized problem, and prove its equivalence to the two-scale model. In Section 7 we retrieve the homogenized problem by  $\Gamma$ -convergence, and show that an analogous procedure may be used to prove the continuity of the dependence of the solution on the operator  $\vec{\alpha}$  (*structural stability*). In Section 8 we illustrate the extension to relaxation dynamics, as an alternative to the

<sup>1</sup> The same remark applies to other standard results; some proofs are displayed whereas other are omitted, trying to compromise between completeness and repetitiveness.

quasi-stationary constitutive relation (1.2). Finally, in Appendix A we review some basic elements of two-scale convergence, for the reader's convenience.

**Literature.** Homogenization, namely the macroscopic representation of the behavior of composite materials, has extensively been studied, starting with the seminal works of Babuška [3], De Giorgi and Spagnolo [30,57], Tartar [58], Bensoussan, Lions and Papanicolaou [7], and so on; see e.g. the monographs [2,5,26,40,45,49,50,55,59]. De Giorgi's notion of  $\Gamma$ -convergence of [29] has also successfully been applied to the homogenization of stationary problems, see e.g. [13,14,20,27].

The notion of two-scale convergence was introduced by Nguetseng [48], and was then further developed by Allaire [1] and others; see also the survey paper [41], and Cioranescu, Damlamian and Griso's reformulation via *periodic unfolding* [24,25]. The homogenization of maximal (possibly non-cyclical) monotone operators was addressed in several works, see e.g. [21–23,28,35,36,56].

After the seminal paper [34] of Fitzpatrick, several authors addressed the representation of monotone operators, see e.g. [18,19,42–44,51–53]. Presently this research is under further expansion, but apparently it has not yet intensively been used in connection with partial differential equations. This author regards this approach as promising, especially in view of the application of variational techniques to certain evolutionary problems. This may also be compared with Ghoussoub's approach [37].

The point of view of the present work rests on a two-scale approach, that was applied to quasilinear models of continuum mechanics, electromagnetism and heat conduction, see e.g. [62,65,66,70,71] and references therein.

Vector free boundary models of nonlinear electromagnetic processes have been studied e.g. in [9–11,60,64]; all in all they have received much less attention than scalar Stefan-type problems. The homogenization of the corresponding univariate model was dealt with in [12]. Homogenization of the quasilinear Maxwell system in a three-dimensional body without symmetry hypotheses was studied in [64]. The setting considered in the present article is less general than that of the latter work; however, the homogenization is here achieved by a different procedure. Moreover, at variance with [64], here the maximal monotone relation is not assumed to be cyclically monotone, and the Fitzpatrick theory and  $\Gamma$ -convergence are applied. The homogenization of a hysteresis-free quasi-stationary model of ferromagnetism was also addressed in [70] assuming the magnetostatic equations, instead of the eddy-current approximation as in this paper. Moreover that work dealt with the homogenization of the relaxation dynamics

$$\mathcal{A}(x)\vec{B}_t + \alpha^{-1}(\vec{B}, x) \ni \vec{H}, \quad (1.8)$$

which is different from both (1.4) and (1.5).

## 2. Fitzpatrick's theory and scale-transformations

In this section we briefly outline the variational formulation of maximal monotone operators due to Fitzpatrick, and review some results of scale-transformations, that we shall use in this work.

**Fitzpatrick's theory.** Let  $B$  be a real Banach space. For any mapping  $\alpha : B \rightarrow \mathcal{P}(B')$  with nonempty graph  $A := \{(\xi, \xi') : \xi' \in \alpha(\xi)\}$ , in [34] Fitzpatrick introduced the convex and lower semicontinuous function

$$\begin{aligned} f_\alpha(\xi, \xi') &:= \langle \xi', \xi \rangle + \sup\{\langle \xi' - \xi'_0, \xi_0 - \xi \rangle : (\xi_0, \xi'_0) \in A\} \\ &= \sup\{\langle \xi', \xi_0 \rangle + \langle \xi'_0, \xi \rangle - \langle \xi'_0, \xi_0 \rangle : (\xi_0, \xi'_0) \in A\} \quad \forall (\xi, \xi') \in B \times B', \end{aligned} \quad (2.1)$$

which is now named after him, and proved the following result.

**Theorem 2.1.** (See [34].)

(i)  $\alpha$  is monotone if and only if

$$f_\alpha(\xi, \xi') = \langle \xi', \xi \rangle \quad \forall (\xi, \xi') \in A; \quad (2.2)$$

(ii)  $\alpha$  is maximal monotone if and only if

$$f_\alpha(\xi, \xi') \geq \langle \xi', \xi \rangle \quad \forall (\xi, \xi') \in B \times B', \quad (2.3)$$

$$f_\alpha(\xi, \xi') = \langle \xi', \xi \rangle \Leftrightarrow (\xi, \xi') \in A. \quad (2.4)$$

Note that, by (2.3),  $f_\alpha(\xi, \xi') = \langle \xi', \xi \rangle$  is tantamount to  $f_\alpha(\xi, \xi') \leq \langle \xi', \xi \rangle$ . By applying this result, in this paper we shall meet several inequalities that are equivalent to the corresponding equality.

The system (2.3), (2.4) extends the following classical statement, that applies to cyclically monotone mappings and is known as the *Fenchel inequality*, see e.g. [32,33,38,39,54]. If  $F : B \rightarrow \mathbf{R} \cup \{+\infty\}$  is a proper function (i.e.,  $F \not\equiv +\infty$ ), then, denoting its convex conjugate function by  $F^*$  and its subdifferential by  $\partial F$ ,

$$F(\xi) + F^*(\xi') \geq \langle \xi', \xi \rangle \quad \forall (\xi, \xi') \in B \times B', \quad (2.5)$$

$$F(\xi) + F^*(\xi') = \langle \xi', \xi \rangle \Leftrightarrow \xi' \in \partial F(\xi).$$

Generalizing Theorem 2.1, nowadays one says that a lower semicontinuous and convex function  $f : B \times B' \rightarrow \mathbf{R} \cup \{+\infty\}$  represents a (necessarily monotone) operator  $\alpha$  whenever it fulfills the system (2.3), (2.4). Any maximal monotone operator  $\alpha$  is thus represented by its Fitzpatrick function  $f_\alpha$ . We may thus provide a variational formulation of the corresponding maximal monotone relation:

$$\begin{aligned} \text{setting } \mathcal{J}(\xi, \xi') &:= f(\xi, \xi') - \langle \xi', \xi \rangle \quad \forall (\xi, \xi') \in B \times B', \\ \xi' \in \alpha(\xi) &\Leftrightarrow \mathcal{J}(\xi, \xi') = \inf \mathcal{J} (= 0). \end{aligned} \quad (2.6)$$

We emphasize that this infimum necessarily vanishes, by Fitzpatrick's Theorem 2.1. This variational approach may be used to prove existence of a solution. It also enables one to apply De Giorgi's notion of  $\Gamma$ -convergence to problems with noncyclically monotone operators; see [68–71].

**Example.** Theorem 2.1 also applies to maximal monotone operators that occur in evolutionary equations, e.g.,

$$u_t + \mathcal{A}(u) \ni g \text{ with } \mathcal{A} \text{ maximal monotone and } g \text{ prescribed.} \quad (2.7)$$

More precisely, let us assume that we are given a triple of real Banach spaces

$$V \subset H = H' \subset V' \text{ with continuous and dense injections.} \quad (2.8)$$

Let us define the operator  $D_t$  as the time-derivative in the sense of vector-valued distributions  $]0, T[ \rightarrow V'$ , for a fixed  $T > 0$ . Let us fix any  $q \in [2, +\infty[$ , and set

$$\begin{aligned} X_0^q &:= \{v \in L^q(0, T; V) \cap W^{1, q'}(0, T; V') : v(0) = 0\} \quad (q' := q/(q-1)), \\ \alpha(v) &= D_t v \quad \forall v \in X_0^q. \end{aligned} \quad (2.9)$$

(The elements of  $X_0^q$  may be identified to continuous functions  $[0, T] \rightarrow H$ .) This operator is clearly linear and positive in  $X_0^q$ , hence maximal monotone, although not cyclically monotone. Let us assume that  $\mathcal{A}$  is a maximal monotone operator  $L^q(0, T; V) \rightarrow \mathcal{P}(L^{q'}(0, T; V'))$  and is defined everywhere, so that

$$D_t + \mathcal{A}: X_0^q \rightarrow \mathcal{P}(L^{q'}(0, T; V')) \subset \mathcal{P}((X_0^q)') \text{ is maximal monotone.} \quad (2.10)$$

We claim that, if  $f_{\mathcal{A}}: L^q(0, T; V) \times L^{q'}(0, T; V') \rightarrow \mathbf{R} \cup \{+\infty\}$  is a representative function of  $\mathcal{A}$ , then

$$\begin{aligned} f: X_0^q \times (X_0^q)' &\rightarrow \mathbf{R} \cup \{+\infty\}: \\ (\xi, \xi') &\mapsto \begin{cases} f_{\mathcal{A}}(\xi, \xi' - D_t \xi) + \frac{1}{2} \|\xi(T)\|_H^2 & \text{if } \xi' \in L^{q'}(0, T; V'), \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \quad (2.11)$$

is a representative function of  $D_t + \mathcal{A}$ , as it may be checked directly. For this example see also, e.g., [16,47,68].

One might also deal with a nonhomogeneous initial condition  $v(0) = v^0$ , for a prescribed  $v^0 \in H$ , at the only expense of replacing  $X_0^q$  by an affine space.

The present discussion may easily be extended to several other equations, e.g.,

$$\int_0^t u(\tau) d\tau + \mathcal{A}(u) \ni g, \quad (2.12)$$

$$u_{tt} + \mathcal{A}(u_t) + \Lambda u \ni g \quad (\text{with } \Lambda: H \rightarrow H \text{ linear and cyclically monotone}), \quad (2.13)$$

$$u_t + u_x + \mathcal{A}(u) \ni g \quad (\text{for } x \in I \subset \mathbf{R}). \quad (2.14)$$

Further examples may be found e.g. in [69].

**Upscaling and downscaling of monotone operators.** In the remainder of this section we review some results of [71], that will be used in this paper. Henceforth we shall assume that  $B$  is a separable reflexive real Banach space, and that  $p, p' \in ]1, +\infty[$  are conjugate indices. Dealing with any space of integrable functions of  $y \in \mathcal{Y}$  (the unit torus, see Appendix A), we shall label the subspace of functions having vanishing mean by appending the index  $*$ , and identify  $B$  ( $B'$ , resp.) with the subspace of  $L^p(\mathcal{Y}; B)$  ( $L^{p'}(\mathcal{Y}; B')$ , resp.) of constant mappings. For any  $v \in L^1(\mathcal{Y}; B)$  let us define the average and oscillating components  $\widehat{v}$  and  $\widetilde{v}$  as in (A.5). We shall denote by  $\langle \cdot, \cdot \rangle$  ( $\langle \cdot, \cdot \rangle$ , resp.) the canonical duality pairing between  $B'$  and  $B$  ( $L^p(\mathcal{Y}; B)$  and  $L^{p'}(\mathcal{Y}; B')$ , resp.). Obviously,

$$\langle \widehat{v}, \widetilde{z} \rangle = \langle \widetilde{v}, \widehat{z} \rangle = 0 \quad \forall v \in L^p(\mathcal{Y}; B), \quad \forall z \in L^{p'}(\mathcal{Y}; B'). \quad (2.15)$$

Let  $\mathcal{V}$  and  $\mathcal{Z}$  be such that

$$\mathcal{V} \text{ (}\mathcal{Z}\text{, resp.) is a subspace of } L_*^p(\mathcal{Y}; B) \text{ (}L_*^{p'}(\mathcal{Y}; B')\text{, resp.)}, \quad (2.16)$$

$$\langle z, v \rangle = 0 \quad \forall v \in \mathcal{V}, \quad \forall z \in \mathcal{Z}. \quad (2.17)$$

(We shall see an example in Section 6.) Let us denote by  $\mathcal{B}(S)$  ( $\mathcal{L}(S)$ , resp.) the  $\sigma$ -algebra of the Borel- (Lebesgue-, resp.) measurable subsets of a topological (measurable, resp.) space over any set  $S \neq \emptyset$ . Let us also denote by  $\mathcal{C}_1 \otimes \mathcal{C}_2$  the  $\sigma$ -algebra generated by any pair of  $\sigma$ -algebras  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . We shall say that  $\varphi$  is a *measurable representative function* (of some monotone operator) whenever

$$\begin{cases} \varphi : B \times B' \times \mathcal{Y} \rightarrow \mathbf{R} \cup \{+\infty\} \text{ is measurable w.r.t. } \mathcal{B}(B \times B') \otimes \mathcal{L}(\mathcal{Y}), \\ \varphi(\cdot, \cdot, y) \text{ is convex and lower semicontinuous for a.e. } y, \\ \varphi(\xi, \xi', y) \geq \langle \xi', \xi \rangle \quad \forall (\xi, \xi') \in B \times B', \text{ for a.e. } y \in \mathcal{Y}. \end{cases} \quad (2.18)$$

We shall denote by  $\|\cdot\|$  the norm of both  $B$  and  $B'$ .

**Proposition 2.2.** (See [71].) Let  $\mathcal{V}$ ,  $\mathcal{Z}$  and  $\varphi$  fulfill (2.16)–(2.18). If

$$\begin{aligned} \exists C > 0, \exists h \in L^1(\mathcal{Y}): \forall (\xi, \xi') \in B \times B', \text{ for a.e. } y \in \mathcal{Y}, \\ \varphi(\xi, \xi', y) \geq C(\|\xi\|^p + \|\xi'\|^{p'}) + h(y), \end{aligned} \quad (2.19)$$

then

$$\varphi_0(\xi, \xi') := \inf_{\mathcal{Y}} \left\{ \int \varphi(\xi + v(y), \xi' + z(y), y) dy : (v, z) \in \mathcal{V} \times \mathcal{Z} \right\} \quad \forall (\xi, \xi') \in B \times B' \quad (2.20)$$

is a representative function in  $B \times B'$  (of some monotone operator), and is coercive, i.e., the set  $\{(\xi, \xi') \in B \times B' : \varphi_0(\xi, \xi') \leq M\}$  is bounded, for any  $M \in \mathbf{R}$ .

We shall refer to  $\varphi_0$  as the *upscaled representative function* of  $\varphi$ .

**Theorem 2.3** (Upscaling and downscaling of representable operators). (See [71].) Let  $\mathcal{V}$ ,  $\mathcal{Z}$  and  $\varphi$  fulfill (2.16)–(2.19); define  $\varphi_0$  as in (2.20), and denote by

$$\alpha : L^p(\mathcal{Y}; B) \times \mathcal{Y} \rightarrow \mathcal{P}(L^{p'}(\mathcal{Y}; B')), \quad \alpha_0 : B \rightarrow \mathcal{P}(B') \quad (2.21)$$

the operators that are respectively represented by  $\varphi$  and  $\varphi_0$ . Then:

(i) (Downscaling or integration) If  $u \in B + \mathcal{V}$  and  $w \in B' + \mathcal{Z}$ , then

$$w(y) \in \alpha(u(y), y) \quad \text{for a.e. } y \in \mathcal{Y} \quad \Rightarrow \quad \widehat{w} \in \alpha_0(\widehat{u}); \quad (2.22)$$

(ii) (Upscaling or disintegration) On the other hand, if

$$\bar{u} \in B, \quad \bar{w} \in B', \quad \bar{w} \in \alpha_0(\bar{u}), \quad (2.23)$$

then there exist  $u \in L^p(\mathcal{Y}; B)$  and  $w \in L^{p'}(\mathcal{Y}; B')$  such that

$$\widetilde{u} \in \mathcal{V}, \quad \widetilde{w} \in \mathcal{Z}, \quad \widehat{u} = \bar{u}, \quad \widehat{w} = \bar{w}, \quad (2.24)$$

$$w(y) \in \alpha(u(y), y) \quad \text{for a.e. } y \in \mathcal{Y}. \quad (2.25)$$

In conclusion, for any  $(\bar{u}, \bar{w}) \in B \times B'$ ,

$$\bar{w} \in \alpha_0(\bar{u}) \Leftrightarrow \begin{cases} \exists (u, w) \in L^p(\mathcal{Y}; B) \times L^{p'}(\mathcal{Y}; B') \text{ such that} \\ w(y) \in \alpha(u(y), y) \text{ for a.e. } y \in \mathcal{Y}, \\ \bar{u} = \int_{\mathcal{Y}} u(y) dy, \quad \bar{w} = \int_{\mathcal{Y}} w(y) dy. \end{cases} \quad (2.26)$$

**Theorem 2.4** (Upscaling of maximal monotone operators). (See [71].) Let  $\mathcal{V}$ ,  $\mathcal{Z}$  and  $\varphi$  fulfill (2.16)–(2.18). Let us assume that

$$\begin{aligned} \exists C_1, C_2 > 0, \exists h_1, h_2 \in L^1(\mathcal{Y}): \text{ for a.e. } y \in \mathcal{Y}, \forall (\xi, \xi') \in B \times B', \\ C_1(\|\xi\|^p + \|\xi'\|^{p'}) + h_1(y) \leq \varphi(\xi, \xi', y) \leq C_2(\|\xi\|^p + \|\xi'\|^{p'}) + h_2(y), \end{aligned} \quad (2.27)$$

and define  $\varphi_0$  as in (2.20). If  $\varphi(\cdot, \cdot, y)$  represents a maximal monotone operator for a.e.  $y \in \mathcal{Y}$ , then  $\varphi_0$  also represents a maximal monotone operator.

### 3. A model of ferromagnetic evolution

In this section we outline a hysteresis-free model of ferromagnetism, and formulate a weak problem in the framework of Sobolev spaces.

**The Maxwell–Ohm equations and the constitutive relation.** Henceforth we shall mark vectors of  $\mathbf{R}^3$  by an arrow, as it often occurs in the physical literature. We assume that a domain  $\Omega$  of  $\mathbf{R}^3$  is occupied by a ferromagnetic metal surrounded by an insulator (e.g., air), denote the magnetic field by  $\vec{H}$ , the magnetization by  $\vec{M}$ , and the magnetic induction by  $\vec{B}$ ; in Gauss units,  $\vec{B} = \vec{H} + 4\pi\vec{M}$ . We also denote the electric field by  $\vec{E}$ , the electric current density by  $\vec{J}$ , an applied electromotive force by  $\vec{E}_a$ , and the speed of light in vacuum by  $c$ . We assume that the electric resistivity is a positive-definite, symmetric  $3 \times 3$ -tensor  $A = A(x)$ , with components  $A_{ij}(x)$ .

Because of the high conductivity, in metals the displacement current  $\vec{D}_t := \partial \vec{D} / \partial t$  is usually dominated by the Ohmic current, provided that the onset of high frequencies is excluded; the term  $\vec{D}_t$  is accordingly neglected (so-called *eddy-current approximation*). The Ampère, Faraday, Gauss and Ohm laws respectively read

$$c \nabla \times \vec{H} = 4\pi \vec{J} \quad \text{in } \Omega_T := \Omega \times ]0, T[, \quad (3.1)$$

$$c \nabla \times \vec{E} = -\vec{B}_t \quad \text{in } \Omega_T, \quad (3.2)$$

$$\nabla \cdot \vec{B} = 0 \quad \text{in } \Omega_T \quad (\nabla \cdot := \text{div}), \quad (3.3)$$

$$A(x) \cdot \vec{J} = \vec{E} + \vec{E}_a \quad \text{in } \Omega_T. \quad (3.4)$$

We also assume the boundary condition

$$\vec{\nu} \times \vec{H} = \vec{0} \quad \text{on } (\partial\Omega) \times ]0, T[, \quad (3.5)$$

here by  $\vec{\nu}$  we denote a normal vector field on  $\partial\Omega$ . In order to simplify the display of formulas, we shall drop the constants  $4\pi$  and  $c$ . By (3.1), (3.2) and (3.4), we thus get

$$\vec{B}_t + \nabla \times [A(x) \cdot \nabla \times \vec{H} - \vec{E}_a] = \vec{0} \quad \text{in } \Omega_T, \quad (3.6)$$

that we couple with (3.5) and with the initial condition

$$\vec{B}(\cdot, 0) = \vec{B}^0 \quad \text{in } \Omega, \quad (3.7)$$

for a prescribed divergence-free field  $\vec{B}^0$ . (In passing, note that the Gauss law (3.3) is then implicit in (3.6).) We set



$$(v * 1)(t) := \int_0^t v(\tau) d\tau \quad \forall t \in \mathbf{R}, \quad \forall v \in L^1(\mathbf{R}), \quad (3.8)$$

$$\vec{G} := \vec{B}^0 + \nabla \times \vec{E}_{\text{app}} * 1 \quad \text{in } \Omega_T;$$

note that the field  $\vec{G}$  is divergence-free. We then reformulate the Cauchy problem (3.6) and (3.7) as

$$\vec{B} + \nabla \times (A(x) \cdot \nabla \times \vec{H} * 1) = \vec{G} \quad \text{in } \Omega_T. \quad (3.9)$$

Let us now come to the constitutive relation between the fields  $\vec{B}$  and  $\vec{H}$ . Soft iron, mild steel and some other ferromagnetic materials are characterized by a very narrow and high hysteresis loop; more precisely, as these fields move along the loop, the order of magnitude of the variation of  $|\vec{B}|$  is much larger than that of  $|\vec{H}|$ . In first approximation, the loop may then be replaced by a maximal monotone graph of the form

$$\vec{B} \in \vec{H} + \vec{\beta}(\vec{H}, x) =: \vec{\alpha}(\vec{H}, x) \quad \text{in } \Omega_T. \quad (3.10)$$

Here  $\vec{\beta}(\cdot, x)$  may be the subdifferential of a lower semicontinuous convex function (e.g., the modulus function), so that  $\vec{\alpha}(\cdot, x)$  is maximal cyclically monotone. However, in this paper we just assume that  $\vec{\alpha}$  is maximal monotone, and use the techniques of Section 2. In Section 8, we shall also consider the alternative relaxation dynamics (1.4) and (1.5).

**Weak formulation.** Let us set

$$V := \{\vec{v} \in L^2(\Omega)^3: \nabla \times \vec{v} \in L^2(\Omega)^3, \vec{v} \times \vec{v} = \vec{0} \text{ on } \partial\Omega\}; \quad (3.11)$$

this trace condition is set in  $H^{-1/2}(\partial\Omega)^3$  (the dual of the fractional Sobolev space  $H^{1/2}(\partial\Omega)^3$ ).  $V$  is a Hilbert space equipped with the graph norm. We claim that the operator

$$L^2(0, T; V) \rightarrow H^1(0, T; V'): \vec{v} \mapsto \nabla \times [A(x) \cdot \nabla \times \vec{v} * 1]$$

is linear, continuous and maximal monotone: it is actually positive and defined on the whole space. Indeed, denoting by  $\langle\langle \cdot, \cdot \rangle\rangle$  the duality product between  $L^2(0, T; V)$  and  $L^2(0, T; V')$ , we have

$$\begin{aligned} \langle\langle \nabla \times (A(x) \cdot \nabla \times \vec{v} * 1), v \rangle\rangle &= \int_0^T dt \int_{\Omega} (A(x) \cdot \nabla \times \vec{v} * 1) \cdot \nabla \times v \, dx \\ &= \frac{1}{2} \int_0^T dt \frac{d}{dt} \int_{\Omega} |A(x)^{1/2} \cdot \nabla \times \vec{v} * 1|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega} |A(x)^{1/2} \cdot \nabla \times \vec{v} * 1|^2(x, T) \, dx \geq 0. \end{aligned} \quad (3.12)$$

We shall be mainly concerned with processes in a periodic inhomogeneous material, and assume that  $\vec{\alpha}$ ,  $A$  and  $\vec{G}$  explicitly depend on  $x$  with period  $\varepsilon$  in each coordinate direction. Equivalently, we prescribe that these fields depend on  $y = x/\varepsilon$ , that we let range through the 3-dimensional (flat) unit torus  $\mathcal{Y}$ . We also assume that

$$\vec{\alpha} : \mathbf{R}^3 \times \mathcal{Y} \rightarrow \mathcal{P}(\mathbf{R}^3) \text{ is } \mathcal{B}(\mathbf{R}^3) \otimes \mathcal{L}(\mathcal{Y})\text{-measurable,} \quad (3.13)$$

$$\vec{\alpha}(\cdot, y) \text{ is maximal monotone for a.e. } y \in \mathcal{Y}, \quad (3.14)$$

$$A \in L^\infty(\mathcal{Y})^{3 \times 3}, \quad A(y) \text{ is symmetric for a.e. } y \in \mathcal{Y}, \quad (3.15)$$

$$\exists C > 0: \sum_{i,j=1}^3 A_{ij}(y) v_i v_j \geq C |\vec{v}|^2 \quad \forall \vec{v} \in \mathbf{R}^3, \text{ for a.e. } y \in \mathcal{Y}, \quad (3.16)$$

$$\vec{G} \in L^2(\mathcal{Y}_T)^3, \quad \nabla \cdot \vec{G} = 0 \quad \text{in } \mathcal{D}'(\mathcal{Y}_T). \quad (3.17)$$

Next we set

$$\vec{\alpha}_\varepsilon(\cdot, x) := \vec{\alpha}(\cdot, x/\varepsilon), \quad A_\varepsilon(x) := A(x/\varepsilon), \quad \vec{G}_\varepsilon(x, \cdot) := \vec{G}(x/\varepsilon, \cdot) \quad \text{for a.e. in } \mathbf{R}^3, \quad (3.18)$$

in place of  $\vec{\alpha}(\cdot, x)$ ,  $A(x)$  and  $\vec{G}(x, \cdot)$ , and provide a weak formulation of the system (3.6) and (3.10).

**Problem 3.1 $_\varepsilon$ .** Find  $\vec{B}_\varepsilon, \vec{H}_\varepsilon \in L^2(\Omega_T)^3$  such that  $\vec{H}_\varepsilon * 1 \in L^2(0, T; V)$  and

$$\vec{B}_\varepsilon \in \vec{\alpha}_\varepsilon(\vec{H}_\varepsilon, x) \quad \text{a.e. in } \Omega_T, \quad (3.19)$$

$$\iint_{\Omega_T} [(\vec{B}_\varepsilon - \vec{G}_\varepsilon) \cdot \vec{v} + A_\varepsilon \cdot \nabla \times \vec{H}_\varepsilon * 1 \cdot \nabla \times \vec{v}] dx dt = 0 \quad \forall \vec{v} \in L^2(0, T; V). \quad (3.20)$$

The inclusion (3.19) is tantamount to the following variational inequality:

$$\begin{aligned} & \iint_{\Omega_T} (T - t)(\vec{B}_\varepsilon - \vec{w}) \cdot (\vec{H}_\varepsilon - \vec{v}) dx dt \geq 0 \\ & \forall (\vec{v}, \vec{w}) \in (L^2(\Omega_T)^3)^2 \text{ such that } \vec{w} \in \vec{\alpha}_\varepsilon(\vec{v}, \cdot) \text{ a.e. in } \Omega_T \end{aligned} \quad (3.21)$$

(the factor  $(T - t)$  might be dropped, but it will turn out to be convenient). Eq. (3.20) is clearly equivalent to

$$\vec{B}_\varepsilon + \nabla \times (A_\varepsilon(x) \cdot \nabla \times \vec{H}_\varepsilon * 1) = \vec{G}_\varepsilon \quad \text{in } V', \text{ for a.e. } t \in ]0, T[. \quad (3.22)$$

**Remark.** The integration in time has lead us to define an especially weak notion of solution, in which just the primitive of the field  $\vec{H}_\varepsilon$  is assumed to belong to the variational space  $V$ . This will allow us to deal with very weak assumptions on the data.

The next statement will be used in the next sections.

**Lemma 3.1.** Any solution  $(\vec{B}_\varepsilon, \vec{H}_\varepsilon)$  of Problem 3.1 $_\varepsilon$  fulfills the equation

$$\iint_{\Omega_T} \left( (T - t)(\vec{B}_\varepsilon - \vec{G}_\varepsilon) \cdot \vec{H}_\varepsilon + \frac{1}{2} |A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2 \right) dx dt = 0. \quad (3.23)$$

**Proof.** We cannot select  $\vec{w} = \vec{H}_\varepsilon$  in (3.20), as this function does not have the necessary regularity. We then use a less direct procedure. For any  $h > 0$  and any function  $v \in L^2(\mathbf{R})$ , first we set

$$D_h^+ v(t) := \frac{v(t+h) - v(t)}{h}, \quad D_h^- v(t) := \frac{v(t) - v(t-h)}{h} \quad \forall t \in \mathbf{R}, \quad (3.24)$$

and note that

$$\int_a^b v(t) D_h^+ v(t) dt \leq \frac{1}{2} [v(b+h)^2 - v(a)^2], \quad \int_a^b v(t) D_h^- v(t) dt \geq \frac{1}{2} [v(b)^2 - v(a-h)^2].$$

Next we extend Eq. (3.22) and the fields  $\vec{B}_\varepsilon$ ,  $\vec{H}_\varepsilon$ ,  $\vec{G}_\varepsilon$  to any  $t < 0$  ( $t > T$ , resp.) with the value that they attain at 0 ( $T$ , resp.). Multiplying (3.22) by  $D_h^+ \vec{H}_\varepsilon * 1$  (which is an element of  $L^2(0, T; V)$ ) and integrating in time, we get

$$\iint_{\Omega_t} (\vec{B}_\varepsilon - \vec{G}_\varepsilon) \cdot D_h^+ \vec{H}_\varepsilon * 1 dx d\tau + \frac{1}{2} \int_{\Omega} |A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2(x, t+h) dx \geq 0$$

for a.e. in  $t \in ]h, T[$ .

(3.25)

A further integration in  $]0, T[$  and then the passage to the limit as  $h \rightarrow 0$  yield

$$\iint_{\Omega_T} \left\{ (T-t)(\vec{B}_\varepsilon - \vec{G}_\varepsilon) \cdot \vec{H}_\varepsilon + \frac{1}{2} |A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2 \right\} dx dt \geq 0. \quad (3.26)$$

Multiplying (3.22) by  $D_h^- \vec{H}_\varepsilon * 1$  and proceeding similarly, we get the opposite inclusion.  $\square$

#### 4. Existence of a solution and reformulation

In this section we show the existence of a solution of Problem 3.1 $_\varepsilon$ , via a classical argument. We then reformulate the constitutive relation (3.19) via Fitzpatrick's theory.

**Proposition 4.1.** Assume that (3.13)–(3.18) are fulfilled and that

$$\exists C_1 > 0, \exists h_1 \in L^2(\mathcal{Y}): \forall \vec{v} \in \mathbf{R}^3, \forall \vec{w} \in \vec{\alpha}(\vec{v}, \cdot),$$

$$\vec{w} \cdot \vec{v} \geq C_1 |\vec{v}|^2 + h_1(y) \quad \text{for a.e. } y, \quad (4.1)$$

$$\exists C_2 > 0, \exists h_2 \in L^2(\mathcal{Y}): \forall \vec{v} \in \mathbf{R}^3, \forall \vec{w} \in \vec{\alpha}(\vec{v}, \cdot),$$

$$|\vec{w}| \leq C_2 |\vec{v}| + h_2(y) \quad \text{for a.e. } y. \quad (4.2)$$

Then there exists a solution of Problem 3.1 $_\varepsilon$ . The field  $\vec{B}$  is uniquely determined; if  $\vec{\alpha}^{-1}(\cdot, y)$  is single-valued for a.e.  $y \in \mathcal{Y}$ , then  $\vec{H}$  is also unique.

The hypotheses (4.1) and (4.2) are consistent with the above model, see (3.10), as the mapping  $\beta(\vec{H}, x)$  represents the magnetization, which is a bounded field.

**Proof.** We display this standard argument, for the sake of completeness. Throughout this proof we shall drop the index  $\varepsilon$ , in order to render formulas more readable. We proceed through approximation, derivation of uniform estimates, and passage to the limit.

(i) *Approximation.* We fix any  $m \in \mathbf{N}$ , set

$$k := \frac{T}{m}, \quad \vec{G}_m^n := \frac{1}{k} \int_{(n-1)k}^{nk} \vec{G}(\cdot, \xi) d\xi \quad \text{for } n = 1, \dots, m,$$

and introduce an implicit time-discretization scheme of Problem 3.1 $_\varepsilon$ .

**Problem 3.1<sub>em</sub>.** Find  $\vec{B}_m^n \in L^2(\Omega)^3$  and  $\vec{H}_m^n \in V$  ( $n = 1, \dots, m$ ) such that

$$\vec{B}_m^n + k \nabla \times \left( A \cdot \nabla \times \sum_{j=1}^n \vec{H}_m^j \right) = \vec{G}_m^n \quad \text{in } V', \text{ for } n = 1, \dots, m, \quad (4.3)$$

$$\vec{B}_m^n(x) \in \alpha(\vec{H}_m^n(x), x) \quad \text{for a.e. } x \in \Omega. \quad (4.4)$$

This system may be solved stepwise. At each step, by eliminating  $\vec{B}_m^n$  we get an equation of the form

$$\Lambda_m^n(\vec{H}_m^n) = \vec{G}_m^n \quad \text{in } V', \text{ for } n = 1, \dots, m, \quad (4.5)$$

for a maximal monotone and coercive operator  $\Lambda_m^n: V \rightarrow V'$ . By a classical theory, see e.g. [6,15,17], this equation has a solution  $\vec{H}_m^n$ . By (4.3), this determines  $\vec{B}_m^n \in V'$ ; moreover, by (4.2) and (4.4),  $\vec{B}_m^n \in L^2(\Omega)^3$ .

(ii) *A priori estimates.* In order to rewrite the time-discretized Problem 3.1<sub>em</sub> in continuous form, for any family  $\{\phi_m^n\}_{n=0, \dots, m}$  we define the piecewise-linear and piecewise-constant interpolate functions:

$$\begin{aligned} \phi_m(t) &:= \phi_m^{n-1} + [t - (n-1)h]\phi_m^n, & \bar{\phi}_m(t) &:= \phi_m^n \\ &\forall t \in [(n-1)h, nh], \text{ for } n = 1, \dots, m. \end{aligned} \quad (4.6)$$

Eq. (4.3) then reads

$$\vec{B}_m + \nabla \times (A \cdot \nabla \times \bar{\vec{H}}_m * 1) = \vec{G}_m \quad \text{in } V', \text{ a.e. in } ]0, T[. \quad (4.7)$$

Let us multiply this equation by  $\bar{\vec{H}}_m$  ( $\in L^2(0, T; V)$ ), and then integrate in time. Denoting by  $A^{1/2}$  the square root of the positive-definite, symmetric tensor-function  $A$ , we get

$$\int_{\Omega_t} (\vec{B}_m - \vec{G}_m) \cdot \bar{\vec{H}}_m dx d\tau + \frac{1}{2} \int_{\Omega} |A(x)^{1/2} \cdot \nabla \times \bar{\vec{H}}_m * 1|^2(x, t) dx = 0 \quad \text{for a.e. } t \in ]0, T[. \quad (4.8)$$

By the hypotheses (3.16), (3.17), (4.1) and (4.2), this yields

$$\|\vec{B}_m\|_{L^2(\Omega_T)^3}, \|\bar{\vec{H}}_m\|_{L^2(\Omega_T)^3}, \|\bar{\vec{H}}_m * 1\|_{L^\infty(0, T; V)} \leq \text{Constant}. \quad (4.9)$$

(iii) *Passage to the limit.*

By the above uniform estimates there exist  $\vec{B}, \vec{H} \in L^2(\Omega_T)^3$  such that  $\vec{H} * 1 \in L^\infty(0, T; V)$  and, as  $m \rightarrow \infty$  along a suitable sequence,<sup>2</sup>

$$\vec{B}_m \rightharpoonup \vec{B}, \quad \vec{H}_m, \bar{\vec{H}}_m \rightharpoonup \vec{H} \quad \text{in } L^2(\Omega_T)^3, \quad (4.10)$$

$$\bar{\vec{H}}_m * 1 \xrightarrow{*} \vec{H} * 1 \quad \text{in } L^\infty(0, T; V). \quad (4.11)$$

<sup>2</sup> We denote the strong, weak, and weak star convergence respectively by  $\rightarrow, \rightharpoonup, \xrightarrow{*}$ .

Eq. (3.20) then follows. In order to derive the inclusion (3.19), we note that (4.4) is equivalent to

$$\begin{aligned} & \iint_{\Omega_T} (T-t)(\vec{B}_m - \vec{w}) \cdot (\vec{H}_m - \vec{v}) \, dx \, dt \geq 0 \\ & \forall (\vec{v}, \vec{w}) \in (L^2(\Omega_T)^3)^2 \text{ such that } \vec{w} \in \vec{\alpha}(\vec{v}, \cdot) \text{ a.e. in } \Omega_T. \end{aligned} \quad (4.12)$$

Moreover, by the lower semicontinuity of convex functionals,

$$\begin{aligned} & \limsup_{h \rightarrow 0} \iint_{\Omega_T} (T-t) \vec{B}_m \cdot \vec{H}_m \, dx \, dt \\ & \stackrel{(4.7)}{=} - \liminf_{m \rightarrow +\infty} \iint_{\Omega_T} \left\{ \int_0^t [(A \cdot \nabla \times \vec{H}_m * 1) \cdot \nabla \times \vec{H}_m] \, d\tau + (T-t) \vec{G}_m \cdot \vec{H}_m \right\} \, dx \, dt \\ & = -\frac{1}{2} \liminf_{m \rightarrow +\infty} \iint_{\Omega_T} |A^{1/2} \cdot \nabla \times \vec{H}_m * 1|^2 \, dx \, dt - \lim_{m \rightarrow +\infty} \iint_{\Omega_T} (T-t) \vec{G}_m \cdot \vec{H}_m \, dx \, dt \\ & \leq -\frac{1}{2} \iint_{\Omega_T} |A^{1/2} \cdot \nabla \times \vec{H} * 1|^2 \, dx \, dt - \iint_{\Omega_T} (T-t) \vec{G} \cdot \vec{H} \, dx \, dt \\ & \stackrel{(3.23)}{=} \iint_{\Omega_T} (T-t) \vec{B} \cdot \vec{H} \, dx \, dt. \end{aligned} \quad (4.13)$$

By passing to the superior limit in (4.12), we thus retrieve (3.21), that is, the inclusion (3.19).

We are left with the proof of uniqueness. Let  $(\vec{B}_i, \vec{H}_i)$  ( $i = 1, 2$ ) be two solutions. We would like to select  $\vec{v} = \vec{H}_1 - \vec{H}_2$  in (3.20), and then use the monotonicity of  $\vec{\alpha}(\cdot, x)$  to infer that

$$\iint_{\Omega_T} |A(x)^{1/2} \cdot \nabla \times (\vec{H}_1 - \vec{H}_2) * 1|^2 \, dx \, dt \leq 0. \quad (4.14)$$

This inequality entails that  $\nabla \times \vec{H}_1 * 1 = \nabla \times \vec{H}_2 * 1$  a.e. in  $\Omega_T$ , whence also  $\vec{B}_1 = \vec{B}_2$  by (3.22). If  $\vec{\alpha}^{-1}(\cdot, y)$  is single-valued for a.e.  $y \in \mathcal{Y}$ , then by (3.19)  $\vec{H}_1 = \vec{H}_2$  a.e. in  $\Omega_T$ .

But this choice of  $\vec{v}$  is not admissible, since a priori  $\vec{H}_1 - \vec{H}_2 \notin L^2(0, T; V)$ . The time-discretization argument of Lemma 3.1 however makes this procedure rigorous. (We omit these details, that closely mimic that proof.)  $\square$

**Remarks.** (i) If we assume that  $\vec{E}_{\text{app } \varepsilon} \in L^2(\Omega_T)^3$  and  $\vec{B}^{0\varepsilon} \in L^2(\Omega)^3$  (so that  $\vec{G}_\varepsilon \in H^1(0, T; V')$ , see (3.8)) and  $\vec{H}^{0\varepsilon} \in V$ , then further estimates can be derived by taking the incremental ratio in time of the approximate equation (4.3), and multiplying it by the incremental ratio in time of  $\vec{H}_m$ . By the monotonicity of  $\vec{\alpha}(\cdot, x)$ , this entails the uniform estimate

$$\|\vec{B}_m\|_{H^1(0, T; L^2(\Omega)^3)}, \|\vec{H}_m\|_{H^1(0, T; L^2(\Omega)^3) \cap L^\infty(0, T; V)} \leq \text{Constant}, \quad (4.15)$$

which corresponds to an extra-regularity for the solution. However we shall refrain from assuming these further hypotheses, and proceed without the corresponding extra-regularity. (The enhanced generality allows one to extend the present results to more general equations.)

(ii) Whenever the mapping  $\vec{\alpha}$  is cyclically monotone, further a priori estimates can be derived by multiplying the approximate equation (4.3) by the incremental ratio in time of  $\vec{H}_m$ .

**Reformulation of (3.19).** Let  $\varphi(\cdot, \cdot, y)$  be the Fitzpatrick function of  $\vec{\alpha}(\cdot, y)$ , see (2.1), for a.e.  $y \in \mathcal{Y}$ ; let us then set  $\varphi_\varepsilon(\cdot, \cdot, x) := \varphi(\cdot, \cdot, x/\varepsilon)$ . Thus

$$\varphi : \mathbf{R}^3 \times \mathbf{R}^3 \times \mathcal{Y} \rightarrow \mathbf{R} \cup \{+\infty\} \text{ is } \mathcal{B}(\mathbf{R}^3 \times \mathbf{R}^3) \otimes \mathcal{L}(\mathcal{Y})\text{-measurable,} \quad (4.16)$$

$$\varphi(\cdot, \cdot, y) \text{ is convex and lower semicontinuous, for a.e. } y, \quad (4.17)$$

$$\varphi(\vec{\xi}, \vec{\xi}', y) \geq \langle \vec{\xi}', \vec{\xi} \rangle \quad \forall (\vec{\xi}, \vec{\xi}') \in \mathbf{R}^3 \times \mathbf{R}^3, \text{ for a.e. } y, \quad (4.18)$$

$$\varphi(\vec{\xi}, \vec{\xi}', y) = \langle \vec{\xi}', \vec{\xi} \rangle \Leftrightarrow \vec{\xi}' \in \alpha(\vec{\xi}, y) \quad \text{for a.e. } y. \quad (4.19)$$

Let us also assume that

$$\begin{aligned} \exists C_3 > 0, \exists h_3 \in L^1(\mathcal{Y}): \forall (\vec{\xi}, \vec{\xi}') \in \mathbf{R}^3 \times \mathbf{R}^3, \\ \varphi(\vec{\xi}, \vec{\xi}', y) \leq C_3(|\vec{\xi}|^2 + |\vec{\xi}'|^2) + h_3(y) \quad \text{for a.e. } y \in \mathcal{Y}, \end{aligned} \quad (4.20)$$

$$\begin{aligned} \exists C_4 > 0, \exists h_4 \in L^1(\mathcal{Y}): \forall (\vec{\xi}, \vec{\xi}') \in \mathbf{R}^3 \times \mathbf{R}^3, \\ \varphi(\vec{\xi}, \vec{\xi}', y) \geq C_4(|\vec{\xi}|^2 + |\vec{\xi}'|^2) + h_4(y) \quad \text{for a.e. } y \in \mathcal{Y}. \end{aligned} \quad (4.21)$$

By (4.18) and (4.19), the inclusion (3.21) is equivalent to

$$\iint_{\Omega_T} (T-t) [\varphi_\varepsilon(\vec{H}_\varepsilon, \vec{B}_\varepsilon) - \vec{H}_\varepsilon \cdot \vec{B}_\varepsilon] dx dt \leq 0, \quad (4.22)$$

and also to the corresponding equality. Lemma 3.1 then yields the next statement.

**Proposition 4.2.** *In Problem 3.1<sub>ε</sub> the inclusion (3.19) may equivalently be replaced either by the inequality*

$$\Psi_\varepsilon(\vec{H}_\varepsilon, \vec{B}_\varepsilon) := \iint_{\Omega_T} \left\{ (T-t) [\varphi_\varepsilon(\vec{H}_\varepsilon, \vec{B}_\varepsilon) - \vec{G}_\varepsilon \cdot \vec{H}_\varepsilon] + \frac{1}{2} |A_\varepsilon^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2 \right\} dx dt \leq 0, \quad (4.23)$$

*or by the corresponding equality.*

(Eq. (3.19) is not equivalent to (4.23): it is rather the system (3.19) and (3.20) that is equivalent to (3.20) and (4.23).)

As  $\Psi_\varepsilon$  is nonnegative, by means of Eq. (3.9) we may eliminate the field  $\vec{B}_\varepsilon$ , and reformulate Problem 3.1<sub>ε</sub> as a minimization principle.

**Proposition 4.3.** *Problem 3.1<sub>ε</sub> is equivalent to the search for a field  $\vec{H}_\varepsilon$  that minimizes the functional*

$$\begin{aligned} \bar{\Psi}_\varepsilon(\vec{v}) := \iint_{\Omega_T} \left\{ (T-t) [\varphi_\varepsilon(\vec{v}, \vec{G}_\varepsilon - \nabla \times (A(x) \cdot \nabla \times \vec{v} * 1)) - \vec{G}_\varepsilon \cdot \vec{v}] + \frac{1}{2} |A_\varepsilon^{1/2} \cdot \nabla \times \vec{v} * 1|^2 \right\} dx dt \\ \forall \vec{v} \in L^2(\Omega_T)^3 \text{ such that } \vec{v} * 1 \in L^2(0, T; V), \end{aligned} \quad (4.24)$$

*and this is tantamount to  $\bar{\Psi}_\varepsilon(\vec{H}_\varepsilon) = 0$ .*

Existence of a solution might thus be proved via the direct method of the calculus of variations.

## 5. Two-scale formulation

In this section we derive a two-scale model by passing to the two-scale limit in Problem 3.1<sub>ε</sub>. This will provide a detailed representation of the physical process, at both the coarse- and fine-length scales, and will be the basis for the homogenization procedure of the next section.

We assume that

$$\vec{G}_\varepsilon \rightarrow \vec{G} \quad \text{in } L^2(\Omega_T)^3, \quad (5.1)$$

and introduce a two-scale model. At the fine scale we shall account for periodic conditions by denoting the 3-dimensional (flat) unit torus by  $\mathcal{Y}$ . We shall denote by  $\widehat{v}$  and  $\widetilde{v}$  the average and oscillating components of any field  $v$  as in (A.5), and define the partial gradients  $\nabla_x$  and  $\nabla_y$  as in Appendix A.

**Problem 5.1.** Find  $\vec{B}, \vec{H}, \vec{H}_\# \in L^2(\Omega_T \times \mathcal{Y})^3$  such that

$$\widehat{H} * 1 \in L^2(0, T; V), \quad \nabla_y \times \vec{H}_\# * 1 \in L^2(\Omega_T \times \mathcal{Y})^3, \quad (5.2)$$

$$\nabla_y \cdot \vec{B} = 0 \quad \text{in } \mathcal{D}'(\Omega_T \times \mathcal{Y}), \quad \nabla_y \times \vec{H} = \vec{0} \quad \text{in } \mathcal{D}'(\Omega_T \times \mathcal{Y})^3, \quad (5.3)$$

$$\nabla_y \cdot \vec{H}_\# = 0 \quad \text{in } \mathcal{D}'(\Omega_T \times \mathcal{Y}), \quad \int_{\mathcal{Y}} \vec{H}_\# dy = \vec{0} \quad \text{a.e. in } \Omega_T, \quad (5.4)$$

$$\vec{B} \in \vec{\alpha}(\vec{H}, y) \quad \text{a.e. in } \Omega_T \times \mathcal{Y}, \quad (5.5)$$

$$\iint_{\Omega_T} (\widehat{B} - \vec{G}) \cdot \vec{w} dx dt + \iiint_{\Omega_T \times \mathcal{Y}} A(y) \cdot [(\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_\#) * 1] \cdot (\nabla_x \times \vec{w} + \nabla_y \times \vec{w}_\#) dx dy dt = 0$$

$$\forall \vec{w} \in L^2(0, T; V), \quad \forall \vec{w}_\# \in L^2(\Omega_T \times \mathcal{Y})^3 \text{ such that } \nabla_y \times \vec{w}_\# \in L^2(\Omega_T \times \mathcal{Y})^3. \quad (5.6)$$

The inclusion (5.5) is equivalent to a variational inequality analogous to (3.19) (here with integration over  $\Omega_T \times \mathcal{Y}$ ). By selecting first  $\vec{w}_\# = \vec{0}$  and then  $\vec{w} = \vec{0}$  in (5.6), we see that this two-scale equation is equivalent to the system of a coarse-scale and a fine-scale equation:

$$\widehat{B} + \nabla_x \times \left\{ \int_{\mathcal{Y}} A(y) \cdot (\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_\#) * 1 dy \right\} = \vec{G} \quad \text{in } L^2(0, T; V'), \quad (5.7)$$

$$\nabla_y \times [A(y) \cdot (\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_\#) * 1] = \vec{0} \quad \text{in } L^2(\Omega_T; H^{-1}(\mathcal{Y})^3). \quad (5.8)$$

In order to reformulate (5.5), we prove the next statement.

**Lemma 5.1.** Any solution  $(\vec{B}, \vec{H}, \vec{H}_\#)$  of Problem 5.1 fulfills the following equation

$$\iint_{\Omega_T} \left\{ (T-t)(\widehat{B} - \vec{G}) \cdot \widehat{H} + \frac{1}{2} \int_{\mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_\#) * 1|^2 dy \right\} dx dt = 0. \quad (5.9)$$

**Proof.** We cannot select  $\vec{w} = \widehat{H}$  and  $\vec{w}_\# = \vec{H}_\#$  in (5.6), as these functions do not have the necessary regularity. We then use the procedure of Lemma 3.1 with the notation (3.24).

First, we extend Eq. (5.6) and the fields  $\vec{B}$ ,  $\vec{H}$ ,  $\vec{H}_\sharp$ ,  $\vec{G}$  to any  $t < 0$  ( $t > T$ , resp.) with the value that they attain at 0 ( $T$ , resp.). By selecting  $\vec{w} = D_h^+ \vec{H} * 1$  and  $\vec{w}_\sharp = D_h^+ \vec{H}_\sharp * 1$  in (5.6) and integrating in  $]0, t[$  for any  $t \in ]0, T[$ , we get

$$\begin{aligned} & \int \int_{\Omega_t} (\vec{B} - \vec{G}) \cdot D_h^+ \vec{H}_\varepsilon * 1 \, dx d\tau \\ & + \frac{1}{2} \int \int_{\Omega \times \mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \widehat{\vec{H}} + \nabla_y \times \vec{H}_\sharp) * 1|^2(x, y, t+h) \, dx dy \geq 0 \end{aligned} \quad (5.10)$$

for a.e. in  $t \in ]0, T[$ . A further integration in  $]0, T[$  and the passage to the limit as  $h \rightarrow 0$  yield

$$\int \int_{\Omega_T} \left\{ (T-t)(\widehat{\vec{B}} - \vec{G}) \cdot \widehat{\vec{H}} + \frac{1}{2} \int_{\mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \widehat{\vec{H}} + \nabla_y \times \vec{H}_\sharp) * 1|^2 dy \right\} dx dt \geq 0. \quad (5.11)$$

Similarly, by selecting  $\vec{w} = D_h^- \vec{H} * 1$  and  $\vec{w}_\sharp = D_h^- \vec{H}_\sharp * 1$  in (5.6), we get the opposite inclusion.  $\square$

**Reformulation of the constitutive relation.** Next we assume that  $\varphi$  is a representative function (of some monotone operator) that fulfills (4.16)–(4.21). The inclusion (5.5) is then tantamount to

$$\int \int_{\Omega_t \times \mathcal{Y}} (\varphi(\vec{H}, \vec{B}, y) - \vec{B} \cdot \vec{H}) \, dx dy d\tau \leq 0 \quad \text{for a.e. } t \in ]0, T[. \quad (5.12)$$

As

$$\begin{aligned} \int \int_{\Omega_t \times \mathcal{Y}} \vec{B} \cdot \vec{H} \, dx dy d\tau & \stackrel{(A.5)}{=} \int \int_{\Omega_t} \widehat{\vec{B}} \cdot \widehat{\vec{H}} \, dx d\tau + \int \int_{\Omega_t \times \mathcal{Y}} \widetilde{\vec{B}} \cdot \widetilde{\vec{H}} \, dx dy d\tau \\ & \stackrel{(5.3)}{=} \int \int_{\Omega_t} \widehat{\vec{B}} \cdot \widehat{\vec{H}} \, dx d\tau \quad \text{for a.e. } t \in ]0, T[, \end{aligned} \quad (5.13)$$

integrating (5.12) in time we get

$$\int \int_{\Omega_T \times \mathcal{Y}} (T-t) \varphi(\vec{H}, \vec{B}, y) \, dx dy dt \leq \int \int_{\Omega_T} (T-t) \widehat{\vec{B}} \cdot \widehat{\vec{H}} \, dx dt; \quad (5.14)$$

by (4.18), conversely (5.14) entails (5.5). By (5.9), (5.14) is tantamount to

$$\begin{aligned} & \int \int_{\Omega_T \times \mathcal{Y}} (T-t) [\varphi(\vec{H}, \vec{B}, y) - \vec{G} \cdot \widehat{\vec{H}}] \, dx dy dt \\ & + \frac{1}{2} \int \int_{\Omega_T \times \mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \widehat{\vec{H}} + \nabla_y \times \vec{H}_\sharp) * 1|^2 \, dx dy dt \leq 0. \end{aligned} \quad (5.15)$$

Note that the inequalities (5.12) and (5.15) cannot be strict, because of (4.18).



We have thus proved that (5.5) is equivalent to (5.15), provided that (5.6) is fulfilled:

**Proposition 5.2.** *In Problem 5.1 the inclusion (5.5) may equivalently be replaced either by (5.15) or by the corresponding equality.*

This problem might also be restated as a variational principle, in analogy with Proposition 4.3.

**Existence and uniqueness of the solution.**

**Theorem 5.3.** *Assume that (3.13)–(3.18) are fulfilled. For any  $\varepsilon > 0$ , let  $(\vec{B}_\varepsilon, \vec{H}_\varepsilon)$  be a solution of Problem 3.1 $_\varepsilon$ . Then there exist  $\vec{B}$ ,  $\vec{H}$ ,  $\vec{H}_\#$  that fulfill (5.2)–(5.4), and such that, as  $\varepsilon \rightarrow 0$  along a suitable subsequence,*

$$\vec{B}_\varepsilon \rightharpoonup \vec{B}, \quad \vec{H}_\varepsilon \rightharpoonup \vec{H} \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3, \quad (5.16)$$

$$\nabla \times \vec{H}_\varepsilon * 1 \rightharpoonup \nabla_x \times \hat{\vec{H}} * 1 + \nabla_y \times \vec{H}_\# * 1 \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3. \quad (5.17)$$

The triple  $(\vec{B}, \vec{H}, \vec{H}_\#)$  then solves Problem 5.1. The fields  $\vec{B}$  and  $\vec{H}_\#$  are uniquely determined; if  $\tilde{\alpha}^{-1}(\cdot, y)$  is single-valued for a.e.  $y \in \mathcal{Y}$ , then  $\vec{H}$  is also unique, and in (5.16) the whole sequences converge.

**Proof.** The argument of Proposition 4.1 yields the following uniform estimates:

$$\|\vec{B}_\varepsilon\|_{L^2(\Omega_T)^3}, \|\vec{H}_\varepsilon\|_{L^2(\Omega_T)^3}, \|\vec{H}_\varepsilon * 1\|_{L^\infty(0,T;V)} \leq \text{Constant}. \quad (5.18)$$

By Propositions A.1 and A.4, then there exist  $(\vec{B}, \vec{H}, \vec{H}_\#) \in L^2(\Omega_T \times \mathcal{Y})^3$  that fulfill (5.2)–(5.4), (5.16), (5.17). Hence, by the definition (A.1) of weak two-scale convergence,

$$\begin{aligned} A_\varepsilon(x) \cdot \nabla \times \vec{H}_\varepsilon * 1 &\rightharpoonup A(y) \cdot (\nabla_x \times \hat{\vec{H}} + \nabla_y \times \vec{H}_\#) * 1 \\ &\quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3. \end{aligned} \quad (5.19)$$

$$A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1 \rightharpoonup A(y)^{1/2} \cdot (\nabla_x \times \hat{\vec{H}} + \nabla_y \times \vec{H}_\#) * 1$$

Let us next select any  $\vec{w}$  and any  $\vec{w}_\#$  as it is prescribed in (5.6), with (say)  $\vec{w}_\# \in C^1(\Omega_T)^3$ . Replacing  $\vec{w}(x, t)$  by  $\vec{w}(x, t) + \varepsilon \vec{w}_\#(x, x/\varepsilon, t)$  in (3.20), and passing to the limit, we then get (5.6).

By two-scale lower semicontinuity, see (A.12),

$$\liminf_{\varepsilon \rightarrow 0} \iint_{\Omega_T} (T-t) \varphi_\varepsilon(\vec{H}_\varepsilon, \vec{B}_\varepsilon, x) dx dt \geq \iiint_{\Omega_T \times \mathcal{Y}} (T-t) \varphi(\vec{H}, \vec{B}, y) dx dy dt. \quad (5.20)$$

Note that

$$A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1 \rightharpoonup A(y)^{1/2} \cdot (\nabla_x \times \hat{\vec{H}} + \nabla_y \times \vec{H}_\#) * 1 \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3, \quad (5.21)$$

whence, applying (A.12) again,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} \iint_{\Omega_T} |A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2 dx dt \\ \geq \iiint_{\Omega_T \times \mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \hat{\vec{H}} + \nabla_y \times \vec{H}_\#) * 1|^2 dx dy dt. \end{aligned} \quad (5.22)$$

By passing to the inferior limit in (4.23) we then get (5.15), namely, (5.5).

In order to show the uniqueness of the solution, let  $(\vec{B}_i, \vec{H}_i, \vec{H}_{\sharp i})$  ( $i = 1, 2$ ) be two solutions. We would like to select  $\vec{w} = \vec{H}_1 - \vec{H}_2$  and  $\vec{w}_{\sharp} = \vec{H}_{\sharp 1} - \vec{H}_{\sharp 2}$  in (5.6), and then use the monotonicity of  $\vec{\alpha}(\cdot, y)$  to infer that

$$\iint_{\Omega_T \times \mathcal{Y}} |A(y)^{1/2} \cdot [\nabla_x \times (\hat{H}_1 - \hat{H}_2) + \nabla_y \times (\vec{H}_{\sharp 1} - \vec{H}_{\sharp 2})] * 1|^2 dx dy dt \leq 0. \quad (5.23)$$

The squared-bracketed term then vanishes a.e. in  $\Omega_T \times \mathcal{Y}$ . Therefore

$$\nabla_x \times \hat{H}_1 = \nabla_x \times \hat{H}_2, \quad \nabla_y \times \vec{H}_{\sharp 1} = \nabla_y \times \vec{H}_{\sharp 2} \quad \text{a.e. in } \Omega_T \times \mathcal{Y}, \quad (5.24)$$

whence  $\vec{B}_1 = \vec{B}_2$  by (5.7), and  $\vec{H}_{\sharp 1} = \vec{H}_{\sharp 2}$  by (5.4). If  $\vec{\alpha}^{-1}(\cdot, y)$  is single-valued for a.e.  $y \in \Omega$ , then by (5.5)  $\vec{H}_1 = \vec{H}_2$  a.e. in  $\Omega_T \times \mathcal{Y}$ .

But this choice of  $\vec{w}$  and  $\vec{w}_{\sharp}$  is not admissible, since a priori

$$\vec{H}_1 - \vec{H}_2 \notin L^2(0, T; V), \quad \nabla_y \times (\vec{H}_{\sharp 1} - \vec{H}_{\sharp 2}) \notin L^2(\Omega_T \times \mathcal{Y})^3.$$

The time-discretization argument of Lemma 3.1 however makes this procedure rigorous. (We omit these details, that closely mimic that proof.)  $\square$

## 6. Homogenization

In this section we derive a coarse-length scale model by integrating the two-scale Problem 5.1 with respect to the fine-scale variable, and show that these two problems are equivalent. First we homogenize the partial differential equation, then the constitutive relation.

**Homogenization of the elliptic operator.** In view of homogenizing the (degenerate) linear elliptic part of Eq. (5.6), first we eliminate the field  $\vec{H}_{\sharp} * 1$  by means of (5.8). We do so by mimicking a standard procedure for the homogenization of elliptic problems in divergence form; see e.g. [1,7,26] and, for the double curl, [64].

The field  $\vec{H}_{\sharp} * 1$  must fulfill (5.4) and the linear elliptic equation (5.8), where it is coupled with  $\hat{H}$  via  $\nabla_x \times \hat{H} * 1$ . The latter may also be regarded as a *cell problem* in  $\mathcal{Y}$ , since the variables  $x, t$  just occur as parameters. Thus

$$\vec{H}_{\sharp} * 1 = \vec{\mathcal{L}}(\nabla_x \times \hat{H} * 1), \quad \vec{\mathcal{L}} \text{ being a linear operator.} \quad (6.1)$$

For  $j = 1, 2, 3$ , let us denote by  $\vec{e}_j$  the unit vector in the direction of the  $j$ th coordinate axis, and set

$$\begin{aligned} R_j &:= (\nabla_x \times \hat{H}) \cdot \vec{e}_j \quad \text{in } H^{-1}(\mathcal{Y})^3, \quad \text{a.e. in } ]0, T[, \\ \vec{z}_j &:= \vec{\mathcal{L}}(\vec{e}_j) \quad (\in \mathbf{R}^3) \quad \text{a.e. in } \mathcal{Y}. \end{aligned} \quad (6.2)$$

Hence, as  $\hat{H} * 1 \in L^2(0, T; V)$ ,

$$\nabla_x \times \hat{H} * 1 = \sum_{i=1}^3 R_j * 1 \vec{e}_j \quad \text{a.e. in } \Omega_T, \quad (6.3)$$

$$\vec{H}_{\sharp} * 1 \stackrel{(6.1)}{=} \vec{\mathcal{L}} \left( \sum_{i=1}^3 R_j * 1 \vec{e}_j \right) = \sum_{i=1}^3 R_j * 1 \vec{z}_j \quad \text{a.e. in } \Omega_T \times \mathcal{Y}, \quad (6.4)$$

$$\sum_{j=1}^3 R_j * 1 \{ \nabla_y \times [A(y) \cdot (\vec{e}_j + \nabla_y \times \vec{z}_j)] \} \stackrel{(5.8)}{=} \vec{0} \quad \text{in } H^{-1}(\mathcal{Y})^3. \quad (6.5)$$

In order to fulfill the last equation, we prescribe

$$\nabla_y \times [A(y) \cdot (\vec{e}_j + \nabla_y \times \vec{z}_j)] = \vec{0} \quad \text{in } H^{-1}(\mathcal{Y})^3, \quad j = 1, 2, 3. \quad (6.6)$$

For any  $j$  this elliptic problem has one and only one divergence-free solution  $\vec{z}_j \in H_*^1(\mathcal{Y})^3$  (the index  $*$  denotes the subspace of functions having vanishing mean). Setting  $\vec{v} \cdot A \cdot \vec{w} := \sum_{i,j} v_i A_{ij} w_j$  for any vectors  $\vec{v}, \vec{w}$ , let us then define the (symmetric and constant) homogenized matrix

$$\begin{aligned} A_{ij} &:= \int_{\mathcal{Y}} [\vec{e}_i + \nabla_y \times \vec{z}_i(y)] \cdot A(y) \cdot [\vec{e}_j + \nabla_y \times \vec{z}_j(y)] dy \\ &\stackrel{(6.6)}{=} \int_{\mathcal{Y}} \vec{e}_i \cdot A(y) \cdot [\vec{e}_j + \nabla_y \times \vec{z}_j(y)] dy \quad \text{for } i, j \in \{1, 2, 3\}. \end{aligned} \quad (6.7)$$

This matrix is determined by the fine-scale fields  $\vec{z}_j$ 's, whose evaluation only needs the integration of three linear elliptic problems in the reference cell  $\mathcal{Y}$ , see (6.6). As, by (6.4) and (6.7),

$$\mathcal{A} \cdot \nabla_x \times \hat{H} * 1 = \int_{\mathcal{Y}} A(y) \cdot (\nabla_x \times \hat{H} * 1 + \nabla_y \times \vec{H}_{\#} * 1) dy \quad \text{a.e. in } \Omega_T,$$

the coarse-scale equation (5.7) yields the homogenized equation

$$\hat{B} + \nabla_x \times (\mathcal{A} \cdot \nabla_x \times \hat{H} * 1) = \vec{G} \quad \text{in } L^2(0, T; V'). \quad (6.8)$$

As  $\hat{H}$  determines  $\vec{H}_{\#}$  via (5.8), in turn this single-scale equation entails the two-scale equation (5.6). We have thus proved the next statement.

**Proposition 6.1.** *Let  $\vec{B}, \vec{H}, \vec{H}_{\#} \in L^2(\Omega_T \times \mathcal{Y})^3$  be such that*

$$\hat{H} * 1 \in L^2(0, T; V), \quad \nabla_y \times \vec{H}_{\#} * 1 \in L^2(\Omega_T \times \mathcal{Y})^3, \quad (6.9)$$

*and define  $\mathcal{A}$  as in (6.7). If  $\vec{B}, \vec{H}, \vec{H}_{\#}$  fulfill (5.6), then  $\hat{B}, \hat{H} \in L^2(\Omega_T)^3$  satisfy (6.8). Conversely, for any  $\hat{B}, \hat{H} \in L^2(\Omega_T)^3$  such that  $\hat{H} * 1 \in L^2(0, T; V)$  and that satisfy (6.8), there exist  $\vec{B}, \vec{H}, \vec{H}_{\#} \in L^2(\Omega_T \times \mathcal{Y})^3$  that fulfill (5.6) and (6.9).*

**Coarse-scale formulation.** Let us first define

$$\mathcal{V}_1 := \{ \nabla \times \psi : \psi \in H^1(\mathcal{Y})^3 \}, \quad \mathcal{Z}_1 := \{ \nabla \theta : \theta \in H^1(\mathcal{Y}) \}; \quad (6.10)$$

this is a pair of orthogonal subspaces of  $L_*^2(\mathcal{Y})^3$ , actually,  $L_*^2(\mathcal{Y})^3 = \mathcal{V}_1 \oplus \mathcal{Z}_1$  (direct sum). Let us then set

$$\varphi_0(\vec{\xi}, \vec{\xi}') := \inf \left\{ \int_{\mathcal{Y}} \varphi(\vec{\xi} + v, \vec{\xi}' + z, y) dy : (v, z) \in \mathcal{V}_1 \times \mathcal{Z}_1 \right\} \quad \forall (\vec{\xi}, \vec{\xi}') \in \mathbf{R}^3 \times \mathbf{R}^3. \quad (6.11)$$

By part (i) of Theorem 2.3, this function represents a maximal monotone operator  $\bar{\alpha}_0: \mathbf{R}^3 \rightarrow \mathcal{P}(\mathbf{R}^3)$ . This is trivially extended to time-dependent functions, namely, for any  $(\bar{\xi}, \bar{\xi}') \in L^2(0, T)^3 \times L^2(0, T)^3$ , just by replacing the functional  $\int_{\mathcal{Y}} \varphi(\dots) dy$  by  $\int_0^T dt \int_{\mathcal{Y}} \varphi(\dots) dy$  and  $\mathcal{V}_1 \times \mathcal{Z}_1$  by  $L^2(0, T; \mathcal{V}_1 \times \mathcal{Z}_1)$ .

Next we state a coarse-scale problem, that we shall then show to be the homogenized formulation of Problem 3.1<sub>ε</sub>. We shall mark by an overbar certain fields that are independent of  $y$ , and use the notation (A.5). For any  $\bar{u} \in B$  and any  $v \in L^1_*(\mathcal{Y})$ , by setting  $u(y) = \bar{u} + v(y)$  for a.e.  $y$ , we then get  $\hat{u} := \int_{\mathcal{Y}} u(y) dy = \bar{u}$  and  $\tilde{u} = v$ . Note the conceptual difference between  $\bar{u}$  (any field) and  $\hat{u}$  (the average of  $u$ ). When selecting any element of  $B$ , we shall accordingly denote it by  $\bar{u}$  rather than  $\hat{u}$ .

**Problem 6.1.** Find  $\bar{\bar{B}}, \bar{\bar{H}} \in L^2(\Omega_T)^3$  such that  $\bar{\bar{H}} * 1 \in L^2(0, T; V)$  and

$$\bar{\bar{B}} \in \bar{\alpha}_0(\bar{\bar{H}}) \quad \text{a.e. in } \Omega_T, \quad (6.12)$$

$$\bar{\bar{B}} + \nabla_x \times (\mathcal{A} \cdot \nabla_x \times \bar{\bar{H}} * 1 - \bar{G}) = \bar{0} \quad \text{in } L^2(0, T; V'). \quad (6.13)$$

In full analogy with Lemma 3.1 and Proposition 4.2, any solution of this problem fulfills the following equation

$$\iint_{\Omega_T} \left\{ (T-t)(\bar{\bar{B}} - \bar{G}) \cdot \bar{\bar{H}} + \frac{1}{2} |\mathcal{A}^{1/2} \cdot \nabla_x \times \bar{\bar{H}} * 1|^2 \right\} dx dt = 0, \quad (6.14)$$

and we have the next statement.

**Proposition 6.2.** In Problem 6.1 the inclusion (6.12) may equivalently be replaced either by the inequality

$$\psi_0(\bar{\bar{H}}, \bar{\bar{B}}) := \iint_{\Omega_T} \left\{ (T-t)[\varphi_0(\bar{\bar{H}}, \bar{\bar{B}}) - \bar{G} \cdot \bar{\bar{H}}] + \frac{1}{2} |\mathcal{A}^{1/2} \cdot \nabla_x \times \bar{\bar{H}} * 1|^2 \right\} dx dt \leq 0, \quad (6.15)$$

or by the corresponding equality.

As  $\psi_0$  is nonnegative, we may eliminate the field  $\bar{\bar{B}}$  via Eq. (6.13), and reformulate Problem 6.1 as a minimization principle, in analogy with Proposition 4.3.

**Proposition 6.3.** Problem 6.1 is equivalent to the search for a field  $\bar{\bar{H}}$  that minimizes the functional

$$\begin{aligned} \bar{\psi}_0(\bar{\bar{v}}) &:= \iint_{\Omega_T} \left\{ (T-t)[\varphi_0(\bar{\bar{v}}, \bar{G} - \nabla_x \times (\mathcal{A} \cdot \nabla_x \times \bar{\bar{v}} * 1)) - \bar{G} \cdot \bar{\bar{v}}] + \frac{1}{2} |\mathcal{A}^{1/2} \cdot \nabla_x \times \bar{\bar{v}} * 1|^2 \right\} dx dt \\ &\forall \bar{\bar{v}} \in L^2(\Omega_T)^3 \text{ such that } \bar{\bar{v}} * 1 \in L^2(0, T; V), \end{aligned} \quad (6.16)$$

and this is tantamount to  $\bar{\psi}_0(\bar{\bar{H}}) = 0$ .

**Full homogenization.** Next we show that the two-scale Problem 5.1 is equivalent to the single-scale Problem 6.1, by an obvious scale-transformation.

**Theorem 6.4** (Upscaling and downscaling). Assume that (3.13)–(3.17) are fulfilled. If  $(\bar{\bar{B}}, \bar{\bar{H}}, \bar{\bar{H}}_x)$  is a solution of Problem 5.1, then  $(\bar{\bar{B}}, \bar{\bar{H}})$  solves Problem 6.1. Conversely, any solution of the latter problem may be represented in this way, provided that  $\bar{\alpha}_0^{-1}$  is single-valued.

(The latter restriction is fulfilled by the model of ferromagnetism that we outlined in Section 3.)

**Proof.** In Proposition 6.1 we have thus seen that Eq. (6.8) is the homogenized formulation of (5.6). By the definition (6.11) of  $\varphi_0$ ,

$$\iint_{\Omega_T} (T-t)\varphi_0(\widehat{H}, \widehat{B}) dx dt \leq \iiint_{\Omega_T \times \mathcal{Y}} (T-t)\varphi(\vec{H}, \vec{B}, y) dx dy dt. \quad (6.17)$$

The inequality (5.15) thus yields (6.15), that is (6.12). The first statement is thus proved.

The converse is a consequence of the uniqueness of the solution of Problem 6.1; this follows from the monotonicity of  $\vec{\alpha}_0$ , provided that  $\vec{\alpha}_0^{-1}$  is single-valued, via the procedure that we saw for Proposition 4.1.  $\square$

By Theorem 5.3, we then infer that Problem 6.1 is the homogenized formulation of Problem 3.1 $_{\varepsilon}$ .

**Theorem 6.5** (Homogenization). *Assume that (3.13)–(3.18) are fulfilled. For any  $\varepsilon > 0$ , let  $(\vec{B}_{\varepsilon}, \vec{H}_{\varepsilon})$  be a solution of Problem 3.1 $_{\varepsilon}$ . Then there exist  $\vec{B}, \vec{H} \in L^2(\Omega_T)^3$  such that  $\vec{H} * 1 \in L^2(0, T; V)$  and, as  $\varepsilon \rightarrow 0$  along a suitable subsequence,*

$$\vec{B}_{\varepsilon} \rightharpoonup \vec{B}, \quad \vec{H}_{\varepsilon} \rightharpoonup \vec{H} \quad \text{in } L^2(\Omega_T)^3, \quad (6.18)$$

$$\vec{H}_{\varepsilon} * 1 \rightarrow \vec{H} * 1 \quad \text{in } L^2(0, T; V). \quad (6.19)$$

The pair  $(\vec{B}, \vec{H})$  then solves Problem 6.1.

## 7. $\Gamma$ -convergence

In this section we retrieve and interpret the homogenization result of Section 6 via De Giorgi's notion of  $\Gamma$ -convergence, see e.g. [13,14,27,29]. More precisely:

- (i) for any  $\varepsilon > 0$  we represent Problem 3.1 $_{\varepsilon}$  by the minimization of a functional  $\Phi_{\varepsilon}$ ,
- (ii) we prove that as  $\varepsilon$  vanishes  $\Phi_{\varepsilon}$   $\Gamma$ -converges to a functional  $\Phi$ ,
- (iii) we show that the minimization of the latter functional is equivalent to the homogenized Problem 6.1.

This corroborates the above two-scale analysis. Anyway it cannot surrogate it, since the form of the homogenized problem was derived via the two-scale approach.

By  $\Gamma$ -convergence we then study the dependence of the solution upon variations of the data, including the monotone operator  $\vec{\alpha}$ . This obviously rests upon the variational formulation of monotonicity due to Fitzpatrick.

Let us first set

$$\mathcal{H} := \{(\vec{B}, \vec{H}) \in (L^2(\Omega_T)^3)^2 : \vec{H} * 1 \in L^2(0, T; V)\}, \quad (7.1)$$

which is a Hilbert space equipped with the graph norm:

$$\|(\vec{B}, \vec{H})\|_{\mathcal{H}}^2 := \|\vec{B}\|_{L^2(\Omega_T)^3}^2 + \|\vec{H}\|_{L^2(\Omega_T)^3}^2 + \|\vec{H} * 1\|_{L^2(0, T; V)}^2.$$

For any  $\varepsilon > 0$ , let us recall the definitions (4.23) and (6.15) of the functionals  $\Psi_\varepsilon$  and  $\Psi_0$ , and set

$$S_\varepsilon := \{(\vec{B}, \vec{H}) \in \mathcal{H}: (3.22) \text{ holds}\}, \quad (7.2)$$

$$\Phi_\varepsilon(\vec{B}, \vec{H}) := \begin{cases} \Psi_\varepsilon(\vec{B}, \vec{H}) & \text{if } (\vec{B}, \vec{H}) \in S_\varepsilon, \\ +\infty & \text{if } (\vec{B}, \vec{H}) \in (L^2(\Omega_T)^3)^2 \setminus S_\varepsilon, \end{cases} \quad (7.3)$$

$$S := \{(\vec{B}, \vec{H}) \in \mathcal{H}: (6.13) \text{ holds}\}, \quad (7.4)$$

$$\Phi_0(\vec{B}, \vec{H}) := \begin{cases} \Psi_0(\vec{B}, \vec{H}) & \text{if } (\vec{B}, \vec{H}) \in S, \\ +\infty & \text{if } (\vec{B}, \vec{H}) \in (L^2(\Omega_T)^3)^2 \setminus S. \end{cases} \quad (7.5)$$

As we saw,  $\Psi_\varepsilon$  and  $\Psi_0$  are nonnegative. We may thus reformulate Propositions 4.3 and 6.3 as minimization principles:

$$(\vec{B}_\varepsilon, \vec{H}_\varepsilon) \in S_\varepsilon \text{ solves Problem 3.1}_\varepsilon \Leftrightarrow \Phi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon) = \inf \Phi_\varepsilon, \quad (7.6)$$

$$(\vec{B}, \vec{H}) \in S \text{ solves Problem 6.1} \Leftrightarrow \Phi_0(\vec{B}, \vec{H}) = \inf \Phi_0. \quad (7.7)$$

Moreover, both infimal values vanish.

**Theorem 7.1.** *If (3.13)–(3.17) are fulfilled, then  $\Phi_\varepsilon$  weakly  $\Gamma$ -converges to  $\Phi_0$  in  $\mathcal{H}$ . That is, for any  $(\vec{B}, \vec{H}) \in S$ ,*

$$\begin{aligned} &\text{for any sequence } \{(\vec{B}_\varepsilon, \vec{H}_\varepsilon)\} \text{ in } (L^2(\Omega_T)^3)^2, \\ &\text{if } (\vec{B}_\varepsilon, \vec{H}_\varepsilon) \rightharpoonup (\vec{B}, \vec{H}) \text{ in } (L^2(\Omega_T)^3)^2, \text{ then } \liminf_{\varepsilon \rightarrow 0} \Phi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon) \geq \Phi_0(\vec{B}, \vec{H}), \end{aligned} \quad (7.8)$$

$$\begin{aligned} &\text{there exists a sequence } \{(\vec{B}_\varepsilon, \vec{H}_\varepsilon)\} \text{ in } (L^2(\Omega_T)^3)^2 \text{ such that} \\ &(\vec{B}_\varepsilon, \vec{H}_\varepsilon) \rightharpoonup (\vec{B}, \vec{H}) \text{ in } (L^2(\Omega_T)^3)^2, \text{ and } \limsup_{\varepsilon \rightarrow 0} \Phi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon) \leq \Phi_0(\vec{B}, \vec{H}). \end{aligned} \quad (7.9)$$

**Proof.** We proceed by two steps.

(i) If a sequence  $\{(\vec{B}_\varepsilon, \vec{H}_\varepsilon)\}$  in  $S_\varepsilon$  is such that

$$\vec{B}_\varepsilon \rightharpoonup \vec{B}, \quad \vec{H}_\varepsilon \rightharpoonup \vec{H} \quad \text{in } L^2(\Omega_T)^3, \quad (7.10)$$

then  $(\vec{B}, \vec{H}) \in S$ , and by Proposition A.1 there exists a pair  $(\vec{B}_0, \vec{H}_0)$  such that, up to subsequences,

$$\vec{B}_\varepsilon \rightharpoonup_2 \vec{B}_0, \quad \vec{H}_\varepsilon \rightharpoonup_2 \vec{H}_0 \quad \text{in } L^2(\Omega_T \times \mathcal{Y}). \quad (7.11)$$

Hence  $\widehat{\vec{B}_0} = \vec{B}$  and  $\widehat{\vec{H}_0} = \vec{H}$  a.e. in  $\Omega$ . Therefore

$$\begin{aligned} &\liminf_{\varepsilon \rightarrow 0} \iint_{\Omega_T} (T-t) \varphi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon, x) dx dt \\ &\stackrel{(A.12)}{\geq} \iiint_{\Omega_T \times \mathcal{Y}} (T-t) \varphi(\vec{B}_0, \vec{H}_0, y) dx dy dt \stackrel{(6.11)}{\geq} \iint_{\Omega_T} (T-t) \varphi_0(\vec{B}, \vec{H}) dx dt. \end{aligned} \quad (7.12)$$

Moreover, as

$$\begin{aligned}
 & \int \int \int_{\Omega_T \times \mathcal{Y}} |A(y)^{1/2} \cdot (\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_{\sharp}) * 1|^2 dx dy dt \\
 &= \int \int \int_{\Omega_T \times \mathcal{Y}} \{(\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_{\sharp}) * 1\} \cdot A(y) \cdot \{(\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_{\sharp}) * 1\} dx dy dt \\
 &\stackrel{(6.7)}{=} \int \int_{\Omega_T} \{\nabla_x \times \widehat{H} * 1\} \cdot \mathcal{A} \cdot \{\nabla_x \times \widehat{H} * 1\} dx dt = \int \int_{\Omega_T} |\mathcal{A}^{1/2} \cdot \{\nabla_x \times \widehat{H} * 1\}|^2 dx dt, \quad (7.13)
 \end{aligned}$$

(5.22) yields

$$\liminf_{\varepsilon \rightarrow 0} \int \int_{\Omega_T} |A_{\varepsilon}^{1/2} \cdot \nabla \times \vec{H}_{\varepsilon} * 1|^2 dx dt \geq \int \int_{\Omega_T} |\mathcal{A}^{1/2} \cdot \nabla_x \times \widehat{H} * 1|^2 dx dy dt. \quad (7.14)$$

By (7.12) and (7.14), we infer that  $\liminf_{\varepsilon \rightarrow 0} \Phi_{\varepsilon}(\vec{B}_{\varepsilon}, \vec{H}_{\varepsilon}) \geq \Phi_0(\vec{B}, \vec{H})$ . The condition (7.8) is thus established.

(ii) With the aim of proving (7.9), let us first fix any  $(\vec{B}, \vec{H}) \in \mathcal{S}$  and note that, by part (ii) of Theorem 2.3,

$$\begin{aligned}
 & \exists(\vec{B}, \vec{H}) \in L^2(0, T; \mathcal{V}_1 \times \mathcal{Z}_1): \\
 & \int \int \int_{\Omega_T \times \mathcal{Y}} (T-t) \varphi(\vec{B} + \vec{B}, \vec{H} + \vec{H}, y) dx dy dt = \int \int_{\Omega_T} (T-t) \varphi_0(\vec{B}, \vec{H}) dx dt. \quad (7.15)
 \end{aligned}$$

By Proposition A.6, there exist sequences  $\{\vec{v}_{\varepsilon}\}$  and  $\{\vec{z}_{\varepsilon}\}$  in  $L^2(\Omega_T)^3$  such that

$$\begin{aligned}
 \vec{B}_{\varepsilon} &:= \vec{B} + \vec{v}_{\varepsilon} \xrightarrow{2} \vec{B} + \vec{B} \\
 \vec{H}_{\varepsilon 0} &:= \vec{H} + \vec{z}_{\varepsilon} \xrightarrow{2} \vec{H} + \vec{H}
 \end{aligned} \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3. \quad (7.16)$$

Moreover, by the second part of Proposition A.4, there exists a sequence  $\{\vec{H}_{\varepsilon 1}\}$  in  $L^2(\mathbf{R}_T^3; H_*^1(\mathcal{Y})^3)$  such that,

$$\begin{aligned}
 \vec{H}_{\varepsilon} &:= \vec{H}_{\varepsilon 0} + \vec{H}_{\varepsilon 1} \xrightarrow{2} \vec{H} + \vec{H} \\
 \nabla_x \times \vec{H}_{\varepsilon} * 1 &\xrightarrow{2} \nabla_x \times \widehat{H} * 1 + \nabla_y \times \vec{H}_{\sharp} * 1
 \end{aligned} \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3. \quad (7.17)$$

Note that  $(\vec{B}_{\varepsilon}, \vec{H}_{\varepsilon}) = (\vec{B}, \vec{H}) \in \mathcal{S}$  for any  $\varepsilon$ . Moreover,

$$\begin{aligned}
 & \int \int_{\Omega_T} (T-t) \varphi_{\varepsilon}(\vec{B}_{\varepsilon}, \vec{H}_{\varepsilon}, x) dx dt \\
 & \xrightarrow{(A.11)} \int \int \int_{\Omega_T \times \mathcal{Y}} (T-t) \varphi(\vec{B} + \vec{B}, \vec{H} + \vec{H}, y) dx dy dt \stackrel{(7.15)}{=} \int \int_{\Omega_T} \varphi_0(\vec{B}, \vec{H}) dx dt, \quad (7.18)
 \end{aligned}$$

$$\begin{aligned}
& \int_{\Omega_T} \int |A_\varepsilon(x)^{1/2} \cdot \nabla \times \vec{H}_\varepsilon * 1|^2 dx dt \\
& \stackrel{(A.11)}{\rightarrow} \int_{\Omega_T \times \mathcal{Y}} \int |A(y)^{1/2} \cdot (\nabla_x \times \widehat{H} + \nabla_y \times \vec{H}_\#) * 1|^2 dx dy dt \\
& \stackrel{(7.13)}{=} \int_{\Omega_T} \int |A^{1/2} \cdot \nabla_x \times \widehat{H} * 1|^2 dx dt.
\end{aligned} \tag{7.19}$$

We then conclude that  $\Phi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon) \rightarrow \Phi_0(\vec{B}, \vec{H})$ . The condition (7.9) is thus established, too.  $\square$

**Remark.** Instead of the functional  $\Phi_\varepsilon(\vec{B}_\varepsilon, \vec{H}_\varepsilon)$  we might have used  $\bar{\Phi}_\varepsilon(\vec{H}_\varepsilon)$ , which (loosely speaking) is retrieved from the former by inserting the partial differential equations into the functional itself. The analysis would have been analogous.

**Structural stability.** Next we replace  $\vec{\alpha}$  by a sequence of maximal monotone mappings  $\{\vec{\alpha}_n(\cdot, x)\}$  as in (3.13) and (3.14), that converges in a sense to be specified, and assume that we are given two sequences  $\{A_n\}$  and  $\{\vec{G}_n\}$  as in (3.15)–(3.17) such that

$$\begin{aligned}
\vec{G}_n &\rightharpoonup \vec{G} \quad \text{in } L^2(\Omega_T)^3, \\
A_n &\rightarrow A \quad \text{in } L^\infty(\Omega_T)^{3 \times 3}.
\end{aligned} \tag{7.20}$$

(The latter convergence might be weakened.) We intend to study the asymptotic behavior of the corresponding solutions of Problem 3.1. We might also let  $\vec{B}$ ,  $\vec{H}_a$ ,  $\vec{g}$  vary; this further extension would be straightforward, but we omit it for the sake of simplicity.

For any  $n$ , we define the Fitzpatrick function  $\varphi_n: \mathbf{R}^3 \times \mathbf{R}^3 \times \Omega \rightarrow \mathbf{R} \cup \{+\infty\}$  of  $\vec{\alpha}_n$ , see (2.1). We then set

$$\begin{aligned}
\psi_n(\vec{H}) &:= \int_{\Omega_T} \int \left\{ (T-t) [\varphi_n(\vec{H}, \vec{G}_n - \nabla \times (A_n(x) \cdot \nabla \times \vec{H}_n * 1), x) - \vec{G}_n \cdot \vec{H}] \right. \\
&\quad \left. + \frac{1}{2} |A_n(x)^{1/2} \cdot \nabla_x \times \vec{H} * 1|^2 \right\} dx dt \quad (\geq 0),
\end{aligned} \tag{7.21}$$

and denote by  $(\vec{B}_n, \vec{H}_n)$  a minimizer. By a statement analogous to Proposition 4.3, this is a solution of the corresponding (coarse-scale) Problem 3.1<sub>n</sub>. We then assume that another convex and lower semicontinuous function  $\varphi: \mathbf{R}^3 \times \mathbf{R}^3 \times \Omega \rightarrow \mathbf{R} \cup \{+\infty\}$  fulfills (4.16), (4.17), and is such that

$$\begin{aligned}
& \forall (\vec{\xi}, \vec{\xi}') \in (L^2(\Omega)^3)^2, \exists \text{ sequence } \{(\vec{\xi}_n, \vec{\xi}'_n)\} \text{ in } (L^2(\Omega)^3)^2 \\
& \text{such that } (\vec{\xi}_n, \vec{\xi}'_n) \rightharpoonup (\vec{\xi}, \vec{\xi}') \text{ in } (L^2(\Omega)^3)^2, \quad \liminf_{n \rightarrow \infty} \langle \vec{\xi}'_n, \vec{\xi}_n \rangle \geq \langle \vec{\xi}', \vec{\xi} \rangle, \quad \text{and} \\
& \liminf_{n \rightarrow \infty} \int_{\Omega} \varphi_n(\vec{\xi}_n, \vec{\xi}'_n, x) dx \leq \int_{\Omega} \varphi(\vec{\xi}, \vec{\xi}', x) dx,
\end{aligned} \tag{7.22}$$

$$\begin{aligned}
& \forall (\vec{\xi}, \vec{\xi}') \in (L^2(\Omega)^3)^2, \forall \text{ sequence } \{(\vec{\xi}_n, \vec{\xi}'_n)\} \text{ in } (L^2(\Omega)^3)^2, \\
& \text{if } (\vec{\xi}_n, \vec{\xi}'_n) \rightharpoonup (\vec{\xi}, \vec{\xi}') \text{ in } (L^2(\Omega)^3)^2 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle \vec{\xi}'_n, \vec{\xi}_n \rangle \leq \langle \vec{\xi}', \vec{\xi} \rangle, \\
& \text{then } \limsup_{n \rightarrow \infty} \int_{\Omega} \varphi_n(\vec{\xi}_n, \vec{\xi}'_n, x) dx \geq \int_{\Omega} \varphi(\vec{\xi}, \vec{\xi}', x) dx.
\end{aligned} \tag{7.23}$$



By (7.22), the inequality (4.18) is preserved as  $n \rightarrow \infty$ ; thus  $\varphi$  is a measurable representative function (of some monotone operator) in  $(L^2(\Omega)^3)^2$ . We may then define  $\psi_0$  as in (7.21), just with  $\varphi$  in place of  $\varphi_n$ .

The argument of Theorem 7.1 yields the next statement.

**Proposition 7.2.** *If the hypotheses (7.22) and (7.23) are fulfilled, then  $\psi_n$  weakly  $\Gamma$ -converges to  $\psi_0$  in  $(L^2(\Omega_T)^3)^2$ .*

## 8. Relaxation dynamics

In this section we replace the equilibrium relation (3.10) by either of the following dynamics of relaxation:

$$\begin{cases} b\vec{H}_t + \vec{\alpha}(\vec{H}, x) \ni \vec{B} & \text{a.e. in } \Omega_T, \\ \vec{H}(\cdot, 0) = \vec{H}^0 & \text{a.e. in } \Omega, \end{cases} \quad (8.1)$$

$$\begin{cases} b\vec{B}_t + \vec{B} \in \vec{\alpha}(\vec{H}, x) & \text{a.e. in } \Omega_T, \\ \vec{B}(\cdot, 0) = \vec{B}^0 & \text{a.e. in } \Omega, \end{cases} \quad (8.2)$$

with  $\vec{H}^0, \vec{B}^0$  prescribed,  $\vec{\alpha}(\cdot, x)$  a maximal monotone mapping for a.e.  $x$ , and  $b$  a positive constant. We shall see that, under suitable assumptions, each of these systems defines a maximal monotone operator between  $\vec{B}$  and  $\vec{H}$  in a space of time-dependent functions. The above variational approach to homogenization might thus be extended to these relaxation dynamics.

**First relaxation mode.** Next we provide a rather weak formulation of (8.1), assuming that (3.13)–(3.17) are fulfilled and that a field  $\vec{H}^0 \in L^2(\Omega)^3$  is prescribed.

**Problem 8.1.** Find  $\vec{B} \in L^2(0, T; V')$  and  $\vec{H} \in L^2(\Omega_T)^3$  such that

$$\vec{H} \in H^1(0, T; V'), \quad \vec{H} * 1 \in L^2(0, T; V), \quad (8.3)$$

$$\begin{cases} \langle \vec{B} - b\vec{H}_t - \vec{w}, \vec{H} - \vec{v} \rangle \geq 0 & \text{a.e. in } ]0, T[, \\ \forall (\vec{v}, \vec{w}) \in (\mathcal{D}(\Omega_T)^3)^2 \text{ such that } \vec{w} \in \vec{\alpha}(\vec{v}, \cdot) \text{ a.e. in } \Omega_T, \end{cases} \quad (8.4)$$

$$\vec{H}(\cdot, 0) = \vec{H}^0 \quad \text{in } V', \quad (8.5)$$

$$\int_0^T \langle \vec{B} - \vec{G}, \vec{v} \rangle dt + \iint_{\Omega_T} A(x) \cdot \nabla \times \vec{H} * 1 \cdot \nabla \times \vec{v} dx dt = 0 \quad \forall \vec{v} \in L^2(0, T; V). \quad (8.6)$$

The variational inequality (8.4) and Eq. (8.6) are respectively equivalent to

$$b\vec{H}_t + \vec{\alpha}(\vec{H}, x) \ni \vec{B} \quad \text{in } V', \text{ for a.e. } t \in ]0, T[, \quad (8.7)$$

$$\vec{B} + \nabla \times (A(x) \cdot \nabla \times \vec{H} * 1) = \vec{G} \quad \text{in } V', \text{ for a.e. } t \in ]0, T[. \quad (8.8)$$

By eliminating the field  $\vec{B}$ , we thus get the hyperbolic inclusion

$$b\vec{H}_t + \nabla \times (A(x) \cdot \nabla \times \vec{H} * 1) + \vec{\alpha}(\vec{H}, x) \ni \vec{G} \quad \text{in } V', \text{ for a.e. } t \in ]0, T[; \quad (8.9)$$

conversely this equation yields the system (8.7), (8.8). Next we shall assume that

$$\vec{H}(\cdot, 0) = \vec{H}^0 = \vec{0} \quad \text{a.e. in } \Omega. \quad (8.10)$$

This condition is not really restrictive: if it is not fulfilled, it may be retrieved by replacing  $\vec{\alpha}(\cdot, x)$  and  $\vec{G}$  by  $\vec{\alpha}(\cdot + \vec{H}^0(x), x)$  and  $\vec{G} - t \nabla \times (A(x) \cdot \nabla \times \vec{H}^0)$ , respectively.

Let us next define the linear operator

$$\begin{aligned} \Lambda \vec{v} &:= b \vec{v}_t + \nabla \times (A(x) \cdot \nabla \times \vec{v} * 1) \quad (\in L^2(0, T; V')), \\ \forall \vec{v} \in Y^0 &:= \{\vec{v} \in H^1(0, T; V') \cap H^{-1}(0, T; V), \vec{v}(\cdot, 0) = \vec{0}\}, \end{aligned} \quad (8.11)$$

and its domain in  $L^2(\Omega_T)^3$ , namely,  $D(\Lambda) := \{\vec{v} \in Y^0 : \Lambda \vec{v} \in L^2(\Omega_T)^3\}$ . Problem 8.1 may then equivalently be reformulated as follows.

**Problem 8.1'.** Find  $\vec{H} \in L^2(\Omega_T)^3 \cap D(\Lambda)$  such that

$$\Lambda \vec{H} + \vec{\alpha}(\vec{H}, x) = \vec{G} \quad \text{a.e. in } \Omega_T. \quad (8.12)$$

We claim that

$$\iint_{\Omega_T} (T-t) \Lambda \vec{v} \cdot \vec{v} \, dx dt = \frac{1}{2} \iint_{\Omega_T} (b |\vec{v}|^2 + |A(x)^{1/2} \cdot \nabla \times \vec{v} * 1|^2) \, dx dt \quad \forall \vec{v} \in D(\Lambda). \quad (8.13)$$

This equality directly follows from a formal calculation, that may be made rigorous via a simple approximation procedure, along the lines of Lemma 3.1.

**Proposition 8.1.** *If (3.13)–(3.17), (4.1) and (4.2) are fulfilled, then there exists one and only one solution of Problem 8.1. If moreover  $\vec{G} \in W^{1,1}(0, T; V')$  and  $\vec{G}(\cdot, 0) \in L^2(\Omega)^3$ , then*

$$\vec{H} \in W^{1,\infty}(0, T; L^2(\Omega)^3) \cap L^\infty(0, T; V), \quad \vec{B} \in W^{1,\infty}(0, T; V'). \quad (8.14)$$

**Outline of the proof.** This result is easily established via a standard argument; we just sketch it, since our main concern stays in the formulation of a variational principle. One may approximate Problem 8.1 by an (implicit) time-discretization scheme, with discretization parameter  $m$ , along the lines of Section 4. Multiplying both  $(8.7)_m$  and  $(8.8)_m$ <sup>3</sup> by  $\vec{H}_m$ , via a standard procedure one gets uniform estimates for  $\vec{H}_m$  in  $L^\infty(0, T; L^2(\Omega)^3)$  and for  $\vec{H}_m * 1$  in  $L^\infty(0, T; V)$ . Comparing the terms of  $(8.8)_m$  one then obtains a uniform estimate for  $\vec{B}_m$  in  $L^2(0, T; V')$ . A comparison in  $(8.7)_m$  in turn yields a uniform estimate for  $\vec{H}_{mt}$  in  $L^2(0, T; V')$ . Therefore there exist  $\vec{B}$  and  $\vec{H}$  such that, as  $m$  diverges along a suitable sequence,

$$\vec{B}_m \rightharpoonup \vec{B} \quad \text{in } L^2(0, T; V'), \quad (8.15)$$

$$\vec{H}_m \overset{*}{\rightharpoonup} \vec{H} \quad \text{in } L^\infty(0, T; L^2(\Omega)^3) \cap H^1(0, T; V'), \quad (8.16)$$

$$\vec{H}_m * 1 \rightharpoonup \vec{H} * 1 \quad \text{in } L^\infty(0, T; V). \quad (8.17)$$

The limit procedure in the nonlinear relation is then performed along the lines of Section 4.

Uniform estimates corresponding to the further regularity (8.14) are easily established by taking the time-increment of Eqs.  $(8.7)_m$  and  $(8.8)_m$ , multiplying them by the time-increment of  $\vec{H}_m$ , and mimicking the above estimate procedure.

By (8.13), the operator  $\Lambda + \vec{\alpha}$  is strictly monotone. By (8.9),  $\vec{H}$  is then uniquely determined; by (8.4),  $\vec{B}$  is then also unique.  $\square$

<sup>3</sup> By appending the index  $m$  we label the corresponding approximated equations and solutions.

**A minimization principle.** Let us assume that (3.13)–(3.17), (4.1) and (4.2) are satisfied, and set

$$\theta(\vec{v}) := \Lambda \vec{v} + \vec{\alpha}(\vec{v}, \cdot) \quad \forall \vec{v} \in D(\theta) = D(\Lambda), \quad (8.18)$$

so that (8.12) reads  $\theta(\vec{H}) = \vec{G}$ . We shall use the Hilbert triple

$$D(\theta) \subset L^2(\Omega_T)^3 = (L^2(\Omega_T)^3)' \subset D(\theta)' \quad (8.19)$$

(with continuous and dense injections). Along the lines of (2.7)–(2.10), it is easily checked that the operator  $\theta : D(\theta) \rightarrow D(\theta)'$  is maximal monotone, although not cyclically monotone.

It is easily checked that, if  $\varphi : \mathbf{R}^3 \times \mathbf{R}^3 \times \Omega \rightarrow \mathbf{R} \cup \{+\infty\}$  is a representative function of  $\vec{\alpha}$ , then

$$\begin{aligned} f : D(\theta) \times L^2(\Omega_T)^3 &\rightarrow \mathbf{R} \cup \{+\infty\}: \\ (\vec{\xi}, \vec{\xi}') &\mapsto \int \int_{\Omega_T} (T-t) \varphi(\vec{\xi}, \vec{\xi}' - \Lambda \vec{\xi}, x) dx dt + \frac{1}{2} \int \int_{\Omega_T} (b|\vec{\xi}|^2 + |A(x)^{1/2} \cdot \nabla \times \vec{\xi} * 1|^2) dx dt \end{aligned} \quad (8.20)$$

is a representative function of  $\theta$ , in the sense that

$$f \text{ is convex and lower semicontinuous,} \quad (8.21)$$

$$f(\vec{\xi}, \vec{\xi}') \geq \int \int_{\Omega_T} (T-t) \vec{\xi}' \cdot \vec{\xi} dt \quad \forall (\vec{\xi}, \vec{\xi}') \in D(\theta) \times L^2(\Omega_T)^3, \quad (8.22)$$

$$f(\vec{\xi}, \vec{\xi}') = \int \int_{\Omega_T} (T-t) \vec{\xi}' \cdot \vec{\xi} dt \quad \Leftrightarrow \quad \vec{\xi}' \in \theta(\vec{\xi}). \quad (8.23)$$

We may thus reformulate our problem as a minimization principle, for any  $\vec{G} \in L^2(\Omega_T)^3$ .

**Problem 8.1'.** Find  $\vec{H} \in D(\theta)$  that minimizes the functional

$$\Psi(\vec{v}) := f(\vec{v}, \vec{G}) - \int \int_{\Omega_T} (T-t) \vec{G} \cdot \vec{v} dx dt, \quad (8.24)$$

and this is equivalent to  $\Psi(\vec{H}) = 0$ .

Under the above hypotheses, it is readily seen that this problem has one and only one solution.

**Second relaxation mode.** Next we discuss the mode (8.2), assuming that (3.13)–(3.17) are fulfilled and that the fields  $\vec{G} \in H^1(0, T; V')$  and  $\vec{B}^0 \in L^2(\Omega)^3$  are prescribed. We thus deal with a system of the form

$$b\vec{B}_t + \vec{B} \in \vec{\alpha}(\vec{H}, x) \quad \text{in } \Omega_T, \quad (8.25)$$

$$\vec{B}_t + \nabla \times (A(x) \cdot \nabla \times \vec{H}) = \vec{G}_t \quad \text{in } \Omega_T, \quad (8.26)$$

$$\vec{B}(\cdot, 0) = \vec{B}^0 \quad \text{in } \Omega. \quad (8.27)$$

Here we shall not dwell on the weak formulation, which might be discussed along the lines of the previous example, and rather go directly to the formulation of a minimization principle. By eliminating the field  $B$  from (8.25) and (8.26), we get the integro-differential inclusion

$$\nabla \times [A(x) \cdot \nabla \times (b\vec{H} + \vec{H} * 1)] + \vec{\alpha}(\vec{H}, x) \ni \vec{G} + b\vec{G}_t, \quad (8.28)$$

that we may assume to hold in  $V'$ , for a.e.  $t \in ]0, T[$ . Let us next define the maximal monotone linear operator

$$\vec{\mu} : L^2(0, T; V) \rightarrow L^2(0, T; V') : \vec{v} \mapsto \nabla \times [A(x) \cdot \nabla \times (b\vec{v} + \vec{v} * 1)], \quad (8.29)$$

so that (8.28) also reads

$$\vec{\gamma}(\vec{H}) := \vec{\mu}(\vec{H}) + \vec{\alpha}(\vec{H}, x) \ni \vec{G} + b\vec{G}_t \quad \text{in } V', \text{ for a.e. } t \in ]0, T[, \quad (8.30)$$

with  $\vec{\gamma}$  maximal monotone. If  $\varphi : \mathbf{R}^3 \times \mathbf{R}^3 \times \Omega \rightarrow \mathbf{R} \cup \{+\infty\}$  is a representative function of  $\vec{\alpha}$ , then

$$\begin{aligned} g : D(\theta) \times L^2(\Omega_T)^3 &\rightarrow \mathbf{R} \cup \{+\infty\}: \\ (\vec{\xi}, \vec{\xi}') &\mapsto \iint_{\Omega_T} (T-t) [\varphi(\vec{\xi}, \vec{\xi}' - \vec{\mu}(\vec{\xi}), x) + \vec{\mu}(\vec{\xi}) \cdot \vec{\xi}] dx dt \\ &= \iint_{\Omega_T} \left\{ (T-t) \varphi(\vec{\xi}, \vec{\xi}' - \vec{\mu}(\vec{\xi}), x) \right. \\ &\quad \left. + \frac{b}{2} (T-t) |A(x)^{1/2} \cdot \nabla \times \vec{\xi}|^2 + \frac{1}{2} |A(x)^{1/2} \cdot \nabla \times \vec{\xi} * 1|^2 \right\} dx dt \end{aligned} \quad (8.31)$$

is a representative function of  $\vec{\gamma}$ , in a sense analogous to (8.21)–(8.23). We may thus reformulate the system (8.25)–(8.27) as a minimization principle.

**Proposition 8.2.** Find  $\vec{H} \in L^2(\Omega_T)^3 \cap D(\mu)$  such that

$$\Psi_1(\vec{v}) := g(\vec{v}, \vec{G}) - \iint_{\Omega_T} (T-t) (\vec{G} + b\vec{G}_t) \cdot \vec{v} dx dt, \quad (8.32)$$

and this is equivalent to  $\Psi_1(\vec{H}) = 0$ .

**Further relaxation modes.** One may consider two further relaxation dynamics:

$$b\vec{B}_t + \vec{\alpha}^{-1}(\vec{B}, x) \ni \vec{H} \quad \text{in } \Omega_T, \quad (8.33)$$

$$b\vec{H}_t + \vec{H} \ni \vec{\alpha}^{-1}(\vec{B}, x) \quad \text{in } \Omega_T. \quad (8.34)$$

Existence of a weak solution may be proved for either of these inclusions coupled with the Maxwell equation (8.8). On the other hand, here the formulation of a corresponding minimization principle seems less natural.

By inserting the displacement current  $\vec{D}_t$  into the Ampère law (3.1), an even richer landscape of models would be obtained.

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## Appendix A. Two-scale convergence

In this section we review some basic properties of two-scale convergence, after [1,24,25,41,48,61].

Let us denote by  $\mathcal{Y}$  the set  $[0, 1]^3$  equipped with the metric and the differential structure of the 3-dimensional unit torus; we may thus identify any  $[0, 1]^3$ -periodic function on  $\mathbf{R}^3$  with a function on  $\mathcal{Y}$ . Let us denote by  $\varepsilon$  a parameter that we assume to vanish along a fixed sequence. For any bounded sequence  $\{u_\varepsilon\}$  of  $L^2(\mathbf{R}^3)$ , and any  $u \in L^2(\mathbf{R}^3 \times \mathcal{Y})$ , we say that  $u_\varepsilon$  *weakly two-scale converges* to  $u$  in  $L^2(\mathbf{R}^3 \times \mathcal{Y})$ , and write  $u_\varepsilon \rightharpoonup_2 u$ , if

$$\int_{\mathbf{R}^3} u_\varepsilon(x) \theta(x, x/\varepsilon) dx \rightarrow \iint_{\mathbf{R}^3 \times \mathcal{Y}} u(x, y) \theta(x, y) dx dy \quad \forall \theta \in \mathcal{D}(\mathbf{R}^3 \times \mathcal{Y}). \quad (\text{A.1})$$

We shall denote the standard (*single-scale*) weak (strong, resp.) convergence by  $\rightharpoonup$  ( $\rightarrow$ , resp.).

We say that  $u_\varepsilon$  *strongly two-scale converges* to  $u$  in  $L^2(\mathbf{R}^3 \times \mathcal{Y})$ , and write  $u_\varepsilon \xrightarrow{2} u$ , whenever (A.1) holds and  $\|u_\varepsilon\|_{L^2(\mathbf{R}^3)} \rightarrow \|u\|_{L^2(\mathbf{R}^3 \times \mathcal{Y})}$ . These definitions are trivially generalized to vector-valued, and also to time-dependent functions, time being here regarded just as a parameter. These constructions and the next two statements take over to functions that are just defined on a subdomain of  $\mathbf{R}^3$ , by extending them with vanishing value outside that domain.

**Proposition A.1.** (See [1,48].) *For any bounded sequence  $\{u_\varepsilon\}$  of  $L^2(\mathbf{R}^3)$ , there exists  $u \in L^2(\mathbf{R}^3 \times \mathcal{Y})$  such that, possibly extracting a subsequence,*

$$u_\varepsilon \rightharpoonup_2 u \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y}). \quad (\text{A.2})$$

**Proposition A.2.** (See [1].) *If  $u_\varepsilon \rightharpoonup_2 u$  in  $L^2(\mathbf{R}^3 \times \mathcal{Y})$ , then*

$$u_\varepsilon \rightharpoonup \widehat{u} := \int_{\mathcal{Y}} u(\cdot, y) dy \quad \text{in } L^2(\mathbf{R}^3), \quad (\text{A.3})$$

$$\liminf_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^2(\mathbf{R}^3)} \geq \|u\|_{L^2(\mathbf{R}^3 \times \mathcal{Y})} \geq \|\widehat{u}\|_{L^2(\mathbf{R}^3)}. \quad (\text{A.4})$$

*If  $u_\varepsilon \rightarrow \widehat{u}$  in  $L^2(\mathbf{R}^3)$  then  $\widehat{u} = u$ , that is,  $u$  does not depend on  $y$ .*

Dealing with functions of  $(x, y)$ , we denote the gradient operator with respect to  $x$  ( $y$ , resp.) by  $\nabla_x$  ( $\nabla_y$ , resp.). We also set  $L^2_{\text{rot}}(\mathbf{R}^3)^3 := \{\tilde{v} \in L^2(\mathbf{R}^3)^3 : \nabla \times \tilde{v} \in L^2(\mathbf{R}^3)^3\}$ , which is a Hilbert space equipped with the graph norm. For any  $v \in L^1_{\text{loc}}(\mathbf{R}^3 \times \mathcal{Y})$  we define the *average* and *fluctuating* components,

$$\widehat{v}(x) := \int_{\mathcal{Y}} v(x, y) dy, \quad \widetilde{v}(x, y) := v(x, y) - \widehat{v}(x) \quad \text{for a.e. } (x, y) \in \mathbf{R}^3 \times \mathcal{Y}. \quad (\text{A.5})$$

We shall use the index  $*$  to label any subspace of functions of  $y \in \mathcal{Y}$  that have vanishing average. For instance,  $H^1_*(\mathcal{Y})$  is a Banach space equipped with the norm  $\|v\|_{H^1_*(\mathcal{Y})} := \|\nabla v\|_{L^2(\mathcal{Y})^3}$ .

**Proposition A.3.** (See [1,25,48].) If  $u_\varepsilon \rightharpoonup u$  in  $H^1(\mathbf{R}^3)$ , then there exists  $u_1 \in L^2(\mathbf{R}^3; H_*^1(\mathcal{Y}))$  such that, as  $\varepsilon \rightarrow 0$  along a suitable subsequence,

$$\nabla u_\varepsilon \rightharpoonup \nabla_x u + \nabla_y u_1 \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \quad (\text{A.6})$$

**Proposition A.4.** (See [61].) Let  $\{\tilde{u}_\varepsilon\}$  be a bounded sequence of  $L_{\text{rot}}^2(\mathbf{R}^3)^3$  such that  $\tilde{u}_\varepsilon \rightharpoonup \tilde{u}$  in  $L^2(\mathbf{R}^3 \times \mathcal{Y})^3$ . Then  $\widehat{\tilde{u}} \in L_{\text{rot}}^2(\mathbf{R}^3)^3$  and  $\nabla_y \times \tilde{u} = \tilde{0}$  in  $\mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3$ . Moreover there exists  $\tilde{u}_1 \in L^2(\mathbf{R}^3; H_*^1(\mathcal{Y})^3)$  such that  $\nabla_y \cdot \tilde{u}_1 = 0$  a.e. in  $\mathbf{R}^3 \times \mathcal{Y}$ , and, as  $\varepsilon \rightarrow 0$  along a suitable subsequence,

$$\nabla \times \tilde{u}_\varepsilon \rightharpoonup \nabla_x \times \widehat{\tilde{u}} + \nabla_y \times \tilde{u}_1 \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \quad (\text{A.7})$$

Conversely, for any  $\tilde{u} \in L^2(\mathbf{R}^3 \times \mathcal{Y})^3$  such that  $\widehat{\tilde{u}} \in L_{\text{rot}}^2(\mathbf{R}^3)^3$  and  $\nabla_y \times \tilde{u} = 0$  in  $\mathcal{D}'(\mathbf{R}^3 \times \mathcal{Y})^3$ , and for any  $\tilde{u}_1 \in L^2(\mathbf{R}^3; H_*^1(\mathcal{Y})^3)$ , there exists a sequence  $\{\tilde{u}_\varepsilon\}$  of  $H^1(\mathbf{R}^3)^3$  such that

$$\tilde{u}_\varepsilon \rightarrow \tilde{u} \quad \text{in } L^2(\mathbf{R}^3)^3, \quad \nabla \times \tilde{u}_\varepsilon \rightharpoonup \nabla \times \widehat{\tilde{u}} + \nabla_y \times \tilde{u}_1 \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \quad (\text{A.8})$$

The two latter results are easily extended to functions defined on any Lipschitz subdomain of  $\mathbf{R}^3$ .

**Proposition A.5.** (See [63].) If

$$\psi : \mathbf{R}^3 \times \mathcal{Y} \rightarrow \mathbf{R} \text{ is measurable w.r.t. } \mathcal{B}(\mathbf{R}^3) \otimes \mathcal{L}(\mathcal{Y}), \quad (\text{A.9})$$

$$\exists c > 0, \exists f \in L^1(\mathcal{Y}): \forall \xi \in \mathbf{R}^3, \text{ for a.e. } (x, y), \quad \psi(\xi, x, y) \geq c|\xi|^2 - f(y), \quad (\text{A.10})$$

then

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } L^2(\Omega \times \mathcal{Y})^3 \Rightarrow \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi(u_\varepsilon(x), x, x/\varepsilon) dx &= \int_{\Omega \times \mathcal{Y}} \psi(u(x, y), x, y) dx dy, \end{aligned} \quad (\text{A.11})$$

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } L^2(\Omega \times \mathcal{Y})^3 \Rightarrow \\ \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi(u_\varepsilon(x), x, x/\varepsilon) dx &\geq \int_{\Omega \times \mathcal{Y}} \psi(u(x, y), x, y) dx dy. \end{aligned} \quad (\text{A.12})$$

**Proposition A.6.** (See [61].)

(i) For any  $u \in L^2(\mathbf{R}^3; H_*^1(\mathcal{Y}))$  there exists a sequence  $\{u_\varepsilon\}$  of  $H^1(\mathbf{R}^3)$  such that

$$u_\varepsilon \rightarrow 0 \quad \text{in } L^2(\mathbf{R}^3), \quad \varepsilon \nabla u_\varepsilon \rightharpoonup \nabla_y u \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \quad (\text{A.13})$$

(ii) For any  $w \in L^2(\mathbf{R}^3; H_*^1(\mathcal{Y})^3)$ , there exists a sequence  $\{w_\varepsilon\}$  of  $H^1(\mathbf{R}^3)^3$  such that

$$w_\varepsilon \rightarrow 0 \quad \text{in } L^2(\mathbf{R}^3)^3, \quad \varepsilon \nabla \times w_\varepsilon \rightharpoonup \nabla_y \times w \quad \text{in } L^2(\mathbf{R}^3 \times \mathcal{Y})^3. \quad (\text{A.14})$$

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