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Regularity analysis for stochastic partial differential equations with nonlinear multiplicative trace class noise

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ABSTRACT

In this article spatial and temporal regularity of the solution process of a stochastic partial differential equation (SPDE) of evolutionary type with nonlinear multiplicative trace class noise is analyzed.

1. Introduction

Spatial and temporal regularity of the solution process of a stochastic partial differential equation (SPDE) of evolutionary type are investigated in this article. More precisely, it is analyzed under which conditions on the noise term of a semilinear SPDE the solution process enjoys values in the domains of fractional powers of the dominating linear operator of the SPDE. It turns out that the essential constituents determining the regularity of the solution process are assumptions on the covariance operator of the driving noise process of the SPDE and appropriate boundary conditions on the diffusion coefficient. While the regularity of (affine) linear SPDEs has been intensively studied in previous results (see, e.g., N.V. Krylov and B.L. Rozovskiĭ [7], B.L. Rozovskiĭ [11], G. Da Prato & J. Zabczyk [4], N.V. Krylov [6], Z. Brzeźniak [1], Z. Brzeźniak and J. van Neerven [2], S. Tindel et al. [13] and Z. Brzeźniak

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et al. [3]), the main purpose of this article is to handle possibly nonlinear diffusion coefficients in SPDEs driven by trace class Brownian noise (see also X. Zhang [19] for a related result).

In order to illustrate the results in this article, we concentrate on the following example SPDE in this introductory section and refer to Section 2 for our general setting and to Section 4 for further examples of SPDEs. Let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $H = L^2((0,1),\mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of square integrable functions from $(0,1)$ to $\mathbb{R}$. Moreover, let $f, b : (0,1)\times\mathbb{R} \to \mathbb{R}$ be two continuously differentiable functions with globally bounded derivatives, let $x_0 : (0,1) \to \mathbb{R}$ be a smooth function with $\lim_{x\searrow 0} x_0(x) = \lim_{x\nearrow 1} x_0(x) = 0$ and let $W : [0,T]\times\Omega \to H$ be a standard Q-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ with a covariance operator $Q : H \to H$. It is a classical result (see, e.g., Theorem VI.3.2 in [18]) that the covariance operator $Q : H \to H$ of the Wiener process $W : [0,T]\times\Omega \to H$ has an orthonormal basis $g_j \in H$, $j \in \{0,1,2,\ldots\}$, of eigenfunctions with summable eigenvalues $\mu_j \in [0,\infty)$, $j \in \{0,1,2,\ldots\}$. In order to have a more concrete example, we consider the choice $g_0(x) = 1$, $g_j(x) = \sqrt{2}\cos(j\pi x)$, $\mu_0 = 0$ and $\mu_j = j^{-r}$ for all $x \in (0,1)$ and all $j \in \mathbb{N}$ with a given real number $r \in (1,\infty)$ in the following and refer to Section 4 for possible further examples. Then we consider the SPDE

$$dX_t(x) = \left[\frac{\partial^2}{\partial x^2} X_t(x) + f\big(x, X_t(x)\big)\right] dt + b\big(x, X_t(x)\big)\, dW_t(x) \tag{1}$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = x_0(x)$ for $t \in [0,T]$ and $x \in (0,1)$. Under the assumptions above the SPDE (1) has a unique mild solution. Specifically, there exists an up to indistinguishability unique adapted stochastic process $X : [0,T]\times\Omega \to H$ with continuous sample paths which satisfies

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s, \tag{2}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ where $A : D(A) \subset H \to H$ is the Laplacian with Dirichlet boundary conditions and where $F : H \to H$ and $B : H \to HS(U_0, H)$ are given by $(F(v))(x) = f(x, v(x))$ and $(B(v)u)(x) = b(x, v(x)) \cdot u(x)$ for all $x \in (0,1)$, $v \in H$ and all $u \in U_0$. Here $U_0 = Q^{1/2}(H)$ with $\langle v, w\rangle_{U_0} = \langle Q^{-1/2}v, Q^{-1/2}w\rangle_H$ for all $v, w \in U_0$ is the image $\mathbb{R}$-Hilbert space of $Q^{\frac{1}{2}}$ (see Appendix C in [9]).

We are then interested to know for which $\gamma \in [0,\infty)$ in dependence on the decay rate $r \in (1,\infty)$ of the eigenfunctions of the covariance operator $Q : H \to H$ the solution process $X : [0,T]\times\Omega \to H$ of (1) takes values in $D((-A)^\gamma)$. For the SPDE (1) it turns out that

$$\mathbb{P}\big[X_t \in D\big((-A)^\gamma\big)\big] = 1 \tag{3}$$

holds for all $t \in [0,T]$ and all $\gamma \in [0, \frac{\min(3,r+1)}{4})$ (see Theorem 1 in Section 3 for the main result of this article and Subsection 4.1 for the SPDE (1)). Under further assumptions on the diffusion coefficient function $b : (0,1)\times\mathbb{R} \to \mathbb{R}$, the solution of (1) has even more regularity which can be seen in Subsection 4.2.

In the following we relate the results in this article with existing regularity results in the literature and also illustrate how (3) can be established. The regularity of linear SPDEs has been intensively analyzed in the literature (see, e.g., [7,11,4,6,1,2,13,3]). For instance, in Theorem 6.19 in [4], Da Prato and Zabczyk already showed for the SPDE (1) in the case $f(x,y) = 0$ for all $x \in (0,1)$, $y \in \mathbb{R}$ and $b : (0,1)\times\mathbb{R} \to \mathbb{R}$ sufficiently small and linear in the second variable that (3) holds for all $t \in [0,T]$ and all $\gamma \in [0, \frac{\min(4,r+1)}{4})$. Their key idea in Theorem 6.19 in [4] was to apply the Banach fixed point theorem in an appropriate Banach space of $D((-A)^\gamma)$-valued stochastic processes for $\gamma \in [0, \frac{\min(4,r+1)}{4})$. This approach is based on the fact that $B : H \to HS(U_0, H)$ is linear and globally Lipschitz continuous from $D((-A)^\gamma) \subset H$ to $HS(U_0, D((-A)^\gamma)) \subset HS(U_0, H)$ for $\gamma \in [0, \frac{\min(2,r-1)}{4})$

since $b:(0,1)\times\mathbb{R}\to\mathbb{R}$ is assumed to be linear in its second variable. Although their method in Theorem 6.19 in [4] works quite well for linear SPDEs, it cannot be generalized to nonlinear SPDEs of the form (1). More formally, in the case of a nonlinear $b:(0,1)\times\mathbb{R}\to\mathbb{R}$, $B:H\to HS(U_0,H)$ is in general not globally Lipschitz continuous from $D((-A)^\gamma)$ to $HS(U_0,D((-A)^\gamma))$ for $\gamma>0$ although $b:(0,1)\times\mathbb{R}\to\mathbb{R}$ is assumed to have globally bounded derivatives. Therefore, a contraction argument as in Theorem 6.19 in [4] (see also J. van Neerven et al. [17] for a related result) in a Banach space of $D((-A)^\gamma)$-valued stochastic processes for $\gamma>\frac{1}{2}$ can in general not be established for nonlinear SPDEs of the form (1). This difficulty is a key problem of regularity analysis for nonlinear SPDEs and has been pointed out in X. Zhang [19] (see p. 456 in [19]).

We now demonstrate our approach to analyze the regularity of (1) which overcomes the lack of Lipschitz continuity of $B:H\to HS(U_0,H)$ with respect to $D((-A)^\gamma)$ and $HS(U_0,D((-A)^\gamma))$ for $\gamma>0$ in the nonlinear case. First of all, by exploiting the smoothing effect of the semigroup of the Laplacian in (2), the existence of an up to modifications unique predictable $D((-A)^\gamma)$-valued solution process $X:[0,T]\times\Omega\to D((-A)^\gamma)$ of (1) with

$$\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t\|^2_{D((-A)^\gamma)}\big]<\infty \tag{4}$$

can be established immediately for all $\gamma\in[0,\frac{1}{2})$ (see J. van Neerven et al. [17] for details). However, we want to show (3) for all $t\in[0,T]$ and all $\gamma\in[0,\frac{\min(3,r+1)}{4})$ instead of $\gamma\in[0,\frac{1}{2})$. To this end a key estimate in our approach is the linear growth bound

$$\big\|B(v)\big\|_{HS(U_0,D((-A)^\alpha))}\leqslant c_\alpha\big(1+\|v\|_{D((-A)^\alpha)}\big) \tag{5}$$

for all $v\in D((-A)^\alpha)$ and all $\alpha\in[0,\frac{\min(1,r-1)}{4})$ with $c_\alpha\in[0,\infty)$, $\alpha\in[0,\frac{\min(1,r-1)}{4})$, appropriate which we sketch below. We would like to point out here that $B:H\to HS(U_0,H)$ fulfills the linear growth bound (5) although it fails to be globally Lipschitz continuous from $D((-A)^\alpha)$ to $HS(U_0,D((-A)^\alpha))$ for $\alpha>0$ in general (see Section 4 for the verification of (5) in the case of SPDEs of the form (1)). Exploiting estimate (5) in an appropriate bootstrap argument will then show (3) for all $t\in[0,T]$ and all $\gamma\in[0,\frac{\min(3,r+1)}{4})$. More formally, using that the semigroup is analytic with $e^{At}(H)\subset D(A)$ for all $t\in(0,T]$ yields

$$\begin{aligned}
&\int_0^t\mathbb{E}\big[\big\|(-A)^\gamma e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H)}\big]\,ds\\
&\quad\leqslant\int_0^t\big\|(-A)^\vartheta e^{A(t-s)}\big\|^2_{L(H)}\mathbb{E}\big[\big\|(-A)^{(\gamma-\vartheta)}B(X_s)\big\|^2_{HS(U_0,H)}\big]\,ds\\
&\quad\leqslant\int_0^t(t-s)^{-2\vartheta}\mathbb{E}\big[\big\|B(X_s)\big\|^2_{HS(U_0,D((-A)^{(\gamma-\vartheta)}))}\big]\,ds
\end{aligned}$$

and using estimate (5) then shows

$$\int_0^t\mathbb{E}\big[\big\|(-A)^\gamma e^{A(t-s)}B(X_s)\big\|^2_{HS(U_0,H)}\big]\,ds$$

$$\leqslant \int_0^t (t-s)^{-2\vartheta} |c_{(\gamma-\vartheta)}|^2 \mathbb{E}\big[\big(1 + \|X_s\|_{D((-A)^{(\gamma-\vartheta)})}\big)^2\big]\, ds$$

$$\leqslant 2|c_{(\gamma-\vartheta)}|^2 \left(\int_0^t s^{-2\vartheta}\, ds\right)\Big(1 + \sup_{s\in[0,T]} \mathbb{E}\big[\|X_s\|^2_{D((-A)^{(\gamma-\vartheta)})}\big]\Big)$$

$$\leqslant \frac{2|c_{(\gamma-\vartheta)}|^2 (T+1)}{(1-2\vartheta)} \Big(1 + \sup_{s\in[0,T]} \mathbb{E}\big[\|X_s\|^2_{D((-A)^{(\gamma-\vartheta)})}\big]\Big) < \infty \tag{6}$$

for all $t \in [0,T]$, $\vartheta \in (\gamma - \frac{\min(1,r-1)}{4}, \frac{1}{2})$ and all $\gamma \in [\frac{1}{2}, \frac{\min(3,r+1)}{4})$. We would like to point out that due to (4) the right-hand side of (6) is indeed finite. Of course, (6) then shows that $\int_0^t e^{A(t-s)} B(X_s)\, dW_s$, $t \in [0,T]$, has a modification with values in $D((-A)^\gamma)$ for all $\gamma \in [0, \frac{\min(3,r+1)}{4})$ and thus, (3) holds for all $t \in [0,T]$ and all $\gamma \in [0, \frac{\min(3,r+1)}{4})$.

Regularities of nonlinear SPDEs as analyzed here have also been investigated in Zhang's instructive paper [19]. In contrast to the results in this article, he investigated which conditions on the coefficients and the noise of an SPDE suffice to ensure that the solution process of the SPDE is infinitely often differentiable in the spatial variable, see Theorem 6.2 in [19]. The solution process of (1) in which we are interested is in general not twice differentiable in the spatial variable and therefore, Theorem 6.2 in [19] can in general not be applied to the SPDE (1) here.

The rest of this article is organized as follows. In Section 2 the setting and assumptions used are formulated. Our main result, Theorem 1, which states existence, uniqueness and regularity of solutions of an SPDE with nonlinear multiplicative trace class noise is presented in Section 3. This result is illustrated by various examples in Section 4. The proof of Theorem 1 is postponed to the final section.

2. Setting and assumptions

Throughout this article assume that the following setting is fulfilled.

Let $T \in (0,\infty)$ be a real number, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be two separable $\mathbb{R}$-Hilbert spaces. Moreover, let $Q : U \to U$ be a trace class operator and let $W : [0,T] \times \Omega \to U$ be a standard Q-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

Assumption 1 *(Linear operator A).* Let $A : D(A) \subset H \to H$ be a closed and densely defined linear operator which generates a strongly continuous analytic semigroup $e^{At} \in L(H)$, $t \in [0,\infty)$.

Let $\eta \in [0,\infty)$ be a nonnegative real number such that $\sigma(A) \subset \{\lambda \in \mathbb{C}\colon \operatorname{Re}(\lambda) < \eta\}$ where $\sigma(A) \subset \mathbb{C}$ denotes as usual the spectrum of the linear operator $A : D(A) \subset H \to H$. Such a real number exists since A is assumed to be a generator of a strongly continuous semigroup (see Assumption 1). By $V_r := D((\eta - A)^r) \subset H$ equipped with the norm $\|v\|_{V_r} := \|(\eta - A)^r v\|_H$ for all $v \in V_r$ and all $r \in [0,\infty)$ we denote the $\mathbb{R}$-Hilbert spaces of domains of fractional powers of the linear operator $\eta - A : D(A) \subset H \to H$ (see, e.g., Subsection 11.4.2 in Renardy and Roggers [10]).

Assumption 2 *(Drift term F).* Let $F : H \to H$ be a globally Lipschitz continuous mapping.

In order to formulate the assumption on the diffusion coefficient of our SPDE, we denote by $(U_0, \langle\cdot,\cdot\rangle_{U_0}, \|\cdot\|_{U_0})$ the separable $\mathbb{R}$-Hilbert space $U_0 := Q^{1/2}(U)$ with $\langle v, w\rangle_{U_0} = \langle Q^{-1/2}v, Q^{-1/2}w\rangle_U$ for all $v, w \in U_0$ (see, for example, Subsection 2.3.2 in [9]). Here $Q^{-1/2} : \operatorname{im}(Q^{1/2}) \subset U \to U$ denotes the pseudo inverse of $Q^{1/2} : U \to U$ (see, e.g., Appendix C in [9] for details).

Assumption 3 *(Diffusion term B).* Let $B : H \to HS(U_0, H)$ be a globally Lipschitz continuous mapping and let $\alpha \in [0, \frac{1}{2})$, $c \in [0, \infty)$ be real numbers such that $B(V_\alpha) \subset HS(U_0, V_\alpha)$ and $\|B(v)\|_{HS(U_0,V_\alpha)} \leqslant c(1 + \|v\|_{V_\alpha})$ for all $v \in V_\alpha$.

Assumption 4 *(Initial value ξ).* Let $\gamma \in [\alpha, \frac{1}{2} + \alpha)$, $p \in [2, \infty)$ and let $\xi : \Omega \to V_\gamma$ be an $\mathcal{F}_0/\mathcal{B}(V_\gamma)$-measurable mapping with $\mathbb{E}[\|\xi\|_{V_\gamma}^p] < \infty$.

Some examples satisfying Assumptions 1–4 are presented in Section 4.

3. Main result

The assumptions in Section 2 suffice to ensure the existence of a unique V_γ-valued solution of the SPDE (7).

Theorem 1 *(Existence and regularity of the solution). Assume that the setting in Section 2 is fulfilled. Then there exists an up to modifications unique predictable stochastic process $X : [0, T] \times \Omega \to V_\gamma$ which fulfills $\sup_{t\in[0,T]} \mathbb{E}[\|X_t\|_{V_\gamma}^p] < \infty$, $\sup_{t\in[0,T]} \mathbb{E}[\|B(X_t)\|_{HS(U_0,V_\alpha)}^p] < \infty$ and*

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s, \tag{7}$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Moreover, we have

$$\sup_{\substack{t_1,t_2\in[0,T]\\ t_1\neq t_2}} \frac{(\mathbb{E}[\|X_{t_2} - X_{t_1}\|_{V_r}^p])^{\frac{1}{p}}}{|t_2 - t_1|^{\min(\gamma-r,\frac{1}{2})}} < \infty \tag{8}$$

for every $r \in [0, \gamma)$. Additionally, the solution process X_t, $t \in [0, T]$, is even continuous with respect to $(\mathbb{E}[\|\cdot\|_{V_\gamma}^p])^{\frac{1}{p}}$.

The proof of Theorem 1 is given in Section 5. The parameters $\alpha \in [0, \frac{1}{2})$, $\gamma \in [\alpha, \frac{1}{2} + \alpha)$ and $p \in [2, \infty)$ used in Theorem 1 are given in Assumptions 3 and 4.

Estimate (8) and the continuity of the solution process X_t, $t \in [0, T]$, with respect to $(\mathbb{E}[\|\cdot\|_{V_\gamma}^p])^{\frac{1}{p}}$ as asserted in Theorem 1 can also be written as

$$X \in \bigcap_{r\in[0,\gamma)} \mathcal{C}^{\min(\gamma-r,\frac{1}{2})}\big([0, T], L^p(\Omega; V_r)\big). \tag{9}$$

Let us complete this section with the following remarks.

In this article we investigate predictable V_γ-valued solution processes of the SPDE (7). For results analyzing continuity of sample paths for H-valued solution processes of SPDEs of the form (7), the reader is referred to P. Kotelenez [5] and L. Tubaro [15], for instance.

If the initial value $X_0 = \xi$ of the SPDE (7) above is H-valued only, then X_t takes values in V_r for all $r < \frac{1}{2} + \alpha$ and all $t \in (0, T]$ nevertheless. More formally, if Assumptions 1–3 are fulfilled and if $\xi : \Omega \to H$ is an $\mathcal{F}_0/\mathcal{B}(H)$-measurable mapping with $\mathbb{E}[\|\xi\|_H^p] < \infty$ for some $p \in [2, \infty)$, then Theorem 1 shows the existence of a predictable solution process $X : [0, T] \times \Omega \to H$ of (7) and this process additionally satisfies $\mathbb{P}[X_t \in V_r] = 1$ with $\mathbb{E}[\|X_t\|_{V_r}^p] < \infty$ for all $r \in [0, \frac{1}{2} + \alpha)$ and all $t \in (0, T]$.

4. Examples

In this section Theorem 1 is illustrated with various examples. To this end let $d \in \mathbb{N}$ and let $H = U = L^2((0,1)^d, \mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable and Lebesgue square integrable functions from $(0,1)^d$ to $\mathbb{R}$. As usual we do not distinguish between a square integrable function from $(0,1)^d$ to $\mathbb{R}$ and its equivalence class in H. For simplicity we restrict our attention to the domain $(0,1)^d$ although more complicated domains in $\mathbb{R}^d$ could be considered. The scalar product and the norm in H and U are given by

$$\langle v, w\rangle_H = \langle v, w\rangle_U = \int\limits_{(0,1)^d} v(x) \cdot w(x)\, dx$$

and

$$\|v\|_H = \|v\|_U = \left(\int\limits_{(0,1)^d} \left|v(x)\right|^2 dx \right)^{\frac{1}{2}}$$

for all $v, w \in H = U$. Moreover, the Euclidean norm $\|x\|_{\mathbb{R}^d} := (|x_1|^2 + \cdots + |x_d|^2)^{\frac{1}{2}}$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is used here. Additionally, the notations

$$\|v\|_{C((0,1)^d,\mathbb{R})} := \sup_{x \in (0,1)^d} \left|v(x)\right| \in [0, \infty]$$

and

$$\|v\|_{C^r((0,1)^d,\mathbb{R})} := \sup_{x \in (0,1)^d} \left|v(x)\right| + \sup_{\substack{x,y \in (0,1)^d \\ x \neq y}} \frac{|v(x) - v(y)|}{\|x - y\|_{\mathbb{R}^d}^r} \in [0, \infty]$$

for all $r \in (0,1]$ and all functions $v : (0,1)^d \to \mathbb{R}$ are used in this section. We also define

$$\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})} := \left(\int\limits_{(0,1)^d} \left|v(x)\right|^2 dx + \int\limits_{(0,1)^d} \int\limits_{(0,1)^d} \frac{|v(x) - v(y)|^2}{\|x - y\|_{\mathbb{R}^d}^{(d+2r)}}\, dx\, dy \right)^{\frac{1}{2}} \in [0, \infty]$$

for all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $v : (0,1)^d \to \mathbb{R}$ and all $r \in (0,1)$. Finally, we denote by $v \cdot w : (0,1)^d \to \mathbb{R}$ the function

$$(v \cdot w)(x) = v(x) \cdot w(x), \quad x \in (0,1)^d,$$

for every $v, w : (0,1)^d \to \mathbb{R}$. Concerning the covariance operator of the Wiener process, let $\mathcal{J}$ be a countable set, let $(g_j)_{j \in \mathcal{J}} \subset U$ be an orthonormal basis of eigenfunctions of $Q : U \to U$ and let $(\mu_j)_{j \in \mathcal{J}} \subset [0, \infty)$ be the corresponding family of eigenvalues (such an orthonormal basis of eigenfunctions exists since $Q : U \to U$ is a trace class operator, see Proposition 2.1.5 in [9]). In particular, we have

$$Qu = \sum_{j \in \mathcal{J}} \mu_j \langle g_j, u\rangle_U g_j$$

for all $u \in U$. Furthermore, we assume in this section that the eigenfunctions $g_j \in U$, $j \in \mathcal{J}$, are continuous and satisfy

$$\sup_{j \in \mathcal{J}} \|g_j\|_{C((0,1)^d,\mathbb{R})} < \infty \quad \text{and} \quad \sum_{j \in \mathcal{J}} \bigl(\mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\bigr) < \infty \tag{10}$$

for some $\delta \in (0,1]$. We will give some concrete examples for $(g_j)_{j \in \mathcal{J}}$ fulfilling (10) later.

For *the linear operator in Assumption* 1, let $\kappa \in (0,\infty)$ be a fixed real number, let $\mathcal{I} = \mathbb{N}^d$ and let $\lambda_i \in \mathbb{R}$, $i \in \mathcal{I}$, and $e_i \in H$, $i \in \mathcal{I}$, be given by

$$\lambda_i = \kappa\pi^2\bigl(|i_1|^2 + \cdots + |i_d|^2\bigr), \qquad e_i(x) = 2^{\frac{d}{2}} \sin(i_1\pi x_1) \cdot \ldots \cdot \sin(i_d \pi x_d)$$

for all $x \in (x_1,\ldots,x_d) \in (0,1)^d$ and all $i = (i_1,\ldots,i_d) \in \mathbb{N}^d$. Next let

$$D(A) = \left\{ v \in H \colon \sum_{i \in \mathcal{I}} |\lambda_i|^2 \bigl|\langle e_i, v\rangle_H\bigr|^2 < \infty \right\}$$

and let

$$Av = \sum_{i \in \mathcal{I}} -\lambda_i \langle e_i, v\rangle_H e_i$$

for all $v \in D(A)$. Hence, the linear operator $A : D(A) \subset H \to H$ in Assumption 1 is nothing else but the Laplacian with Dirichlet boundary conditions times the constant $\kappa \in (0,\infty)$, i.e.

$$Av = \kappa \cdot \Delta v = \kappa\left(\left(\frac{\partial^2}{\partial x_1^2}\right)v + \cdots + \left(\frac{\partial^2}{\partial x_d^2}\right)v\right) \tag{11}$$

holds for all $v \in D(A)$ in this subsection (see, for instance, Subsection 3.8.1 in [12]).

In view of *the drift term in Assumption* 2, let $f : (0,1)^d \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{B}((0,1)^d \times \mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable function with $\int_{(0,1)^d} |f(x,0)|^2\,dx < \infty$ and

$$\sup_{x \in (0,1)^d} \sup_{\substack{y_1,y_2 \in \mathbb{R} \\ y_1 \neq y_2}} \left(\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|}\right) < \infty. \tag{12}$$

Then the (in general nonlinear) operator $F : H \to H$ given by

$$\bigl(F(v)\bigr)(x) = f\bigl(x, v(x)\bigr), \quad x \in (0,1)^d, \tag{13}$$

for all $v \in H$ satisfies Assumption 2, i.e.

$$\sup_{\substack{v,w \in H \\ v \neq w}} \frac{\|F(v) - F(w)\|_H}{\|v - w\|_H} < \infty \tag{14}$$

holds.

We now describe a class of *diffusion terms satisfying Assumption* 3. To this end let $q \in [0,\infty)$ be a real number and let $b : (0,1)^d \times \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$\bigl|b(x_1,y_1) - b(x_2,y_2)\bigr| \leqslant q\bigl(\|x_1 - x_2\|_{\mathbb{R}^d} + |y_1 - y_2|\bigr) \tag{15}$$

for all $x_1, x_2 \in (0,1)^d$ and all $y_1, y_2 \in \mathbb{R}$. In addition, we assume for simplicity that $\int_{(0,1)^d} |b(x,0)|^2 \, dx \leqslant q^2$. Then let $B : H \to HS(U_0, H)$ be the (in general nonlinear) operator given by

$$\big(B(v)u\big)(x) = \big(b(\cdot, v) \cdot u\big)(x) = b\big(x, v(x)\big) \cdot u(x), \quad x \in (0,1)^d, \tag{16}$$

for all $v \in H$ and all $u \in U_0 \subset U$. We now check step by step that $B : H \to HS(U_0, H)$ given by (16) satisfies Assumption 3. First of all, B is well defined. Indeed, we obviously have $U_0 \subset L^\infty((0,1)^d, \mathbb{R})$ continuously due to (10) and therefore, $B(v) : U_0 \to H$ is a bounded linear operator from U_0 to H for every $v \in H$. Moreover, we have

$$\begin{aligned}
\big\|B(v)\big\|^2_{HS(U_0,H)} &= \sum_{j\in\mathcal{J}} \big\|B(v)\sqrt{\mu_j} g_j\big\|^2_H = \sum_{j\in\mathcal{J}} \mu_j \big\|B(v) g_j\big\|^2_H \\
&= \sum_{j\in\mathcal{J}} \mu_j \bigg(\int_{(0,1)^d} \big|b\big(x, v(x)\big) \cdot g_j(x)\big|^2 \, dx \bigg) \\
&\leqslant \sum_{j\in\mathcal{J}} \mu_j \bigg(\int_{(0,1)^d} \big|b\big(x, v(x)\big)\big|^2 \, dx \bigg) \Big(\sup_{x\in(0,1)^d} \big|g_j(x)\big|^2 \Big)
\end{aligned}$$

and hence

$$\begin{aligned}
\big\|B(v)\big\|_{HS(U_0,H)} &\leqslant \big\|b(\cdot, v)\big\|_H \bigg(\sum_{j\in\mathcal{J}} \mu_j \bigg)^{\frac{1}{2}} \Big(\sup_{j\in\mathcal{J}} \|g_j\|_{C((0,1)^d,\mathbb{R})} \Big) \\
&= \big\|b(\cdot, v)\big\|_H \sqrt{\operatorname{Tr}(Q)} \Big(\sup_{j\in\mathcal{J}} \|g_j\|_{C((0,1)^d,\mathbb{R})} \Big) < \infty
\end{aligned}$$

for all $v \in H$ which shows that $B : H \to HS(U_0, H)$ is well defined. Moreover, $B : H \to HS(U_0, H)$ is globally Lipschitz continuous. More precisely, we have

$$\begin{aligned}
&\big\|B(v) - B(w)\big\|^2_{HS(U_0,H)} \\
&\quad = \sum_{j\in\mathcal{J}} \mu_j \big\|\big(B(v) - B(w)\big) g_j\big\|^2_H \\
&\quad = \sum_{j\in\mathcal{J}} \mu_j \bigg(\int_{(0,1)^d} \big|b\big(x, v(x)\big) - b\big(x, w(x)\big)\big|^2 \big|g_j(x)\big|^2 \, dx \bigg) \\
&\quad \leqslant \bigg(\sum_{j\in\mathcal{J}} \mu_j \bigg) \bigg(\int_{(0,1)^d} \big|b\big(x, v(x)\big) - b\big(x, w(x)\big)\big|^2 \, dx \bigg) \Big(\sup_{j\in\mathcal{J}} \|g_j\|^2_{C((0,1)^d,\mathbb{R})} \Big)
\end{aligned}$$

and therefore

$$\begin{aligned}
\big\|B(v) - B(w)\big\|_{HS(U_0,H)} &\leqslant q\|v - w\|_H \bigg(\sum_{j\in\mathcal{J}} \mu_j \bigg)^{\frac{1}{2}} \Big(\sup_{j\in\mathcal{J}} \|g_j\|_{C((0,1)^d,\mathbb{R})} \Big) \\
&= q\sqrt{\operatorname{Tr}(Q)} \Big(\sup_{j\in\mathcal{J}} \|g_j\|_{C((0,1)^d,\mathbb{R})} \Big) \|v - w\|_H
\end{aligned}$$

for all $v, w \in H$. Hence, it remains to check

$$B(V_\alpha) \subset HS(U_0, V_\alpha) \quad \text{and} \quad \left\| B(v) \right\|_{HS(U_0,V_\alpha)} \leqslant c\left(1 + \|v\|_{V_\alpha}\right) \tag{17}$$

for every $v \in V_\alpha$ for appropriate $\alpha \in [0, \frac{1}{2})$, $c \in [0, \infty)$. In order to verify (17), several preparations are needed. First, we review appropriate characterizations of the spaces $(V_r, \|\cdot\|_{V_r})$, $r \in (0, \frac{1}{2})$, from the literature. More formally, it is known that

$$V_r = \left\{ v \in H \colon \|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} < \infty \right\} \tag{18}$$

holds for all $r \in (0, \frac{1}{4})$, that

$$V_r = \left\{ v \in H \colon \|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} < \infty,\ v|_{\partial(0,1)^d} \equiv 0 \right\} \tag{19}$$

holds for all $r \in (\frac{1}{4}, \frac{1}{2})$ and that there are real numbers $C_r \in [1, \infty)$, $r \in (0, \frac{1}{2})$, such that

$$\frac{1}{C_r} \|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} \leqslant \|v\|_{V_r} \leqslant C_r \|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} \tag{20}$$

holds for all $v \in V_r$ and all $r \in (0, \frac{1}{2})$ (see, e.g., A. Lunardi [8] or also (A.46) in [4]). In particular, (18) shows

$$\|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} < \infty \quad \Longrightarrow \quad v \in V_r \tag{21}$$

for all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $v : (0,1)^d \to \mathbb{R}$ and all $r \in (0, \frac{1}{4})$. We remark that (21) does not hold for all $r \in (\frac{1}{4}, \frac{1}{2})$ instead of $r \in (0, \frac{1}{4})$ since a $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable function $v : (0,1)^d \to \mathbb{R}$ with $\|v\|_{W^{2r,2}((0,1)^d,\mathbb{R})} < \infty$ for some $r \in (0, \frac{1}{2})$ does, in general, not fulfill the Dirichlet boundary conditions in (19). In the next step observe that

$$\begin{aligned}
&\|v \cdot w\|^2_{W^{r,2}((0,1)^d,\mathbb{R})} \\
&\quad \leqslant \int\limits_{(0,1)^d} \left|v(x) \cdot w(x)\right|^2 dx + \int\limits_{(0,1)^d} \int\limits_{(0,1)^d} \frac{|v(x) \cdot w(x) - v(y) \cdot w(y)|^2}{\|x-y\|^{(d+2r)}_{\mathbb{R}^d}}\, dx\, dy \\
&\quad \leqslant \|v\|^2_H \|w\|^2_{C((0,1)^d,\mathbb{R})} + 2\|w\|^2_{C((0,1)^d,\mathbb{R})} \int\limits_{(0,1)^d} \int\limits_{(0,1)^d} \frac{|v(x) - v(y)|^2}{\|x-y\|^{(d+2r)}_{\mathbb{R}^d}}\, dx\, dy \\
&\qquad + 2 \int\limits_{(0,1)^d} \int\limits_{(0,1)^d} \frac{|v(y)|^2 |w(x) - w(y)|^2}{\|x-y\|^{(d+2r)}_{\mathbb{R}^d}}\, dx\, dy \\
&\quad \leqslant 2\|v\|^2_{W^{r,2}((0,1)^d,\mathbb{R})} \|w\|^2_{C((0,1)^d,\mathbb{R})} \\
&\qquad + 2\|v\|^2_H \Bigg(\sup_{\substack{x,y \in (0,1)^d \\ x \neq y}} \frac{|w(x) - w(y)|^2}{\|x-y\|^{2\delta}_{\mathbb{R}^d}} \Bigg) \Bigg(\int\limits_{(-1,1)^d} \|y\|^{(2\delta - d - 2r)}\, dy \Bigg)
\end{aligned}$$

for all $v \in H$, $r \in (0, 1)$ and all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $w : (0,1)^d \to \mathbb{R}$. The estimate

$$\int\limits_{(-1,1)^d} \|x\|_{\mathbb{R}^d}^z \, dx \leqslant \int\limits_{\{y\in\mathbb{R}^d:\ \|y\|_{\mathbb{R}^d}\leqslant\sqrt{d}\}} \|x\|_{\mathbb{R}^d}^z \, dx = \frac{\pi^{\frac{d}{2}} d}{\Gamma(\frac{d}{2}+1)} \int\limits_0^{\sqrt{d}} r^{(z+d-1)} \, dr \leqslant 3^d \int\limits_0^{\sqrt{d}} r^{(z+d-1)} \, dr$$
$$= \frac{3^d d^{\frac{(z+d)}{2}}}{(z+d)} \leqslant \frac{(3d)^d}{(d+z)} \tag{22}$$

for all $z \in (-d, d)$ therefore gives

$$\begin{aligned}
&\|v \cdot w\|_{W^{r,2}((0,1)^d,\mathbb{R})} \\
&\leqslant \sqrt{2}\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}\left(\|w\|_{C((0,1)^d,\mathbb{R})} + \sup_{\substack{x,y\in(0,1)^d \\ x\neq y}} \frac{|w(x)-w(y)|}{\|x-y\|_{\mathbb{R}^d}^{\delta}} \cdot \frac{(3d)^{\frac{d}{2}}}{\sqrt{2\delta-2r}}\right) \\
&\leqslant \left(\frac{(3d)^{\frac{d}{2}}}{\sqrt{\delta-r}}\right)\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}\|w\|_{C^{\delta}((0,1)^d,\mathbb{R})}
\end{aligned} \tag{23}$$

for all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $v, w : (0,1)^d \to \mathbb{R}$ and all $r \in (0, \delta)$ (see also Section 4.2 in H. Triebel [14]). In addition, note that the estimate $(a+b)^2 \leqslant 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$ and inequality (15) imply

$$\begin{aligned}
&\big\|b(\cdot, v)\big\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 \\
&= \int\limits_{(0,1)^d} \big|b\big(x, v(x)\big)\big|^2 \, dx + \int\limits_{(0,1)^d}\int\limits_{(0,1)^d} \frac{|b(x, v(x)) - b(y, v(y))|^2}{\|x-y\|_{\mathbb{R}^d}^{(d+2r)}} \, dx\, dy \\
&\leqslant \int\limits_{(0,1)^d} \big(q\big|v(x)\big| + \big|b(x,0)\big|\big)^2 \, dx + 2\int\limits_{(0,1)^d}\int\limits_{(0,1)^d} \frac{|b(x, v(x)) - b(x, v(y))|^2}{\|x-y\|_{\mathbb{R}^d}^{(d+2r)}} \, dx\, dy \\
&\quad + 2\int\limits_{(0,1)^d}\int\limits_{(0,1)^d} \frac{|b(x, v(y)) - b(y, v(y))|^2}{\|x-y\|_{\mathbb{R}^d}^{(d+2r)}} \, dx\, dy \\
&\leqslant 2q^2\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 + 2q^2 + 2q^2 \int\limits_{(0,1)^d}\int\limits_{(0,1)^d} \|x-y\|_{\mathbb{R}^d}^{(2-d-2r)} \, dx\, dy
\end{aligned}$$

for all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $v : (0,1)^d \to \mathbb{R}$ and all $r \in (0,1)$. Inequality (22) therefore shows

$$\begin{aligned}
\big\|b(\cdot, v)\big\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 &\leqslant 2q^2\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 + 2q^2 + q^2\frac{(3d)^d}{(1-r)} \\
&\leqslant q^2\left(2\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 + \frac{2(3d)^d}{(1-r)}\right) \\
&\leqslant \left(\frac{q^2\, 2(3d)^d}{(1-r)}\right)\big(\|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}^2 + 1\big)
\end{aligned}$$

this finally yields

$$\left\|b(\cdot, v)\right\|_{W^{r,2}((0,1)^d,\mathbb{R})} \leqslant \left(\frac{q(3d)^d}{(1-r)}\right)\left(1 + \|v\|_{W^{r,2}((0,1)^d,\mathbb{R})}\right) \tag{24}$$

for all $\mathcal{B}((0,1)^d)/\mathcal{B}(\mathbb{R})$-measurable functions $v:(0,1)^d \to \mathbb{R}$ and all $r \in (0,1)$. Combining (20) and (24) then, in particular, shows

$$\left\|b(\cdot, v)\right\|_{W^{2r,2}((0,1)^d,\mathbb{R})} \leqslant \left(\frac{qC_r(3d)^d}{(1-2r)}\right)\left(1 + \|v\|_{V_r}\right) < \infty \tag{25}$$

for all $v \in V_r$ and all $r \in (0, \frac{1}{2})$. Next we combine (23), (25) and (10) to obtain

$$\begin{aligned}
&\left(\sum_{j\in\mathcal{J}} \mu_j \left\|B(v)g_j\right\|^2_{W^{2r,2}((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \\
&\quad\leqslant \left(\frac{(3d)^{\frac{d}{2}}}{\sqrt{\delta-2r}}\right)\left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \left\|b(\cdot, v)\right\|_{W^{2r,2}((0,1)^d,\mathbb{R})} \\
&\quad\leqslant \left(\frac{q\,C_r(3d)^{2d}}{(\delta-2r)^2}\right)\left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \left(1 + \|v\|_{V_r}\right) < \infty
\end{aligned} \tag{26}$$

for all $v \in V_r$ and all $r \in (0, \frac{\delta}{2})$. The Cauchy–Schwartz inequality and estimate (26) then imply

$$\begin{aligned}
\left\|B(v)u\right\|_{W^{2r,2}((0,1)^d,\mathbb{R})} &= \left\|B(v)\left(\sum_{j\in\tilde{\mathcal{J}}} \mu_j \langle g_j, u\rangle_{U_0} g_j\right)\right\|_{W^{2r,2}((0,1)^d,\mathbb{R})} \\
&\leqslant \sum_{j\in\tilde{\mathcal{J}}} \left(\mu_j \left|\langle g_j, u\rangle_{U_0}\right| \left\|B(v)g_j\right\|_{W^{2r,2}((0,1)^d,\mathbb{R})}\right) \\
&\leqslant \left(\sum_{j\in\tilde{\mathcal{J}}} \left|\langle \sqrt{\mu_j} g_j, u\rangle_{U_0}\right|^2\right)^{\frac{1}{2}} \left(\sum_{j\in\tilde{\mathcal{J}}} \mu_j \left\|B(v)g_j\right\|^2_{W^{2r,2}((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \\
&\leqslant \left(\frac{q\,C_r(3d)^{2d}}{(\delta-2r)^2}\right)\left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \left(1 + \|v\|_{V_r}\right)\|u\|_{U_0} \\
&< \infty
\end{aligned} \tag{27}$$

for all $v \in V_r$, $r \in (0, \frac{\delta}{2})$, $u \in U_0$ with $u = \sum_{j\in\tilde{\mathcal{J}}} \mu_j \langle g_j, u\rangle_{U_0} g_j$ and all finite subsets $\tilde{\mathcal{J}} \subset \mathcal{J}$ of $\mathcal{J}$. This and (21) then show that $B(v)u \in V_r$ and that

$$\left\|B(v)u\right\|_{V_r} \leqslant \left(\frac{q|C_r|^2(3d)^{2d}}{(\delta-2r)^2}\right)\left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \left(1 + \|v\|_{V_r}\right)\|u\|_{U_0} \tag{28}$$

for all $v \in V_r$, $r \in (0, \min(\frac{1}{4}, \frac{\delta}{2}))$ and all $u \in U_0$. Therefore, we obtain that $B(v) \in L(U_0, V_r)$ for all $v \in V_r$ and all $r \in (0, \min(\frac{1}{4}, \frac{\delta}{2}))$. Hence, (20) and (26) give

$$\sum_{j\in\mathcal{J}}\bigl(\mu_j\bigl\|B(v)g_j\bigr\|_{V_r}^2\bigr)\leqslant\left(\frac{q|C_r|^2(3d)^{2d}}{(\delta-2r)^2}\right)^2\left(\sum_{j\in\mathcal{J}}\mu_j\|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)\bigl(1+\|v\|_{V_r}\bigr)^2<\infty \tag{29}$$

for all $v\in V_r$ and all $r\in(0,\min(\frac{1}{4},\frac{\delta}{2})$. Therefore, we obtain $B(v)\in HS(U_0,V_r)$ and

$$\bigl\|B(v)\bigr\|_{HS(U_0,V_r)}\leqslant\left(\frac{q|C_r|^2(3d)^{2d}}{(\delta-2r)^2}\right)\left(\sum_{j\in\mathcal{J}}\mu_j\|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}}\bigl(1+\|v\|_{V_r}\bigr)<\infty \tag{30}$$

for all $v\in V_r$ and all $r\in(0,\min(\frac{1}{4},\frac{\delta}{2}))$. This finally shows that Assumption 3 is fulfilled for all $\alpha\in[0,\min(\frac{1}{4},\frac{\delta}{2}))$.

Concerning *the initial value in Assumption* 4, let $x_0:[0,1]^d\to\mathbb{R}$ be a twice continuously differentiable function with $x_0|_{\partial(0,1)^d}\equiv 0$. Then the $\mathcal{F}_0/\mathcal{B}(V_\gamma)$-measurable mapping $\xi:\Omega\to V_\gamma$ given by $\xi(\omega)=x_0$ for all $\omega\in\Omega$ fulfills Assumption 4 for all $\gamma\in[\alpha,\frac{1}{2}+\alpha)$ and all $p\in[2,\infty)$.

Having constructed examples of Assumptions 1–4, we now formulate the SPDE (7) in the setting of this section. More formally, under the setting above the SPDE (7) reduces to

$$dX_t(x)=\bigl[\kappa\,\Delta X_t(x)+f\bigl(x,X_t(x)\bigr)\bigr]\,dt+b\bigl(x,X_t(x)\bigr)\,dW_t(x) \tag{31}$$

with $X_t|_{\partial(0,1)^d}\equiv 0$ and $X_0(x)=x_0(x)$ for $t\in[0,T]$ and $x\in(0,1)^d$. Moreover, we define a family $\beta^j:[0,T]\times\Omega\to\mathbb{R}$, $j\in\{k\in\mathcal{J}\mid\mu_k\neq 0\}$, of independent standard Brownian motions by

$$\beta_t^j(\omega):=\frac{1}{\sqrt{\mu_j}}\bigl\langle g_j,W_t(\omega)\bigr\rangle_U$$

for all $\omega\in\Omega$, $t\in[0,T]$ and all $j\in\mathcal{J}$ with $\mu_j\neq 0$. Using this notation, the SPDE (31) can be written as

$$dX_t(x)=\bigl[\kappa\,\Delta X_t(x)+f\bigl(x,X_t(x)\bigr)\bigr]\,dt+\sum_{\substack{j\in\mathcal{J}\\ \mu_j\neq 0}}\bigl[\sqrt{\mu_j}b\bigl(x,X_t(x)\bigr)g_j(x)\bigr]\,d\beta_t^j \tag{32}$$

with $X_t|_{\partial(0,1)^d}\equiv 0$ and $X_0(x)=x_0(x)$ for $t\in[0,T]$ and $x\in(0,1)^d$. Finally, due to (30), Theorem 1 shows the existence of an up to modifications unique predictable stochastic process $X:[0,T]\times\Omega\to V_\gamma$ fulfilling (32) for any $\gamma\in[0,\frac{\min(3,2\delta+2)}{4})$.

At this point we would like to thank an anonymous referee for pointing out to us that Theorem 1 can be generalized to SPDEs on UMD Banach spaces with type 2 by exploiting the results in van Neerven et al. [16]. In such a Banach space framework the state space $L^q((0,1)^2,\mathbb{R})$ with possibly large $q\in[2,\infty)$ can be considered instead of the Hilbert space $H=L^2((0,1)^d,\mathbb{R})$. By using appropriate Sobolev embeddings we then expect that one can even show that the solution process of the SPDE (32) enjoys values in the space $C^{2\gamma}((0,1)^d,\mathbb{R})$ of continuous differentiable functions from $(0,1)^d$ to $\mathbb{R}$ with $(2\gamma-1)$-Hölder continuous derivatives for any $\gamma\in(\frac{1}{2},\frac{\min(3,2\delta+2)}{4})$. The precise regularity study of the SPDE (32) in such a Banach space framework instead of the Hilbert space framework considered here remains an open question for future research.

In the next step we illustrate Theorem 1 using (27) and (30) in the following three more concrete examples.

4.1. A one dimensional stochastic reaction diffusion equation

Consider the situation described above in the case $d = 1$. In this subsection we want to give a concrete example for $(g_j)_{j\in\mathcal{J}}$ and $(\mu_j)_{j\in\mathcal{J}}$ so that (10) is fulfilled and all above applies. Let $\mathcal{J} = \{0, 1, 2, \ldots\}$, let $g_0(x) = 1$ and let $g_j(x) = \sqrt{2}\cos(j\pi x)$ for all $x \in (0, 1)$ and all $j \in \mathbb{N}$. Moreover, let $\rho \in (1, \infty)$ and $\nu \in (0, \infty)$ be given real numbers, let $\mu_0 = 0$ and let $\mu_j = \frac{\nu}{j^\rho}$ for all $j \in \mathbb{N}$. This choice ensures that (10) is fulfilled. Indeed, we have

$$\begin{aligned}\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})} &= \sum_{j=1}^{\infty} \frac{\nu}{j^\rho} \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})} \\ &= \sum_{j=1}^{\infty} \frac{2\nu}{j^\rho}\left(1 + \sup_{\substack{x,y\in(0,1)\\ x\neq y}} \frac{|\cos(j\pi x) - \cos(j\pi y)|}{|x-y|^\delta}\right)^2 \\ &\leqslant \sum_{j=1}^{\infty} \frac{2\nu}{j^\rho}\left(1 + \sup_{\substack{x,y\in(0,1)\\ x\neq y}} \frac{2^{(1-\delta)}|\cos(j\pi x) - \cos(j\pi y)|^\delta}{|x-y|^\delta}\right)^2\end{aligned}$$

and hence

$$\begin{aligned}\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})} &\leqslant \sum_{j=1}^{\infty} \frac{2\nu}{j^\rho}\bigl(1 + 2^{(1-\delta)}(j\pi)^\delta\bigr)^2 \\ &\leqslant \sum_{j=1}^{\infty} \frac{2\nu}{j^\rho}\bigl(1 + \pi\, j^\delta\bigr)^2 \\ &\leqslant 8\nu\pi^2\left(\sum_{j=1}^{\infty} j^{(2\delta-\rho)}\right) < \infty \end{aligned} \tag{33}$$

for all $\delta \in (0, \frac{\rho-1}{2})$. Assumption 3 is thus fulfilled for every $\alpha \in (0, \min(\frac{1}{4}, \frac{\rho-1}{4})) = (0, \frac{\min(1,\rho-1)}{4})$ (see (30)). Here the SPDE (32) reduces to

$$dX_t(x) = \left[\kappa \frac{\partial^2}{\partial x^2} X_t(x) + f\bigl(x, X_t(x)\bigr)\right] dt + \sum_{j=1}^{\infty}\left[\frac{\sqrt{2\nu}}{j^{\frac{\rho}{2}}}\, b\bigl(x, X_t(x)\bigr)\, \cos(j\pi x)\right] d\beta_t^j \tag{34}$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = x_0(x)$ for $t \in [0, T]$ and $x \in (0, 1)$. Theorem 1 finally yields the existence of an up to modifications unique stochastic process $X : [0, T] \times \Omega \to V_\gamma$ fulfilling (34) for any $\gamma \in [0, \frac{\min(3,\rho+1)}{4})$. Under further assumptions on $b : (0, 1) \times \mathbb{R} \to \mathbb{R}$, the solution of (34) enjoys even more regularity which is demonstrated in the following subsection.

4.2. More regularity for a one dimensional stochastic reaction diffusion equation

Consider the situation of Subsection 4.1 with $\rho = 3$. Hence, (33) shows that (10) holds for all $\delta \in (0, 1)$. Therefore, (30) gives that Assumption 3 is fulfilled for all $\alpha \in [0, \frac{1}{4})$. However, we now additionally assume that the diffusion coefficient $b : (0, 1) \times \mathbb{R} \to \mathbb{R}$ respects the Dirichlet boundary conditions in (32), i.e., we assume that

$$\lim_{x\searrow 0} b(x, x) = \lim_{x\nearrow 1} b(x, x-1) = 0 \tag{35}$$

holds. Under this additional assumption more regularity for the solution process of (32) can be established. More precisely, (35), (19) and (27) yield that $B(v)u \in V_r$ and that

$$\left\| B(v)u \right\|_{V_r} \leqslant \left(\frac{q|C_r|^2 (3d)^{2d}}{(\delta - 2r)^2} \right) \left(\sum_{j \in \mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d, \mathbb{R})} \right)^{\frac{1}{2}} \left(1 + \|v\|_{V_r}\right) \|u\|_{U_0} \tag{36}$$

for all $v \in V_r$, $r \in (\frac{1}{4}, \frac{1}{2})$ and all $u \in U_0$. This implies that $B(v) \in L(U_0, V_r)$ for all $v \in V_r$ and all $r \in (\frac{1}{4}, \frac{1}{2})$. Hence, (20) and (26) give

$$\sum_{j \in \mathcal{J}} \left(\mu_j \left\| B(v) g_j \right\|^2_{V_r} \right) \leqslant \left(\frac{q|C_r|^2 (3d)^{2d}}{(\delta - 2r)^2} \right)^2 \left(\sum_{j \in \mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d, \mathbb{R})} \right) \left(1 + \|v\|_{V_r}\right)^2 < \infty \tag{37}$$

for all $v \in V_r$ and all $r \in (\frac{1}{4}, \frac{1}{2})$. Thus, Assumption 3 is here even fulfilled for all $\alpha \in [0, \frac{1}{2})$. Theorem 1 finally shows that, under condition (35), the SPDE

$$dX_t(x) = \left[\kappa \frac{\partial^2}{\partial x^2} X_t(x) + f\left(x, X_t(x)\right) \right] dt + \sum_{j=1}^{\infty} \left[\frac{\sqrt{2\nu}}{j^{\frac{3}{2}}} b\left(x, X_t(x)\right) \cos(j\pi x) \right] d\beta_t^j$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = x_0(x)$ for $t \in [0, T]$ and $x \in (0, 1)$ admits an up to modifications unique predictable solution process $X : [0, T] \times \Omega \to V_\gamma$ for any $\gamma \in [0, 1)$.

4.3. *Stochastic reaction diffusion equations with commutative noise*

Consider the situation before Subsection 4.1 and assume that the eigenfunctions of the linear operator $A : D(A) \subset H \to H$ and of the covariance operator $Q : U = H \to H$ coincide. More formally, let $\mathcal{J} = \mathcal{I} = \mathbb{N}^d$, let $g_j = e_j$ for all $j \in \mathcal{J}$, let $\rho \in (d, d + 2)$ and $\nu \in (0, \infty)$ be given real numbers and let $\mu_j = \nu (j_1 + \cdots + j_d)^{-\rho}$ for all $j \in (j_1, \ldots, j_d) \in \mathcal{J} = \mathbb{N}^d$. We now check condition (10). To this end note that

$$\begin{aligned}
\left\| g_j'(x) \right\|_{L(\mathbb{R}^d, \mathbb{R})} &= \sup_{\substack{v \in \mathbb{R}^d \\ \|v\|_{\mathbb{R}^d} \leqslant 1}} \left| g_j'(x) v \right| \leqslant \sup_{\substack{v \in \mathbb{R}^d \\ \|v\|_{\mathbb{R}^d} \leqslant 1}} \left(\sum_{k=1}^d \left| \left(\frac{\partial g_j}{\partial x_k} \right)(x) \right| \cdot |v_k| \right) \\
&\leqslant \left(\sum_{k=1}^d \left| \left(\frac{\partial g_j}{\partial x_k} \right)(x) \right|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{k=1}^d \pi^2 |j_k|^2 2^d \right)^{\frac{1}{2}} \\
&= 2^{\frac{d}{2}} \pi \left(\sum_{k=1}^d |j_k|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

holds for all $x \in (0, 1)^d$ and all $j \in (j_1, \ldots, j_d) \in \mathcal{J}$. This implies

$$\begin{aligned}
\left| g_j(x) - g_j(y) \right| &\leqslant \int_0^1 \left| g_j'\left(x + r(y - x)\right)(y - x) \right| dr \\
&\leqslant 2^{\frac{d}{2}} \pi \left(\sum_{k=1}^d |j_k|^2 \right)^{\frac{1}{2}} \|x - y\|_{\mathbb{R}^d}
\end{aligned} \tag{38}$$

for all $x, y \in (0,1)^d$ and all $j \in \mathcal{J}$. Hence, we obtain

$$\|g_j\|_{C^\delta((0,1)^d,\mathbb{R})} \leqslant \|g_j\|_{C((0,1)^d,\mathbb{R})} + \sup_{\substack{x,y\in(0,1)^d\\ x\neq y}} \frac{|g_j(x)-g_j(y)|}{\|x-y\|^\delta_{\mathbb{R}^d}}$$
$$\leqslant 2^{\frac{d}{2}} + \sup_{\substack{x,y\in(0,1)^d\\ x\neq y}} \frac{(2\cdot 2^{\frac{d}{2}})^{(1-\delta)}|g_j(x)-g_j(y)|^\delta}{\|x-y\|^\delta_{\mathbb{R}^d}}$$

and

$$\|g_j\|_{C^\delta((0,1)^d,\mathbb{R})} \leqslant 2^{\frac{d}{2}} + 2^{(\frac{d}{2}+1)(1-\delta)}\left(2^{\frac{d}{2}}\pi\left(\sum_{k=1}^d |j_k|^2\right)^{\frac{1}{2}}\right)^\delta$$
$$\leqslant 2^{\frac{d}{2}} + 2^{\frac{d}{2}}\pi\left(\sum_{k=1}^d |j_k|^2\right)^{\frac{\delta}{2}} \leqslant 2^{(\frac{d}{2}+1)}\pi\left(\sum_{k=1}^d |j_k|^2\right)^{\frac{\delta}{2}} \tag{39}$$

for all $\delta \in (0,1]$ and all $j \in \mathcal{J}$. Therefore, we get

$$\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})} \leqslant \sum_{j\in\mathbb{N}^d} \nu(j_1+\cdots+j_d)^{-\rho} 2^{(d+2)}\pi^2\left(\sum_{k=1}^d |j_k|^2\right)^\delta$$
$$= \nu\, 2^{(d+2)}\pi^2\left(\sum_{j\in\mathbb{N}^d} \frac{(|j_1|^2+\cdots+|j_d|^2)^\delta}{(j_1+\cdots+j_d)^\rho}\right) < \infty$$

for all $\delta \in (0, \frac{\rho-d}{2})$ and hence, (10) holds for all $\delta \in (0, \frac{\rho-d}{2})$. Furthermore, since $g_j|_{\partial(0,1)^d} = e_j|_{\partial(0,1)^d} = 0$ for all $j \in \mathcal{J}$ here, (18), (19) and (27) yield that $B(v)u \in V_r$ and that

$$\big\|B(v)u\big\|_{V_r} \leqslant \left(\frac{q|C_r|^2(3d)^{2d}}{(\delta-2r)^2}\right)\left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)^{\frac{1}{2}} \big(1+\|v\|_{V_r}\big)\|u\|_{U_0} \tag{40}$$

for all $v \in V_r$, $r \in (0, \frac{\rho-d}{4})\backslash\{\frac{1}{4}\}$ and all $u \in U_0$. This implies that $B(v) \in L(U_0, V_r)$ for all $v \in V_r$ and all $r \in (0, \frac{\rho-d}{4})\backslash\{\frac{1}{4}\}$. Hence, (20) and (26) give

$$\sum_{j\in\mathcal{J}} \big(\mu_j \big\|B(v)g_j\big\|^2_{V_r}\big) \leqslant \left(\frac{q|C_r|^2(3d)^{2d}}{(\delta-2r)^2}\right)^2 \left(\sum_{j\in\mathcal{J}} \mu_j \|g_j\|^2_{C^\delta((0,1)^d,\mathbb{R})}\right)\big(1+\|v\|_{V_r}\big)^2 < \infty \tag{41}$$

for all $v \in V_r$ and all $r \in (0, \frac{\rho-d}{4})\backslash\{\frac{1}{4}\}$. Assumption 3 is thus fulfilled for all $\alpha \in [0, \frac{\rho-d}{4})\backslash\{\frac{1}{4}\}$ here. Theorem 1 therefore yields that the SPDE

$$dX_t(x) = \big[\kappa\,\Delta X_t(x) + f\big(x, X_t(x)\big)\big]\,dt$$
$$+ \sum_{j\in\mathbb{N}^d}\left[\frac{\sqrt{\nu 2^d}\,\sin(j_1\pi x_1)\cdot\ldots\cdot\sin(j_d\pi x_d)}{(j_1+\cdots+j_d)^{\frac{\rho}{2}}}\, b\big(x, X_t(x)\big)\right]d\beta^j_t \tag{42}$$

with $X_t|_{\partial(0,1)^d} \equiv 0$ and $X_0(x) = x_0(x)$ for all $t \in [0,T]$ and $x \in (0,1)^d$ enjoys an up to modifications unique predictable solution process $X : [0,T] \times \Omega \to V_\gamma$ fulfilling (42) for any $\gamma \in [0, \frac{\rho-d+2}{4})$.

5. Proof of Theorem 1

Throughout this section the notation

$$\|Z\|_{L^p(\Omega;E)} := \left(\mathbb{E}\left[\|Z\|_E^p\right]\right)^{\frac{1}{p}} \in [0,\infty]$$

is used for an $\mathbb{R}$-Banach space $(E, \|\cdot\|_E)$ and an $\mathcal{F}/\mathcal{B}(E)$-measurable mapping $Z : \Omega \to E$. The real number $p \in [2,\infty)$ is as given in Assumption 4. Next a well-known estimate for analytic semigroups is presented (see, e.g., Lemma 11.36 in Renardy and Roggers [10])

Lemma 1. *Assume that the setting in Section* 2 *is fulfilled. Then there exist real numbers $c_r \in [1,\infty)$, $r \in [0,1]$, such that*

$$\left\|\left(t(\eta - A)\right)^r e^{At}\right\|_{L(H)} \leqslant c_r \tag{43}$$

and

$$\left\|\left(t(\eta - A)\right)^{-r}\left(e^{At} - I\right)\right\|_{L(H)} \leqslant c_r \tag{44}$$

for all $t \in (0,T]$ and all $r \in [0,1]$.

Moreover, we would like to note the following remark.

Remark 1. Assume that the setting in Section 2 is fulfilled and let $Y : [0,T] \times \Omega \to HS(U_0,H)$ be a predictable stochastic process. Then we obtain $e^{At} Y_s(\omega) \in \bigcap_{u \in [0,\infty)} V_u$ for all $\omega \in \Omega$, $s \in [0,T]$ and all $t \in (0,T]$ since the semigroup $e^{At} \in L(H)$, $t \in [0,\infty)$, is analytic (see Assumption 1). In particular, if $\int_0^t \mathbb{E}[\|e^{A(t-s)} Y_s\|_{HS(U_0,V_r)}^2]\,ds < \infty$ for all $t \in [0,T]$ and some $r \in [0,\infty)$, then the stochastic process $\int_0^t e^{A(t-s)} Y_s\,dW_s$, $t \in [0,T]$, has a V_r-valued adapted modification.

Using Lemma 1 and Remark 1 we now present the proof of Theorem 1.

Proof of Theorem 1. The real number $R \in (0,\infty)$ given by

$$\begin{aligned} R := 1 + \left\|(\eta - A)^{-1}\right\|_{L(H)} + \left\|F(0)\right\|_H + \sup_{\substack{v,w \in H \\ v \neq w}} \left(\frac{\|F(v) - F(w)\|_H}{\|v - w\|_H}\right) \\ + \left\|B(0)\right\|_{HS(U_0,H)} + \sup_{\substack{v,w \in H \\ v \neq w}} \left(\frac{\|B(v) - B(w)\|_{HS(U_0,H)}}{\|v - w\|_H}\right) \end{aligned} \tag{45}$$

is used throughout this proof. Due to Assumptions 1–3 the number R is indeed finite. Moreover, let $\mathcal{V}_r$ for $r \in [0,\infty)$ be the $\mathbb{R}$-vector space of equivalence classes of V_r-valued predictable stochastic processes $Y : [0,T] \times \Omega \to V_r$ that satisfy

$$\sup_{t \in [0,T]} \mathbb{E}\left[\|Y_t\|_{V_r}^p\right] < \infty \tag{46}$$

where two stochastic processes lie in one equivalence class if and only if they are modifications of each other. As usual we do not distinguish between a predictable stochastic process $Y : [0,T] \times \Omega \to V_r$ satisfying (46) and its equivalence class in $\mathcal{V}_r$ for $r \in [0,\infty)$. Then we equip these spaces with the norms

$$\|Y\|_{\mathcal{V}_r,u} := \sup_{t\in[0,T]} \left(e^{ut}\|Y_t\|_{L^p(\Omega;V_r)}\right)$$

for all $Y \in \mathcal{V}_r$, $u \in \mathbb{R}$ and all $r \in [0,\infty)$. Note that the pair $(\mathcal{V}_r, \|\cdot\|_{\mathcal{V}_r,u})$ is an $\mathbb{R}$-Banach space for every $u \in \mathbb{R}$ and every $r \in [0,\infty)$. In the next step we consider the mapping $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ given by

$$(\Phi Y)_t := e^{At}\xi + \int_0^t e^{A(t-s)}F(Y_s)\,ds + \int_0^t e^{A(t-s)}B(Y_s)\,dW_s \tag{47}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and all $Y \in \mathcal{V}_\alpha$. In the following we show that $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ given by (47) is well defined.

To this end note that Assumptions 1 and 4 yield that $e^{At}\xi$, $t \in [0,T]$, is an adapted V_γ-valued stochastic process with continuous sample paths. Hence, $e^{At}\xi$, $t \in [0,T]$, is a $V_\gamma \subset V_\alpha$-valued predictable stochastic process (see Proposition 3.6(ii) in [4]). Additionally, we have

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\left[\left\|e^{At}\xi\right\|_{V_\gamma}^p\right] &\leqslant \left(\sup_{t\in[0,T]}\left\|e^{At}\right\|_{L(H)}^p\right)\mathbb{E}\left[\|\xi\|_{V_\gamma}^p\right] \\ &\leqslant |c_0|^p\,\mathbb{E}\left[\|\xi\|_{V_\gamma}^p\right] < \infty,\end{aligned} \tag{48}$$

which shows that $e^{At}\xi$, $t \in [0,T]$, is indeed in $\mathcal{V}_\gamma \subset \mathcal{V}_\alpha$.

We now concentrate on the second summand on the right-hand side of (47). First observe that the mapping $F|_{V_\alpha} : V_\alpha \to H$ given by $F|_{V_\alpha}(v) = F(v)$ for all $v \in V_\alpha$ is $\mathcal{B}(V_\alpha)/\mathcal{B}(H)$-measurable. Indeed, the Kuratowski theorem gives $V_\alpha \in \mathcal{B}(H)$ and $\mathcal{B}(V_\alpha) = \mathcal{B}(H) \cap V_\alpha$ which in turn implies the asserted Borel measurability of $F|_{V_\alpha}$. Next Lemma 1 and Jensen's inequality yield

$$\begin{aligned}\int_0^t \mathbb{E}\left[\left\|e^{A(t-s)}F(Y_s)\right\|_{V_\gamma}\right]ds &\leqslant \int_0^t \left\|(\eta - A)^\gamma e^{A(t-s)}\right\|_{L(H)}\mathbb{E}\left[\left\|F(Y_s)\right\|_H\right]ds \\ &\leqslant Rc_\gamma \int_0^t (t-s)^{-\gamma}\left(1 + \mathbb{E}\left[\|Y_s\|_H\right]\right)ds \\ &\leqslant \frac{Rc_\gamma T^{(1-\gamma)}}{(1-\gamma)}\left(1 + \sup_{s\in[0,T]}\mathbb{E}\left[\|Y_s\|_H\right]\right) \\ &\leqslant \frac{Rc_\gamma T^{(1-\gamma)}}{(1-\gamma)}\left(1 + \sup_{s\in[0,T]}\|Y_s\|_{L^p(\Omega;H_\alpha)}\right) < \infty\end{aligned}$$

for all $t \in [0,T]$ and all $Y \in \mathcal{V}_\alpha$. This shows that $\int_0^t e^{A(t-s)}F(Y_s)\,ds$, $t \in [0,T]$, is a well-defined V_γ-valued (and in particular V_α-valued) adapted stochastic process for every $Y \in \mathcal{V}_\alpha$. Moreover, we have

$$\left\|\int_0^{t_2} e^{A(t_2-s)}F(Y_s)\,ds - \int_0^{t_1} e^{A(t_1-s)}F(Y_s)\,ds\right\|_{L^p(\Omega;V_r)}$$

$$\leqslant \left\| \int_{t_1}^{t_2} e^{A(t_2-s)} F(Y_s)\, ds \right\|_{L^p(\Omega; V_r)} + \left\| \left(e^{A(t_2-t_1)} - I\right) \int_0^{t_1} e^{A(t_1-s)} F(Y_s)\, ds \right\|_{L^p(\Omega; V_r)}$$

$$\leqslant \int_{t_1}^{t_2} \left\| (\eta - A)^r e^{A(t_2-s)} \right\|_{L(H)} \left\| F(Y_s) \right\|_{L^p(\Omega; H)} ds$$

$$+ \left\| (\eta - A)^{(r-\gamma-\varepsilon)} \left(e^{A(t_2-t_1)} - I\right) \right\|_{L(H)} \int_0^{t_1} \left\| e^{A(t_1-s)} F(Y_s) \right\|_{L^p(\Omega; V_{\gamma+\varepsilon})} ds$$

and Lemma 1 thus shows

$$\left\| \int_0^{t_2} e^{A(t_2-s)} F(Y_s)\, ds - \int_0^{t_1} e^{A(t_1-s)} F(Y_s)\, ds \right\|_{L^p(\Omega; V_r)}$$

$$\leqslant c_r \int_{t_1}^{t_2} (t_2 - s)^{-r} \left\| F(Y_s) \right\|_{L^p(\Omega; H)} ds$$

$$+ c_{(\gamma+\varepsilon-r)} c_{(\gamma+\varepsilon)} (t_2 - t_1)^{(\gamma+\varepsilon-r)} \int_0^{t_1} (t_1 - s)^{-(\gamma+\varepsilon)} \| F(Y_s) \|_{L^p(\Omega; H)}\, ds$$

$$\leqslant R \left(\frac{c_r (t_2 - t_1)^{(1-r)}}{(1-r)} + \frac{c_{(\gamma+\varepsilon-r)} c_{(\gamma+\varepsilon)} (t_2 - t_1)^{(\gamma+\varepsilon-r)} T^{(1-\gamma-\varepsilon)}}{(1-\gamma-\varepsilon)} \right)$$

$$\cdot \left(1 + \sup_{s \in [0,T]} \|Y_s\|_{L^p(\Omega; H)}\right)$$

for all $t_1, t_2 \in [0, T]$ with $t_1 \leqslant t_2$, $\varepsilon \in [0, 1-\gamma)$, $r \in [0, \gamma]$ and all $Y \in \mathcal{V}_\alpha$. This finally shows

$$\left\| \int_0^{t_2} e^{A(t_2-s)} F(Y_s)\, ds - \int_0^{t_1} e^{A(t_1-s)} F(Y_s)\, ds \right\|_{L^p(\Omega; V_r)}$$

$$\leqslant R \left(\frac{T^{(1-\gamma-\varepsilon)}}{(1-\gamma-\varepsilon)} \right) \left(1 + \sup_{s \in [0,T]} \|Y_s\|_{L^p(\Omega; H)}\right) (c_r + c_{(\gamma+\varepsilon-r)} c_{(\gamma+\varepsilon)}) (t_2 - t_1)^{(\gamma+\varepsilon-r)} \tag{49}$$

for all $t_1, t_2 \in [0, T]$ with $t_1 \leqslant t_2$, $\varepsilon \in [0, 1-\gamma)$, $r \in [0, \gamma]$ and all $Y \in \mathcal{V}_\alpha$. Proposition 3.6(ii) in [4] thus yields that the stochastic process $\int_0^t e^{A(t-s)} F(Y_s)\, ds$, $t \in [0, T]$, has a modification in $\mathcal{V}_\gamma \subset \mathcal{V}_\alpha$ for every $Y \in \mathcal{V}_\alpha$.

In the sequel we concentrate on the third summand on the right-hand side of (47). First observe that Kuratowski's theorem shows $V_\alpha \in \mathcal{B}(H)$, $HS(U_0, V_\alpha) \in \mathcal{B}(HS(U_0, H))$, $\mathcal{B}(V_\alpha) = \mathcal{B}(H) \cap V_\alpha$ and $\mathcal{B}(HS(U_0, V_\alpha)) = \mathcal{B}(HS(U_0, H)) \cap HS(U_0, V_\alpha)$. This implies that the mapping $\tilde{B} : V_\alpha \to HS(U_0, V_\alpha)$ given by $\tilde{B}(v) = B(v)$ for all $v \in V_\alpha$ is $\mathcal{B}(V_\alpha)/\mathcal{B}(HS(U_0, V_\alpha))$-measurable. Next Lemma 1 gives

$$\int_0^t \mathbb{E}\big[\big\|e^{A(t-s)}B(Y_s)\big\|^2_{HS(U_0,V_\gamma)}\big]\,ds$$
$$\leqslant \int_0^t \big\|(\eta-A)^{(\gamma-\alpha)}e^{A(t-s)}\big\|^2_{L(H)}\mathbb{E}\big[\big\|B(Y_s)\big\|^2_{HS(U_0,V_\alpha)}\big]\,ds$$
$$\leqslant 2c^2|c_{(\gamma-\alpha)}|^2\int_0^t (t-s)^{(2\alpha-2\gamma)}\big(1+\mathbb{E}\big[\|Y_s\|^2_{V_\alpha}\big]\big)\,ds$$
$$\leqslant \left(\frac{2c^2|c_{(\gamma-\alpha)}|^2T^{(1+2\alpha-2\gamma)}}{(1+2\alpha-2\gamma)}\right)\Big(1+\sup_{s\in[0,T]}\mathbb{E}\big[\|Y_s\|^2_{V_\alpha}\big]\Big)<\infty$$

for all $t\in[0,T]$ and all $Y\in\mathcal{V}_\alpha$. Therefore, Remark 1 shows that $\int_0^t e^{A(t-s)}B(Y_s)dW_s$, $t\in[0,T]$, is a well-defined V_γ-valued (and in particular V_α-valued) adapted stochastic process for every $Y\in\mathcal{V}_\alpha$ (cf. the heuristic calculation (6) in the introduction). Moreover, the Burkholder–Davis–Gundy type inequality in Lemma 7.7 in [4] gives

$$\left\|\int_0^{t_2} e^{A(t_2-s)}B(Y_s)\,dW_s-\int_0^{t_1} e^{A(t_1-s)}B(Y_s)\,dW_s\right\|_{L^p(\Omega;V_r)}$$
$$\leqslant \left\|\int_{t_1}^{t_2} e^{A(t_2-s)}B(Y_s)\,dW_s\right\|_{L^p(\Omega;V_r)}+\left\|\big(e^{A(t_2-t_1)}-I\big)\int_0^{t_1} e^{A(t_1-s)}B(Y_s)\,dW_s\right\|_{L^p(\Omega;V_r)}$$
$$\leqslant p\left(\int_{t_1}^{t_2}\big\|e^{A(t_2-s)}B(Y_s)\big\|^2_{L^p(\Omega;HS(U_0,V_r))}\,ds\right)^{\frac{1}{2}}$$
$$+\,p\big\|e^{A(t_2-t_1)}-I\big\|_{L(H,V_{(r-\gamma-\varepsilon)})}\left(\int_0^{t_1}\big\|e^{A(t_1-s)}B(Y_s)\big\|^2_{L^p(\Omega;HS(U_0,V_{\gamma+\varepsilon}))}\,ds\right)^{\frac{1}{2}}$$

and Lemma 1 therefore shows

$$\left\|\int_0^{t_2} e^{A(t_2-s)}B(Y_s)\,dW_s-\int_0^{t_1} e^{A(t_1-s)}B(Y_s)\,dW_s\right\|_{L^p(\Omega;V_r)}$$
$$\leqslant p\left(\int_{t_1}^{t_2}\big\|(\eta-A)^{(r-\alpha)}e^{A(t_2-s)}\big\|^2_{L(H)}\big\|B(Y_s)\big\|^2_{L^p(\Omega;HS(U_0,V_\alpha))}\,ds\right)^{\frac{1}{2}}$$
$$+\,pc_{(\gamma+\varepsilon-r)}c_{(\gamma+\varepsilon-\alpha)}(t_2-t_1)^{(\gamma+\varepsilon-r)}$$
$$\cdot\left(\int_0^{t_1}(t_1-s)^{(2\alpha-2\gamma-2\varepsilon)}\big\|B(Y_s)\big\|^2_{L^p(\Omega;HS(U_0,V_\alpha))}\,ds\right)^{\frac{1}{2}} \qquad (50)$$

for all $t_1, t_2 \in [0,T]$ with $t_1 \leqslant t_2$, $\varepsilon \in [0, \frac{1}{2} + \alpha - \gamma)$, $r \in [0,\gamma]$ and all $Y \in \mathcal{V}_\alpha$. In the case $r \in [\alpha, \gamma]$ we have

$$\left\| (\eta - A)^{(r-\alpha)} e^{As} \right\|_{L(H)} \leqslant c_{(r-\alpha)} s^{(\alpha - r)} \tag{51}$$

for all $s \in (0,T]$ (see Lemma 1) and in the case $r \in [0,\alpha)$ we have

$$\left\| (\eta - A)^{(r-\alpha)} e^{As} \right\|_{L(H)} \leqslant \left\| (\eta - A)^{(r-\alpha)} \right\|_{L(H)} c_0 \leqslant c_0 R \tag{52}$$

for all $s \in (0,T]$. Combining (51) and (52) shows

$$\begin{aligned} \left(\int_0^t \left\| (\eta - A)^{(r-\alpha)} e^{As} \right\|_{L(H)}^2 ds \right)^{\frac{1}{2}} &\leqslant \left(\int_0^t \left(|c_{\max(r-\alpha,0)}|^2 s^{(2\alpha - 2r)} + |c_{\max(r-\alpha,0)}|^2 R^2 \right) ds \right)^{\frac{1}{2}} \\ &\leqslant c_{\max(r-\alpha,0)} R \left(\frac{t^{(1/2+\alpha-r)}}{\sqrt{1+2\alpha-2r}} + t^{1/2} \right) \end{aligned} \tag{53}$$

and hence

$$\begin{aligned} \left(\int_0^t \left\| (\eta - A)^{(r-\alpha)} e^{As} \right\|_{L(H)}^2 ds \right)^{\frac{1}{2}} &\leqslant \frac{c_{\max(r-\alpha,0)} R (t^{(1/2+\alpha-r)} + t^{1/2})}{\sqrt{1+2\alpha-2\gamma-2\varepsilon}} \\ &\leqslant \frac{c_{\max(r-\alpha,0)} R (T^{(1/2+\alpha-\gamma-\varepsilon)} t^{(\gamma+\varepsilon-r)} + t^{1/2})}{\sqrt{1+2\alpha-2\gamma-2\varepsilon}} \end{aligned} \tag{54}$$

for all $t \in [0,T]$, $\varepsilon \in [0, \frac{1}{2} + \alpha - \gamma)$ and all $r \in [0,\gamma]$. Using (54) in (50) then gives

$$\begin{aligned} &\left\| \int_0^{t_2} e^{A(t_2 - s)} B(Y_s)\, dW_s - \int_0^{t_1} e^{A(t_1 - s)} B(Y_s)\, dW_s \right\|_{L^p(\Omega; V_r)} \\ &\quad \leqslant \left(\left(\frac{2p c_{\max(r-\alpha,0)} R \max(T,1)(t_2 - t_1)^{\min(\gamma+\varepsilon-r, \frac{1}{2})}}{\sqrt{1+2\alpha-2\gamma-2\varepsilon}} \right) \right. \\ &\qquad \left. + \left(\frac{p c_{(\gamma+\varepsilon-r)} c_{(\gamma+\varepsilon-\alpha)} \max(T,1)(t_2 - t_1)^{\min(\gamma+\varepsilon-r, \frac{1}{2})}}{\sqrt{1+2\alpha-2\gamma-2\varepsilon}} \right) \right) \\ &\qquad \cdot \left(\sup_{t \in [0,T]} \left\| B(Y_t) \right\|_{L^p(\Omega; HS(U_0, V_\alpha))} \right) \end{aligned} \tag{55}$$

for all $t_1, t_2 \in [0,T]$ with $t_1 \leqslant t_2$, $\varepsilon \in [0, \frac{1}{2} + \alpha - \gamma)$, $r \in [0,\gamma]$ and all $Y \in \mathcal{V}_\alpha$. Proposition 3.6(ii) in [4] thus yields that $\int_0^t e^{A(t-s)} B(Y_s)\, dW_s$, $t \in [0,T]$, has a modification in $\mathcal{V}_\gamma \subset \mathcal{V}_\alpha$ for every $Y \in \mathcal{V}_\alpha$ and this finally shows the well definedness of $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ in (47) (see (48), (49) and (55)).

In the next step we show that $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ is a contraction with respect to $\| \cdot \|_{\mathcal{V}_\alpha, u}$ for an appropriate $u \in \mathbb{R}$. The Banach fixed point theorem will then yield the existence of a unique fixed point for $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$. More formally, Lemma 7.7 in [4] gives

$$\begin{aligned}
&\big\|(\Phi Y)_t - (\Phi Z)_t\big\|_{L^p(\Omega; V_\alpha)} \\
&\quad \leqslant \left\| \int_0^t e^{A(t-s)}\big(F(Y_s) - F(Z_s)\big)\, ds \right\|_{L^p(\Omega;V_\alpha)} + \left\| \int_0^t e^{A(t-s)}\big(B(Y_s) - B(Z_s)\big)\, dW_s \right\|_{L^p(\Omega;V_\alpha)} \\
&\quad \leqslant \int_0^t \big\|(\eta - A)^\alpha e^{A(t-s)}\big\|_{L(H)} \big\|F(Y_s) - F(Z_s)\big\|_{L^p(\Omega;H)}\, ds \\
&\qquad + p\left(\int_0^t \big\|(\eta - A)^\alpha e^{A(t-s)}\big\|_{L(H)}^2 \big\|B(Y_s) - B(Z_s)\big\|_{L^p(\Omega;HS(U_0,H))}^2\, ds \right)^{\frac{1}{2}}
\end{aligned}$$

and the definition of R and Lemma 1 yield

$$\begin{aligned}
\big\|(\Phi Y)_t - (\Phi Z)_t\big\|_{L^p(\Omega;V_\alpha)} &\leqslant Rc_\alpha \int_0^t (t-s)^{-\alpha} \|Y_s - Z_s\|_{L^p(\Omega;H)}\, ds \\
&\quad + pRc_\alpha \left(\int_0^t (t-s)^{-2\alpha} \|Y_s - Z_s\|_{L^p(\Omega;H)}^2\, ds \right)^{\frac{1}{2}} \\
&\leqslant Rc_\alpha \left(\int_0^t (t-s)^{-\alpha} e^{-us}\, ds \right) \|Y - Z\|_{\mathcal{V}_0,u} \\
&\quad + pRc_\alpha \left(\int_0^t (t-s)^{-2\alpha} e^{-2us}\, ds \right)^{\frac{1}{2}} \|Y - Z\|_{\mathcal{V}_0,u}
\end{aligned}$$

for all $t \in [0,T]$, $Y, Z \in \mathcal{V}_\alpha$ and all $u \in \mathbb{R}$. The Cauchy–Schwartz inequality therefore implies

$$\begin{aligned}
\big\|\Phi(Y) - \Phi(Z)\big\|_{\mathcal{V}_\alpha,u} &\leqslant Rc_\alpha(\sqrt{T} + p) \left(\int_0^T s^{-2\alpha} e^{2us}\, ds \right)^{\frac{1}{2}} \|Y - Z\|_{\mathcal{V}_0,u} \\
&\leqslant Rc_\alpha(\sqrt{T} + p) \left(\int_0^T \frac{e^{2us}}{s^{2\alpha}}\, ds \right)^{\frac{1}{2}} \|Y - Z\|_{\mathcal{V}_\alpha,u}
\end{aligned} \tag{56}$$

for all $Y, Z \in \mathcal{V}_\alpha$ and all $u \in \mathbb{R}$. This shows that $\Phi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha$ is a contraction with respect to $\|\cdot\|_{\mathcal{V}_\alpha,u}$ for a sufficiently small $u \in (-\infty, 0)$. Hence, there is an up to modifications unique predictable stochastic process $Y : [0,T] \times \Omega \to V_\alpha \in \mathcal{V}_\alpha$ with $\Phi(Y) = Y$, i.e.

$$Y_t = e^{At}\xi + \int_0^t e^{A(t-s)} F(Y_s)\, ds + \int_0^t e^{A(t-s)} B(Y_s)\, dW_s \tag{57}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. Moreover, (48), (49), (55) and Proposition 3.6 (ii) in [4] then show that there exists a predictable modification $X : [0,T] \times \Omega \to V_\gamma$ of $Y : [0,T] \times \Omega \to V_\alpha$.

Additionally, note that the inequality $\|B(v)\|^p_{HS(U_0,H_\alpha)} \leqslant 2^p c^p (1 + \|v\|^p_{H_\alpha})$ for all $v \in H_\alpha$ (see Assumption 3) implies

$$\sup_{t\in[0,T]} \mathbb{E}\big[\big\|B(X_t)\big\|^p_{HS(U_0,V_\alpha)}\big] \leqslant 2^p c^p \Big(1 + \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^p_{V_\alpha}\big]\Big) < \infty. \tag{58}$$

It remains to establish the temporal continuity properties asserted in Theorem 1. To this end note that Lemma 1 gives

$$\begin{aligned}
\big\|e^{At_2}\xi - e^{At_1}\xi\big\|_{L^p(\Omega;V_r)} &= \big\|e^{At_1}(\eta - A)^{(r-\gamma)}\big(e^{A(t_2-t_1)} - I\big)(\eta - A)^\gamma \xi\big\|_{L^p(\Omega;H)} \\
&\leqslant \big\|e^{At_1}\big\|_{L(H)} \big\|(\eta - A)^{(r-\gamma)}\big(e^{A(t_2-t_1)} - I\big)\big\|_{L(H)} \|\xi\|_{L^p(\Omega;V_\gamma)} \\
&\leqslant c_0 c_{(\gamma-r)} \|\xi\|_{L^p(\Omega;V_\gamma)} (t_2 - t_1)^{(\gamma-r)}
\end{aligned} \tag{59}$$

for all $t_1, t_2 \in [0,T]$ with $t_1 \leqslant t_2$ and all $r \in [0,\gamma]$. Combining (49), (55) and (59) then yields (8). Finally, (48), (49) and (55) show that X_t, $t \in [0,T]$, is continuous with respect to $(\mathbb{E}[\|\cdot\|^p_{V_\gamma}])^{\frac{1}{p}}$. This completes the proof of Theorem 1. $\square$

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