



Linear Schrödinger evolution equations with moving Coulomb singularities

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ABSTRACT

The Cauchy problem for Schrödinger evolution equations with a finite number of moving Coulomb singularities is investigated. The case of a single singularity has been studied by several authors (see, e.g., Baudouin et al. (2005) [1] and Okazawa et al. (2010) [19]). However, it seems to be no previous work on plural singularities. We shall show that the problem has a unique (classical) solution by using a time-dependent linear transformation of the unknown function which locally freezes the motion of the whole singularities under the simplest collisionless condition. In fact, a new existence and uniqueness theorem is available for the transformed problem. Such an abstract framework is established from the viewpoint of linear evolution equations of hyperbolic type in a Hilbert space as an innovative modification of those in Okazawa (1998) [17] and Okazawa and Yoshii (2011) [20].

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1. Introduction

Given an integer $N \geq 3$, let $H^s(\mathbb{R}^N)$ be the usual Sobolev space and define

$$\begin{aligned}\Sigma^s(\mathbb{R}^N) &:= H^s(\mathbb{R}^N) \cap H_s(\mathbb{R}^N), \\ H_s(\mathbb{R}^N) &:= \{u \in L^2(\mathbb{R}^N); |x|^s u \in L^2(\mathbb{R}^N)\}, \quad s > 0.\end{aligned}$$

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Put $I := [0, T]$. Then the present paper concerns the Cauchy problem for the Schrödinger equation with moving Coulomb singularities in $L^2(\mathbb{R}^N)$:

$$\begin{cases} i \frac{\partial u}{\partial t} - \Delta u + V_0(t, x)u + \sum_{j=1}^m \frac{e_j u}{|x - c_j(t)|} + \sum_{1 \leq j < k \leq m} \frac{e_j e_k u}{|c_j(t) - c_k(t)|} = f(t, x), \\ (t, x) \in I \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 \in \Sigma^2 = \Sigma^2(\mathbb{R}^N), \end{cases} \quad (\text{P})$$

where $u : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ is an unknown function and $c_j : I \rightarrow \mathbb{R}^N$ expresses the center of j -th singularity $|x - c_j(t)|^{-1}$ with charge e_j ($1 \leq j \leq m$), while $V_0 : I \times \mathbb{R}^N \rightarrow \mathbb{R}$ is another potential with singularity at infinity and $f : I \times \mathbb{R}^N \rightarrow \mathbb{C}$ is an external force. From the physical viewpoint (P) with $N = 3$ describes the situation where the wave function of an electron, satisfying the Schrödinger evolution equation, is influenced by m nuclei. Thus the last term on the left-hand side of the equation is regarded as the interactions among the nuclei (see, e.g., Lieb [13, Part I]).

The term of the repulsive potential among the nuclei does not generate singularities since we assume that $c_j(t) \neq c_k(t)$ ($j \neq k$) as a collisionless condition. Setting

$$V(t, x) := V_0(t, x) + \sum_{1 \leq j < k \leq m} \frac{e_j e_k}{|c_j(t) - c_k(t)|},$$

we obtain our problem

$$\begin{cases} i \frac{\partial u}{\partial t} - \Delta u + V(t, x)u + \sum_{j=1}^m \frac{e_j u}{|x - c_j(t)|} = f(t, x), & (t, x) \in I \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 \in \Sigma^2 = \Sigma^2(\mathbb{R}^N). \end{cases} \quad (\text{SE})$$

Therefore the main interest in (SE) is how to deal with the plural singularities.

It seems to be difficult to solve (SE) directly by applying a known theorem on abstract evolution equations because the singularities are not fixed. In fact, put

$$L := -\Delta + V(t, x) + \sum_{j=1}^m \frac{e_j}{|x - c_j(t)|}.$$

If $c_j(t) \equiv c_j^0$, where c_j^0 is a constant vector ($j = 1, \dots, m$), then Yajima [24,25] has given a sufficient condition to guarantee the selfadjointness of L . This suggests that we have to fix every local singularity. Actually, we could find such a time-dependent transformation. However, applying the transformation, L is mapped into the general elliptic operator in divergence form

$$\widehat{L} := -\operatorname{div}(a(t, y)\nabla) + W(t, y) + \sum_{j=1}^m \frac{e_j}{|y - c_j(0)|}.$$

In this way, we are led to construct a new abstract theorem because \widehat{L} is outside of Yajima's framework in [24,25].

The above mentioned transformation may be called a “local pseudo-Galilean transformation”. This is an analogous notion to the “local pseudo-Lorentz transformation” introduced by Kato and Yajima [12] to deal with the Dirac equation with Liénard–Wiechert singularities (the moving Coulomb potential is relativistically replaced with the Liénard–Wiechert potential). Here the transformation is a

mapping from the original unknown function u to a new unknown function v . The basic idea of such transformations goes back to Hunziker [3].

Now we want to state our main theorem from which one can observe how to define the transformation.

Theorem 1.1. *Let $\{c_j\}$, V and f satisfy the conditions:*

$$c_j \in W^{2,1}(I; \mathbb{R}^N) \quad (j = 1, 2, \dots, m), \quad (\text{c1})$$

$$c_j(t) \neq c_k(t) \quad (t \in I, j \neq k), \quad (\text{c2})$$

$$(1 + |x|^2)^{-1} V \in W^{1,1}(I; L^\infty(\mathbb{R}^N)), \quad (\text{V1})$$

$$V \in L^1(I; W_{\text{loc}}^{1,\infty}(\mathbb{R}^N)), \quad (\text{V2})$$

$$f \in C(I; L^2(\mathbb{R}^N)) \cap L^1(I; \Sigma^2(\mathbb{R}^N)). \quad (\text{f1})$$

Then for every initial value $u_0 \in \Sigma^2$ problem (SE) has a unique (classical) solution

$$u(\cdot) \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; \Sigma^2(\mathbb{R}^N)).$$

First we note that (c2) is a collisionless condition. In fact, conditions (c1) and (c2) yield the following two constants:

$$\begin{aligned} \|c'\|_{L^\infty(I)} &:= \max\{\|(d/dt)c_j\|_{L^\infty(I)^N}; j = 1, \dots, m\} < \infty, \\ \varepsilon_0 &:= \frac{1}{4} \min\{|c_j(t) - c_k(t)|; t \in I, 1 \leq j < k \leq m\} > 0. \end{aligned} \quad (1.1)$$

Therefore we can choose a short time interval $[0, T_0]$, where

$$T_0 := \frac{\varepsilon_0}{2\|c'\|_{L^\infty(I)}}. \quad (1.2)$$

1.1. Definition of the transformation

Let $\{I_k\}_{k=1}^{k_0}$ be a division of the time interval I and u_k the restriction of u to I_k :

$$u_k := u|_{I_k} \quad (1 \leq k \leq k_0).$$

Here, using T_0 in (1.2), $k_0 \in \mathbb{N}$ may be determined as

$$I_k := [(k-1)T_0, kT_0] \quad (k = 1, \dots, k_0-1), \quad I_{k_0} := [(k_0-1)T_0, T]$$

with $|I_{k_0}| \leq T_0 = |I_k|$ ($1 \leq k \leq k_0-1$). In this connection note that if u is a solution to (SE), then the family $\{u_k\}$ has the following property:

$$u_k((k-1)T_0, \cdot) = \begin{cases} u_0, & k = 1, \\ u_{k-1}((k-1)T_0, \cdot), & 2 \leq k \leq k_0. \end{cases} \quad (1.3)$$

The desired transformation is now given by a family

$$\{\Phi_k(t); t \in I_k, 1 \leq k \leq k_0\} \quad (1.4)$$

of unitary operators on $L^2(\mathbb{R}^N)$. Set

$$v_k := \Phi_k(t)u_k, \quad t \in I_k, 1 \leq k \leq k_0.$$

Next let $\varphi_k \in C^1(I_k \times \mathbb{R}^N; \mathbb{R}^N)$ ($1 \leq k \leq k_0$). More precisely, we assume that $\varphi_k(t, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is C^3 -diffeomorphic for a fixed $t \in I_k$ ($1 \leq k \leq k_0$). Then we can define the k -th unitary operator on the family (1.4) by

$$v_k(t, y) = (\Phi_k(t)u_k)(t, y) := (J_y \varphi_k(t, y))^{1/2} u_k(t, \varphi_k(t, y)), \quad (t, y) \in I_k \times \mathbb{R}^N. \quad (1.5)$$

Here $J_y \varphi_k(t, y)$ is the Jacobian of φ_k :

$$J_y \varphi_k(t, y) := \det(\text{Jac}_y \varphi_k(t, y)), \quad \text{Jac}_y \varphi_k(t, y) := \left(\frac{\partial(\varphi_k)_p}{\partial y_q}(t, y) \right)_{pq};$$

we shall construct the Jacobian to keep the sign positive. As is well-known in calculus, (1.5) yields that for $t \in I_k$ and $1 \leq k \leq k_0$,

$$\int_{\mathbb{R}^N} |v_k(t, y)|^2 dy = \int_{\mathbb{R}^N} (J_y \varphi_k(t, y)) |u_k(t, \varphi_k(t, y))|^2 dy = \int_{\mathbb{R}^N} |u_k(t, x)|^2 dx; \quad (1.6)$$

moreover, the positivity of Jacobian on each I_k guarantees the existence of the inverse function $\psi_k(t, \cdot)$ which defines $\Phi_k^{-1}(t)$:

$$(\Phi_k^{-1}(t)v_k)(t, x) = (J_x \psi_k(t, x))^{1/2} v_k(t, \psi_k(t, x)), \quad (t, x) \in I_k \times \mathbb{R}^N.$$

Here we can give a concrete form of φ_k . However, it is possible and simple to describe the transformed problem without using the concrete form of φ_k . By a careful computation we shall show that (SE) restricted on I_k is converted into the following Cauchy problem in $L^2(\mathbb{R}^N)$:

$$\begin{aligned} i \frac{\partial v_k}{\partial t} + \left(\frac{1}{i} \text{div} - b_k(t, y) \right) a_k(t, y) \left(\frac{1}{i} \nabla - b_k(t, y) \right) v_k + r_k(t, y) v_k \\ + V(t, \varphi_k(t, y)) v_k + \sum_{j=1}^m \frac{e_j v_k}{|\varphi_k(t, y) - c_j(t)|} = g_k(t, y), \quad (t, y) \in I_k \times \mathbb{R}^N, \end{aligned} \quad (1.7)$$

$$v_k((k-1)T_0, \cdot) = \begin{cases} \Phi_1(0)u_0, & k=1, \\ \Phi_k((k-1)T_0)u_{k-1}((k-1)T_0, \cdot), & 2 \leq k \leq k_0; \end{cases} \quad (1.8)$$

note that the initial value of $v_k(\cdot)$ is given in terms of $u_{k-1}(\cdot)$ (see also (1.14) below). In Eq. (1.7) the coefficient functions are given as follows:

$$a_k(t, y) := (\text{Jac}_y \varphi_k(t, y))^{-1} {}^t (\text{Jac}_y \varphi_k(t, y))^{-1}, \quad (1.9)$$

$$b_k(t, y) := \frac{1}{2} {}^t (\text{Jac}_y \varphi_k(t, y)) \frac{\partial \varphi_k}{\partial t}(t, y), \quad (1.10)$$

$$r_k(t, y) := -\frac{1}{4} \left| \frac{\partial \varphi_k}{\partial t}(t, y) \right|^2 - (J_y \varphi_k(t, y))^{1/2} [\Delta_x (J_x \psi_k(t, x))^{1/2} |_{x=\varphi_k}], \quad (1.11)$$

$$g_k(t, y) := (J_y \varphi_k(t, y))^{1/2} f(t, \varphi_k(t, y)), \quad (1.12)$$

where $\Delta_x := \sum_{p=1}^N (\partial^2 / \partial x_p^2)$ and t denotes the transposed. We have used $\psi_k(t, x)$ to shorten the second term on the right-hand side of (1.11) though it can be written down in terms of only $\varphi_k(t, y)$. The detailed computation will be found in the proof of Lemma 5.2 given at the beginning of Appendix A.

For every k the function $\varphi_k(t, y)$ is expected to fix the whole singularities:

$$\sum_{j=1}^m \frac{e_j}{|\varphi_k(t, y) - c_j(t)|} = h_k(t, y) + \sum_{j=1}^m \frac{e_j}{|y - c_j(0)|}, \quad t \in I_k, \quad (1.13)$$

where h_k is bounded and smooth enough.

Remark 1. In particular, if $m = 1$, then a translation suffices for the transformation:

$$(\Phi_I u)(t, y) := u(t, y + c_1(t)),$$

$$\varphi_I(t, y) := y + c_1(t), \quad t \in I;$$

note that in this case Φ_I is defined on the whole interval I . Thus (1.7) is simplified as

$$i \frac{\partial v}{\partial t} + \left(\frac{1}{i} \nabla - \frac{1}{2} c_1'(t) \right)^2 v - \frac{1}{4} |c_1'(t)|^2 v + V(t, y + c_1(t)) v + \frac{e_1}{|y|} v = f(t, y + c_1(t)),$$

where $c_1' = (d/dt)c_1$. Namely, $a(t, y)$ is the unit matrix and $b(t, y) = c_1'(t)/2$. This case has already been solved by Baudouin, Kavian and Puel [1] partly with formal computation. The argument in [1] is modified with rigorous proofs by Okazawa, Yokota and Yoshii [19] (for C^1 -solvability see Yoshii [26]; cf. also Yajima [25]).

Now let us introduce a final form of φ_k for $(t, y) \in I_k \times \mathbb{R}^N$:

$$\varphi_k(t, y) := y + \sum_{j=1}^m \zeta \left(\frac{|y - c_j((k-1)T_0)|}{\varepsilon_0} \right) (c_j(t) - c_j((k-1)T_0)),$$

where $\varepsilon_0 > 0$ is defined by (1.1) and ζ is a cut-off function as defined in Lemma 5.1. In particular, we see that

$$\varphi_k((k-1)T_0, y) = y$$

and hence $\Phi_k((k-1)T_0)$ is the identity. Accordingly, (1.8) is simplified as the initial value for Eq. (1.7):

$$v_k((k-1)T_0, \cdot) = \begin{cases} u_0, & k = 1, \\ u_{k-1}((k-1)T_0, \cdot), & 2 \leq k \leq k_0. \end{cases} \quad (1.14)$$

Furthermore, this φ_k realizes (1.13) (for detail see the proof of Lemma 5.3) together with the positivity of its Jacobian.

The main purpose of this paper is to show that Eq. (1.7) with initial value (1.14) has a unique (classical) solution

$$v_k(\cdot) \in C^1(I_k; L^2(\mathbb{R}^N)) \cap C(I_k; \Sigma^2(\mathbb{R}^N));$$

in fact, the continuation of this kind of family $\{v_k(\cdot); 1 \leq k \leq k_0\}$ leads us to our main theorem (Theorem 1.1) through the inverse transformation. Since $-\Delta$ in (SE) is replaced with the general elliptic operator in divergence form as in (1.7), we have to construct a new abstract theorem which will be described in the next subsection.

1.2. Abstract setting: linear evolution equations of hyperbolic type

Let $\{A(t); t \in I\}$ be a family of closed linear operators in a complex Hilbert space X . Then we consider the abstract Cauchy problem for linear evolution equations of the form

$$\begin{cases} (d/dt)u(t) + A(t)u(t) = f(t), & t \in I, \\ u(0) = u_0. \end{cases} \quad (\text{ACP})$$

Here the initial value u_0 is selected as follows. Let S_0 be a selfadjoint operator in X satisfying

$$(u, S_0 u) \geq \|u\|^2 \quad \text{for } u \in D(S_0), \quad (1.15)$$

where (\cdot, \cdot) and $\|\cdot\|$ are the inner product and norm of X . Then the square root $S_0^{1/2}$ is well-defined and

$$Y := D(S_0^{1/2})$$

is also a Hilbert space, with norm $\|v\|_Y := \|S_0^{1/2}v\|$ and inner product $(u, v)_Y := (S_0^{1/2}u, S_0^{1/2}v)$, embedded continuously and densely in X . Y plays the role of the space of initial values as in Theorem 1.3 (see below).

To introduce our assumption on $\{A(t); t \in I\}$ we need one more family $\{S(t); t \in I\}$ of auxiliary operators in X .

Assumption on $\{S(t)\}$. The family $\{S(t)\}$ satisfies the following three conditions:

(S1) For every $t \in I$, $S(t)$ is positive selfadjoint in X , with $D(S(t)^{1/2}) := Y = D(S_0^{1/2})$ and there exists a constant $K \geq 1$ such that

$$K^{-1} \|S_0^{1/2}u\|^2 \leq (u, S(t)u) \leq K \|S_0^{1/2}u\|^2, \quad u \in D(S(t)), \quad t \in I.$$

(S2) $S(\cdot)^{1/2} \in C_*(I; B(Y, X))$, where $B(Y, X)$ is the space of all bounded linear operators on Y to X , with norm $\|\cdot\|_{B(Y, X)}$, while the subscript $*$ is used to refer the strong operator topology in $B(Y, X)$ (for this notation see Kato [10]).

(S3) There exists a nonnegative function $\sigma \in L^1(I)$ such that

$$\|S(t)^{1/2}v\| - \|S(s)^{1/2}v\| \leq \left| \int_s^t \sigma(r) dr \right| \max_{r \in [s, t]} \|S(r)^{1/2}v\|, \quad v \in Y, \quad t, s \in I.$$

In connection with the symbol $B(Y, X)$ we shall also use the abbreviation: $B(X) := B(X, X)$, $B(Y) := B(Y, Y)$.

Let S_0 and $\{S(t)\}$ be as defined above. Then we may introduce the following

Assumption on $\{A(t)\}$. The family $\{A(t)\}$ satisfies the following four conditions:

(A1) There exists a constant $\alpha \geq 0$ such that

$$|\operatorname{Re}(A(t)v, v)| \leq \alpha \|v\|^2, \quad v \in D(A(t)), \quad t \in I. \quad (1.16)$$

(A2) $Y \subset D(A(t))$, $t \in I$.

(A3) There exists a constant $\beta \geq \alpha$ such that

$$|\operatorname{Re}(A(t)u, S(t)u)| \leq \beta \|S(t)^{1/2}u\|^2, \quad u \in D(S(t)) \subset Y, \quad t \in I. \quad (1.17)$$

(A4) $A(\cdot) \in C_*(I; B(Y, X))$.

Remark 2. Let $\{A(t)\}$ be a family of closed linear operators in X . Then $\{A(t)\}$ satisfies conditions (A1)–(A4) if and only if $\{-A(t)\}$ does. This is the reason why we employ the term, hyperbolic type. In fact, the above-mentioned equivalence does not in general hold for linear evolution equations of “hyperbolic” type introduced by Kato [6,7] (see also Remark 4 below).

Theorem 1.2. Let S_0 be selfadjoint in X , with (1.15). Suppose that Assumptions on $\{A(t)\}$ and $\{S(t)\}$ are satisfied. Then there exists a unique evolution operator $\{U(t, s); (t, s) \in \Delta_+\}$ for (ACP), where $\Delta_+ := \{(t, s); 0 \leq s \leq t \leq T\}$, having the following properties in terms of K, α, β and σ in Assumptions on $\{A(t)\}$ and $\{S(t)\}$:

(i) $U(\cdot, \cdot)$ is strongly continuous on Δ_+ to $B(X)$, with

$$\|U(t, s)\|_{B(X)} \leq e^{\alpha(t-s)}, \quad (t, s) \in \Delta_+.$$

(ii) $U(t, r)U(r, s) = U(t, s)$ on Δ_+ and $U(s, s) = 1$ (the identity).

(iii) $U(t, s)Y \subset Y$ and $U(\cdot, \cdot)$ is strongly continuous on Δ_+ to $B(Y)$, with

$$\|U(t, s)\|_{B(Y)} \leq K \exp\left(\int_s^t \tilde{\sigma}(r) dr\right), \quad (t, s) \in \Delta_+, \quad (1.18)$$

where $\tilde{\sigma}(t) := \beta + K\sigma(t)$.

Furthermore, let $v \in Y$. Then $U(\cdot, \cdot)v \in C^1(\Delta_+; X)$, with

(iv) $(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$, $(t, s) \in \Delta_+$, and

(v) $(\partial/\partial s)U(t, s)v = U(t, s)A(s)v$, $(t, s) \in \Delta_+$.

The equation in (ACP) is naturally interpreted if the solution has an additional property $u(\cdot) \in C(I; Y)$. In fact, it is guaranteed by condition (A2) that $u(t) \in Y \subset D(A(t))$ for every $t \in I$.

Theorem 1.3. Let $\{U(t, s)\}$ be the evolution operator for (ACP) as in Theorem 1.2 above. For $u_0 \in Y$ and

$$f(\cdot) \in C(I; X) \cap L^1(I; Y)$$

define $u(\cdot)$ as

$$u(t) := U(t, 0)u_0 + \int_0^t U(t, s)f(s) ds.$$

Then (ACP) has a unique (classical) solution

$$u(\cdot) \in C^1(I; X) \cap C(I; Y).$$

Remark 3. Abstract Theorems 1.2 and 1.3 are nothing but generalizations of the corresponding theorems in [17] in which $S(t) \equiv S_0$ is denoted simply by S (see also [20]). In other words, the role of S in [17] is divided into two parts respectively played by S_0 and $\{S(t)\}$ in this paper. On the one hand, because of the simplicity of S_0 it is easily seen how to define the initial value space Y . On the other hand, because of the time-dependence of $\{S(t)\}$ it is not so difficult to compute the inner product in (1.17). For example, in our application in Section 5, $S(t)$ and $S(t)^{1/2}$ can be explicitly written down in term of $A(t)$ (see (5.21), (5.24) and (5.29) below). The original idea of employing time-dependent auxiliary operators goes back to Kato [6,7].

Remark 4. In addition to Remark 2, it should be noted that $\{U(t, s); (t, s) \in \Delta_+\}$ in Theorem 1.2 can be extended from Δ_+ to $\Sigma := [0, T] \times [0, T] = \Delta_+ \cup \Delta_-$, where $\Delta_- := \{(t, s); 0 \leq t \leq s \leq T\}$. In fact, let $\{\bar{A}(t)\}$ and $\{\bar{S}(t)\}$ be a new pair of families of closed linear operators in X such that

$$\bar{A}(t) := -A(-t), \quad \bar{S}(t) := S(-t) \quad \text{on } I_- := [-T, 0].$$

Then $\{\bar{A}(t)\}$ and $\{\bar{S}(t)\}$ also satisfy assumptions of Theorem 1.2, with I replaced with I_- . Consequently, there exists a unique evolution operator $\{V(t, s); -T \leq s \leq t \leq 0\}$ for (ACP) with $A(\cdot)$ and I replaced with $\bar{A}(\cdot)$ and I_- , respectively. Thus we can define $U(t, s) := V(-t, -s)$ for $(t, s) \in \Delta_-$ (see [17, Remark 1.3 and Section 5.2]). Accordingly, the notion of our *hyperbolicity* is more restrictive and different from the “hyperbolicity” in the sense of Kato [6,7]. For example, Kato’s theorems are applied to both of the (linearized) Euler and Navier–Stokes equations (see Kato [8, Section 14] and Kato and Ponce [11]) though our theorems are applied only to the (linearized) Euler equation (see Okazawa [16]).

After preparing several lemmas in Section 2, we shall prove Theorems 1.2 and 1.3 in Sections 3 and 4, respectively. In Section 5, we apply Theorem 1.3 to Cauchy problem (1.7) with (1.14) for the family $\{v_k\}$ of new unknown functions. After that, we return to the family $\{u_k\}$ of original unknown functions and complete the proof of Theorem 1.1. However, the proofs of two lemmas in Section 5 are relatively long so that the detailed computations are given in Appendix A.

2. Preliminaries

Let X be a (complex) Hilbert space. In this section we prepare some useful lemmas.

2.1. Lemmas on time-independent operators

In this subsection we consider a pair $\{A, S\}$ of closed linear operators in X . Let A be *quasi-accretive* in the sense of Kato [5, Section V.3.10]:

$$\operatorname{Re}(Av, v) \geq -\alpha \|v\|^2, \quad v \in D(A), \quad (2.1)$$

for some constant $\alpha \geq 0$; in other words, $\alpha + A$ is accretive. Let S be a positive-definite selfadjoint operator in X , with $D(S) \subset D(A)$. Now we can state a condition connecting A and S : assume that there exists a constant $\beta \geq 0$ such that

$$\operatorname{Re}(Au, Su) \geq -\beta(u, Su), \quad u \in D(S) \subset D(A). \quad (2.2)$$

Lemma 2.1. *Let A and S be as in (2.1) and (2.2), respectively. Then*

- (a) $\alpha + A$ is m -accretive in X ;
- (b) $D(S)$ is a core for A .

(a) was first proved by Okazawa [14], while (b) was later noted by Kato [9] (for a complete proof see Tanabe [23, Section 7.7] or Ouhabaz [21, Section 1.3.3]).

Given the pair $\{A, S\}$ as in Lemma 2.1, let $\{A_n; n > \alpha\}$ and $\{S_\varepsilon; \varepsilon > 0\}$ be Yosida approximations of A and S , respectively:

$$A_n := A J_n = n(1 - J_n), \quad J_n := (1 + n^{-1}A)^{-1}, \quad (2.3)$$

$$S_\varepsilon := S J_\varepsilon = \varepsilon^{-1}(1 - J_\varepsilon), \quad J_\varepsilon := (1 + \varepsilon S)^{-1}. \quad (2.4)$$

Then the pair $\{A_n, S_\varepsilon\}$ satisfies conditions in Lemma 2.1 with α and β replaced with $\alpha(1 - n^{-1}\alpha)^{-1}$ and $\beta(1 - n^{-1}\beta)^{-1}$, respectively.

The quasi-accretivity of A is invariant under taking its Yosida approximation.

Lemma 2.2. (See [17, Lemma 2.2].) Given A as in Lemma 2.1, let $\{A_n\}$ and $\{J_n\}$ be as in (2.3). Then $\|J_n\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ ($n > \alpha$) and $\|A_n\|_{B(X)} \leq n$ ($n \geq 2\alpha$), with

$$\operatorname{Re}(A_n w, w) \geq -\alpha(1 - n^{-1}\alpha)^{-1} \|w\|^2, \quad w \in X, n > \alpha. \quad (2.1)_n$$

The condition connecting A and S is translated into that connecting A_n and S_ε .

Lemma 2.3. (See [20, Lemmas 2.7].) Given the pair $\{A, S\}$ as in Lemma 2.1, let $\{A_n\}$ and $\{S_\varepsilon\}$ be as in (2.3) and (2.4). Then

$$\operatorname{Re}(A_n w, S_\varepsilon w) \geq -\beta(1 - n^{-1}\beta)^{-1} \|S_\varepsilon^{1/2} w\|^2, \quad w \in X, n > \beta \geq \alpha. \quad (2.2)_{n,\varepsilon}$$

The next lemma concerns the Yosida approximation of (nonnegative) selfadjoint operators, which was prepared in Ikehata and Okazawa [4, Section 3].

Lemma 2.4. Let S be nonnegative and selfadjoint in X . Put $S_\varepsilon := S(1 + \varepsilon S)^{-1}$ for $\varepsilon > 0$. Then $S_\varepsilon^{1/2} = S^{1/2}(1 + \varepsilon S)^{-1/2}$. If $\{S_\varepsilon^{1/2} v\}$ ($v \in X$) is bounded in X , then $v \in D(S^{1/2})$ and $S_\varepsilon^{1/2} v \rightarrow S^{1/2} v$ ($\varepsilon \downarrow 0$) strongly in X .

In fact, the boundedness of $\{S_\varepsilon^{1/2} v\}$ yields that $v \in D(S^{1/2})$ and $S_\varepsilon^{1/2} v \rightarrow S^{1/2} v$ ($\varepsilon \downarrow 0$) weakly in X , with $\|S^{1/2} v\| \leq \liminf_{\varepsilon \downarrow 0} \|S_\varepsilon^{1/2} v\|$. On the other hand, we have $\limsup_{\varepsilon \downarrow 0} \|S_\varepsilon^{1/2} v\| \leq \|S^{1/2} v\|$. This concludes the strong convergence because $\|S^{1/2} v\| = \lim_{\varepsilon \downarrow 0} \|S_\varepsilon^{1/2} v\|$.

Remark 5. In Lemma 2.4 the square root $(1 + \varepsilon S)^{-1/2}$ has a fairly simple expression similar to Euler's beta function:

$$(1 + \varepsilon S)^{-1/2} = \frac{1}{\pi} \int_0^1 \xi^{-1/2} (1 - \xi)^{-1/2} (1 + \xi(\varepsilon S))^{-1} d\xi.$$

In fact, if $\varepsilon = 0$, then the integral on the right-hand side is nothing but $B(1/2, 1/2)$, where $B(p, q)$, $p > 0$, $q > 0$, is Euler's beta function (see Okazawa [18]).

2.2. Lemmas on time-dependent operators

Since we need conditions (A1), (A3) and (S3) as a whole only in the last step of the proof of Theorem 1.2, we may introduce a set of weaker conditions:

(A1)₊ There exists a constant $\alpha \geq 0$ such that

$$\operatorname{Re}(A(t)v, v) \geq -\alpha \|v\|^2, \quad v \in D(A(t)), \quad t \in I.$$

(A3)₊ There exists a constant $\beta \geq \alpha$ such that

$$\operatorname{Re}(A(t)v, S(t)v) \geq -\beta \|S(t)^{1/2}v\|^2, \quad v \in D(S(t)) \subset Y, \quad t \in I.$$

(S3)₊ There exists a nonnegative function $\sigma \in L^1(I)$ such that

$$\|S(t)^{1/2}u\| - \|S(s)^{1/2}u\| \leq \left(\int_s^t \sigma(r) dr \right) \|S(s)^{1/2}u\|, \quad u \in Y, \quad 0 \leq s \leq t \leq T.$$

By virtue of Lemma 2.1, it follows from conditions (A1)₊, (A2) and (A3)₊ that $\alpha + A(t)$ is m -accretive in X .

Let S_0 be a selfadjoint operator in X , satisfying (1.15). Then we have

Proposition 2.5. *Let $\{S(t)\}$ be a family of selfadjoint operators in X , $\{A(t)\}$ a family of closed linear quasi-accretive operators in X , satisfying conditions (S1), (S2), (S3)₊ for $\{S(t)\}$, and conditions (A1)₊, (A2), (A3)₊, (A4) for $\{A(t)\}$. Then there exist Yosida approximations $\{A_n(t); n > \alpha\}$ and $\{S_\varepsilon(t); \varepsilon > 0\}$ of $\{A(t)\}$ and $\{S(t)\}$, respectively:*

$$A_n(t) := A(t)J_n(t) = n(1 - J_n(t)), \quad J_n(t) := (1 + n^{-1}A(t))^{-1}, \quad (2.5)$$

$$S_\varepsilon(t) := S(t)J_\varepsilon(t) = \varepsilon^{-1}(1 - J_\varepsilon(t)), \quad J_\varepsilon(t) := (1 + \varepsilon S(t))^{-1}. \quad (2.6)$$

Put

$$\alpha_n := \alpha(1 - n^{-1}\alpha)^{-1}, \quad \beta_n := \beta(1 - n^{-1}\beta)^{-1} \quad (n > \beta \geq \alpha). \quad (2.7)$$

Then the pair of $\{A_n(t)\}$ and $\{S_\varepsilon(t)\}$ satisfies Assumptions on $\{A(t)\}$ and $\{S(t)\}$ with Y, α and β replaced with X, α_n and β_n , respectively:

(S_ε1) $D(S_\varepsilon(t)) = X, t \in I$.

(S_ε2) $S_\varepsilon(\cdot) \in C_*(I; B(X))$.

(S_ε3)₊ For $w \in X$,

$$\|S_\varepsilon(t)^{1/2}w\|^2 - \|S_\varepsilon(s)^{1/2}w\|^2 \leq 2K \int_s^t \sigma(r) dr \|S_\varepsilon(s)^{1/2}w\|^2, \quad 0 \leq s \leq t \leq T.$$

(A_n1)₊ For $n > \alpha, t \in I, \|J_n(t)\|_{B(X)} \leq (1 - n^{-1}\alpha)^{-1}$ and

$$\operatorname{Re}(A_n(t)w, w) \geq -\alpha_n \|w\|^2, \quad w \in X.$$

(A_n2) $D(A_n(t)) = X, t \in I$.

(A_n3)₊ For $n > \beta, t \in I$,

$$\operatorname{Re}(A_n(t)w, S_\varepsilon(t)w) \geq -\beta_n \|S_\varepsilon(t)^{1/2}w\|^2, \quad w \in X.$$

(A_n4) $A_n(\cdot) \in C_*(I; B(X))$ ($n > \beta$), with $\|A_n(t)\|_{B(X)} \leq n$ ($n \geq 2\alpha, t \in I$).

Proof. $(S_\varepsilon \mathbf{1})$ and $(A_n \mathbf{2})$ are trivial. $(A_n \mathbf{1})_+$ and $(A_n \mathbf{3})_+$ follow from Lemmas 2.2 and 2.3, respectively. $(A_n \mathbf{4})$ was proved in [17, Lemma 3.1(a), (b)].

To prove $(S_\varepsilon \mathbf{2})$ put $Q^+(t) := iS(t)^{1/2}$, $Q^-(t) := -iS(t)^{1/2}$, for $t \in I$. Then we see from condition $(S \mathbf{2})$ that

$$Q^+(\cdot), Q^-(\cdot) \in C_*(I; B(Y, X)).$$

Since $S(t)^{1/2}$ is selfadjoint, we can define

$$\begin{aligned} Q_{\sqrt{\varepsilon}}^+(t) &:= Q^+(t)(1 + \sqrt{\varepsilon}Q^+(t))^{-1}, \\ Q_{\sqrt{\varepsilon}}^-(t) &:= Q^-(t)(1 + \sqrt{\varepsilon}Q^-(t))^{-1} = -Q^+(t)(1 - \sqrt{\varepsilon}Q^+(t))^{-1}. \end{aligned}$$

We see from conditions $(S \mathbf{1})$ and $(S \mathbf{2})$ that

$$Q_{\sqrt{\varepsilon}}^+(\cdot), Q_{\sqrt{\varepsilon}}^-(\cdot) \in C_*(I; B(X)). \quad (2.8)$$

Since $S(t) = -Q^+(t)^2$, it follows from the commutativity that

$$Q_{\sqrt{\varepsilon}}^-(t)Q_{\sqrt{\varepsilon}}^+(t) = -Q^+(t)^2(1 - \varepsilon Q^+(t)^2)^{-1} = S_\varepsilon(t).$$

Therefore $(S_\varepsilon \mathbf{2})$ is a consequence of (2.8).

$(S_\varepsilon \mathbf{3})_+$ is a consequence of $(S \mathbf{1})$ and $(S \mathbf{3})_+$. First we remind of the definition of $S_\varepsilon(\cdot)$: $w = J_\varepsilon(t)w + \varepsilon S_\varepsilon(t)w$ for $w \in X$. Then the symmetry and positivity of $J_\varepsilon(\cdot)$ yield that

$$\begin{aligned} &(w, S_\varepsilon(t)w - S_\varepsilon(s)w) \\ &= (w, S_\varepsilon(t)(J_\varepsilon(s) + \varepsilon S_\varepsilon(s))w - (J_\varepsilon(t) + \varepsilon S_\varepsilon(t))S_\varepsilon(s)w) \\ &= (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) + ((J_\varepsilon(t) - J_\varepsilon(s))w, (S(t) - S(s))J_\varepsilon(s)w) \\ &= (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) - \varepsilon(J_\varepsilon(t)(S(t) - S(s))J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) \\ &\leq (J_\varepsilon(s)w, (S(t) - S(s))J_\varepsilon(s)w) \\ &= \|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2. \end{aligned} \quad (2.9)$$

On the other hand, it follows from condition $(S \mathbf{1})$ that

$$\|S(t)^{1/2}J_\varepsilon(s)w\| \leq K^{1/2}\|S_0^{1/2}J_\varepsilon(s)w\| \leq K\|S_\varepsilon(s)^{1/2}w\|, \quad (2.10)$$

where we have used $\|J_\varepsilon(s)^{1/2}\|_{B(X)} \leq 1$ ($\varepsilon > 0$). We see from (2.10) and $(S \mathbf{3})_+$ that

$$\begin{aligned} &\|S(t)^{1/2}J_\varepsilon(s)w\|^2 - \|S(s)^{1/2}J_\varepsilon(s)w\|^2 \\ &= (\|S(t)^{1/2}J_\varepsilon(s)w\| + \|S(s)^{1/2}J_\varepsilon(s)w\|)(\|S(t)^{1/2}J_\varepsilon(s)w\| - \|S(s)^{1/2}J_\varepsilon(s)w\|) \\ &\leq (1 + K)\|S_\varepsilon(s)^{1/2}w\| \cdot \left(\int_s^t \sigma(r) dr \right) \|S_\varepsilon(s)^{1/2}w\|. \end{aligned} \quad (2.11)$$

Noting that $K \geq 1$ and then combining (2.11) with (2.9), we obtain $(S_\varepsilon \mathbf{3})_+$. \square

The next proposition is essential in the proof of Theorem 1.2 in which it is required to differentiate $S_\varepsilon(\cdot)$. For that purpose $\{S_\varepsilon(t); t \in I\}$ is replaced with a new family

$$S_\varepsilon^h(t) := \frac{1}{h} \int_t^{t+h} S_\varepsilon(s) ds, \quad h > 0, t \in I, \quad (2.12)$$

where we define $S_\varepsilon(s) := S_\varepsilon(T)$ ($s > T$). Then, in view of $(S_\varepsilon 2)$, we have

$$\frac{d}{dt} S_\varepsilon^h(t) w = \frac{1}{h} (S_\varepsilon(t+h) w - S_\varepsilon(t) w), \quad (2.13)$$

$$s\text{-}\lim_{h \downarrow 0} S_\varepsilon^h(t) w = S_\varepsilon(t) w, \quad w \in X. \quad (2.14)$$

Proposition 2.6. Assume that $\{S(t)\}$ satisfies conditions $(S1)$, $(S2)$ and $(S3)_+$. Let $\{S_\varepsilon^h(t); t \in I\}$ be as in (2.12). Then $S_\varepsilon^h(\cdot) \in C_*^1(I; B(X))$, with

$$\left(w, \frac{d}{ds} S_\varepsilon^h(s) w \right) \leq 2K \left(\frac{1}{h} \int_s^{s+h} \sigma(r) dr \right) \|S_\varepsilon(s)^{1/2} w\|^2, \quad w \in X, s \in I, \quad (S_\varepsilon^h 3)_+$$

where one defines $\sigma(s) := 0$ ($s > T$). Moreover, for $w \in X$ one has

$$\lim_{h \downarrow 0} \int_{t_0}^t \left(w, \frac{d}{ds} S_\varepsilon^h(s) w \right) ds \leq 2K \int_{t_0}^t \sigma(s) \|S_\varepsilon(s)^{1/2} w\|^2 ds, \quad t_0 \leq t. \quad (2.15)$$

To prove Proposition 2.6 we need the following lemma in which we use the integration by parts and Lebesgue's convergence theorem.

Lemma 2.7. (See [26, Lemma 2.2].) Let $\varphi \in L^1(I)$ and $\psi \in L^\infty(I)$. Then

$$\lim_{h \downarrow 0} \int_s^t \left(\frac{1}{h} \int_r^{r+h} \varphi(\tau) d\tau \right) \psi(r) dr = \int_s^t \varphi(r) \psi(r) dr, \quad 0 \leq s \leq t \leq T,$$

where one defines $\varphi(s) := 0$ ($s > T$).

Proof of Proposition 2.6. $(S_\varepsilon^h 3)_+$ follows from (2.13) and $(S_\varepsilon 3)_+$. On the other hand, (2.15) is a consequence of Lemma 2.7 with $\varphi := \sigma$. \square

3. Construction of evolution operators

In this section we shall prove Theorem 1.2. Let S_0 be a selfadjoint operator in a Hilbert space X , satisfying (1.15). The major part of the assertions in Theorem 1.2 is contained in the following

Theorem 3.1. Let $\{S(t)\}$ be a family of selfadjoint operators in X , $\{A(t)\}$ a family of closed linear quasi-accretive operators in X , satisfying conditions $(S1)$, $(S2)$, $(S3)_+$ for $\{S(t)\}$, and conditions $(A1)_+$, $(A2)$, $(A3)_+$, $(A4)$ for $\{A(t)\}$. Then there exists a unique two-parameter family $\{U(t, s); (t, s) \in \Delta_+\}$ in $B(X)$, having the properties:

- (iii)_w $U(t, s)Y \subset Y$ and $U(\cdot, \cdot)$ is weakly continuous on Δ_+ to $B(Y)$, with (1.18);
 (iv)_w $U(\cdot, \cdot)v \in W^{1,1}(\Delta_+; X)$, with $(\partial/\partial t)U(t, s)v = -A(t)U(t, s)v$, a.a. $t \in (s, T)$, $v \in Y$;

in addition to properties (i), (ii) and (v) in Theorem 1.2.

Therefore the first purpose of this section is to prove Theorem 3.1. To replace (iii)_w and (iv)_w with (iii) and (iv) in the final step we need the whole conditions (S1)–(S3) and (A1)–(A4) in Theorem 1.2.

To prove Theorem 3.1 we consider the approximate problem:

$$\begin{cases} (d/dt)u_n(t) + A_n(t)u_n(t) = 0, & t \in [s, T], \\ u_n(s) = w \in X, \end{cases} \quad (\text{ACP})_n$$

where $\{A_n(t); n > \alpha\}$ is the Yosida approximation as in (2.5).

According to Pazy [22, Theorems 5.1.1 and 5.1.2] (in which $A_n(\cdot) \in C(I; B(X))$ is assumed, however, it can be replaced by $A_n(\cdot) \in C_*(I; B(X))$ (condition (A_n4)) with appropriate modification of the conclusion), we obtain the following

Proposition 3.2. *Let $s \in [0, T)$ and $n > 2\beta$, where β is defined in (A3)₊. Then the approximate problem (ACP)_n has a unique classical solution $u_n(\cdot) \in C^1([s, T]; X)$. Accordingly, there exists a unique evolution operator $\{U_n(t, s); (t, s) \in \Delta_+\}$ for (ACP)_n having the following properties:*

- (i)_n $U_n(\cdot, \cdot)$ is strongly continuous on Δ_+ to $B(X)$, with

$$\|U_n(t, s)\|_{B(X)} \leq e^{n(t-s)}, \quad (t, s) \in \Delta_+.$$

- (ii)_n $U_n(t, r)U_n(r, s) = U_n(t, s)$ on Δ_+ and $U_n(s, s) = 1$ (the identity).

- (iii)_n $U_n(t, s)$ is uniformly continuous on Δ_+ .

- (iv)_n $(\partial/\partial t)U_n(t, s)w = -A_n(t)U_n(t, s)w$, $w \in X$, $(t, s) \in \Delta_+$.

- (v)_n $(\partial/\partial s)U_n(t, s)v = U_n(t, s)A_n(s)v$, $w \in X$, $(t, s) \in \Delta_+$.

For the limiting procedure for $\{U_n(t, s)\}$ we need several estimates of $\{U_n(t, s)\}$ which are independent of n .

Lemma 3.3. *Let $\{U_n(t, s)\}$ be as in Proposition 3.2, α_n, β_n as in (2.7) and σ as in condition (S3)₊. If $n > 2\beta$, then for $(t, s) \in \Delta_+$,*

- (a) $\|U_n(t, s)\|_{B(X)} \leq e^{\alpha_n(t-s)}$.

- (b) $U_n(t, s)Y \subset Y$ and $\|U_n(t, s)\|_{B(Y)} \leq Ke^{\beta_n(t-s)} \exp(K \int_s^t \sigma(r) dr)$, with

$$\|S(t)^{1/2}U_n(t, s)v\| \leq e^{\beta_n(t-s)} \exp\left(K \int_s^t \sigma(r) dr\right) \|S(s)^{1/2}v\|, \quad v \in Y. \quad (3.1)$$

- (c) *There exists a constant $c \geq 0$ such that*

$$\|A(t)v\| \leq c \|S_0^{1/2}v\|, \quad v \in Y, \quad (3.2)$$

and hence

$$\|A_n(t)U_n(t, s)v\| \leq M \|v\|_Y, \quad v \in Y, \quad (3.3)$$

where $M := 2cKe^{2\beta T} \exp(K\|\sigma\|_{L^1(0,T)})$.

Proof. (a) We see from property $(\mathbf{iv})_n$ and $(\mathbf{A}_n \mathbf{1})_+$ that for $w \in X$,

$$\begin{aligned} (\partial/\partial r) \|U_n(r, s)w\|^2 &= -2 \operatorname{Re}(A_n(r)U_n(r, s)w, U_n(r, s)w) \\ &\leq 2\alpha_n \|U_n(r, s)w\|^2, \quad s \leq r \leq t. \end{aligned}$$

Integrating this inequality, we obtain the assertion.

(b) Let $\{S_\varepsilon(t)\}$ and $\{S_\varepsilon^h(t)\}$ be as in (2.6) and (2.12). Since $S_\varepsilon^h(t)^{1/2}$ is bounded and symmetric on X , it follows from property $(\mathbf{iv})_n$ that for $v \in Y$,

$$\begin{aligned} (\partial/\partial r) \|S_\varepsilon^h(r)^{1/2}U_n(r, s)v\|^2 &= -2 \operatorname{Re}(A_n(r)U_n(r, s)v, S_\varepsilon^h(r)U_n(r, s)v) \\ &\quad + (U_n(r, s)v, ((d/dr)S_\varepsilon^h(r))U_n(r, s)v). \end{aligned} \quad (3.4)$$

Integrating this equality on $[s, t]$, we see from $(S_\varepsilon^h \mathbf{3})_+$ that

$$\begin{aligned} \|S_\varepsilon^h(t)^{1/2}U_n(t, s)v\|^2 &\leq \|S_\varepsilon^h(s)^{1/2}v\|^2 - 2 \int_s^t \operatorname{Re}(A_n(r)U_n(r, s)v, S_\varepsilon^h(r)U_n(r, s)v) dr \\ &\quad + 2K \int_s^t \left(\frac{1}{h} \int_r^{r+h} \sigma(\tau) d\tau \right) \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr. \end{aligned}$$

Passing to the limit as $h \downarrow 0$, we see from (2.14), (2.15) and $(\mathbf{A}_n \mathbf{3})_+$ that

$$\begin{aligned} \|S_\varepsilon(t)^{1/2}U_n(t, s)v\|^2 &\leq \|S_\varepsilon(s)^{1/2}v\|^2 - 2 \int_s^t \operatorname{Re}(A_n(r)U_n(r, s)v, S_\varepsilon(r)U_n(r, s)v) dr \\ &\quad + 2K \int_s^t \sigma(r) \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr \\ &\leq \|S(s)^{1/2}v\|^2 + 2 \int_s^t (\beta_n + K\sigma(r)) \|S_\varepsilon(r)^{1/2}U_n(r, s)v\|^2 dr. \end{aligned}$$

Applying the Gronwall lemma, we obtain

$$\|S_\varepsilon(t)^{1/2}U_n(t, s)v\|^2 \leq \exp\left(2 \int_s^t (\beta_n + K\sigma(r)) dr\right) \|S(s)^{1/2}v\|^2.$$

It then follows from Lemma 2.4 that $U_n(t, s)Y \subset Y$, with estimate (3.1). Noting that $\|v\|_Y = \|S_0^{1/2}v\|$, the remaining assertion is a consequence of condition $(\mathbf{S1})$.

(c) The existence of such a constant c is guaranteed by conditions $(\mathbf{A2})$ and $(\mathbf{A4})$. Therefore we see from $(\mathbf{A}_n \mathbf{1})_+$ and (3.2) that

$$\|A_n(t)v\| \leq (1 - n^{-1}\alpha)^{-1} \|A(t)v\| \leq 2c\|v\|_Y, \quad v \in Y.$$

Thus we obtain (3.3) as a consequence of (b). \square

Lemma 3.4 (Existence of evolution operator). Let $\{U_n(t, s)\}$ be the evolution operator for $(ACP)_n$. Then there exists a new family $\{U(t, s); (t, s) \in \Delta_+\}$ such that $U(t, s) := s\text{-}\lim_{n \rightarrow \infty} U_n(t, s)$, where the convergence is uniform on Δ_+ , and has properties (i) and (ii) in Theorem 1.2, with

$$\|U(t, s)v - U_n(t, s)v\| \leq \sqrt{\frac{2T}{n-2\alpha}} M e^{2\alpha T} \|v\|_Y, \quad v \in Y, \quad n > 2\beta. \quad (3.5)$$

Proof. Let $v \in Y$. Then we shall show that for all $n, m > 2\beta$ ($\geq 2\alpha$),

$$\|U_n(t, s)v - U_m(t, s)v\|^2 \leq 2M^2 a_{nm} |t-s| e^{4\alpha(t-s)} \|v\|_Y^2, \quad (3.6)$$

where we have defined

$$a_{nm} := \left| \frac{1}{\sqrt{n-2\alpha}} - \frac{1}{\sqrt{m-2\alpha}} \right|^2. \quad (3.7)$$

Let $r \in [s, t]$ for a fixed $(t, s) \in \Delta_+$. Then put

$$u_{nm}(r) := U_n(r, s)v - U_m(r, s)v, \quad w_{nm}(r) := J_n(t)U_n(r, s)v - J_m(t)U_m(r, s)v.$$

Then we have

$$u_{nm}(r) - w_{nm}(r) = \frac{1}{n} A_n(r) U_n(r, s)v - \frac{1}{m} A_m(r) U_m(r, s)v. \quad (3.8)$$

The proof of (3.6) is essentially the same as in [17]. However, we want to modify the proof given in [17, Lemma 3.4(a)] following the idea in [20, Lemma 3.5(a)].

Property (iv)_n yields that $u_{nm}(\cdot) \in C^1([s, t]; X)$, with

$$\begin{aligned} \frac{1}{2} \frac{d}{dr} \|u_{nm}(r)\|^2 &= -\operatorname{Re}(A_n(r)U_n(r, s)v - A_m(r)U_m(r, s)v, u_{nm}(r)) \\ &= I_1 + I_2, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} I_1 &:= -\operatorname{Re}(A(r)w_{nm}(r), w_{nm}(r)), \\ I_2 &:= -\operatorname{Re}(A_n(r)U_n(r, s)v - A_m(r)U_m(r, s)v, u_{nm}(r) - w_{nm}(r)). \end{aligned}$$

On the one hand, we see from $(A1)_+$ and (3.8) that

$$\begin{aligned} I_1 &\leq \alpha \|w_{nm}(r)\|^2 = \alpha \left\| u_{nm}(r) - \left(\frac{1}{n} A_n(r) U_n(r, s)v - \frac{1}{m} A_m(r) U_m(r, s)v \right) \right\|^2 \\ &\leq 2\alpha \|u_{nm}(r)\|^2 + 2\alpha \left\| \frac{1}{n} A_n(r) U_n(r, s)v - \frac{1}{m} A_m(r) U_m(r, s)v \right\|^2 \\ &= 2\alpha \|u_{nm}(r)\|^2 - \frac{4\alpha}{nm} \operatorname{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v) \\ &\quad + \frac{2\alpha}{n^2} \|A_n(r)U_n(r, s)v\|^2 + \frac{2\alpha}{m^2} \|A_m(r)U_m(r, s)v\|^2. \end{aligned}$$

On the other hand, (3.8) yields that

$$\begin{aligned} I_2 &= -\operatorname{Re}\left(A_n(r)U_n(r, s)v - A_m(r)U_m(r, s)v, \frac{1}{n}A_n(r)U_n(r, s)v - \frac{1}{m}A_m(r)U_m(r, s)v\right) \\ &= \left(\frac{1}{n} + \frac{1}{m}\right)\operatorname{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v) \\ &\quad - \frac{1}{n}\|A_n(r)U_n(r, s)v\|^2 - \frac{1}{m}\|A_m(r)U_m(r, s)v\|^2. \end{aligned}$$

Combining I_2 with the estimate of I_1 , we see from (3.9) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}\|u_{nm}(r)\|^2 - 2\alpha\|u_{nm}(r)\|^2 \\ &= \left(\frac{1}{n} + \frac{1}{m} - \frac{4\alpha}{nm}\right)\operatorname{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v) \\ &\quad - \left(\frac{1}{n} - \frac{2\alpha}{n^2}\right)\|A_n(r)U_n(r, s)v\|^2 - \left(\frac{1}{m} - \frac{2\alpha}{m^2}\right)\|A_m(r)U_m(r, s)v\|^2 \\ &= -\left\|\frac{\sqrt{n-2\alpha}}{n}A_n(r)U_n(r, s)v - \frac{\sqrt{m-2\alpha}}{m}A_m(r)U_m(r, s)v\right\|^2 \\ &\quad + \frac{1}{nm}(\sqrt{n-2\alpha} - \sqrt{m-2\alpha})^2\operatorname{Re}(A_n(r)U_n(r, s)v, A_m(r)U_m(r, s)v) \\ &\leq \frac{(\sqrt{n-2\alpha} - \sqrt{m-2\alpha})^2}{(n-2\alpha)(m-2\alpha)}\|A_n(r)U_n(r, s)v\| \cdot \|A_m(r)U_m(r, s)v\| \\ &\leq M^2 a_{nm}\|v\|_Y^2, \end{aligned}$$

where we have used (3.3) and (3.7). Integrating this inequality, we obtain (3.6). Since Y is dense in X , it follows from Lemma 3.3(a) that the family $\{U(t, s); (t, s) \in \Delta_+\}$ in $B(X)$ with property (i) is defined: for $w \in X$,

$$U_n(\cdot, \cdot)w \rightarrow U(\cdot, \cdot)w \quad \text{in } C(\Delta_+; X) \text{ as } n \rightarrow \infty.$$

To derive (3.5) it suffices to pass to the limit as $m \rightarrow \infty$ in (3.6). Property (ii) is a consequence of property (ii)_n. \square

Lemma 3.5 (Uniqueness of evolution operator). *Let $\{U(t, s)\}$ be as in Lemma 3.4 and $v \in Y$. Then*

(a) $U(t, s)Y \subset Y$ and $S_0^{1/2}U(t, s)v = w\text{-}\lim_{n \rightarrow \infty} S_0^{1/2}U_n(t, s)v$, with (1.18) and

$$\|S(t)^{1/2}U(t, s)v\| \leq \exp\left(\int_s^t \tilde{\sigma}(r) dr\right)\|S(s)^{1/2}v\|, \quad (t, s) \in \Delta_+, \quad v \in Y, \quad (3.10)$$

where $\tilde{\sigma}$ is defined as in Theorem 1.2(iii).

(b) $S_0^{1/2}U(t, s)v$ is weakly continuous on Δ_+ .

(c) $U(\cdot, \cdot)v \in W^{1, \infty}(\Delta_+; X)$, with properties (iv)_w and (v).

(d) $\{U(t, s); (t, s) \in \Delta_+\}$ is unique: $U(t, s) \equiv V(t, s)$ on Δ_+ if $\{V(t, s); (t, s) \in \Delta_+\}$ is another family in $B(X)$ with properties (i), (ii) and (v).

Proof. (a) is a consequence of Lemmas 3.4 and 3.3(b).

(b) Set $v_\varepsilon(t, s) := S_0^{1/2}(1 + \varepsilon S_0)^{-1/2}U(t, s)v$. Then by property (i), v_ε is continuous on Δ_+ . Next we can show that $S_0^{1/2}U(t, s)v = w\text{-}\lim_{\varepsilon \downarrow 0} v_\varepsilon(t, s)$ uniformly on Δ_+ :

$$\begin{aligned} & |(S_0^{1/2}U(t, s)v, z) - (v_\varepsilon(t, s)v, z)| \\ &= |(S_0^{1/2}U(t, s)v, [1 - (1 + \varepsilon S_0)^{-1/2}]z)| \\ &\leq \|S_0^{1/2}U(t, s)v\| \cdot \|z - (1 + \varepsilon S_0)^{-1/2}z\| \\ &\leq K \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|v\|_Y \|z - (1 + \varepsilon S_0)^{-1/2}z\| \rightarrow 0 \quad (\varepsilon \downarrow 0), \quad z \in X, \end{aligned} \quad (3.11)$$

where we have used (1.18). This completes the proof of assertion (b).

(c) Let $v \in Y$. Then, as is proved in [17, Lemmas 3.6 and 3.7], it follows from properties (iv)_n and (v)_n, Lemmas 3.3(c) and 3.4 that

$$A(\cdot)U(\cdot, s)v \in L^\infty(s, T; X) \quad (3.12)$$

and $U(t, \cdot)A(\cdot)v \in C([0, t]; X)$, with

$$\begin{aligned} U(t, s)v - v &= - \int_s^t A(r)U(r, s)v \, dr, \quad t \in [s, T], \\ v - U(t, s)v &= \int_s^t U(t, r)A(s)v \, dr, \quad s \in [0, t]. \end{aligned}$$

These equalities lead us to the assertion.

(d) We can employ the equality

$$U(t, s)v - V(t, s)v = \int_s^t \frac{\partial}{\partial r} [V(t, r)U(r, s)v] \, dr = 0, \quad v \in Y,$$

which follows from properties (iv)_w and (v) with U replaced with V . \square

This completes the proof of Theorem 3.1 because property (iii)_w is a combination of (a) and (b) in Lemma 3.5.

The purpose of the second half of this section is to prove properties (iii) and (iv) in Theorem 1.2. In addition to (3.11), we see from Lemma 2.4 that for a fixed $(t, s) \in \Delta_+$,

$$S_0^{1/2}U(t, s)v = \lim_{\varepsilon \downarrow 0} S_0^{1/2}(1 + \varepsilon S_0)^{-1/2}U(t, s)v, \quad v \in Y.$$

But it seems to be difficult to show that this strong convergence is uniform on Δ_+ . This is the reason why we shall prove (3.17) in Lemma 3.9 below in stead of (3.12). What is new in Lemmas 3.6 and 3.7 below is very little, however, we need them in the proof of Lemma 3.9.

Lemma 3.6. Let $\{U(t, s)\}$ be as in Lemma 3.4 and $v \in Y$. Assume that S_0 and $S(t)$ satisfy (1.15) and conditions (S1)–(S3). Then

- (a) $S(t)^{1/2}U(t, s)v$ is weakly continuous on Δ_+ .
- (b) $S(t)^{1/2}U(t, s)v \rightarrow S(t_0)^{1/2}v$ as $(t, s) \rightarrow (t_0, t_0)$.

- (b') $S_0^{1/2}U(t, s)v \rightarrow S_0^{1/2}v$ as $(t, s) \rightarrow (t_0, t_0)$.
 (c) For $t \in (0, T]$, $U(t, \cdot)v \in C([0, t]; Y)$.

Proof. (a) Since $S_\varepsilon(\cdot)^{1/2} \in C_*(I; B(X))$ (see **(S_ε2)**), $S_\varepsilon(t)^{1/2}U(t, s)v$ is continuous on Δ_+ . Next we can modify (3.11):

$$\begin{aligned} & |(S(t)^{1/2}U(t, s)v, w) - (S_\varepsilon(t)^{1/2}U(t, s)v, w)| \\ & \leq \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|S(s)^{1/2}v\| \cdot \|w - (1 + \varepsilon S(t))^{-1/2}w\| \\ & \leq K\varepsilon \exp(\|\tilde{\sigma}\|_{L^1(0, T)}) \|v\|_Y \|w\|_Y \rightarrow 0 \quad (\varepsilon \downarrow 0), \quad w \in Y. \end{aligned}$$

Since Y is dense in X , we can conclude (a).

(b) is easily proved. In fact, (a) yields that

$$\|S(t_0)^{1/2}v\| \leq \liminf_{(t, s) \rightarrow (t_0, t_0)} \|S(t)^{1/2}U(t, s)v\|.$$

On the other hand, it follows from (3.10) that

$$\limsup_{(t, s) \rightarrow (t_0, t_0)} \|S(t)^{1/2}U(t, s)v\| \leq \|S(t_0)^{1/2}v\|.$$

(b') is a consequence of (b) and conditions **(S1)**, **(S2)** because we have

$$\begin{aligned} & \|S_0^{1/2}U(t, s)v - S_0^{1/2}w\| \\ & \leq K^{1/2} \|S(t)^{1/2}U(t, s)v - S(t)^{1/2}w\| \\ & \leq K^{1/2} \|S(t)^{1/2}U(t, s)v - S(t_0)^{1/2}w\| + K^{1/2} \|S(t_0)^{1/2}w - S(t)^{1/2}w\|. \end{aligned} \quad (3.13)$$

In fact, we see from (3.13) with $w = v$ that $S_0^{1/2}v = \lim_{(t, s) \rightarrow (t_0, t_0)} S_0^{1/2}U(t, s)v$.

(c) follows from (b') (see Kato [6, Lemma 5.2(f)]). \square

Now we are in a position to prove **(iii)** and **(iv)** of Theorem 1.2. As is easily seen, the proof of **(iii)** and **(iv)** is based on Lemmas 3.7 and 3.8 below. In other words, we need the whole assumptions on $\{A(t)\}$ and $\{S(t)\}$.

Lemma 3.7. (See [17, Lemma 3.9].) Let $\{A(t)\}$ and $\{S(t)\}$ be as in Theorem 1.2. Assume that conditions **(A1)**, **(A2)** and **(A3)** are satisfied. Then

$$|\operatorname{Re}(A(t)v, S_\varepsilon(t)v)| \leq \beta \|S_\varepsilon(t)^{1/2}v\|^2, \quad v \in Y, \quad t \in I. \quad (3.14)$$

Under conditions **(S1)**–**(S3)** Proposition 2.6 is modified as follows.

Lemma 3.8. Assume that $\{S(t)\}$ satisfies conditions **(S1)**–**(S3)**. Let $\{S_\varepsilon^h(t)\}$ be as in Proposition 2.6. Then condition **(S_ε^h3)₊** is replaced with

$$\left| \left(w, \frac{d}{ds} S_\varepsilon^h(s) w \right) \right| \leq 2K \left(\frac{1}{h} \int_s^{s+h} \sigma(r) dr \right) \max_{r \in [s, s+h]} \|S_\varepsilon(r)^{1/2}w\|^2 \quad (\mathbf{S}_\varepsilon^h \mathbf{3})$$

for $w \in X$, $h > 0$ and $s \in I$. Consequently, for $t, t_0 \in I$ one has

$$\lim_{h \downarrow 0} \left| \int_{t_0}^t \left(w, \frac{d}{ds} S_\varepsilon^h(s) w \right) ds \right| \leq 2K \left| \int_{t_0}^t \sigma(s) \|S_\varepsilon(s)^{1/2} w\|^2 ds \right|. \quad (3.15)$$

The next lemma completes the proof of Theorem 1.2.

Lemma 3.9 (Continuity in Y -norm of evolution operator). For $\{A(t)\}$ and $\{S(t)\}$ as in Theorem 1.2 let $\{U(t, s)\}$ be as defined in Lemma 3.4 and $v \in Y$. Then

- (a) $S(\cdot)^{1/2} U(\cdot, s) v \in C([s, T]; X)$, $s \in [0, T]$.
- (b) $U(\cdot, s) v \in C([s, T]; Y)$, $s \in [0, T]$.
- (c) $U(\cdot, \cdot) v \in C(\Delta_+; Y)$; this establishes property (iii) in Theorem 1.2.
- (d) $U(\cdot, \cdot) v \in C^1(\Delta_+; X)$, with property (iv) in Theorem 1.2.

Proof. We follow the idea in [17, Lemma 3.10].

(a) By virtue of Lemma 3.6(a), it suffices to show that

$$\|S(\cdot)^{1/2} U(\cdot, s) v\| \in C[s, T]. \quad (3.16)$$

We trace the proof of Lemma 3.3(b). Let us $t, t_0 \in [s, T]$. Integrating the inequality (3.4) from $r = t_0$ to $r = t$ and passing to the limit as $n \rightarrow \infty$, we have

$$\begin{aligned} \|S_\varepsilon^h(t)^{1/2} U(t, s) v\|^2 - \|S_\varepsilon^h(t_0)^{1/2} U(t_0, s) v\|^2 &= -2 \int_{t_0}^t \operatorname{Re}(A(r) U(r, s) v, S_\varepsilon^h(r) U(r, s) v) dr \\ &\quad + \int_{t_0}^t \left(U(r, s) v, \left(\frac{d}{dr} S_\varepsilon^h(r) \right) U(r, s) v \right) dr. \end{aligned}$$

Passing to the limit as $h \downarrow 0$, we see from (3.15) that

$$\begin{aligned} \left| \|S_\varepsilon(t)^{1/2} U(t, s) v\|^2 - \|S_\varepsilon(t_0)^{1/2} U(t_0, s) v\|^2 \right| &\leq 2 \left| \int_{t_0}^t \operatorname{Re}(A(r) U(r, s) v, S_\varepsilon(r) U(r, s) v) dr \right| \\ &\quad + 2K \left| \int_{t_0}^t \sigma(r) \|S_\varepsilon(r)^{1/2} U(r, s) v\|^2 dr \right|. \end{aligned}$$

Therefore (3.14) yields that

$$\left| \|S_\varepsilon(t)^{1/2} U(t, s) v\|^2 - \|S_\varepsilon(t_0)^{1/2} U(t_0, s) v\|^2 \right| \leq 2 \left| \int_{t_0}^t \tilde{\sigma}(r) \|S(r)^{1/2} U(r, s) v\|^2 dr \right|,$$

where $\tilde{\sigma}(r) = \beta + K\sigma(r)$. By virtue of (3.10) and condition (S1), we have

$$\begin{aligned} & \left| \|S_\varepsilon(t)^{1/2}U(t, s)v\|^2 - \|S_\varepsilon(t_0)^{1/2}U(t_0, s)v\|^2 \right| \\ & \leq K \left| \int_{t_0}^t \frac{d}{dr} \left[\exp \left(2 \int_s^r \tilde{\sigma}(\tau) d\tau \right) dr \right] \right| \|v\|_Y^2 = K |F(t) - F(t_0)| \cdot \|v\|_Y^2, \end{aligned}$$

where $F(t) := \exp(2 \int_s^t \tilde{\sigma}(r) dr)$. Passing to the limit as $\varepsilon \downarrow 0$, we obtain (3.16).

(b) We see from (3.13) with $w = U(t_0, s)v$ that

$$\begin{aligned} & K^{-1/2} \|S_0^{1/2}U(t, s)v - S_0^{1/2}U(t_0, s)v\| \\ & \leq \|S(t)^{1/2}U(t, s)v - S(t_0)^{1/2}U(t_0, s)v\| + \|S(t_0)^{1/2}U(t_0, s)v - S(t)^{1/2}U(t_0, s)v\|. \end{aligned}$$

The assertion is a consequence of (a) and condition (S2).

(c) Let $(t_0, s_0) \in \Delta_+$ be fixed. Then we may assume by Lemma 3.6(b') that $t_0 > s_0$. In this context, Kato [6, Remark 5.4] noticed that the assertion is a combination of (b) and Lemma 3.6(c) (see also [20, Lemma 3.11(b)]). Together with the estimate in Lemma 3.5(a), this completes the proof of property (iii).

(d) By virtue of Lemma 3.5(c), it suffices to replace (3.12) with

$$A(\cdot)U(\cdot, s)v \in C([s, T]; X). \quad (3.17)$$

Let $t, t_0 \in [s, T]$. Then we see from (3.2) that the desired continuity is reduced to those of $S_0^{1/2}U(\cdot, s)v$ and $A(\cdot)U(t_0, s)v$:

$$\begin{aligned} & \|A(t)U(t, s)v - A(t_0)U(t_0, s)v\| \\ & \leq \|A(t)U(t, s)v - A(t)U(t_0, s)v\| + \|A(t)U(t_0, s)v - A(t_0)U(t_0, s)v\| \\ & \leq c \|S_0^{1/2}U(t, s)v - S_0^{1/2}U(t_0, s)v\| + \|A(t)U(t_0, s)v - A(t_0)U(t_0, s)v\|. \end{aligned}$$

Therefore the conclusion follows from (b) and condition (A4). Finally, property (iv) is a consequence of property (iv)_w. \square

4. Variation-of-constant formula

In this section we prove Theorem 1.3. Suppose that Assumptions on $\{A(t)\}$ and $\{S(t)\}$ are satisfied. Let $\{U(t, s); (t, s) \in \Delta_+\}$ be the evolution operator for (ACP) with the properties stated in Theorem 1.2. Then for $u_0 \in Y$,

$$(d/dt)U(t, 0)u_0 + A(t)U(t, 0)u_0 = 0 \quad \text{on } I. \quad (4.1)$$

Let $f(\cdot) \in C(I; X) \cap L^1(I; Y)$ and put

$$v(t) := \int_0^t U(t, s)f(s) ds.$$

Then we see from [17, Lemma 4.5] that

$$v(\cdot) \in C^1(I; X) \cap C(I; Y) \quad (4.2)$$

and

$$(d/dt)v(t) = -A(t)v(t) + f(t), \quad t \in I. \quad (4.3)$$

Now we are in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let $u_0 \in Y$, $f(\cdot) \in C(I; X) \cap L^1(I; Y)$ and put

$$u(t) := U(t, 0)u_0 + v(t). \quad (4.4)$$

Then we see from Lemma 3.9 and (4.2) that

$$u(\cdot) \in C^1(I; X) \cap C(I; Y).$$

By virtue of (4.1) and (4.3), $u(\cdot)$ is a (classical) solution of (ACP). On the other hand, let $u(\cdot) \in W^{1,1}(I; X) \cap L^1(I; Y)$. If $u(\cdot)$ satisfies (ACP) almost everywhere, then for a fixed $t \in (0, T]$ we have

$$(\partial/\partial s)U(t, s)u(s) = U(t, s)f(s), \quad \text{a.a. } s \in (0, t).$$

Integrating this equality over the interval $(0, t)$, we see that $u(\cdot)$ is exactly given by (4.4). Therefore (ACP) has a unique (classical) solution. \square

5. Application to the Schrödinger equations

In this section we shall apply abstract theorems in the previous sections to the Cauchy problem for the Schrödinger equations with moving Coulomb singularities in $L^2(\mathbb{R}^N)$:

$$\begin{cases} i \frac{\partial u}{\partial t} - \Delta u + V(t, x)u + \sum_{j=1}^m \frac{e_j u}{|x - c_j(t)|} = f(t, x), & (t, x) \in I \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 \in \Sigma^2 = \Sigma^2(\mathbb{R}^N) = H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N) \end{cases} \quad (\text{SE})_I$$

with $N \geq 3$. Under conditions (c1), (c2), (V1), (V2) and (f1) we will prove Theorem 1.1, i.e., that $(\text{SE})_I$ has a unique (classical) solution $u(\cdot) \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; \Sigma^2(\mathbb{R}^N))$.

In what follows we use the abbreviations $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^N)}$ and $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\mathbb{R}^N)}$.

5.1. Transformation to the new problem

Let ε_0 and T_0 be the constants defined as in (1.1) and (1.2). Then we obtain the division $\{I_k\}_{k=1}^{k_0}$ of I . Accordingly, $(\text{SE})_I$ is divided into $\{(\text{SE})_{I_k}\}_{k=1}^{k_0}$. Here $\{(\text{SE})_{I_k}\}_{k=1}^{k_0}$ is the time-restricted problem with initial value (1.3):

$$\begin{cases} i \frac{\partial u_k}{\partial t} - \Delta u_k + V(t, x)u_k + \sum_{j=1}^m \frac{e_j u_k}{|x - c_j(t)|} = f(t, x), & (t, x) \in I_k \times \mathbb{R}^N, \\ u_k((k-1)T_0, \cdot) = \begin{cases} u_0, & k=1, \\ u_{k-1}((k-1)T_0, \cdot), & 2 \leq k \leq k_0. \end{cases} \end{cases} \quad (\text{SE})_{I_k}$$

First we have to solve $(\text{SE})_{I_1}$. Then the solvability of $(\text{SE})_{I_k}$, $2 \leq k \leq k_0$, can be discussed in a way similar to $(\text{SE})_{I_1}$. We may assume without loss of generality that $I_1 = I$. In fact, the continuation of solutions from u_k to u_{k+1} is verified at the end of this section. In this context we concentrate to solve $(\text{SE})_I$ on the short time interval $I = [0, T_0]$.

The first lemma concerns the positivity of Jacobian.

Lemma 5.1. *Take*

$$\varphi(t, y) = y + \sum_{j=1}^m \zeta \left(\frac{|y - c_j(0)|}{\varepsilon_0} \right) (c_j(t) - c_j(0)), \quad (5.1)$$

where $\zeta \in C^\infty([0, \infty); [0, 1])$ satisfies the following conditions:

$$\zeta(r) = 1 \quad (r \leq 1), \quad \zeta(r) = 0 \quad (r \geq 2), \quad |\zeta'(r)| \leq 3/2. \quad (5.2)$$

Then the Jacobian of φ has a positive sign:

$$\frac{1}{4} \leq J_y \varphi := \det(\text{Jac}_y \varphi) = \det \left(\left(\frac{\partial \varphi_p}{\partial y_q} \right)_{pq} \right) \leq \frac{7}{4}.$$

Consequently, the eigenvalues of $\text{Jac}_y \varphi$ are contained in the interval $[1/4, 7/4]$.

Proof. (5.1) yields that

$$\text{Jac}_y \varphi(t, y) = E_N + \sum_{j=1}^m \zeta' \left(\frac{|y - c_j(0)|}{\varepsilon_0} \right) \left((c_j(t) - c_j(0)) \otimes \frac{y - c_j(0)}{\varepsilon_0 |y - c_j(0)|} \right), \quad (5.3)$$

where the two kinds of matrices are defined by $E_N := (\delta_{pq})_{pq}$ and $w_1 \otimes w_2 := (w_{1p} w_{2q})_{pq}$ for $w_1, w_2 \in \mathbb{R}^N$. Put

$$\zeta_j(y) := \zeta(\varepsilon_0^{-1} |y - c_j(0)|). \quad (5.4)$$

Then we see from (5.2) that

$$\text{supp } \zeta_j \subset B(c_j(0); 2\varepsilon_0). \quad (5.5)$$

It follows from (1.1) and (5.5) that

$$\text{supp } \zeta_j \cap \text{supp } \zeta_k = \emptyset \quad (j \neq k). \quad (5.6)$$

This means that (5.3) can be expressed as

$$\text{Jac}_y \varphi(t, y) = \begin{cases} E_N + \zeta' \left(\frac{|y - c_j(0)|}{\varepsilon_0} \right) \left((c_j(t) - c_j(0)) \otimes \frac{y - c_j(0)}{\varepsilon_0 |y - c_j(0)|} \right), & y \in B(c_j(0); 2\varepsilon_0), \\ E_N, & \text{otherwise.} \end{cases} \quad (5.7)$$

Let $w_1 \cdot w_2$ denote the usual inner product of $w_1, w_2 \in \mathbb{R}^N$. It is not so difficult to show that $w_1 \otimes w_2$ has the eigenvalues one of which is $w_1 \cdot w_2$ and the others are 0 so that $E_N + w_1 \otimes w_2$ has the eigenvalues one of which is $1 + w_1 \cdot w_2$ and the others are 1. Thus we have

$$\det(E_N + w_1 \otimes w_2) = 1 + \text{tr}(w_1 \otimes w_2) = 1 + w_1 \cdot w_2.$$

Therefore it follows from (5.3) and (5.6) that

$$\begin{aligned}
J_y \varphi(t, y) &= \det(\text{Jac}_y \varphi(t, y)) \\
&= \begin{cases} 1 + \zeta' \left(\frac{|y - c_j(0)|}{\varepsilon_0} \right) \frac{(c_j(t) - c_j(0)) \cdot (y - c_j(0))}{\varepsilon_0 |y - c_j(0)|}, & y \in B(c_j(0); 2\varepsilon_0), \\ 1, & \text{otherwise} \end{cases} \\
&= 1 + \sum_{j=1}^m \zeta' \left(\frac{|y - c_j(0)|}{\varepsilon_0} \right) \frac{(c_j(t) - c_j(0)) \cdot (y - c_j(0))}{\varepsilon_0 |y - c_j(0)|}.
\end{aligned}$$

In particular, the eigenvalues of $\text{Jac}_y \varphi(t, y)$ consist of 1 and $J_y \varphi(t, y)$. In this way we are led to the desired conclusion:

$$|1 - J_y \varphi(t, y)| \leq \frac{3}{4} \quad \text{on } I = [0, T_0]. \quad (5.8)$$

In fact, let T_0 be given by (1.2). Then we have

$$|c_j(t) - c_j(0)| \leq \frac{\varepsilon_0}{2} \quad \text{on } I \quad (1 \leq j \leq m). \quad (5.9)$$

Combining this estimate with (5.2) and (5.6), we can verify (5.8). This completes the proof of the whole assertion. \square

Definition 1 (Local pseudo-Galilean transformation). Let $\varphi \in C^1(I \times \mathbb{R}^N; \mathbb{R}^N)$. Suppose that $\varphi(t, \cdot) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a homeomorphism for fixed $t \in I$ with $J_y \varphi > 0$. Let $\Phi(\cdot) : I \rightarrow B(L^2(\mathbb{R}^N))$ be given by

$$\Phi(t)f = (J_y \varphi(t, \cdot))^{1/2} (f \circ \varphi)(t, \cdot), \quad f \in L^2(\mathbb{R}^N).$$

Then Φ is called a “local pseudo-Galilean transformation” associated with $\varphi(\cdot)$ if $\varphi(\cdot)$ is given by (5.1). In this case note that $\Phi(0) = E_N$, the unit matrix.

Next we construct the transformed problem.

Lemma 5.2. Given a local pseudo-Galilean transformation $\Phi(t)$ associated with C^3 -diffeomorphism $\varphi(t, \cdot)$, define a new unknown function as

$$v(t, y) := (\Phi(t)u)(t, y), \quad t \in I.$$

Then $(\text{SE})_I$ is converted into the following Cauchy problem in $L^2(\mathbb{R}^N)$:

$$\begin{cases} i \frac{\partial v}{\partial t} + \left(\frac{1}{i} \text{div} - b(t, y) \right) a(t, y) \left(\frac{1}{i} \nabla - b(t, y) \right) v + r(t, y) v \\ \quad + V(t, \varphi(t, y)) v + \sum_{j=1}^m \frac{e_j v}{|\varphi(t, y) - c_j(t)|} = g(t, y), \\ v(0, \cdot) = v_0 := \Phi(0)u_0 = u_0 \in \Sigma^2(\mathbb{R}^N), \end{cases} \quad (\text{SE-}v)$$

where the expressions of a, b, r and g are given by (1.9)–(1.12) in the Introduction.

Here we want to explain the outline of the proof. The detailed computation will be given in Appendix A. The principal part of the equation in (SE- v) is transformed by $\Phi(t)$ as

$$\Phi(t) \left(i \frac{\partial u}{\partial t} - \Delta_x u \right) = i \frac{\partial v}{\partial t} + \left(\frac{1}{i} \operatorname{div}_y - b(t, y) \right) a(t, y) \left(\frac{1}{i} \nabla_y - b(t, y) \right) v + r(t, y) v. \quad (5.10)$$

In fact, we have

$$\begin{aligned} \Phi(t) \left(\frac{\partial u}{\partial t} \right) - \frac{\partial v}{\partial t} &= - \left((\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t} \right) \cdot \nabla_y v - \frac{1}{2} \left[\operatorname{div}_y \left((\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t} \right) \right] v \\ &= -2a(t, y)b(t, y) \cdot \nabla_y v - [\operatorname{div}_y(a(t, y)b(t, y))]v \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} \Phi(t)(-\Delta_x u) &= -\operatorname{div}_y((\operatorname{Jac}_y \varphi)^{-1}(\operatorname{Jac}_y \varphi)^{-1} \nabla_y v) - (J_y \varphi)^{1/2} [\Delta_x (J_x \psi)^{1/2} \big|_{x=\varphi}] v \\ &= -\operatorname{div}_y(a(t, y) \nabla_y v) + r(t, y) + \frac{1}{4} \left| \frac{\partial \varphi}{\partial t}(t, y) \right|^2 \\ &= -\operatorname{div}_y(a(t, y) \nabla_y v) + r(t, y) + b(t, y) \cdot (a(t, y)b(t, y)). \end{aligned} \quad (5.12)$$

Thus we obtain (5.10).

Reminding of (1.13), we may define

$$h(t, y) := \sum_{j=1}^m e_j h_j(t, y), \quad (t, y) \in I \times \mathbb{R}^N, \quad (5.13)$$

where

$$h_j(t, y) := \frac{1}{|\varphi(t, y) - c_j(t)|} - \frac{1}{|y - c_j(0)|}, \quad (t, y) \in I \times \mathbb{R}^N. \quad (5.14)$$

Then we have

Lemma 5.3. *Let φ and h be defined as in (5.1) and (5.13). Then the given functions of the equation in (SE-v) satisfy the following conditions:*

(C1) (ellipticity) *For $t \in I$, $a(t, \cdot) \in C^1(\mathbb{R}^N; \mathbb{R}^{N \times N})$ is symmetric and satisfies strong ellipticity condition with time-independent ellipticity constant $\lambda > 0$, that is,*

$$\sum_{j,k=1}^N a_{jk}(t, y) \xi_j \xi_k \geq \lambda |\xi|^2, \quad y, \xi \in \mathbb{R}^N.$$

(C2) (regularity of coefficients) $a_{jk}, b_j, r, h \in W^{1,1}(I; W^{1,\infty}(\mathbb{R}^N))$.

(C3) (singularity of potentials at infinity) $\langle y \rangle^{-2} V(\cdot, \varphi(\cdot, y)) \in W^{1,1}(I; L^\infty(\mathbb{R}^N))$, where $\langle y \rangle := (1 + |y|^2)^{1/2}$.

(C4) (inhomogeneous term) $g \in C(I; L^2(\mathbb{R}^N)) \cap L^1(I; \Sigma^2(\mathbb{R}^N))$.

The proof of Lemma 5.3 is given in Appendix A. In particular, to verify (C2) and (C3) we need the boundedness and smoothness of $\varphi(t, y) - y$ and $\langle \varphi(t, y) \rangle / \langle y \rangle$, respectively. Since $B(c_j(t); 2\varepsilon_0) \cap B(c_k(t); 2\varepsilon_0) = \emptyset$ ($j \neq k, t \in I$), we see from (5.1) and (5.9) that

$$\varphi(t, y) - y \in W^{2,1}(I; C_0^\infty(\Omega)), \quad \Omega := \sum_{j=1}^m \{B(c_j(t); 2\varepsilon_0); t \in I\}, \quad (5.15)$$

where \sum means the union of disjoint sets, and hence

$$|\varphi(t, y) - y| \leq \frac{\varepsilon_0}{2}, \quad y \in \mathbb{R}^N. \quad (5.16)$$

As a direct consequence we have

$$\left(1 + \frac{\varepsilon_0}{2}\right)^{-1} \leq \frac{\langle \varphi(t, y) \rangle}{\langle y \rangle} \leq 1 + \frac{\varepsilon_0}{2}. \quad (5.17)$$

The last estimate is also useful in the proof of Lemma 5.7.

5.2. Concrete setting of abstract framework

Put $X := L^2(\mathbb{R}^N)$ and $Y := \Sigma^2(\mathbb{R}^N)$. By virtue of the previous subsection, we are ready to fix both the families $\{A(t)\}$ and $\{S(t)\}$ in addition to S_0 . In X we define

$$\begin{aligned} (\mathcal{A}(t)u)(y) &:= i^{-1}(i^{-1} \operatorname{div} - b(t, y))a(t, y)(i^{-1} \nabla - b(t, y))u + i^{-1}r(t, y)u \\ &\quad + i^{-1}V(t, \varphi(t, y))u + i^{-1}h(t, y)u + i^{-1} \sum_{j=1}^m \frac{e_j}{|y - c_j(0)|}u, \\ D(\mathcal{A}(t)) &:= C_0^\infty(\mathbb{R}^N). \end{aligned}$$

Since $i\mathcal{A}(t)$ is symmetric, $\mathcal{A}(t)$ is closable. Then we take $A(t)$ in X as the closure $\widetilde{\mathcal{A}}(t)$ of $\mathcal{A}(t)$:

$$A(t) := \widetilde{\mathcal{A}}(t), \quad D(A(t)) := \overline{C_0^\infty(\mathbb{R}^N)}^{A(t)}. \quad (5.18)$$

Define a closed linear operator $H(t)$ in $X = L^2(\mathbb{R}^N)$ as

$$\begin{aligned} (H(t)u)(y) &:= (i^{-1} \operatorname{div} - b(t, y))a(t, y)(i^{-1} \nabla - b(t, y))u + r(t, y)u \\ &\quad + V(t, \varphi(t, y))u + h(t, y)u + \sum_{j=1}^m \frac{e_j}{|y - c_j(0)|}u + M\langle y \rangle^2u \end{aligned} \quad (5.19)$$

for $u \in D(H(t)) := \Sigma^2(\mathbb{R}^N)$, where we have set

$$M := 1 + 2 \max_{t \in I} \|\langle y \rangle^{-2} V(t, \varphi(t, \cdot))\|_\infty. \quad (5.20)$$

It is helpful to note that $H(t)$ is nothing but $iA(t)$ perturbed by $M\langle y \rangle^2$:

$$H(t) = iA(t) + M\langle y \rangle^2. \quad (5.21)$$

It is not so difficult to show that $H(t)$ is bounded below. That is, there is a constant $K_0 > 0$ such that

$$(u, (K_0 + H(t))u) \geq \|u\|^2, \quad u \in D(H(t)). \quad (5.22)$$

So we can define S_0 and $\{S(t)\}$ in X as

$$S_0 := (-\Delta + \langle y \rangle^2)^2, \quad D(S_0) := \Sigma^4(\mathbb{R}^N), \quad (5.23)$$

$$S(t) := (K_0 + H(t))^2, \quad D(S(t)) := \{u \in \Sigma^2(\mathbb{R}^N); H(t)u \in \Sigma^2(\mathbb{R}^N)\}. \quad (5.24)$$

In the next step we want to show that $H(t)$ is selfadjoint in $X = L^2(\mathbb{R}^N)$. To this end we need the following lemma, concerning the m -accretivity of the sum of two m -accretive operators in an abstract Hilbert space, in addition to the celebrated Kato–Rellich perturbation theorem (see, e.g., Kato [5, Theorem V.4.3]).

Lemma 5.4. (See [15, Theorem 5.4].) *Let A and B be two linear m -accretive operators in X and $\{B_n\}$ the Yosida approximation of B . Assume that there exists a constant $c \geq 0$ such that for all $n \in \mathbb{N}$,*

$$\operatorname{Re}(Au, B_n u) \geq -c\|u\|^2, \quad u \in D(A). \quad (5.25)$$

Then $A + B$ is also m -accretive in X and $D(A + B)$ is a core for both A and B , with

$$\|Au\|^2 + \|Bu\|^2 \leq \|(A + B)u\|^2 + 2c\|u\|^2, \quad u \in D(A + B)$$

as a simple consequence of (5.25).

Now we can prove the selfadjointness of $H(t)$. Here we abbreviate $a(t, \cdot)$, $b(t, \cdot)$, $r(t, \cdot)$, $V(t, \varphi(t, \cdot))$ and $h(t, \cdot)$ as a , b , r , $V \circ \varphi$ and h , respectively.

Lemma 5.5. *Assume that conditions (C1), (C2) and (C3) are satisfied. Then for a fixed $t \in I$ the following operators are selfadjoint in $L^2(\mathbb{R}^N)$:*

- (a) $H_0 u := -\operatorname{div}(a \nabla u)$ for $u \in D(H_0) = H^2(\mathbb{R}^N)$.
- (b) $P u := H_0 u + Q_0 u = (i^{-1} \operatorname{div} - b)a(i^{-1} \nabla - b)u$ for $u \in D(P) = H^2(\mathbb{R}^N)$, where

$$Q_0 u := i b a \nabla u + i \operatorname{div}(a b u) + b a b u.$$

- (c) $P + M \langle y \rangle^2$ with $D(P + M \langle y \rangle^2) = \Sigma^2(\mathbb{R}^N)$, where $M > 0$ is defined as (5.20) (note that $P + M \langle y \rangle^2 = -\Delta + \langle y \rangle^2$ when $a = E_N$, $b = 0$ and $V = 0$).
- (d) $H(t) = P + r + V \circ \varphi + h + \sum_{j=1}^m \frac{e_j}{|y - c_j(0)|} + M \langle y \rangle^2$ with $D(H(t)) = \Sigma^2(\mathbb{R}^N)$.

Proof. (a) follows from conditions (C1) and (C2) (Brezis [2, Chapter 9, Remark 24]).

(b) Clearly, Q_0 is symmetric. Moreover, we can show that Q_0 is H_0 -bounded with H_0 -bound = 0. In fact, by virtue of conditions (C1) and (C2), for any $\varepsilon > 0$ there exists a constant $C_\varepsilon = C_\varepsilon(\|a\|_{1,\infty}, \|b\|_{1,\infty}) > 0$, where $\|\cdot\|_{1,\infty} := \|\cdot\|_{W^{1,\infty}}$, such that for $u \in H^1(\mathbb{R}^N)$,

$$\|Q_0 u\| \leq \operatorname{const.}(\|\nabla u\| + \|u\|) \leq \varepsilon \|H_0 u\| + C_\varepsilon \|u\|.$$

Therefore the Kato–Rellich theorem yields the assertion.

(c) To apply Lemma 5.4 put $A := M^{-1}P$ and $B_n := \langle y \rangle^2(1 + n^{-1}\langle y \rangle^2)^{-1}$. Then the key of the proof is the computation of $\operatorname{Re}(Av, B_nv)$ as in (5.25). Noting

$$B_n^{1/2} = \langle y \rangle (1 + n^{-1}\langle y \rangle^2)^{-1/2} \geq |y| (1 + n^{-1}\langle y \rangle^2)^{-2},$$

we have for $v \in H^2(\mathbb{R}^N)$,

$$\begin{aligned} M \operatorname{Re}(Av, B_nv) &= \operatorname{Re}(Pv, B_nv) \\ &= \operatorname{Re}(a(i^{-1}\nabla - b)v, B_n(i^{-1}\nabla - b)v + 2i^{-1}y(1 + n^{-1}\langle y \rangle^2)^{-2}v) \\ &\geq \lambda \|B_n^{1/2}(i^{-1}\nabla - b)v\|^2 - 2\|a\|_\infty \|B_n^{1/2}(i^{-1}\nabla - b)v\| \cdot \|v\| \\ &= \lambda (\|B_n^{1/2}(i^{-1}\nabla - b)v\| - \lambda^{-1}\|a\|_\infty \|v\|)^2 - \lambda^{-1}\|a\|_\infty^2 \|v\|^2 \\ &\geq -\lambda^{-1}\|a\|_\infty^2 \|v\|^2. \end{aligned}$$

This concludes the selfadjointness of $P + M\langle y \rangle^2 = M(A + \langle y \rangle^2)$ on $\Sigma^2(\mathbb{R}^N)$, with

$$\|Pu\|^2 + \|M\langle y \rangle^2 u\|^2 \leq \|(P + M\langle y \rangle^2)u\|^2 + 2M\lambda^{-1}\|a\|_\infty^2 \|u\|^2. \quad (5.26)$$

(d) follows again from the Kato–Rellich theorem. In fact, by virtue of conditions (C2) and (C3), we see from (5.20) and the Hardy inequality that for $u \in \Sigma^2(\mathbb{R}^N)$,

$$\begin{aligned} \|ru\| &\leq \|r\|_\infty \|u\|, \quad \|hu\| \leq \|h\|_\infty \|u\|, \\ \|(V \circ \varphi)u\| &\leq \|\langle y \rangle^{-2}(V \circ \varphi)\|_\infty \|\langle y \rangle^2 u\| \leq \frac{1}{2} \|M\langle y \rangle^2 u\|, \\ \left\| \frac{e_j}{|y - c_j(0)|} u \right\| &\leq \frac{2|e_j|}{N-2} \|\nabla u\| \leq \frac{2|e_j|}{N-2} (\|(i^{-1}\nabla - b)u\| + \|b\|_\infty \|u\|). \end{aligned}$$

Since $\lambda \|(i^{-1}\nabla - b)u\|^2 \leq (Pu, u) \leq \|Pu\| \cdot \|u\|$, we see that there exists a constant $c = c(\|r\|_\infty, \|h\|_\infty, \|b\|_\infty, \lambda) > 0$ such that

$$\|ru\| + \|(V \circ \varphi)u\| + \|hu\| + \sum_{j=1}^m \left\| \frac{e_j}{|y - c_j(0)|} u \right\| \leq \frac{1}{2} (\|Pu\| + \|M\langle y \rangle^2 u\|) + c\|u\|.$$

This implies by (5.26) that $r + V \circ \varphi + h + \sum e_j |y - c_j(0)|^{-1}$ is $(P + M\langle y \rangle^2)$ -bounded with $(P + M\langle y \rangle^2)$ -bound $\leq 1/2$. Therefore $H(t)$ is also selfadjoint on $\Sigma^2(\mathbb{R}^N)$. \square

Since $-\Delta + \langle y \rangle^2$ is positive-definite, we see from (5.23) and Lemma 5.5(c) that

$$S_0^{1/2} = -\Delta + \langle y \rangle^2, \quad Y := D(S_0^{1/2}) = \Sigma^2(\mathbb{R}^N), \quad (5.27)$$

with

$$(S_0^{1/2}v, v) \geq \|v\|^2, \quad v \in Y. \quad (5.28)$$

On the other hand, it follows from (5.22), (5.24) and Lemma 5.5(d) that

$$S(t)^{1/2} = K_0 + H(t), \quad D(S(t)^{1/2}) = D(H(t)) = \Sigma^2(\mathbb{R}^N). \quad (5.29)$$

Lemma 5.6. Let $X = L^2(\mathbb{R}^N)$ and $\{A(t)\}$, S_0 and $\{S(t)\}$ be defined as in (5.18), (5.23) and (5.24), respectively. Assume that conditions (C1)–(C3) are satisfied. Then S_0 and $\{S(t)\}$ satisfy (1.15) and (S1)–(S3). Moreover, $\{A(t)\}$ satisfies the following conditions:

(A1) $\operatorname{Re}(A(t)v, v) = 0$, $v \in D(A(t))$, $t \in I$.

(A2) $Y = \Sigma^2(\mathbb{R}^N) \subset D(A(t))$.

(A3) There exists a constant $\beta \geq 0$ such that for $v \in D(S(t))$,

$$|\operatorname{Re}(A(t)v, S(t)v)| \leq \beta \|S(t)^{1/2}v\|^2, \quad t \in I.$$

(A4) $A(\cdot) \in C_*(I; B(Y, L^2(\mathbb{R}^N)))$.

Proof. (A1) is clear because $A(t)$ is skew-symmetric. (1.15) is noting but (5.28). (S1) follows from Lemmas 5.3 and 5.5.

(A2) is shown in a way similar to Lemma 5.5. It is easy to see that for $u \in Y$,

$$\|A(t)u\| \leq \|Pu\| + \|M\langle y \rangle^2 u\| + \|r + h\|_\infty \|u\| + \sum_{j=1}^m \frac{2|e_j|}{N-2} \|\nabla u\|.$$

Since $\|u\|_{H^2} \leq c(\|\Delta u\| + \|u\|)$ for $u \in H^2(\mathbb{R}^N)$ (see [2, Theorem 9.25]), we obtain

$$\|A(t)u\| \leq C(\|\Delta u\| + \|\langle y \rangle^2 u\|), \quad u \in Y = \Sigma^2(\mathbb{R}^N) \subset D(A(t)).$$

To prove (A3) let $v \in D(S(t)) \subset Y$. Then we see from (5.22) and Lemma 5.5(d) that

$$\operatorname{Re}(A(t)v, S(t)v) = -M \operatorname{Im}((H(t)\langle y \rangle^2 - \langle y \rangle^2 H(t))v, S(t)^{1/2}v). \quad (5.30)$$

In fact, it follows from (5.21) and (5.24) that

$$\begin{aligned} \operatorname{Re}(A(t)v, S(t)v) &= \operatorname{Im}(iA(t)v, S(t)v) \\ &= \operatorname{Im}((H(t) - M\langle y \rangle^2)v, (K_0 + H(t))^2 v) \\ &= -\operatorname{Im}((K_0 + M\langle y \rangle^2)v, (K_0 + H(t))^2 v). \end{aligned}$$

Since $K_0 + H(t)$ is selfadjoint, we see from (5.22) that

$$\begin{aligned} \operatorname{Re}(A(t)v, S(t)v) &= -M \operatorname{Im}((K_0 + H(t))\langle y \rangle^2 v, (K_0 + H(t))v) \\ &= -M \operatorname{Im}(\langle y \rangle^2 (K_0 + H(t))v, (K_0 + H(t))v) \\ &\quad - M \operatorname{Im}((H(t)\langle y \rangle^2 - \langle y \rangle^2 H(t))v, S(t)^{1/2}v). \end{aligned}$$

Since $\langle y \rangle^2$ is positive-definite, we can obtain (5.30). It remains to compute the first factor of the inner product on the right-hand side of (5.30):

$$\begin{aligned} H(t)\langle y \rangle^2 v - \langle y \rangle^2 H(t)v &= 2i^{-1}y(a(t, y) + {}^t a(t, y))(i^{-1}\nabla - b(t, y))v \\ &\quad - 2 \sum_{j,k=1}^N \frac{\partial a_{jk}}{\partial y_j}(t, y)y_k v - 2(\operatorname{tr} a(t, y))v. \end{aligned}$$

By virtue of condition **(C2)**, we see from (5.30) that

$$|\operatorname{Re}(A(t)v, S(t)v)| \leq C(\| |y|\nabla v \|^2 + \| |y|v \|^2 + \| v \|^2)^{1/2} \| S(t)^{1/2} v \|$$

for some constant $C > 0$. Since $\operatorname{Re}((1 - \Delta)v, |y|^2 v) \leq (1/2)\| -\Delta v + (1 + |y|^2)v \|^2$, it follows from (5.27) and (1.15) that

$$\begin{aligned} \| |y|\nabla v \|^2 + \| |y|v \|^2 + \| v \|^2 &= \operatorname{Re}((1 - \Delta)v, |y|^2 v) + (N + 1)\| v \|^2 \\ &\leq (N + (3/2))\| S_0^{1/2} v \|^2. \end{aligned}$$

Thus **(S1)** yields the assertion.

(A4) follows from conditions **(C2)** and **(C3)**; note that $W^{1,1}(I) \subset C(I)$.

To prove **(S2)** and **(S3)** let $v \in Y$. It suffices to show that

$$\| H(t)v - H(s)v \| \leq \left| \int_s^t \sigma(\tau) d\tau \right| (\| \Delta v \| + \| \langle y \rangle^2 v \|). \quad (5.31)$$

We see from (5.19), **(C2)**, **(C3)** and (5.13) that

$$\begin{aligned} H(t)v - H(s)v &= \int_s^t \left[(i^{-1} \operatorname{div} - b(\tau, \cdot)) \left(\frac{\partial a(\tau, \cdot)}{\partial \tau} (i^{-1} \nabla - b(\tau, \cdot))v \right) \right. \\ &\quad \left. - 2 \frac{\partial b(\tau, \cdot)}{\partial \tau} a(\tau, \cdot) (i^{-1} \nabla - b(\tau, \cdot))v \right. \\ &\quad \left. - (i^{-1} \operatorname{div} - b(\tau, \cdot)) \left(a(\tau, \cdot) \frac{\partial b(\tau, \cdot)}{\partial \tau} v \right) + \frac{\partial r(\tau, \cdot)}{\partial \tau} v \right. \\ &\quad \left. + \left(\frac{\partial}{\partial \tau} V(\tau, \varphi(\tau, \cdot)) \right) v + \frac{\partial h(\tau, \cdot)}{\partial \tau} v \right] d\tau. \end{aligned}$$

It follows from condition **(C2)** that

$$\left\| \frac{\partial b(\tau)}{\partial \tau} a(\tau) (i^{-1} \nabla - b(\tau))v \right\| \leq \left\| \frac{\partial b(\tau)}{\partial \tau} \right\|_{\infty} \| a(\tau) \|_{\infty} (\| \nabla v \| + \| b(\tau) \|_{\infty} \| v \|).$$

In the same way we can estimate the other terms. Thus we obtain (5.31). \square

By virtue of Lemmas 5.5 and 5.6, applying Theorems 1.2 and 1.3, we can conclude that the Cauchy problem (SE- v) has a unique (classical) solution

$$v(\cdot) \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; \Sigma^2(\mathbb{R}^N))$$

under conditions **(C1)**–**(C4)** stated in Lemma 5.3.

5.3. Return to the original problem (the proof of the main theorem)

To show that (SE)_I also has a unique (classical) solution, we prepare the next

Lemma 5.7. Let $v(\cdot) \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; \Sigma^2(\mathbb{R}^N))$. If $u(\cdot)$ is given by

$$u(t, x) := (\Phi^{-1}(t)v)(t, x) = (J\psi(t, x))^{1/2} v(t, \psi(t, x)),$$

then $u(\cdot) \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; \Sigma^2(\mathbb{R}^N))$. Here $\psi(t, \cdot)$ is the inverse function of $\varphi(t, \cdot)$.

Proof. Let $u(t, x) := (\Phi(t)^{-1}v)(t, x)$. First we show that

$$v \in C(I; L^2(\mathbb{R}^N)) \Leftrightarrow u \in C(I; L^2(\mathbb{R}^N)). \quad (5.32)$$

Since ψ has the same smoothness as φ , (5.32) is reduced to

$$\|v(\cdot)\| \in C(I) \Leftrightarrow \|u(\cdot)\| \in C(I), \quad (5.33)$$

$$v \in C_w(I; L^2(\mathbb{R}^N)) \Leftrightarrow u \in C_w(I; L^2(\mathbb{R}^N)). \quad (5.34)$$

Here C_w means weak continuity. (5.33) is a consequence of (1.6). On the other hand, let $w, z \in L^2(\mathbb{R}^N)$. Since Φ is unitary, we have

$$\begin{aligned} (u(t), w) &= (\Phi(t)u(t), \Phi(t)w) = (v(t), \Phi(t)w), \\ (v(t), z) &= (\Phi(t)^{-1}v(t), \Phi(t)^{-1}z) = (u(t), \Phi(t)^{-1}z). \end{aligned}$$

Therefore (5.34) is a consequence of $\Phi(\cdot) \in C_*(I; B(X))$. Next we show that

$$v \in C(I; H^2(\mathbb{R}^N)) \Rightarrow u \in C(I; H^2(\mathbb{R}^N)), \quad (5.35)$$

$$v \in C(I; H_2(\mathbb{R}^N)) \Rightarrow u \in C(I; H_2(\mathbb{R}^N)). \quad (5.36)$$

To prove (5.35) and (5.36) we proceed in the same way as above.

In fact, (5.12) yields the weak continuity of $\Delta_x u(\cdot)$:

$$\begin{aligned} (\Delta_x u(t), w) &= (\operatorname{div}_y(a(t, y)\nabla_y v), \Phi(t)w) \\ &\quad + ((J_y\varphi(t, y))^{1/2} [\Delta_x(J_x\psi(t, x))^{1/2}|_{x=\varphi}] v, \Phi(t)w), \quad w \in L^2(\mathbb{R}^N). \end{aligned}$$

On the other hand, writing as

$$\begin{aligned} \|\Delta_x u(t)\| &= \|\Phi(t)(\Delta_x u(t))\| \\ &= \|\Phi(t)(\Delta_x \Phi^{-1}(t)v(t))\| \\ &= \|\operatorname{div}_y(a(t, y)\nabla_y v(t)) + (J_y\varphi(t, y))^{1/2} [\Delta_x(J_x\psi(t, x))^{1/2}|_{x=\varphi}] v(t)\|, \end{aligned} \quad (5.37)$$

we see from $v \in C(I; H^2(\mathbb{R}^N))$ that

$$\|\Delta_x u(\cdot)\| \in C(I).$$

This proves (5.35). Next, the proof of (5.36) is based on the equality $\Phi(t)(|x|^2 u(t)) = |\varphi(t, \cdot)|^2 v(t)$ because we have

$$(|x|^2 u(t), w) = (|\varphi(t, \cdot)|^2 v(t), \Phi(t)w), \quad w \in L^2(\mathbb{R}^N).$$

Applying (5.17) to the right-hand side, we obtain the implication

$$v \in C_w(I; H_2(\mathbb{R}^N)) \Rightarrow u \in C_w(I; H_2(\mathbb{R}^N)).$$

In a way similar to (5.37) we see from $v(\cdot) \in C(I; H_2(\mathbb{R}^N))$ that

$$\| |x|^2 u(\cdot) \| \in C(I).$$

Therefore we obtain (5.36). Finally we show that $u \in C^1(I; L^2(\mathbb{R}^N))$. It remains to show that

$$\frac{\partial u}{\partial t} \in C(I; L^2(\mathbb{R}^N)). \quad (5.38)$$

This is a consequence of

$$v \in C^1(I; L^2(\mathbb{R}^N)) \cap C(I; H^1(\mathbb{R}^N)).$$

In fact, by virtue of (5.11) the proof of (5.38) is reduced to

$$\frac{\partial u}{\partial t} \in C_w(I; L^2(\mathbb{R}^N)), \quad \left\| \frac{\partial u}{\partial t}(\cdot) \right\| \in C(I).$$

Therefore we obtain the assertion. \square

Proof of Theorem 1.1. We go back to the original Cauchy problem (SE) which has been divided into the family $\{(SE)_{I_k}\}$. What has been proved in the previous subsection imply that problem (SE- v_k) on I_k has a unique (classical) solution

$$v_k(\cdot) \in C^1(I_k; L^2(\mathbb{R}^N)) \cap C(I_k; \Sigma^2(\mathbb{R}^N)).$$

Put $u_k(t, x) := (\Phi^{-1}(t)v_k)(t, x)$. Then, noting that (SE- v_k) is mapped into (SE) $_{I_k}$, we see that u_k satisfies (SE) $_{I_k}$. Moreover, Lemma 5.7 shows that u_k belongs to the same class as v_k . By virtue of the unitarity of $\Phi(t)$, the uniqueness of u_k follows from that of v_k . Therefore (SE) $_{I_k}$ has a unique (classical) solution

$$u_k(\cdot) \in C^1(I_k; L^2(\mathbb{R}^N)) \cap C(I_k; \Sigma^2(\mathbb{R}^N)).$$

Let us define

$$u(t) := u_k(t), \quad t \in I_k.$$

Then $u(\cdot) \in C(I; \Sigma^2(\mathbb{R}^N))$ and is piecewise smooth on $L^2(\mathbb{R}^N)$. The differentiability at $t = kT_0$ ($k = 1, 2, \dots, k_0 - 1$) is not clear. However, we can overcome the difficulty by the following argument. In fact, there exists the left-derivative $(\partial^-/\partial t)u(kT_0) = (\partial^-/\partial t)u_k(kT_0)$. Since $u_k(\cdot)$ satisfies (SE) $_{I_k}$, we have

$$(\partial^-/\partial t)u_k(kT_0) = f(kT_0) - A(kT_0)u_k(kT_0).$$

In the same way we obtain

$$(\partial^+/\partial t)u(kT_0) = f(kT_0) - A(kT_0)u_{k+1}(kT_0).$$

We see from (1.3) that $u_k(kT_0) = u_{k+1}(kT_0)$. This concludes that $u(\cdot)$ is differentiable at $t = kT_0$ and therefore $u(\cdot) \in C^1(I; L^2(\mathbb{R}^N))$. \square

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Appendix A. Proofs of Lemmas 5.2 and 5.3

Proof of Lemma 5.2. The initial condition is kept the same. In fact, we see from (5.1) that $\varphi(0, y) = y$ and hence $\Phi(0)$ is the identity, that is, $\Phi(0)u_0 = u_0$.

Next we need an equality for nonsingular matrix-valued functions

$$\frac{1}{\det Z(t)} \frac{d}{dt} \det Z(t) = \operatorname{tr} \left(Z(t)^{-1} \frac{d}{dt} Z(t) \right), \quad (\text{A.1})$$

where $Z(\cdot)$ is assumed to be differentiable, with elementwise derivative $(d/dt)Z(t)$.

Now we can carry out the computation from (SE)_I to (SE-v). First of all we have

$$\Phi(t)(Vu) = (\operatorname{Jac}_y \varphi)^{1/2} (V \circ \varphi)(u \circ \varphi) = (V \circ \varphi) \Phi(t)u = (V \circ \varphi)v.$$

The central part of the proof is to verify (5.10). We prepare several basic equalities.

Differentiating two identities $\psi(t, \varphi(t, y)) = y$, $\varphi(t, \psi(t, x)) = x$ with respect to t and x , respectively, we obtain

$$\frac{\partial \psi}{\partial t} \circ \varphi = -((\operatorname{Jac}_x \psi) \circ \varphi) \frac{\partial \varphi}{\partial t} = -(\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t}, \quad (\text{A.2})$$

$$((\operatorname{Jac}_y \varphi) \circ \psi) \operatorname{Jac}_x \psi_{qj} = \sum_{p=1}^N \left(\frac{\partial \varphi_q}{\partial y_p} \circ \psi \right) \frac{\partial \psi_p}{\partial x_j} = \delta_{jq}. \quad (\text{A.3})$$

By virtue of (A.1), we have

$$\frac{1}{J_x \psi} \left(\frac{\partial (J_x \psi)}{\partial x_j} \right) = \operatorname{tr} \left((\operatorname{Jac}_x \psi)^{-1} \frac{\partial}{\partial x_j} (\operatorname{Jac}_x \psi) \right) = \sum_{p,q=1}^N \left(\frac{\partial \varphi_q}{\partial y_p} \circ \psi \right) \frac{\partial^2 \psi_p}{\partial x_j \partial x_q}, \quad (\text{A.4})$$

where $j = 0, 1, \dots, N$ and $x_0 := t$. Setting $x = \varphi(t, y)$ in (A.3) and (A.4), we have

$$\sum_{p=1}^N \left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} = \delta_{jq}, \quad (\text{A.5})$$

$$(J_y \varphi) \left(\frac{\partial (J_x \psi)}{\partial x_j} \circ \varphi \right) = \sum_{p,q=1}^N \left(\frac{\partial^2 \psi_p}{\partial x_j \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p}, \quad 0 \leq j \leq N. \quad (\text{A.6})$$

In particular, we see from (A.6) with $j = 0$ and (A.2) that

$$\begin{aligned} (J_y \varphi) \left(\frac{\partial (J_x \psi)}{\partial t} \circ \varphi \right) &= \sum_{p,q=1}^N \left(\frac{\partial^2 \psi_p}{\partial t \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} \\ &= \operatorname{div}_y \left(\frac{\partial \psi}{\partial t} \circ \varphi \right) = -\operatorname{div}_y \left((\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t} \right). \end{aligned} \quad (\text{A.7})$$

By definition we have

$$u(t, x) = (\Phi^{-1}(t)v)(t, x) = (J_x \psi(t, x))^{1/2} v(t, \psi(t, x)).$$

Thus we see that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t} [(J_x \psi)^{1/2} (v \circ \psi)] \\ &= \frac{1}{2} \frac{1}{(J_x \psi)^{1/2}} \frac{\partial (J_x \psi)}{\partial t} (v \circ \psi) + (J_x \psi)^{1/2} \left[\left(\frac{\partial v}{\partial t} \circ \psi \right) + \frac{\partial \psi}{\partial t} \cdot (\nabla_y v \circ \psi) \right] \\ &= \frac{1}{2} \frac{1}{J_x \psi} \frac{\partial (J_x \psi)}{\partial t} (\Phi^{-1}(t)v) + \Phi^{-1}(t) \frac{\partial v}{\partial t} + \frac{\partial \psi}{\partial t} \cdot (\Phi^{-1}(t) \nabla_y v). \end{aligned}$$

This proves (5.11):

$$\begin{aligned} \Phi(t) \left(\frac{\partial u}{\partial t} \right) &= (J_y \varphi)^{1/2} \left(\frac{\partial u}{\partial t} \circ \varphi \right) \\ &= \frac{\partial v}{\partial t} + \left(\frac{\partial \psi}{\partial t} \circ \varphi \right) \cdot \nabla_y v + \frac{1}{2} (J_y \varphi) \left(\frac{\partial (J_x \psi)}{\partial t} \circ \varphi \right) v \\ &= \frac{\partial v}{\partial t} - \left((\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t} \right) \cdot \nabla_y v - \frac{1}{2} \left[\operatorname{div}_y \left((\operatorname{Jac}_y \varphi)^{-1} \frac{\partial \varphi}{\partial t} \right) \right] v; \end{aligned}$$

in the final step we have used (A.2) and (A.7). In the same way we have

$$\begin{aligned} \Phi(t)(\Delta_x u) &= \sum_{j,k,p=1}^N \left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial^2 v}{\partial y_p \partial y_k} + \sum_{j,k=1}^N \left(\frac{\partial^2 \psi_k}{\partial x_j^2} \circ \varphi \right) \frac{\partial v}{\partial y_k} \\ &\quad + (J_y \varphi) \sum_{j,k=1}^N \left(\frac{\partial (J_x \psi)}{\partial x_j} \circ \varphi \right) \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial v}{\partial y_k} \\ &\quad + (J_y \varphi)^{1/2} [\Delta_x (J_x \psi)^{1/2}]_{|x=\varphi} v. \end{aligned}$$

On the other hand, we see that

$$\begin{aligned} &\operatorname{div}_y ((\operatorname{Jac}_y \varphi)^{-1} (\operatorname{Jac}_y \varphi)^{-1} \nabla_y v) \\ &= \sum_{j,k,p=1}^N \frac{\partial}{\partial y_p} \left[\left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial v}{\partial y_k} \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{j,k,p=1}^N \left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial^2 v}{\partial y_p \partial y_k} + \sum_{j,k,p,q=1}^N \left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \left(\frac{\partial^2 \psi_k}{\partial x_j \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} \frac{\partial v}{\partial y_k} \\
 &\quad + \sum_{j,k,p,q=1}^N \left(\frac{\partial^2 \psi_p}{\partial x_j \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial v}{\partial y_k}.
 \end{aligned}$$

Therefore it follows from (A.5) and (A.6) that

$$\begin{aligned}
 &\Phi(t)(\Delta_x u) - \operatorname{div}_y (\operatorname{Jac}_y \varphi)^{-1t} (\operatorname{Jac}_y \varphi)^{-1} \nabla_y v) \\
 &= \sum_{j,k=1}^N \left(\left(\frac{\partial^2 \psi_k}{\partial x_j^2} \circ \varphi \right) - \sum_{p,q=1}^N \left(\frac{\partial \psi_p}{\partial x_j} \circ \varphi \right) \left(\frac{\partial^2 \psi_k}{\partial x_j \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} \right) \frac{\partial v}{\partial y_k} \\
 &\quad + \sum_{j,k=1}^N \left((J_y \varphi) \left(\frac{\partial (J_x \psi)}{\partial x_j} \circ \varphi \right) - \sum_{p,q=1}^N \left(\frac{\partial^2 \psi_p}{\partial x_j \partial x_q} \circ \varphi \right) \frac{\partial \varphi_q}{\partial y_p} \right) \left(\frac{\partial \psi_k}{\partial x_j} \circ \varphi \right) \frac{\partial v}{\partial y_k} \\
 &\quad + (J_y \varphi)^{1/2} [\Delta_x (J_x \psi)^{1/2}|_{x=\varphi}] v = (J_y \varphi)^{1/2} [\Delta_x (J_x \psi)^{1/2}|_{x=\varphi}] v.
 \end{aligned}$$

This proves (5.12) with a , b and r given by (1.9)–(1.11), respectively. Thus we obtain (SE- v) with g given by (1.12). \square

Proof of Lemma 5.3. Since $a(t, y)$ is given by (1.9), $a(t, y)$ is symmetric and hence

$$\sum_{j,k=1}^N a_{jk} \xi_j \xi_k = |{}^t(\operatorname{Jac}_y \varphi)^{-1} \xi|^2, \quad \xi \in \mathbb{R}^N.$$

Now $\operatorname{Jac}_y \varphi(t, y)$ is simply written as $E_N + w_1 \otimes w_2$ (see (5.7)). Let $\eta \in \mathbb{R}^N$. Then, since $(w_1 \otimes w_2)\eta = w_1(w_2 \cdot \eta)$, we see that

$$|(E_N + w_1 \otimes w_2)\eta| \leq |\eta| + |w_1| \cdot |w_2 \cdot \eta| \leq (1 + |w_1| \cdot |w_2|)|\eta|.$$

Here we know that $|w_1| \cdot |w_2| \leq 3/4$ (see the proof of Lemma 5.1). Thus we have

$$|(\operatorname{Jac}_y \varphi)\eta| \leq \frac{7}{4}|\eta|.$$

Therefore ellipticity condition (C1) is satisfied with $\lambda = (7/4)^{-2} = 16/49$. Next we verify that condition (C2) is satisfied. We see from (5.15) and (5.16) that $a(t, y)$, $b(t, y)$ and $r(t, y)$ satisfy condition (C2). Next we show that $h(t, y) = \sum_{j=1}^m e_j h_j(t, y)$ satisfies condition (C2). It suffices to show that

$$h_j \in W^{1,1}(I; W^{1,\infty}(\mathbb{R}^N)). \quad (\text{A.8})$$

First we note that

$$\varphi(t, y) - c_j(t) = y - c_j(0) + (\zeta_j(y) - 1)(c_j(t) - c_j(0)) + \sum_{k \neq j} \zeta_k(y)(c_k(t) - c_k(0)),$$

where ζ_j is defined as (5.4). It follows from (5.6) that

$$\begin{aligned}
& (\varphi(t, y) - c_j(t)) - (y - c_j(0)) \\
&= \begin{cases} 0, & |y - c_j(0)| \leq \varepsilon_0, \\ (\zeta_j(y) - 1)(c_j(t) - c_j(0)), & \varepsilon_0 < |y - c_j(0)| \leq 2\varepsilon_0, \\ -(c_j(t) - c_j(0)) + \zeta_{k_0}(y)(c_{k_0}(t) - c_{k_0}(0)), & 2\varepsilon_0 < |y - c_j(0)|. \end{cases}
\end{aligned}$$

Obviously, we have $|\varphi(t, y) - c_j(t)| = |y - c_j(0)|$ if $|y - c_j(0)| \leq \varepsilon_0$. On the other hand, by virtue of (5.2) and (5.9) we see that

$$|\varphi(t, y) - c_j(t)| > \begin{cases} \varepsilon_0/2 & (\varepsilon_0 < |y - c_j(0)| \leq 2\varepsilon_0), \\ \varepsilon_0 & (2\varepsilon_0 < |y - c_j(0)|). \end{cases}$$

This implies that $h_j \in L^\infty(I \times \mathbb{R}^N)$. The derivatives of h_j are similarly estimated. Thus we obtain (A.8). Now we show that $V \circ \varphi$ satisfies condition (C3). (5.17) leads us to the following inequalities:

$$\begin{aligned}
\|\langle y \rangle^{-2} V(t, \varphi(t, \cdot))\|_\infty &\leq \left(1 + \frac{\varepsilon_0}{2}\right)^2 \|\langle x \rangle^{-2} V(t, \cdot)\|_\infty, \\
\left\| \frac{\partial}{\partial t} \left(\frac{V(t, \varphi(t, \cdot))}{\langle y \rangle^2} \right) \right\|_\infty &\leq \left(1 + \frac{\varepsilon_0}{2}\right)^2 \left\| \frac{1}{\langle x \rangle^2} \frac{\partial V}{\partial t}(t, \cdot) \right\|_\infty + \|c'\|_{L^\infty(I)} \|\nabla_x V(t, \cdot)\|_{L^\infty(\Omega)}
\end{aligned}$$

(for Ω see (5.15)). The first is straightforward. To derive the second we use

$$\frac{\partial}{\partial t} V(t, \varphi(t, y)) = \frac{\partial V}{\partial t}(t, \varphi(t, y)) + (\nabla_x V)(t, \varphi(t, y)) \frac{\partial \varphi}{\partial t}(t, y) \quad (\text{A.9})$$

and $(\partial \varphi / \partial t)(t, y) = \sum_{j=1}^m \zeta_j(y) c'_j(t)$. Each term on the right-hand side of (A.9) is respectively estimated by conditions (V1) and (V2). Therefore $V \circ \varphi$ satisfies condition (C3).

Remark 6. To show that V satisfies condition (C3), it is enough to estimate ∇V on Ω . So, we can replace condition (V2) with, for instance,

$$V(t, \cdot) \in L^1(I; W^{1,\infty}(\Omega)). \quad (\text{V2}')$$

Proof of Lemma 5.3, concluded. It remains to show that $g(t, y)$ satisfies condition (C4). Let us remind that $g(t, y)$ is given by

$$g(t, y) = (\Phi(t)f)(t, y) = (J_y \varphi(t, y))^{1/2} f(t, \varphi(t, y)).$$

We have to show that

$$f \in C(I; L^2(\mathbb{R}^N)) \cap L^1(I; \Sigma^2(\mathbb{R}^N)) \Rightarrow g \in C(I; L^2(\mathbb{R}^N)) \cap L^1(I; \Sigma^2(\mathbb{R}^N)).$$

However, this can be similarly proved as in the proof of Lemma 5.7. \square

References

- [1] L. Baudouin, O. Kavian, J.-P. Puel, Regularity for a Schrödinger equation with singular potentials and application to bilinear optimal control, J. Differential Equations 216 (2005) 188–222.
- [2] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011.
- [3] W. Hunziker, Distortion analyticity and molecular resonance curves, Ann. Inst. H. Poincaré, Phys. Théor. 45 (1986) 241–258.
- [4] R. Ikehata, N. Okazawa, Yosida approximation and nonlinear hyperbolic equation, Nonlinear Anal. 15 (1990) 479–495.

- [5] T. Kato, Perturbation Theory for Linear Operators, Grundlehren Math. Wiss., vol. 132, Springer-Verlag, Berlin, New York, 1966; 2nd ed., 1976.
- [6] T. Kato, Linear evolution equations of “hyperbolic” type, J. Fac. Sci. Univ. Tokyo, Sec. I. 17 (1970) 241–258.
- [7] T. Kato, Linear evolution equations of “hyperbolic” type, II, J. Math. Soc. Japan 25 (1973) 648–666.
- [8] T. Kato, Quasi-Linear Equations of Evolution, with Applications to Partial Differential Equations, Lecture Notes in Math., vol. 448, Springer-Verlag, Berlin, New York, 1975, pp. 25–70.
- [9] T. Kato, Singular Perturbation and Semigroup Theory, Lecture Notes in Math., vol. 565, Springer-Verlag, Berlin, New York, 1976, pp. 104–112.
- [10] T. Kato, Abstract Differential Equations and Nonlinear Mixed Problems, Lezioni Fermiane [Fermi Lectures], Accad. Naz. Lincei, Scuola Normale Superiore, Pisa, 1985.
- [11] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier–Stokes equations, Comm. Pure Appl. Math. 41 (1988) 891–907.
- [12] T. Kato, K. Yajima, Dirac equations with moving nuclei, Ann. Inst. H. Poincaré, Phys. Théor. 54 (1991) 209–221.
- [13] E.H. Lieb, The stability of matter: from atoms to stars, Bull. Amer. Math. Soc. 22 (1990) 1–49.
- [14] N. Okazawa, Remarks on linear m -accretive operators in a Hilbert space, J. Math. Soc. Japan 27 (1975) 160–165.
- [15] N. Okazawa, On the perturbation of linear operators in Banach and Hilbert spaces, J. Math. Soc. Japan 34 (1982) 677–701.
- [16] N. Okazawa, The Euler equation on a bounded domain as a quasilinear evolution equation, Comm. Appl. Nonlinear Anal. 3 (1996) 107–113.
- [17] N. Okazawa, Remarks on linear evolution equations of hyperbolic type in Hilbert space, Adv. Math. Sci. Appl. 8 (1998) 399–423.
- [18] N. Okazawa, Gauss hypergeometric functions of operators unifying fractional powers and logarithms, in: Semigroups of Operators: Theory and Applications, Rio de Janeiro, 2001, Optimization Software, New York, 2002, pp. 209–219.
- [19] N. Okazawa, T. Yokota, K. Yoshii, Remarks on linear Schrödinger evolution equations with Coulomb potential with moving center, SUT J. Math. 46 (2010) 155–176.
- [20] N. Okazawa, K. Yoshii, Linear evolution equations with strongly measurable families and application to the Dirac equation, Discrete Contin. Dyn. Syst. Ser. S 4 (2011) 723–744.
- [21] E.M. Ouhabaz, Analysis of Heat Equations on Domains, London Math. Soc. Monogr. Ser., Princeton Univ. Press, Princeton, Oxford, 2005.
- [22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci., vol. 44, Springer-Verlag, Berlin, New York, 1983.
- [23] H. Tanabe, Functional Analytic Methods for Partial Differential Equations, Pure Appl. Math., vol. 204, Marcel Dekker, New York, 1997.
- [24] K. Yajima, On time dependent Schrödinger equations, in: Dispersive Nonlinear Problems in Mathematical Physics, in: Quad. Mat., vol. 15, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 267–329.
- [25] K. Yajima, Schrödinger equations with time-dependent unbounded singular potentials, Rev. Math. Phys. 23 (2011) 828–838.
- [26] K. Yoshii, Classical solutions to a linear Schrödinger evolution equation involving a Coulomb potential with a moving center of mass, Funkcial. Ekvac. 54 (2011) 485–493.