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Traveling waves solutions of isothermal chemical systems with decay

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Abstract

This article studies propagating traveling waves in a class of reaction–diffusion systems which include a model of microbial growth and competition in a flow reactor proposed by Smith and Zhao [17], and isothermal autocatalytic systems in chemical reaction of order m with a decay order n, where m and n are positive integers and $m \neq n$. A typical system in autocatalysis is $A + 2B \to 3B$ (with rate k_1ab^2) and $B \to C$ (with rate k_2b), where $m = 2$ and $n = 1$, involving two chemical species, a reactant A and an auto-catalyst B whose diffusion coefficients, D_A and D_B, are unequal due to different molecular weights and/or sizes. Here a is the concentration density of A, b that of B and C an inert chemical species. The two constants k_1 and k_2 are material constants measuring the relative strength of respective reactions.

It is shown that there exist traveling waves when $m > 1$ and $n = 1$ with suitable relation between the ratio D_B/D_A, traveling speed c and rate constants k_1, k_2. On the other hand, it is proved that there exists no traveling wave when one of the chemical species is immobile, $D_B = 0$ or $n > m$ for all choices of other parameters.

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1. Introduction

In this paper we study reaction–diffusion systems of the form

$$\text{(I)} \quad \begin{cases} u_t = D_A u_{xx} - f(u,v), \\ v_t = D_B v_{xx} + f(u,v) - g(v), \end{cases} \tag{1.1}$$

where f is a C^1 function defined on $[0,\infty) \times [0,\infty)$, g a C^1 function defined on $[0,\infty)$ with properties

$$f(u,0) = f(0,v) = 0, \quad \text{and} \quad f(u,v) > 0 \quad \text{on } (0,\infty) \times (0,\infty),$$
$$g(0) = 0 \quad \text{and} \quad g(v) > 0 \quad \text{on } (0,\infty),$$

where D_A, D_B are positive constants representing the diffusion coefficients of two different species. The particular feature we are interested in is the existence and non-existence of traveling waves. Without loss of generality, we shall assume in what follows that $D_A = 1$ and use d in place of D_B, since the general case can be transformed to this one by a simple non-dimensional scaling.

Many interesting phenomena in population dynamics, bio-reactors and chemical reactions can be modeled by a system of the form as in (1.1). For example, a system modeling microbial growth and competition in a flow reactor was first studied in [1] and [17], where a special case is $f(u,v) = F(u)v$, $g(v) = Kv$, K a positive constant, and $F(0) = 0$ and $F'(0) > 0$. In that context, u is the density of nutrient and v that of microbial population. $g(v)$ is the death rate of microbial. Subsequent works with emphasis on traveling waves appeared later in [18] and more recently in [10]. Furthermore, when $F(u) = u$, it is reduced to a classical diffusive epidemic model of Kermack and Mckendric [11].

Another interesting case arises from isothermal autocatalytic chemical reaction between two chemical spices A and B taking the form:

$$A + mB \longrightarrow (m+1)B \quad \text{with rate } r[A][B]^m,$$

where m is an integer and $r > 0$ is a rate constant. In that situation, $f(u,v) = uv^m$ with u the concentration density of A and v that of B. If there is no decay, then $g(v) = 0$. The resulting system is

$$\begin{cases} u_t = u_{xx} - uv^m, \\ v_t = dv_{xx} + uv^m, \end{cases} \tag{1.2}$$

after a simple non-dimensional transformation. The global dynamics of the Cauchy problem as well as existence of traveling wave, sharp estimate of minimum speed and stability were investigated in [3,4,12,13,15] for $m > 1$ case.

Furthermore, it was demonstrated in [2] by asymptotic analysis and numerical computation, and rigorously proved for $m = 1$ in [5] that any small amount of B introduced locally with uniform initial distribution of A can generate traveling wave. The feature seems to be contradictory to the fact that in the relevant experimental result of chemical reactions, the initiation of traveling wave calls for sufficient amount of B to be added [20]. To overcome this lacking of threshold phenomenon, Gray [6] made the observation that the auto-catalyst B cannot be stable indefinitely and should be used to produce other chemicals. In particular, it was suggested in [14] that B decays to an inert product C at a rate of order n,

$$B \longrightarrow C \quad \text{with rate } kv^n.$$

The resulting PDE system is

$$\text{(II)} \qquad \begin{cases} u_t = u_{xx} - uv^m \\ v_t = dv_{xx} + uv^m - kv^n, \end{cases} \tag{1.3}$$

where $m, n \geq 1$ and $k > 0$ is a rate constant, after a simple scaling. It is also being suggested in [11] as simple model for the spread of infectious diseases. We assume throughout that $m \neq n$.

This is the system of our main interest in this paper. Furthermore, in the context of microbial growth and competition in a flow reactor model, we have $f(u, v) = F(u)v^m$, $g(v) = kv^n$. In addition to the fact that (II) is important in many applications, it is mathematically rather challenging. The system has very different property than related systems which have been studied in the literature. In particular, unlike the system (1.2), where a traveling wave means both u and v are fronts, the traveling wave for (1.3) has u being a front, but v being a pulse, which increases the difficulty in analysis. We make it clear in what follows.

Traveling wave: $(u(x,t), v(x,t))$ is called a traveling wave solution to (1.3) if $u(x,t) = a(z)$, $v(x,t) = b(z)$, with $z = x - ct$ and the positive functions $(a, b) \in C^2(R)$ satisfy

$$\text{(TW)} \quad \begin{cases} a''(z) + ca'(z) - a(z)b^m(z) = 0, & -\infty < z < \infty, \\ db''(z) + cb'(z) + a(z)b^m(z) - kb^n(z) = 0, & -\infty < z < \infty, \\ \lim\limits_{z \to -\infty}(a, b) = (a_0, 0), & \lim\limits_{z \to \infty}(a, b) = (a_1, 0), \end{cases} \tag{1.4}$$

where $a_0 < a_1$ are two positive numbers and the positive constant c is the wave speed.

The case of $m = n$ with $m = 1$ in (1.4) is a representative case of the system being studied in [1,17,18] and later in [10]. Other related results appeared in [8,9]. Whereas $m = n$ with $m > 1$ case has been studied in [7] and [19]. The existence of traveling wave is proved for various cases in those works.

But, the arguments in above works cannot be carried out to cover the more general case of $m \neq n$. In particular, it is fairly easy to prove, as was done by many authors that the graph of v is bell-shaped when $m = n$, but we are unable to verify it for our case of $m \neq n$ in general.

We also note that related works on steady-states solution are obtained for the system (1.2) in [16] and for the system (1.3) in [21].

Our main result is as follows.

Theorem 1.

(1) *Suppose $m > 1 = n$ and $d > 1$. There exists a traveling wave solution to (II) if $(d-1)c^2 = k$, $d^2 > (8m-1)(d-1)^2 + 6(d-1)d$ provided either $m \geq 2$, $(m-2)(d-1) < 1$ or $1 < m < 2$ and $2 - m \ll 1$.*

(2) *There exists no traveling wave solution to (II) if either $1 \leq m < n$ or $m > n$ and $d = 0$.*

(3) *Fix* $a_1 > a_0 > 0$, *there exists no traveling wave solution to (II) when* $m > 1 = n$ *and* $d > 0$ *if*

$$c \le \frac{k^{(m+1)/2(m-1)} d^{1/2}}{a_1^{1/(m-1)}(a_1 - a_0)}.$$

The organization of the paper is as follows. In Section 2 we derive some preliminary results which shows the non-existence part of Theorem 1. In addition, they serve as the basis of more important results to be proved in subsequent sections. In Section 3, we treat the case of $m > n$ and $n = 1$, with $d = 0$ to show interesting behavior of the solutions. In Section 4, we prove the existence of traveling waves.

2. Preliminary results

In this section we discuss some basic facts about the system (1.4). In particular, we shall derive some simple properties of traveling wave solutions.

Proposition 1. *Suppose* (a, b) *is a positive solution of (1.4) on* $(-\infty, X)$ *with* X *either a fixed number or* ∞. *Then, for* $z \le X$,

(i) *the following identities hold:*

$$a'(z) + c\big(a(z) - a_0\big) = \int_{-\infty}^{z} b^m(s)a(s)\, ds,$$

$$db'(z) + cb(z) = \int_{-\infty}^{z} \big(kb^n(s) - b^m(s)a(s)\big)\, ds. \tag{2.5}$$

Consequently, $a' > 0$ *on* $(-\infty, X)$. *In particular, if* $X = \infty$, *and* (a, b) *is a traveling wave solution, then*

$$c(a_1 - a_0) = \int_{-\infty}^{\infty} kb^n(s)\, ds = \int_{-\infty}^{\infty} b^m(s)a(s)\, ds. \tag{2.6}$$

Therefore, both integrals in (2.6) are finite when (a, b) *is a traveling wave solution of (1.4).*

(ii) *Let* (a, b) *be a traveling wave solution. If* b *achieves a local maxima at* z_1, *with value* $b_{\max}$, *then*

$$b_{\max} < a_1 - a(z_1) < a_1 - a_0,$$

and

$$a(z_1)\big(a_1 - a(z_1)\big)^{m-n} > k$$

(iii) $e^{cz/d}(a(z) + b(z) - a_0)$ *is an increasing function if* $d \ge 1$.

Proof. The two identities in (2.5) are a direct consequence of integrating the first and second equations in (1.4) on $(-\infty, z)$, respectively. The addition of the two identities in (2.5) gives

$$a'(z) + db'(z) + c\big(a(z) + b(z) - a_0\big) = k \int_{-\infty}^{z} b^n(s)\, ds \tag{2.7}$$

and let $z \to \infty$, we derive (2.6) by using the boundary conditions at $z = \pm\infty$.

For (ii), from the equation

$$db'(z) + cb(z) = \int_{-\infty}^{z} \big(kb^n(s) - b^m(s)a(s)\big)\, ds,$$

we have

$$cb_{\max} < c(a_1 - a_0) - c\big(a(z_1) - a_0\big) = c\big(a_1 - a(z_1)\big)$$

by using (2.5) and (2.6). The other inequality follows from the equation and the above estimate of $b_{\max}$.

The proof of (iii) is based on writing (2.7) as

$$d\big\{e^{cz/d}\big(a(z) + b(z) - a_0\big)\big\}' = (d-1)a'(z)e^{cz/d} + ke^{cz/d} \int_{-\infty}^{z} b^n(s)\, ds.$$

An integration yields

$$d(a + b - a_0) = (d-1) \int_{-\infty}^{z} a'(s)e^{c(s-z)/d}\, ds + \frac{d}{c}k \int_{-\infty}^{z} b^n(s)\big(1 - e^{c(s-z)/d}\big)\, ds.$$

This completes the proof of the proposition. □

A direct corollary of the above is the following two non-existence results.

Corollary 1. *There exist no traveling wave solution if either (i) $m < n$ or (ii) $m > n$ and $d = 0$.*

Proof. It follows directly from the fact that, for a traveling wave solution, kb^n dominates ab^m at $-\infty$ for case (i), and at ∞ in case of (ii). □

Remark. It is of interest to know what exactly happens when $d = 0$, which will be done in next section, especially in comparison to the case of autocatalysis without decay. Our analysis reveals some interesting phenomena. From now on, we shall assume $m > n$. Simple integration shows

$$\begin{cases} a(z) - a_0 = \dfrac{1}{c} \displaystyle\int_{-\infty}^{z} \big(1 - e^{-c(z-s)}\big) a(s) b^m(s)\, ds, \\ b(z) = \dfrac{1}{c} \displaystyle\int_{-\infty}^{z} \big(1 - e^{-c(z-s)/d}\big)\big(kb^n(s) - a(s)b^m(s)\big)\, ds. \end{cases}$$

Consequently, any non-trivial non-negative solution $(a(z), b(z))$ coming out of $z = -\infty$ will be positive and increasing until $a(z)b^m(z) > kb^n(z)$.

In the following, we shall look at the special case of $n = 1$ more closely. Most results in what follows will be on this case unless otherwise stated. The system (1.4) now takes the form

$$\text{(TW1)} \quad \begin{cases} a''(z) + ca'(z) - a(z)b^m(z) = 0, & -\infty < z < \infty, \\ db''(z) + cb'(z) + a(z)b^m(z) - kb(z) = 0, & -\infty < z < \infty, \\ \lim\limits_{z\to-\infty} (a, b) = (a_0, 0), \quad \lim\limits_{z\to\infty} (a, b) = (a_1, 0). \end{cases} \tag{2.8}$$

Let

$$\lambda_0 = \frac{c + \sqrt{c^2 + 4dk}}{2d},$$

which is the positive root of quadratic equation $d\lambda^2 - c\lambda - k = 0$. Easy computation shows

$$\int_{-\infty}^{z} e^{\lambda_0 s} b''(s)\, ds = e^{\lambda_0 z}\big(b'(z) - \lambda_0 b(z)\big) + \lambda_0^2 \int_{-\infty}^{z} e^{\lambda_0 s} b(s)\, ds,$$

$$\int_{-\infty}^{z} e^{\lambda_0 s} b'(s)\, ds = e^{\lambda_0 z} b(z) - \lambda_0 \int_{-\infty}^{z} e^{\lambda_0 s} b(s)\, ds.$$

Hence,

$$\begin{aligned} \int_{-\infty}^{z} e^{\lambda_0 s}\big[db''(s) + cb'(s) - kb(s)\big]\, ds &= e^{\lambda_0 z}\big(db'(z) - d\lambda_0 b(z) + cb(z)\big) \\ &= -\int_{-\infty}^{z} e^{\lambda_0 s} a(s) b^m(s)\, ds. \end{aligned}$$

This implies

$$db'(z) + (c - d\lambda_0) b(z) < 0 \quad \text{on } (-\infty, X),$$

if $b > 0$ on the same interval. An integration then shows

$$\frac{d\lambda_0 - c}{d} \int_{-\infty}^{z} b(s)\, ds > b(z).$$

If (a, b) are traveling waves solution, then since

$$\frac{d\lambda_0 - c}{d} < \min\left(\sqrt{\frac{k}{d}}, \frac{k}{c}\right), \qquad \sqrt{\frac{k}{d}} \int_{-\infty}^{\infty} b(s)\, ds > b_{\max} \quad \text{and} \quad k \int_{-\infty}^{\infty} b(s)\, ds = c(a_1 - a_0),$$

we get $\dfrac{\sqrt{kd}}{c} b_{\max} < (a_1 - a_0)$.

This, in combination with $a_1 b_{\max}^{m-1} > k$ gives

$$c > \frac{k^{(m+1)/2(m-1)} d^{1/2}}{a_1^{1/(m-1)}(a_1 - a_0)}.$$

This is a low bound of speed c.

Proposition 2. *Suppose (a, b) is a traveling waves solution of (2.8), then the following statements hold.*

(1) $b' + \lambda_0 b > 0$ *on* $(-\infty, \infty)$.
(2) $db' - lb < 0$ *on* $(-\infty, \infty)$, *where* $l = (\sqrt{c^2 + 4kd} - c)/2$.
(3) *Furthermore, if* $0 < d \le 1$, *or* $d > 1$ *and* $k + (1 - d)c\lambda_0 \ge 0$, *then* $a' + db' > 0$ *on* $(-\infty, \infty)$.
(4) *The linearized system at* $(a_0, 0)$ *when* $z \to -\infty$ *has a unique solution satisfying, with* $\lambda_3 = l/d$,

$$\big(a(z) - a_0\big)e^{-\lambda_3 z} \to 0, \qquad b(z)e^{-\lambda_3 z} \to 1.$$

Proof. It is easy to check that

$$d\big(b' + \lambda_0 b\big)' = (d\lambda_0 - c)\big(b' + \lambda_0 b\big) - ab^m.$$

If there exists z_0 such that $db' + \lambda_0 b(z_0) = 0$, then $db' + \lambda_0 b < 0$ and $(db' + \lambda_0 b)' < 0$ on (z_0, ∞), which implies b must reach zero at a finite z. We reach a contradiction. Thus, the first statement holds true.

The second statement was already proved above. For the third, if $0 < d \le 1$, $a' + db'$ solves

$$\big(a' + db'\big)' = -\frac{c}{d}\big(a' + db'\big) + \frac{c(1 - d)}{d} a' + kb,$$

a simple integration then yields the desired result using the fact that $a' + db' > 0$ when $z \ll -1$. But, if $d > 1$ and $k + (1 - d)c\lambda_0 \ge 0$,

$$\begin{aligned}\big(a'+db'\big)' &= -\frac{c}{d}\big(a'+db'\big)+\frac{c(1-d)}{d}a'+kb\\ &= -\frac{c}{d}\big(a'+db'\big)+\frac{c(1-d)}{d}\big(a'+db'\big)+\big(k+(1-d)c\lambda_0\big)b+c(d-1)\big(b'+\lambda_0 b\big)\\ &\geq \frac{c}{d}\big(a'+db'\big)+\frac{c(1-d)}{d}\big(a'+db'\big),\end{aligned}$$

using the result of the first statement. Again, this shows $a'+db'>0$ on $(-\infty,\infty)$. Finally, let $\bar a = a'$, $\bar b = b'$, an equivalent dynamical system to (1.4) is

$$\begin{cases} a' = \bar a,\\ \bar a' = -c\bar a + ab^m,\\ b' = \bar b,\\ \bar b' = -\frac{c}{d}\bar b - \frac{1}{d}ab^m + \frac{k}{d}b. \end{cases}$$

It is easy to verify that the linearized system at $(a_0,0,0,0)$ with $\alpha\sim a-a_0$, $\zeta\sim\bar a$, $\beta\sim b$, $\eta\sim\bar b$ is

$$\begin{cases} \alpha' = \zeta,\\ \zeta' = -c\zeta,\\ \beta' = \eta,\\ \eta' = -\frac{c}{d}\beta' + \frac{k}{d}\beta. \end{cases}$$

Computation shows that the four eigenvalues of the matrix A associated with the above system and corresponding eigenvectors are

$$\begin{aligned} &\lambda_1 = 0, \quad \mathbf{e_1} = [1,0,0,0]^T; \qquad \lambda_2 = -c, \quad \mathbf{e_2} = [1,-c,0,0]^T;\\ &\lambda_3 = \frac{-c+\sqrt{c^2+4kd}}{2d}, \quad \mathbf{e_3} = [0,0,1,\lambda_3]^T;\\ &\lambda_4 = \frac{-c-\sqrt{c^2+4kd}}{2d}, \quad \mathbf{e_4} = [0,0,1,\lambda_4]^T. \end{aligned}$$

Therefore, the last statement is true. □

Remark. An interesting fact which follows directly from the proposition is that any positive solution (a,b) of (2.8) coming out of $x=-\infty$ must go along the direction of $\mathbf{e_3}$ with asymptotic behavior of

$$a(z)-a_0 = o\big(e^{\lambda_3 z}\big), \qquad b(z) = O\big(e^{\lambda_3 z}\big).$$

This is because the unique solution along the direction of $\mathbf{e_1}$ must be $(a(z),b(z))\equiv(a_0,0)$, $\forall z$. Therefore, there is a unique positive solution (a,b) of (2.8) which exists on $(-\infty,Z)$ for any given (a_0,c,k), where Z is either finite or ∞. We shall give a complete answer to the case of $d=0$ in next section, and also derive the asymptotic behavior of $(a(z),b(z))$ as $z\to\infty$.

3. The case of $d = 0$

We study in more details the case of $d = 0$ in this section. This is because it is the only case we know that b is bell-shaped and also because (a, b) is always positive solution on $(-\infty, \infty)$ with interesting asymptotic behavior at $z = \infty$. In particular, it has very different behavior from the case of $m = n$, or with the autocatalytic reaction system without decay. For simplicity, we shall treat only the case of $n = 1$.

The main result of this section is stated as follows.

Theorem 2. *Suppose $d = 0$. Then, any positive solution (a, b) of (2.8) which exists on some interval $(-\infty, Z)$ is positive on $(-\infty, \infty)$. Furthermore, $b(z)$ is bell-shaped which tends to zero as $z \to \infty$, and $a(z)$ is strictly increasing and tends to ∞ as $z \to \infty$. Moreover, the exact asymptotic behavior of (a, b) is given by:*

$$a(z)z^{-(m-1)/m} \to A_0, \qquad b(z)z^{1/m} \to B_0, \quad \text{as } z \to \infty,$$

where

$$B_0 = \left(\frac{c(m-1)}{m}\right)^{1/m}, \qquad A_0 = kB_0^{-(m-1)}.$$

Remark. The detailed study of asymptotic behavior of (a, b) for $d = 0$ case proves to be important for the study of other cases. First, the asymptotic behavior is universal no matter what the value of d is, under the conditions that (a, b) are positive on $(-\infty, \infty)$ and $a(z) \to \infty$ as $z \to \infty$. Second, it plays an important role in establishing the existence of traveling wave in next section.

We shall prove the theorem through a series of lemmas.

Lemma 1. *Suppose $d = 0$. Then, any positive solution (a, b) of (2.8) which exists on some interval $(-\infty, Z)$, is positive on $(-\infty, \infty)$. Moreover, b has the property that b increases up to a point z_0 at which $kb(z) = a(z)b^m(z)$ and then it decreases for $z > z_0$ to zero as $z \to \infty$. Moreover, $a(z) \to \infty$ as $z \to \infty$ and both $\int_{-\infty}^{\infty} b^m(s)a(s)\,ds$ and $\int_{-\infty}^{\infty} b(s)\,ds$ are divergent.*

Proof. It is clear that $b' > 0$ as long as $kb(z) > a(z)b^m(z)$. If $kb(z) > a(z)b^m(z)$ holds for all z, then $a(z) \to \infty$ as $z \to \infty$ and $kb(z) > a(z)b^m(z)$ must be violated. Therefore, there will be a point where $kb(z) = a(z)b^m(z)$. At such a point, $b'(z) = 0$ but $b''(z) < 0$, and b starts to decrease immediately and b' can never change sign again. This rules out the possibility that $b(z) = 0$ at some finite point, since right before that b' must turn positive. Hence, $b(z) \searrow$ to a finite value L as $z \to \infty$. If $L > 0$, then again, $a(z) \to \infty$ as $z \to \infty$, which in turn yields $b'(z) < -L$ for all $z \gg 1$, a contradiction. Thus, $b(z) \searrow 0$ as $z \to \infty$. Consequently, $a(z) \to \infty$ as $z \to \infty$, because a finite limit would yield $b'(z) > 0$ for all $z \gg 1$.

If one integral is finite, then the other must be finite from the equation. But, that would imply $a(z) \nearrow$ to a finite limit as $z \to \infty$. We reach a contradiction. This completes the proof of the lemma. □

Lemma 2. *Suppose $d = 0$. Then, any positive solution (a, b) of (2.8) has the property that*

$$\frac{a'(z)}{a(z)} \to 0 \quad \text{as } z \to \infty.$$

Furthermore,

$$a(z)b^{m-1}(z) \to k \quad \text{as } z \to \infty.$$

Proof. Let $f(z) = a'(z)/a(z)$. Then,

$$f'(z) = \frac{a''(z)a(z) - (a'(z))^2}{a^2(z)} = \frac{-ca(z)a'(z) + a^2(z)b^m(z)}{a^2(z)} - f^2(z)$$
$$= -cf(z) - f^2(z) + b^m(z).$$

It is clear that $f(z) \to 0$ as $z \to \infty$ since $a(z) \to \infty$ and $b(z) \to 0$ as $z \to \infty$.

Let $g(z) = Lb(z) - a(z)b^m(z)$, with $L > k$ a constant.

$$g' = Lb' - a'b^m - mab^{m-1}b' = \frac{1}{c}\big(L - mab^{m-1}\big)\big(kb - ab^m\big) - a'b^m.$$

At the point where $g = 0$, $ab^{m-1} = L$,

$$g' = \frac{1-m}{c}ab^m(k - L) - a'b^m = b^m\left(\frac{m-1}{c}a(L-k) - a'\right) > 0$$

if $z \gg 1$. Therefore, either $g(z) > 0$, $\forall z \gg 1$ or $g(z) < 0$, $\forall z \gg 1$. If $g(z) < 0$, $\forall z \gg 1$, then

$$cb' = kb - ab^m < -(L-k)b$$

and an integration would yield $b(z)$ decay exponentially as $z \to \infty$, which contradicts the fact that $\int^{\infty} b(s)\, ds = \infty$. Hence, $g(z) > 0$, $\forall z \gg 1$. Similarly, if $L < k$, $Lb(z) - a(z)b^m(z) < 0$, $\forall z \gg 1$. This shows $a(z)b^{m-1}(z) \to k$ as $z \to \infty$. $\square$

Lemma 3. *Suppose $d = 0$ and (a, b) is a positive solution of (2.8). Then, there exist positive constants $M_1, M_2 > 0$ such that*

$$M_1\big(1 + |z|\big)^{(m-1)/m} < a(z) < M_2\big(1 + |z|\big)^{(m-1)/m} \quad \text{on } (-\infty, \infty).$$

Consequently,

$$c_1\big(1 + |z|\big)^{-1/m} < b(z) < c_2\big(1 + |z|\big)^{-1/m} \quad \text{on } (-\infty, \infty)$$

for some positive constants c_1, c_2.

Proof. An integration of the first equation in (2.8) on (z_0, z) shows

$$a' + ca\,|_{z_0}^{z} = \int_{z_0}^{z} a(s)b^m(s)\,ds > \int_{z_0}^{z} \frac{3k}{4}b(s)\,ds > \frac{3k}{4}b(z)(z - z_0),$$

by Lemma 2, if z_0 is sufficiently large. Using the fact that $a'/a \to 0$ as $z \to \infty$, there holds

$$ca(z) > \frac{3}{5}kzb(z) > \frac{1}{2}k^{m/(m-1)}za^{-1/(m-1)}, \quad \forall z \gg 1,$$

which is the same as $a(z) > M_1(1 + |z|)^{(m-1)/m}$ on $(-\infty, \infty)$ for some positive constant M_1. On the other hand, define $h(z) = a'(z) - Lcb(z)$, $L > 0$ a constant.

$$\begin{aligned} h'(z) &= a''(z) - Lcb'(z) = -ca'(z) + a(z)b^m(z) - Lkb(z) + La(z)b^m(z) \\ &= -c\big(a'(z) - Lcb(z)\big) - Lc^2b(z) - Lkb(z) + (1 + L)a(z)b^m(z). \end{aligned}$$

Since $a(z)b^{m-1}(z) \to k$ as $z \to \infty$, it follows that

$$h'(z) = -ch(z) + \big(k - Lc^2\big)b(z) + h.o.t.$$

If $L > k/c^2$, an integration of the above equation on (z_0, z) yields

$$h(z)e^{cz} - h(z_0)e^{cz_0} < \frac{1}{2}\big(k - Lc^2\big) \int_{z_0}^{z} e^{cs}b(s)\,ds$$

if z_0 is sufficiently large. Then, $a'(z_1) - Lcb(z_1) < 0$ at some $z_1 \gg 1$, and hence $h(z) < 0$ for all $z > z_1$. This is because at any point $z > z_1$ with $h(z) = 0$ we must have $h'(z) < 0$, a contradiction. Thus,

$$a'(z) < Lcb(z) < Lc(k + 1)a^{-1/(m-1)}(z).$$

An integration then implies

$$a(z) < M(1 + z)^{(m-1)/m} \quad \forall z \gg 1,$$

where $M > 0$ is a constant. This proves the bounds of $a(z)$. The bounds of $b(z)$ are easy consequences of $a(z)b^{m-1}(z) \to k$ as $z \to \infty$, and those of a. $\square$

Lemma 4. *Let $d = 0$. The asymptotic behavior of $a(z)$ and $b(z)$ as $z \to \infty$ is given as follows:*

$$a(z)z^{-(m-1)/m} \to A_0, \qquad b(z)z^{1/m} \to B_0, \quad \text{as } z \to \infty,$$

where

$$B_0 = \left(\frac{c(m-1)}{m}\right)^{1/m}, \qquad A_0 = kB_0^{-(m-1)}.$$

Proof. It was shown in Lemma 3 that

$$a'(z) - Lcb(z) > 0, \quad \text{if } Lc^2 > k, \ \forall z \gg 1.$$

Similarly,

$$a'(z) - Lcb(z) < 0, \quad \text{if } Lc^2 < k, \ \forall z \gg 1.$$

Hence,

$$\frac{a'(z)}{b(z)} \to \frac{k}{c} \quad \text{as } z \to \infty.$$

We use this fact to prove $(ab^m)'(z) < 0 \ \forall z \gg 1$. Let $f(z) = a(z)b^m(z)$.

$$f'(z) = a'(z)b^m(z) + ma(z)b^{m-1}(z)b'(z),$$
$$f'' = a''b^m + 2mb^{m-1}a'b' + m(m-1)ab^{m-2}(b')^2 + mab^{m-1}b''$$
$$f'' + cf' = b^m f + mb^{m-1}b'(2a' + ca) + m(m-1)ab^{m-2}(b')^2 + \frac{m}{c}ab^{m-1}(kb' - f').$$

Thus,

$$f'' + \left(c + \frac{m}{c}ab^{m-1}\right)f' = b^m f + mb^{m-1}b'\left(2a' + \left(c + \frac{k}{c}\right)a\right) + m(m-1)ab^{m-2}(b')^2.$$

If $f'(z_1) = 0$, then $a'b = -mab'$ at the point, and

$$f'' = b^m\left(ab^m - \left(c + \frac{k}{c}\right)a'\right) + mb^{m-2}b'\left(2a'b - \frac{m-1}{m}a'b\right).$$

Since

$$a(z)b^m(z) \sim kb(z), \qquad a'(z) \sim \frac{k}{c}b(z), \quad \forall z \gg 1,$$

at $z = z_1$,

$$a(z)b^m(z) - \left(c + \frac{k}{c}\right)a'(z) \sim kb(z) - kb(z) - \left(\frac{k}{c}\right)^2 b(z) < 0$$

if z_1 is sufficiently large. Consequently, $f''(z_1) < 0$. Thus, $(ab^m)'(z) < 0$ and it follows that $a''(z) < 0$, $\forall z \gg 1$, because $(ab^m)(z) \to 0$ as $z \to \infty$.

We can now derive the exact asymptotic behavior of $a(z)$ and $b(z)$ as $z \to \infty$. Since

$$ca'(z) \sim kb(z), \qquad a(z)b^{m-1}(z) \sim k \quad \text{as } z \to \infty,$$

$$ca'(z) \sim k\left(\frac{k}{a(z)}\right)^{1/(m-1)}, \quad \text{that is } ca'(z)a^{1/(m-1)}(z) = k^{m/(m-1)} + o(1).$$

An integration on $[1, z]$ then yields

$$\frac{c(m-1)}{m}a^{m/(m-1)}(z) = k^{m/(m-1)}z + o(z).$$

This, and $a(z)b^{m-1}(z) \to k$ as $z \to \infty$, yields the result of lemma. □

Remark. It can be shown using the above argument that $a(z)b^l(z)$ is strictly increasing (decreasing) if $0 < l < m-1$ ($l > m-1$) for all $z \gg 1$.

Proof of Theorem 2. It follows directly from the above lemmas. □

4. Existence of traveling wave

In this section, we show the existence of traveling wave solutions for a particular combination of c, d and k.

Let $d > 1$, $l = (\sqrt{c^2 + 4kd} - c)/2$. It is easy to verify that

$$db'' + cb' - kb = db'' - lb' + \lambda_0\big(db' - lb\big).$$

If

$$(d-1)c^2 = k, \tag{4.9}$$

then $\lambda_0 = c$. This is the case we shall focus our attention on in this section. That is, we assume

$$d > 1, \qquad (d-1)c^2 = k$$

throughout this section. The main result in this section is the following existence result.

Theorem 3. *Suppose $m > 1$, $d > 1$ but close to 1 is the sense that $(8m-1)(d-1)^2 + 6(d-1)d < d^2$ and (4.9) is satisfied. Then, there exists a positive solution (a, b) of (2.8) with the property that $b(z) \to 0$ as $z \to \infty$ for any a_0 suitably large. Furthermore, they are traveling wave solutions provided $m \geq 2$ and $(m-2)(d-1) < 1$.*

Remark. The only requirement for a_0 is in Lemma 6 below and it is met if

$$a_0 > \left(\frac{d}{m-1}\right)^{(m-1)/m}\big((m\sigma + c)d\sigma\big)^{1/m},$$

where $\sigma = l/d$.

It is easy to check that under (4.9), the identity

$$a'' + ca' + db'' + cb' - kb = 0$$

can be written as

$$\big(e^{cz}a'(z) + \big(db'(z) - lb(z)\big)e^{cz}\big)' = 0.$$

An integration then gives

$$a'(z) + db'(z) - lb(z) = 0. \tag{4.10}$$

We now do a change of variables, taking advantage of the monotonicity of a and the 1st order equation (4.10) to replace the second equation in (2.8) to arrive at a 2nd order non-autonomous system.

Let $a = a(z)$ be the independent variable, $P(a) = a'(z)$, $Q(a) = b(z)$, then an equivalent system to (2.8), using its 1st equation and (4.10) is

$$\text{(III)} \qquad \begin{cases} PP' + cP = aQ^m, & \text{for } a > a_0, \\ PQ' = \sigma Q - \frac{1}{d}P, & \text{for } a > a_0, \\ P(a_0) = Q(a_0) = 0, \end{cases}$$

where $\sigma = l/d$. It is easy to prove there exist positive solutions (P, Q) with the property $P(a_0) = Q(a_0) = 0$ on $a \in (a_0, a_0 + \delta]$ with $\delta > 0$ small. Furthermore, computation shows, when $0 < a - a_0 \ll 1$,

$$P(a) = \eta_1(a - a_0) + \eta_2(a - a_0)^\alpha + h.o.t., \qquad Q(a) = \xi_1(a - a_0)^{1/m} + \xi_2(a - a_0) + h.o.t.$$

with

$$\alpha = \frac{2m-1}{m}, \qquad \eta_1 = m\sigma, \qquad \xi_1 = \left(\frac{m\sigma(c + m\sigma)}{a_0}\right)^{1/m},$$

$$\xi_2 = -\frac{m\sigma(c + (1+\alpha)m\sigma)}{d(m\sigma(c + m\sigma) + (m-1)\sigma(c + (1+\alpha)m\sigma))}, \qquad \eta_2 = m\frac{\xi_2}{\xi_1}\frac{m\sigma(c + m\sigma)}{c + (1+\alpha)m\sigma}.$$

In addition, P is positive before Q reaches zero.

Lemma 5. *Let $0 < \delta < a_0/(m\sigma + c)$. Suppose $(P(a), Q(a))$ are positive in (a_0, A), for some $A > a_0$, then $P(a) - \delta Q^m(a) > 0$ in the same interval. Moreover,*

$$Q^{m-1}(a) \le \frac{\sigma d}{a_0}(m\sigma + c) \quad \textit{on } [a_0, A].$$

Proof. It is clear that $P(a) > \delta Q^m(a)$ when a is close to a_0. We consider the quantity $P(a) - \delta Q^m(a)$. Using the system (III) we have

$$P(a)\big[P(a) - \delta Q^m(a)\big]' = -cP(a) - m\delta Q^{m-1}(a)\left(\sigma Q(a) - \frac{1}{d}P(a)\right) + aQ^m(a)$$
$$= \left(-c + \frac{m\delta}{d}Q^{m-1}(a)\right)\big(P(a) - \delta Q^m(a)\big)$$
$$+ Q^m(a)\left(a - (m\sigma + c)\delta + \frac{m\delta^2}{d}Q^{m-1}(a)\right).$$

Consequently, when δ is in the given range, $P(a) - \delta Q^m(a) > 0$.

Next, substitute $P(a) - \delta Q^m(a) > 0$ into the second equation of (III), we get

$$P(a)Q'(a) = \sigma Q(a) - \frac{1}{d}P(a) \le \sigma Q(a) - \frac{\delta}{d}Q^m(a) < 0$$

if $\delta Q^{m-1}(a) > d\sigma$. This implies

$$Q^{m-1}(a) \le \frac{d\sigma}{\delta} \quad \text{for any } \delta < \frac{a_0}{m\sigma + c}.$$

Taking

$$\delta \to \frac{a_0}{m\sigma + c},$$

we derive the desired inequality. This completes the proof of lemma. □

Lemma 6. *Let (P, Q) be as in Lemma 5. Then,*

$$P(a) - \varepsilon a Q^m(a) > 0 \quad \text{on } [a_0, A) \text{ if } \varepsilon < \frac{1}{m\sigma + c} \tag{4.11}$$

and $Q(a) \le ma/d$ on $[a_0, A)$. Consequently, assuming

$$a_0 > \big[\sigma d(m\sigma + c)\big]^{1/m}\left(\frac{d}{m-1}\right)^{m/(m-1)},$$

$$aQ^{m-1}(a) \le 2\sigma d(m\sigma + c).$$

Proof. The inequality (4.11) holds at $a = a_0$ by Lemma 5. If there exists $\bar{a}$ such that the inequality holds in $[a_0, \bar{a})$ but at $a = \bar{a}$, $P(a) - \varepsilon a Q^m(a) = 0$, it must be true that

$$\big[P(a) - \varepsilon a Q^m(a)\big]' \le 0$$

at this point. But, detailed calculation shows

$$
\begin{aligned}
I(a) &\equiv P(a)\big[P(a) - \varepsilon a Q^m(a)\big]' \\
&= -cP(a) - m\varepsilon a Q^{m-1}(a)\left(\sigma Q(a) - \frac{1}{d}P(a)\right) + aQ^m(a) - \varepsilon P(a)Q^m(a) \\
&= Q^m(a)\left(a - \varepsilon a(m\sigma + c) + \frac{m}{d}\varepsilon^2 a^2 Q^{m-1}(a) - \varepsilon^2 a Q^m(a)\right) > 0
\end{aligned}
$$

at $a = \bar{a}$. A contradiction. Therefore, (4.11) holds. The bound on $Q(a)$ is proved in what follows.

Consider $Q(a) - Ma^{-1/(m-1)}$, with $M = (2d\sigma(m\sigma + c))^{1/(m-1)}$. The function is negative if $0 < a - a_0 \ll 1$. Suppose there exists $\bar{a} > a_0$ such that the function is zero at $\bar{a}$. Then, $[Q(a) - Ma^{-1/(m-1)}]' \geq 0$ at $\bar{a}$. But,

$$
\begin{aligned}
P(a)\big[Q(a) - Ma^{-1/(m-1)}\big]' &= \sigma Q(a) - \frac{1}{d}P(a) + \frac{M}{m-1}a^{-m/(m-1)}P(a) \\
&= \frac{M}{m-1}a^{-m/(m-1)}P(a) + \sigma M a^{-1/(m-1)} - \frac{1}{d}P(a) < 0,
\end{aligned}
$$

since (4.11) yields $\sigma M a^{-1/(m-1)} < P(a)/2d$ and the assumption on a_0 and Lemma 5 imply

$$
\frac{M}{m-1}a^{-m/(m-1)} = \frac{M^{-(m-1)}}{m-1}Q^m(a) < P(a)/2d.
$$

We reach a contradiction! □

Next, we show that the resulting solution cannot be the one with the property that $Q(a) = 0$ at some point, while P stays positive.

Lemma 7. *Let (P, Q) be as in Lemma 6. Then, there exists $\delta > 0$ such that $Q(a) - \delta P(a) \geq 0$ on $[a_0, A]$, if $d - 1$ is close to zero in the sense that*

$$
(8m - 1)(d - 1)^2 + 6(d - 1)d < d^2. \tag{4.12}
$$

Proof. It is clear that the inequality holds at $a = a_0$ for any $\delta > 0$. Furthermore,

$$
\begin{aligned}
P[Q - \delta P]' &= \sigma Q - \frac{P}{d} - \delta\big(-cP + aQ^m\big) \\
&= \sigma(Q - \delta P) - \delta a Q^{m-1}(Q - \delta P) + P\left(\delta(\sigma + c) - \delta^2 a Q^{m-1} - \frac{1}{d}\right).
\end{aligned}
$$

Using the bound of Lemma 6 on Q, we derive

$$
\delta(\sigma + c) - \delta^2 a Q^{m-1} - \frac{1}{d} \geq \delta(\sigma + c) - \delta^2 2\sigma d(m\sigma + c) - \frac{1}{d} > 0
$$

for some $\delta > 0$ provided

$$
(\sigma + c)^2 > 8\sigma(m\sigma + c),
$$

which is exactly the condition as in (4.12). □

Now, we switch back to the original formulation (2.8). It is clear that under the assumptions of Lemma 6 and Lemma 7, (a, b) is a positive solution of (2.8) on $(-\infty, \infty)$. We derive further properties of such solutions before proving the main result.

Lemma 8. *Let (a, b) be a positive solution of (2.8) with the property that*

$$M_1 a'(z) < b(z) < M_2 \left(\frac{a'(z)}{a(z)}\right)^{1/m} \quad on\ (-\infty, \infty),$$

where M_1, M_2 are positive constants, then $b(z) \to 0$ as $z \to \infty$.

Proof. First, it is impossible that $\liminf_{z\to\infty} b(z) > 0$. For, otherwise, we would have $\lim_{z\to\infty} a(z) = \infty$, and

$$a'(z) = e^{-cz} \int_{-\infty}^{z} e^{cs} a(s) b^m(s)\, ds \to \infty \quad \text{as } z \to \infty.$$

This is clearly absurd. If there exists no limit of $b(z)$ as $z \to \infty$, then there exist sequences $\{x_n\}$ and $\{y_n\}$ with the property that $x_n, y_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} b(x_n) = b_0 > 0, \qquad \lim_{n\to\infty} b(y_n) = 0,$$

where b_0 is a positive constant. But, since $|b'(z)| < Mb(z)$ for some $M > 0$, we would have

$$\int_{-\infty}^{\infty} b^{m+1}(z)\, dz = \infty$$

as well as $a(z) \to \infty$ as $z \to \infty$. On the other hand, by the assumption of the lemma,

$$\int_{-\infty}^{z} b^{m+1}(s)\, ds < M \int_{-\infty}^{z} \frac{a'(s)}{a^{1+1/m}(s)}\, ds < \infty,$$

a clear contradiction. Hence, $b(z) \to 0$ as $z \to \infty$. $\square$

A direct consequence of the above lemma is the following result.

Corollary 2. *Let (a, b) as in Lemma 8, then*

$$\int_{-\infty}^{\infty} b^{m+\delta}(z)\, dz < \infty$$

for any $\delta > 0$.

Remark. We are now ready to prove Theorem 3. Due to the lengthy procedure, we give a sketch of the line of arguments. First, we suppose (a, b) is not a traveling wave solution. We then show it must have the asymptotic behavior as given in Theorem 2, which is the leading order expansion of a positive solution as $z \to \infty$. Second, we proceed to find higher order asymptotics of such a solution and reach a contradiction when some coefficients in the asymptotic expansion of it determined by system (2.8) proves to be inconsistent with (4.9).

Proof of Theorem 3. Suppose to the contrary that (a, b) is not a traveling wave solution. Then,

$$a(z) \to \infty, \quad \text{and} \quad \int_A^z a(s)b^m(s)\,ds \to \infty \quad \text{as } z \to \infty,\ \forall A > 0.$$

Since

$$db'(s) + cb(s)|_A^z = \int_A^z \big(kb(s) - a(s)b^m(s)\big)\,ds$$

and $db'(z) + cb(z) \to 0$ as $z \to \infty$, the integral on the right is convergent for any $A > 0$ fixed. That is,

$$\int_A^z b(s)\,ds \to \infty \quad \text{as } z \to \infty,\ \forall A > 0.$$

We proceed to show that $a(z)b^{m-1}(z) \to k$ as $z \to \infty$. If there exists $A > 0$ such that $a(z)b^{m-1}(z) \le k$, $\forall z \ge A$, then

$$db'(A) + cb(A) = \int_A^\infty \big(a(s)b^m(s) - kb(s)\big)\,ds \le 0 \quad \text{and} \quad db'(z) + cb(z) \le 0, \quad \forall x \ge A.$$

Hence, $b(z) \searrow 0$ exponentially as $z \to \infty$. This, in turn, would imply $a'(z) \searrow 0$ exponentially as $z \to \infty$, and $a(z) \nearrow a_1 > 0$, as $z \to \infty$. A clear contradiction to our assumption that (a, b) are not traveling wave solutions. Therefore, $\forall A > 0$, there exists $z > A$ such that $a(z)b^{m-1}(z) > k$.

If $a(z)b^{m-1}(z) \ge k$ for all $z \gg 1$, we easily deduce, using

$$\text{the convergence of } \int_A^\infty \big(kb(s) - a(s)b^m(s)\big)\,ds \text{ and the divergence of } \int_A^\infty b(s)\,ds$$

that

$$a(z)b^{m-1}(z) \to k \quad \text{as } z \to \infty.$$

Otherwise, there exist a sequence of local maxima $\{x_n\}$ and a sequence of local minima $\{y_n\}$ with $x_n > y_n$, $a(x_n)b^{m-1}(x_n) > k$, $a(y_n)b^{m-1}(y_n) < k$, $x_n, y_n \to \infty$ as $n \to \infty$. We show

$$a(z)b^{m-1}(z) \to k \quad \text{as } z \to \infty \tag{4.13}$$

by proving

$$\lim_{n\to\infty} a(x_n)b^{m-1}(x_n) = \lim_{n\to\infty} a(y_n)b^{m-1}(y_n) = k.$$

Next, consider $f(z) = a(z)b^l(z)$, with $l > 0$.

$$\begin{aligned}
f' &= a'b^l + lab'b^{l-1}, \\
f'' &= a''b^l + 2la'b'b^{l-1} + l(l-1)ab^{l-2}(b')^2 + lab^{l-1}b''. \\
f'' + cf' &= (a'' + ca')b^l + b'b^{l-1}(cla + 2la') + l(l-1)ab^{l-2}(b')^2 \\
&\quad + \frac{lab^{l-1}}{d}(-cb' + kb - ab^m) \\
&= ab^m\left(b^l - \frac{lab^{l-1}}{d}\right) + \frac{klab^l}{d} + lb'b^{l-1}\left(\frac{c(d-1)}{d}a + 2a'\right) \\
&\quad + l(l-1)ab^{l-2}(b')^2.
\end{aligned}$$

Let $l = m - 1$, then the equation becomes

$$f'' + cf' = -\frac{l}{d}f^2 + \frac{kl}{d}f + fb^{l+1} + alb'b^{l-2}\left(\frac{c(d-1)}{d}b + (l-1)b' + 2\frac{a'}{a}b\right). \tag{4.14}$$

Define

$$E(x) = \frac{1}{2}(f')^2 - \frac{kl}{2d}f^2 + \frac{l}{3d}f^3.$$

$$\begin{aligned}
E'(z) &= \left(f'' - \frac{kl}{d}f + \frac{l}{d}f^2\right)f' \\
&= -c(f')^2 + f'fb^{l+1} + alb'b^{l-2}\left(\frac{c(d-1)}{d}b + (l-1)b' + 2\frac{a'}{a}b\right)f' \\
&= (f')^2\left(-\frac{c}{d} + \frac{(m-2)b'}{b}\right) - \frac{c(d-1)}{d}a'b^lf' - (3l-1)a'b'b^{l-1}f' + f'fb^{l+1} \\
&\le (f')^2\left(-\frac{c}{d} + |m-2|\sigma\right) - \frac{c(d-1)}{d}a'b^lf' - (3l-1)a'b'b^{l-1}f' + f'fb^{l+1}.
\end{aligned}$$

It is clear by our assumption $|m-2|(d-1) < 1$, the bound $a' = O(b)$, Corollary 2 and Young's inequality that for any $\varepsilon > 0$,

$$E(z_2) - E(z_1) < \varepsilon$$

provided $z_2 > z_1 \gg 1$. Let the local maximum value at x_n be $k + \eta_2$ and the local minimum value at y_n be $k - \eta_1$, then $E(x_n) < E(y_n) + \varepsilon$ means

$$-\frac{kl}{2}\big((k+\eta_2)^2 - (k-\eta_1)^2\big) + \frac{l}{3}\big((k+\eta_2)^3 - (k-\eta_1)^3\big) < \varepsilon,$$

which can be simplified to

$$l(\eta_1+\eta_2)\left(\frac{k}{2}(\eta_2-\eta_1) + \frac{1}{3}\big(\eta_2^2+\eta_1^2-\eta_1\eta_2\big)\right) < \varepsilon.$$

If $\eta_2 \ge \eta_1$, then we deduce

$$\frac{l}{12}(\eta_1+\eta_2)^2 < \varepsilon.$$

If $\eta_2 < \eta_1$ but $\eta_2 > \mu\eta_1$ with

$$\mu = \frac{6k}{3k + \sqrt{9k^2 + 24\eta_1 k}},$$

$$\frac{k}{2}(\eta_2-\eta_1) + \frac{l}{3}\big(\eta_2^2+\eta_1^2-\eta_1\eta_2\big) \ge \frac{1}{3}(1-\mu)\eta_1 = \frac{(\sqrt{9k^2+24\eta_1 k}-3k)^2}{72k}.$$

Again, we have

$$(\eta_1+\eta_2)^2 = O(\varepsilon).$$

Otherwise, $\eta_2 < \frac{6k}{3k+\sqrt{9k^2+24\eta_1 k}}\eta_1$. We note that the difference between η_1 and η_2 is bounded below by a constant multiple of η_1 and tends to zero only when η_1 tends to zero.

Now, if $z_1 < z_2$ are the locations of two local minimum values $k - \eta_1$ and $k - \eta_3$, we have

$$-\frac{kl}{2}\big((k-\eta_3)^2 - (k-\eta_1)^2\big) + \frac{l}{3}\big((k-\eta_3)^3 - (k-\eta_1)^3\big) < \varepsilon,$$

which is equivalent to

$$l(\eta_1-\eta_3)\left(-\frac{k}{2}(\eta_3+\eta_1) + \frac{1}{3}\big(\eta_3^2+\eta_1^2+\eta_1\eta_3\big)\right) < \varepsilon.$$

If $\eta_3 \le \eta_1$, it is fine. If $\eta_3 > \eta_1$, then

$$\begin{aligned}
&\frac{k}{2}(\eta_3+\eta_1) - \frac{1}{3}\big(\eta_3^2+\eta_1^2+\eta_1\eta_3\big)\\
&\quad = \frac{k}{2}(\eta_3+\eta_1) - \frac{1}{2}\big(\eta_3^2+\eta_1^2\big) + \frac{1}{6}(\eta_1-\eta_3)^2\\
&\quad \ge \frac{l}{6}(\eta_1-\eta_3)^2.
\end{aligned}$$

Hence,

$$\frac{1}{6}(\eta_1 - \eta_3)^3 < \varepsilon.$$

This implies $\eta_3 \leq \eta_1 + O(\varepsilon^{1/3})$. That is, the worst case for local minimum values is non-increasing, and for local maximum values it is comparatively fast reduction provided the local minimum values do not converge to k. This imbalance will cause non-convergence of the integral $\int_A^\infty (kb(s) - a(s)b^m(s))\, ds$ if local maximum values do not converge to k.

Therefore,

$$\lim_{n\to\infty} a(x_n)b^{m-1}(x_n) = \lim_{n\to\infty} a(y_n)b^{m-1}(y_n) = k.$$

We prove (4.13).

It can be proved that

$$\frac{b'(z)}{b(z)} \to 0 \quad \text{as } z \to \infty$$

using the equation

$$d\left(\frac{b'(z)}{b(z)}\right)' = -c\frac{b'(z)}{b(z)} - d\left(\frac{b'(z)}{b(z)}\right)^2 + k - a(z)b^{m-1}(z).$$

In particular, at any point where

$$-\frac{b'(z)}{b(z)} = \delta_0 < \frac{c}{2d}, \qquad \frac{c\delta_0}{4} > k - ab^{m-1} \quad \text{on } (z, \infty),$$
$$d\left(\frac{b'(z)}{b(z)}\right)' > -\frac{c}{2}\left(\frac{b'(z)}{b(z)}\right) + k - a(z)b^{m-1}(z) > 0.$$

Hence,

$$\frac{b'(z)}{b(z)} \to 0 \quad \text{and} \quad \frac{b''(z)}{b(z)} \to 0 \quad \text{as } z \to \infty.$$

We can now proceed exactly as in Section 3 to have the precise asymptotic behavior of (a, b) as in Lemma 3.

To reach a contradiction, we need to do further expansion of (a, b) to get the coefficients of next two orders. Let $z > 0$ and

$$a(z) = \big(A_0 + r(t)\big)z^{(m-1)/m}, \qquad b(z) = \big(B_0 + s(t)\big)z^{-1/m}, \quad t = \log z,$$

where A_0, B_0 are as described in Theorem 2, with $r(t),\ s(t) \to 0$ as $t \to \infty$.

$$a'(z) = \frac{m-1}{m}\big(A_0 + r(t)\big)z^{-1/m} + \frac{dr}{dt}z^{-1/m},$$

$$a''(z) = -\frac{m-1}{m^2}\big(A_0 + r(t)\big)z^{-(1+m)/m} + \frac{(m-2)}{m}\frac{dr}{dt}\cdot z^{-(1+m)/m} + \frac{d^2r}{dt^2}\cdot z^{-(1+m)/m},$$

$$\begin{aligned}
a(z)b^m(z) &= z^{-1/m}\big(A_0 + r(t)\big)\big(B_0 + s(t)\big)^m \\
&= A_0 B_0^m z^{-1/m}\left(1 + \frac{r(t)}{A_0}\right)\left(1 + m\frac{s(t)}{B_0} + m(m-1)\frac{s(t)^2}{2B_0^2}\right) + h.o.t. \\
&= A_0 B_0^m z^{-1/m} + \big(mks(t) + B_0^m r(t)\big)z^{-1/m} \\
&\quad + \left(mB_0^{m-1} r(t)s(t) + \frac{m(m-1)}{2}A_0 B_0^{m-2}s^2(t)\right)z^{-1/m} + h.o.t.
\end{aligned}$$

Then, the first equation in (2.8) becomes

$$\begin{aligned}
&\frac{d^2r}{dt^2} + \left(ce^t + \frac{m-2}{m}\right)\frac{dr}{dt} - mks(t)e^t - \frac{m-1}{m^2}\big(A_0 + r(t)\big) \\
&\quad - \left(mB_0^{m-1}r(t)s(t) + \frac{m(m-1)}{2}A_0 B_0^{m-2}s^2(t)\right)e^t + h.o.t. = 0.
\end{aligned} \tag{4.15}$$

Similarly,

$$b'(z) = -\frac{1}{m}\big(B_0 + s(t)\big)z^{-(1+m)/m} + \frac{ds}{dt}z^{-(1+m)/m},$$

$$b''(z) = \frac{1+m}{m^2}\big(B_0 + s(t)\big)z^{-(1+2m)/m} - \frac{(2+m)}{m}\frac{ds}{dt}z^{-(1+2m)/m} + \frac{d^2s}{dt^2}z^{-(1+2m)/m}$$

and the second equation in (2.8) turns to

$$\begin{aligned}
&d\frac{d^2s}{dt^2} + \left(ce^t - d\frac{(m+2)}{m}\right)\frac{ds}{dt} - ks(t)e^{2t} - \frac{c}{m}\big(B_0 + s(t)\big)e^t + \big(mks(t) + B_0^m r(t)\big)e^{2t} \\
&\quad + \left(mB_0^{m-1}r(t)s(t) + \frac{m(m-1)}{2}A_0 B_0^{m-2}s^2(t)\right)e^{2t} \\
&\quad + \frac{d(m+1)}{m^2}\big(B_0 + s(t)\big) + h.o.t. = 0.
\end{aligned} \tag{4.16}$$

Suppose

$$r(t) = e^{-t}\bar{r}(t) + e^{-2t}\underline{r}(t) + h.o.t., \quad \bar{r}(t) = \sum_{i=0}^{N} r_i t^i,\ \underline{r}(t) = \sum_{j=0}^{M} \underline{r}_j t^j,$$

$$s(t) = e^{-t}\bar{s}(t) + e^{-2t}\underline{s}(t) + h.o.t., \quad \bar{s}(t) = \sum_{i=0}^{N} s_i t^i,\ \underline{s}(t) = \sum_{j=0}^{M} \underline{s}_j t^j,$$

with r_i, s_i, $i = 1, \cdots, N$ and $\underline{r}_j, \underline{s}_j$, $j = 1, \cdots, M$ constants, then

$$\frac{dr(t)}{dt} = -e^{-t}\bar{r}(t) + e^{-t}\frac{d\bar{r}(t)}{dt} - 2e^{-2t}\underline{r}(t) + e^{-2t}\frac{d\underline{r}(t)}{dt} + h.o.t.$$
$$= e^{-t}\left(-\sum_{i=0}^{N} r_i t^i + \sum_{i=1}^{N} i r_i t^{i-1}\right) + e^{-2t}\left(\frac{d\underline{r}(t)}{dt} - 2\underline{r}(t)\right) + h.o.t.$$
$$= \left(-r_N t^N + \sum_{i=0}^{N-1}\big[(i+1)r_{i+1} - r_i\big]t^i\right)e^{-t} + e^{-2t}\left(\frac{d\underline{r}(t)}{dt} - 2\underline{r}(t)\right) + h.o.t.,$$
$$\frac{ds(t)}{dt} = e^{-t}\left(\frac{d\bar{s}(t)}{dt} - \bar{s}(t)\right) + e^{-2t}\left(\frac{d\underline{s}(t)}{dt} - 2\underline{s}(t)\right) + h.o.t.$$
$$= \left(-s_N t^N + \sum_{i=0}^{N-1}\big[(i+1)s_{i+1} - s_i\big]t^i\right)e^{-t} + e^{-2t}\left(\frac{d\underline{s}(t)}{dt} - 2\underline{s}(t)\right) + h.o.t.$$

From (4.15), for order $O(1)$, we have

$$e^t\left(c\frac{d\bar{r}(t)}{dt} - mk\bar{s}(t)\right) = \frac{m-1}{m^2}A_0,$$

which yields a linear system

$$\begin{aligned} cr_N &= -mks_N, \\ c\big((i+1)r_{i+1} - r_i\big) &= mks_i, \quad i = 1, \cdots, N-1, \\ c(r_1 - r_0) - mks_0 &= \frac{m-1}{m^2}A_0. \end{aligned} \tag{4.17}$$

Similarly, from (4.16), for order $O(e^t)$, we obtain

$$(m-1)k\bar{s}(t) + B_0^m \bar{r}(t) = \frac{c}{m}B_0,$$

which implies a linear system, since $B_0^m = c(m-1)/m$,

$$\begin{aligned} (m-1)ks_i + B_0^m r_i = 0 \quad &\Longleftrightarrow \quad mks_i + cr_i = 0, \quad i = 1, \cdots, N, \\ mks_0 + cr_0 &= \frac{c}{m-1}B_0. \end{aligned} \tag{4.18}$$

Together, (4.17) and (4.18) yield,

$$\begin{aligned} r_2 = \cdots = r_N = 0, &\qquad s_2 = \cdots = s_N = 0, \\ cr_1 = \frac{m-1}{m^2}A_0 + \frac{c}{m-1}B_0 &= \frac{cB_0}{m-1}\left(\frac{m-1}{m}(d-1) + 1\right) \\ s_1 = -\frac{cr_1}{mk} &= -\frac{cB_0}{km(m-1)}\left(\frac{m-1}{m}(d-1) + 1\right) \\ cr_0 + mks_0 &= \frac{c}{m-1}B_0. \end{aligned} \tag{4.19}$$

To get the coefficients of $\underline{r}_j, \underline{s}_j$, $j = 1, \cdots, M$, we look at the next order from the (4.15) and (4.16).

From (4.15), for order $O(e^{-t})$, we have

$$\begin{aligned}
&e^t\big(c\big(e^{-2t}\underline{r}(t)\big)' - mke^{-2t}\underline{s}(t)\big) \\
&= e^{-t}\left(\frac{(m-1)}{m^2}\bar{r}(t)\right) + e^{-t}\left(mB_0^{m-1}\bar{r}\cdot\bar{s} + \frac{m(m-1)}{2}A_0B_0^{m-2}\bar{s}^2\right) \\
&\quad - \left(\big(\bar{r}(t)e^{-t}\big)'' - \frac{m-2}{m}\big(\bar{r}(t)e^{-t}\big)'\right).
\end{aligned}$$

This implies

$$\begin{aligned}
c\big(\underline{r}' - 2\underline{r}\big) - mk\underline{s} &= \frac{(m-1)}{m^2}\bar{r} - \big(\bar{r}'' - 2\bar{r}' + \bar{r}\big) + \left(mB_0^{m-1}\bar{r}\cdot\bar{s} + \frac{m(m-1)}{2}A_0B_0^{m-2}\bar{s}^2\right) \\
&\quad - \frac{m-2}{m}\big(\bar{r}' - \bar{r}\big).
\end{aligned} \tag{4.20}$$

In a similar fashion, from (4.16), for order $O(1)$, we get

$$\begin{aligned}
(m-1)k\underline{s}(t) + B_0^m\underline{r}(t) &= \frac{c}{m}\bar{s} - c\big(\bar{s}' - \bar{s}\big) - \left(mB_0^{m-1}\bar{r}\cdot\bar{s} + \frac{m(m-1)}{2}A_0B_0^{m-2}\bar{s}^2\right) \\
&\quad + \frac{d(m+1)}{m^2}B_0.
\end{aligned} \tag{4.21}$$

It is clear that we can assume $M = 2$, and we shall work out $(\underline{r}_2, \underline{s}_2)$, $(\underline{r}_1, \underline{s}_1)$ and $(\underline{r}_0, \underline{s}_0)$ consecutively to reach a contradiction when we have the explicit expression of $(\underline{r}_0, \underline{s}_0)$. It is easy to check that

$$\begin{gathered}
\underline{r}(t) = \sum_{j=0}^{2} r_j t^j, \qquad \underline{s}(t) = \sum_{j=0}^{2} s_j t^j, \\
\underline{r}'(t) - 2\underline{r}(t) = -2\underline{r}_2 t^2 + (2\underline{r}_2 - 2\underline{r}_1)t + (\underline{r}_1 - 2\underline{r}_0), \\
\bar{r}''(t) - 2\bar{r}'(t) + \bar{r}(t) = r_1 t - 2r_1 + r_0, \qquad \bar{r}'(t) - \bar{r}(t) = -r_1 t + r_1 - r_0.
\end{gathered}$$

The linear system satisfied by $(\underline{r}_2, \underline{s}_2)$ is, as follows directly from (4.20) and (4.21),

$$\begin{aligned}
-2c\underline{r}_2 - mk\underline{s}_2 &= \left(mB_0^{m-1}r_1s_1 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_1^2\right), \\
(m-1)k\underline{s}_2 + B_0^m\underline{r}_2 &= -\left(mB_0^{m-1}r_1s_1 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_1^2\right).
\end{aligned}$$

Hence,

$$c\underline{r}_2 = \frac{1}{m-1}\left(mB_0^{m-1}r_1s_1 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_1^2\right),$$

$$mk\underline{s}_2 = -\frac{m+1}{m-1}\left(mB_0^{m-1}r_1s_1 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_1^2\right).$$

In the same fashion, the linear system satisfied by $(\underline{r}_1, \underline{s}_1)$ is,

$$2c\underline{r}_1 + mk\underline{s}_1 = 2c\underline{r}_2 + \frac{m+1}{m^2}r_1 - \big(mB_0^{m-1}(r_1s_0 + r_0s_1) + m(m-1)A_0B_0^{m-2}s_1s_0\big),$$
$$(m-1)k\underline{s}_1 + B_0^m\underline{r}_1 = \frac{1+m}{m}s_1 - \big(mB_0^{m-1}(r_1s_0 + r_0s_1) + m(m-1)A_0B_0^{m-2}s_1s_0\big),$$

which in turn yields,

$$\begin{aligned} c\underline{r}_1 &= 2c\underline{r}_2 + \frac{m+1}{m^2}r_1 - \frac{m+1}{m-1}s_1 \\ &\quad + \frac{1}{m-1}\big(mB_0^{m-1}(r_1s_0 + r_0s_1) + m(m-1)A_0B_0^{m-2}s_1s_0\big), \\ mk\underline{s}_1 &= -2c\underline{r}_2 - \frac{m+1}{m^2}r_1 + \frac{2(m+1)}{m-1}s_1 \\ &\quad - \frac{m+1}{m-1}\big(mB_0^{m-1}(r_1s_0 + r_0s_1) + m(m-1)A_0B_0^{m-2}s_1s_0\big). \end{aligned}$$

At last, the linear system of $(\underline{r}_0, \underline{s}_0)$ takes the form

$$\begin{aligned} 2c\underline{r}_0 + mk\underline{s}_0 &= c\underline{r}_1 + \frac{m+1}{m^2}r_0 - \frac{m+2}{m}r_1 - \left(mB_0^{m-1}r_0s_0 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_0^2\right), \\ c\underline{r}_0 + mk\underline{s}_0 &= \frac{c(m+1)}{m-1}s_0 - \frac{cm}{m-1}s_1 + \frac{d(m+1)}{m(m-1)}B_0 \\ &\quad - \frac{m}{m-1}\left(mB_0^{m-1}r_0s_0 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_0^2\right). \end{aligned}$$

The solutions are

$$\begin{aligned} c\underline{r}_0 &= c\underline{r}_1 + \frac{m+1}{m^2}r_0 - \frac{m+2}{m}r_1 - c\frac{(m+1)}{m-1}s_0 \\ &\quad + c\frac{m}{m-1}s_1 - \frac{d(m+1)}{m(m-1)}B_0 + \frac{1}{m-1}\left(mB_0^{m-1}r_0s_0 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_0^2\right), \\ mk\underline{s}_0 &= -c\underline{r}_1 - \frac{m+1}{m^2}r_0 + \frac{m+2}{m}r_1 + 2c\frac{(m+1)}{m-1}s_0 - 2c\frac{m}{m-1}s_1 \\ &\quad + \frac{2d(m+1)}{m(m-1)}B_0 - \frac{m+1}{m-1}\left(mB_0^{m-1}r_0s_0 + \frac{m(m-1)}{2}A_0B_0^{m-2}s_0^2\right). \end{aligned} \tag{4.22}$$

But, the relation (4.10) yields

$$-\frac{m-1}{m}\underline{r}_0 - (\underline{r}_1 - 2\underline{r}_0) + \frac{k}{c}\underline{s}_0 = d\left(s_1 - s_0 - \frac{1}{m}s_0\right),$$

or, equivalently,

$$(m+1)c\underline{r}_0 + mk\underline{s}_0 = mc\underline{r}_1 + dmcs_1 - d(1+m)cs_0.$$

This, combined with (4.22), gives

$$(d-1)mcs_1 - (d-1)c(1+m)s_0 + (m+2)r_1 - \frac{m+1}{m}r_0 + \frac{2d(m+1)}{m}B_0 = 0.$$

But, using the explicit values of (r_1, s_1) and the relation $cr_0 + mks_0 = cB_0/(m-1)$, one would derive $d = 0$. This is absurd. Therefore, (a, b) must be a traveling wave. This completes the proof of theorem. □

Remark. The above asymptotic expansion of solutions can be made rigorous using boot-strap type of argument. But, for simplicity, we shall not do it here.

Remark. It is to verify that if $1 < m < 2$, the above proof of existence of traveling wave is valid provided $(2-m) < c\delta/d$, where δ is as in Lemma 7.

Proof of Theorem 1. It is a direct consequence of non-existence results proved in Section 2 and Theorem 3. □

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