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A non-existence result for low energy sign-changing solutions of the Brezis–Nirenberg problem in dimensions 4, 5 and 6

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Abstract

We consider the Brezis–Niremberg problem:

$$(P_\varepsilon) \qquad \begin{cases} -\Delta u = |u|^{p-1}u + \varepsilon u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n = 4, 5, 6$, $p + 1 = \frac{2n}{n-2}$ is the critical Sobolev exponent and ε is a positive parameter. The main result of the paper generalizes the result of A. Iacopetti and F. Pacella [10]. Precisely we show that there are no low energy sign-changing solutions u_ε with $\max u_\varepsilon / \min u_\varepsilon \to 0$ or $-\infty$ as ε goes to zero.

Keywords: Blow-up analysis; Sign-changing solutions; Lack of compactness; Critical exponent

1. Introduction

In this paper, we study the following semi-linear elliptic problem:

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$$(P_\varepsilon) \qquad \begin{cases} -\Delta u = |u|^{p-1}u + \varepsilon u & \text{in } \Omega, \quad \text{(a)} \\ u = 0 & \text{on } \partial\Omega, \quad \text{(b)} \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$, $p+1 = \frac{2n}{n-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$ and ε is a real positive parameter.

The problem is known as "the Brezis–Nirenberg problem" because the first fundamental results about the existence of positive solutions were obtained by H. Brezis and L. Nirenberg in 1983. The authors explain in [7] that dimension plays a crucial role in the study of (P_ε). They proved that if $n \geq 4$ there exists a positive solution of (P_ε) for every $\varepsilon \in (0, \lambda_1(\Omega))$, $\lambda_1(\Omega)$ being the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions. While for $n = 3$, there are positive solutions only for $\varepsilon \in (\lambda^*, \lambda_1)$, where $\lambda^* := \lambda^*(\Omega)$ is a positive constant dependent on Ω.

Concerning the case of sign-changing solutions, the existence results hold for $n \geq 4$ both for $\varepsilon \in (0, \lambda_1(\Omega))$ and $\varepsilon > \lambda_1(\Omega)$ as shown in [1], [8] and [9]. Furthermore, in [13], the authors proved that, if Ω is symmetric and $n = 4, 5$, there exists a sign-changing solution whose positive part concentrates and blows-up at the center of symmetry of the domain, while the negative part vanishes, as $\varepsilon \to \lambda_1(\Omega)$. Note that the small dimensions $n = 4, 5, 6$ are specific to this problem. Indeed, Atkinson, Brezis and Peletier show in [2] that if Ω is a ball, then there exists $\widetilde{\lambda} := \widetilde{\lambda}(n)$ so there are no radial sign-changing solutions of (P_ε) for $\varepsilon \in (0, \widetilde{\lambda})$. While, in [11], the authors gave asymptotic profile of the positive and negative part of radial solution u_ε in dimensions $n = 3, 4, 5, 6$ as ε tends to some limit value.

However, for $n \geq 7$, Schechter and Zou have shown in [15] that in any bounded smooth domain, there is an infinity of sign-changing solutions for any $\varepsilon > 0$.

Concerning the low energy sign-changing solutions of (P_ε), a study has been carried out in [6] concerning the solutions u_ε satisfying

$$\frac{1}{c_1} \leq -\frac{\max u_\varepsilon}{\min u_\varepsilon} \leq c_1. \tag{2}$$

The authors were able to prove the axial symmetry results for the same kinds of solutions in a ball. Next, A. Iacopetti and G. Vaira built in [12] solutions in the form of:

$$u_\varepsilon = \delta_{a,\lambda_1} - \delta_{a,\lambda_2} + v_\varepsilon \text{ with } \lambda_1/\lambda_2 \to 0 \text{ or } +\infty, \tag{3}$$

where

$$\delta(x) := \delta_{a,\lambda}(x) = c_0 \frac{\lambda^{(n-2)/2}}{(1 + \lambda^2|x-a|^2)^{(n-2)/2}}, \qquad \lambda > 0, \quad a \in \mathbb{R}^n,$$

$c_0 := (n(n-2))^{\frac{n-2}{4}}$, describe all regular positive solutions of the Yamabe problem

$$-\Delta u = u^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n.$$

This result has been proved only for large dimensions $n \geq 7$. Note that the size $n \geq 7$ is optimal, since in [10], A. Iacopetti and F. Pacella showed that, in dimension $n = 4, 5, 6$ the sign-changing solutions of the form (3) do not exist in any bounded smooth domain.

In [10], the authors have imposed $a_1 = a_2$, this choice of points is compulsory for their argument based on the Pohazaev identity. In this paper, we have considered a general case of low energy sign-changing solutions whose the positive and the negative parts blow-up with different speeds. This kind of solutions u_ε have to satisfy

$$\|u_\varepsilon\|^2 := \int_\Omega |\nabla u_\varepsilon|^2 \to 2S_1^{n/2}, \quad \text{and} \tag{4}$$
$$\max u_\varepsilon / \min u_\varepsilon \to 0 \text{ or } -\infty, \quad \text{as } \varepsilon \to 0,$$

where S_1 is the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$, that is,

$$S_1 := \inf\left\{ \frac{\|u\|^2_{H_0^1(\Omega)}}{\|u\|^2_{L^{2n/(n-2)}(\Omega)}}, u \in H_0^1(\Omega), u \neq 0 \right\}.$$

Note that, according to [6], if there exists a sign-changing solutions u_ε of (P_ε) satisfying (4), then there exist two points $a_{i,\varepsilon}$'s, two reals $\lambda_{i,\varepsilon}$'s and a function v_ε such that

$$\begin{cases} u_\varepsilon := P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}} - P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}} + v_\varepsilon \text{ with} \\ \|v_\varepsilon\| \to 0,\ \lambda_{i,\varepsilon} d(a_{i,\varepsilon}, \partial\Omega) \to +\infty \text{ and } \langle P\delta_{a_{1,\varepsilon},\lambda_{1,\varepsilon}}, P\delta_{a_{2,\varepsilon},\lambda_{2,\varepsilon}}\rangle \to 0 \text{ as } \varepsilon \to 0. \end{cases} \tag{5}$$

Hence, Eqs. (4) and (5) are equivalent.

Observe that, by using the blow-up analysis, we can assume that one of the concentration points is the global extremum point of $|u_\varepsilon|$, for example $|u_\varepsilon(a_{2,\varepsilon})| = \max |u_\varepsilon|$. In addition, the corresponding speed $\lambda_{2,\varepsilon}$ can be taken equal to $\lambda_{2,\varepsilon} = (|u_\varepsilon(a_{2,\varepsilon})|/c_0)^{2/(n-2)}$. Moreover, according to [6], if (2) holds then we get $\lambda_{1,\varepsilon}/\lambda_{2,\varepsilon}$ is bounded below and above. Furthermore, an easy computation implies that, if $\lambda_{1,\varepsilon}/\lambda_{2,\varepsilon}$ is bounded then, there exists $b \in B(a_{1,\varepsilon}, 1/\lambda_{1,\varepsilon})$ such that $u_\varepsilon(b) \geq c\lambda_{1,\varepsilon}^{(n-2)/2}$ which gives that $\max u_\varepsilon \geq c\lambda_{1,\varepsilon}^{(n-2)/2}$, and therefore $-\max u_\varepsilon / \min u_\varepsilon$ is bounded.

Our argument is carried out by contradiction. It is based on the analysis of Euler functional gradient related to this problem (P_ε). The main difficulty of our proof comes from the v_ε. Following the ideas introduced by A. Bahri and Xu in [4], we managed to decompose this function v_ε into two parts $v_\varepsilon := v_1^\varepsilon + v_2^\varepsilon$. Some accurate estimates shown on these two functions, allowed us to improve the remaining of certain formulas. This improvement led to a contradiction justifying the non-existence of such a family of solutions.

To state precisely our result, we denote by $P\delta := P\delta_{a,\lambda}$ the projection of $\delta_{a,\lambda}$ onto $H_0^1(\Omega)$, i.e.

$$-\triangle P\delta = \delta^{\frac{n+2}{n-2}} \quad \text{in } \Omega, \qquad P\delta = 0 \quad \text{on } \partial\Omega.$$

We also denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part i.e., for $x \in \Omega$,

$$-\triangle G(x,.) = c_n \delta_x \quad \text{in } \Omega, \qquad G(x,.) = 0 \qquad \text{on } \partial\Omega,$$
$$H(x_1,x_2) = |x_1 - x_2|^{2-n} - G(x_1,x_2), \ \ \forall (x_1,x_2) \in \Omega^2.$$

To simplify the presentation, we denote by: $H_{ij} := H(a_i, a_j)$, by $G_{ij} := G(a_i, a_j)$ and $d_i := d(a_i, \partial\Omega)$.

Now, we state our results.

Theorem 1.1. *Let $n = 5$, there are no sign-changing solutions u_ε of (P_ε) in the form (5) such that $\lambda_{1,\varepsilon}/\lambda_{2,\varepsilon} \to 0$ or $+\infty$ as $\varepsilon \to 0$.*

For $n = 6$, Theorem 1.1 holds true if we assume that the concentration points satisfy $|a_{1,\varepsilon} - a_{2,\varepsilon}|$ is small. In the general case, we need a geometric assumption on the domain Ω.

Theorem 1.2. *Let $n = 6$, there are no sign-changing solutions of (P_ε) in the form (5) such that $(i)\quad \lambda_{1,\varepsilon}/\lambda_{2,\varepsilon} \to 0$ or $+\infty \quad$ and $\quad (ii) \quad |a_{1,\varepsilon} - a_{2,\varepsilon}| \to 0$.*

Furthermore, if Ω verifies the following hypothesis:

$$\textit{For each critical point } \overline{y} \textit{ of the function } x \mapsto R(x) := H(x,x) \textit{ and for each}$$
$$\textit{critical point } \overline{z} \textit{ of the function } x \mapsto G(\overline{y}, x) \textit{ we have: } G(\overline{y}, \overline{z}) \neq H(\overline{y}, \overline{y}),$$

then we can cancel the assumption (ii).

Concerning $n = 4$, we need to assume that the concentration points are in a compact set of Ω.

Theorem 1.3. *Let $n = 4$, there are no sign-changing solutions of (P_ε) in the form (5) such that $\lambda_{1,\varepsilon}/\lambda_{2,\varepsilon} \to 0$ or $+\infty$ and $d(a_{i,\varepsilon}, \partial\Omega) \ge c > 0$, $i = 1, 2$.*

The remainder of this paper is organized as follows. In Sections 2 and 3 we collect estimates of some necessary integrals following our work. The main result of Section 4 is to prove that if $n = 5, 6$, the concentration point $a_{1,\varepsilon}$ is not close to the boundary of Ω. However, Section 5 is devoted to studying the v_ε-part. We decompose it into two functions $v_{1,\varepsilon}$ and $v_{2,\varepsilon}$. We find a punctual estimate of $v_{1,\varepsilon}$, then we deduce the estimate of $\|v_{2,\varepsilon}\|$. In Section 6, we prove that if $n = 5, 6$, the concentration point $a_{2,\varepsilon}$ is not close to the boundary of Ω. Finally, Section 7 is devoted to the proofs of the main theorems.

Note, in all this paper, for sake of simplicity, we will omit the index ε of the points $a_{i,\varepsilon}$, $\lambda_{i,\varepsilon}$ and v_ε. Furthermore, without loss of the generality, we will assume that $\lambda_1/\lambda_2 \to 0$ (since if u_ε is a solution then $-u_\varepsilon$ is too).

2. Some a priori estimates

In this section, we collect estimates of some necessary integrals following our work. Most of these integrals have already been evaluated (of instance in [3], [4], [6] and [14]), but we need to improve some of them in order to get the adequate estimates to our situation.

Lemma 2.1. *[14, pp. 29–30] Let $n \ge 3$. We have the following estimates*

$$(a) \qquad \int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d)^n}\right), \qquad \textit{where } \; S = \int_{\mathbb{R}^n} \delta_{0,1}^{\frac{2n}{n-2}},$$

(b) $$\int_\Omega \delta_{a,\lambda}^{\frac{n+2}{n-2}}\theta_{a,\lambda} = O\left(\frac{1}{(\lambda d)^{n-2}}\right), \qquad \textit{where } \theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda},$$

(c) $$\int_\Omega P\delta_{a,\lambda}^{\frac{2n}{n-2}} = S + O\left(\frac{1}{(\lambda d)^{n-2}}\right).$$

Lemma 2.2. *Let $n \geq 3$ and let δ_{a_1,λ_1} and δ_{a_2,λ_2} be such that* $\max(\lambda_1/\lambda_2, \lambda_2/\lambda_1, \lambda_1\lambda_2|a_1 - a_2|^2)$ *is very large. The following estimates hold*

(a) $$\int_{\mathbb{R}^n} \delta_1^{\frac{n+2}{n-2}}\delta_2 = \int_{\mathbb{R}^n} \delta_2^{\frac{n+2}{n-2}}\delta_1 = c\varepsilon_{12} + O(\varepsilon_{12}^{\frac{n}{n-2}}), \textit{ where } \varepsilon_{12} := \left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} + \lambda_1\lambda_2|a_1 - a_2|^2\right)^{\frac{2-n}{2}}$$

(b) $$\int_{\mathbb{R}^n} (\delta_1\delta_2)^{\frac{n}{n-2}} = O\left(\varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1})\right), \quad (c) \int_\Omega \delta_1\delta_2 = O\left(\varepsilon_{12}\right), \quad (d) \int_\Omega \delta_1^\alpha\delta_2^\beta = O\left(\varepsilon_{12}^{min(\alpha,\beta)}\right),$$

for all positive real numbers α and β checking $\alpha \neq \beta$ and $\alpha + \beta = \frac{2n}{n-2}$.

(In this lemma, we do not require that $\lambda_1/\lambda_2 \to 0$. It is a general result under the hypothesis ε_{12} which is very small and λ_i very large.)

Proof. The affirmations (a) and (b) are extracted from [3, Estimate 1,2, p. 4]. As for (d), it is estimated in [3, Estimate 3, p. 4], but the presence of the term $\ln(\varepsilon_{12}^{-1})$ annoys us in some steps. Therefore, we have pushed the calculation to eliminate it. As for the (c), by using Hölder inequality and the affirmation (b), it is easily estimated, but in the presence of $\ln(\varepsilon_{12}^{-1})$.

(d) Without loss of the generality, we assume $\alpha < \beta$. Three cases may occur.

The first case: If $\max\left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}, \lambda_1\lambda_2|a_1 - a_2|^2\right) = \frac{\lambda_2}{\lambda_1}$. In this case, it is easy to check that $\varepsilon_{12} \geq c\,(\lambda_1/\lambda_2)^{(n-2)/2}$. Also, we have

$$I := \int_\Omega \delta_1^\alpha\delta_2^\beta \leq c\lambda_1^{\frac{n-2}{2}\alpha}\int_{\mathbb{R}^n} \frac{\lambda_2^{\frac{n-2}{2}\beta}}{(1 + \lambda_2^2|x - a_2|^2)^{\frac{n-2}{2}\beta}}dx \leq c\frac{\lambda_1^{\frac{n-2}{2}\alpha}}{\lambda_2^{n-\frac{n-2}{2}\beta}}\int_0^{+\infty} \frac{r^{n-1}}{(1 + r^2)^{\frac{n-2}{2}\beta}}dr.$$

Since $\alpha < \beta$, then $\beta > \frac{n}{n-2}$, which implies that $\frac{n-2}{2}\beta > \frac{n}{2}$. So the last integral is convergent.

Note that, $n - \frac{n-2}{2}\beta = n - \frac{n-2}{2}(\frac{2n}{n-2} - \alpha) = \frac{n-2}{2}\alpha$. Then, $I \leq c\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n-2}{2}\alpha} \leq c\varepsilon_{12}^\alpha$.

The second case: If $\max\left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}, \lambda_1\lambda_2|a_1 - a_2|^2\right) = \frac{\lambda_1}{\lambda_2}$. In this case we have $\varepsilon_{12} \geq c\,(\lambda_2/\lambda_1)^{(n-2)/2}$ and $\lambda_2 << \lambda_1$. We define the set

$$\Omega_1 := \left\{x \in \Omega : \lambda_2\varepsilon_{12}^{\frac{2}{n-2}}\lambda_1|x - a_1|^2 < \frac{1}{32}\right\}. \tag{6}$$

This set is introduced for the first time by A. Bahri in [3]. We split I in two parts.

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We notice that, if $x \in \Omega \setminus \Omega_1$, then $\lambda_1^2|x-a_1|^2 \geq c(\lambda_1/\lambda_2)\varepsilon_{12}^{-2/(n-2)}$. Consequently, we obtain:

$$\frac{1}{1+\lambda_1^2|x-a_1|^2} \leq c\frac{\lambda_2}{\lambda_1}\varepsilon_{12}^{2/(n-2)} \quad \text{and} \quad \delta_1(x) \leq c\left(\lambda_2\varepsilon_{12}^{2/(n-2)}\right)^{(n-2)/2}.$$

Hence, we get:

$$I_1 := \int_{\Omega\setminus\Omega_1} \delta_1^\alpha \delta_2^\beta \leq \lambda_2^{\frac{n-2}{2}\alpha}\varepsilon_{12}^\alpha \int_{\mathbb{R}^n} \delta_2^\beta \leq c\varepsilon_{12}^\alpha. \tag{7}$$

Concerning the second integral, we have:

$$\begin{aligned} I_2 &:= \int_{\Omega_1} \delta_1^\alpha \delta_2^\beta \\ &= \frac{\lambda_1^{\frac{n-2}{2}\alpha}\lambda_2^{\frac{n-2}{2}\beta}}{\lambda_1^n \gamma^{\frac{n-2}{2}\beta}} \int_{\tilde{\Omega}_1} \frac{1}{(1+|y|^2)^{\frac{n-2}{2}\alpha}} \frac{1}{\left(1+\frac{1}{\gamma}(\frac{\lambda_2}{\lambda_1})^2|y|^2 - 2(\frac{\lambda_2}{\lambda_1})^2\frac{1}{\gamma}\langle y, \lambda_1(a_2-a_1)\rangle\right)^{\frac{n-2}{2}\beta}}dy, \end{aligned}$$

where $\gamma := 1+\lambda_2^2|a_1-a_2|^2$ and

$$\tilde{\Omega}_1 = \left\{ y : |y|^2 < \frac{1}{32}\frac{\lambda_1}{\lambda_2}\varepsilon_{12}^{-\frac{2}{n-2}} \right\}. \tag{8}$$

By replacing γ by its value and by using the fact that $y \in \tilde{\Omega}_1$, we obtain:

$$\left| \frac{1}{\gamma}(\frac{\lambda_2}{\lambda_1})^2|y|^2 - 2(\frac{\lambda_2}{\lambda_1})^2\frac{1}{\gamma}\langle y, \lambda_1(a_2-a_1)\rangle \right| \leq \left(\frac{1}{32}+o(1)\right) + \frac{2}{\sqrt{32}}(1+o(1)) \leq \frac{1}{2}.$$

Hence,

$$I_2 \leq c\left(\frac{\lambda_2}{\lambda_1\gamma}\right)^{\frac{n-2}{2}\beta} \int_{\tilde{\Omega}_1} \frac{dy}{(1+|y|^2)^{\frac{n-2}{2}\alpha}} \leq c\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{n}{2}} \frac{1}{\gamma^{\frac{n-2}{2}\beta}}\varepsilon_{12}^{\alpha-\frac{n}{n-2}}.$$

We also notice that in this case we have: $\varepsilon_{12} \simeq \left(\lambda_1/\lambda_2 + \lambda_1\lambda_2|a_1-a_2|^2\right)^{(2-n)/2}$, ($f \simeq g$ means that $f/g \to 1$), then

$$I_2\varepsilon_{12}^{-\alpha} \leq c\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{n}{2}} \frac{1}{\gamma^{\frac{n-2}{2}\beta}}\varepsilon_{12}^{-\frac{n}{n-2}} \leq c\left(1+\lambda_2^2|a_1-a_2|^2\right)^{\frac{n-(n-2)\beta}{2}} \leq c, \tag{9}$$

since $\beta > n/(n-2)$, which implies that $\frac{n-(n-2)\beta}{2} < 0$.

(7) and (9) complete the proof in this case.

The third case: If $\max\left(\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}, \lambda_1\lambda_2|a_1-a_2|^2\right) = \lambda_1\lambda_2|a_1-a_2|^2$. The proof is the same as the proof of the second case by using the fact that $\varepsilon_{12} \simeq \left(\lambda_1\lambda_2|a_1-a_2|^2\right)^{(2-n)/2}$. This completes the proof in this case.

The proof of claim (c) is similar to the proof of claim (d). $\square$

Lemma 2.3. *We assume that ε_{12} is very small. For all $v \in H_0^1(\Omega)$, we have*

$$(a) \qquad \int_\Omega P\delta_1^{\frac{4}{n-2}} P\delta_2|v| = \begin{cases} O\left(\|v\|\varepsilon_{12}\right) & \text{if } n=4,5 \\ O\left(\|v\|\varepsilon_{12}\ln^{2/3}(\varepsilon_{12}^{-1})\right) & \text{if } n=6. \end{cases}$$

$$(b) \qquad \int_\Omega P\delta_1 P\delta_2^{\frac{6-n}{n-2}} v^2 = \begin{cases} O\left(\|v\|^2\varepsilon_{12}\ln^{1/2}(\varepsilon_{12}^{-1})\right) & \text{if } n=4 \\ O\left(\|v\|^2\varepsilon_{12}^{1/3}\right) & \text{if } n=5. \end{cases}$$

$$(c) \qquad \int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}\theta_{a,\lambda}|v| = \begin{cases} O\left(\frac{\|v\|}{(\lambda d)^{n-2}}\right) & \text{if } n=4,5 \\ O\left(\frac{\|v\|}{(\lambda d)^4}(\ln(\lambda d))^{2/3}\right) & \text{if } n=6, \end{cases}$$

where $d := d(a, \partial\Omega)$.

Proof. (a) and (b) follow from Lemma 2.2 by using Hölder's inequality. Concerning claim (c), we decompose the integral on the ball $B := B_{(a,d)}$ and on $\Omega \setminus B$.

On B, we use the fact that $\|\theta_{a,\lambda}\|_\infty \le c\lambda^{(2-n)/2}d^{(2-n)}$, then we apply the Hölder's inequality.

On $\Omega \setminus B$, we increase $\theta_{a,\lambda}$ by $\delta_{a,\lambda}$, then we apply the Hölder's inequality. $\square$

Lemma 2.4. *[14, p. 34] For $n = 4, 5, 6$ and for all $v \in \langle P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a^1}, \dots, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a^n}\rangle^\perp$ we have:*

$$\int_\Omega P\delta_{a,\lambda}^{\frac{n+2}{n-2}} v = \left(\frac{\|v\|}{(\lambda d)^{n-2}}(1 + \underbrace{(\ln(\lambda d))^{2/3}}_{\text{if } n=6})\right),$$

where $d := d(a, \partial\Omega)$.

Lemma 2.5. *For $n = 4, 5, 6$ and for all $v \in H_0^1(\Omega)$, we have:*

$$(a) \quad \int_\Omega \delta_{a,\lambda}^{\frac{2n}{n-2}-\beta}|v|^\beta = O\left(\|v\|^\beta\right) \quad \text{for all real } \beta \text{ verifying } 0 < \beta \le \frac{2n}{n-2},$$

$$(b) \quad \int_\Omega \delta_{a,\lambda}|v| = O\left(\frac{\|v\|}{\lambda^{\frac{n-2}{2}}}(1 + \underbrace{(\ln\lambda)^{2/3}}_{\text{if } n=6})\right).$$

Proof. The proof follows immediately from Hölder's inequality. $\square$

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Standard computation implies that

Lemma 2.6. *We have:*

$$\int_\Omega \delta_{a,\lambda}^2 = O\left(\frac{1}{\lambda^2}\ln\lambda\right) \quad \text{if } n=4 \quad , \quad \int_\Omega \delta_{a,\lambda}^2 = O\left(\frac{1}{\lambda^2}\right) \quad \text{if } n=5,6.$$

Lemma 2.7. [3, Proposition 3.1, p. 64] *Let $n \geq 3$. There exists $\beta_0 > 0$ such that for each $v \in F^\perp$ we have:*

$$\|v\|_{H_0^1}^2 - \frac{n+2}{n-2}\sum_{i=1}^{2}\int_\Omega P\delta_i^{4/n-2}v^2 \geq \beta_0\|v\|_{H_0^1}^2,$$

where $F := \langle P\delta_1, \lambda_1\frac{\partial P\delta_1}{\partial\lambda_1}, \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1^1}, ..., \frac{1}{\lambda_1}\frac{\partial P\delta_1}{\partial a_1^n}, P\delta_2, \lambda_2\frac{\partial P\delta_2}{\partial\lambda_2}, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^1}, ..., \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^n}\rangle$.

The proof of the following Lemmas 2.8–2.10 are inspired from [4] where the authors deal with the case of sphere S^3. Some additional difficulties emerge in our case. The first stems from the boundary of Ω. As for the second the authors in [4] assume that ε_{12} behaves as $\frac{1}{(\lambda_1\lambda_2|a_1-a_2|^2)^{1/2}}$, but in our case, this assumption is deleted. The third difficulty that we came across in the dimension $n=4,5,6$, where certain integrals present a good behavior for $n=3$, but this behavior is not adequate for the dimensions $n=4,5,6$.

Lemma 2.8. *For $n=4,5,6$, and for all $y \in \Omega$ we have:*

$$\int_\Omega \delta_{a,\lambda}(x)G(x,y)dx \leq c\delta_{a,\lambda}(y).$$

Proof. Let $y \in \Omega$. We split two cases.

The first case: if $\lambda|y-a| \geq C$, where C is a large constant. Let $\widetilde{c}$ be such that $0 < \widetilde{c} < 1$, and let $B := B_{(y,\widetilde{c}|y-a|)}$. Observe that, for $x \in B$, we deduce that:

$$\delta_{a,\lambda}(x) \leq c'\delta_{a,\lambda}(y) \quad \text{and} \quad \int_B \frac{dx}{|x-y|^{n-2}} \leq c|y-a|^2. \tag{10}$$

Hence,

$$\int_B \delta_{a,\lambda}(x)G(x,y)dx \leq c\delta_{a,\lambda}(y). \tag{11}$$

For $x \in \Omega\setminus B$, we have: $|x-y|^{2-n} \leq |y-a|^{2-n}$, which implies

$$\int_{\Omega\setminus B} \delta_{a,\lambda}(x)G(x,y)dx \leq \frac{c}{|y-a|^{n-2}}\int_{B_{(a,R)}}\delta(x)dx \leq \frac{c}{\lambda^{\frac{n-2}{2}}|y-a|^{n-2}} \leq c\delta(y), \tag{12}$$

since we have $\lambda|y-a| \geq C$, which implies that $\lambda^{n-2}|y-a|^{n-2} \geq c\left(1+\lambda^2|y-a|^2\right)^{(n-2)/2}$.

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(11) and (12) prove the lemma in this case.

The second case: if $\lambda|y-a| \le C$. In this case, it is easy to see that $\delta_{a,\lambda}(y) \ge c\lambda^{(n-2)/2}$ and therefore, for all x in Ω, we have $\delta_{a,\lambda}(x) \le c'\delta_{a,\lambda}(y)$. Hence:

$$I \le c\delta_{a,\lambda}(y) \int_\Omega \frac{1}{|x-y|^{n-2}} dx \le c\delta_{a,\lambda}(y).$$

Lemma 2.8 follows. □

Lemma 2.9. *Let $n = 4, 5, 6$. For all $y \in \Omega$ and for all $\alpha \in \left(\frac{n-4}{n-2}, 1\right)$, there exists a positive constant c_α such that*

$$\int_\Omega \delta_{a,\lambda}^{\frac{4}{n-2}}(x)\theta_{a,\lambda}(x)G(x,y)dx \le \frac{c_\alpha}{(\lambda d)^{(n-2)(1-\alpha)}}\delta_{a,\lambda}(y),$$

where $d := d(a, \partial\Omega)$.

Proof.

$$I := \int_\Omega \delta^{\frac{4}{n-2}}(x)\theta(x)G(x,y)dx \le \|\theta\|_\infty^{1-\alpha} \int_\Omega \delta^{\frac{4}{n-2}+\alpha}(x)G(x,y)dx. \tag{13}$$

We split two cases.

The first case: if $\lambda|y-a| \ge C$, where C is a large constant. Arguing exactly as the proof of the analogous case of the proof of Lemma 2.8, we derive that

$$I \le c\|\theta\|_\infty^{1-\alpha} \frac{1}{\lambda^{\frac{n-2}{2}(1-\alpha)}}\delta_{a,\lambda}(y) \le \frac{c}{(\lambda d)^{(n-2)(1-\alpha)}}\delta_{a,\lambda}(y).$$

The second case: if $\lambda|y-a| \le C$. Observe that

$$\begin{aligned} &\int_\Omega \delta^{\frac{4}{n-2}+\alpha}(x)G(x,y)dx \\ &\quad \le c\lambda^{\frac{n-2}{2}}\delta(y) \int_{\lambda|x-y|\ge C} \delta^{\frac{4}{n-2}+\alpha}(x)dx + c\delta^{\frac{4}{n-2}+\alpha}(y) \int_{\lambda|x-y|\le C} \frac{1}{|x-y|^{n-2}}dx, \end{aligned}$$

where we have used the fact that $|x-y|^{2-n} \le c\lambda^{(n-2)/2}\delta(y)$ if $\lambda|x-y| \ge C$ and $\delta(x) \le c\delta(y)$ since $\lambda|y-a| \le C$. Thus the result follows by using standard computations and the estimate of $\|\theta\|_\infty$. □

Lemma 2.10. *For $n = 4, 5, 6$, and for all $y \in \Omega$, we have:*

$$\int_\Omega \delta_{a,\lambda}^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}}G(x,y)dx \le c\|v\|^{\frac{2}{n-2}}\delta_{a,\lambda}(y).$$

Proof. We split two cases.

The first case: if $\lambda|y-a| \geq C$, where C is a large constant. We divide the integral into two parts. For the first one, observe that for $|x-y| \leq 1/(2\lambda)$, we have $|x-y| \leq |y-a|/2$, and therefore $\delta(x) \leq c\delta(y)$. Hence

$$\int\limits_{|x-y|\leq\frac{1}{2\lambda}} \delta^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}} G(x,y)dx \leq c\delta^{\frac{n}{n-2}}(y) \int\limits_{|x-y|\leq\frac{1}{2\lambda}} \frac{|v(x)|^{\frac{2}{n-2}}}{|x-y|^{n-2}} dx \leq c\delta(y)\|v\|^{\frac{2}{n-2}}, \tag{14}$$

by using Hölder's inequality and the fact $\delta(y) \leq c_0\lambda^{(n-2)/2}$. Concerning the second integral, we have $|x-y| \geq 1/(2\lambda)$. In this region, we get

$$\frac{1}{|x-y|^{n-2}} \leq c\frac{\lambda^{n-2}}{\left(1+\lambda^2|x-y|^2\right)^{(n-2)/2}} \leq c\lambda^{(n-2)/2}\delta_{y,\lambda}(x),$$

which implies that

$$\int\limits_{|x-y|\geq\frac{1}{2\lambda}} \delta^{\frac{n}{n-2}}|v|^{\frac{2}{n-2}} G \leq c\lambda^{\frac{n-2}{2}} \int \delta_{a,\lambda}^{\frac{n}{n-2}} \delta_{y,\lambda}|v|^{\frac{2}{n-2}} \leq c\lambda^{\frac{n-2}{2}} \|v\|^{\frac{2}{n-2}} \left(\int \delta_{a,\lambda}^{\frac{n}{n-2}\frac{n}{n-1}} \delta_{y,\lambda}^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}}.$$

Note that $\lambda|y-a|$ is very large, so the hypotheses of Lemma 2.2 are satisfied. Hence Lemma 2.2 implies that:

$$\int\limits_{|x-y|\geq\frac{1}{2\lambda}} \delta^{\frac{n}{n-2}}|v|^{\frac{2}{n-2}} G \leq c\lambda^{\frac{n-2}{2}} \|v\|^{\frac{2}{n-2}} \widetilde{\varepsilon}_{12},$$

where $\widetilde{\varepsilon}_{12} := \left(1+1+\lambda^2|y-a|^2\right)^{(2-n)/2} \leq c\left(1+\lambda^2|y-a|^2\right)^{(2-n)/2}$. Hence

$$\int\limits_{|x-y|\geq\frac{1}{2\lambda}} \delta^{\frac{n}{n-2}}|v|^{\frac{2}{n-2}} G \leq c\delta(y)\|v\|^{\frac{2}{n-2}}. \tag{15}$$

(14) and (15) prove the lemma in this case.

The second case: if $\lambda|y-a| \leq C$. Note that, in this case we have $\delta_{a,\lambda} \geq c\lambda^{(n-2)/2}$. We divide the integral into two parts. For the first one, we have

$$\int\limits_{|x-a|\leq\frac{2C}{\lambda}} \delta^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}} G(x,y)dx \leq c\lambda^{n/2} \int\limits_{|x-y|\leq\frac{3C}{\lambda}} \frac{|v(x)|^{\frac{2}{n-2}}}{|x-y|^{n-2}} dx \leq c\delta(y)\|v\|^{\frac{2}{n-2}}, \tag{16}$$

by using Hölder's inequality. Concerning the second integral, we have $|x-a| \geq 2C/\lambda$ which implies that $|x-y| \geq |x-a|/2$. Hence

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$$\int_{|x-a|\geq \frac{2C}{\lambda}} \delta^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}} G(x,y)dx \leq c \int_{|x-a|\geq \frac{2C}{\lambda}} \frac{\delta^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}}}{|x-a|^{n-2}} dx.$$

Since $\lambda|x-a| \geq 2C$, then $|x-a|^{2-n} \leq c\lambda^{(n-2)/2}\delta(x)$. Now, by using Hölder's inequality, we obtain

$$\int_{|x-a|\geq \frac{2C}{\lambda}} \delta^{\frac{n}{n-2}}(x)|v(x)|^{\frac{2}{n-2}} G(x,y)dx \leq c\lambda^{\frac{n-2}{2}}\|v\|^{\frac{2}{n-2}} \leq c\delta(y)\|v\|^{\frac{2}{n-2}}. \tag{17}$$

(16) and (17) prove the lemma in this case. □

Lemma 2.11. *[5, Lemmas A5 and A6, pp. 571–572] For $n = 4, 5, 6$, we have:*

(a) $$\int_\Omega P\delta_i^{\frac{n+2}{n-2}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = 2\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle + O\left(\frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^n}\right),$$

(b) $$\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\rangle = \frac{n-2}{2}\widetilde{c}\frac{H_{ii}}{\lambda_i^{n-2}} + O\left(\frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^n}\right),$$

where $d := d(a, \partial\Omega)$ and $\widetilde{c} := c_0^{2n/n-2} \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n+2}{2}}}$.

The following three lemmas correspond to the lemmas A7–A9 of [5]. The rest of the terms presented in [5] are inadequate in our case. We do not like the term $(\lambda_1 d_1)$ to appear by itself in the rest, where $d_i := d(a_i, \partial\Omega)$. Therefore, we repeat the proofs to present the rest in another way.

Lemma 2.12. *We have:*

$$\int_\Omega P\delta_i^{\frac{n+2}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} = \langle P\delta_i, \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}\rangle + \begin{cases} O\left(\frac{1}{(\lambda_i d_i)^{(n-2)/2}}\varepsilon_{12}\right) & \text{if } n = 4, 5, \\ O\left(\frac{1}{\lambda_i d_i}\varepsilon_{12}\right) & \text{if } n = 6. \end{cases}$$

Proof. Note that $P\delta_i = \delta_i - \theta_i$ and $\theta_i \leq \delta_i$, then we have

$$\int_\Omega P\delta_i^{\frac{n+2}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} = \int_\Omega \delta_i^{\frac{n+2}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} + O\left(\int_\Omega \delta_i^{\frac{4}{n-2}} \theta_i \delta_j\right).$$

For $n = 4, 5$, it stems from the Hölder's inequality that:

$$\left(\int_\Omega \delta_i^{\frac{4}{n-2}} \theta_i \delta_j\right) \leq \|\theta_i\|_{L^{\frac{2n}{n-2}}} \left(\int_\Omega \delta_i^{\frac{4}{n-2}\frac{2n}{n+2}} \delta_j^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \leq \frac{c}{(\lambda_i d_i)^{\frac{n-2}{2}}}\varepsilon_{12},$$

by using Lemma 2.2 and the fact that $\|\theta_i\|_{L^{\frac{2n}{n-2}}} \le c(\lambda_i d_i)^{(2-n)/2}$. This proves the lemma in this case.

For $n = 6$, it stems from Lemma 2.2 that

$$\left(\int_\Omega \delta_i \theta_i \delta_j\right) \le \left(\int_\Omega \theta_i^{\frac{1}{2}} \delta_i^{\frac{3}{2}} \delta_j\right) \le \|\theta_i\|_{L^3}^{\frac{1}{2}} \left(\int_\Omega \delta_i^{\frac{9}{5}} \delta_j^{\frac{6}{5}}\right)^{\frac{5}{6}} \le \frac{c}{\lambda_i d_i} \varepsilon_{12}.$$

Hence the lemma is proved. □

Lemma 2.13. *For $n = 4, 5, 6$, and $i \ne j$, we have:*

$$\int_\Omega P\delta_i P\delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} = \frac{n-2}{n+2} \langle P\delta_i, \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}\rangle + O\left(\varepsilon_{12}^{n/n-2} \ln(\varepsilon_{12}^{-1}) + \frac{\ln(\lambda_j d_j)}{(\lambda_j d_j)^n}\right).$$

Proof. Note that $P\delta_i \le \delta_i$ and $|\lambda_j \partial\theta_j/\partial\lambda_j| \le c\delta_j$, then

$$\begin{aligned} I := \int_\Omega P\delta_i P\delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} &= \int_\Omega P\delta_i \left(\delta_j - \theta_j\right)^{\frac{4}{n-2}} \lambda_j \left(\frac{\partial \delta_j}{\partial \lambda_j} - \frac{\partial \theta_j}{\partial \lambda_j}\right) \\ &= \int_\Omega P\delta_i \delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial \delta_j}{\partial \lambda_j} - \int_\Omega P\delta_i \delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial \theta_j}{\partial \lambda_j} + O\left(\int_\Omega \delta_i \delta_j^{\frac{4}{n-2}} \theta_j\right). \end{aligned} \tag{18}$$

For the first integral of (18), observe that we have:

$$\int_\Omega \lambda_j \frac{\partial(\delta_j^{\frac{n+2}{n-2}})}{\partial \lambda_j} P\delta_i = \int_\Omega \lambda_j \frac{\partial(-\Delta P\delta_j)}{\partial \lambda_j} P\delta_i = \int_\Omega \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} (-\Delta P\delta_i) = \langle P\delta_i, \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}\rangle.$$

Concerning the second integral of (18), for $n = 4, 5$, by Hölder's inequality and Lemma 2.2 we get

$$\left|\int_\Omega P\delta_i \delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right| \le \left(\int_\Omega \left|\lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right|^{\frac{n}{2}} \delta_j^{\frac{6-n}{n-2}\frac{n}{2}}\right)^{2/n} \varepsilon_{12} (\ln \varepsilon_{12}^{-1})^{\frac{n-2}{n}}. \tag{19}$$

We notice that $B_j := B_{(a_j, d_j)}$, we have:

$$\int_\Omega \left|\lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right|^{\frac{n}{2}} \delta_j^{\frac{6-n}{n-2}\frac{n}{2}} \le \left\|\lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right\|_\infty^{\frac{n}{2}} \int_{B_j} \delta_j^{\frac{6-n}{n-2}\frac{n}{2}} + \int_{\Omega \setminus B_j} \delta_j^{\frac{2n}{n-2}} \le c \frac{\ln(\lambda_j d_j)}{(\lambda_j d_j)^n}. \tag{20}$$

(19)–(20) imply that:

$$\left|\int_\Omega P\delta_i \delta_j^{\frac{4}{n-2}} \lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right| \le c\frac{\left(\ln(\lambda_j d_j)\right)^{2/n}}{(\lambda_j d_j)^2}\varepsilon_{12}(\ln \varepsilon_{12}^{-1})^{\frac{n-2}{n}} \le c\frac{\ln(\lambda_j d_j)}{(\lambda_j d_j)^n} + c\varepsilon_{12}^{\frac{n}{n-2}} \ln \varepsilon_{12}^{-1}.$$

For $n = 6$, by Hölder's inequality and Lemma 2.2 we get:

$$\left|\int_\Omega P\delta_i \delta_j \lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right| \le \left\|\lambda_j \frac{\partial \theta_j}{\partial \lambda_j}\right\|_{L^3} \left(\int_\Omega (\delta_i\delta_j)^{\frac{3}{2}}\right)^{\frac{2}{3}} \le c\frac{\varepsilon_{12}(\ln \varepsilon_{12}^{-1})^{\frac{2}{3}}}{(\lambda_j d_j)^2} \le \frac{c}{(\lambda_j d_j)^n} + c\varepsilon_{12}^{\frac{n}{n-2}} \ln \varepsilon_{12}^{-1}.$$

This completes the estimate of the second integral of (18).

The third integral of (18) is dealt with in the same way as the previous one. Thus the proof of Lemma 2.13 follows. □

Lemma 2.14. *For $n = 4, 5, 6$, if $d_1 \nrightarrow 0$, then we have:*

$$\langle P\delta_1, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2}\rangle = \widetilde{c}\left(\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) + O\left(\varepsilon_{12}^{\frac{n}{n-2}} \ln(\varepsilon_{12}^{-1}) + \frac{\ln(\lambda_2 d_2)}{(\lambda_2 d_2)^2}\varepsilon_{12} + \frac{\varepsilon_{12}}{\lambda_1^2}\right).$$

Proof. Recall that $P\delta_1 = \delta_1 - \theta_1$, then there holds

$$\begin{aligned} I := \langle P\delta_1, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2}\rangle &= \int_{\mathbb{R}^n} \delta_1 \lambda_2 \frac{\partial}{\partial \lambda_2}\left(\delta_2^{\frac{n+2}{n-2}}\right) - \frac{n+2}{n-2}\int_{\mathbb{R}^n\setminus\Omega} \delta_1 \delta_2^{\frac{4}{n-2}} \lambda_2 \frac{\partial \delta_2}{\partial \lambda_2} \\ &\quad - \frac{n+2}{n-2}\int_{\Omega\setminus B_2} \theta_1 \delta_2^{\frac{4}{n-2}} \lambda_2 \frac{\partial \delta_2}{\partial \lambda_2} - \frac{n+2}{n-2}\int_{B_2} \theta_1 \delta_2^{\frac{4}{n-2}} \lambda_2 \frac{\partial \delta_2}{\partial \lambda_2} \\ &:= I_1 - I_2 - I_3 - I_4. \end{aligned}$$

By using [3, F16, p. 23], we have:

$$I_1 = \int_{\mathbb{R}^n} \delta_1(-\Delta)\left(\lambda_2 \frac{\partial \delta_2}{\partial \lambda_2}\right) = \int_{\mathbb{R}^n} \delta_1^{\frac{n+2}{n-2}} \lambda_2 \frac{\partial \delta_2}{\partial \lambda_2} = \widetilde{c}\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} + O\left(\varepsilon_{12}^{n/(n-2)} \ln(\varepsilon_{12}^{-1})\right), \qquad (21)$$

where $\widetilde{c}$ is defined in Lemma 2.11.

For I_2, note that, since $d_1 \nrightarrow 0$ then for all $x \in \mathbb{R}^n \setminus \Omega$, we have: $\delta_1(x) \le c\lambda_1^{(2-n)/2}$, hence

$$|I_2| \le \frac{c}{\lambda_1^{(n-2)/2}}\int_{\mathbb{R}^n\setminus\Omega} \delta_2^{\frac{n+2}{n-2}} \le \frac{c}{\lambda_1^{(n-2)/2}}\int_{B^c_{(a_2,d_2)}} \delta_2^{\frac{n+2}{n-2}} \le \frac{c}{(\lambda_1\lambda_2)^{(n-2)/2}(\lambda_2 d_2)^2}. \qquad (22)$$

For I_3, there holds

$$|I_3| \le \|\theta_1\|_\infty \int_{\Omega\setminus B_2} \delta_2^{\frac{n+2}{n-2}} \le \frac{c}{(\lambda_1\lambda_2)^{(n-2)/2}(\lambda_2 d_2)^2}. \qquad (23)$$

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For I_4, using the fact that θ_1 is a harmonic function and the evenness of δ_2 and its derivative, we get

$$\begin{aligned}
I_4 &= \frac{n+2}{n-2}\theta_1(a_2)\int_{B_2}\delta_2^{\frac{4}{n-2}}\lambda_2\frac{\partial\delta_2}{\partial\lambda_2} + O\left(\sup_{y\in B_2}\left|D^2\theta_1(y)\right|\int_{B_2}\delta_2^{\frac{n+2}{n-2}}|x-a_2|^2\right)\\
&= \frac{n+2}{n-2}\theta_1(a_2)\left(\int_{\mathbb{R}^n}\delta_2^{\frac{4}{n-2}}\lambda_2\frac{\partial\delta_2}{\partial\lambda_2} - \int_{\mathbb{R}^n\setminus B_2}\delta_2^{\frac{4}{n-2}}\lambda_2\frac{\partial\delta_2}{\partial\lambda_2}\right) + O\left(\frac{\ln(\lambda_2 d_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2}\right)\\
&= -\widetilde{c}\frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + O\left(\frac{1}{\lambda_1^{\frac{n+2}{2}}\lambda_2^{\frac{n-2}{2}}}\right) + O\left(\frac{\ln(\lambda_2 d_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2}\right).
\end{aligned}\tag{24}$$

By (21)–(24) and the fact that $(\lambda_1\lambda_2)^{(2-n)/2} = O(\varepsilon_{12})$, the proof of Lemma 2.14 follows. □

Lemma 2.15. *[5, Lemmas A10 and A11, p. 572] For $n = 4, 5, 6$, there hold:*

(a) $$\int_\Omega P\delta_2^{\frac{n+2}{n-2}}\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} = 2\langle P\delta_2, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle + O\left(\frac{1}{(\lambda_2 d_2)^n}\right),$$

(b) $$\langle P\delta_2, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle = -\frac{1}{2}\frac{\widetilde{c}}{\lambda_2^{n-1}}\frac{\partial H(a_2,a_2)}{\partial a_2} + O\left(\frac{1}{(\lambda_2 d_2)^n}\right).$$

Lemma 2.16. *[6, 3.33, p. 785] For $n = 4, 5, 6$, we have*

$$\int_\Omega P\delta_2\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} = O\left(\frac{1}{(\lambda_2 d_2)^{n-1}}\right).$$

The following two lemmas correspond to the ones A14 and A12 of [5]. The remaining presented in [5] are inadequate in our case, therefore, we repeat the proofs to present the rest differently.

Lemma 2.17. *For $n = 4, 5, 6$, we have*

$$\frac{n+2}{n-2}\int_\Omega P\delta_1 P\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} = \langle P\delta_1, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle + O\left(\frac{1}{(\lambda_2 d_2)^{\frac{n-2}{2}}}\varepsilon_{12}\left(\left(\ln\varepsilon_{12}^{-1}\right)^{\frac{2}{3}}\ \textit{if}\ n=6\right)\right).$$

Proof. Recall that $P\delta_2 = \delta_2 - \theta_2$, then we have

$$\begin{aligned}
I := \frac{n+2}{n-2}\int_\Omega P\delta_1 P\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} &= \frac{n+2}{n-2}\int_\Omega P\delta_1\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2} - \frac{n+2}{n-2}\int_\Omega P\delta_1\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\\
&\quad + O\left(\int_\Omega\delta_1\delta_2^{\frac{6-n}{n-2}}\theta_2\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}\right) + O\left(\int_\Omega\delta_1\delta_2^{\frac{6-n}{n-2}}\theta_2\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right)\\
&:= I_1 + I_2 + O(I_3) + O(I_4).
\end{aligned}\tag{25}$$

The first integral of the right hand side of (25),

$$I_1 = \int_\Omega P\delta_1 \frac{1}{\lambda_2}\frac{\partial(\delta_2^{\frac{n+2}{n-2}})}{\partial a_2} = \int_\Omega P\delta_1 \frac{1}{\lambda_2}\frac{\partial(-\triangle P\delta_2)}{\partial a_2} = \langle P\delta_1, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle. \tag{26}$$

Concerning the second integral of the right hand side, by using Hölder's inequality and Lemma 2.2 we deduce that:

$$|I_2| \le \left(\int_\Omega (\delta_1\delta_2^{\frac{4}{n-2}})^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \left\| \frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right\|_{L^{\frac{2n}{n-2}}} \le \frac{c\varepsilon_{12}}{(\lambda_2 d_2)^{n/2}}(1+\underbrace{(\ln\varepsilon_{12}^{-1})^{\frac{2}{3}}}_{\text{if } n=6}). \tag{27}$$

For the third integral, using Lemma 2.2, Hölder's inequality and the fact that: $|(1/\lambda_2)(\partial\delta_2/\partial a_2)| \le c\delta_2$, we obtain,

$$|I_3| \le c\int_\Omega \delta_1\delta_2^{\frac{4}{n-2}}\theta_2 \le c\|\theta_2\|_{L^{\frac{2n}{n-2}}} \left(\int_\Omega (\delta_1\delta_2^{\frac{4}{n-2}})^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \le \frac{c\varepsilon_{12}}{(\lambda_2 d_2)^{\frac{n-2}{2}}}(1+\underbrace{(\ln\varepsilon_{12}^{-1})^{\frac{2}{3}}}_{\text{if } n=6}). \tag{28}$$

Finally, I_4 is handled in the same way as I_2. Thus the proof of Lemma 2.17 follows. □

Lemma 2.18. *For $n = 4, 5, 6$, if $d_1 := d(a_1, \partial\Omega) \nrightarrow 0$ then*

$$\langle P\delta_1, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle = \frac{\widetilde{c}}{\lambda_2}\left(\frac{\partial\varepsilon_{12}}{\partial a_2} - \frac{1}{(\lambda_1\lambda_2)^{(n-2)/2}}\frac{\partial H}{\partial b}(a_1,a_2)\right)$$
$$+ O\left(\frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2} + \lambda_1|a_1-a_2|\varepsilon_{12}^{\frac{n+1}{n-2}} + \frac{1}{\lambda_1^{\frac{n+2}{2}}\lambda_2^{\frac{n}{2}}d_2}\right).$$

Proof. Let us denote by $B_2' := B_{(a_2,d_2/2)}$. There holds

$$I := \int_\Omega P\delta_1\frac{1}{\lambda_2}\frac{\partial\left(\delta_2^{\frac{n+2}{n-2}}\right)}{\partial a_2} = \int_{\mathbb{R}^n} \delta_1\frac{1}{\lambda_2}\frac{\partial\left(\delta_2^{\frac{n+2}{n-2}}\right)}{\partial a_2} - \frac{n+2}{n-2}\int_{\mathbb{R}^n\setminus\Omega}\delta_1\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}$$
$$- \frac{n+2}{n-2}\int_{B_2'}\theta_1\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2} - \frac{n+2}{n-2}\int_{\Omega\setminus B_2'}\theta_1\delta_2^{\frac{4}{n-2}}\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}$$
$$:= I_1 - I_2 - I_3 - I_4.$$

By using [3, (2.206), p. 58], we have:

$$I_1 = \frac{\widetilde{c}}{\lambda_2}\frac{\partial\varepsilon_{12}}{\partial a_2} + O\left(\lambda_1|a_1-a_2|\varepsilon_{12}^{\frac{n+1}{n-2}}\right). \tag{29}$$

For I_2, note that $d_1 \nrightarrow 0$ then for all $x \in \mathbb{R}^n \setminus \Omega$, we have: $\delta_1(x) \leq c\lambda_1^{(2-n)/2}$, then:

$$|I_2| \leq \frac{c}{\lambda_1^{(n-2)/2}} \int\limits_{\mathbb{R}^n \setminus \Omega} \delta_2^{\frac{n+2}{n-2}} \leq \frac{c}{\lambda_1^{(n-2)/2}} \int\limits_{B^c_{(a_2,d_2)}} \delta_2^{\frac{n+2}{n-2}} \leq \frac{c}{(\lambda_1\lambda_2)^{(n-2)/2}(\lambda_2 d_2)^2}. \tag{30}$$

For I_3, let: $\widetilde{\theta}_1 := \theta_1 - c_0 H(a_1, .)/\lambda_1^{(n-2)/2}$, then

$$\begin{aligned} I_3 =& \frac{n+2}{n-2} \frac{c_0}{\lambda_1^{\frac{n-2}{2}}} \int\limits_{B_2'} H(a_1,x)\delta_2^{\frac{4}{n-2}}(x) \frac{1}{\lambda_2} \frac{\partial \delta_2(x)}{\partial a_2} + \frac{n+2}{n-2} \int\limits_{B_2'} \widetilde{\theta}_1(x)\delta_2^{\frac{4}{n-2}}(x) \frac{1}{\lambda_2} \frac{\partial \delta_2(x)}{\partial a_2} \\ =& (n+2)c_0^{\frac{2n}{n-2}} \frac{\lambda_2^{\frac{n+4}{2}}}{\lambda_1^{\frac{n-2}{2}}} \int\limits_{B_2'} \left(\sum_{i=1}^n \frac{\partial H}{\partial b_i}(a_1,a_2)(x-a_2)_i \right) \frac{(x-a_2)}{(1+\lambda_2^2|x-a_2|^2)^{\frac{n+4}{2}}} dx \\ &+ O\left(\frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2} \right) + O\left(\frac{1}{\lambda_1^{\frac{n+2}{2}} \lambda_2^{\frac{n}{2}} d_2} \right), \end{aligned} \tag{31}$$

where we have used Taylor expansions of $H(a_1,x)$ and $\widetilde{\theta}_1(x)$ and the evenness of δ_2 and the oddness of $\partial\delta_2/\partial a_2$. Note that the integral in (31) is a vector. To calculate its jth component, we need to estimate the following integral

$$\begin{aligned} &\int\limits_{B_2'} \left(\sum_{i=1}^n \frac{\partial H}{\partial b_i}(a_1,a_2)(x-a_2)_i \right) \frac{(x-a_2)_j}{(1+\lambda_2^2|x-a_2|^2)^{\frac{n+4}{2}}} \\ &= \frac{\partial H}{\partial b_j}(a_1,a_2) \int\limits_{B_2'} \frac{(x-a_2)_j^2}{(1+\lambda_2^2|x-a_2|^2)^{\frac{n+4}{2}}} \\ &= \frac{1}{n\lambda_2^{n+2}} \frac{\partial H}{\partial b_j}(a_1,a_2) \left(\frac{n\widetilde{c}}{(n+2)\omega_n} + O\left(\frac{1}{(\lambda_2 d_2)^2} \right) \right). \end{aligned}$$

Then:

$$I_3 = \frac{\widetilde{c}}{\lambda_2} \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} \frac{\partial H}{\partial b}(a_1,a_2) + O\left(\frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2} \right) + O\left(\frac{1}{\lambda_1^{\frac{n+2}{2}} \lambda_2^{\frac{n-2}{2}}} \right). \tag{32}$$

Finally

$$|I_4| \leq c \int\limits_{\Omega \setminus B_2'} \theta_1 \delta_2^{\frac{n+2}{n-2}} \leq c\|\theta_1\|_\infty \int\limits_{\mathbb{R}^n \setminus B_2'} \delta_2^{\frac{n+2}{n-2}} \leq \frac{c}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2}. \tag{33}$$

(29), (30), (32) and (33) prove the lemma. □

Lemma 2.19. *For all $v \in H_0^1(\Omega)$ we have:*

$$\int_\Omega |v| \frac{1}{\lambda_2} \left| \frac{\partial P\delta_2}{\partial a_2} \right| \leq \begin{cases} \frac{C\|v\|}{\lambda_2^2} (\ln \lambda_2)^{3/4} + \frac{C\|v\|}{(\lambda_2 d_2)^2} & \textit{if } n = 4 \\ \frac{C\|v\|}{\lambda_2^2} + \frac{C\|v\|}{(\lambda_2 d_2)^{n/2}} & \textit{if } n = 5, 6. \end{cases}$$

Proof. It follows immediately from the Hölder's inequality by using the fact that $\partial P\delta_2/\partial a_2 = \partial\delta_2/\partial a_2 + \partial\theta_2/\partial a_2$. $\square$

3. Some known results

To show the main theorem, we reason by the contradiction. We assume that there exists a family of solutions under the form

$$u_\varepsilon := P\delta_{a_1,\lambda_1} - P\delta_{a_2,\lambda_2} + v_\varepsilon.$$

Noting that, according to A. Bahri, the solution u_ε can be written as:

$$u_\varepsilon := \alpha_1 P\delta_{a_1,\lambda_1} - \alpha_2 P\delta_{a_2,\lambda_2} + v \tag{34}$$

where α_i is close to 1 and v satisfies: $\|v\|_{H_0^1} \to 0$, $v \in F^\perp$, where F is defined in Lemma 2.7.

In this section, we propose presenting some estimates concerning the variables v, α_i, λ_i and a_i.

Proposition 3.1. *[6, Lemma 3.3, p. 778] For $n = 4, 5, 6$, the function v defined in (34) satisfies the following estimate:*

$$\|v\| \leq c \sum_{k=1}^{2} \frac{1}{(\lambda_k d_k)^{n-2}} (1 + \underbrace{(\ln \lambda_k d_k)^{2/3}}_{\textit{if } n=6}) + \sum_{k=1}^{2} \frac{\varepsilon}{(\lambda_k)^{(n-2)/2}} (1 + \underbrace{(\ln \lambda_k)^{2/3}}_{\textit{if } n=6}) + \varepsilon_{12} (\ln \varepsilon_{12}^{-1})^{(n-2)/2}.$$

Proposition 3.2. *For $i = 1, 2$ and for $n = 4, 5, 6$, we have:*

$$A_i := 1 - \alpha_i^{\frac{4}{n-2}} = O\left(\frac{1}{(\lambda_i d_i)^{n-2}} + \varepsilon_{12} + \|v\|^2 + R_i \right), \textit{ where}$$

$$R_i = \frac{\varepsilon}{\lambda_i^2} (1 + \underbrace{\ln \lambda_i}_{\textit{if } n=4}) + \varepsilon \frac{\|v\|}{\lambda_i^{(n-2)/2}} (1 + \underbrace{(\ln \lambda_i)^{2/3}}_{\textit{if } n=6}) + \underbrace{\|v\| \frac{\ln(\lambda_i d_i)}{(\lambda_i d_i)^4} + \|v\| \varepsilon_{12} (\ln\ \varepsilon_{12}^{-1})^{2/3}}_{\textit{if } n=6}.$$

Proof. We prove the proposition for $i = 1$. Let u_ε be a solution for (P_ε), we multiply the equation (1)(a) by $P\delta_1$, and we integrate it on Ω, we get:

$$\int_\Omega -\Delta(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)P\delta_1 = \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)P\delta_1 + \varepsilon\alpha_1\int_\Omega P\delta_1^2 - \varepsilon\alpha_2\int_\Omega P\delta_1 P\delta_2 + \varepsilon\int_\Omega P\delta_1 v. \tag{35}$$

Lemmas 2.1 and 2.2 imply that the left hand side of (35) is equal to

$$\alpha_1\int_\Omega \delta_1^{\frac{2n}{n-2}} - \alpha_1\int_\Omega \delta_1^{\frac{n+2}{n-2}}\theta_1 + \alpha_2\int_\Omega \delta_2^{\frac{n+2}{n-2}}P\delta_1 = \alpha_1 S + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) + O(\varepsilon_{12}). \tag{36}$$

The first integral of the right hand side of (35) is equal to:

$$\begin{aligned} I_1 :=& \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}}(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)P\delta_1 + \frac{n+2}{n-2}\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{4}{n-2}} vP\delta_1 \\ &+ O\left(\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|^{\frac{6-n}{n-2}} v^2 P\delta_1 (if\ \ n=4,5)\right) + O\left(\int_\Omega |v|^{\frac{n+2}{n-2}}P\delta_1\right) \\ =&\alpha_1^{\frac{n+2}{n-2}}\int_\Omega P\delta_1^{\frac{2n}{n-2}} + O\left(\int_\Omega \delta_1^{\frac{n+2}{n-2}}\delta_2 + \delta_2^{\frac{n+2}{n-2}}\delta_1\right) + \frac{n+2}{n-2}\alpha_1^{\frac{4}{n-2}}\int_\Omega P\delta_1^{\frac{n+2}{n-2}}v + O\left(\int_\Omega \delta_1^{\frac{4}{n-2}}\delta_2|v|\right) \\ &+ O\left(\int_\Omega \delta_1\delta_2^{\frac{4}{n-2}}|v|\right) + O\left(\int_\Omega \delta_1^{\frac{4}{n-2}}v^2 + \delta_1\delta_2^{\frac{6-n}{n-2}}v^2\right) + O\left(\int_\Omega |v|^{\frac{n+2}{n-2}}P\delta_1\right). \end{aligned} \tag{37}$$

Lemmas 2.1–2.5 imply that:

$$\begin{aligned} I_1 =&\alpha_1^{\frac{n+2}{n-2}}S + O\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) + O(\varepsilon_{12}) + O\left(\frac{\|v\|}{(\lambda_1 d_1)^{n-2}}\left((\ln(\lambda_1 d_1))^{2/3}(\,if\ \ n=6)\right)\right) \\ &+ O\left(\|v\|\varepsilon_{12}\left((\ln(\varepsilon_{12}^{-1}))^{2/3}(\text{ if } n=6)\right)\right) + O\left(\|v\|^2\right) + O\left(\|v\|^{\frac{n+2}{n-2}}\right). \end{aligned} \tag{38}$$

By using (35)–(38) and Lemmas 2.2, 2.5 and 2.6, the proof of Proposition 3.2 follows. □

Proposition 3.3. *[6, Proposition 3.5, p. 781] For $i = 1, 2$ and for $n = 4, 5, 6$, we have:*

$$\widetilde{c}\frac{n-2}{2}\frac{H_{ii}}{\lambda_i^{n-2}} - \widetilde{c}\left(\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - \varepsilon\Gamma_i = O(R_{\lambda_i}),$$

where $\widetilde{c}$ is defined in Lemma 2.11,

$$R_{\lambda_i} = \sum_{k=1}^{2}\frac{\ln(\lambda_k d_k)}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}}\ln(\varepsilon_{12}^{-1}) + \varepsilon\varepsilon_{12} + \frac{\varepsilon}{(\lambda_i d_i)^{n-2}} + \frac{\varepsilon^2}{\lambda_i^{n-2}}((\ln\lambda_i)^2\ if\, n=6),\ and$$

$$\Gamma_i := - \int_\Omega \alpha_i P\delta_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$

If $n = 5, 6$, then $\Gamma_i = \frac{C_2}{\lambda_i^2} + O\left(\frac{1}{(\lambda_i d_i)^{n-2}}\right)$, where $C_2 = \frac{n-2}{2} c_0^2 \int_{\mathbb{R}^n} \frac{|x|^2 - 1}{(1+|x|^2)^{(n+2)/2}} > 0$ and if $n = 4$, assuming that $d_i \geq c > 0$ for $i = 1, 2$, then Γ_i satisfies:

$$\Gamma_i = \frac{C_2}{\lambda_i^2} \ln(\lambda_i) + O\left(\frac{1}{\lambda_i^2}\right).$$

Proposition 3.4. [6, Proposition 3.6, p. 784] *For $i = 1, 2$, and for $n = 5, 6$, we have:*

$$\frac{1}{\lambda_i^{n-1}} \frac{\partial H(a_i, a_i)}{\partial a_i} + \frac{2}{\lambda_i}\left(\frac{\partial \varepsilon_{12}}{\partial a_i} - \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} \frac{\partial H}{\partial a_i}(a_1, a_2)\right) = O(R_{a_i}),$$

where

$$R_{a_i} = \sum_{k=1}^{2} \frac{1}{(\lambda_k d_k)^n} + \varepsilon_{12}^{\frac{n}{n-2}} \ln(\varepsilon_{12}^{-1}) + \frac{\varepsilon}{(\lambda_i d_i)^{n-1}} + \frac{\varepsilon}{\lambda_i} \varepsilon_{12} (\ln \varepsilon_{12}^{-1})^{\frac{n-2}{n}}$$
$$+ \sum_{k=1}^{2} \frac{\varepsilon^2}{\lambda_k^{n-2}} ((\ln \lambda_k)^2 \text{ if } n = 6) + \lambda_j |a_1 - a_2| \varepsilon_{12}^{\frac{n+1}{n-2}} (j \neq i).$$

Note that the two terms $\frac{\varepsilon^2}{\lambda_j^{n-2}}((\ln \lambda_j)^2$ if $n = 6)$ and $\lambda_j |a_1 - a_2| \varepsilon_{12}^{\frac{n+1}{n-2}} (j \neq i)$ do not appear in [6, Proposition 3.6, p. 784], because if the λ_i's are comparable, then $\varepsilon^2/\lambda_j^{n-2} \sim \varepsilon^2/\lambda_i^{n-2}$ and $\lambda_j |a_1 - a_2| \varepsilon_{12}^{(n+1)/(n-2)} = O(\varepsilon_{12}^{n/(n-2)})$ ($f \sim g$ means that f/g is bounded).

4. The concentration point a_1

Let u_ε be a solution of (P_ε) in the form: $u_\varepsilon = \alpha_1 P\delta_{a_1,\lambda_1} - \alpha_2 P\delta_{a_2,\lambda_2} + v$ where $\lambda_1/\lambda_2 \to 0$, $\|v\| \to 0$ and $v \in F^\perp$. The main objective of this section is to prove that if $n = 5, 6$, the concentration point a_1 (associated with the least concentrated bubble) is not close to the boundary of Ω.

Lemma 4.1. *For $n = 5, 6$, we have: $\lambda_1 d_1 \ll \lambda_2 d_2$, (this means that $\lambda_1 d_1/(\lambda_2 d_2) \to 0$).*

Proof. Suppose the contrary, there exists a constant $m > 1$ such that $\lambda_2 d_2 \leq m\lambda_1 d_1$. Note that $\lambda_1 \ll \lambda_2$, then we deduce that $d_2 \ll d_1$. Therefore $d_2 \to 0$ and $|a_1 - a_2| \geq c d_1 \geq c d_2$. Hence

$$\varepsilon_{12} \simeq \frac{1}{(\lambda_1 \lambda_2 |a_1 - a_2|^2)^{\frac{n-2}{2}}}. \tag{39}$$

Also we have

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$$\lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} = -\frac{n-2}{2}\varepsilon_{12}\left(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{12}^{2/(n-2)}\right). \tag{40}$$

Thus,

$$\lambda_2 \frac{\partial \varepsilon_{12}}{\partial \lambda_2} = -\frac{n-2}{2}\varepsilon_{12}\left(1 + O\left(\varepsilon_{12}^{2/(n-2)}\right)\right), \text{(since } \lambda_1 \le \lambda_2) \tag{41}$$

$$\lambda_1 \frac{\partial \varepsilon_{12}}{\partial \lambda_1} = -\frac{n-2}{2}\varepsilon_{12}(1+o(1)), \text{(since } |a_1 - a_2| \ge cd_1 \ge cd_2). \tag{42}$$

Proposition 3.3 (for $n = 5, 6$), (39) and the fact that $G_{12} = \dfrac{1}{|a_1 - a_2|^{n-2}} - H_{12}$ imply that:

$$\begin{cases} \widetilde{c}\frac{n-2}{2}\frac{H_{11}}{\lambda_1^{n-2}} + \widetilde{c}\frac{n-2}{2}\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} - C_2\frac{\varepsilon}{\lambda_1^2} = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \frac{\varepsilon}{\lambda_1^2}\right) & \text{(a)} \\ \widetilde{c}\frac{n-2}{2}\frac{H_{22}}{\lambda_2^{n-2}} + \widetilde{c}\frac{n-2}{2}\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} - C_2\frac{\varepsilon}{\lambda_2^2} = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \frac{\varepsilon}{\lambda_2^2}\right). & \text{(b)} \end{cases} \tag{43}$$

$2m^{n-2}2^{n-2} \times$ (43)(a) $-$ (43)(b) gives

$$\widetilde{c}\frac{n-2}{2}\left(2m^{n-2}2^{n-2}\frac{H_{22}}{\lambda_2^{n-2}} - \frac{H_{11}}{\lambda_1^{n-2}}\right) + \widetilde{c}\frac{n-2}{2}(2^{n-1}m^{n-2} - 1)\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + C_2\frac{\varepsilon}{\lambda_1^2}$$
$$= o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \frac{\varepsilon}{\lambda_1^2}\right).$$

Note that $d_2 \to 0$, then $H_{22} \simeq \frac{1}{2^{n-2}d_2^{n-2}}$ and $H_{11} \le \frac{1}{d_1^{n-2}}$, hence

$$2m^{n-2}2^{n-2}\frac{H_{22}}{\lambda_2^{n-2}} - \frac{H_{11}}{\lambda_1^{n-2}} \ge \frac{2m^{n-2}}{(\lambda_2 d_2)^{n-2}} - \frac{1}{(\lambda_1 d_1)^{n-2}} \ge \frac{m^{n-2}}{(\lambda_2 d_2)^{n-2}}.$$

Thus

$$\frac{m^{n-2}}{(\lambda_2 d_2)^{n-2}} + \frac{\overline{c}\, G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + C_2\frac{\varepsilon}{\lambda_1^2} = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12} + \frac{\varepsilon}{\lambda_1^2}\right), \tag{44}$$

which gives a contradiction, since by (39) we have

$$\varepsilon_{12} = \frac{G_{12} + H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = \frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + O\left(\frac{1}{(\lambda_2 d_2)^{n-2}}\right). \tag{45}$$

Hence Lemma 4.1 follows. □

Lemma 4.2. *For $n = 5, 6$, we have*

$$(i)\ \widetilde{c}\frac{n-2}{2}\frac{H_{11}}{\lambda_1^{n-2}} \simeq C_2\frac{\varepsilon}{\lambda_1^2}, \qquad (ii)\ \varepsilon_{12} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right),$$

where $\widetilde{c}$ is defined in Lemma 2.11 and C_2 is defined in Proposition 3.3.

Proof. By using (41) and Proposition 3.3 (for $i = 2$), we deduce that

$$\widetilde{c}\frac{n-2}{2}\frac{H_{22}}{\lambda_2^{n-2}} + \widetilde{c}\frac{n-2}{2}\left(\varepsilon_{12} - \frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - C_2\frac{\varepsilon}{\lambda_2^2} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \varepsilon_{12} + \frac{\varepsilon}{\lambda_2^2}\right). \tag{46}$$

Note that $H_{12} \le c/d_1^{n-2}$, and $H_{22} \le c/d_2^{n-2}$. Using the fact that $\lambda_1 \ll \lambda_2$ and $\lambda_1 d_1 \ll \lambda_2 d_2$ (according to Lemma 4.1), we get

$$\frac{H_{22}}{\lambda_2^{n-2}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right) \quad \text{and} \quad \frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}}\right). \tag{47}$$

(46)–(47) imply that

$$\varepsilon_{12} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \frac{\varepsilon}{\lambda_1^2}\right). \tag{48}$$

Now, it is easy to get from (40) that $|\lambda_1 \partial\varepsilon_{12}/\partial\lambda_1| \le c\varepsilon_{12}$ and therefore Proposition 3.3 (for $i = 1$), (47) and (48) imply that

$$\widetilde{c}\frac{n-2}{2}\frac{H_{11}}{\lambda_1^{n-2}} - C_2\frac{\varepsilon}{\lambda_1^2} = o\left(\frac{1}{(\lambda_1 d_1)^{n-2}} + \frac{\varepsilon}{\lambda_1^2}\right). \tag{49}$$

This completes the proof of claim (i) of the lemma.

The second claim follows immediately from (48) and (49). □

Lemma 4.3. *For $n = 5, 6$, we have*

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right| = o\left(\frac{1}{(\lambda_1 d_1)^{n-1}}\right).$$

Proof. We split two cases.

The first case: if $\dfrac{|a_1 - a_2|}{d_1} \to 0$.

Note that in this case we have $d_1 \sim d_2$ and $|a_1 - a_2| \ll d_1$, hence $|a_1 - a_2| \ll d_2$.

Then easy computations imply the following affirmations

$$(a)\ \frac{H_{22}}{\lambda_2^{n-2}} = o(\varepsilon_{12}), \qquad (b)\ \frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = o(\varepsilon_{12}), \qquad (c)\ \frac{\varepsilon}{\lambda_2^2} = o(\varepsilon_{12}), \tag{50}$$

where we have used the estimate of ε given by Lemma 4.2 for (c).

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Now, we need to estimate ε_{12}. Proposition 3.3 (for $i=2$), (41) and (50) imply that

$$\varepsilon_{12}=O\left(\sum_{k=1}^{2}\frac{\ln(\lambda_k d_k)}{(\lambda_k d_k)^n}+\frac{\varepsilon}{(\lambda_2 d_2)^{n-2}}+\frac{\varepsilon}{\lambda_2^{n-2}}\left((\ln\lambda_2)^2 \text{ if } n=6\right)\right)=o\left(\frac{1}{(\lambda_1 d_1)^{n-1}}\right). \tag{51}$$

Finally, note that

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right|=\frac{(n-2)\lambda_2|a_1-a_2|}{\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+\lambda_1\lambda_2|a_1-a_2|^2}\varepsilon_{12}\leq\frac{(n-2)\varepsilon_{12}}{\frac{1}{\lambda_1|a_1-a_2|}+\lambda_1|a_1-a_2|}\leq c\varepsilon_{12}. \tag{52}$$

(51) and (52) prove the lemma in this case.

The second case: if $\dfrac{|a_1-a_2|}{d_1}\nrightarrow 0$. In this case, there exists a constant $c>0$ such that $|a_1-a_2|\geq cd_1$. Then

$$\frac{1}{\lambda_1}\left|\frac{\partial\varepsilon_{12}}{\partial a_1}\right|\leq\frac{(n-2)\lambda_2|a_1-a_2|}{\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+\lambda_1\lambda_2|a_1-a_2|^2}\varepsilon_{12}\leq\frac{c}{\lambda_1|a_1-a_2|}\varepsilon_{12}\leq\frac{c}{\lambda_1 d_1}\varepsilon_{12}. \tag{53}$$

Lemma 4.2 and (53) prove the lemma in this case. □

Lemma 4.4. *For $n=5,6$, we have*

$$d_1\nrightarrow 0.$$

Proof. We argue by contradiction, we assume that $d_1\to 0$. It is easy to get that

$$\frac{\partial H}{\partial a_1}(a_1,a_1)\cdot\nu_1\simeq\frac{1}{2^{n-1}d_1^{n-1}}\quad\text{and}\quad\frac{1}{\lambda_1}\frac{\partial H}{\partial a_1}(a_1,a_2)\cdot\nu_1\frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}=o\left(\frac{1}{(\lambda_1 d_1)^{n-1}}\right). \tag{54}$$

Now, combining Proposition 3.4 (for $i=1$), Lemmas 4.2–4.3, we get a contradiction. Hence our result follows. □

5. Study of the v-part

Before stating this section, we are going to explain the reasons why we are going to conduct an elaborate study of the function v. We notice that in the Proposition 3.3 (for $i=1$), every term of the remaining is small with respect to the main terms of the proposition. In contrast, the term $\ln(\lambda_1 d_1)/(\lambda_1 d_1)^n$ of the same proposition (for $i=2$) is not small with respect to the main terms of the proposition. So this proposition such as the one written is not very important. This term comes mainly from $\|v\|$, therefore, we have thought about decomposing the function v in sums of two functions v_1 and v_2 such that the function v_1 contains all the disturbing terms and v_2 has a good estimate. The main objective of this section is to find a punctual estimate of v_1, then we deduce the estimate of $\|v_2\|$.

Let u_ε be a solution of (P_ε) under the form (34), then we have: $u_\varepsilon:=\alpha_1 P\delta_1-\alpha_2 P\delta_2+v$. It holds that

$$-\Delta v = f := -\alpha_1 \delta_1^{\frac{n+2}{n-2}} + \alpha_2 \delta_2^{\frac{n+2}{n-2}} + |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon + \varepsilon u_\varepsilon. \tag{55}$$

Let

$$f_1(w) := -\alpha_1 \delta_1^{\frac{n+2}{n-2}} + |\alpha_1 P\delta_1 + w|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 + w) + \varepsilon \alpha_1 P\delta_1 + \varepsilon w, \quad \text{and let} \tag{56}$$

$$(\varphi_1, ..., \varphi_{n+2}) \text{ be an orthonormal basis of } E_1 := \langle P\delta_1, \lambda_1 \frac{\partial P\delta_1}{\partial \lambda_1}, \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1^1}., ., ., \frac{1}{\lambda_1} \frac{\partial P\delta_1}{\partial a_1^n} \rangle. \tag{57}$$

We define the function v_1 by

$$\begin{cases} -\Delta v_1 = f_1\left((\inf(|v_1|, \mu\delta_1)) sign(v_1)\right) + \sum_{k=1}^{n+2} \left[\int_\Omega f_1\left((\inf(|v_1|, \mu\delta_1)) sign(v_1)\right) \varphi_k \right] \Delta\varphi_k \text{ in } \Omega \\ v_1 = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{58}$$

where μ is a small positive constant.

The idea of introducing v_1 in this form is inspired from [4] where the authors use equations of the type (58) in order to get punctual estimates of some functions similar to our function v.

We start by the following result.

Proposition 5.1. *For $n = 4, 5, 6$, the function v_1 satisfies*

$$(a)\ v_1 \in E_1^\perp \quad \textit{and} \quad (b)\ \|v_1\| = O\left(\frac{1}{(\lambda_1 d_1)^{n-2}} (1 + \underbrace{(\ln(\lambda_1 d_1))^{2/3}}_{if\, n=6}) + \frac{\varepsilon}{\lambda_1^{\frac{n-2}{2}}} (1 + \underbrace{(\ln \lambda_1)^{2/3}}_{if\, n=6}) \right).$$

Proof. (a) To simplify the presentation, we denote by $v_1^* := \inf(|v_1|, \mu\delta_1) sign(v_1)$. Let $(\varphi_1, ..., \varphi_{n+2})$ be an orthonormal basis of E_1 and let $j \in \{1, ..., n+2\}$, it holds

$$\langle v_1, \varphi_j \rangle_{H_0^1} = \int_\Omega -\Delta v_1 \varphi_j = \int_\Omega f_1 \varphi_j - \sum_{i=1}^{n+2} \left(\int_\Omega f_1 \varphi_i \right) \langle \varphi_i, \varphi_j \rangle_{H_0^1} = 0.$$

(b) We multiply (58) by v_1, then we integrate it on Ω, we obtain

$$\|v_1\|_{H_0^1}^2 \le \int_\Omega (\alpha_1 P\delta_1)^{\frac{n+2}{n-2}} v_1 + \frac{n+2}{n-2} \int_\Omega (\alpha_1 P\delta_1)^{\frac{4}{n-2}} v_1^2$$
$$+ (c\mu + c\mu^{\frac{4}{n-2}}) \int_\Omega \delta_1^{\frac{4}{n-2}} |v_1|^2 + c\varepsilon \int_\Omega \delta_1 |v_1|. \tag{59}$$

(59) and Lemmas 2.3, 2.5 imply that

$$
\begin{aligned}
\|v_1\|^2_{H^1_0} - \frac{n+2}{n-2}\int_\Omega (P\delta_1)^{\frac{4}{n-2}} v_1^2 &\le (c\mu + c\mu^{\frac{4}{n-2}} + c|A_1|)\|v_1\|^2 \\
&+ c\frac{\varepsilon\|v_1\|}{\lambda_1^{\frac{n-2}{2}}}(1+\underbrace{(\ln\lambda_1)^{2/3}}_{\text{if } n=6}) + c\frac{\|v_1\|}{(\lambda_1 d_1)^{n-2}}(1+\underbrace{(\ln(\lambda_1 d_1))^{2/3}}_{\text{if } n=6}).
\end{aligned}
$$

Note that $v_1 \in E_1^\perp$, Lemma 2.4 implies that there exists $\beta_0 > 0$ such that

$$
\begin{aligned}
&(\beta_0 - c\mu - c\mu^{\frac{4}{n-2}} - c|A_1|)\|v_1\|^2 \\
&\le \left[\frac{c}{(\lambda_1 d_1)^{n-2}}(1+\underbrace{(\ln(\lambda_1 d_1))^{2/3}}_{\text{if } n=6}) + \frac{c\varepsilon}{\lambda_1^{\frac{n-2}{2}}}(1+\underbrace{(\ln\lambda_1)^{2/3}}_{\text{if } n=6})\right]\|v_1\|.
\end{aligned}
$$

We choose μ such that $c\mu + c\mu^{\frac{4}{n-2}} + c|A_1| \le \frac{\beta_0}{2}$, the proof of Proposition 5.1 follows. □

Proposition 5.2. *Let $n = 4, 5, 6$. For all y in Ω and for all α satisfying $\frac{n-4}{n-2} < \alpha < 1$, we have:*

$$
|v_1(y)| \le C\Lambda_1\delta_1(y), \quad \textit{where} \quad \Lambda_1 := |A_1| + \varepsilon + \frac{1}{(\lambda_1 d_1)^{(n-2)(1-\alpha)}}.
$$

Proof. Note that v_1 satisfies (58), then

$$
\begin{aligned}
|v_1(y)| \le &\left|\int_\Omega f_1(sign(v_1)\inf(|v_1|, \mu\delta_1))G(x,y)dx\right| \\
&+ \left|\sum_{k=1}^{n+2}\left[\left(\int_\Omega f_1(sign(v_1)\inf(|v_1|, \mu\delta_1))\varphi_k(x)dx\right)(-\varphi_k(y))\right]\right| \\
&:= |I_1| + |I_2|.
\end{aligned} \tag{60}
$$

Moreover, note that for all $i \in \{1, ..., n+2\}$, we have $\varphi_i = \sum_{j=1}^{n+2} C_j\overline{\varphi_j}$, where $\overline{\varphi_j}$ is one of the functions $P\delta_1, \lambda_1\partial P\delta_1/\partial\lambda_1, (1/\lambda_1)(\partial P\delta_1/\partial a_1^k); k \in \{1, ..., n\}$. Hence easy computation implies that $|\varphi_k(y)| \le c\delta_1(y)$. Thus

$$
|I_2| \le cK\delta_1(y), \quad \text{where} \quad K := \left|\sum_{k=1}^{n+2}\left(\int_\Omega f_1(sign(v_1)\inf(|v_1|, \mu\delta_1))\varphi_k(x)dx\right)\right|. \tag{61}
$$

Also we have:

$$
|f_1(w)| \le \alpha_1|A_1|\delta_1^{\frac{n+2}{n-2}} + c\theta_1\delta_1^{\frac{4}{n-2}} + c\delta_1^{\frac{4}{n-2}}|w| + c|w|^{\frac{n+2}{n-2}} + \varepsilon\delta_1 + \varepsilon|w|. \tag{62}
$$

(62), Lemmas 2.1 and 2.6 imply that

$$K \le \alpha_1|A_1|\int_\Omega \delta_1^{\frac{2n}{n-2}} + c\int_\Omega \theta_1\delta_1^{\frac{n+2}{n-2}} + c\int_\Omega \delta_1^{\frac{n+2}{n-2}}|v_1| + c\varepsilon\int_\Omega \delta_1^2$$
$$\le c|A_1| + \frac{c}{(\lambda_1 d_1)^{n-2}} + c\|v_1\| + \frac{c\varepsilon}{\lambda_1^2}(1+\underbrace{\ln\lambda_1}_{\text{if } n=4}). \tag{63}$$

(61), (63) and Proposition 5.1 implies that:

$$|I_2| \le c\left(|A_1| + \frac{1}{(\lambda_1 d_1)^{n-2}}(1+\underbrace{\ln(\lambda_1 d_1)}_{\text{if } n=6}) + \frac{\varepsilon}{\lambda_1^{\frac{n-2}{2}}}(1+\underbrace{\ln\lambda_1}_{\text{if } n=6})\right)\delta_1(y). \tag{64}$$

We now estimate I_1. By (62) we have

$$|I_1| \le \int_\Omega \alpha_1|A_1|\delta_1^{\frac{n+2}{n-2}}(x)G(x,y)dx + c\int_\Omega \theta_1(x)\delta_1^{\frac{4}{n-2}}(x)G(x,y)dx$$
$$+ c\mu^{\frac{n-4}{n-2}}\int_\Omega \delta_1^{\frac{n}{n-2}}(x)|v_1(x)|^{\frac{2}{n-2}}G(x,y)dx + c\varepsilon\int_\Omega \delta_1(x)G(x,y)dx.$$

Lemmas 2.8–2.10 and Proposition 5.1 imply that

$$|I_1| \le C\left(|A_1| + \varepsilon + \frac{1}{(\lambda_1 d_1)^{(n-2)(1-\alpha)}}\right)\delta_1(y). \tag{65}$$

(64) and (65) prove Proposition 5.2. □

Note that Λ_1, defined in Proposition 5.2 is a very small constant. Then we deduce that $|v_1(y)| \le \mu\delta_1(y)$ for all $y \in \Omega$, hence we obtain

Corollary 5.3. *The function v_1, defined in (58), satisfies*

$$\begin{cases} -\Delta v_1 = f_1(v_1) - \sum_{k=1}^{n+2}\left(\int_\Omega f_1(v_1)\varphi_k\right)(-\Delta\varphi_k) & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$v_2 = v - v_1. \tag{66}$$

The fact that $v \in E_1^\perp$, $v_1 \in E_1^\perp$, $\|v\| = o(1)$ and $\|v_1\| = o(1)$, implies that:

$$v_2 \in E_1^\perp \quad \text{and } \|v_2\| = o(1). \tag{67}$$

The following proposition is crucial in our proof. In fact, it is an estimate of $\|v_2\|$ in which we stipulate that λ_1 does not appear alone. This piece of information will be very vital in the Propositions 6.1 and 6.3.

Proposition 5.4. *For $n = 4, 5, 6$ there holds*

$$\|v_2\| \le c\varepsilon_{12}(1 + \underbrace{(\ln \varepsilon_{12}^{-1})^{2/3}}_{if\, n=6}) + \frac{c}{(\lambda_2 d_2)^{n-2}}(1 + \underbrace{\ln(\lambda_2 d_2)}_{if\, n=6}) + \frac{c\varepsilon}{\lambda_2^{\frac{n-2}{2}}}(1 + \underbrace{(\ln \lambda_2)^{2/3}}_{if\, n=6}) + c|A_2|.$$

Proof. The idea of the proof is the following: At the beginning, we introduce the equation that verifies v_2 (see (68)). Then, by multiplying the obtained equation by v_2 and by integrating it on Ω, we get a piece of information on $\|v_2\|$ (see (71)). As v_2 is not necessarily in $F^{\perp}$, we need to decompose it into two functions: $v_2 = \overline{v}_2 + \underline{v}_2$, so that $\overline{v}_2 \in F^{\perp}$ and what follows, the obtained quadratic form in (71) becomes defined positive. Concerning $\underline{v}_2$, we prove that its norm is very small with respect to certain well chosen terms.

From (55), Corollary 5.3 and (66) some computations imply that

$$\begin{aligned} -\triangle v_2 =& \alpha_2 \delta_2^{\frac{n+2}{n-2}} - (\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} + \frac{n+2}{n-2}(\alpha_1 P\delta_1)^{\frac{4}{n-2}} v_2 + \frac{n+2}{n-2}(\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} v_2 \\ & - \varepsilon\alpha_2 P\delta_2 + \varepsilon v_2 + O\left(\delta_1^{\frac{4}{n-2}}\delta_2 + \delta_1\delta_2^{\frac{4}{n-2}} + \underbrace{\delta_1^{\frac{6-n}{n-2}}|v_1 v_2| + \delta_1^{\frac{6-n}{n-2}}\delta_2|v_2| + \delta_1\delta_2^{\frac{6-n}{n-2}}|v_2|}_{\text{if } n=4,5}\right) \\ & + O\left(\underbrace{\delta_1^{\frac{6-n}{n-2}} v_2^2 + \delta_2^{\frac{6-n}{n-2}} v_2^2}_{\text{if } n=4,5} + \underbrace{\sqrt{\delta_1\delta_2}|v_2|}_{\text{if } n=6} + |v_2|^{\frac{n+2}{n-2}}\right) + \sum_{k=1}^{n+2}\left(\int_\Omega f_1(v_1)\varphi_k\right)(-\triangle\varphi_k). \end{aligned} \tag{68}$$

We multiply (68) by v_2, and we integrate it on Ω, by Proposition 5.2 and the fact that $v_2 \in E_1^{\perp}$, we get

$$\begin{aligned} &\|v_2\|^2 - \frac{n+2}{n-2}\int_\Omega P\delta_1^{\frac{4}{n-2}} v_2^2 - \frac{n+2}{n-2}\int_\Omega P\delta_2^{\frac{4}{n-2}} v_2^2 \\ &\le \alpha_2|A_2|\langle P\delta_2, v_2\rangle + c|A_1|\int_\Omega \delta_1^{\frac{4}{n-2}} v_2^2 + c|A_2|\int_\Omega \delta_2^{\frac{4}{n-2}} v_2^2 + O\left(\int_\Omega \theta_2\delta_2^{\frac{4}{n-2}}|v_2|\right) \\ &\quad + c\varepsilon\int_\Omega P\delta_2 v_2 + \varepsilon\|v_2\|^2 + O\left(\int_\Omega \delta_1^{\frac{4}{n-2}}\delta_2|v_2| + \int_\Omega \delta_1\delta_2^{\frac{4}{n-2}}|v_2|\right) \\ &\quad + c\|v_2\|^{\frac{2n}{n-2}} + c\underbrace{\int_\Omega \sqrt{\delta_1\delta_2} v_2^2}_{\text{if } n=6} \end{aligned}$$

$$+ O\left(\underbrace{\Lambda_1\int_\Omega \delta_1^{\frac{4}{n-2}} v_2^2 + \int_\Omega \delta_1^{\frac{6-n}{n-2}}\delta_2 v_2^2 + \int_\Omega \delta_1\delta_2^{\frac{6-n}{n-2}} v_2^2 + \int_\Omega \delta_1^{\frac{6-n}{n-2}}|v_2|^3 + \int_\Omega \delta_2^{\frac{6-n}{n-2}}|v_2|^3}_{\text{if } n=4,5}\right). \tag{69}$$

Observe that

$$|\langle P\delta_2, v_2\rangle| = |\langle P\delta_2, v\rangle - \langle P\delta_2, v_1\rangle| = \left|\int_\Omega \Delta(P\delta_2)v_1\right| \le \Lambda_1\varepsilon_{12}. \tag{70}$$

(69), (70), Lemmas 2.3 and 2.5 imply that

$$\begin{aligned}\|v_2\|^2 - \frac{n+2}{n-2}\int_\Omega P\delta_1^{\frac{4}{n-2}} v_2^2 - \frac{n+2}{n-2}\int_\Omega P\delta_2^{\frac{4}{n-2}} v_2^2 \le \frac{c\|v_2\|}{(\lambda_2 d_2)^{n-2}}(1+\underbrace{(\ln(\lambda_2 d_2))^{\frac{2}{3}}}_{\text{if } n=6})\\ + c|A_2|\Lambda_1\varepsilon_{12} + \frac{c\varepsilon\|v_2\|}{\lambda_2^{\frac{n-2}{2}}}(1+\underbrace{(\ln\lambda_2)^{\frac{2}{3}}}_{\text{if } n=6}) + c\varepsilon_{12}\|v_2\|(1+\underbrace{(\ln\varepsilon_{12}^{-1})^{\frac{2}{3}}}_{\text{if } n=6}) + o\left(\|v_2\|^2\right).\end{aligned} \tag{71}$$

Let

$$v_2 := \underline{v}_2 + \overline{v}_2 \quad \text{where} \quad \underline{v}_2 \in F \text{ and } \overline{v}_2 \in F^\perp. \tag{72}$$

Before keeping on proving the proposition, we introduce the following lemma (its proof will be presented at the end of this section).

Lemma 5.5. *The function $\underline{v}_2$ defined in (72), satisfies*

$$\|\underline{v}_2\| = O\left(\Lambda_1\varepsilon_{12}\right),$$

where Λ_1 is defined in Proposition 5.2.

By Hölder's inequality we have

$$\int_\Omega P\delta_i^{\frac{4}{n-2}} v_2^2 = \int_\Omega P\delta_i^{\frac{4}{n-2}}(\overline{v}_2^2 + \underline{v}_2^2 + 2\underline{v}_2\overline{v}_2) = \int_\Omega P\delta_i^{\frac{4}{n-2}}\overline{v}_2^2 + O\left(\|\underline{v}_2\|^2 + \|\underline{v}_2\|\|\overline{v}_2\|\right). \tag{73}$$

(71) and (73) imply that

$$\|\overline{v}_2\|^2 - \frac{n+2}{n-2}\int_\Omega P\delta_1^{\frac{4}{n-2}}\overline{v}_2^2 - \frac{n+2}{n-2}\int_\Omega P\delta_2^{\frac{4}{n-2}}\overline{v}_2^2 + o\left(\|\overline{v}_2\|^2\right) \le \gamma_1\|\overline{v}_2\| + \gamma_2, \text{ where} \tag{74}$$

$$\gamma_1 :=c\|\underline{v}_2\| + \frac{c}{(\lambda_2 d_2)^{n-2}}(1+\underbrace{((\ln(\lambda_2 d_2))^{\frac{2}{3}}}_{\text{if } n=6}) + \frac{c\varepsilon}{\lambda_2^{\frac{n-2}{2}}}(1+\underbrace{(\ln\lambda_2)^{\frac{2}{3}}}_{\text{if } n=6}) + c\varepsilon_{12}(1+\underbrace{(\ln\varepsilon_{12}^{-1})^{\frac{2}{3}}}_{\text{if } n=6}),$$

$$\gamma_2 :=c\left(|A_2|\Lambda_1\varepsilon_{12} + \|\underline{v}_2\|^3\right) + \frac{c\|\underline{v}_2\|}{(\lambda_2 d_2)^{n-2}}(1+\underbrace{((\ln(\lambda_2 d_2))^{\frac{2}{3}}}_{\text{if } n=6})$$

$$+ \frac{c\varepsilon\|\underline{v}_2\|}{\lambda_2^{\frac{n-2}{2}}}(1+\underbrace{(\ln\lambda_2)^{\frac{2}{3}}}_{\text{if } n=6}) + c\varepsilon_{12}\|\underline{v}_2\|(1+\underbrace{(\ln\varepsilon_{12}^{-1})^{\frac{2}{3}}}_{\text{if } n=6}) + c\|\underline{v}_2\|^2.$$

Note that $\overline{v}_2 \in F^{\perp}$, then Lemma 2.7 implies that there exists $\beta_0 > 0$ such that

$$(\beta_0 + o(1))\,\|\overline{v}_2\|^2 \le \gamma_1\|\overline{v}_2\| + \gamma_2. \tag{75}$$

Hence, we obtain

$$\|\overline{v}_2\| \le c\gamma_1 + c\sqrt{\gamma_2}\,. \tag{76}$$

Lemma 5.5 and the fact that $\|v_2\| \le \|\overline{v}_2\| + \|\underline{v}_2\|$ complete the proof of Proposition 5.4. □

We notice that A_2 appears in the estimate of $\|v_2\|$. But the estimate of A_2, proved in Proposition 3.2, is a defect. In what follows, we suggest improving this estimate by using the information already found.

Proposition 5.6. *For $n = 4, 5, 6$, the variable $A_2 := 1 - \alpha_2^{4/(n-2)}$ satisfies*

$$|A_2| = O\left(\varepsilon_{12} + \frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2}(1+\underbrace{\ln\lambda_2}_{\text{if } n=4})\right).$$

Proof. Note that, Equations (35)–(37) hold true by permuting 1 and 2. Now we need to replace v by $v_1 + v_2$ and use the fact that $|v_1| \le c\delta_1$. We get

$$I_1 := -\alpha_2^{\frac{n+2}{n-2}}S + O\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \varepsilon_{12}\right) + \frac{n+2}{n-2}\alpha_2^{\frac{4}{n-2}}\int_\Omega P\delta_2^{\frac{n+2}{n-2}}v_2$$

$$+ O\left(\int_\Omega \delta_1\delta_2^{\frac{4}{n-2}}|v_2| + \int_\Omega \delta_1^{\frac{4}{n-2}}\delta_2|v_2| + \int_\Omega \delta_1^{\frac{6-n}{n-2}}\delta_2 v_2^2 + \int_\Omega \delta_2^{\frac{4}{n-2}}v_2^2 + \int_\Omega \delta_2|v_2|^{\frac{n+2}{n-2}}\right).$$

Observe that, we have

$$\int P\delta_2^{\frac{n+2}{n-2}}v_2 = \langle P\delta_2, v_2\rangle + O\left(\int \delta_2^{\frac{4}{n-2}}\theta_2|v_2|\right). \tag{77}$$

The result follows from Lemmas 2.3, 2.5, 2.6 and Proposition 5.4. □

Proof of Lemma 5.5. Let $(\varphi_1, ..., \varphi_{n+2})$ and $(\psi_1, ..., \psi_{n+2})$ two orthonormal bases of E_1 and E_2 respectively, where E_1 is defined in (57) and $E_2 := \langle P\delta_2, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2}, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^1}, ., ., ., \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2^n}\rangle$. Thus the function $\underline{v}_2$ can be written as

$$\underline{v}_2 = \sum_{k=1}^{n+2} \nu_k \varphi_k + \sum_{k=1}^{n+2} \mu_k \psi_k.$$

Note that, for all $k \in \{1, ..., n+2\}$, we claim that

$$(a)\ \ \mu_k = O\left(\Lambda_1 \varepsilon_{12}\right), \qquad \text{and} \qquad (b)\ \ \nu_k = O\left(\Lambda_1 \varepsilon_{12}^2\right). \tag{78}$$

In fact, on the one hand, for all $\psi \in E_2$, we have

$$\langle \psi, v_2\rangle = \langle \psi, v - v_1\rangle = -\langle \psi, v_1\rangle = \int_\Omega \Delta\psi\, v_1. \tag{79}$$

Moreover, for $g \in \{P\delta_2, \lambda_2 \partial P\delta_2/\partial\lambda_2, (1/\lambda_2)(\partial\delta_2/\partial a_2^k)\}$, we have $|\Delta g| \le c\delta_2^{(n+2)/(n-2)}$ and therefore by using Proposition 5.2 and Lemma 2.2 we obtain

$$\left|\int_\Omega \Delta g v_1\right| \le c\Lambda_1 \int \delta_1 \delta_2^{\frac{n+2}{n-2}} \le c\Lambda_1 \varepsilon_{12}.$$

Thus, on the other hand, for all $k \in \{1, ..., n+2\}$, we have $\langle \psi_k, v_2\rangle = O(\Lambda_1\varepsilon_{12})$, note that $\overline{v}_2 \in F^\perp$, then

$$\langle \psi_k, v_2\rangle = \langle \psi_k, \underline{v}_2\rangle = \mu_k + \sum_{i=1}^{n+2} \nu_i \langle \varphi_i, \psi_k\rangle,$$

which implies that

$$\mu_k = -\sum_{i=1}^{n+2} \nu_i \langle \varphi_i, \psi_k\rangle + O(\Lambda_1\varepsilon_{12}) = \sum_{i=1}^{n+2} O(\nu_i \varepsilon_{12}) + O(\Lambda_1 \varepsilon_{12}). \tag{80}$$

Since $v_2 \in E_1^\perp$ and $\overline{v}_2 \in F^\perp$, then for all $k \in \{1, ..., n+2\}$ we have

$$0 = \langle \varphi_k, v_2\rangle = \sum_{i=1}^{n+2} \nu_i \langle \varphi_i, \varphi_k\rangle + \sum_{i=1}^{n+2} \mu_i \langle \psi_i, \varphi_k\rangle = \nu_k + \sum_{i=1}^{n+2} O(\varepsilon_{12}\mu_i). \tag{81}$$

From (80) and (81), we deduce that for all $k \in \{1, ..., n+2\}$ we have

$$\mu_k = \sum_{j=1}^{n+2} o(\mu_j) + O(\Lambda_1 \varepsilon_{12}). \tag{82}$$

Thus, we obtain the following linear system: $AX = B$ on the variables $\mu_1, ..., \mu_{n+2}$, where $A := (m_{ij})_{1\le i,j\le n+2}$, $m_{ii} = 1 + o(1)$, $m_{ij} = o(1)$ for $i \neq j$, $X := (\mu_1, ..., \mu_{n+2})^t$ and $B := (O(\Lambda_1\varepsilon_{12}), ..., O(\Lambda_1\varepsilon_{12}))^t$. Thus for all $k \in \{1, ..., n+2\}$, we have $\mu_k = O(\Lambda_1\varepsilon_{12})$, by (81), we deduce that $\nu_k = O\left(\Lambda_1\varepsilon_{12}^2\right)$ and our claim follows.

Finally (78) implies that

$$\|\underline{v}_2\|^2 = c\sum_{k=1}^{n+2}\nu_k^2 + c\sum_{k=1}^{n+2}\mu_k^2 + c\sum_{j,k}\nu_j\mu_k\langle\varphi_j, \psi_k\rangle = O\left(\Lambda_1^2\varepsilon_{12}^2\right).$$

The proof of the lemma follows. □

6. Improvement of Propositions 3.3 and 3.4

Since the remaining of Propositions 3.3 and 3.4 are inadequate to our situation, we aim to use, in this section, the results already acquired in order to improve them. These new propositions will serve us, at the end of this section, to show that, for $n = 5, 6$, the point a_2 is not close to the boundary of Ω.

Proposition 6.1. *We assume that $d_1 := d(a_1, \partial\Omega) \ge c > 0$. For $n = 4, 5, 6$ we have*

$$\frac{n-2}{2}\widetilde{c}\frac{H_{22}}{\lambda_2^{n-2}} - \widetilde{c}\left(\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) - \varepsilon\Gamma_2 = o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2}\right) + R,$$

where $\widetilde{c}$ is defined in Lemma 2.14, Γ_2 is defined in Proposition 3.3 and $R = o(\varepsilon_{12})$.

Besides, when $|a_1 - a_2| \ge c > 0$, we can improve the estimate of R and obtain:

$$R = O\left(\varepsilon_{12}^{\frac{2(n-1)}{n}}(1 + \underbrace{\ln\varepsilon_{12}^{-1}}_{if\, n=6}) + \frac{1}{\lambda_1^{\frac{n+2}{2}}\lambda_2^{\frac{n-2}{2}}} + \frac{\ln(\lambda_2 d_2)}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2} + \frac{\varepsilon_{12}d_2}{\lambda_1^{\frac{n-2}{2}}} + \frac{\varepsilon d_2}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right).$$

The proof of Proposition 6.1 requires the introduction of the following lemma whose proof will be presented at the end of this section.

Lemma 6.2. *We assume that $d_1 := d(a_1, \partial\Omega) \ge c > 0$ et $|a_1 - a_2| \ge c > 0$. For $n = 4, 5, 6$ we have:*

$$\int_\Omega \delta_1^\alpha P\delta_2 \le \left(d_2 + \frac{1}{(\lambda_2 d_2)^2}\right)\frac{c}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}, \ \forall\, 1 \le \alpha \le \frac{n+2}{n-2}.$$

Proof of Proposition 6.1. We multiply the equation (1)(a) by $\varphi_2 := \lambda_2\frac{\partial P\delta_2}{\partial\lambda_2}$, and we integrate it on Ω, and by using the fact that v is orthogonal to φ_2, we get:

$$\alpha_1 \langle P\delta_1, \varphi_2\rangle - \alpha_2 \langle P\delta_2, \varphi_2\rangle = \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)\varphi_2 + \varepsilon\alpha_1 \int_\Omega P\delta_1\varphi_2 - \varepsilon\alpha_2 \int_\Omega P\delta_2\varphi_2 + \varepsilon \int_\Omega v\varphi_2. \tag{83}$$

First we introduce the following claims which follow from the maximum principle and some proprieties of the harmonic functions.

Let τ be a positive constant, we have

$$\left|\lambda \frac{\partial P\delta_{a,\lambda}}{\partial\lambda}\right| \leq cP\delta_{a,\lambda} \text{ on } \Omega \quad \text{and} \quad P\delta_{a,\lambda}(x) = O\left(\frac{d}{\lambda^{\frac{n-2}{2}}} + \frac{1}{\lambda^{\frac{n-2}{2}}(\lambda d)^2}\right) \text{ on } \Omega\backslash B_{(a,\tau)} \tag{84}$$

$$\theta_{a,\lambda}(x) = \frac{c_0}{\lambda^{\frac{n-2}{2}}} H(a,x) + O\left(\frac{H(a,x)}{\lambda^{\frac{n+2}{2}} d^2}\right) \quad \text{and} \quad \left|\frac{\partial\theta}{\partial a}\right| \leq \frac{c}{d}\theta \text{ where } \theta_{a,\lambda} := \delta_{a,\lambda} - P\delta_{a,\lambda}. \tag{85}$$

The first integral of the right hand side of (83) is equal to

$$\begin{aligned}
I :&= \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v|^{\frac{4}{n-2}} (\alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v)\varphi_2 \\
&= \int_\Omega (\alpha_1 P\delta_1)^{\frac{n+2}{n-2}} \varphi_2 - \int_\Omega (\alpha_2 P\delta_2)^{\frac{n+2}{n-2}} \varphi_2 + \frac{n+2}{n-2} \int_\Omega \alpha_1 P\delta_1 (\alpha_2 P\delta_2)^{\frac{4}{n-2}} \varphi_2 \\
&\quad + O\left(\int_{\delta_1 \leq \delta_2} \delta_1^2 \delta_2^{\frac{4}{n-2}} + \int_{\delta_2 \leq \delta_1} \delta_1^{\frac{4}{n-2}} \delta_2^2\right) + \frac{n+2}{n-2} \int_\Omega (\alpha_1 P\delta_1)^{\frac{4}{n-2}} v\varphi_2 \\
&\quad + \frac{n+2}{n-2} \int_\Omega (\alpha_2 P\delta_2)^{\frac{4}{n-2}} v\varphi_2 + O\left(\int_{\delta_2 \leq \delta_1} \delta_1^{\frac{6-n}{n-2}} \delta_2^2 |v| + \int_{\delta_1 \leq \delta_2} \delta_1 \delta_2^{\frac{4}{n-2}} |v|\right) \\
&\quad + O\left(\int_\Omega \delta_1^{\frac{6-n}{n-2}} \delta_2 |v|^2 + \int_\Omega \delta_2^{\frac{4}{n-2}} |v|^2 + \int_\Omega |v|^{\frac{n+2}{n-2}} \delta_2\right) \\
:&= I_1 - I_2 + I_3 + O(I_4 + I_5) + I_6 + I_7 + O(I_8 + I_9 + I_{10} + I_{11} + I_{12}).
\end{aligned} \tag{86}$$

For I_1 we have:

$$I_1 = \alpha_1^{\frac{n+2}{n-2}} \langle P\delta_1, \lambda_2 \frac{\partial P\delta_2}{\partial\lambda_2}\rangle + O\left(\int_\Omega \delta_1^{\frac{4}{n-2}} \theta_1 P\delta_2\right). \tag{87}$$

Note that $\|\theta_1\|_\infty \leq c/\lambda_1^{(n-2)/2}$ because $(d_1 \geq c > 0)$ then we get

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$$\int_\Omega \delta_1^{\frac{4}{n-2}}\theta_1 P\delta_2 \le \frac{c}{\lambda_1^{\frac{n-2}{2}}}\int_\Omega \delta_1^{\frac{4}{n-2}}\delta_2 \le \frac{c\varepsilon_{12}}{\lambda_1^{\frac{n-2}{2}}}. \tag{88}$$

Furthermore $|a_1 - a_2| \ge c > 0$, by Lemma 6.2, we can be more precise and then we get:

$$\int_\Omega \delta_1^{\frac{4}{n-2}}\theta_1 P\delta_2 = \|\theta_1\|_\infty \int_\Omega \delta_1^{\frac{4}{n-2}} P\delta_2 \le \frac{c}{\lambda_1^{\frac{n-2}{2}}} d_2\varepsilon_{12} + \frac{c\varepsilon_{12}}{\lambda_1^{\frac{n-2}{2}}(\lambda_2 d_2)^2}. \tag{89}$$

By using Lemmas 2.11 and 2.13 respectively, it is easy to estimate I_2 and I_3.

Lemma 2.2 implies that

$$I_k \le \int_\Omega (\delta_1\delta_2)^{n/(n-2)} = O(\varepsilon_{12}^{\frac{n}{n-2}} \ln \varepsilon_{12}^{-1}), \text{ for } k = 4, 5. \tag{90}$$

Note that $v := v_1 + v_2$ (refer to (66)). Lemmas 2.2, 2.3, Propositions 5.2, 5.4 and 5.6 imply that

$$I_6 \le c\int_\Omega \delta_1^{\frac{4}{n-2}} P\delta_2(|v_1| + |v_2|) \le c\Lambda_1\langle P\delta_1, P\delta_2\rangle + c\|v_2\|\varepsilon_{12}(1 + \underbrace{(\ln \varepsilon_{12}^{-1})^{2/3}}_{\text{if } n=6}). \tag{91}$$

$$\begin{aligned} I_7 &= \frac{n+2}{n-2}\alpha_2^{\frac{4}{n-2}} \int_\Omega \delta_2^{\frac{4}{n-2}} v\lambda_2 \frac{\partial P\delta_2}{\partial\lambda_2} + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}}\theta_2|v|\right) \\ &= O\left(\int_\Omega \delta_2^{\frac{4}{n-2}}|v|\left|\lambda_2\frac{\partial\theta_2}{\partial\lambda_2}\right|\right) + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}}\theta_2|v|\right), \end{aligned}$$

by using the fact that $v \in F^\perp$. Thus,

$$\begin{aligned} I_7 &= o(1)\int_\Omega \delta_1\delta_2^{\frac{4}{n-2}}\left(\theta_2 + \left|\lambda_2\frac{\partial\theta_2}{\partial\lambda_2}\right|\right) + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}}\theta_2|v_2|\right) + O\left(\int_\Omega \delta_2^{\frac{4}{n-2}}\left|\lambda_2\frac{\partial\theta_2}{\partial\lambda_2}\right||v_2|\right) \\ &:= o(J_1) + J_2 + J_3. \end{aligned}$$

Note that, by using Hölder's inequality and (d) of Lemma 2.2, we get

$$\begin{aligned} J_1 &\le \frac{c}{(\lambda_2 d_2)^{\frac{n-2}{2}}}\left(\int_\Omega \left(\delta_1\delta_2^{\frac{4}{n-2}}\right)^{\frac{2n}{n+2}}\right)^{\frac{n+2}{2n}} \le c\varepsilon_{12}^2(1 + \underbrace{(\ln \varepsilon_{12}^{-1})^{2/3}}_{\text{if } n=6}) + \frac{c}{(\lambda_2 d_2)^{n-2}}, \\ J_k &\le c\frac{\|v_2\|}{(\lambda_2 d_2)^{\frac{n-2}{2}}} = O\left(\frac{1}{(\lambda_2 d_2)^{n-1}} + \|v_2\|^{\frac{2(n-1)}{n}}\right), \text{ for } k = 2, 3. \end{aligned}$$

Thus

$$I_7 = o\left(\varepsilon_{12}^2 + \frac{1}{(\lambda_2 d_2)^{n-2}}\right) + O\left(\|v_2\|^{\frac{2(n-1)}{n}}\right). \tag{92}$$

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Concerning the other integrals, and by using Hölder's inequality, Lemmas 2.2, 2.5, Propositions 5.2, 5.4 and 5.6 we deduce that

$$I_8 \le \int_{\delta_2 \le \delta_1} \delta_1^{\frac{6-n}{n-2}} \delta_2^2 |v_1| + \int_{\delta_2 \le \delta_1} \delta_1^{\frac{6-n}{n-2}} \delta_2^2 |v_2| = O\left(\varepsilon_{12}^{\frac{n}{n-2}} \left(\ln \varepsilon_{12}^{-1} \right)^{\frac{n+2}{2n}} + \|v_2\| \varepsilon_{12}^{\frac{2n}{n-2}} \right), \tag{93}$$

$$I_9 \le \int_{\delta_1 \le \delta_2} \delta_1 \delta_2^{\frac{4}{n-2}} |v_1| + \int_{\delta_1 \le \delta_2} \delta_1 \delta_2^{\frac{4}{n-2}} |v_2| = o\left(\varepsilon_{12}^{\frac{n}{n-2}} \left(\ln \varepsilon_{12}^{-1} \right)^{\frac{n+2}{2n}} + \|v_2\|^{\frac{n}{2}} \right), \tag{94}$$

$$I_{10} \le \int_\Omega \delta_1^{\frac{6-n}{n-2}} \delta_2 |v_1|^2 + \int_\Omega \delta_1^{\frac{6-n}{n-2}} \delta_2 |v_2|^2 \le c \Lambda_1^2 \langle P\delta_1, P\delta_2 \rangle + O\left(\|v_2\|^2 \varepsilon_{12}^{1/3} \right), \tag{95}$$

$$I_{11} \le \int_\Omega \delta_2^{\frac{4}{n-2}} |v_1|^2 + \int_\Omega \delta_2^{\frac{4}{n-2}} |v_2|^2 = o\left(\varepsilon_{12}^{\frac{4}{n-2}} \left(\ln \varepsilon_{12}^{-1} (\text{ if } n = 4) \right) + O\left(\|v_2\|^2 \right), \tag{96}$$

$$I_{12} \le \int_\Omega \delta_2 |v_1|^{\frac{n+2}{n-2}} + \int_\Omega \delta_2 |v_2|^{\frac{n+2}{n-2}} \le c \Lambda_1^{\frac{n+2}{n-2}} \langle P\delta_1, P\delta_2 \rangle + O\left(\|v_2\|^{\frac{n+2}{n-2}} \right). \tag{97}$$

(86)–(97) imply that

$$I = \left(\alpha_1^{\frac{n+2}{n-2}} + \alpha_1 \alpha_2^{\frac{4}{n-2}} \right) \langle P\delta_1, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} \rangle - 2\alpha_2^{\frac{n+2}{n-2}} \langle P\delta_2, \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} \rangle + o(1) \langle P\delta_1, P\delta_2 \rangle$$
$$+ o\left(\frac{1}{(\lambda_2 d_2)^{n-2}} + \underbrace{\varepsilon_{12}^{\frac{4}{n-2}} \left(\ln \varepsilon_{12}^{-1} \text{ if } n = 4 \right)}_{\text{if } n=4,5} \right) + O\left(\frac{d_2 \varepsilon_{12}}{\lambda_1^{(n-2)/2}} + \varepsilon_{12}^{\frac{n}{n-2}} \ln \varepsilon_{12}^{-1} + \|v_2\|^{\frac{2(n-1)}{n}} \right). \tag{98}$$

On the other hand we have

$$\varepsilon \alpha_1 \int_\Omega P\delta_1 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} = o(\varepsilon_{12}), \tag{99}$$

$$\varepsilon \int_\Omega v \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} = \varepsilon \int_\Omega v_1 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} + \varepsilon \int_\Omega v_2 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} = o\left(\varepsilon_{12} + \frac{1}{(\lambda_2 d_2)^{n-2}} + \frac{\varepsilon}{\lambda_2^2} \right) \tag{100}$$

and if we have $|a_1 - a_2| \ge c > 0$, then we get:

$$\left| \varepsilon \alpha_1 \int_\Omega P\delta_1 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} \right| \le \left| \int_\Omega \varepsilon \delta_1 P\delta_2 \right| = O\left(\frac{\varepsilon d_2}{(\lambda_1 \lambda_2)^{(n-2)/2}} + \frac{\varepsilon \varepsilon_{12}}{(\lambda_2 d_2)^2} \right), \tag{101}$$

$$\varepsilon \int_\Omega v \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} = \varepsilon \int_\Omega v_1 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2} + \varepsilon \int_\Omega v_2 \lambda_2 \frac{\partial P\delta_2}{\partial \lambda_2}$$
$$= o\left(\frac{\varepsilon d_2}{(\lambda_1 \lambda_2)^{(n-2)/2}} + \frac{\varepsilon \varepsilon_{12}}{(\lambda_2 d_2)^2} + \frac{1}{\lambda_2^{n-2}} + \|v_2\|^2 \right). \tag{102}$$

(98)–(102) imply that:

$$\begin{aligned}&\alpha_1(A_1+A_2-1)\langle P\delta_1,\lambda_2\frac{\partial P\delta_2}{\partial\lambda_2}\rangle+\alpha_2(1-2A_2)\langle P\delta_2,\lambda_2\frac{\partial P\delta_2}{\partial\lambda_2}\rangle-\varepsilon\Gamma_2\\&=o(1)\langle P\delta_1,P\delta_2\rangle+o\left(\frac{1}{(\lambda_2 d_2)^{n-2}}+\varepsilon_{12}^{4/n-2}\left(\ln\varepsilon_{12}^{-1}\text{ if }n=4\right)+\frac{\varepsilon}{\lambda_2^2}\right)\\&+O\left(\frac{d_2\varepsilon_{12}}{\lambda_1^{(n-2)/2}}+\varepsilon_{12}^{\frac{n}{n-2}}\ln\varepsilon_{12}^{-1}+\|v_2\|^{\frac{2(n-1)}{n}}+\frac{\varepsilon d_2}{(\lambda_1\lambda_2)^{(n-2)/2}}\right).\end{aligned}\tag{103}$$

(83)–(103), Lemmas 2.11, 2.14 and Proposition 5.4 complete the proof. □

Proposition 6.3. *Let $n=6$ and $u_\varepsilon=\alpha_1 P\delta_1-\alpha_2 P\delta_2+v$ be a solution of (P_ε) satisfying $|a_1-a_2|\geq c>0$, $d_1:=d(a_1,\partial\Omega)\geq c>0$. Then, we have*

$$\begin{aligned}&\frac{1}{\lambda_2^5}\frac{\partial H}{\partial b}(a_2,a_2)+\frac{1}{\lambda_2(\lambda_1\lambda_2)^2}\frac{\partial G}{\partial b}(a_1,a_2)=O\left(\varepsilon_{12}^{3/2}\ln\varepsilon_{12}^{-1}\right)+o\left(\frac{1}{(\lambda_2 d_2)^5}+\frac{\|v_2\|}{(\lambda_2 d_2)^2}\right)\\&\qquad+o\left(\frac{\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3}}{(\lambda_2 d_2)^2}+\frac{\lambda_2 d_2}{(\lambda_1\lambda_2)^4}+\frac{\varepsilon_{12}}{\lambda_2 d_2}+\|v_2\|^2+\frac{\varepsilon d_2}{\lambda_1^2\lambda_2^3}+\varepsilon\varepsilon_{12}+\frac{\varepsilon\|v_2\|}{\lambda_2^2}\right).\end{aligned}$$

The proof of Proposition 6.3 requires the introduction of the following lemma whose proof will be presented at the end of this section.

Lemma 6.4. *For $n=6$, if $|a_1-a_2|\geq c>0$, $d_1:=d(a_1,\partial\Omega)\geq c>0$. Then we have:*

$$\int_\Omega P\delta_1^2\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}=\langle P\delta_1,\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle+o\left(\frac{1}{\lambda_1^2\lambda_2^3}+\frac{\varepsilon_{12}}{\lambda_1^2(\lambda_2 d_2)}\right).$$

Proof of Proposition 6.3. We multiply the equation (1)(a) by $\psi_2:=\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}$, and we integrate it on Ω, we obtain

$$\begin{aligned}\int_\Omega -\Delta(\alpha_1 P\delta_1-\alpha_2 P\delta_2+v)\psi_2&=\int_\Omega |\alpha_1 P\delta_1-\alpha_2 P\delta_2+v|(\alpha_1 P\delta_1-\alpha_2 P\delta_2+v)\psi_2\\&\quad+\varepsilon\alpha_1\int_\Omega P\delta_1\psi_2-\varepsilon\alpha_2\int_\Omega P\delta_2\psi_2+\varepsilon\int_\Omega v\psi_2.\end{aligned}\tag{104}$$

The integral of the left hand side of (104) is equal to:

$$\alpha_1\langle P\delta_1,\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle-\alpha_2\langle P\delta_2,\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle.\tag{105}$$

Lemmas 6.4, 2.15 and 2.17 imply that the first integral of the right hand side of (104) is equal to

$$I := \int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2|(\alpha_1 P\delta_1 - \alpha_2 P\delta_2)\psi_2 + 2\int_\Omega |\alpha_1 P\delta_1 - \alpha_2 P\delta_2| v\psi_2 + O\left(\int_\Omega |v|^2\psi_2\right)$$
$$:= I_1 + 2I_2 + O(I_3). \tag{106}$$

Let $\Omega_1 := \{x \in \Omega : \alpha_1 P\delta_1(x) \geq \alpha_2 P\delta_2(x)\}$ and $\Omega_2 := \{x \in \Omega : \alpha_2 P\delta_2(x) \geq \alpha_1 P\delta_1(x)\}$, then

$$I_1 = \alpha_1^2 \int_\Omega P\delta_1^2\psi_2 - 2\alpha_1^2 \int_{\Omega_2} P\delta_1^2\psi_2 - \alpha_2^2 \int_\Omega P\delta_2^2\psi_2$$
$$+ 2\alpha_2^2 \int_{\Omega_1} P\delta_2^2\psi_2 + 2\alpha_1\alpha_2 \int_\Omega P\delta_1 P\delta_2\psi_2 - 4\alpha_1\alpha_2 \int_{\Omega_1} P\delta_1 P\delta_2 \psi_2$$

Lemmas 2.15, 2.17 and 6.4 imply that

$$I_1 = \alpha_1^2 \langle P\delta_1, \psi_2\rangle - 2\alpha_2^2 \langle P\delta_2, \psi_2\rangle + \alpha_1\alpha_2 \langle P\delta_1, \psi_2\rangle + O\left(\varepsilon_{12}^{3/2} \ln \varepsilon_{12}^{-1}\right)$$
$$+ o\left(\frac{1}{\lambda_1^2\lambda_2^3} + \frac{\varepsilon_{12}}{\lambda_1^2(\lambda_2 d_2)}\right). \tag{107}$$

The second integral of the right hand side of (106) is equal to

$$I_2 = \int_{\Omega_1} \alpha_1 P\delta_1 v\psi_2 - \int_{\Omega_1} \alpha_2 P\delta_2 v\psi_2 + \int_{\Omega_2} \alpha_2 P\delta_2 v\psi_2 - \int_{\Omega_2} \alpha_1 P\delta_1 v\psi_2$$
$$= \int_\Omega \alpha_2 P\delta_2 v\psi_2 + O\left(\int_\Omega P\delta_1 |v||\psi_2|\right)$$
$$:= J_1 + O(J_2). \tag{108}$$

Note that

$$J_1 = \int_\Omega \alpha_2\delta_2 v \frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2} - \int_\Omega \alpha_2\delta_2 v \frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2} - \int_\Omega \alpha_2\theta_2 v \frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2} + \int_\Omega \alpha_2\theta_2 v \frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}. \tag{109}$$

On the other hand we have

$$\int_\Omega \delta_2 v \frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2} = \frac{1}{2}\langle v, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle = 0. \tag{110}$$

By Lemmas 2.2 and 2.5, since $|v_1| \leq c\delta_1$, we deduce that:

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$$\left|\int_\Omega \delta_2 v \frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right| \le \int_\Omega \delta_2|v_1|\left|\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right| + \int_\Omega \delta_2|v_2|\left|\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right|$$
$$\le \frac{c}{(\lambda_2 d_2)^3}\left(\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3} + \|v_2\|\right), \qquad (111)$$

$$\left|\int_\Omega \theta_2 v \frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}\right| \le c\,\|\theta_2\|_{L^3}\,\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3} + c\,\|\theta_2\|_{L^3}\,\|v_2\|$$
$$\le \frac{c}{(\lambda_2 d_2)^2}\left(\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3} + \|v_2\|\right), \qquad (112)$$

$$\left|\int_\Omega \theta_2 v \frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right| \le c\,\|\theta_2\|_{L^3}\left\|\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right\|_{L^3}\|v\| = o\left(\frac{1}{(\lambda_2 d_2)^5}\right). \qquad (113)$$

(110)–(113) imply that:

$$J_1 = o\left(\frac{1}{(\lambda_2 d_2)^5} + \frac{\|v_2\|}{(\lambda_2 d_2)^2} + \frac{\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3}}{(\lambda_2 d_2)^2}\right). \qquad (114)$$

For J_2, Proposition 5.2 implies that:

$$|J_2| \le c\Lambda_1\int_\Omega \delta_1^2\left|\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}\right| + c\Lambda_1\int_\Omega \delta_1^2\left|\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right| + c\int_\Omega \delta_1\delta_2|v_2|. \qquad (115)$$

Let $B_2 := B_{(a_2,d_2)}$, then we have

$$\Lambda_1\int_\Omega \delta_1^2\left|\frac{1}{\lambda_2}\frac{\partial\delta_2}{\partial a_2}\right| \le \Lambda_1\int_{B_2}\delta_1^2\delta_2\frac{\lambda_2|x-a_2|}{1+\lambda_2^2|x-a_2|^2} + \Lambda_1\int_{\Omega\setminus B_2}\delta_1^2\delta_2\frac{\lambda_2|x-a_2|}{1+\lambda_2^2|x-a_2|^2}$$
$$\le \frac{c\Lambda_1}{\lambda_1^4}\int_{B_2}\frac{\lambda_2^3|x-a_2|}{(1+\lambda_2^2|x-a_2|^2)^3} + \frac{c\Lambda_1}{\lambda_2 d_2}\int_{\Omega\setminus B_2}\delta_1^2\delta_2$$
$$= o\left(\frac{\lambda_2 d_2}{(\lambda_1\lambda_2)^4} + \frac{\varepsilon_{12}}{\lambda_2 d_2}\right), \qquad (116)$$

$$\Lambda_1\int_\Omega \delta_1^2\left|\frac{1}{\lambda_2}\frac{\partial\theta_2}{\partial a_2}\right| \le c\Lambda_1\frac{1}{\lambda_2 d_2}\int_\Omega \delta_1^2\delta_2 = o\left(\frac{\varepsilon_{12}}{\lambda_2 d_2}\right). \qquad (117)$$

By Hölder's inequality we deduce that:

$$\int_\Omega \delta_1\delta_2|v_2| \le c\varepsilon_{12}\left(\ln\varepsilon_{12}^{-1}\right)^{2/3}\|v_2\|. \qquad (118)$$

(116)–(118) prove that:

$$J_2 = o\left(\frac{\lambda_2 d_2}{(\lambda_1\lambda_2)^4} + \frac{\varepsilon_{12}}{\lambda_2 d_2} + \varepsilon_{12}\big(\ln \varepsilon_{12}^{-1}\big)^{2/3}\|v_2\|\right). \tag{119}$$

(114) and (119) imply that:

$$I_2 = o\left(\frac{1}{(\lambda_2 d_2)^5} + \frac{\|v_2\|}{(\lambda_2 d_2)^2} + \frac{\varepsilon_{12}\big(\ln \varepsilon_{12}^{-1}\big)^{2/3}}{(\lambda_2 d_2)^2} + \frac{\lambda_2 d_2}{(\lambda_1\lambda_2)^4} + \frac{\varepsilon_{12}}{\lambda_2 d_2} + \varepsilon_{12}\big(\ln \varepsilon_{12}^{-1}\big)^{2/3}\|v_2\|\right). \tag{120}$$

The third integral of the right hand side of (106) is equal to:

$$I_3 = \int_\Omega |v_1|^2\psi_2 + \int_\Omega |v_2|^2\psi_2 = o\left(\frac{\lambda_2 d_2}{(\lambda_1\lambda_2)^4} + \frac{\varepsilon_{12}}{\lambda_2 d_2} + \|v_2\|^2\right), \tag{121}$$

where we have used Proposition 5.2 and (116)–(117).

This achieves the estimate of I defined in (106).

Arguing as in the proof of (116) and (117) we deduce that

$$\varepsilon\int_\Omega \delta_1 \frac{1}{\lambda_2}\left|\frac{\partial P\delta_2}{\partial a_2}\right| \le \varepsilon\int_\Omega \delta_1 \frac{1}{\lambda_2}\left|\frac{\partial \delta_2}{\partial a_2}\right| + \varepsilon\int_\Omega \delta_1 \frac{1}{\lambda_2}\left|\frac{\partial \theta_2}{\partial a_2}\right| = O\left(\frac{\varepsilon d_2}{\lambda_1^2\lambda_2^3} + \frac{\varepsilon\varepsilon_{12}}{\lambda_2 d_2}\right). \tag{122}$$

Finally, since $|v_1| \le c\delta_1$, (122) and Lemma 2.19 imply that

$$\begin{aligned}\left|\varepsilon\int_\Omega v\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\right| &\le \varepsilon\int_\Omega |v_1|\frac{1}{\lambda_2}\left|\frac{\partial P\delta_2}{\partial a_2}\right| + \varepsilon\int_\Omega |v_2|\frac{1}{\lambda_2}\left|\frac{\partial P\delta_2}{\partial a_2}\right| \\ &= O\left(\varepsilon\varepsilon_{12} + \frac{\varepsilon\|v_2\|}{\lambda_2^2} + \frac{\varepsilon\|v_2\|}{(\lambda_2 d_2)^3}\right).\end{aligned} \tag{123}$$

The result follows from (104)–(105), (120)–(123), Lemmas 2.15, 2.16, 2.18 and the fact $|a_1 - a_2| \ge c$. □

Corollary 6.5. *Let $n = 6$ and $u_\varepsilon = \alpha_1 P\delta_1 - \alpha_2 P\delta_2 + v$ be a solution of (P_ε) satisfying $|a_1 - a_2| \ge c > 0$, $d_1 := d(a_1, \partial\Omega) \ge c > 0$ and $d_2 := d(a_2, \partial\Omega) \to 0$. Then, we have:*

$$\frac{1}{\lambda_2^5}\frac{\partial H}{\partial b}(a_2, a_2) = o\left(\frac{1}{(\lambda_2 d_2)^5} + \frac{1}{\lambda_2(\lambda_1\lambda_2)^2}\right).$$

Proof. The proof is based on the estimates of the variables ε and ε_{12}. For this part, we start by the following claim. For $n = 4, 5, 6$, we have

$$G_{12} \sim c d_2. \tag{124}$$

In fact, it is easy to see that this relation is an evidence if $d_2 \nrightarrow 0$ since $|a_1 - a_2| \ge c > 0$. If $d_2 \to 0$, there exists $t_0 \in (0, 1)$ and $a_0 = t_0 a_2 + (1 - t_0)\overline{a}_2$ where $\overline{a}_2$ denotes the orthogonal projection of a_2 on $\partial\Omega$ such that

$$G(a_1,a_2)=\frac{\partial G}{\partial b}(a_1,a_0)(a_2-\overline{a}_2)=-\frac{\partial G}{\partial b}(a_1,a_0)\cdot \nu_{\overline{a}_2}d_2.$$

Note that, $\frac{\partial G}{\partial b}(a_1,a_0)=\frac{\partial G}{\partial b}(a_1,\overline{a}_2)+O(|a_0-\overline{a}_2|)$, which implies that

$$G_{12}=-\frac{\partial G}{\partial \nu}(a_1,\overline{a}_2)d_2+O(d_2^2).$$

Thus, (124) follows from Hopf Lemma and the fact that $\partial\Omega$ is a compact set and $|a_1-a_2|\geq c$.

Proposition 6.1 for $n=6$ implies that:

$$2\widetilde{c}\frac{H_{22}}{\lambda_2^4}-\widetilde{c}\left(\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2}+2\frac{H_{12}}{(\lambda_1\lambda_2)^2}\right)-C_2\frac{\varepsilon}{\lambda_2^2}=o\left(\frac{1}{(\lambda_2 d_2)^4}+\varepsilon_{12}+\frac{\varepsilon}{\lambda_2^2}\right).$$

Note that $|a_1-a_2|\geq c>0$, then (39), (41) and (42) hold true and therefore

$$\frac{c}{(\lambda_2 d_2)^4}+\frac{cG_{12}}{(\lambda_1\lambda_2)^2}-C_2\frac{\varepsilon}{\lambda_2^2}=o\left(\frac{1}{(\lambda_2 d_2)^4}+\frac{1}{(\lambda_1\lambda_2)^2}+\frac{\varepsilon}{\lambda_2^2}\right). \tag{125}$$

From another part, since $\varepsilon_{12}\ll\lambda_1^{2-n}$, Proposition 3.3 (with $i=1$) implies

$$\varepsilon=\frac{n-2}{2}\frac{\widetilde{c}}{C_2}\frac{H_{11}}{\lambda_1^{n-4}}(1+o(1)). \tag{126}$$

Observe that, for $n=6$, by (124) and (126) if $d_2\to 0$, we get that

$$\frac{G_{12}}{(\lambda_1\lambda_2)^2}=o\left(\frac{\varepsilon}{\lambda_2^2}\right). \tag{127}$$

Thus, (125)–(127) imply that $\frac{1}{(\lambda_2 d_2)^4}=\frac{c}{(\lambda_1\lambda_2)^2}(1+o(1))$, thus $\lambda_1\simeq\lambda_2 d_2^2$.

Finally, by using the estimates of λ_1, ε and ε_{12}, it is easy to prove that the remaining terms in Proposition 6.3 are less than those in the Corollary. □

Proposition 6.6. *For $n=5,6$, we have: $d_2:=d(a_2,\partial\Omega)\nrightarrow 0$.*

Proof. Suppose the contrary, which is $d_2\to 0$. Since $d_1\nrightarrow 0$, then $|a_1-a_2|\geq c>0$.

Observe that for $n=6$, $\partial H/\partial b(a_2,a_2)\simeq c d_2^{1-n}$ and therefore Corollary gives a contradiction.

For $n=5$, Proposition 6.1, (39), (41)–(42) and (126) imply that:

$$\frac{c}{(\lambda_2 d_2)^3}+c'\frac{G_{12}}{(\lambda_1\lambda_2)^{3/2}}=\frac{1}{\lambda_1\lambda_2^2}(1+o(1)),$$

which implies that

$$\frac{1}{(\lambda_2 d_2)^3}\leq\frac{1}{\lambda_1\lambda_2^2}\quad\text{and}\quad\frac{G_{12}}{(\lambda_1\lambda_2)^{3/2}}\leq\frac{1}{(\lambda_1\lambda_2)^2}$$

thus, $\lambda_1\leq\lambda_2 d_2^3$ and $\lambda_2 d_2^2\leq\lambda_1$, hence $\lambda_2 d_2^2\leq\lambda_2 d_2^3$, this is a contradiction because $d_2\to 0$. □

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Proof of Lemma 6.2. Let $\tau = \frac{1}{2}\min\{d_1, |a_1 - a_2|\}$, then we have:

$$\int_\Omega \delta_1^\alpha P\delta_2 = \int_{\Omega\setminus B_{(a_1,\tau)}} \delta_1^\alpha P\delta_2 + \int_{B_{(a_1,\tau)}} \delta_1^\alpha P\delta_2 := I_1 + I_2.$$

Concerning I_1, using (84), we obtain

$$I_1 \le \frac{c}{\lambda_1^{\frac{n-2}{2}\alpha}} \int_{B_2} \delta_2 + \frac{c}{\lambda_1^{\frac{n-2}{2}\alpha}} \int_{\Omega\setminus B_2} P\delta_2 = O\left(\frac{d_2}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2}\right). \tag{128}$$

Concerning I_2, note that $B_{(a_2,\tau)} \subset \Omega\setminus B_{(a_1,\tau)}$ then (84) implies that:

$$I_2 \le \left(\frac{cd_2}{\lambda_2^{\frac{n-2}{2}}} + \frac{c}{\lambda_2^{\frac{n-2}{2}}(\lambda_2 d_2)^2}\right) \int_\Omega \delta_1^\alpha(x)dx = O\left(\frac{d_2}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + \frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}(\lambda_2 d_2)^2}\right), \tag{129}$$

(128) and (129) prove the lemma. □

Proof of Lemma 6.4.

$$\begin{aligned}\int_\Omega P\delta_1^2 \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} &= \int_\Omega (\delta_1 - \theta_1)^2 \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2} \\ &= \langle P\delta_1, \frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\rangle + O\left(\int_\Omega \delta_1\theta_1 \left|\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\right|\right). \end{aligned}\tag{130}$$

Note that, using (85) we have

$$\begin{aligned}\int_\Omega \delta_1\theta_1 \left|\frac{1}{\lambda_2}\frac{\partial P\delta_2}{\partial a_2}\right| \le& \|\theta_1\|_\infty \left(\int_{B_1} \delta_1 \left|\frac{1}{\lambda_2}\frac{\partial \delta_2}{\partial a_2}\right| + \int_{B_1^c} \delta_1 \left|\frac{1}{\lambda_2}\frac{\partial \delta_2}{\partial a_2}\right| + \int_\Omega \delta_1 \left|\frac{1}{\lambda_2 d_2}\theta_2\right|\right) \\ \le& \frac{c}{\lambda_1^2\lambda_2^3}\int_{B_1} \frac{\lambda_1^2}{(1+\lambda_1^2|x-a_1|^2)^2} + \frac{c}{\lambda_1^4}\int_{B_{(a_2,R)}} \frac{\lambda_2^3|x-a_2|}{(1+\lambda_2^2|x-a_2|^2)^3} + \frac{\varepsilon_{12}}{\lambda_1^2(\lambda_2 d_2)} \\ =& o\left(\frac{1}{\lambda_1^2\lambda_2^3} + \frac{\varepsilon_{12}}{\lambda_1^2(\lambda_2 d_2)}\right), \end{aligned}\tag{131}$$

the proof of the lemma follows. □

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7. Proof of the main theorems

The main objective of this section is to show the main theorems. Arguing by contradiction, we assume that there exists a solution u_ε of (P_ε) under the form (34) with $\lambda_1/\lambda_2 \to 0$.

Note that $d_i := d(a_i, \partial\Omega) \geq c > 0$ (according to Lemma 4.4 and Proposition 6.6 for $n = 5, 6$ and according to the hypothesis of Theorem 1.3 for $n = 4$). This piece of information implies that the quantities H_{11} and H_{12} are bounded. We then distinguish three cases and in each one, we end up with a contradiction.

The first case: there exists $M > 0$ such that $\lambda_1|a_1 - a_2| \leq M$. In this case we have:

$$\frac{1}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = o(\varepsilon_{12}) \quad \text{and} \quad \frac{1}{\lambda_2^{n-2}} = o(\varepsilon_{12}). \tag{132}$$

On the other hand, the fact that $\lambda_1|a_1 - a_2|$ is bounded implies that, there exists $c > 0$ such that

$$\varepsilon_{12} \geq c\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n-2}{2}} \qquad \text{and} \qquad \frac{1}{\lambda_2^2}\big(\ln\lambda_2 \text{ if } n = 4\big) = o(\varepsilon_{12}). \tag{133}$$

Note that $\lambda_1 \ll \lambda_2$, then, we have:

$$-\lambda_2\frac{\partial\varepsilon_{12}}{\partial\lambda_2} = \frac{n-2}{2}\varepsilon_{12}\left(1 - 2\frac{\lambda_1}{\lambda_2}\varepsilon_{12}^{2/n-2}\right) = \frac{n-2}{2}\varepsilon_{12}\left(1 + o\left(\varepsilon_{12}^{2/n-2}\right)\right). \tag{134}$$

By injecting this information into Proposition 6.1, we get $\varepsilon_{12} = o(\varepsilon_{12})$, which presents a contradiction by justifying that this case does not appear.

The second case: $|a_1 - a_2| \to 0$ and $\lambda_1|a_1 - a_2| \to \infty$. In this case (133) is satisfied, $G_{12} \to +\infty$ and we have:

$$\varepsilon_{12} = \frac{1 + o(1)}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{\frac{n-2}{2}}}, \quad \frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} = \varepsilon_{12}(1 + o(1)) \quad \text{and}$$
$$-\lambda_1\frac{\partial\varepsilon_{12}}{\partial\lambda_1} = \frac{n-2}{2}\varepsilon_{12}(1 + o(1)). \tag{135}$$

Which implies that:

$$\begin{aligned}-\lambda_i\frac{\partial\varepsilon_{12}}{\partial\lambda_i} + \frac{n-2}{2}\frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} &= \frac{n-2}{2}\left(-\frac{1}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{\frac{n-2}{2}}} + \frac{H_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}}\right) + o(\varepsilon_{12})\\ &= -\frac{n-2}{2}\frac{G_{12}}{(\lambda_1\lambda_2)^{\frac{n-2}{2}}} + o(\varepsilon_{12}).\end{aligned} \tag{136}$$

Proposition 3.3, Proposition 6.1 and (135)–(136) imply that:

$$\begin{cases}\tilde{c}\frac{H_{11}}{\lambda_1^2} + \tilde{c}\frac{G_{12}}{\lambda_1\lambda_2} - \frac{C_2\varepsilon}{\lambda_1^2}\xi_1 = o\left(\frac{1}{\lambda_1^{n-2}} + \frac{\varepsilon}{\lambda_1^2} + \varepsilon_{12}\right) & \text{(a)}\\ \tilde{c}\frac{H_{22}}{\lambda_2^2} + \tilde{c}\frac{G_{12}}{\lambda_1\lambda_2} - \frac{C_2\varepsilon}{\lambda_2^2}\xi_2 = o\left(\frac{1}{\lambda_2^{n-2}} + \frac{\varepsilon}{\lambda_2^2} + \varepsilon_{12}\right), & \text{(b)}\end{cases} \tag{137}$$

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where $\xi_i = \frac{2}{n-2}$ if $n = 5, 6$ and $\xi_i = \ln \lambda_i$ if $n = 4$. (137)(a)–(137)(b) imply that:

$$\varepsilon = \frac{\widetilde{c}}{C_2} \frac{1}{\xi_1} \frac{H_{11}}{\lambda_1^{n-4}}(1 + o(1)) + o\left(\frac{\lambda_1^2 \varepsilon_{12}}{\xi_1}\right). \tag{138}$$

Now, by (138), (132) and (135), (137)(b) becomes:

$$\widetilde{c} \frac{G_{12}}{(\lambda_1\lambda_2)^{(n-2)/2}}(1 + o(1)) = \widetilde{c} \frac{\xi_2}{\xi_1} \frac{H_{11}}{\lambda_1^{n-4}} \frac{1 + o(1)}{\lambda_2^2}. \tag{139}$$

The last equation is equivalent to

$$G_{12} = \frac{\xi_2}{\xi_1} H_{11} \frac{(\lambda_1\lambda_2)^{(n-2)/2}}{\lambda_1^{n-4}\lambda_2^2}(1 + o(1)), \tag{140}$$

which presents a contradiction since $G_{12} \to \infty$ (because $|a_1 - a_2| \to 0$) and the term on the right hand is close to 0 if $n = 4, 5$ and it is bounded for $n = 6$. Thus the second case does not appear.

The third case: $|a_1 - a_2| \geq c > 0$. In this case, we have $0 < c'' \leq G_{12} \leq c'$ and

$$\varepsilon_{12} = \frac{(1 + o(1))}{(\lambda_1\lambda_2|a_1 - a_2|^2)^{\frac{n-2}{2}}} \quad \text{and} \quad -\lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} = \frac{n-2}{2} \varepsilon_{12}(1 + o(1)). \tag{141}$$

Thus, (137)(a) and (137)(b) are satisfied in this case. By (137)(a) we get:

$$\varepsilon = \frac{\widetilde{c}}{C_2} \frac{1}{\xi_1} \frac{H_{11}}{\lambda_1^{n-4}}(c + o(1)). \tag{142}$$

By (142) and (137)(b) we obtain:

$$G_{12} = \frac{\xi_2}{\xi_1} \frac{H_{11}}{\lambda_1^{n-4}} \frac{(\lambda_1\lambda_2)^{(n-2)/2}}{\lambda_2^2}(1 + o(1)). \tag{143}$$

For $n = 4, 5$ (143) presents a contradiction since the term on the right is close to 0 and $G_{12} \geq c > 0$ (since these points a_1 and a_2 are far from the boundary). Thus the third case does not appear for $n = 4, 5$. This complete the proof of Theorems 1.1 and 1.3.

For $n = 6$, Proposition 3.4 ($i = 1$), (141) and (142) imply that

$$\frac{1}{\lambda_1^5} \frac{\partial H}{\partial b}(a_1, a_1) + \frac{2}{\lambda_1} \frac{1}{(\lambda_1\lambda_2)^2} \frac{\partial G}{\partial b}(a_1, a_2) = o\left(\frac{1}{\lambda_1^5}\right), \tag{144}$$

hence, $\dfrac{\partial H}{\partial b}(a_1, a_1) = o(1)$.

So $a_1 \to \overline{y}$ which is fixed, a critical point of the function $R : x \mapsto H(x, x)$.

Proposition 6.3 and (138) imply that: $\dfrac{\partial G}{\partial b}(a_1, a_2) = o(1)$, thus $\dfrac{\partial G}{\partial b}(\overline{y}, a_2) = o(1)$, hence $a_2 \to \overline{z}$ which is fixed, a critical point of the function $x \mapsto G(\overline{y}, x)$.

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(143) implies that:

$$G_{12}(1+o(1)) = H_{11}(1+o(1)),$$

note that $a_1 \to \overline{y}$ and $a_2 \to \overline{z}$, then we deduce that $G(\overline{y}, \overline{z}) = H(\overline{y}, \overline{y})$. This equality contradicts the hypothesis introduced in Theorem 1.2. Thus the third case does not appear. Which completes the proof of Theorem 1.2. □

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