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Wong–Zakai approximation for the stochastic Landau–Lifshitz–Gilbert equations

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Abstract

In this work we study stochastic Landau–Lifshitz–Gilbert equations (SLLGEs) in one dimension, with non-zero exchange energy only. Firstly, by introducing a suitable transformation, we convert the SLLGEs to a highly nonlinear time dependent partial differential equation with random coefficients, which is not fully parabolic. We then prove that there exists a pathwise unique solution to this equation and that this solution enjoys the maximal regularity property. Following regular approximation of the Brownian motion and using reverse transformation, we show existence of strong solution of SLLGEs taking values in a two-dimensional unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. The construction of the solution and its corresponding convergence results are based on Wong–Zakai approximation.

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Keywords: Stochastic Landau–Lifshitz–Gilbert equations; Wong–Zakai approximation; Maximal regularity; Ferromagnetism

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1. Introduction

The study of the theory of magnetic behaviour in ferromagnetic materials was initiated by Weiss [65], see also [11], and references therein, and further developed by Landau and Lifshitz [47] and Gilbert [35]. This theory suggests that the magnetization $\mathbf{M}$ of a ferromagnetic material occupying an open bounded region $D \subset \mathbb{R}^d$, $d = 1, 2, 3$ at temperatures below the critical, the so-called Curie temperature satisfies, for $t > 0$ and $x \in D$, the following Landau–Lifshitz–Gilbert equations (LLGEs):

$$\frac{\partial \mathbf{M}}{\partial t}(t,x) = \lambda_1 \mathbf{M}(t,x) \times H(t,x) - \lambda_2 \mathbf{M}(t,x) \times (\mathbf{M}(t,x) \times H(t,x)), \tag{1.1}$$

where $\times$ is the vector cross product in $\mathbb{R}^3$ and H is the so-called effective field, which is the negative of the gradient (with respect to $\mathbf{M}$) of the total magnetic energy functional $\mathcal{E}$, which is the sum of the anisotropy energy, exchange energy and electronic energy, see Visintin [63]. There are evidences that a large part of the mathematical difficulty of the problem seems to stem from the exchange energy. In such situation, when the energy functional consists of the exchange energy only, $\mathcal{E} = \frac{1}{2}\int_D |\nabla \mathbf{M}(x)|^2\,dx$, we have $H = \Delta \mathbf{M}$ and we obtain the following version of the LLGEs:

$$\begin{cases} \frac{\partial \mathbf{M}}{\partial t}(t,x) = \lambda_1 \mathbf{M}(t,x) \times \Delta \mathbf{M}(t,x) - \lambda_2 \mathbf{M}(t,x) \times (\mathbf{M}(t,x) \times \Delta \mathbf{M}(t,x)), & t > 0, x \in D, \\ \frac{\partial \mathbf{M}}{\partial n}(t,x) = 0, & t > 0, x \in \partial D, \\ \mathbf{M}(0,x) = \mathbf{M}_0(x), & x \in D. \end{cases} \tag{1.2}$$

Here $\mathbf{M} : [0,T] \times D \to \mathbb{S}^2$ denote the magnetism of a ferromagnetic material, where $\mathbb{S}^2$ represents the two dimensional unit sphere in $\mathbb{R}^3$, with the assumption that the material is saturated at the initial time, i.e.

$$|\mathbf{M}_0(x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D. \tag{1.3}$$

The parameters $\lambda_1 \neq 0, \lambda_2 > 0$ are constants. n is the outer unit normal vector at the boundary ∂D.

1.1. Stochastic Landau–Lifshitz–Gilbert equations

It is known in the literature that the stationary solutions of equations (1.2) corresponding to the equilibrium states of the ferromagnet are not unique in general, and depend upon the dimensionality of the domain. An important physical question in the theory of ferromagnetism is the description of phase transitions between different equilibrium states induced by thermal fluctuations of the effective field H. In order to understand this, the LLGEs need to be suitably modified in order to incorporate random fluctuations of the field H into the dynamics of the magnetization $\mathbf{M}$ and to describe noise-induced transitions between equilibrium states of the ferromagnet. A simple way to incorporate the noise into the LLGEs, see e.g. Brzeźniak et al. [18] is to perturb the effective field by a Gaussian noise, that is to replace H in (1.1) (or $\Delta \mathbf{M}$ in (1.2)) by $H + \eta$ (resp. by $\Delta \mathbf{M} + \eta$), where informally η is a space-time white noise. It is noteworthy, see for e.g. [45], [11], [34], [18] etc., that the main technical issue rests in the fact that the noise must preserve the invariance property under coordinate transformation and this plays an important role in preserving the non-convex constraint condition (1.3). Moreover, following [46] and [34], assumption on the smallness of λ_2 in physical problems justifies the negligance of noise in the second term on the right hand side of (1.2). Thus the stochastic version of (1.2) become

$$\begin{cases} d\mathbf{M}(t) = (\lambda_1 \mathbf{M}(t) \times \Delta\mathbf{M}(t) - \lambda_2 \mathbf{M}(t) \times (\mathbf{M}(t) \times \Delta\mathbf{M}(t)))\, dt + (\mathbf{M}(t) \times g) \circ dW(t), \\ \frac{\partial \mathbf{M}}{\partial n}(t,x) = 0, \text{ on } (0,\infty) \times \partial D, \\ \mathbf{M}(0,x) = \mathbf{M}_0(x), \text{ on } D, \end{cases} \tag{1.4}$$

where $\circ\, dW(t)$ stands for the Stratonovich differential, W is a real-valued Wiener process, $g : D \to \mathbb{R}^3$ is a given function with certain regularity.

If both the exchange and anisotropy energies are present, the total energy $\mathcal{E}$ of the LLGEs takes the form

$$\mathcal{E}(\mathbf{M}) = \mathcal{E}_{an}(\mathbf{M}) + \mathcal{E}_{ex}(\mathbf{M}) = \int_D \Big(\psi(\mathbf{M}(x)) + \frac{1}{2}|\nabla \mathbf{M}(x)|^2\Big) dx$$

where $\mathcal{E}_{an}(\mathbf{M}) := \int_D \psi(\mathbf{M}(x))dx$ stands for the anisotropy energy and $\mathcal{E}_{ex}(\mathbf{M}) := \int_D |\nabla \mathbf{M}(x)|^2 dx$ stands for the exchange energy, the effective field H takes the form $\Delta\mathbf{M} - \nabla\psi(\mathbf{M})$.

Brzeźniak et al. in [18] have introduced the Gaussian noise with $\psi = 0$ into three dimensional bounded domain for LLGEs in the Stratonovich sense, and proved existence of weak martingale solutions taking values in a sphere $\mathbb{S}^2$. In recent years, Brzeźniak and Li in [20] have generalized the results with non-zero anisotropy energy and multidimensional noise. Finite dimensional analysis of this kind has been studied in [46], [48]. In [17], Brzeźniak et al. have considered the one-dimensional situation and prove the large deviations principle for small noise asymptotic of solutions to the SLLGEs, and the noise by $\sqrt{\varepsilon} dW(t)$. The main ingredients of their proof are the pathwise uniqueness in one dimension, maximal regularity property of the solution and weak convergence techniques. In this context it is worth mentioning that the first two authors of this paper, in [21–23], have initiated studies on phase transition between different equilibrium states under the effect of random fluctuations of Lévy or jump type in the Marcus canonical form. To be little more precise, they have proved existence of weak martingale solution for SLLGEs in three dimensions perturbed by jump noise in the Marcus canonical form with non-zero exchange energy only, see [21], and with non-zero anisotropy energy, see [22] (see also [19] and [24] for related works by the authors), whereas, in [23], the authors have established large deviations principle for small noise asymptotic of solutions for a one-dimensional problem.

In this context, we now mention certain significant numerical studies of SLLGEs. [7] proposes a convergent finite element approximation to prove the long-time dynamics of both the finite and the infinite ensembles of ferromagnetic spins with space time white noise. [8] constructs a fully discrete finite-element-based discretization of SLLGEs whose solutions approximate weak martingale solutions of SLLGEs for vanishing discretization parameters, see [9] also. The authors of the paper [36] have reformulated SLLGEs into a partial differential equation (PDE), but without Itô terms, to obtain time-differentiable solutions. They have employed the θ-linear scheme for the numerical solution of the transformed equation. In our present paper, we have adopted some techniques from [36] to obtain the solvability of the time dependent transformed PDEs.

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Another linear finite element scheme to solve the SLLGEs has been proposed in [2] to announce a convergent time semi-discrete scheme to work with the Itô form. The last paper differs from all previous papers as it deals with the LLGEs in the so called Gilbert form, see [35] and [3] for some related deterministic results, and with an infinite dimensional noise (correlated in space).

1.2. Problem description

For the sake of reader's convenience, let us fix some notations before we describe the problem. For a domain D, we will use the notation $\mathbb{L}^p$ for the space $L^p(D;\mathbb{R}^3)$, $\mathbb{W}^{m,p}$ for the Sobolev space $W^{m,p}(D;\mathbb{R}^3)$, and so on. We will often write $\mathbb{H}^m$ instead of $\mathbb{W}^{m,2}$.

In this paper we consider the case $d=1$ and we assume that D to be a bounded open interval in $\mathbb{R}$. In particular, we take $D=(0,1)$. The main reason for considering a one-dimensional problem is the availability of pathwise uniqueness and maximal regularity properties, which seem to be absent in multidimensional domain (see Subsection 1.5). The SLLGEs in consideration in this paper is of the form

$$d\mathbf{M} = \Big(\lambda_1 \mathbf{M}\times\mathbf{M}_{xx} - \lambda_2\mathbf{M}\times(\mathbf{M}\times\mathbf{M}_{xx})\Big)dt + (\mathbf{M}\times g)\circ dW(t), \quad \text{in} \quad (0,T)\times D, \tag{1.5}$$

$$\mathbf{M}_x(t,0) = 0 = \mathbf{M}_x(t,1), \quad \forall t\in(0,T), \tag{1.6}$$

$$\mathbf{M}(0,x) = \mathbf{M}_0(x), \quad \forall x\in D, \tag{1.7}$$

where $g: D\to\mathbb{R}^3$ is a given function such that $g\in\mathbb{W}^{2,\infty}$, $T>0$ is fixed and $W(t)$, $t\in[0,T]$ is the standard real-valued Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\mathcal{F}_t = \sigma\{W(s), s\in[0,t]\}$ and $\circ\, dW(t)$ stands for the Stratonovich differential.

One of the most fundamental questions related to problems similar to the above is the question about robustness, i.e., whether the solutions depend in a continuous way of the coefficients (the Wiener process in our case). Let us describe our approach to this question, see also [13], [27] and others.

Let $W^n(t)$, $t\in[0,T]$, $n\in\mathbb{N}$ be a sequence of pathwise continuously differentiable stochastic processes on $(\Omega,\mathcal{F},\mathbb{P})$ such that

$$W^n(t,\omega)\to W(t,\omega) \quad \text{uniformly in } t\in[0,T], \quad \text{a.e. on } \Omega. \tag{1.8}$$

Let $\mathcal{F}_t^n = \sigma\{W^n(s): s\in[0,t]\}$.

Let us now consider the following stochastic system:

$$d\mathbf{M}_n = \Big(\lambda_1\mathbf{M}_n\times(\mathbf{M}_n)_{xx} - \lambda_2\mathbf{M}_n\times\Big(\mathbf{M}_n\times(\mathbf{M}_n)_{xx}\Big)\Big)dt + (\mathbf{M}_n\times g)dW_n(t), \text{ in } (0,T)\times D, \tag{1.9}$$

$$(\mathbf{M}_n)_x(t,0) = 0 = (\mathbf{M}_n)_x(t,1) \quad \forall t\in(0,T), \tag{1.10}$$

$$\mathbf{M}_n(0,x) = \mathbf{M}_0(x), \quad \forall x\in D. \tag{1.11}$$

Our goal is to prove (1.5)–(1.7) has a unique strong solution in some suitable space which is a limit as $n\to\infty$ of the solutions of sequence of an approximated system (1.9)–(1.11). Convergence results of this kind are well known in the literature, see [13] and references therein. Although the motivation of [13] differs from the present paper as former one deals with the linear

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equations. We use ideas from [17] and [36] to prove that the time dependent PDEs (1.13)–(1.15) are solvable and the solutions depend continuously with respect to the approximation. The technique is based on Wong–Zakai approximation. It is worth mentioning that the higher order non-linear terms and complicated non-linearity structure of F in the transformed equation (1.13) are hard to tackle.

1.3. Wong–Zakai approximation for related problems

Let us mention here some of the growing literature rationalized to the establishment of Wong–Zakai approximation of stochastic evolution type equations. Starting from the celebrated work of Wong and Zakai [66], which involved piecewise linear approximations of one-dimensional Wiener process, Clark in his Phd thesis [25], see also [26], McShane in [54], Stroock–Varadhan in [60] had expanded the area by dealing with multi-dimensional Wiener process. For a quick survey, see Doss [31], Malliavin [52,53], Sussman [59], Ikeda–Watanabe [44], Elworthy [33], Moulinier [55], Bismut [10], to name a few. Since then the result has been generalised in many directions and in varied areas. One should also mention the papers of Brzeźniak and Carroll [14], Brzeźniak, Capinski and Flandoli [13], Brzeźniak and Flandoli [16], Gyöngy [37], Gyöngy and Pröhle [38], Gyöngy and Shmatkov [39], Nowak [56], Tessitore and Zabczyk [64], Hausenblas [40], Dawidowicz and Twardowska [28].

On a related note, motivated by the SLLGEs, Hocquet [41] studied the well-posedness of the two-dimensional Stochastic Harmonic Map flow (similar to the case when $\lambda_1 = 0$ in LLGEs) and proved a stochastic counterpart of the so-called "Struwe solutions" [61] of the deterministic model. On another related note, the rough paths theory (see for e.g. the seminal paper by Lyons [50] and the recent monograph by Lyons et al. [51]) is intimately related to Wong–Zakai results as it essentially allows to construct solutions as limits of Wong–Zakai type approximations. In this connection, a few recent results, e.g. Bailleul and Gubinelli [6], Deya, Gubinelli, Hofmanová and Tindel [29], Hocquet, Hofmanová [42], Hofmanová, Leahy and Nilssen [43], on various rough PDEs and construction of their weak solutions using energy methods (similarly to the paper under consideration) are worth mentioning. See Section 1.5 for further discussion.

1.4. Contribution of the paper and the main ideas

By introducing a suitable transformation, we define a new process $\mathbf{m}$ from $\mathbf{M}$ by:

$$\mathbf{m}(t,x) = e^{-W(t)G}\mathbf{M}(t,x), \quad \forall t \in [0,T], \text{ a.e. } x \in D, \tag{1.12}$$

where $G : \mathbb{L}^2 \to \mathbb{L}^2$ is bounded linear map which is also skew symmetric (see Lemma 2.2). Exploiting properties of the linear map G, we observe that (see Goldys et al. [36] for the derivation), the SLLGEs (1.5)–(1.7) can be converted to highly non-linear time dependent partial differential equation (PDE) (called often the robust equation) with random coefficients (but without Itô terms):

$$\frac{\partial \mathbf{m}}{\partial t} = \lambda_1(\mathbf{m}\times\mathbf{m}_{xx}) - \lambda_2\mathbf{m}\times(\mathbf{m}\times\mathbf{m}_{xx}) + F(t,\mathbf{m}), \quad \text{in} \quad (0,T)\times D, \tag{1.13}$$

$$\mathbf{m}_x(t,0) = 0 = \mathbf{m}_x(t,1), \quad \forall t \in (0,T), \tag{1.14}$$

$$\mathbf{m}(0,x) = \mathbf{m}_0(x), \quad \forall x \in D, \tag{1.15}$$

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where

$$F(t,\mathbf{m}) = \lambda_1 \mathbf{m} \times \tilde{C}(W(t), \mathbf{m}(t,\cdot)) - \lambda_2 \mathbf{m} \times \Big(\mathbf{m} \times \tilde{C}(W(t), \mathbf{m}(t,\cdot))\Big), \tag{1.16}$$

$$\tilde{C}(W(t), \mathbf{m}(t,\cdot)) = e^{-W(t)G}\Big(\sin(W(t))C\mathbf{m} + \Big[1-\cos(W(t))\Big](GC\mathbf{m} + CG\mathbf{m})\Big), \tag{1.17}$$

$$C\mathbf{m} = \mathbf{m} \times g_{xx} + 2\mathbf{m}_x \times g_x, \tag{1.18}$$

$$\text{and} \quad G\mathbf{m} = \mathbf{m} \times g. \tag{1.19}$$

By introducing a regular approximation of the Brownian Motion, see (1.8), it is natural to consider a family of time dependent equations (1.9)–(1.11) containing the time derivative of these approximations. These equations are, by means of a similar exponential transformation (1.12), converted into equations (2.15)–(2.17), which turn out to be, as one would usually expect, approximations of the PDEs (1.13)–(1.15). However, this verification is not straightforward and involves technicality and hard work. It is worth to emphasise here that this work is the first result of this kind for PDEs with constraints, and we are hopeful that similar ideas may be borrowed to other constraint PDEs.

Using the Faedo–Galerkin approximation, energy and compactness methods, the existence of solutions $\mathbf{m}^n$ and $\mathbf{m}$ for the approximated system (i.e., (2.15)–(2.17)) and the transformed system (i.e. (1.13)–(1.15)) have been proved respectively in Theorem 3.2 and Theorem 4.1. Pathwise uniqueness of the corresponding solutions follow from Theorem 3.16. Moreover, these evolution PDEs (1.13)–(1.15) have solutions which are differentiable with respect to time variable. We then prove in Theorem 5.1, that the solution of (2.15)–(2.17) converge to the solution of (1.13)–(1.15) in the natural topology of $\mathrm{L}^\infty(0,T;\mathbb{L}^2)$. Taking into account the ultra-contractivity property of the heat semigroup, i.e. the semigroup generated by $\mathrm{A} := -\dfrac{d^2}{dx^2}$ in the space $\mathbb{L}^2$ and owing to the special structure of the equation, we see in Theorem 6.1, that both the solutions $\mathbf{m}^n$ and $\mathbf{m}$ have the so called maximal regularity, i.e. belong to the space $\mathrm{L}^\infty(0,T;\mathbb{H}^1) \cap \mathrm{L}^2(0,T;D(\mathrm{A}))$. Furthermore, in Theorem 6.4, the convergence of $\mathbf{m}^n$ to $\mathbf{m}$ have been established in the maximal regularity space. In this respect, we have used some ideas from a paper [17] of the first named author about the Large Deviation Principle (LDP) for the SLLGEs confirming thus a belief that there is a deeper relationship between the LDP and the Wong–Zakai approximation.

In Appendix C we present some comments about our proof and possible, not yet verified, links with the theory of quasilinear parabolic equations.

Referring here to that discussion of a possible use of the quasilinear structure of equation (C.3), we just say that it might be possible to prove Theorem 6.4 exploiting the maximal regularity and the principle of continuous dependence of solutions on coefficients, see e.g. [12]. But even in the linear case studied in [13], where, the variational theory of evolution equations and semigroup theory are used to study the robust equation (1.13), this has not been so easy to achieve, see also [27]. We hope that this matter will be further investigated.

Finally, again using the reverse transformation,

$$\mathbf{M}(t,x) = e^{W(t)G}\mathbf{m}(t,x), \quad \mathbf{M}_n(t,x) = e^{W^n(t)G}\mathbf{m}^n(t,x), \quad \forall t \in [0,T], \text{ a.e. } x \in D,$$

one can observe that the SLLGEs and its corresponding approximated system have unique solutions $\mathbf{M}$ and $\mathbf{M}_n$ in the maximal regular space and in addition, the convergence of $\mathbf{M}_n$ to $\mathbf{M}$ has

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been proved in the maximal regular space. This is the main result of this paper, which has been proved in Theorem 7.1.

We now briefly describe the content of the paper. We start with Section 2, containing some auxiliary facts that will be useful in the later course of analysis. In addition, by introducing a suitable transformation, we convert the SLLGEs to a highly nonlinear time dependent partial differential equation with random coefficients. In Section 3, we prove the existence of a pathwise unique solution stated in Theorem 3.2 and Theorem 3.16. We devote Section 4 to state the corresponding existence and uniqueness theorem for the approximated system. In Section 5, we first prove the convergence result in the space $L^\infty(0,T;\mathbb{L}^2)$ and then obtain the progressively measurability of both the approximated and limiting processes with respect to the corresponding filtrations. In Section 6, we present Theorem 6.1 and Theorem 6.4 which contains the proof of solution in maximal regular space and the convergence result in this space. We return to the study of SLLGEs and prove existence of strong solution and its corresponding convergence result in the maximal regular space in Section 7. Finally, we have listed a basic result and some vector algebraic identities in the Appendix.

1.5. Remarks and open questions

(1) We believe that the results from this paper can be generalised to the case when the noise coefficient in equation (1.6) is not linear with respect to **M**. For instance, if the smallness of λ_2 is not assumed, then the noise term takes the following form

$$\Big[\mathbf{M}\times(\mathbf{M}\times g_1)+\mathbf{M}\times g_2\Big]\circ dW(t).$$

It is straightforward to see that transformation like (1.12) will not work for noises with nonlinear coefficient. Therefore the difficulty lies in identifying a suitable transformation for this kind of noise, which is unknown till date. We believe that following the work [59] by Sussman, this may be achieved for real-valued Brownian Motion, and this work is under investigation by the authors.

(2) Is it possible to extend our technique, which is based on Wong–Zakai approximation, to a more general setting where transformation (1.12) is not applicable, i.e. when the Wiener process is no longer one-dimensional (and nor the corresponding vector fields commute)? For instance, when the noise term is of the form

$$\mathbf{M}\times(\mathbf{M}\times g_1)\circ dW_1(t)+(\mathbf{M}\times g_2)\circ dW_2(t)$$

where W_1 and W_2 are two independent Wiener processes.

(3) Can the rough path theory of Terry Lyons be of help with respect to the two previous questions?

(4) Is the Wong–Zakai theorem true also for stochastic LLGEs in multidimensional domains where there is, at least so far, no uniqueness result?

(5) Is the Wong–Zakai theorem true also for stochastic LLGEs (in the Marcus form) driven by Lévy processes?

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2. The auxiliary equations

In this section, we provide some basic results on the operators and the spaces required and in the next subsection, following [36] we define new processes $\mathbf{m}$ and $\mathbf{m}^n$ using the operator G and the regular approximation of the Brownian motion respectively.

2.1. Preliminaries

We define the Laplacian with the Neumann boundary conditions by

$$\begin{cases} D(\mathrm{A}) & := \{\mathbf{m} \in \mathbb{H}^2 : \mathbf{m}_x(0) = \mathbf{m}_x(1) = 0\}, \\ \mathrm{A}\mathbf{m} & := -\Delta \mathbf{m} = -\mathbf{m}_{xx}, \quad \mathbf{m} \in D(\mathrm{A}). \end{cases} \tag{2.1}$$

We note that the operator A is self-adjoint and nonnegative in $\mathbb{L}^2$. Define $\mathrm{A}_1 := I + \mathrm{A}$. We note that $\mathrm{V} := Dom(\mathrm{A}_1^{1/2})$ when endowed with the graph norm coincides with $\mathbb{H}^1$. Also the operator A_1^{-1} is compact. Later on we will use that $\mathrm{V} \hookrightarrow \mathbb{L}^2 \hookrightarrow \mathrm{V}'$ is a Gelfand triple.

For any real number $\beta \geq 0$, we write X^β for the domain of the fractional power operator $D(\mathrm{A}_1^\beta)$ endowed with the norm $|x|_{X^\beta} := |\mathrm{A}_1^\beta x|$ and $X^{-\beta}$ denotes the dual space of X^β so that $X^\beta \subset \mathbb{L}^2 \subset X^{-\beta}$ is a Gelfand triple. Note that for $\beta \in [0, \frac{3}{4})$,

$$X^\beta = \mathbb{H}^{2\beta}.$$

Now we specify the following, well known, interpolation inequality which will be useful in the course of analysis in later subsections.

$$|\mathbf{u}|^2_{\mathbb{L}^\infty} \leq k^2 |\mathbf{u}|_{\mathbb{L}^2} |\mathbf{u}|_{\mathbb{H}^1}, \ \forall \mathbf{u} \in \mathbb{H}^1, \tag{2.2}$$

where the optimal value of the constant k is

$$k = 2 \max\left(1, \frac{1}{\sqrt{|D|}}\right).$$

We recall a simple result from [17].

Lemma 2.1. *Let* $\mathbf{u}$ *be any element of* $\mathbb{H}^1$ *such that*

$$|\mathbf{u}(x)|_{\mathbb{R}^3} = 1 \quad \forall\, x \in D.$$

Then in $(\mathbb{H}^1)'$, *we have*

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{u}_{xx}) = -|\mathbf{u}_x|^2_{\mathbb{R}^3} \mathbf{u} - \mathbf{u}_{xx}. \tag{2.3}$$

Lemma 2.2. *Assume that* $g \in \mathbb{L}^\infty$. *Let* $G : \mathbb{L}^2 \to \mathbb{L}^2$ *be a map defined by*

$$[G\mathbf{u}](x) = \mathbf{u}(x) \times g(x), \quad \forall\, \mathbf{u} \in \mathbb{L}^2, \ x \in D.$$

Then G *is a well defined and bounded linear map. Further,* $G^* = -G$.

For proof see Lemma 3.1 of Goldys et al. [36].

2.2. New processes $\mathbf{m}$ and $\mathbf{m}^n$

Following the discussion in Section 1.4, we recall that by using the operator G, we can define a new process $\mathbf{m}$ from $\mathbf{M}$ by:

$$\mathbf{m}(t,x) = e^{-W(t)G}\mathbf{M}(t,x), \quad \forall t \in [0,T], \text{ a.e. } x \in D.$$

Let $g \in \mathbb{W}^{2,\infty}$. Using the transformation (1.12), we observe that if $\mathbf{M}$ is a solution of (1.5)–(1.7), then $\mathbf{m}$ is a solution of (1.13)–(1.15) and vice versa. Note that the condition (1.3) about the initial data $\mathbf{M}_0$ is equivalent to an analogous one for $\mathbf{m}_0$, i.e.

$$|\mathbf{m}_0(x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D. \tag{2.4}$$

Note that, one could have also considered the transformation (1.12) for all $t \geq 0$, rather than $t \in [0,T]$, with a fixed $T > 0$. Moreover, as one can easily prove $|[e^{tG}(u)](x)|_{\mathbb{R}^3} = |u(x)|_{\mathbb{R}^3}$ for a.a. $x \in D$, for all $t \in [0,T]$, we see that the following saturation conditions for $\mathbf{M}$ and $\mathbf{m}$ are equivalent:

$$\begin{aligned} |\mathbf{M}(t,x)|_{\mathbb{R}^3} = 1 \quad &\text{for a.e. } x \in D, \text{ for all } t \in [0,T], \\ |\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1 \quad &\text{for a.e. } x \in D, \text{ for all } t \in [0,T]. \end{aligned}$$

It is worth mentioning here that the equivalence of the (weak) solutions to the original and the transformed system has been proved in Section 4 of [36]. In fact, this idea has been exploited later in our main result, Theorem 7.1.

For convenience of the reader, we repeat below some of the basic algebraic calculations from [36]. Using (1.18)–(1.19) we have

$$\begin{aligned} GC\mathbf{m} + CG\mathbf{m} &= (C\mathbf{m} \times g) + (G\mathbf{m} \times g_{xx}) + (G\mathbf{m})_x \times g_x \\ &= (\mathbf{m} \times g_{xx}) \times g + 2(\mathbf{m}_x \times g_x) \times g + (\mathbf{m} \times g) \times g_{xx} + 2(\mathbf{m}_x \times g) \times g_x \\ &\quad + (\mathbf{m} \times g) \times g_{xx} + 2(\mathbf{m}_x \times g) \times g_x + 2(\mathbf{m} \times g_x) \times g_x \\ &= (\mathbf{m} \times g_{xx}) \times g + (\mathbf{m} \times g) \times g_{xx} + 2\Big((\mathbf{m}_x \times g_x) \times g + (\mathbf{m}_x \times g) \times g_x \\ &\quad + (\mathbf{m} \times g_x) \times g_x\Big). \end{aligned} \tag{2.5}$$

Hence, substituting (2.5) in (1.17), (1.16) becomes:

$$\begin{aligned} F(t,\mathbf{m}) = \lambda_1 \mathbf{m} \times e^{-W(t)G}\Big\{ &\sin(W(t))\Big(\mathbf{m} \times g_{xx} + 2\mathbf{m}_x \times g_x\Big) \\ &+ [1-\cos(W(t))]\Big((\mathbf{m} \times g_{xx}) \times g + (\mathbf{m} \times g) \times g_{xx} + 2\Big((\mathbf{m}_x \times g_x) \times g \\ &+ (\mathbf{m}_x \times g) \times g_x + (\mathbf{m} \times g_x) \times g_x\Big)\Big)\Big\} \end{aligned}$$

$$
\begin{aligned}
&- \lambda_2 \mathbf{m} \times \Big(\mathbf{m} \times e^{-W(t)G} \Big\{ \sin(W(t)) \Big(\mathbf{m} \times g_{xx} + 2\mathbf{m}_x \times g_x \Big) \\
&+ [1 - \cos(W(t))] \Big((\mathbf{m} \times g_{xx}) \times g + (\mathbf{m} \times g) \times g_{xx} + 2 \Big((\mathbf{m}_x \times g_x) \times g \\
&+ (\mathbf{m}_x \times g) \times g_x + (\mathbf{m} \times g_x) \times g_x \Big) \Big) \Big\} \Big). \qquad (2.6)
\end{aligned}
$$

Thus in simplified form, we can rewrite F as:

$$F(t, \mathbf{m}) = \lambda_1 \mathbf{m} \times \hat{F}(t, \mathbf{m}) - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \hat{F}(t, \mathbf{m})), \qquad (2.7)$$

where

$$
\begin{aligned}
\hat{F}(t, \mathbf{m}) = {} & e^{-W(t)G} \sin(W(t)) \Big(\mathbf{m} \times g_{xx} + 2\mathbf{m}_x \times g_x \Big) \\
& + e^{-W(t)G} [1 - \cos(W(t))] \Big((\mathbf{m} \times g_{xx}) \times g + (\mathbf{m} \times g) \times g_{xx} + 2 \Big((\mathbf{m}_x \times g_x) \\
& \times g + (\mathbf{m}_x \times g) \times g_x + (\mathbf{m} \times g_x) \times g_x \Big) \Big) \\
:= {} & S(W)\mathfrak{S}(\mathbf{m}) + \mathcal{C}(W)\mathfrak{C}(\mathbf{m}), \qquad (2.8)
\end{aligned}
$$

and

$$
\begin{aligned}
S(W) &:= e^{-W(t)G} \sin(W(t)), && (2.9) \\
\mathfrak{S}(\mathbf{m}) &:= \mathbf{m} \times g_{xx} + 2\mathbf{m}_x \times g_x, && (2.10) \\
\mathcal{C}(W) &:= e^{-W(t)G} [1 - \cos(W(t))], && (2.11) \\
\mathfrak{C}(\mathbf{m}) &:= (\mathbf{m} \times g_{xx}) \times g + (\mathbf{m} \times g) \times g_{xx} + 2 \Big((\mathbf{m}_x \times g_x) \times g \\
&\qquad + (\mathbf{m}_x \times g) \times g_x + (\mathbf{m} \times g_x) \times g_x \Big). && (2.12)
\end{aligned}
$$

We note that $\mathfrak{S}$ and $\mathfrak{C}$ are linear in $\mathbf{m}$.

Remark 2.3. We note that $G : \mathbb{L}^2 \to \mathbb{L}^2$ is bounded linear map which is also skew symmetric (see Lemma 2.2). Thus, $\{e^{-sG}\}_{s \in \mathbb{R}}$ is uniformly continuous group of unitary linear maps on $\mathbb{L}^2$. It follows (see Corollary 1.4 (d) and Corollary 4.4 in [57]) that the map $\mathbb{R} \ni s \mapsto e^{-sG}(1 - \cos s)$ is of C^∞-class. In particular, it is Lipschitz on bounded sets, i.e., for every bounded set B, there exists a positive constant L_1 (depending on B) such that

$$\|e^{-sG}(1 - \cos s) - e^{-rG}(1 - \cos r)\|_{\mathcal{L}(\mathbb{L}^2)} \le L_1 |s - r|, \;\; s, r \in B,$$

where $\mathcal{L}(\mathbb{L}^2)$ denotes the space of all bounded linear operators from $\mathbb{L}^2$ to $\mathbb{L}^2$. This implies that $\mathcal{C}$ is locally Lipschitz, i.e.,

$$\|\mathcal{C}(W_2) - \mathcal{C}(W_1)\|_{\mathcal{L}(\mathbb{L}^2)} \le L_1 \, |W_2(t) - W_1(t)|. \qquad (2.13)$$

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In similar manner we can prove that S is locally Lipschitz, i.e., there exists a positive constant L_2 (depending on B)

$$\|S(W_2) - S(W_1)\|_{\mathcal{L}(\mathbb{L}^2)} \le L_2 \, |W_2(t) - W_1(t)|. \tag{2.14}$$

Before moving to next subsection, let us consider the corresponding approximated system which is similar to (1.13)–(1.15), where the Brownian motion W is replaced by the regular approximation W^n of W, i.e. we will consider the Wong–Zakai approximated system of (1.13)–(1.15) which is:

$$\frac{\partial \mathbf{m}^n}{\partial t} = \lambda_1(\mathbf{m}^n \times \mathbf{m}^n_{xx}) - \lambda_2 \mathbf{m}^n \times (\mathbf{m}^n \times \mathbf{m}^n_{xx}) + F^n(t, \mathbf{m}^n), \tag{2.15}$$

$$\mathbf{m}^n_x(t, 0) = 0 = \mathbf{m}^n_x(t, 1), \tag{2.16}$$

$$\mathbf{m}^n(0) = \mathbf{m}_0, \tag{2.17}$$

where

$$F^n(t, \mathbf{m}^n) = \lambda_1 \mathbf{m}^n \times \tilde{C}(W^n(t), \mathbf{m}^n(t, \cdot)) - \lambda_2 \mathbf{m}^n \times (\mathbf{m}^n \times \tilde{C}(W^n(t), \mathbf{m}^n(t, \cdot))), \tag{2.18}$$

where $\tilde{C}$ is given in (1.17).

3. Existence and uniqueness of a solution to problem (1.13)–(1.15)

In this section, we state definition of a weak solution to (1.13)–(1.15) and prove existence theorem followed by pathwise uniqueness theorem. The construction of the solution is based on the Faedo–Galerkin approximation, energy and compactness methods.

Definition 3.1. (Weak solution) Let $T > 0$ be fixed and W be any continuous function on $[0, T]$. A function $\mathbf{m} \in \mathrm{L}^2(0, T; \mathbb{H}^1)$ is said to be a weak solution of the system (1.13)–(1.15) if the following hold:

1. $\sup_{t\in[0,T]} |\mathbf{m}(t)|_{\mathbb{H}^1} < \infty$;
2. For almost every $t \in [0, \infty)$, $\mathbf{m}(t) \times \mathbf{m}_{xx}(t) \in \mathbb{L}^2$ we have

$$\int_0^T |\mathbf{m}(t) \times \mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2} dt < \infty; \tag{3.1}$$

3. $\mathbf{m}$ satisfies the following saturation condition

$$|\mathbf{m}(t, x)|_{\mathbb{R}^3} = 1 \quad \text{for a.e. } x \in D, \text{ for all } t \in [0, T]; \tag{3.2}$$

4. For all $\phi \in \mathbb{H}^1$,

$$\langle \mathbf{m}(t), \phi \rangle_{\mathbb{L}^2} = \langle \mathbf{m}(0), \phi \rangle_{\mathbb{L}^2} - \lambda_1 \int_0^t \int_D \langle \mathbf{m}_x(s,x), \phi_x(x) \times \mathbf{m}(s,x) \rangle_{\mathbb{R}^3} dx ds$$
$$- \lambda_2 \int_0^t \int_D \langle \mathbf{m}_x(s,x), (\mathbf{m} \times \phi)_x(s,x) \times \mathbf{m}(s,x) \rangle_{\mathbb{R}^3} dx ds$$
$$+ \int_0^t \int_D \langle F(s, \mathbf{m}(s,x)), \phi(x) \rangle_{\mathbb{R}^3} dx ds \tag{3.3}$$

holds for all $t \in [0, T]$.

We now introduce the following notation for the set of all $\mathbb{R}^3$-valued functions defined on the domain D which belong to the Sobolev space $\mathbb{H}^1 := H^1(D; \mathbb{R}^3)$ and satisfy the saturation condition (3.2) (or (1.3)):

$$\mathbb{H}^1(D; \mathbb{S}^2) := \left\{ \mathbf{m} \in \mathbb{H}^1 \text{ such that } |\mathbf{m}(x)|_{\mathbb{R}^3} = 1 \text{ for a.a. } x \in D \right\}. \tag{3.4}$$

In other words, $\mathbb{H}^1(D; \mathbb{S}^2)$ is the set of all functions belonging to the Sobolev space $\mathbb{H}^1$ whose values are in the sphere. Since D is one-dimensional, $\mathbb{H}^1$ is embedded in $C(D; \mathbb{R}^3)$, the 'a.a.' condition in (3.4) can be replaced by 'all'.

Theorem 3.2. *Let* $\mathbf{m}_0 \in \mathbb{H}^1(D; \mathbb{S}^2)$. *Then there exists a weak solution* $\mathbf{m}$ *to the system* (1.13)–(1.15) *satisfying the following:*

1. *There exists a positive constant* C*, depending on* $T, \lambda_1, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}$*, such that*

$$\sup_{t \in [0,T]} |\mathbf{m}(t)|_{\mathbb{H}^1} \leq C. \tag{3.5}$$

2. *For almost every* $t \in [0, \infty)$, $\mathbf{m}(t) \times \mathbf{m}_{xx}(t) \in \mathbb{L}^2$ *and for every* $T > 0$*, there exists a positive constant* C*, depending on* $T, \lambda_1, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}$*, such that we have*

$$\int_0^T |\mathbf{m}(t) \times \mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2} dt \leq C. \tag{3.6}$$

Remark 3.3. A direct consequence of the above theorem is that the solution $\mathbf{m}(t)$ is weakly continuous with values in $\mathbb{H}^1$. In fact, we will prove later in Proposition 6.3 that $\mathbf{m} \in C([0, T]; \mathbb{H}^1)$.

Remark 3.4. A different approach to prove Theorem 3.2 would be via the finite element approximation, see [36]. However, since the formulation of our result is different that the one in [36], see Proposition 6.7 therein, for the sake of reader's convenience and completeness, we provide a proof here, though by a different but classical approach, namely the Faedo–Galerkin approximation. Another important reason for including the proof is that in our setting, the function $t \mapsto W(t)$ is only assumed to be a continuous function, without any Hölder regularity. This is important as it should make possible to use the current approach in order to study the SLLGEs driven by a one dimensional fractional Brownian Motion, see e.g. Duncan et al. [32].

Remark 3.5. The paper [17] deals with a more general SLLGEs in the presence of anisotropy energy in the case when the space dimension $d = 1$ and the authors established the existence, the uniqueness, the regularity of the solutions. Moreover they also proved the large deviation principle. However, the results from [17] can not be directly applied here in order to prove our Theorem 3.2 due to two facts. Firstly, here we do not work with the SLLGEs and rather we work with the transformed quasilinear PDE (1.13)–(1.15), which has a very involved structure of the nonlinear term denoted by F; secondly, as mentioned in Remark 3.4, here W is not simply a trajectory of a Brownian motion but rather any real-valued continuous function. On the other hand, in the present paper we use some ideas from [17], see the proof of Theorems 6.1, and especially the proof of Theorem 6.4.

To prove this theorem, we first construct the Faedo–Galerkin solution.

3.1. Faedo–Galerkin approximation and energy estimate for the approximating sequence

For each $k \in \mathbb{N}$, let $\mathbb{L}^2_k$ be the linear span of the first k elements of the orthonormal basis of $\mathbb{L}^2$ composed of eigenvectors of the operator A defined earlier and let

$$P_k : \mathbb{L}^2 \to \mathbb{L}^2_k \tag{3.7}$$

be the corresponding orthogonal projection. Define the maps

$$\begin{aligned}
Q^1_k &: \mathbb{L}^2_k \ni \mathbf{m}_k \mapsto P_k(\mathbf{m}_k \times (\mathbf{m}_k)_{xx}) \in \mathbb{L}^2_k,\\
Q^2_k &: \mathbb{L}^2_k \ni \mathbf{m}_k \mapsto P_k(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})) \in \mathbb{L}^2_k,\\
Q^3_k &: \mathbb{L}^2_k \ni \mathbf{m}_k \mapsto P_k(F(t, \mathbf{m}_k)) \in \mathbb{L}^2_k.
\end{aligned}$$

For each $k \in \mathbb{N}$, let $\mathbf{m}_k : [0, T] \times \Omega \to \mathbb{L}^2_k$ be a solution of the following ordinary differential equation on $\mathbb{L}^2_k$:

$$\frac{d\mathbf{m}_k}{dt} = \lambda_1 Q^1_k - \lambda_2 Q^2_k + Q^3_k, \tag{3.8}$$

$$\mathbf{m}_k(0) = P_k\mathbf{m}(0), \tag{3.9}$$

$$(\mathbf{m}_k)_x(t, 0) = (\mathbf{m}_k)_x(t, 1) = 0, \forall\, t \in (0, T). \tag{3.10}$$

Lemma 3.6. *The maps Q^i_k; $i = 1, 2, 3$ are Lipschitz on balls and for all $\mathbf{m}_k \in \mathbb{L}^2_k$, and $i = 1, 2, 3$,*

$$\langle Q^i_k(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} = 0.$$

Proof. Q_k^i for $i = 1, 2$ are polynomials in $\mathbf{m}_k$ variable, hence continuously differentiable functions. Therefore they are Lipschitz on balls in $\mathbb{L}_k^2$. Q_k^3 is polynomial in $\mathbf{m}_k$ with bounded coefficients, hence continuously differentiable with respect to $\mathbf{m}_k$. Therefore it is Lipschitz on balls in $\mathbf{m}_k$ variable on $\mathbb{L}_k^2$.

Since $\mathbf{m}_k \in \mathbb{L}_k^2$, $\mathbf{m}_k = P_k \mathbf{m}_k \in \mathbb{L}_k^2$. Now using the identity $\langle a \times b, b\rangle_{\mathbb{R}^3} = 0$ and the self adjoint property of P_k, we have

$$\begin{aligned}\langle Q_k^1(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} &= \langle P_k(\mathbf{m}_k \times (\mathbf{m}_k)_{xx}), \mathbf{m}_k\rangle_{\mathbb{L}^2} = \langle \mathbf{m}_k \times (\mathbf{m}_k)_{xx}, P_k\mathbf{m}_k\rangle_{\mathbb{L}^2} \\ &= \langle \mathbf{m}_k \times (\mathbf{m}_k)_{xx}, \mathbf{m}_k\rangle_{\mathbb{L}^2} = \int_D \langle \mathbf{m}_k(x) \times (\mathbf{m}_k)_{xx}(x), \mathbf{m}_k(x)\rangle_{\mathbb{R}^3} dx = 0.\end{aligned}$$

In similar way we can prove $\langle Q_k^i(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} = 0$ for $i = 2, 3$. $\square$

We note that from Lemma 3.6 the coefficients $Q_k^i, i = 1, 2, 3$ are locally Lipschitz on $\mathbb{L}_k^2$. Hence by the deterministic version of Theorem 3.1 of Albeverio et al. [1], (3.8)–(3.10) has unique global strong solution in $\mathbb{L}_k^2$.

Lemma 3.7. *Let $\mathbf{m}_k \in \mathbb{L}_k^2$ be such that $\mathbf{m}_k$ satisfies (3.8)–(3.10). Then $|\mathbf{m}_k(t)|_{\mathbb{L}^2} = |P_k\mathbf{m}(0)|_{\mathbb{L}^2}$, for all $t \in [0, T]$.*

Proof. Taking inner product of (3.8) with $\mathbf{m}_k$ and using Lemma 3.6 we get

$$\frac{1}{2}\frac{d}{dt}|\mathbf{m}_k(t)|^2_{\mathbb{L}^2} = \langle Q_k^1(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} + \langle Q_k^2(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} + \langle Q_k^3(\mathbf{m}_k), \mathbf{m}_k\rangle_{\mathbb{L}^2} = 0,$$

and this implies $|\mathbf{m}_k(t)|_{\mathbb{L}^2} = |P_k\mathbf{m}(0)|_{\mathbb{L}^2}$, for all $t \in [0, T]$. $\square$

Lemma 3.8. *For all $\mathbf{m}_k \in \mathbb{L}_k^2$, such that $\mathbf{m}_k$ satisfies (3.8)–(3.10), we have the following estimates:*

1.

$$\langle Q_k^1(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2} = 0, \quad \langle Q_k^2(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2} = -|\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2}. \tag{3.11}$$

2. *For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$|\langle Q_k^3(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2}| \le \varepsilon|\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_\varepsilon(|(\mathbf{m}_k)_x|^2_{\mathbb{L}^2} + 1). \tag{3.12}$$

Proof. Both the equalities in (3.11) can be followed from Lemma 3.3 of [18]. Since $g \in \mathbb{W}^{2,\infty}$, using vector product $\langle a \times b, a\rangle_{\mathbb{R}^3} = 0$, $\langle a \times b, c\rangle_{\mathbb{R}^3} = \langle b \times c, a\rangle_{\mathbb{R}^3} = \langle c \times a, b\rangle_{\mathbb{R}^3}$, Young's inequality and from (2.6), we have

$$\begin{aligned}|\langle Q_k^3(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2}| &= |\langle P_k F(t, \mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2}| = |\langle F(t, \mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2}| \\ &= \left|\lambda_1 \int_D \Big\langle \mathbf{m}_k(x) \times (\mathbf{m}_k(x))_{xx}, \hat{F}(t, \mathbf{m}_k(x))\Big\rangle_{\mathbb{R}^3} dx\end{aligned}$$

$$\begin{aligned}
&- \lambda_2 \int_D \Big\langle \mathbf{m}_k(x) \times (\mathbf{m}_k(x))_{xx}, \Big(\mathbf{m}_k(x) \times \hat{F}(t, \mathbf{m}_k(x))\Big)\Big\rangle_{\mathbb{R}^3} dx \Big| \\
&= \Big| \lambda_1 \Big\langle \mathbf{m}_k \times (\mathbf{m}_k)_{xx}, \hat{F}(t, \mathbf{m}_k)\Big\rangle_{\mathbb{L}^2} - \lambda_2 \Big\langle \mathbf{m}_k \times (\mathbf{m}_k)_{xx}, \Big(\mathbf{m}_k \times \hat{F}(t, \mathbf{m}_k)\Big)\Big\rangle_{\mathbb{L}^2} \Big| \\
&\leq |\lambda_1| \Big\{ \frac{\varepsilon}{2|\lambda_1|} |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_\varepsilon |\lambda_1| |\hat{F}(t, \mathbf{m}_k)|^2_{\mathbb{L}^2} \Big\} \\
&+ |\lambda_2| \Big\{ \frac{\varepsilon}{2|\lambda_2|} |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_\varepsilon |\lambda_2| |\hat{F}(t, \mathbf{m}_k)|^2_{\mathbb{L}^2} \Big\} \\
&\leq |\lambda_1| \Big\{ \frac{\varepsilon}{2|\lambda_1|} |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_\varepsilon |\lambda_1| \Big(1 + |(\mathbf{m}_k)_x|^2_{\mathbb{L}^2}\Big) \Big\} \\
&\quad + |\lambda_2| \Big\{ \frac{\varepsilon}{2|\lambda_2|} |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_\varepsilon |\lambda_2| \Big(1 + |(\mathbf{m}_k)_x|^2_{\mathbb{L}^2}\Big) \Big\} \\
&\leq \varepsilon |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|^2_{\mathbb{L}^2} + C_{\varepsilon, \lambda_1, \lambda_2} \Big(1 + |(\mathbf{m}_k)_x|^2_{\mathbb{L}^2}\Big). \quad \square
\end{aligned}$$

Lemma 3.9. *Let $\mathbf{m}_k$ be as before. Then there exists a positive constant $C = C(T, \lambda_1, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1})$ such that for every $k \in \mathbb{N}$,*

$$|\mathbf{m}_k|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)} \leq C; \tag{3.13}$$

$$\text{and} \quad |\mathbf{m}_k \times (\mathbf{m}_k)_{xx}|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \leq C. \tag{3.14}$$

Proof. Taking $\mathbb{L}^2$-inner product of (3.8) with $(\mathbf{m}_k)_{xx}$, we get

$$-\frac{1}{2}\frac{d}{dt} |(\mathbf{m}_k)_x(t)|^2_{\mathbb{L}^2} = \langle Q_k^1(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2} + \langle Q_k^2(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2} + \langle Q_k^3(\mathbf{m}_k), (\mathbf{m}_k)_{xx}\rangle_{\mathbb{L}^2}.$$

Using Lemma 3.8 in the above equation, using part 2 of Lemma 3.8 with $\varepsilon = \lambda_2$ and then integrating over $[0, t]$ we get,

$$\begin{aligned}
&|(\mathbf{m}_k)_x(t)|^2_{\mathbb{L}^2} + 2\lambda_2 \int_0^t |\mathbf{m}_k(s) \times (\mathbf{m}_k)_{xx}(s)|^2_{\mathbb{L}^2} ds \\
&= |(\mathbf{m}_k)_x(0)|^2_{\mathbb{L}^2} - 2 \int_0^t \langle Q_k^3(\mathbf{m}_k(s)), (\mathbf{m}_k(s))_{xx}\rangle_{\mathbb{L}^2} ds \\
&\leq |\mathbf{m}(0)|^2_{\mathbb{H}^1} + \lambda_2 \int_0^t |\mathbf{m}_k(s) \times (\mathbf{m}_k)_{xx}(s)|^2_{\mathbb{L}^2} ds + C \int_0^t (1 + |(\mathbf{m}_k)_x(s)|^2_{\mathbb{L}^2}) ds.
\end{aligned}$$

Hence we have

$$|(\mathbf{m}_k)_x(t)|^2_{\mathbb{L}^2} + \lambda_2 \int_0^t |\mathbf{m}_k(s) \times (\mathbf{m}_k)_{xx}(s)|^2_{\mathbb{L}^2} ds \leq |\mathbf{m}(0)|^2_{\mathbb{H}^1} + C \int_0^t (1 + |(\mathbf{m}_k)_x(s)|^2_{\mathbb{L}^2}) ds.$$

Now using Gronwall's inequality, we have, for every $T > 0$,

$$\sup_{t\in[0,T]} |\mathbf{m}_k(t)|_{\mathbb{H}^1} \le C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}), \quad \text{for each} \quad k, \tag{3.15}$$

and

$$\int_0^t |\mathbf{m}_k(s) \times (\mathbf{m}_k)_{xx}(s)|^2_{\mathbb{L}^2} ds \le C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}) \quad \text{for each} \quad k. \tag{3.16}$$

Hence (3.13) and (3.14) are obtained. □

Lemma 3.10. *Let $\mathbf{m}_k$ be as before. Then there exists a positive constant $C = C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1})$ such that for every $k \in \mathbb{N}$ and for every $q \in [1, 2]$,*

$$|\mathbf{m}_k|_{H^1(0,T;\mathbb{L}^q)} \le C. \tag{3.17}$$

Proof. Using the embedding in one dimension $\mathbb{H}^1 \hookrightarrow \mathbb{L}^r$ for any $r \in [1, \infty]$, Hölder's inequality with $\frac{1}{q} = \frac{1}{2} + \frac{1}{r}$ for $r \in [1, \infty]$ and Lemma 3.9, we get

$$|\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})|^2_{L^2(0,T;\mathbb{L}^q)} = \int_0^T |\mathbf{m}_k(t) \times (\mathbf{m}_k(t) \times (\mathbf{m}_k(t))_{xx})|^2_{\mathbb{L}^q} dt$$
$$\le \int_0^T |\mathbf{m}_k(t)|^2_{\mathbb{L}^r} |\mathbf{m}_k(t) \times (\mathbf{m}_k(t))_{xx}|^2_{\mathbb{L}^2} dt \le \Big(\sup_{t\in[0,T]} |\mathbf{m}_k(t)|^2_{\mathbb{L}^r}\Big)\Big(\int_0^T |\mathbf{m}_k(t) \times (\mathbf{m}_k(t))_{xx}|^2_{\mathbb{L}^2} dt\Big)$$
$$\le \Big(\sup_{t\in[0,T]} |\mathbf{m}_k(t)|^2_{\mathbb{H}^1}\Big)\Big(\int_0^T |\mathbf{m}_k(t) \times (\mathbf{m}_k(t))_{xx}|^2_{\mathbb{L}^2} dt\Big) \le C. \tag{3.18}$$

Now $\frac{1}{q} = \frac{1}{2} + \frac{1}{r}$ for $r \in [1, \infty]$ gives $q \in [1, 2]$, thus we have

$$\Big|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big) - \mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big|_{\mathbb{L}^q}$$
$$\le C\Big|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big) - \mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big|_{\mathbb{L}^2} \to 0 \quad \text{as} \quad k \to \infty.$$

Thus by Dominated Convergence Theorem, we have

$$\Big|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big) - \mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big|_{L^2(0,T;\mathbb{L}^q)} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.19}$$

Hence using (3.18) and (3.19) we have

$$\begin{aligned}
&\left|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \\
&\leq \left|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big) - \mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \\
&\quad + \left|\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \leq C. \qquad (3.20)
\end{aligned}$$

Similarly we can prove that $\left|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k)_{xx}\Big)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \leq C$. Now for $q \in [1,2]$, we have

$$\begin{aligned}
|F(t,\mathbf{m}_k)|^2_{\mathrm{L}^2(0,T;\mathbb{L}^q)} &\leq \int_0^T |F(t,\mathbf{m}_k(t))|^2_{\mathbb{L}^2} dt \leq \int_0^T (1 + |\mathbf{m}_k(t)|^2_{\mathbb{L}^2}) dt \\
&\leq T + T \sup_{t\in[0,T]} |\mathbf{m}_k(t)|^2_{\mathbb{H}^1} \leq C. \qquad (3.21)
\end{aligned}$$

Using similar techniques as in (3.19) and (3.20) we have from (3.21),

$$\left|P_k F(t,\mathbf{m}_k)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \leq C.$$

Therefore, by (3.8), we have

$$\begin{aligned}
\left|\frac{d\mathbf{m}_k}{dt}\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} &\leq |\lambda_1|\left|P_k(\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} + \left|P_k F(t,\mathbf{m}_k)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \\
&\quad + |\lambda_2|\left|P_k\Big(\mathbf{m}_k \times (\mathbf{m}_k \times (\mathbf{m}_k)_{xx})\Big)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \\
&\leq C.
\end{aligned}$$

Together with (3.13) we finally conclude that there exists a constant $C > 0$ such that

$$|\mathbf{m}_k|_{H^1(0,T;\mathbb{L}^q)} \leq |\mathbf{m}_k|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} + \left|\frac{d\mathbf{m}_k}{dt}\right|_{\mathrm{L}^2(0,T;\mathbb{L}^q)} \leq C.$$

This completes the proof. □

3.1.1. Existence of weak solution

Using Banach Alaoglu Theorem and (3.13) and (3.17) we have the following:
there exist $\mathbf{m}$, $\mathfrak{M}$, $\mathfrak{F}$ such that for some subsequence $\mathbf{m}_k$ (still denoted by the same) and for every $q \in [1,2]$,

$$(i) \quad \mathbf{m}_k \to \mathbf{m} \text{ weakly star in } \mathrm{L}^\infty(0,T;\mathbb{H}^1) \text{ and weakly in } H^1(0,T;\mathbb{L}^q), \qquad (3.22)$$

$$(ii) \quad \mathbf{m}_k \times (\mathbf{m}_k)_{xx} \to \mathfrak{M} \text{ weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2), \text{ and} \qquad (3.23)$$

$$(iii) \quad F(t,\mathbf{m}_k) \to \mathfrak{F} \text{ weakly in } \mathrm{L}^2(0,T;\mathbb{L}^2). \qquad (3.24)$$

Lemma 3.11. *For* $2 \le r < \infty$, $\lim_{k\to\infty} |\mathbf{m}_k - \mathbf{m}|_{\mathrm{L}^r(Q)} = 0$, *where* $Q := [0, T] \times D$.

Proof. By compact embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^p$ for $1 \le p \le \infty$. Let $X_0 = \mathbb{H}^1$, $X = \mathbb{L}^p$, $X_1 = \mathbb{L}^q$ for $1 < q \le 2$. We note that X_0 and X_1 are reflexive. So, $X_0 \hookrightarrow X \hookrightarrow X_1$ and $X_0 \hookrightarrow X$ is compact for $1 < q \le 2$, $q \le p \le \infty$. Let $\alpha_0 = r > 1, \alpha_2 = 2$ and $Y = \{v \in \mathrm{L}^r(0, T; \mathbb{H}^1), \dfrac{dv}{dt} \in \mathrm{L}^r(0, T; \mathbb{L}^q)\}$. Then by Theorem 3.2.1 of Temam [62], for $T < \infty$, the embedding $Y \hookrightarrow \mathrm{L}^r(0, T; \mathbb{L}^p)$ is compact. Take in particular, $r = p \ge q$, we have $Y \hookrightarrow \mathrm{L}^r(Q)$ is compact. As $\mathbb{H}^1 \hookrightarrow \mathbb{L}^r$ continuously, we have for $r \ge q$,

$$|\mathbf{m}_k|_{\mathrm{L}^\infty(0,T;\mathbb{L}^r)} \le C, \quad \forall k \ge 1. \tag{3.25}$$

By (3.25) and (3.17) we have, $|\mathbf{m}_k|_Y \le C, \quad \forall k \ge 1$. Hence $\{\mathbf{m}_k\}_{k\ge1}$ has a convergent subsequence (still denoted by the same) in $\mathrm{L}^r(Q)$. Let us assume that the limit be $\tilde{\mathbf{m}}$, i.e., $\mathbf{m}_k \to \tilde{\mathbf{m}}$ in $\mathrm{L}^r(Q)$. We will show that $\mathbf{m} = \tilde{\mathbf{m}}$ in $\mathrm{L}^r(Q)$. Now we need to assume $r \ge 2 \ge q \ge 1$. As $\mathrm{L}^r(Q) \hookrightarrow \mathrm{L}^2(Q)$ continuously, $\mathbf{m}_k \to \tilde{\mathbf{m}}$ in $\mathrm{L}^2(Q)$. So,

$$\begin{aligned}
|\mathbf{m} - \tilde{\mathbf{m}}|^2_{\mathrm{L}^2(Q)} &= \langle \mathbf{m} - \tilde{\mathbf{m}}, \mathbf{m} - \tilde{\mathbf{m}}\rangle_{\mathrm{L}^2(Q)} = \lim_{k\to\infty} \langle \mathbf{m} - \mathbf{m}_k, \mathbf{m} - \tilde{\mathbf{m}}\rangle_{\mathrm{L}^2(Q)} \\
&= \lim_{k\to\infty} \int_0^T \int_D \langle \mathbf{m}(t,x) - \mathbf{m}_k(t,x), \mathbf{m}(t,x) - \tilde{\mathbf{m}}(t,x)\rangle_{\mathbb{R}^3} dx dt \\
&= \lim_{k\to\infty} {}_{\mathrm{L}^\infty(0,T;\mathbb{H}^{-1})}\langle \mathbf{m} - \mathbf{m}_k, \mathbf{m} - \tilde{\mathbf{m}}\rangle_{\mathrm{L}^1(0,T;\mathbb{H}^{-1})} = 0.
\end{aligned}$$

So $\mathbf{m} = \tilde{\mathbf{m}}$ a.e. in Q. Hence $\mathbf{m} = \tilde{\mathbf{m}}$ in $\mathrm{L}^r(Q)$. Thus $\lim_{k\to\infty} |\mathbf{m}_k - \mathbf{m}|_{\mathrm{L}^r(Q)} = 0$ for $r \ge 2$. □

Lemma 3.12. *For almost every* $t \in [0, T]$, $\phi \in C_c^\infty(D)$,

$$\begin{aligned}
\int_D \langle \mathbf{m}(t) - \mathbf{m}_0, \phi\rangle \, dx = \lambda_1 \int_0^t \int_D \langle \mathfrak{M}, \phi\rangle dx\, ds - \lambda_2 \int_0^t \int_D \langle \mathfrak{M}, \phi \times \mathbf{m}\rangle \, dx\, ds \\
+ \int_0^t \int_D \langle \mathfrak{F}, \mathbf{m}\rangle \, dx\, ds.
\end{aligned}$$

We omit the proof of the above Lemma and refer a reader to the proof of Proposition 3.27 of [49].

Lemma 3.13. *Let* $\mathbf{m}_0 \in \mathbb{H}^1(D; \mathbb{S}^2)$. *Let* $\mathbf{m}(t)$ *be weakly continuous with values in* $\mathbb{H}^1$ *such that* $\mathbf{m} \in C([0, T]; \mathbb{L}^2)$ *satisfying* (3.5), (3.6) *and* (3.3). *Then*

$$\int_0^T \left| \mathbf{m}(s) \times \left(\mathbf{m}(s) \times \mathbf{m}_{xx}(s) \right) \right|^2_{\mathbb{L}^2} ds < \infty \tag{3.26}$$

and $\mathbf{m}$ *satisfy* (1.13)–(1.15) *in the strong sense i.e., for all* $t \in [0,T]$ *we have*

$$\mathbf{m}(t) = \mathbf{m}(0) + \lambda_1 \int_0^t \mathbf{m}(s) \times \mathbf{m}_{xx}(s) ds - \lambda_2 \int_0^t \mathbf{m}(s) \times \Big(\mathbf{m}(s) \times \mathbf{m}_{xx}(s)\Big) ds$$
$$+ \int_0^t F(s, \mathbf{m}(s)) ds. \tag{3.27}$$

Proof of Lemma 3.13. Let us first prove (3.26). By the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^\infty$ in one-dimension, from (3.5) we have

$$\sup_{t\in[0,T]} |\mathbf{m}(t)|_{\mathbb{L}^\infty} \le \sup_{t\in[0,T]} |\mathbf{m}(t)|_{\mathbb{H}^1} < \infty. \tag{3.28}$$

Now using (3.6) and (3.28) we have

$$\begin{aligned}\int_0^T \Big|\mathbf{m}(t) \times \Big(\mathbf{m}(t) \times \mathbf{m}_{xx}(t)\Big)\Big|^2_{\mathbb{L}^2} dt &\le \int_0^T \int_D \Big|\mathbf{m}(t,x) \times \Big(\mathbf{m}(t,x) \times \mathbf{m}_{xx}(t,x)\Big)\Big|^2_{\mathbb{R}^3} dx\, dt \\ &\le \int_0^T \int_D |\mathbf{m}(t,x)|^2_{\mathbb{R}^3} |\mathbf{m}(t,x) \times \mathbf{m}_{xx}(t,x)|^2_{\mathbb{R}^3} dx\, dt \\ &\le \int_0^T |\mathbf{m}(t)|^2_{\mathbb{L}^\infty} \int_D |\mathbf{m}(t,x) \times \mathbf{m}_{xx}(t,x)|^2_{\mathbb{R}^3} dx\, dt \\ &\le \sup_{t\in[0,T]} |\mathbf{m}(t)|^2_{\mathbb{H}^1} \Big(\int_0^T |\mathbf{m}(t) \times \mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2} dt\Big) < \infty.\end{aligned}$$

Also recalling $|F(t,\mathbf{m}(t))|^2_{\mathbb{L}^2} \le C\Big(|\mathbf{m}(t)|^2_{\mathbb{L}^2} + |\mathbf{m}_x(t)|^2_{\mathbb{L}^2}\Big)$, we have

$$\int_0^T |F(t,\mathbf{m}(t))|^2_{\mathbb{L}^2} dt = C \int_0^T |\mathbf{m}(t)|^2_{\mathbb{H}^1} dt \le CT \sup_{t\in[0,T]} |\mathbf{m}(t)|^2_{\mathbb{H}^1} < \infty.$$

Since the equation (3.3) is satisfied for all $\phi \in \mathbb{H}^1$, so in particular for $\phi \in C_c^\infty(D)$, using integration by parts in (3.3) we have for all $t \in [0,T]$

$$\mathbf{m}(t) = \mathbf{m}(0) + \lambda_1 \int_0^t \mathbf{m}(s) \times \mathbf{m}_{xx}(s) ds - \lambda_2 \int_0^t \mathbf{m}(s) \times \Big(\mathbf{m}(s) \times \mathbf{m}_{xx}(s)\Big) ds$$

$$+\int_0^t F(s,\mathbf{m}(s))ds \quad \text{in } \mathbb{L}^2.$$

This completes the proof of Lemma 3.13. □

Lemma 3.14. *Let* $\mathbf{m}$ *be as before. Then for any bounded real valued measurable function* $\phi : D \to \mathbb{R}$ *we have for all* $s \in [0,T]$,

1. $\displaystyle\int_D \langle \mathbf{m}(s,x) \times \mathbf{m}_{xx}(s,x), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx = 0$
2. $\displaystyle\int_D \langle \mathbf{m}(s,x) \times \Big(\mathbf{m}(s,x) \times \mathbf{m}_{xx}(s,x)\Big), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx = 0$
3. $\displaystyle\int_D \langle F(s,\mathbf{m}(s,x)), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx = 0$

For proof see Lemma 4.7 in [18].

Lemma 3.15. *Let* $\mathbf{m}$ *be as before satisfying* (3.5)–(3.6) *and* (3.27). *Then* $|\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1$, *for a.e.* $x \in D$ *and for all* $t \in [0,T]$.

Proof of Lemma 3.15. Let $\phi \in C_c^\infty(D;\mathbb{R})$. Let us define a function $I : \mathbb{L}^2 \ni \mathbf{m} \mapsto \langle \mathbf{m}, \phi\mathbf{m}\rangle_{\mathbb{L}^2} \in \mathbb{R}$. Using the calculus in Hilbert space we have

$$\langle DI(\mathbf{m}), v\rangle_{\mathbb{L}^2} = 2\langle \phi\mathbf{m}, v\rangle_{\mathbb{L}^2}.$$

By fundamental theorem of calculus,

$$I(\mathbf{m}(t)) - I(\mathbf{m}(0)) = \int_0^t \frac{d}{ds} I(\mathbf{m}(s))ds = \int_0^t \langle DI(\mathbf{m}(s)), \mathbf{m}'(s)\rangle_{\mathbb{L}^2} ds = 2\int_0^t \langle \phi\mathbf{m}(s), \mathbf{m}'(s)\rangle_{\mathbb{L}^2} ds \tag{3.29}$$

where $'$ denotes the derivative with respect to time variable. Using (3.27) we note that $\mathbf{m}$ satisfies

$$\begin{aligned}\mathbf{m}'(t,x) = {}& \lambda_1\Big(\mathbf{m}(t,x) \times \mathbf{m}_{xx}(t,x)\Big) - \lambda_2 \mathbf{m}(t,x) \times \Big(\mathbf{m}(t,x) \times \mathbf{m}_{xx}(t,x)\Big) \\ & + F(t,\mathbf{m}(t,x)).\end{aligned} \tag{3.30}$$

Multiplying (3.30) with $\phi\mathbf{m}$ and integrating in D and using (3.26) we get

$$\int_D \langle \mathbf{m}'(s,x), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx = \lambda_1 \int_D \langle \mathbf{m}(s,x) \times \mathbf{m}_{xx}(s,x), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx$$

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$$- \lambda_2 \int_D \langle \mathbf{m}(s,x) \times \Big(\mathbf{m}(s,x) \times \mathbf{m}_{xx}(s,x)\Big), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx$$
$$+ \int_D \langle F(s,\mathbf{m}(s,x)), \phi\mathbf{m}(s,x)\rangle_{\mathbb{R}^3} dx.$$

This implies

$$\begin{aligned}\langle \mathbf{m}'(s), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} = \lambda_1 &\langle \mathbf{m}(s) \times \mathbf{m}_{xx}(s), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} \\ &- \lambda_2 \langle \mathbf{m}(s) \times \Big(\mathbf{m}(s) \times \mathbf{m}_{xx}(s)\Big), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} \\ &+ \langle F(s,\mathbf{m}(s)), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2}.\end{aligned} \tag{3.31}$$

Using (3.31) from (3.29) we have

$$\begin{aligned} I(\mathbf{m}(t)) - I(\mathbf{m}(0)) = &\lambda_1 \int_0^t \langle \mathbf{m}(s) \times \mathbf{m}_{xx}(s), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} ds \\ &- \lambda_2 \int_0^t \langle \mathbf{m}(s) \times \Big(\mathbf{m}(s) \times \mathbf{m}_{xx}(s)\Big), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} ds \\ &+ \int_0^t \langle F(s,\mathbf{m}(s)), \phi\mathbf{m}(s)\rangle_{\mathbb{L}^2} ds. \end{aligned} \tag{3.32}$$

Now using Lemma 3.14 we have first, second and third term of the right hand side of (3.32) vanish. Hence $I(\mathbf{m}(t)) = I(\mathbf{m}(0))$, $\forall t \in [0,T]$, i.e., $\langle \mathbf{m}(t), \phi\mathbf{m}(t)\rangle_{\mathbb{L}^2} = \langle \mathbf{m}(0), \phi\mathbf{m}(0)\rangle_{\mathbb{L}^2}$, $\forall t \in [0,T]$. Since ϕ is arbitrary and $|\mathbf{m}_0(x)|_{\mathbb{R}^3} = 1$ for a.e. $x \in D$, we have $|\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1$, for a.e. $x \in D$, $\forall t \in [0,T]$. This completes the proof of Lemma 3.15. □

3.1.2. Proof of Theorem 3.2

Properties (3.5) and (3.6) follow from (3.13) and (3.14) respectively. From Lemma 3.15 we conclude that $|\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1$, for a.e. $x \in D$, $\forall t \in [0,T]$. From Lemma 3.12 it follows that $\mathbf{m}$ satisfies (1.13)–(1.15) in weak sense. Therefore, finally we conclude that $\mathbf{m}$ is a weak solution to (1.13)–(1.15).

3.1.3. Pathwise uniqueness

Theorem 3.16. *Assume* $\mathbf{m} \in \mathrm{L}^4(0,T;\mathbb{H}^1)$ *satisfying property 3 of Definition 3.1, is a solution to (1.13)–(1.15). Then* $\mathbf{m}' \in \mathrm{L}^2(0,T;\mathrm{V}')$ *and* $\mathbf{m}$ *solve, together with (1.14)–(1.15), the following equation (in the weak sense with respect to the Gelfand triple* $\mathrm{V} \subset \mathbb{L}^2 \subset \mathrm{V}'$*):*

$$\frac{\partial \mathbf{m}(t)}{\partial t} + \lambda_2 A\mathbf{m}(t) = \lambda_2 |\mathbf{m}(t)_x|^2_{\mathbb{R}^3}\mathbf{m}(t) + \lambda_1(\mathbf{m}(t) \times (\mathbf{m}(t)_{xx})) + F(t,\mathbf{m}(t)), \quad t \in (0,T). \tag{3.33}$$

Moreover, if $\mathbf{m}_i \in \mathrm{L}^4(0,T;\mathbb{H}^1)$*, for* $i=1,2$*, are solutions to* (1.13)–(1.15) *with the same initial data* $\mathbf{m}_0 \in \mathbb{H}^1(D;\mathbb{S}^2)$*, then* $\mathbf{m}_1(t)=\mathbf{m}_2(t)$*,* $\forall t\in[0,T]$.

Before we embark on a proof let us recall [17, Lemma 2.3].

Lemma 3.17. *Let* $\mathbf{m}$ *be an element of* $\mathbb{H}^1(D;\mathbb{S}^2)$*. Then, in* V'*, we have*

$$\mathbf{m}\times(\mathbf{m}\times\ \mathbf{m}_{xx}) = -|\mathbf{m}_x|^2_{\mathbb{R}^3}\mathbf{m} - \mathbf{m}_{xx}. \tag{3.34}$$

If additionally $\mathbf{m}\in\mathbb{H}^2$*, then (with* $\cdot$ *on the LHS denoting the scalar product in* $\mathbb{R}^3$*)*

$$\mathbf{m}(x)\cdot\mathbf{m}_{xx}(x) = -|\mathbf{m}_x(x)|^2_{\mathbb{R}^3}, \qquad \textit{for a.a. } x\in D.$$

Proof of Theorem 3.16. We assume that $\mathbf{m}\in\mathrm{L}^4(0,T;\mathbb{H}^1)$ satisfying property 3 of Definition 3.1 is a solution of (1.13)–(1.15). Then by employing Lemma 3.17, as in the proof of [17, Theorem 4.1], we can show that each $\mathbf{m}$ solves (3.33) with (1.14)–(1.15). We note also that since $\mathbf{m}\in\mathrm{L}^4(0,T;\mathbb{H}^1)$ and $\mathbf{m}$ satisfies (3.2), both terms $A\mathbf{m}$ and $|\mathbf{m}_x|^2_{\mathbb{R}^3}\mathbf{m}$ belong to $\mathrm{L}^2(0,T;\mathrm{V}')$. Moreover, by (3.1), $\mathbf{m}\times\mathbf{m}_{xx}\in\mathrm{L}^2(0,T;\mathrm{V}')$. Since it's obvious that $F(\cdot,\mathbf{m}(\cdot))\in\mathrm{L}^2(0,T;\mathrm{V}')$, we infer that $\mathbf{m}'\in\mathrm{L}^2(0,T;\mathrm{V}')$, and this concludes the proof of the 1st part of the Theorem.

Let $\mathbf{m}_1,\mathbf{m}_2\in\mathrm{L}^4(0,T;\mathbb{H}^1)$ be two solutions of (1.13)–(1.15) for which we do not assume at this moment that $\mathbf{m}_1(0)=\mathbf{m}_2(0)$.

Let $z=\mathbf{m}_2-\mathbf{m}_1$. Then, by the first part $z\in\mathrm{L}^2(0,T;\mathrm{V})$ and $z'\in\mathrm{L}^2(0,T;\mathrm{V}')$ and z is a solution of

$$\begin{aligned}\frac{\partial z}{\partial t}+\lambda_2 Az &= \lambda_2\Big[|(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}\mathbf{m}_2 - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\mathbf{m}_1\Big] + \lambda_1\Big((\mathbf{m}_2\times(\mathbf{m}_2)_{xx}) - (\mathbf{m}_1\times(\mathbf{m}_1)_{xx})\Big)\\ &\quad + F(t,\mathbf{m}_2) - F(t,\mathbf{m}_1).\end{aligned} \tag{3.35}$$

Hence, by applying using Lemma 1.2 Chapter-III in Temam [62] to $|z|^2_{\mathbb{L}^2}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|z(t)|^2_{\mathbb{L}^2} &= -\lambda_2\langle Az,z\rangle_{\mathbb{L}^2} + \lambda_2\langle|(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}\mathbf{m}_2 - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\mathbf{m}_1, z\rangle_{\mathbb{L}^2}\\ &\quad + \lambda_1\Big(\langle(\mathbf{m}_2\times z_{xx}),z\rangle_{\mathbb{L}^2} - \langle(z\times(\mathbf{m}_1)_{xx}),z\rangle_{\mathbb{L}^2}\Big) + \langle F(t,\mathbf{m}_2)-F(t,\mathbf{m}_1),z\rangle_{\mathbb{L}^2}\\ &:= \sum_{i=1}^4 J_i(t).\end{aligned} \tag{3.36}$$

Now we estimate each J_i's for $i=1,2,3,4$. Using definition of A, we have

$$J_1(t) = -\lambda_2\langle Az(t),z(t)\rangle_{\mathbb{L}^2} = -\lambda_2|z_x(t)|^2_{\mathbb{L}^2}. \tag{3.37}$$

For simplicity we take $\lambda_2=1$ in $J_2(t)$.

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$$
\begin{aligned}
J_2(t) &= \langle |(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}\mathbf{m}_2 - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\mathbf{m}_1, z\rangle_{\mathbb{L}^2} \\
&= \langle |(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}(\mathbf{m}_2-\mathbf{m}_1) + \mathbf{m}_1\Big((\mathbf{m}_2)_x|^2_{\mathbb{R}^3} - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\Big), z\rangle)_{\mathbb{L}^2} \\
&= \langle |(\mathbf{m}_2)_x|^2_{\mathbb{R}^3} z, z\rangle_{\mathbb{L}^2} + \Big(((\mathbf{m}_2)_x + (\mathbf{m}_1)_x)((\mathbf{m}_2)_x - (\mathbf{m}_1)_x)\Big)\mathbf{m}_1, z\rangle)_{\mathbb{L}^2} \\
&= \langle |(\mathbf{m}_2)_x|^2_{\mathbb{R}^3} z, z\rangle_{\mathbb{L}^2} + \langle\Big((\mathbf{m}_1)_x + (\mathbf{m}_2)_x\Big) z_x \mathbf{m}_1, z\rangle_{\mathbb{L}^2} \\
&:= \sum_{i=1}^{3} J_2^i(t).
\end{aligned} \tag{3.38}
$$

We now derive estimates for fixed $\eta > 0$, and a generic constant $C > 0$. Using (2.2) and Young's inequality, we have

$$
\begin{aligned}
J_2^1(t) &= \langle |(\mathbf{m}_2)_x|^2_{\mathbb{R}^3} z, z\rangle_{\mathbb{L}^2} \le |(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^\infty} \le k^2|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|_{\mathbb{L}^2}|z|_{\mathbb{H}^1} \\
&\le k^2|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|_{\mathbb{L}^2}\Big(|z|_{\mathbb{L}^2} + |z_x|_{\mathbb{L}^2}\Big) \le k^2|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + k^2|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|_{\mathbb{L}^2}|z_x|_{\mathbb{L}^2} \\
&\le k^2|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{k^4}{2\eta^2}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \eta^2|z_x|_{\mathbb{L}^2}.
\end{aligned} \tag{3.39}
$$

Again using (2.2), Hölder's inequality and Young's inequality we have

$$
\begin{aligned}
J_2^2(t) &= \langle (\mathbf{m}_1)_x \mathbf{m}_1 z_x, z\rangle_{\mathbb{L}^2} \le |(\mathbf{m}_1)_x|_{\mathbb{L}^2}|\mathbf{m}_1|_{\mathbb{L}^\infty}|z_x|_{\mathbb{L}^2}|z|_{\mathbb{L}^\infty} \\
&\le |(\mathbf{m}_1)_x|_{\mathbb{L}^2}|z_x|_{\mathbb{L}^2}|z|_{\mathbb{L}^\infty} \le k|(\mathbf{m}_1)_x|_{\mathbb{L}^2}|z_x|_{\mathbb{L}^2}|z|^{1/2}_{\mathbb{L}^2}(|z|^{1/2}_{\mathbb{L}^2} + |z_x|^{1/2}_{\mathbb{L}^2}) \\
&\le k|(\mathbf{m}_1)_x|_{\mathbb{L}^2}|z_x|_{\mathbb{L}^2}|z|_{\mathbb{L}^2} + k|(\mathbf{m}_1)_x|_{\mathbb{L}^2}|z|^{1/2}_{\mathbb{L}^2}|z_x|^{3/2}_{\mathbb{L}^2} \\
&\le \frac{k^2}{\eta^2}|(\mathbf{m}_1)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \eta^2|z_x|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_1)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{3}{4}\eta^2|z_x|^2_{\mathbb{L}^2} \\
&\le \frac{k^2}{\eta^2}|(\mathbf{m}_1)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_1)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{7}{4}\eta^2|z_x|^2_{\mathbb{L}^2}.
\end{aligned} \tag{3.40}
$$

Proceeding in similar manner as in $J_2^2(t)$, we have

$$
J_2^3(t) \le \frac{k^2}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{7}{4}\eta^2|z_x|^2_{\mathbb{L}^2}. \tag{3.41}
$$

Substituting (3.39)–(3.41) in (3.38) we achieve

$$
\begin{aligned}
J_2(t) \le{}& k^2\Big(|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} + \frac{k^2}{2\eta^2}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2} + \frac{1}{\eta^2}\sum_{i=1}^{2}|(\mathbf{m}_i)_x|^2_{\mathbb{L}^2} + \frac{k^2}{4\eta^6}\sum_{i=1}^{2}|(\mathbf{m}_i)_x|^4_{\mathbb{L}^2}\Big)|z|^2_{\mathbb{L}^2} \\
&+ \frac{9\eta^2}{2}|z_x|^2_{\mathbb{L}^2}.
\end{aligned} \tag{3.42}
$$

In order to estimate $J_3(t)$, we note that $\langle z \times (\mathbf{m}_1)_{xx}, z\rangle_{\mathbb{L}^2} = 0$, hence using Hölder's inequality, (2.2) and Young's inequality we estimate $\langle \mathbf{m}_2 \times z_{xx}, z\rangle_{\mathbb{L}^2}$.

$$\begin{aligned}\langle \mathbf{m}_2 \times z_{xx}, z\rangle_{\mathbb{L}^2} &= -\langle z \times (\mathbf{m}_2)_x, z_x\rangle_{\mathbb{L}^2} \le |z|_{\mathbb{L}^\infty}|(\mathbf{m}_2)_x|_{\mathbb{L}^2}|z_x|_{\mathbb{L}^2}\\ &\le \frac{k^2}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \eta^2|z_x|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{3}{4}\eta^2|z_x|^2_{\mathbb{L}^2}\\ &\le \frac{k^2}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{7}{4}\eta^2|z_x|^2_{\mathbb{L}^2}.\end{aligned}$$

Finally, $J_3(t)$ becomes:

$$J_3(t) \le \frac{k^2}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{k^4}{4\eta^6}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{7}{4}\eta^2|z_x|^2_{\mathbb{L}^2}. \tag{3.43}$$

Now using (2.7) and simple vector algebraic identity, we estimate $J_4(t)$ as:

$$\begin{aligned}J_4(t) &= \langle F(t,\mathbf{m}_2) - F(t,\mathbf{m}_1), z\rangle_{\mathbb{L}^2}\\ &\le |\langle \lambda_1\mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2) - \lambda_2(\mathbf{m}_2 \times (\mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2)))\\ &\quad - \lambda_1(\mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1)) + \lambda_2(\mathbf{m}_1 \times (\mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1))), z\rangle_{\mathbb{L}^2}|\\ &\le |\lambda_1||\langle \mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2) - (\mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1)), z\rangle_{\mathbb{L}^2}|\\ &\quad + |\lambda_2||\langle(\mathbf{m}_2 \times (\mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2))) - (\mathbf{m}_1 \times (\mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1))), z\rangle_{\mathbb{L}^2}|\\ &:= \sum_{i=1}^{2} J_4^i(t).\end{aligned}$$

Using Young's inequality, $\langle z \times \hat{F}(t,\mathbf{m}_2), z\rangle_{\mathbb{L}^2} = 0$, and $|\mathbf{m}_1(x)_{\mathbb{R}^3} = 1$, for a.e. $x \in D$, $J_4^1(t)$ becomes:

$$\begin{aligned}J_4^1(t) &= |\lambda_1||\langle \mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2) - (\mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1)), z\rangle_{\mathbb{L}^2}|\\ &\le |\lambda_1||(\mathbf{m}_2 - \mathbf{m}_1) \times \hat{F}(t,\mathbf{m}_2) + \mathbf{m}_1 \times (\hat{F}(t,\mathbf{m}_2) - \hat{F}(t,\mathbf{m}_1)), z\rangle_{\mathbb{L}^2}|\\ &\le |\lambda_1||\langle z \times \hat{F}(t,\mathbf{m}_2), z\rangle_{\mathbb{L}^2}| + |\lambda_1||\mathbf{m}_1 \times (\hat{F}(t,\mathbf{m}_2) - \hat{F}(t,\mathbf{m}_1)), z\rangle_{\mathbb{L}^2}|\\ &\le |\lambda_1|\int_0^1 |\mathbf{m}_1(x)|_{\mathbb{R}^3}|\hat{F}(t,\mathbf{m}_2(x)) - \hat{F}(t,\mathbf{m}_1(x))|_{\mathbb{R}^3}|z(x)|_{\mathbb{R}^3}dx\\ &\le |\lambda_1||\hat{F}(t,\mathbf{m}_2) - \hat{F}(t,\mathbf{m}_1)|_{\mathbb{L}^2}|z|_{\mathbb{L}^2}.\end{aligned} \tag{3.44}$$

Proceeding as in $J_4^1(t)$, and using the standard vector algebraic identities, we have,

$$J_4^2(t) \le |\lambda_2||\langle z \times (\mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2)), z\rangle_{\mathbb{L}^2}| + |\lambda_2||\mathbf{m}_1 \times (\mathbf{m}_2 \times \hat{F}(t,\mathbf{m}_2) - \mathbf{m}_1 \times \hat{F}(t,\mathbf{m}_1)), z\rangle_{\mathbb{L}^2}|$$

$$\leq |\lambda_2| \int_0^1 \left| z \times \hat{F}(t, \mathbf{m}_2(x)) - \mathbf{m}_1 \times \left(\hat{F}(t, \mathbf{m}_2(x)) - \hat{F}(t, \mathbf{m}_1(x)) \right) \right|_{\mathbb{R}^3} |z(x)|_{\mathbb{R}^3} dx$$

$$= |\lambda_2| \int_0^1 |\hat{F}(t, \mathbf{m}_2(x))|_{\mathbb{R}^3} |z(x)|^2_{\mathbb{R}^3} dx + |\lambda_2| |\hat{F}(t, \mathbf{m}_2) - \hat{F}(t, \mathbf{m}_1)|_{\mathbb{L}^2} |z|_{\mathbb{L}^2}. \tag{3.45}$$

Now using the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^\infty$, (2.8) $|\hat{F}(t, \mathbf{m}_2)|_{\mathbb{L}^2} \leq C(1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2}$, and Young's inequality, the first term on the right hand side of (3.45) becomes:

$$\int_0^1 |\hat{F}(t, \mathbf{m}_2(x))|_{\mathbb{R}^3} |z(x)|^2_{\mathbb{R}^3} dx \leq C\left(1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2}\right) |z|^2_{\mathbb{L}^2} + \eta^2 |z_x|^2_{\mathbb{L}^2} + \frac{C}{\eta^2}\left(1 + |(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}\right) |z|^2_{\mathbb{L}^2}. \tag{3.46}$$

In order to estimate right hand side of (3.44) and second term on right hand side of (3.45), we now use (2.8) and Remark 2.3 and estimate the following term.

$$\begin{aligned}
&|\hat{F}(t, \mathbf{m}_2) - \hat{F}(t, \mathbf{m}_1)|_{\mathbb{L}^2} \\
&= \Big| e^{-W(t)G} \sin(W(t)) \Big(z \times g_{xx} + 2 z_x \times g_x \Big) \\
&\quad + e^{-W(t)G} [1 - \cos(W(t))] \Big((z \times g_{xx}) \times g + (z \times g) \times g_{xx} + 2 \Big((z_x \times g_x) \\
&\qquad \times g + (z_x \times g) \times g_x + (z \times g_x) \times g_x \Big) \Big) \Big|_{\mathbb{L}^2} \\
&\leq \| e^{-W(t)G} \sin(W(t)) \|_{\mathcal{L}(\mathbb{L}^2)} |z \times g_{xx} + 2 z_x \times g_x|_{\mathbb{L}^2} \\
&\quad + \| e^{-W(t)G} [1 - \cos(W(t))] \|_{\mathcal{L}(\mathbb{L}^2)} \Big| (z \times g_{xx}) \times g + (z \times g) \times g_{xx} + 2 \Big((z_x \times g_x) \\
&\qquad \times g + (z_x \times g) \times g_x + (z \times g_x) \times g_x \Big) \Big) \Big|_{\mathbb{L}^2} \\
&\leq C \Big(|z|_{\mathbb{L}^2} + |z_x|_{\mathbb{L}^2} \Big).
\end{aligned} \tag{3.47}$$

Using Young's inequality and (3.47), from (3.44) we have

$$J_4^1(t) \leq C\left(|z|^2_{\mathbb{L}^2} + |z|_{\mathbb{L}^2} |z_x|_{\mathbb{L}^2} \right) \leq C |z|^2_{\mathbb{L}^2} + \frac{C}{\eta^2} |z|^2_{\mathbb{L}^2} + \eta^2 |z_x|^2_{\mathbb{L}^2}. \tag{3.48}$$

Substituting (3.46) and (3.47) in (3.45) we have

$$J_4^2(t) \leq C\left(1 + \frac{1}{\eta^2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2} |(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} \right) |z|^2_{\mathbb{L}^2} + 2\eta^2 |z_x|^2_{\mathbb{L}^2}. \tag{3.49}$$

Hence, combining (3.48) and (3.49) we have

$$J_4(t) \leq C\left(1 + \frac{1}{\eta^2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2} |(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} \right) |z|^2_{\mathbb{L}^2} + 3\eta^2 |z_x|^2_{\mathbb{L}^2}.$$

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Therefore, combining all these estimates of J_i's for $i = 1, \ldots, 4$ and substituting back in (3.36), we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|z(t)|^2_{\mathbb{L}^2} \leq\ & -\lambda_2|z_x|^2_{\mathbb{L}^2} + k^2\Big(|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} + \frac{k^2}{2\eta^2}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2} + \frac{1}{\eta^2}\sum_{i=1}^{2}|(\mathbf{m}_i)_x|^2_{\mathbb{L}^2} \\
& + \frac{k^2}{4\eta^6}\sum_{i=1}^{2}|(\mathbf{m}_i)_x|^4_{\mathbb{L}^2}\Big)|z|^2_{\mathbb{L}^2} + \frac{9\eta^2}{2}|z_x|^2_{\mathbb{L}^2} + \frac{k^2}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} \\
& + \frac{k^4}{4\eta^6}|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2}|z|^2_{\mathbb{L}^2} + \frac{7}{4}\eta^2|z_x|^2_{\mathbb{L}^2} \\
& + C(1 + \frac{1}{\eta^2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2})|z|^2_{\mathbb{L}^2} + 3\eta^2|z_x|^2_{\mathbb{L}^2} \\
=\ & -\lambda_2|z_x|^2_{\mathbb{L}^2} + \varphi_C(t)|z|^2_{\mathbb{L}^2} + \frac{29}{4}\eta^2|z_x|^2_{\mathbb{L}^2},
\end{aligned}
\tag{3.50}
$$

where

$$
\begin{aligned}
\varphi_C(t) =\ & C\Big(1 + \frac{1}{\eta^2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}\Big) + k^2(1 + \frac{1}{\eta^2})|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} \\
& + k^4(\frac{1}{2\eta^2} + \frac{1}{4\eta^6})|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2} + \sum_{i=1}^{2}\Big(\frac{k^2}{\eta^2}|(\mathbf{m}_i)_x|^2_{\mathbb{L}^2} + \frac{k^2}{4\eta^6}|(\mathbf{m}_i)_x|^4_{\mathbb{L}^2}\Big).
\end{aligned}
$$

We note that as $\mathbf{m}_i \in \mathrm{L}^4(0, T; \mathbb{H}^1)$ for $i = 1, 2$, so φ_C is integrable on $[0, T]$, i.e., $\int_0^t \varphi_C(s)ds < \infty$. Choosing η so that $\frac{37}{4}\eta^2 = \lambda_2$, i.e., $\eta = \left(\frac{4\lambda_2}{37}\right)^{1/2}$. Then we have from (3.50),

$$
\frac{1}{2}\frac{d}{dt}|z(t)|^2_{\mathbb{L}^2} \leq \varphi_C(t)|z(t)|^2_{\mathbb{L}^2}.
$$

Integrating in $(0, t)$ we get

$$
|z(t)|^2_{\mathbb{L}^2} \leq |z(0)|^2_{\mathbb{L}^2} + 2\int_0^t \varphi_C(s)|z(s)|^2_{\mathbb{L}^2}ds.
$$

Using Gronwall's inequality we get

$$
|z(t)|^2_{\mathbb{L}^2} \leq |z(0)|^2_{\mathbb{L}^2} e^{2\int_0^t \varphi_C(s)ds}. \tag{3.51}
$$

Thus if $\mathbf{m}_1(0) = \mathbf{m}_2(0)$ then we have $z(0) = \mathbf{m}_1(0) - \mathbf{m}_2(0) = 0$. Hence from (3.51) we have $|z(t)|^2_{\mathbb{L}^2} = 0$, $\forall t \in [0, T]$. This implies $\int_0^1 |z(t, x)|^2_{\mathbb{R}^3}dx = 0$, $\forall t \in [0, T]$. Thus we have $z(t, x) = 0$, $\forall t \in [0, T]$, a.e. $x \in (0, 1)$. $\square$

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4. Existence and uniqueness of solutions to (2.15)–(2.17)

In this section we return to the study of the system of equations (2.15)–(2.17) and state existence and uniqueness theorem for $\mathbf{m}^n$.

Analogous to Definition 3.1 we can define weak solution for the system of equations (2.15)–(2.17).

Theorem 4.1. *Let* $\mathbf{m}_0 \in \mathbb{H}^1(D; \mathbb{S}^2)$. *Then there exists a unique solution* $\mathbf{m}^n \in C([0,T]; \mathbb{H}^1)$ *to the equation* (2.15)–(2.17), *i.e., it satisfies the following:*

1. *For every* $T > 0$,

$$\sup_{t\in[0,T]} |\mathbf{m}^n(t)|_{\mathbb{H}^1} \leq C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}); \tag{4.1}$$

2. *For almost every* $t \in [0,\infty)$, $\mathbf{m}^n(t) \times \mathbf{m}^n_{xx}(t) \in \mathbb{L}^2$ *and every* $T > 0$ *we have*

$$\int_0^T |\mathbf{m}^n(t) \times \mathbf{m}^n_{xx}(t)|^2_{\mathbb{L}^2} dt \leq C(T, \lambda_2, |\mathbf{m}_0|_{\mathbb{H}^1}); \tag{4.2}$$

3. $|\mathbf{m}^n(t,x)|_{\mathbb{R}^3} = 1$, *a.e.* $x \in D$ *and for all* $t \in [0,T]$;
4. *For all* $\phi \in \mathbb{H}^1$,

$$\begin{aligned}\langle \mathbf{m}^n(t), \phi\rangle_{\mathbb{L}^2} = \langle \mathbf{m}(0), \phi\rangle_{\mathbb{L}^2} &- \lambda_1 \int_0^t \int_D \langle \mathbf{m}^n_x(s,x), \phi_x(x) \times \mathbf{m}^n(s,x)\rangle_{\mathbb{R}^3} dx\, ds \\ &- \lambda_2 \int_0^t \int_D \langle \mathbf{m}^n_x(s,x), (\mathbf{m}^n \times \phi)_x(s,x) \times \mathbf{m}^n(s,x)\rangle_{\mathbb{R}^3} dx\, ds \\ &+ \int_0^t \int_D \langle F^n(s, \mathbf{m}^n(s,x)), \phi(x)\rangle_{\mathbb{R}^3} dx\, ds,\end{aligned}$$

holds for all $t \in [0,T]$.

Remark 4.2. We note that proof of Theorem 4.1 is completely analogous to the proof of Theorem 3.2. The only difference here is that because of the term $F^n(t, \mathbf{m}^n)$ (see (2.18) which involves the approximation W^n) present in the equation (2.15), constants in (4.1) and (4.2) will be dependent on n, say C_n. Then exploiting (1.8) we can choose a large constant C (independent of n) such that $\sup_{n\in\mathbb{N}} C_n := C < \infty$.

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5. Convergence of solution of the auxiliary equation

In this section we first prove the convergence result in the space $\mathrm{L}^\infty(0,T;\mathbb{L}^2)$ and then obtain the progressively measurability of both the approximated and limiting processes with respect to the corresponding filtrations.

Theorem 5.1. *If* $\mathbf{m}$ *and* $\mathbf{m}^n$ *are the unique solutions to the equations* (1.13) *and* (2.15) *respectively, then we have the following convergence:*

$$\mathbf{m}^n(\cdot,\omega) \to \mathbf{m}(\cdot,\omega) \quad \text{as} \quad n\to\infty \quad \text{for almost all} \quad \omega\in\Omega \tag{5.1}$$

in the natural topology of $\Phi := \mathrm{L}^\infty(0,T;\mathbb{L}^2)\cap \mathrm{L}^2(0,T;\mathrm{V})$.

Proof. We split the proof in three steps. In Step I, we will prove the following Lemma 5.2, while in Step II, we will apply this Lemma specifically to our case. In Step III, we will apply Step II and prove the required convergence in $\mathrm{L}^2(0,T;V)$.

Step I:

Lemma 5.2. *Let* $\mathbf{m}_i : [0,T]\to\mathbb{H}^1$ *be two weak solutions of*

$$\frac{\partial \mathbf{m}_i}{\partial t} + \lambda_2 A\mathbf{m}_i = \lambda_2|(\mathbf{m}_i)_x|^2_{\mathbb{R}^3}\mathbf{m}_i + \lambda_1(\mathbf{m}_i\times(\mathbf{m}_i)_{xx}) + F_i(t,\mathbf{m}_i) \tag{5.2}$$

for $i=1,2$ *where*

$$F_i(t,\mathbf{m}_i) = \lambda_1\mathbf{m}_i\times\hat{F}_i(t,\mathbf{m}_i) - \lambda_2\mathbf{m}_i\times(\mathbf{m}_i\times\hat{F}_i(t,\mathbf{m}_i))$$

and $\hat{F}_i$ *is given by* $\hat{F}_i(t,\mathbf{m}_i) = S(W_i)\mathfrak{S}(\mathbf{m}_i) + \mathcal{C}(W_i)\mathfrak{C}(\mathbf{m}_i)$ *and* $S,\mathfrak{S},\mathcal{C},\mathfrak{C}$ *are given by* (2.9)–(2.12)*. Then there exists a constant* $C>0$ *and an integrable function* φ_C*, so that we have the following estimate*

$$|\mathbf{m}_2(t)-\mathbf{m}_1(t)|^2_{\mathbb{L}^2} \le \Big(|\mathbf{m}_2(0)-\mathbf{m}_1(0)|^2_{\mathbb{L}^2} + C\int_0^t |W_2(s)-W_1(s)|ds\Big)e^{2\int_0^t\varphi_C(s)ds},\ t\in[0,T].$$

Proof of Lemma 5.2. We note that $\mathfrak{S}$ and $\mathfrak{C}$ are linear in $\mathbf{m}$. Let $z=\mathbf{m}_2-\mathbf{m}_1$. Then z is a weak solution of

$$\begin{aligned}\frac{\partial z}{\partial t} + \lambda_2 Az = \lambda_2\Big[|(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}\mathbf{m}_2 - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\mathbf{m}_1\Big] + \lambda_1\Big((\mathbf{m}_2\times(\mathbf{m}_2)_{xx}) - (\mathbf{m}_1\times(\mathbf{m}_1)_{xx})\Big)\\ + F_2(t,\mathbf{m}_2) - F_1(t,\mathbf{m}_1).\end{aligned} \tag{5.3}$$

Now again using Lemma 1.2 Chapter-III in Temam [62] to $|z|^2_{\mathbb{L}^2}$, we have

$$\frac{1}{2}\frac{d}{dt}|z(t)|^2_{\mathbb{L}^2} = -\lambda_2\langle Az,z\rangle_{\mathbb{L}^2} + \lambda_2\langle|(\mathbf{m}_2)_x|^2_{\mathbb{R}^3}\mathbf{m}_2 - |(\mathbf{m}_1)_x|^2_{\mathbb{R}^3}\mathbf{m}_1, z\rangle_{\mathbb{L}^2}$$

$$
\begin{aligned}
&+\lambda_1\Big(\langle(\mathbf{m}_2\times z_{xx}),z\rangle_{\mathbb{L}^2}-\langle(z\times(\mathbf{m}_1)_{xx}),z\rangle_{\mathbb{L}^2}\Big)+\langle F_2(t,\mathbf{m}_2)-F_1(t,\mathbf{m}_1),z\rangle_{\mathbb{L}^2}\\
&:=\sum_{i=1}^{4}L_i(t). \qquad (5.4)
\end{aligned}
$$

Now we estimate each L_i for $i=1,2,3,4$. We note that $L_i=J_i$ for $i=1,2,3$. So the estimates of $L_1,\ L_2,\ L_3$ follow from (3.37), (3.42) and (3.43) respectively. We now derive estimate $L_4(t)$ for fixed $\eta>0$, and a generic constant $C>0$.

$$
\begin{aligned}
L_4(t)&=\langle F_2(t,\mathbf{m}_2)-F_1(t,\mathbf{m}_1),z\rangle_{\mathbb{L}^2}\\
&\le|\langle\lambda_1(\mathbf{m}_2\times\hat{F}_2(t,\mathbf{m}_2))-\lambda_2(\mathbf{m}_2\times(\mathbf{m}_2\times\hat{F}_2(t,\mathbf{m}_2)))\\
&\quad-\lambda_1(\mathbf{m}_1\times\hat{F}_1(t,\mathbf{m}_1))+\lambda_2(\mathbf{m}_1\times(\mathbf{m}_1\times\hat{F}_1(t,\mathbf{m}_1))),z\rangle_{\mathbb{L}^2}|\\
&\le|\lambda_1||\langle\mathbf{m}_2\times\hat{F}_2(t,\mathbf{m}_2)-(\mathbf{m}_1\times\hat{F}_1(t,\mathbf{m}_1)),z\rangle_{\mathbb{L}^2}|\\
&\quad+|\lambda_2||\langle(\mathbf{m}_2\times(\mathbf{m}_2\times\hat{F}_2(t,\mathbf{m}_2)))-(\mathbf{m}_1\times(\mathbf{m}_1\times\hat{F}_1(t,\mathbf{m}_1))),z\rangle_{\mathbb{L}^2}|\\
&:=\sum_{i=1}^{2}L_4^i(t). \qquad (5.5)
\end{aligned}
$$

Using (2.8) and proceeding similarly as in (3.47), we have

$$
\begin{aligned}
\Big|\hat{F}_2(t,\mathbf{m}_2)-\hat{F}_1(t,\mathbf{m}_1)\Big|_{\mathbb{L}^2}&\le\Big|S(W_2)\Big[\mathfrak{S}(\mathbf{m}_2-\mathbf{m}_1)\Big]\Big|_{\mathbb{L}^2}+\Big|\Big[S(W_2)-S(W_1)\Big]\mathfrak{S}(\mathbf{m}_1)\Big|_{\mathbb{L}^2}\\
&\quad+\Big|\mathcal{C}(W_2)\mathfrak{C}(\mathbf{m}_2-\mathbf{m}_1)\Big|_{\mathbb{L}^2}\\
&\quad+\Big|\Big[\mathcal{C}(W_2)-\mathcal{C}(W_1)\Big]\mathfrak{S}(\mathbf{m}_1)\Big|_{\mathbb{L}^2}\\
&\le C\Big[|z|_{\mathbb{L}^2}+|z_x|_{\mathbb{L}^2}+|W_2-W_1|\Big(1+|(\mathbf{m}_1)_x|_{\mathbb{L}^2}\Big)\Big]. \qquad (5.6)
\end{aligned}
$$

Using (5.6), Hölder's inequality, $|(\mathbf{m}_1)_x|_{\mathbb{L}^2}|z|_{\mathbb{L}^2}\le\frac{1}{2}+\frac{|z|^2_{\mathbb{L}^2}|(\mathbf{m}_1)_x|^2_{\mathbb{L}^2}}{2}$ and $|z|_{\mathbb{L}^2}\le\frac{1}{2}+\frac{|z|^2_{\mathbb{L}^2}}{2}$ and arguments similar to (3.44) (as in $J_4^1(t)$ in Theorem 3.16), we get

$$
\begin{aligned}
L_4^1(t)&\le|\lambda_1||\langle z\times\hat{F}_2(t,\mathbf{m}_2),z\rangle_{\mathbb{L}^2}|+|\lambda_1||\mathbf{m}_1\times(\hat{F}_2(t,\mathbf{m}_2)-\hat{F}_1(t,\mathbf{m}_1)),z\rangle_{\mathbb{L}^2}|\\
&\le|\lambda_1||\hat{F}_2(t,\mathbf{m}_2)-\hat{F}_1(t,\mathbf{m}_1)|_{\mathbb{L}^2}|z|_{\mathbb{L}^2}\\
&\le C\Big[|z|_{\mathbb{L}^2}+|z_x|_{\mathbb{L}^2}+|W_2-W_1|(1+|(\mathbf{m}_1)_x|_{\mathbb{L}^2})\Big]|z|_{\mathbb{L}^2}\\
&\le\eta^2|z_x|^2_{\mathbb{L}^2}+C\Big[|z|^2_{\mathbb{L}^2}+\frac{1}{\eta^2}|z|^2_{\mathbb{L}^2}+|W_2-W_1|\Big(1+\frac{1}{2}\Big(1+|(\mathbf{m}_1)_x|^2_{\mathbb{L}^2}\Big)|z|^2_{\mathbb{L}^2}\Big]. \qquad (5.7)
\end{aligned}
$$

Using $|\hat{F}_2(t,\mathbf{m}_2)|_{\mathbb{L}^2}\le C(1+|(\mathbf{m}_2)_x|_{\mathbb{L}^2})$, $|\mathbf{m}_1(x)_{\mathbb{R}^3}=1$, for a.e. $x\in D$, and (5.6) and proceeding similar manner as in (5.7) and also following arguments in (3.45) (as in $J_4^2(t)$ in Theorem 3.16), we have

$$
\begin{aligned}
L_4^2(t) &\le |\lambda_2| \left| \left\langle z \times \Big(\mathbf{m}_2 \times \hat{F}_2(t, \mathbf{m}_2) \Big), z \right\rangle_{\mathbb{L}^2} \right| \\
&\quad + |\lambda_2| \int_D \left| \left\langle \mathbf{m}_1(x) \times \Big(\mathbf{m}_2(x) \times \hat{F}_2(t, \mathbf{m}_2(x)) - \mathbf{m}_1(x) \times \hat{F}_1(t, \mathbf{m}_1(x)) \Big), z(x) \right\rangle_{\mathbb{R}^3} \right| dx \\
&\le |\lambda_2| \int_D \Big(|z(x)|_{\mathbb{R}^3} |\hat{F}_2(t, \mathbf{m}_2(x))|_{\mathbb{R}^3} + |\mathbf{m}_1(x)|_{\mathbb{R}^3} |\hat{F}_2(t, \mathbf{m}_2(x)) - \hat{F}_1(t, \mathbf{m}_1(x))|_{\mathbb{R}^3} \Big) \\
&\quad \times |z(x)|_{\mathbb{R}^3} dx \\
&\le |\lambda_2| |z|_{\mathbb{L}^\infty} |z|_{\mathbb{L}^2} |\hat{F}_2(t, \mathbf{m}_2)|_{\mathbb{L}^2} + |\lambda_2| |\hat{F}_2(t, \mathbf{m}_2) - \hat{F}_1(t, \mathbf{m}_1)|_{\mathbb{L}^2} |z|_{\mathbb{L}^2} \\
&\le C \Big\{ |z|_{\mathbb{L}^2} |z|_{\mathbb{L}^\infty} (1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2}) + |z|_{\mathbb{L}^2} \Big(|z|_{\mathbb{L}^2} + |z_x|_{\mathbb{L}^2} + |W_2 - W_1| \Big(1 + |(\mathbf{m}_1)_x|_{\mathbb{L}^2} \Big) \Big) \Big\} \\
&:= L_4^{21}(t) + L_4^{22}(t). \qquad (5.8)
\end{aligned}
$$

Now, using the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^\infty$ and Young's inequality, the first term on right hand side of (5.8) becomes:

$$
\begin{aligned}
L_4^{21}(t) &= C |z|_{\mathbb{L}^2} |z|_{\mathbb{L}^\infty} \Big(1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} \Big) \\
&\le C |z|_{\mathbb{L}^2} \Big(|z|_{\mathbb{L}^2} + |z_x|_{\mathbb{L}^2} \Big) \Big(1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} \Big) \\
&= C \Big(1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2} (1 + |(\mathbf{m}_2)_x|_{\mathbb{L}^2}^2) \Big) |z|_{\mathbb{L}^2}^2 + \eta^2 |z_x|_{\mathbb{L}^2}^2. \qquad (5.9)
\end{aligned}
$$

Using Young's inequality, the second term on right hand side of (5.8) becomes:

$$
\begin{aligned}
L_4^{22}(t) &= C |z|_{\mathbb{L}^2} \Big(|z|_{\mathbb{L}^2} + |z_x|_{\mathbb{L}^2} + |W_2 - W_1| \Big(1 + |(\mathbf{m}_1)_x|_{\mathbb{L}^2} \Big) \Big) \\
&\le C \Big(1 + \frac{1}{\eta^2} + |W_2 - W_1| \Big(1 + |(\mathbf{m}_1)_x|_{\mathbb{L}^2}^2 \Big) \Big) |z|_{\mathbb{L}^2}^2 + C |W_2 - W_1| + \eta^2 |z_x|_{\mathbb{L}^2}^2. \qquad (5.10)
\end{aligned}
$$

Substituting (5.9), (5.10) in (5.8) we have

$$
\begin{aligned}
L_4^2(t) &\le C \Big\{ 1 + \frac{1}{\eta^2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{1}{\eta^2} |(\mathbf{m}_1)_x|_{\mathbb{L}^2}^2 + |W_2 - W_1| \Big(1 + |(\mathbf{m}_1)_x|_{\mathbb{L}^2}^2 \Big) \Big\} |z|_{\mathbb{L}^2}^2 \\
&\quad + C |W_2 - W_1| + 2\eta^2 |z_x|_{\mathbb{L}^2}^2.
\end{aligned}
$$

Hence, substituting back (5.9) and (5.10) in (5.8) and then using (5.7), $L_4(t)$ becomes:

$$
\begin{aligned}
L_4(t) \le \Big\{ & 3\eta^2 |z_x|_{\mathbb{L}^2}^2 + C(1 + \frac{1}{\eta^2}) + |(\mathbf{m}_2)_x|_{\mathbb{L}^2}^2 + \frac{1}{\eta^2} |(\mathbf{m}_1)_x|_{\mathbb{L}^2}^2 \qquad (5.11) \\
& + |W_2 - W_1| \Big(1 + |(\mathbf{m}_1)_x|_{\mathbb{L}^2}^2 \Big) \Big\} |z|_{\mathbb{L}^2}^2 + C |W_2 - W_1|.
\end{aligned}
$$

We choose η so that $\frac{37}{2}\eta^2 = \lambda_2$, i.e., $\eta = \left(\frac{2\lambda_2}{37}\right)^{1/2}$. Thus combining the estimates of L_i, $i = 1, 2, 3$ (see (3.37), (3.42), (3.43) respectively) and (5.11) and substituting back in (5.4), we have

$$\frac{1}{2}\frac{d}{dt}|z(t)|^2_{\mathbb{L}^2} + \frac{\lambda_2}{2}|z_x(t)|^2_{\mathbb{L}^2} \le \varphi_C(t)|z\mathbf{u}(t)|^2_{\mathbb{L}^2} + C|W_2(t) - W_1(t)|, \tag{5.12}$$

where

$$\begin{aligned}\varphi_C(t) = {} & C\Big(1 + \frac{37}{2\lambda_2} + |(\mathbf{m}_2)_x|_{\mathbb{L}^2} + \frac{37}{2\lambda_2}|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2}\Big) + k^2\Big(1 + \frac{37}{2\lambda_2}\Big)|(\mathbf{m}_2)_x|^2_{\mathbb{L}^2} \\ & + k^4\Big(\frac{37}{4\lambda_2} + \frac{1}{4}\Big(\frac{37}{2\lambda_2}\Big)^3\Big)|(\mathbf{m}_2)_x|^4_{\mathbb{L}^2} + \sum_{i=1}^{2}\Big(\frac{k^2}{\eta^2}|(\mathbf{m}_i)_x|^2_{\mathbb{L}^2} + \frac{k^2}{4\eta^6}|(\mathbf{m}_i)_x|^4_{\mathbb{L}^2}\Big) \\ & + C\Big(|(\mathbf{m}_1)_x|^2_{\mathbb{L}^2} + 1\Big)|W_2(t) - W_1(t)|.\end{aligned}$$

We note that as $\mathbf{m}_i \in \mathrm{L}^4(0, T; \mathbb{H}^1)$ for $i = 1, 2$, φ_C is integrable on $[0, T]$, i.e. $\int_0^t \varphi_C(s)ds < \infty$, for $t \in [0, T]$. Using Gronwall's inequality we get

$$|z(t)|^2_{\mathbb{L}^2} \le \Big(|z(0)|^2_{\mathbb{L}^2} + C\int_0^t |W_2(s) - W_1(s)|ds\Big)e^{2\int_0^t \varphi_C(s)ds}. \tag{5.13}$$

Step II: Choosing $\mathbf{m}_2 = \mathbf{m}^n$, $\mathbf{m}_1 = \mathbf{m}$, $W_2 = W^n$ and $W_1 = W$ in Lemma 5.2 and using (2.17) and (1.8) i.e., the fact that the Brownian motion W is approximated by sequence of suitable regular stochastic process W^n, we infer that (5.1) holds in the space $\mathrm{L}^\infty(0, T; \mathbb{L}^2)$.

Step III: From Step II and inequality (5.12), it can be also deduced by standard methods the convergence of z_n to 0 in the space $\mathrm{L}^2(0, T; \mathrm{V})$.

This completes the proof. □

Theorem 5.3. $\mathbf{m}$ *and* $\mathbf{m}^n$ *as processes from* $[0, T] \times \Omega$ *to* $\mathbb{L}^2$ *are progressively measurable with respect to the filtration* $\mathcal{F}_t$ *and* $\mathcal{F}_t^n$.

Proof. Since for each i, each Q_k^i's are polynomials in $\mathbf{m}_k$ with bounded coefficients and $\mathbf{m}_k$'s are solutions of (3.8)–(3.9), hence $\mathbf{m}_k$'s are progressively measurable. Hence $(\mathbf{m}_k(t), \phi)$ is progressively measurable for any $\phi \in \mathbb{L}^2$. Since $\mathbf{m}_k(t, \omega) \to \mathbf{m}(t, \omega)$ weakly in $\mathbb{L}^2$ for a.e. $\omega \in \Omega$, for each $t \in [0, T]$, thus it follows that $(\mathbf{m}(t), \phi)$ is progressively measurable for any $\phi \in \mathbb{L}^2$. Since strong and weak measurability are equivalent in a separable Hilbert space H, so $\mathbf{m}$ is progressively measurable process. The proof for $\mathbf{m}^n$ is analogous. □

6. Convergence in stronger topology via the maximal regularity

This section is devoted to prove maximal regularity of solutions $\mathbf{m}$ and $\mathbf{m}^n$ to both the respective system of equations. The key ingredients of the proof are the maximal regularity and ultracontractivity properties of the semigroup generated by the Laplace operator with the Neumann boundary conditions and the estimates for weak solutions of equations (1.13)–(1.15) and (2.15)–(2.17) obtained in Theorems 4.1 and 3.2.

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Secondly, we prove the convergence of the solutions in maximal regular space, stated as in Theorem 6.4.

We note that using part 3 of Theorem 3.2 we have $|\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1,\ \forall t \in [0,T]$, a.e. $x \in D$.

Using (2.3), (1.13) becomes:

$$\frac{\partial \mathbf{m}}{\partial t} = \lambda_1(\mathbf{m} \times \mathbf{m}_{xx}) + \lambda_2(\mathbf{m}_{xx} + |\mathbf{m}_x|^2_{\mathbb{R}^3}\mathbf{m}) + F(t,\mathbf{m}), \quad \text{in} \quad (0,1)\times(0,T). \tag{6.1}$$

Using the definition of A, (6.1) reduces to:

$$\frac{\partial \mathbf{m}}{\partial t} + \lambda_2 \mathrm{A}\mathbf{m} = \lambda_2|\mathbf{m}_x|^2_{\mathbb{R}^3}\mathbf{m} + \lambda_1(\mathbf{m} \times \mathbf{m}_{xx}) + F(t,\mathbf{m}), \quad \text{in} \quad (0,1)\times(0,T). \tag{6.2}$$

Theorem 6.1. *(Maximal Regularity Theorem) Assume $d = 1$. Suppose $\mathbf{m} : [0,T] \to \mathbb{H}^1$ is a weak solution of (1.13), $\mathbf{m}^n : [0,T] \to \mathbb{H}^1$ is a weak solution of (2.15). Then for every $\mathbf{m}_0 \in \mathbb{H}^1(D;\mathbb{S}^2)$, there exist positive constants $C_i = C_i(T, |\mathbf{m}_0|_{\mathbb{H}^1}, \lambda_1, \lambda_2)$, $i = 1, 2$, such that for every $n \in \mathbb{N}$,*

$$\int_0^T \left(|\mathbf{m}^n_{xx}(s)|^2_{\mathbb{L}^2} + |\mathbf{m}^n_x(s)|^4_{\mathbb{L}^4}\right) ds \le C_1, \tag{6.3}$$

and

$$\int_0^T \left(|\mathbf{m}_{xx}(s)|^2_{\mathbb{L}^2} + |\mathbf{m}_x(s)|^4_{\mathbb{L}^4}\right) ds \le C_2. \tag{6.4}$$

Proof. Let us fix $T > 0$, $r > 0$ and $\delta \in (\frac{5}{8}, \frac{3}{4})$. Let us choose $\lambda_1 = \lambda_2 = 1$.

Let $\{e^{-t\mathrm{A}}\}_{t\ge 0}$ be the semigroup generated by A. Hence, $\mathbf{m}$ can be written in mild form as:

$$\begin{aligned}\mathbf{m}(t) =& e^{-t\mathrm{A}}\mathbf{m}_0 + \int_0^t e^{-(t-s)\mathrm{A}}(\mathbf{m}(s) \times \mathbf{m}_{xx}(s))ds + \int_0^t e^{-(t-s)\mathrm{A}}|\mathbf{m}_x(s)|^2_{\mathbb{R}^3}\mathbf{m}(s)ds \\ &+ \int_0^t e^{-(t-s)\mathrm{A}} F(s,\mathbf{m}(s))ds \qquad (6.5)\\ &:= \sum_{i=1}^4 I_i(t). \qquad (6.6)\end{aligned}$$

We note that $e^{-t\mathrm{A}}$ is ultracontractive [see [5]], i.e., there exists a positive constant $C = C(p,q)$ such that for $1 \le p \le q \le \infty$,

$$|e^{-t\mathrm{A}} f|_{\mathbb{L}^q} \le \frac{C}{t^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}}|f|_{\mathbb{L}^p}, \qquad f \in \mathbb{L}^p, t > 0, \tag{6.7}$$

and A has maximal regularity property, i.e., there exists a $C > 0$ such that for any $f \in \mathrm{L}^2(0, T; \mathbb{L}^2)$ and

$$u(t) = \int_0^t e^{-(t-s)\mathrm{A}} f(s) ds, \ t \in [0, T],$$

we have

$$\int_0^t |\mathrm{A}u(t)|^2_{\mathbb{L}^2} dt \le C \int_0^t |f(t)|^2_{\mathbb{L}^2} dt. \tag{6.8}$$

Using Theorem 1.1 in Pazy [57], we conclude that $\mathrm{A}_1 = \mathrm{A} + I$ generates a semigroup denoted by $e^{-t\mathrm{A}_1}$. Furthermore using (6.7) we observe that

$$|e^{-t\mathrm{A}_1} f|_{\mathbb{L}^q} = |e^{-t\mathrm{A}} e^{-tI} f|_{\mathbb{L}^q} \le C |e^{-t\mathrm{A}} f|_{\mathbb{L}^q} \le \frac{C}{t^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}} |f|_{\mathbb{L}^p}, \quad f \in \mathbb{L}^p, \ t > 0. \tag{6.9}$$

We split our proof in two steps. In Step I we will show the second part of the inequality (6.4) i.e., we will show

$$\int_0^T |\mathbf{m}(t)|^4_{\mathbb{W}^{1,4}} dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}),$$

which will give us

$$\int_0^T |\mathbf{m}_x(t)|^4_{\mathbb{L}^4} dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

In Step II we deduce the first part of the inequality (6.4), i.e., $\int_0^T |\mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2} dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1})$.

Step I: We will first show that

$$\int_0^T |\mathbf{m}(t)|^4_{\mathbb{W}^{1,4}} dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

From Section 2, recall the definition of X^δ for $\delta \ge 0$, and note that due to the Sobolev imbedding $X^\delta \hookrightarrow \mathbb{W}^{1,4}$ holds for $\delta \in (\frac{5}{8}, \frac{3}{4})$ (see Appendix A). Hence it is sufficient to prove the following stronger estimate:

$$\int_0^T |\mathrm{A}_1^\delta \mathbf{m}(t)|^4_{\mathbb{L}^2} dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

We note from part (c) of Theorem 6.13 in Pazy [57], that for every $t > 0$ the operator $\mathrm{A}^\delta e^{-t\mathrm{A}}$ and $\mathrm{A}_1^\delta e^{-t\mathrm{A}_1}$ are bounded and

$$\|\mathrm{A}^\delta e^{-t\mathrm{A}}\| \leq \frac{C}{t^\delta}, \tag{6.10}$$

$$\|\mathrm{A}_1^\delta e^{-t\mathrm{A}_1}\| \leq \frac{C}{t^\delta}, \tag{6.11}$$

where $\|\cdot\|$ is the operator norm.

Now we will estimate each of I_i's, $i = 1, 2, 3, 4$, from (6.6).

Estimate of I_1: Recall $I_1(t) := e^{-t\mathrm{A}}\mathbf{m}_0$. For each $t \in [0, T]$, using (6.11) we have

$$\begin{aligned}
|\mathrm{A}_1^\delta I_1(t)|^4_{\mathbb{L}^2} &= |(I+\mathrm{A})^\delta e^{-t(I+\mathrm{A})} e^{tI}\mathbf{m}_0|^4_{\mathbb{L}^2} \leq Ce^t |\mathrm{A}_1^\delta e^{-t\mathrm{A}_1}\mathbf{m}_0|^4_{\mathbb{L}^2} \\
&\leq Ce^T |\mathrm{A}_1^{\delta-1/2} e^{-t\mathrm{A}_1}\mathrm{A}_1^{1/2}\mathbf{m}_0|^4_{\mathbb{L}^2} \leq C\|\mathrm{A}_1^{\delta-1/2}e^{-t\mathrm{A}_1}\|^4 |\mathrm{A}_1^{1/2}\mathbf{m}_0|^4_{\mathbb{L}^2} \\
&\leq \frac{C}{t^{4\delta-2}}|\mathrm{A}_1^{1/2}\mathbf{m}_0|^4_{\mathbb{L}^2} = \frac{C}{t^{4\delta-2}}|\mathbf{m}_0|^4_{\mathbb{H}^1},
\end{aligned}$$

and therefore, since $\delta < \frac{3}{4}$ so that $2 - 4\delta > -1$ we infer that

$$\int_0^T |\mathrm{A}_1^\delta I_1(t)|^4_{\mathbb{L}^2} dt \leq C|\mathbf{m}_0|^4_{\mathbb{H}^1} \int_0^T \frac{1}{t^{4\delta-2}} dt = C|\mathbf{m}_0|^4_{\mathbb{H}^1} \int_0^T t^{2-4\delta} dt \leq C|\mathbf{m}_0|^4_{\mathbb{H}^1}.$$

Estimate of I_2: We move to $I_2(t) := \int_0^t e^{-(t-s)\mathrm{A}}(\mathbf{m}(s) \times \mathbf{m}_{xx}(s))ds$. Let us take $f = \mathbf{m} \times \mathbf{m}_{xx}$ and substituting it in (6.9) and using (6.11) we have,

$$|\mathrm{A}_1^\delta e^{-(t-s)\mathrm{A}_1} f(s)|_{\mathbb{L}^2} \leq \|\mathrm{A}_1^\delta e^{-(t-s)\mathrm{A}_1}\| |f(s)|_{\mathbb{L}^2} \leq \frac{C}{(t-s)^\delta}|f(s)|_{\mathbb{L}^2}, \ 0 < s < t < T. \tag{6.12}$$

Hence using (6.12) we have

$$\begin{aligned}
\int_0^T |\mathrm{A}_1^\delta I_2(t)|^4_{\mathbb{L}^2} dt &\leq \int_0^T \Big(\int_0^t |\mathrm{A}_1^\delta e^{-(t-s)\mathrm{A}} f(s)|_{\mathbb{L}^2} ds\Big)^4 dt \\
&= \int_0^T \Big(\int_0^t |\mathrm{A}_1^\delta e^{-(t-s)\mathrm{A}_1} f(s) e^{(t-s)I}|_{\mathbb{L}^2} ds\Big)^4 dt \\
&\leq C \int_0^T \Big(\int_0^t (t-s)^{-\delta} |f(s)|_{\mathbb{L}^2} ds\Big)^4 dt.
\end{aligned}$$

Now exploiting Young's inequality, $\delta < \frac{3}{4}$ and using (3.6) we obtain

$$\int_0^T |A_1^\delta I_2(t)|_{\mathbb{L}^2}^4 dt \le C\Big(\int_0^T s^{\frac{-4\delta}{3}}\Big)^3 \Big(\int_0^T |f(s)|_{\mathbb{L}^2}^2 ds\Big)^2 \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Estimate of I_3: We move to $I_3(t) =: \int_0^t e^{-(t-s)A} |\mathbf{m}_x(s)|_{\mathbb{R}^3}^2 \mathbf{m}(s) ds$. We note that for $f = |\mathbf{m}_x|_{\mathbb{R}^3}^2 \mathbf{m}$, using (3.5) we have

$$\begin{aligned} \sup_{0\le s\le t} |f(s)|_{\mathbb{L}^1} &= \sup_{0\le s\le t} \Big(\int_0^1 |\mathbf{m}_x(s,x)|_{\mathbb{R}^3}^2 |\mathbf{m}(s,x)|_{\mathbb{R}^3} dx\Big) \\ &= \sup_{0\le s\le t} |\mathbf{m}_x(s)|_{\mathbb{L}^2}^2 \le \sup_{0\le s\le t} |\mathbf{m}(s)|_{\mathbb{H}^1}^2 < \infty. \end{aligned} \tag{6.13}$$

Hence using $f = |\mathbf{m}_x|_{\mathbb{R}^3}^2 \mathbf{m} \in L^\infty(0, T; \mathbb{L}^1)$, and putting it in (6.9) with $p = 1, q = 2$, we see that there exists a positive constant C (depending on p, q) such that using (6.13) we achieve

$$\begin{aligned} |A_1^\delta e^{-(t-s)A} f(s)|_{\mathbb{L}^2} &\le |A_1^\delta e^{-(t-s)A_1} e^{(t-s)I} f(s)|_{\mathbb{L}^2} \le C|A_1^\delta e^{\frac{-(t-s)A_1}{2}} e^{\frac{-(t-s)A_1}{2}} f(s)|_{\mathbb{L}^2} \\ &\le \|A_1^\delta e^{\frac{-(t-s)A_1}{2}}\| |e^{\frac{-(t-s)A_1}{2}} f(s)|_{\mathbb{L}^2} \\ &\le \frac{C}{(t-s)^{\delta+\frac{1}{4}}} |f(s)|_{\mathbb{L}^1} \le \frac{C}{(t-s)^{\delta+\frac{1}{4}}} \sup_{s\in[0,t]} |f(s)|_{\mathbb{L}^1} \\ &\le \frac{C}{(t-s)^{\delta+\frac{1}{4}}} \sup_{r\in[0,T]} |\mathbf{m}(r)|_{\mathbb{H}^1}^2, \ 0 < s < t < T. \end{aligned}$$

Therefore,

$$\int_0^T \Big|\int_0^t A_1^\delta e^{-(t-s)A} f(s) ds\Big|_{\mathbb{L}^2}^4 dt \le C \sup_{r\in[0,T]} |\mathbf{m}(r)|_{\mathbb{H}^1}^8 \int_0^T \Big(\int_0^t \frac{ds}{(t-s)^{\delta+\frac{1}{4}}}\Big)^4 dt.$$

Using $\delta + \frac{1}{4} < 1$ and (3.5) we achieve

$$\int_0^T |A_1^\delta I_3(t)|_{\mathbb{L}^2}^4 dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Estimate of I_4: We move to $I_4(t) =: \int_0^t e^{-(t-s)A} F(s, \mathbf{m}(s)) ds$. Now using (2.6), $g \in \mathbb{W}^{2,\infty}$, the semigroup generated by G, $e^{-W(t)G}$ is bounded in $\mathbb{L}^2$, implementing simple vector product rule $|a \times b|_{\mathbb{R}^3} = |a|_{\mathbb{R}^3} |b|_{\mathbb{R}^3} |\sin\theta|$ and also using $|\mathbf{m}(t,x)|_{\mathbb{R}^3} = 1$, $\forall t \ge 0$, a.e. $x \in D$, we have

$$|F(s, \mathbf{m}(s))|_{\mathbb{L}^2} \le C(1 + |\mathbf{m}_x(s)|_{\mathbb{L}^2}) \le C|\mathbf{m}(s)|_{\mathbb{H}^1}. \tag{6.14}$$

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Hence using (6.11), (6.14) and (6.9) we have

$$\begin{aligned}|\mathrm{A}_1^{\delta} e^{-(t-s)\mathrm{A}} F(s,\mathbf{m}(s))|_{\mathbb{L}^2} &\le |\mathrm{A}_1^{\delta} e^{-(t-s)\mathrm{A}_1} e^{(t-s)I} F(s,\mathbf{m}(s))|_{\mathbb{L}^2} \\ &\le C\|\mathrm{A}_1^{\delta} e^{-(t-s)\mathrm{A}_1}\| |F(s,\mathbf{m}(s))|_{\mathbb{L}^2} \\ &\le \frac{C}{(t-s)^{\delta}} |F(s,\mathbf{m}(s))|_{\mathbb{L}^2} \\ &\le \frac{C}{(t-s)^{\delta}} \sup_{r\in[0,T]} |\mathbf{m}(r)|_{\mathbb{H}^1},\ 0<s<t<T.\end{aligned}$$

Thus as $\delta < 1$ and using Theorem 3.2

$$\begin{aligned}&\int_0^T |\mathrm{A}_1^{\delta} I_4(t)|_{\mathbb{L}^2}^4 dt \le C \sup_{r\in[0,T]} |\mathbf{m}(r)|_{\mathbb{H}^1}^4 \Big[\int_0^T \Big(\int_0^t (t-s)^{-\delta} ds\Big)^4 dt\Big] \\ &\le C \sup_{r\in[0,T]} |\mathbf{m}(r)|_{\mathbb{H}^1}^4 \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).\end{aligned}$$

Step II: Now we will show that

$$\int_0^T |\mathbf{m}_{xx}(t)|_{\mathbb{L}^2}^2 dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}). \tag{6.15}$$

We note that $I_1(t) = e^{-t\mathrm{A}}\mathbf{m}_0$ satisfies the equation $\frac{dI_1(t)}{dt} + \mathrm{A}I_1(t) = 0,\ I(0) = I_0$. Multiplying this equation with $\mathrm{A}I_1(t)$ we see that

$$\int_0^T |\mathrm{A}I_1(t)|_{\mathbb{L}^2}^2 dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Now using maximal inequality (6.8) and (3.6) we have

$$\int_0^T |\mathrm{A}I_2(t)|_{\mathbb{L}^2}^2 dt \le C \int_0^T |\mathbf{m}(t)\times\mathbf{m}_{xx}(t)|_{\mathbb{L}^2}^2 dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Again by substituting $u = I_3(t),\ f = |\mathbf{m}_x|_{\mathbb{R}^3}^2 \mathbf{m}$ in maximal inequality (6.8), Step I and using $|\mathbf{m}(s)|_{\mathbb{L}^2}^2 = \int_0^1 |\mathbf{m}(s,x)|_{\mathbb{R}^3}^2 dx = 1$, we get

$$\int_0^T |\mathrm{A}I_3(t)|_{\mathbb{L}^2}^2 dt \le C \int_0^T |\mathbf{m}_x(t)|_{\mathbb{L}^2}^4 |\mathbf{m}(t)|_{\mathbb{L}^2}^2 dt \le C \int_0^T |\mathbf{m}(t)|_{\mathbb{H}^1}^4 dt \le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Again by substituting $u = I_4(t),\ f = F(t, \mathbf{m}(t))$ in maximal inequality (6.8) we have

$$\int_0^T |\mathrm{A} I_4(t)|^2_{\mathbb{L}^2} dt \le C \int_0^T |F(t, \mathbf{m}(t))|^2_{\mathbb{L}^2} dt \le C \int_0^T |\mathbf{m}(t)|^2_{\mathbb{H}^1} dt \le C \sup_{0\le s\le T} |\mathbf{m}(s)|^2_{\mathbb{H}^1}$$
$$\le C(T, \|\mathbf{m}_0\|_{\mathbb{H}^1}).$$

Hence (6.15) is obtained. This completes the proof of (6.4).

Now (6.3) can be obtained by proceeding in the similar manner as in the proof of the inequality (6.4). Note that, following Remark 4.2, we infer that in this case, the constant at first may depend on n, which can later be made independent of n. □

Corollary 6.2. *Using (3.5) in Theorem 3.2 and (6.4) we have* $\mathbf{m} \in \mathrm{L}^\infty(0, T; \mathbb{H}^1) \cap \mathrm{L}^2(0, T; D(\mathrm{A}))$. *Again using (4.1) in Theorem 4.1 and (6.3) we can conclude that* $\mathbf{m}^n \in \mathrm{L}^\infty(0, T; \mathbb{H}^1) \cap \mathrm{L}^2(0, T; D(\mathrm{A}))$.

Proposition 6.3. *The solution* $\mathbf{m}$ *lie in the space* $C([0, T]; \mathbb{H}^1)$.

Proof. Using the fact that $\mathbf{m} \in \mathrm{L}^2(0, T; \mathbb{H}^2)$ and $\dfrac{d\mathbf{m}}{dt} \in \mathrm{L}^2(0, T; \mathbb{L}^2)$, we have $\mathbf{m} \in C([0, T]; \mathbb{H}^1)$. □

Finally, as a consequence of Theorem 5.1 and Corollary 6.2 we have the following key result.

Theorem 6.4. *If* $\mathbf{m}$ *and* $\mathbf{m}^n$ *are the unique solutions to the equations (1.13) and (2.15) respectively such that* $\mathbf{m}, \mathbf{m}^n \in \mathrm{L}^4(0, T; \mathbb{H}^1)$, *then we have the following convergence:*

$$\mathbf{m}^n(\cdot, \omega) \to \mathbf{m}(\cdot, \omega) \quad \textit{as} \quad n \to \infty \quad \textit{for almost all} \quad \omega \in \Omega \tag{6.16}$$

in the natural topology of $\Phi := \mathrm{L}^\infty(0, T; \mathbb{H}^1) \cap \mathrm{L}^2(0, T; D(\mathrm{A}))$.

Proof. Let us choose $\lambda_1 = \lambda_2 = 1$. Our aim is to prove (omitting ω for simplicity of notations)

$$\lim_{n\to\infty} \Big[\sup_{s\in[0,T]} |\mathbf{m}^n(t) - \mathbf{m}(t)|^2_{\mathbb{H}^1} + \int_0^T |(\mathbf{m}^n_{xx}(t) - \mathbf{m}_{xx}(t))|^2_{\mathbb{L}^2} dt \Big] = 0.$$

Similarly to equation (3.35), if we substitute $z^n := \mathbf{m}^n - \mathbf{m}$ then $z^n(t)$ satisfies

$$z^n(t) = \int_0^t z^n_{xx}(s) ds + \int_0^t \Big(|\mathbf{m}^n_x(s)|^2_{\mathbb{R}^3} \mathbf{m}^n(s) - |\mathbf{m}_x(s)|^2_{\mathbb{R}^3} \mathbf{m}(s) \Big) ds$$
$$+ \int_0^t \Big(\mathbf{m}^n(s) \times \mathbf{m}^n_{xx}(s) - \mathbf{m}(s) \times \mathbf{m}_{xx}(s) \Big) + \int_0^t \Big(F^n(s, \mathbf{m}^n) - F(s, \mathbf{m}) \Big) ds,$$

for all $t \in [0, T]$. Taking inner product in $\mathbb{L}^2$ with z^n_{xx}, (i.e., by applying Lemma 1.2 Chapter-III in Temam [62] to $|z^n_{xx}|^2_{\mathbb{L}^2}$) and integrating by parts, we have for all $t \in [0, T]$,

$$
\begin{aligned}
&|z^n_x(t)|^2_{\mathbb{L}^2} + 2\int_0^t |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds \\
&= -2\int_0^t \left\langle \Big(|\mathbf{m}^n_x(s)|_{\mathbb{R}^3} - |\mathbf{m}_x(s)|_{\mathbb{R}^3}\Big)\Big(|\mathbf{m}^n_x(s)|_{\mathbb{R}^3} + |\mathbf{m}_x(s)|_{\mathbb{R}^3}\Big)\mathbf{m}^n(s), z^n_{xx}(s)\right\rangle_{\mathbb{L}^2} ds \\
&\quad - 2\int_0^t \left\langle (|\mathbf{m}_x(s)|^2_{\mathbb{R}^3} z^n(s), z^n_{xx}(s)\right\rangle_{\mathbb{L}^2} ds - 2\int_0^t \left\langle z^n(s) \times \mathbf{m}^n_{xx}(s), z^n_{xx}(s)\right\rangle_{\mathbb{L}^2} ds \\
&\quad - 2\int_0^t \left\langle F^n(s, \mathbf{m}^n(s)) - F(s, \mathbf{m}(s)), z^n_{xx}(s)\right\rangle_{\mathbb{L}^2} ds \\
&:= \sum_{i=1}^4 \mathcal{I}_i(t)
\end{aligned}
\tag{6.17}
$$

In order to estimate the terms $\mathcal{I}_i(t)$ for $i = 1, 2, 3$, we follow the steps due to [17], see Step 3 of the proof of Lemma 6.3, and [23], and obtain the following estimates: for fixed $\eta > 0$, and a generic constant $C > 0$, we have for all $t \in [0, T]$,

$$
\mathcal{I}_1(t) \le \frac{3}{\eta^2}\int_0^t |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds + \left(C\eta^2 + \frac{\eta^6}{2}\right)\int_0^t \Big(|\mathbf{m}^n_x(s)|^4_{\mathbb{L}^4} + |\mathbf{m}_x(s)|^4_{\mathbb{L}^4}\Big)|z^n(s)|^2_{\mathbb{H}^1} ds;
\tag{6.18}
$$

$$
\mathcal{I}_2(t) \le \frac{1}{\eta^2}\int_0^t |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds + 4\eta^2\int_0^t |\mathbf{m}_x(s)|^4_{\mathbb{L}^4}|z^n(s)|^2_{\mathbb{L}^2} ds;
\tag{6.19}
$$

$$
\mathcal{I}_3(t) \le \frac{1}{\eta^2}\int_0^t |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds + k^2\eta^2\int_0^t |\mathbf{m}^n_x(s)|^2_{\mathbb{L}^2}|z^n(s)|^2_{\mathbb{H}^1} ds.
\tag{6.20}
$$

We will now estimate the term $\mathcal{I}_4$. Note,

$$
\begin{aligned}
\mathcal{I}_4(t) &\le \int_0^t |\langle \mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)) - (\mathbf{m}(s) \times \hat{F}(s, \mathbf{m}(s))), z^n_{xx}(s)\rangle_{\mathbb{L}^2}| \\
&\quad + |\langle (\mathbf{m}^n(s) \times (\mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)))) - (\mathbf{m}(s) \times (\mathbf{m}(s) \times \hat{F}(s, \mathbf{m}(s)))), z^n_{xx}(s)\rangle_{\mathbb{L}^2}| ds \\
&:= 2\int_0^t [\mathcal{I}^1_4(s) + \mathcal{I}^2_4(s)] ds.
\end{aligned}
\tag{6.21}
$$

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Following similar steps involved in estimating $L_4(t)$ in Lemma 5.2, in order to estimate $\mathcal{I}_4^i(s)$, $i = 1, 2$ we have for all $s \in [0, T]$,

$$\mathcal{I}_4^1(s) \leq \frac{2}{\eta^2} |z_{xx}^n(s)|_{\mathbb{L}^2}^2 + 4C\eta^2 \Big[|z^n(s)|_{\mathbb{L}^2}^2 + |z_x^n(s)|_{\mathbb{L}^2}^2 + |W^n(s) - W(s)|^2 (1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}^2) \Big] \\ + 4\eta^2 |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}^2 (|z_x(s)|_{\mathbb{L}^2}^2 + |z_x^n(s)|_{\mathbb{L}^2}^2), \tag{6.22}$$

and

$$\mathcal{I}_4^2(s) := \Big| \Big\langle z^n(s) \times \Big(\mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)) \Big), z_{xx}^n(s) \Big\rangle_{\mathbb{L}^2} \Big| \\ + \Big| \Big\langle \mathbf{m}^n(s) \times \Big(\mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)) - \mathbf{m} \times \hat{F}(s, \mathbf{m}(s)) \Big), z_{xx}^n(s) \Big\rangle_{\mathbb{L}^2} \Big| \\ := \mathcal{I}_4^{21}(s) + \mathcal{I}_4^{22}(s). \tag{6.23}$$

Hence using $|\hat{F}^n(s, \mathbf{m}^n(s))|_{\mathbb{L}^2} \leq 1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}$, $|\mathbf{m}^n(x)|_{\mathbb{R}^3} = 1$, for a.e. $x \in D$, and Young's inequality, we have for all $s \in [0, T]$,

$$\mathcal{I}_4^{21}(s) = \Big| \Big\langle z^n(s) \times \Big(\mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)) \Big), z_{xx}^n(s) \Big\rangle_{\mathbb{L}^2} \Big| \\ \leq C\, |z^n(s)|_{\mathbb{L}^\infty} \Big(1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2} \Big) |z_{xx}^n(s)|_{\mathbb{L}^2} \\ \leq \frac{1}{\eta^2} |z_{xx}^n(s)|_{\mathbb{L}^2}^2 + 4\eta^2 (1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}^2) |z^n(s)|_{\mathbb{H}^1}^2. \tag{6.24}$$

Using $|\mathbf{m}^n(x)|_{\mathbb{R}^3} = 1$, for a.e. $x \in D$, and Young's inequality, $\mathcal{I}_4^{22}(s)$ becomes:

$$\mathcal{I}_4^{22}(s) \\ = \Big| \Big\langle \mathbf{m}^n(s) \times \Big(\mathbf{m}^n(s) \times \hat{F}^n(s, \mathbf{m}^n(s)) - \mathbf{m}(s) \times \hat{F}(s, \mathbf{m}(s)) \Big), z_{xx}^n(s) \Big\rangle_{\mathbb{L}^2} \Big| \\ \leq \int_D |\mathbf{m}^n(s,x)|_{\mathbb{R}^3} \Big| \mathbf{m}^n(s,x) \times \hat{F}^n(s, \mathbf{m}^n(s,x)) - \mathbf{m}(s,x) \times \hat{F}(s, \mathbf{m}(s,x)) |_{\mathbb{R}^3} |z_{xx}^n(s,x) \Big|_{\mathbb{R}^3} dx \\ \leq \int_D \Big| \Big(\mathbf{m}^n(s,x) - \mathbf{m}(s,x) \Big) \times \hat{F}^n(s, \mathbf{m}^n(s,x)) \Big|_{\mathbb{R}^3} |z_{xx}^n(s,x)|_{\mathbb{R}^3} dx \\ + \int_D \Big| \mathbf{m}(s,x) \times \Big(\hat{F}^n(s, \mathbf{m}^n(s,x)) - \hat{F}(s, \mathbf{m}(s,x)) \Big) \Big|_{\mathbb{R}^3} |z_{xx}^n(s,x)|_{\mathbb{R}^3} dx \\ \leq \frac{1}{\eta^2} |z_{xx}^n(s)|_{\mathbb{L}^2}^2 + 4\eta^2 \Big(1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}^2 \Big) |z^n(s)|_{\mathbb{H}^1}^2 + 4\eta^2 |W^n(s) - W(s)|^2 \Big(1 + |\mathbf{m}_x^n(s)|_{\mathbb{L}^2}^2 \Big). \tag{6.25}$$

Combining (6.22)–(6.25) in (6.21), we get for all $t \in [0, T]$,

$$\mathcal{I}_4(t) \le \frac{5}{\eta^2}|z^n_{xx}(t)|^2_{\mathbb{L}^2} + C\eta^2\Big\{\Big(1+|\mathbf{m}^n_x(t)|^2_{\mathbb{L}^2}\Big)|z^n(t)|^2_{\mathbb{H}^1} + |W^n(t)-W(t)|^2\Big(1+|\mathbf{m}^n_x(t)|^2_{\mathbb{L}^2}\Big)\Big\}. \tag{6.26}$$

Choosing $\eta > 0$ such that $\frac{15}{\eta^2} = 1$, and substituting (6.18), (6.19), (6.20), (6.26) in (6.17) we get

$$|z^n_x(t)|^2_{\mathbb{L}^2} + \int_0^t |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds \le \int_0^t \phi_n(s)|z^n(s)|^2_{\mathbb{H}^1} ds + \int_0^t \psi_n(s) ds$$

where

$$\psi_n(s) = C\Big(1+|\mathbf{m}_x(s)|^2_{\mathbb{L}^2}\Big)|W^n(s)-W(s)|^2,$$

and

$$\phi_n(s) = C\Big(|\mathbf{m}^n_x(s)|^4_{\mathbb{L}^4} + |\mathbf{m}_x(s)|^4_{\mathbb{L}^4}\Big) + 15k^2|\mathbf{m}^n_{xx}(s)|^2_{\mathbb{L}^2} + C\Big(1+|\mathbf{m}^n_x(s)|^2_{\mathbb{L}^2}\Big).$$

Using Gronwall's inequality we have

$$\sup_{t\in[0,T]} |z^n_x(t)|^2_{\mathbb{L}^2} + \int_0^T |z^n_{xx}(s)|^2_{\mathbb{L}^2} ds \le e^{\int_0^T \phi_n(s)ds}\Big(\int_0^T \psi_n(s)ds\Big). \tag{6.27}$$

We note that $\psi_n \in \mathrm{L}^1(0,T)$ and using Theorem 4.1 and equation (1.8) we have

$$\int_0^T \psi_n(s)ds \le C|W^n - W|^2_{\mathrm{L}^\infty(0,T)} \int_0^T (1+|\mathbf{m}^n_x(s)|^2_{\mathbb{L}^2})ds \to 0 \quad \text{as} \quad n\to\infty. \tag{6.28}$$

Using $\mathbf{m}, \mathbf{m}^n \in \mathrm{L}^4(0,T;\mathbb{H}^1)$, and Theorem 6.1, we infer that

$$\sup_{n\in\mathbb{N}} \int_0^T \phi_n(s)ds < \infty. \tag{6.29}$$

Hence using (6.28), (6.29) in (6.27), the convergence result (6.16) follows. This completes the proof. □

Corollary 6.5. *Suppose that $\mathbf{m}^n \to \mathbf{m}$ in the sense of* (6.16)*. Then if F is defined by* (1.16)*, then $F^n(\cdot,\mathbf{m}^n) \to F(\cdot,\mathbf{m})$ in $\mathrm{L}^2(0,T;\mathbb{L}^2)$.*

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Proof. Recalling (2.7) and (2.8), first we see that

$$\begin{aligned}
&|F^n(\cdot,\mathbf{m}^n) - F(\cdot,\mathbf{m})|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\leq \left|\lambda_1\Big(\mathbf{m}^n \times \hat{F}^n(\cdot,\mathbf{m}^n) - \mathbf{m}\times\hat{F}(\cdot,\mathbf{m})\Big)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\quad + \left|\lambda_2\Big(\mathbf{m}^n \times (\mathbf{m}^n \times \hat{F}^n(\cdot,\mathbf{m}^n)) - \mathbf{m}\times(\mathbf{m}\times\hat{F}(\cdot,\mathbf{m}))\Big)\right|_{\mathrm{L}^2(0,T;\mathbb{L}^2)}.
\end{aligned} \tag{6.30}$$

Now using (2.8), and proceeding as in (5.6), and using the assumption that $g \in \mathbb{W}^{2,\infty}$, we obtain

$$\begin{aligned}
&|\hat{F}^n(\cdot,\mathbf{m}^n) - \hat{F}(\cdot,\mathbf{m})|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\leq \Big|e^{-W(t)G}\sin(W(t))\Big((\mathbf{m}^n-\mathbf{m})\times g_{xx} + 2(\mathbf{m}^n-\mathbf{m})_x \times g_x\Big) \\
&\quad + e^{-W(t)G}[1-\cos(W(t))]\Big(((\mathbf{m}^n-\mathbf{m})\times g_{xx})\times g + ((\mathbf{m}^n-\mathbf{m})\times g)\times g_{xx} \\
&\quad + 2\Big(((\mathbf{m}^n-\mathbf{m})_x \times g_x)g + ((\mathbf{m}^n-\mathbf{m})_x\times g)\times g_x + ((\mathbf{m}^n-\mathbf{m})\times g_x)\times g_x\Big)\Big)\Big|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\leq C|\mathbf{m}^n-\mathbf{m}\Big|_{\mathrm{L}^2(0,T;\mathbb{H}^1)}.
\end{aligned} \tag{6.31}$$

Also from the definition of $\hat{F}^n$ in (2.8), we note that

$$|\hat{F}^n(\cdot,\mathbf{m}^n)|_{\mathrm{L}^2(0,T;\mathbb{H}^1)} \leq C|\mathbf{m}^n|_{\mathrm{L}^2(0,T;\mathbb{H}^1)}. \tag{6.32}$$

Again using simple algebraic identities, (6.31), (6.32) and (6.16), we have

$$\begin{aligned}
&\Big|\lambda_1\Big(\mathbf{m}^n\times\hat{F}^n(\cdot,\mathbf{m}^n) - \mathbf{m}\times\hat{F}(\cdot,\mathbf{m})\Big)|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\leq |\lambda_1||\mathbf{m}^n\times\hat{F}^n(\cdot,\mathbf{m}^n) - \mathbf{m}\times\hat{F}^n(\cdot,\mathbf{m}^n)|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\quad + |\lambda_1||\mathbf{m}\times\hat{F}^n(\cdot,\mathbf{m}^n) - \mathbf{m}\times\hat{F}(\cdot,\mathbf{m})|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \\
&\leq C|\lambda_1||\mathbf{m}^n-\mathbf{m}|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)}(1+|\mathbf{m}^n|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)}) + |\lambda_1||\mathbf{m}|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)}|\mathbf{m}^n-\mathbf{m}|_{\mathrm{L}^2(0,T;\mathbb{H}^1)} \\
&\leq C|\mathbf{m}^n-\mathbf{m}|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)} \quad \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned} \tag{6.33}$$

Similarly, we can prove that

$$|\lambda_2\Big(\mathbf{m}^n\times(\mathbf{m}^n\times\hat{F}^n(\cdot,\mathbf{m}^n)) - \mathbf{m}\times(\mathbf{m}\times\hat{F}(\cdot,\mathbf{m}))\Big)|_{\mathrm{L}^2(0,T;\mathbb{L}^2)} \to 0 \quad \text{as} \quad n\to\infty. \tag{6.34}$$

Thus combining (6.33) and (6.34), from (6.30) we have $F^n(\cdot,\mathbf{m}^n) \to F(\cdot,\mathbf{m})$ in $\mathrm{L}^2(0,T;\mathbb{L}^2)$. □

7. Proof of the main result

The main result of this paper is the following theorem:

Theorem 7.1. *Let* $\mathbf{m}$ *and* $\mathbf{m}^n$ *be the solutions of the system of equations* (1.13)–(1.15) *and* (2.15)–(2.17) *respectively. Let* $W(t)$ *be the one dimensional Brownian motion and* $W^n(t)$ *be the regular approximation of* $W(t)$ *given by* (1.8). *Let* G *be the bounded linear operator given in Lemma* 2.2. *Then there exist unique solutions* $\mathbf{M}$ *and* $\mathbf{M}_n$ *of* (1.5)–(1.7) *and* (1.9)–(1.11) *respectively and are given by*

$$\mathbf{M}(t) = e^{W(t)G}\mathbf{m}(t), \quad \mathbf{M}_n(t) = e^{W^n(t)G}\mathbf{m}^n(t), \quad \forall t \in [0, T]. \tag{7.1}$$

Moreover for almost all $\omega \in \Omega$, *we have the following convergence:*

$$\mathbf{M}_n(\cdot, \omega) \to \mathbf{M}(\cdot, \omega) \quad in \quad C([0, T]; \mathbb{H}^1) \cap \mathrm{L}^2(0, T; D(\mathrm{A})), \ \ as\ n \to \infty.$$

Remark 7.2. As in this work, the authors in [36] transformed the SLLGEs into a partial differential equation with random coefficients. They proposed a convergent θ-linear scheme for the numerical solution of the reformulated equation, and as a consequence, they proved the existence of weak martingale solutions to the original SLLGEs. The key difference between the current work and [36] lies on the type of approximations – Wong–Zakai in our case and finite element for [36]. Let us point out that in the very first paper on the numerical approximations to the SLLGEs, see Remark (iii) on page 505 in [8], the authors pointed out that the Wong–Zakai approximation for the SLLGEs is an open problem. Through this current work we resolve this open problem in one dimension. Moreover, here we are able to achieve the uniqueness and the convergence in a stronger topology than the one used in [36], which might be due to the one-dimensionality of the space domain as compared to two and three dimensional domains in [36].

Proof. Step 1 – Existence of M and $\mathbf{M}_n$:
We note that $\mathbf{m}$ satisfies (3.3), $\forall\, \psi \in \mathbb{H}^1$. Using vector product rule $\langle a, b \times c\rangle_{\mathbb{R}^3} = \langle b, c \times a\rangle_{\mathbb{R}^3}$, it can be shown that $\mathbf{m}$ satisfies

$$\begin{aligned}\langle \mathbf{m}_t, \psi\rangle_{\mathrm{L}^2(D_T)} &+ \lambda_1 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla \psi\rangle_{\mathrm{L}^2(D_T)} + \lambda_2 \langle \mathbf{m} \times \nabla \mathbf{m}, \nabla(\mathbf{m} \times \psi)\rangle_{\mathrm{L}^2(D_T)} \\ &- \langle F(t, \mathbf{m}), \psi\rangle_{\mathrm{L}^2(D_T)} = 0 \quad \forall \psi \in \mathrm{L}^2(0, T; \mathbb{W}^{1,\infty}),\end{aligned} \tag{7.2}$$

where $D_T := (0, T) \times D$.

Exploiting the result, see Lemma 4.1 of [36], that weak solutions to the original and the transformed system are equivalent, existence of $\mathbf{M}$ can be proved. Existence of $\mathbf{M}_n$ can also be proved analogously.

Uniqueness of $\mathbf{M}$ and $\mathbf{M}_n$ follow from the uniqueness of $\mathbf{m}$ and $\mathbf{m}^n$ (see Theorem 3.16) respectively.

Step 2 – Convergence: We note that

$$\begin{aligned}\mathbf{M}_n(t) - \mathbf{M}(t) &= e^{W^n(t)G}\mathbf{m}^n(t) - e^{W(t)G}\mathbf{m}(t) \\ &= e^{W^n(t)G}\Big(\mathbf{m}^n(t) - \mathbf{m}(t)\Big) + \Big(e^{W^n(t)G}\mathbf{m}(t) - e^{W(t)G}\mathbf{m}(t)\Big) \\ &:= \mathbb{I}_1(t) + \mathbb{I}_2(t). \end{aligned} \tag{7.3}$$

We will now show that $\mathbb{I}_1 \to 0$ in $C([0,T];\mathbb{H}^1)$ as $n \to \infty$.

Using (6.16), Theorem 5.1, Theorem 6.4, and $\{e^{-sG}\}_{s\in\mathbb{R}}$ and $\{e^{-sG_x}\}_{s\in\mathbb{R}}$ are uniformly continuous group of unitary linear maps on $\mathbb{L}^2$, we have

$$\begin{aligned}&\int_D |(\mathbb{I}_1(t))_x|^2_{\mathbb{R}^3}dx \\ &\leq 2\int_D \Big|e^{W^n(t)G}\Big(\mathbf{m}^n_x(t) - \mathbf{m}_x(t)\Big)\Big|^2_{\mathbb{R}^3}dx + 2|W^n(t)|^2\int_D \Big|e^{W^n(t)G_x}\Big(\mathbf{m}^n(t) - \mathbf{m}(t)\Big)\Big|^2_{\mathbb{R}^3}dx \\ &\leq 2\|e^{W^n(t)G}\|_{\mathcal{L}(\mathbb{L}^2)}|\mathbf{m}^n_x(t) - \mathbf{m}_x(t)|^2_{\mathbb{L}^2} + 2|W^n(t)|^2\|e^{W^n(t)G_x}\|_{\mathcal{L}(\mathbb{L}^2)}|\mathbf{m}^n(t) - \mathbf{m}(t)|^2_{\mathbb{L}^2} \\ &\leq C\Big\{\sup_{t\in[0,T]}|\mathbf{m}^n(t) - \mathbf{m}(t)|^2_{\mathbb{H}^1} + \sup_{t\in[0,T]}|\mathbf{m}^n(t) - \mathbf{m}(t)|^2_{\mathbb{L}^2}\Big\} \\ &\leq C\Big\{|\mathbf{m}^n - \mathbf{m}|^2_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)} + |\mathbf{m}^n - \mathbf{m}|^2_{\mathrm{L}^\infty(0,T;\mathbb{L}^2)}\Big\} \to 0 \quad \text{as} \quad n \to \infty,\end{aligned}$$

and

$$\begin{aligned}\int_D |\mathbb{I}_1(t)|^2_{\mathbb{R}^3}dx &\leq \int_D \Big|e^{W^n(t)G}\Big(\mathbf{m}^n(t) - \mathbf{m}(t)\Big)\Big|^2_{\mathbb{R}^3}dx \\ &\leq \|e^{W^n(t)G}\|_{\mathcal{L}(\mathbb{L}^2)}\sup_{t\in[0,T]}|\mathbf{m}^n(t) - \mathbf{m}(t)|^2_{\mathbb{L}^2} \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

This implies that $\sup_{t\in[0,T]}|\mathbb{I}_1(t)|^2_{\mathbb{H}^1} \to 0 \quad \text{as} \quad n \to \infty$. Now, using Fundamental Theorem of Calculus, we have

$$\mathbb{I}_2(t) := e^{W^n(t)G}\mathbf{m}(t) - e^{W(t)G}\mathbf{m}(t) = \int_{W^(t)}^{W^n(t)} e^{sG}G\mathbf{m}(t)ds.$$

Using boundedness of $W^n(t)$ and $W(t)$, we have

$$\begin{aligned}\int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} s^2 ds &\leq \Big(W(t) \vee W^n(t)\Big)^2|W^n(t) - W(t)| \\ &\leq |W^n(t) - W(t)|\Big([W^n(t)]^2 + [W(t)]^2 + 2|W^n(t)||W(t)|\Big) \\ &\leq C|W^n(t) - W(t)| \end{aligned}\tag{7.4}$$

Using similar arguments, $|G\mathbf{m}(t)|_{\mathbb{L}^2} \le C|\mathbf{m}(t)|_{\mathbb{L}^2}$ and $|G\mathbf{m}_x(t)|_{\mathbb{L}^2} \le C|\mathbf{m}_x(t)|_{\mathbb{L}^2}$, $|G_x\mathbf{m}(t)|_{\mathbb{L}^2} \le C|\mathbf{m}(t)|_{\mathbb{L}^2}$, (1.8) and Theorem 6.4, we have

$$\begin{aligned}
\int_D |(\mathbb{I}_2)_x(t)|^2_{\mathbb{R}^3} dx &\le 2\Big[\int_D \int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} \Big(|e^{sG} G\mathbf{m}_x(t)|^2_{\mathbb{R}^3} + |s e^{sG_x} G\mathbf{m}(t)|^2_{\mathbb{R}^3} \\
&\quad + |e^{sG} G_x \mathbf{m}(t)|^2_{\mathbb{R}^3}\Big) ds\, dx\Big] \times |W^n(t) - W(t)| \\
&\le \Big[\int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} \Big(\|e^{sG}\|^2_{\mathcal{L}(\mathbb{L}^2)} |G\mathbf{m}_x(t)|^2_{\mathbb{L}^2} + s^2 \|e^{sG_x}\|^2_{\mathcal{L}(\mathbb{L}^2)} |G\mathbf{m}(t)|^2_{\mathbb{L}^2} \\
&\quad + |s| \|e^{sG}\|^2_{\mathcal{L}(\mathbb{L}^2)} |G_x\mathbf{m}(t)|^2_{\mathbb{L}^2}\Big) ds\Big] \times |W^n(t) - W(t)| \\
&\le C\Big(\sup_{t\in[0,T]} |\mathbf{m}_x(t)|^2_{\mathbb{L}^2}\Big) |W^n(t) - W(t)|^2 + \Big(C|\mathbf{m}(t)|^2_{\mathbb{L}^2} \int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} s^2 ds\Big) \\
&\quad \times |W^n(t) - W(t)| + \Big(C|\mathbf{m}(t)|^2_{\mathbb{L}^2} \int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} |s| ds\Big) \times |W^n(t) - W(t)| \\
&\le C|\mathbf{m}|_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)} \times |W^n(t) - W(t)|^2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Also using similar arguments as above, we have

$$\begin{aligned}
\int_D |\mathbb{I}_2(t)|^2_{\mathbb{R}^3} dx &= \int_D \Big| \int_{W(t)}^{W^n(t)} e^{sG} G\mathbf{m}(t) ds \Big|^2_{\mathbb{R}^3} dx \\
&\le \int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} \|e^{sG}\|^2_{\mathcal{L}(\mathbb{L}^2)} |G\mathbf{m}(t)|^2_{\mathbb{L}^2} ds \times |W^n(t) - W(t)| \\
&\le C|W^n(t) - W(t)|^2 \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

This implies that $\sup_{t\in[0,T]} |\mathbb{I}_2(t)|^2_{\mathbb{H}^1} \to 0$ as $n \to \infty$. Hence we have $\mathbf{M}_n \to \mathbf{M}$ in $C([0,T];\mathbb{H}^1)$.

Now to show $\mathbf{M}_n \to \mathbf{M}$ in $\mathrm{L}^2(0,T;D(\mathrm{A}))$, it is enough to show that $\mathbb{I}_1$ and $\mathbb{I}_2$ converge to zero in $\mathrm{L}^2(0,T;D(\mathrm{A}))$. Let us calculate $(\mathbb{I}_1)_{xx}(t)$.

$$\begin{aligned}
(\mathbb{I}_1)_{xx}(t) &= e^{W^n(t)G}\Big(\mathbf{m}^n_{xx}(t) - \mathbf{m}_{xx}(t)\Big) + 2W^n(t) e^{W^n(t)G_x}\Big(\mathbf{m}^n_x(t) - \mathbf{m}_x(t)\Big) \\
&\quad + [W^n(t)]^2 e^{W^n(t)G_{xx}}\Big(\mathbf{m}^n(t) - \mathbf{m}(t)\Big).
\end{aligned}$$

Hence we have,

$$
\begin{aligned}
|\mathrm{A}\mathbb{I}_1(t)|_{\mathbb{L}^2}^2 \leq C\Big\{&\Big|e^{W^n(t)G}\Big(\mathbf{m}^n_{xx}(t)-\mathbf{m}_{xx}(t)\Big)\Big|_{\mathbb{L}^2}^2 + |W^n(t)|^2\Big|e^{W^n(t)G_x}\Big(\mathbf{m}^n_x(t)-\mathbf{m}_x(t)\Big)\Big|_{\mathbb{L}^2}^2 \\
&+|W^n(t)|^4\Big|e^{W^n(t)G_{xx}}\Big(\mathbf{m}^n(t)-\mathbf{m}(t)\Big)\Big|_{\mathbb{L}^2}^2\Big\} \\
\leq C\Big\{&|\mathrm{A}(\mathbf{m}^n-\mathbf{m})(t)|_{\mathbb{L}^2}^2 + |\mathbf{m}^n(t)-\mathbf{m}(t)|_{\mathbb{H}^1}^2 + |\mathbf{m}^n(t)-\mathbf{m}(t)|_{\mathbb{L}^2}^2\Big\}.
\end{aligned}
$$

Integrating on $[0,T]$ and exploiting (6.16), we have

$$
\begin{aligned}
\int_0^T |\mathrm{A}\mathbb{I}_1(t)|_{\mathbb{L}^2}^2 dt &\leq C\Big\{\int_0^T |\mathrm{A}(\mathbf{m}^n-\mathbf{m})(t)|_{\mathbb{L}^2}^2 dt + \int_0^T |\mathbf{m}^n(t)-\mathbf{m}(t)|_{\mathbb{H}^1}^2 dt \\
&\qquad + \int_0^T |\mathbf{m}^n(t)-\mathbf{m}(t)|_{\mathbb{L}^2}^2 dt\Big\} \\
&\leq C\Big\{|\mathbf{m}^n-\mathbf{m}|^2_{\mathrm{L}^2(0,T;D(\mathrm{A}))} + T\,|\mathbf{m}^n-\mathbf{m}|^2_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)}\Big\} \\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

This proves that $\mathbb{I}_1$ converges to zero in $\mathrm{L}^2(0,T;D(\mathrm{A}))$.

Now we will estimate $\mathbb{I}_2(t)$.

$$
\begin{aligned}
(\mathbb{I}_2)_{xx}(t) = \int_{W(t)}^{W^n(t)} \Big[&se^{sG_x}G\mathbf{m}_x(t) + e^{sG}G_x\mathbf{m}_x(t) + e^{sG}G\mathbf{m}_{xx}(t) + se^{sG_x}G_x\mathbf{m}(t) \\
&+ s^2e^{sG_{xx}}G\mathbf{m}(t) + se^{sG_x}G\mathbf{m}_x(t) + e^{sG}G_{xx}\mathbf{m}(t) + se^{sG_x}G_x\mathbf{m}(t) \\
&+ e^{sG}G_x\mathbf{m}_x(t)\Big]ds.
\end{aligned}
$$

Now using Tonelli's theorem, (7.4), arguments similar in obtaining (7.4), $|G\mathbf{m}(t)|_{\mathbb{L}^2} \leq C|\mathbf{m}(t)|_{\mathbb{L}^2}$, $|G\mathbf{m}_x(t)|_{\mathbb{L}^2} \leq C|\mathbf{m}_x(t)|_{\mathbb{L}^2}$, $|G_x\mathbf{m}(t)|_{\mathbb{L}^2} \leq C|\mathbf{m}(t)|_{\mathbb{L}^2}$, $|G_x\mathbf{m}_x(t)|_{\mathbb{L}^2} \leq C|\mathbf{m}_x(t)|_{\mathbb{L}^2}$, $|G_{xx}\mathbf{m}(t)|_{\mathbb{L}^2} \leq C|\mathbf{m}(t)|_{\mathbb{L}^2}$, (1.8) and using Theorem 6.4 we have

$$
\begin{aligned}
|\mathbb{I}_2(t)|^2_{D(\mathrm{A})} := |\mathrm{A}\mathbb{I}_2(t)|^2_{\mathbb{L}^2} = \int_D |(\mathbb{I}_2)_{xx}(t)|^2_{\mathbb{R}^3} dx \\
\leq C\Big\{\int_{W(t)\wedge W^n(t)}^{W(t)\vee W^n(t)} \int_D \Big(&|se^{sG_x}G\mathbf{m}_x(t)|^2_{\mathbb{R}^3} + |e^{sG}G_x\mathbf{m}_x(t)|^2_{\mathbb{R}^3} \\
&+ |e^{sG}G\mathbf{m}_{xx}(t)|^2_{\mathbb{R}^3} + |se^{sG_x}G_x\mathbf{m}(t)|^2_{\mathbb{R}^3} + |s^2e^{sG_{xx}}G\mathbf{m}(t)|^2_{\mathbb{R}^3} \\
&+ |se^{sG_x}G\mathbf{m}_x(t)|^2_{\mathbb{R}^3} + |e^{sG}G_{xx}\mathbf{m}(t)|^2_{\mathbb{R}^3} + |se^{sG_x}G_x\mathbf{m}(t)|^2_{\mathbb{R}^3}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |e^{sG} G_x \mathbf{m}_x(t)|^2_{\mathbb{R}^3}\Big) dx\, ds\Big\} \times |W^n(t) - W(t)| \\
&\le C\Big\{|\mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2} + |\mathbf{m}_x(t)|^2_{\mathbb{L}^2} + |\mathbf{m}(t)|^2_{\mathbb{L}^2}\Big\} \times |W^n(t) - W(t)|^2 \\
&\le C\Big\{|\mathbf{m}(t)|^2_{D(\mathrm{A})} + |\mathbf{m}(t)|^2_{\mathbb{H}^1}\Big\} \times |W^n(t) - W(t)|^2.
\end{aligned}
$$

Integrating this on $[0, T]$ and using (1.8), we get

$$
\begin{aligned}
\int_0^T |\mathbb{I}_2(t)|^2_{D(\mathrm{A})} &\le C\Big\{|\mathbf{m}|^2_{\mathrm{L}^2(0,T;D(\mathrm{A}))} + T|\mathbf{m}|^2_{\mathrm{L}^\infty(0,T;\mathbb{H}^1)}\Big\} \times \sup_{t\in[0,T]} |W^n(t) - W(t)|^2 \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This implies that $\mathbb{I}_2 \to 0$ in $\mathrm{L}^2(0, T; D(\mathrm{A}))$.

Hence $\mathbf{M}_n \to \mathbf{M}$ in $\mathrm{L}^2(0, T; D(\mathrm{A}))$. This completes the proof. □

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Appendix A. Basic result

In this Section we recall some results which are used in the course to prove our main theorems.

Result 1. We have the following embedding

$$
\mathbb{W}^{2\delta,2} \hookrightarrow \mathbb{W}^{1,4}. \tag{A.1}
$$

Proof. We know from the general Sobolev embedding theorem that

$$
\mathbb{W}^{s,p} \hookrightarrow \mathbb{L}^{\frac{np}{n-sp}}, \quad \text{if} \quad s < \frac{n}{p}.
$$

Thus, if we take $n = 1$, $p = 2$ and $s = 2\delta - 1$, then we infer that

$$
\mathbb{W}^{2\delta-1,2} \hookrightarrow \mathbb{L}^{\frac{2}{1-2(2\delta-1)}} \hookrightarrow \mathbb{L}^4, \tag{A.2}
$$

and the first embedding is true if $1 - 2(2\delta - 1) > 0$, i.e., if $\delta < \frac{3}{4}$ and the second embedding is true if $\frac{2}{1-2(2\delta-1)} > 4$, i.e., if $\delta > \frac{5}{8}$. Hence, in summary for $\delta \in (\frac{5}{8}, \frac{3}{4})$, we have the embedding $\mathbb{W}^{2\delta-1,2} \hookrightarrow \mathbb{L}^4$.

Using $X^\delta = \mathbb{H}^{2\delta} = \mathbb{W}^{2\delta,2}$ if $\delta < \frac{3}{4}$, we deduce (A.1). □

Appendix B. Some algebraic identities

Here we will list all algebraic identities used in this paper. Assume that $a, b, c, d \in \mathbb{R}^3$. Then

$$a \times b = -b \times a, \tag{B.1}$$

$$\langle a \times (b \times c), d\rangle = \langle c, (d \times a) \times b\rangle, \tag{B.2}$$

$$\langle a \times b, c\rangle = \langle b, c \times a\rangle, \tag{B.3}$$

$$\langle a \times b, b\rangle = 0, \tag{B.4}$$

$$-\langle a \times b, c\rangle = \langle b, a \times c\rangle, \tag{B.5}$$

$$a \times (b \times c) = \langle a, c\rangle b - \langle a, b\rangle c \tag{B.6}$$

$$|a \times b| \leq |a||b|. \tag{B.7}$$

In particular, if $\langle a, b\rangle = 0$, then $(a \times b) \times b = b \times (b \times a) = \langle b, a\rangle b - \langle b, b\rangle a = -|b|^2 a$ and $a \times (a \times b) = \langle a, b\rangle a - \langle a, a\rangle b = -|a|^2 b$, i.e.

$$(a \times b) \times b = -|b|^2 a, \text{ if } \langle a, b\rangle = 0. \tag{B.8}$$

$$a \times (a \times b) = -|a|^2 b, \text{ if } \langle a, b\rangle = 0. \tag{B.9}$$

Corollary B.1.

$$\langle a \times (a \times b), b\rangle = -|a \times b|^2 \tag{B.10}$$

Proof. Apply, (B.3) and then (B.1). □

Appendix C. Comments about our method of proving Theorems 6.1 and 6.4

Although equation (1.13) is *a-priori* a quasilinear "parabolic" since the coefficients in front of $\mathbf{m}_{xx}$ depend only on $\mathbf{m}$, it does **not** satisfy usual assumption that for fixed $\mathbf{m}$, the map

$$u \mapsto \lambda_1(\mathbf{m} \times u_{xx}) - \lambda_2 \mathbf{m} \times (\mathbf{m} \times u_{xx}) \tag{C.1}$$

is a generator of an analytic or C_0-semigroup, see e.g. a vague definition from the Introduction Chapter of Amann's monograph [4], Theorem 5.1.1 from a very recent monograph [58] by Prüss and Simonett and/or Definition 4.1 from Chapter 6.4 of Pazy's book [57]. Though in our paper, as well as other papers on the SLLGEs, we assume that the initial data $\mathbf{m}_0$ takes values in the unit sphere $\mathbb{S}^2$ (i.e., satisfies the saturation condition (1.3)), we believe that following [63] it can be proved that for every[1] $\mathbf{m}_0 \in \mathbb{H}^1 := H^1(D; \mathbb{R}^3)$, problem (1.13)–(1.15) has a weak solution $\mathbf{m}$ which satisfies

$$|\mathbf{m}(t, x)|_{\mathbb{R}^3} = |\mathbf{m}_0(x)|_{\mathbb{R}^3}, \text{ for all } t \geq 0, \text{ for a.a. } x \in D. \tag{C.2}$$

[1] Our Theorem 3.2 is formulated for $\mathbf{m}_0 \in \mathbb{H}^1(D; \mathbb{S}^2)$.

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It is to be noted that only when $\mathbf{m}_0$ satisfies the constraint condition (1.3), one can show that the weak solution to the problem (1.13), satisfying additionally the condition (C.2), solves the following equation (see (3.33) in Theorem 3.16 later on)

$$\frac{\partial \mathbf{m}}{\partial t} + \lambda_2 \mathbf{m}_{xx} = \lambda_2 |\mathbf{m}_x|^2_{\mathbb{R}^3} \mathbf{m} + \lambda_1 (\mathbf{m} \times (\mathbf{m}_{xx})) + F(t, \mathbf{m}(t)), \quad t \in (0, T), \tag{C.3}$$

in the weak sense with respect to the Gelfand triple $\mathrm{V} \hookrightarrow \mathbb{L}^2 \hookrightarrow \mathrm{V}'$.

Coming back to the quasilinear equations approach, for a fixed $\mathbf{m} \in \mathbb{H}^1(D; \mathbb{S}^2)$, in view of Lemma 3.17, the map C.1 can be written as

$$u \mapsto -\lambda_2 u_{xx} + \lambda_1 (\mathbf{m} \times u_{xx}) - \lambda_2 |\mathbf{m}_x|^2_{\mathbb{R}^3} u, \tag{C.4}$$

which is a bounded linear map from $\mathbb{H}^2$ to $\mathbb{L}^2$, but it is not clear whether this map satisfies relevant assumptions, see for instance [58]. One way to do so would be to employ technique of [30]. But the authors of that paper in Section 2 have worked with different spaces and moreover, only have proved the existence of a local regular solution. Our approach here is different. We only consider the linear map consisting of the first term in (C.4) and then we use (3.1) and (6.3) (i.e. that $\int_0^T |\mathbf{m}(t) \times \mathbf{m}_{xx}(t)|^2_{\mathbb{L}^2}\, ds < \infty$ and $\int_0^T |\mathbf{m}^n_x(s)|^4_{\mathbb{L}^4}\, ds < \infty$, which are the consequences of *a-priori estimates* and utracontractivity of the heat semigroup), in order to deduce the maximal regularity of the solutions in Theorem 6.1.

To conclude, it might be possible to exploit the quasilinear structure of the equation but some additional work needs to be done. For instance, the map enjoys nice property not for every $\mathbf{m} \in \mathbb{H}^1$, but only for every $\mathbf{m}$ in a closed submanifold $\mathbb{H}^1(D; \mathbb{S}^2)$. Further, although we cannot prove this, but see Section 3 of [15] for a related result concerning different equation with constraints, we suspect that the problem (C.3) may not be even well-posed for all initial data $\mathbf{m}_0 \in \mathbb{H}^1$ (although it is well-posed for initial data $\mathbf{m}_0 \in \mathbb{H}^1(D; \mathbb{S}^2)$).

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