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J. Differential Equations 267 (2019) 3797–3826

Journal of Differential Equations

www.elsevier.com/locate/jde

Singularity formation to the two-dimensional compressible non-isothermal nematic liquid crystal flows in a bounded domain ☆

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Received 21 October 2018
Available online 3 May 2019

Abstract

We study the singularity formation of strong solutions to the two-dimensional (2D) compressible non-isothermal nematic liquid crystal flows in a bounded domain. Under a geometric condition of the initial orientation field, we show that the strong solution exists globally if the temporal integral of the maximum norm of the divergence of the velocity is bounded. Our method relies on critical Sobolev inequalities of logarithmic type and delicate energy estimates.

MSC: 76W05; 76N10

Keywords: Compressible non-isothermal nematic liquid crystal flows; Strong solutions; Blow-up criterion

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a domain, the motion of compressible nematic liquid crystal flows in Ω can be governed by the following simplified version of the Ericksen-Leslie equations (see [4,9]):

☆ Supported by Fundamental Research Funds for the Central Universities (No. XDJK2019B031), Chongqing Research Program of Basic Research and Frontier Technology (No. cstc2018jcyjAX0049), the Postdoctoral Science Foundation of Chongqing (No. xm2017015), and China Postdoctoral Science Foundation (Nos. 2018T110936, 2017M610579).
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https://doi.org/10.1016/j.jde.2019.04.025
0022-0396/

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P = -\operatorname{div}\left(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathbb{I}_n\right), \\ c_\nu[(\rho\theta)_t + \operatorname{div}(\rho\mathbf{u}\theta)] + P\operatorname{div}\mathbf{u} - \kappa\Delta\theta = 2\mu|\mathfrak{D}(\mathbf{u})|^2 + \lambda(\operatorname{div}\mathbf{u})^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2, \\ \mathbf{d}_t + (\mathbf{u}\cdot\nabla)\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}, \\ |\mathbf{d}| = 1. \end{cases} \tag{1.1}$$

Here, $t \geq 0$ is the time, $x \in \Omega$ is the spatial coordinate, and $\rho, \mathbf{u}, P = R\rho\theta$ $(R > 0), \theta, \mathbf{d}$ are the density, velocity, pressure, absolute temperature, and the macroscopic average of the nematic liquid crystal orientation field, respectively; $\mathbb{I}_n$ is the identity matrix of order n and $\mathfrak{D}(\mathbf{u})$ denotes the deformation tensor given by

$$\mathfrak{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{tr}).$$

The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu > 0,\ 2\mu + n\lambda \geq 0. \tag{1.2}$$

Positive constants c_ν and κ are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity, respectively.

Liquid crystals can form and remain in an intermediate phase of matter between liquids and solids. When a solid melts, if the energy gain is enough to overcome the positional order but the shape of the molecules prevents the immediate collapse of orientational order, liquid crystals are formed. The nematic liquid crystals exhibit long—range ordering in the sense that their rigid rod-like molecules arrange themselves with their long axes parallel to each other. Their molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation. The continuum theory of the nematic liquid crystals was first developed by Ericksen [2] and Leslie [19] during the period of 1958 through 1968. For more results on the simplified Ericksen-Leslie system modeling incompressible liquid crystal flows, refer to [6,7,18,21–24,26] and references therein.

Recently, there is an increasing interest in the theory of well-posedness of solutions to the Cauchy problem and the initial boundary value problem (IBVP) for the compressible nematic liquid crystal flows due to the physical importance, complexity, rich phenomena, and mathematical challenges, refer to [4,9,14–17,20,27,28,31,34] and references therein. Adapting the standard three-level approximation scheme and the weak convergence arguments used in the compressible Navier-Stokes equations [5,25], Jiang-Jiang-Wang [16,17] proved the global existence of weak solutions with finite energy for multi-dimensional compressible nematic liquid crystal flows. Li-Xu-Zhang [20] established the unique global classical solutions to the Cauchy problem for the isentropic compressible nematic liquid crystal flows in 3D with smooth initial data which are of small energy but possibly large oscillations and vacuum, which is analogous to the result of compressible Navier-Stokes equations obtained by Huang-Li-Xin [11]. Later, Wang [31] obtained the unique global strong solutions to the 2D compressible nematic liquid crystal flows provided that the smooth initial data were of small total energy. Very recently, for the 3D non-isothermal nematic liquid crystal flows (1.1), the local well-posedness of strong solutions to the initial boundary value problem has been investigated by Fan-Li-Nakamura [4], while Guo-Xi-Xie [9] proved the global existence of smooth solutions for the Cauchy problem provided that the initial datum is close to a steady state. By delicate weighted energy estimates and a Hardy-type inequality, Zhong [34] showed the local existence of strong solutions to the 2D Cauchy problem of the system (1.1) with $\kappa = 0$. It should be noted that the 2D case is different from the 3D case when the far field is vacuum. The main reason is that it seems impossible to control $L^p(\mathbb{R}^2)$-norm of the velocity field $\mathbf{u}$ in terms of $\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)}$ only due to the criticality of the standard Sobolev embedding theorem in two dimensions, while if $\lim_{|x|\to\infty}\mathbf{u} = \mathbf{0}$ and $\nabla\mathbf{u} \in L^2(\mathbb{R}^2)$, then $\mathbf{u} \in L^6(\mathbb{R}^3)$.

However, many physical important and mathematical fundamental problems are still unknown due to the lack of smoothing mechanism and the strong nonlinearity. Up to now, the regularity and uniqueness of weak solutions and the global well-posedness of the strong solutions to compressible nematic liquid crystal flows for general initial data are still open and challenging even in two dimensions. Therefore, it is important to study the mechanism of blow-up and structure of possible singularities of strong (or classical) solutions to the compressible nematic liquid crystal flows.

For the Cauchy problem and IBVP of 3D compressible nematic liquid crystal flows, Huang-Wang-Wen [14] obtained the following criterion

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla\mathbf{d}\|_{L^3(0,T;L^\infty)}\right) = \infty \tag{1.3}$$

under the assumption

$$7\mu > 9\lambda. \tag{1.4}$$

Without the artificial restriction (1.4), the authors [15] showed that

$$\lim_{T\to T^*}\left(\|\mathfrak{D}(\mathbf{u})\|_{L^1(0,T;L^\infty)} + \|\nabla\mathbf{d}\|_{L^2(0,T;L^\infty)}\right) = \infty. \tag{1.5}$$

Later on, Huang-Wang [12] established the following Serrin type criterion

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\mathbf{u}\|_{L^{s_1}(0,T;L^{r_1})} + \|\nabla\mathbf{d}\|_{L^{s_2}(0,T;L^{r_2})}\right) = \infty \tag{1.6}$$

with s_i and r_i satisfying

$$\frac{2}{s_i} + \frac{3}{r_i} \le 1,\ \ 3 < r_i \le \infty,\ \ i = 1, 2. \tag{1.7}$$

Recently, for the Cauchy problem of 2D compressible nematic liquid crystal flows, Wang [32] proved that

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla\mathbf{d}\|_{L^s(0,T;L^r)}\right) = \infty \tag{1.8}$$

with s and r satisfying

$$\frac{2}{s} + \frac{2}{r} \le 1,\ \ 2 < r \le \infty. \tag{1.9}$$

However, if the initial orientation $\mathbf{d_0} = (d_{01}, d_{02}, d_{03})$ satisfies a geometric condition

$$d_{03} \geq \varepsilon_0 \tag{1.10}$$

for some positive constant ε_0, Liu-Wang [27] extended (1.8) to a refiner form

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty. \tag{1.11}$$

Very recently, for two-dimensional case of the system (1.1) with $\kappa = 0$, the author [35,36] showed that

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|P\|_{L^\infty(0,T;L^\infty)} \right) = \infty, \tag{1.12}$$

provided that (1.10) holds true. It is worth mentioning that the key idea in [36] is different from that of in [35]. Roughly speaking, weighted energy estimates and Hardy-type inequalities play a crucial role in [35], while the key ingredient of the analysis in [36] is critical Sobolev inequalities of logarithmic type. Our goal in this paper is to give a blow-up criterion of strong solutions to the two-dimensional IBVP of the system (1.1).

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, and without loss of generality, we take $c_\nu = R = 1$. Noting that

$$\operatorname{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d}) = \nabla \mathbf{d} \cdot \Delta \mathbf{d} + \nabla \left(\frac{|\nabla \mathbf{d}|^2}{2} \right), \tag{1.13}$$

the system (1.1) can be rewritten as

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla P = -\nabla \mathbf{d} \cdot \Delta \mathbf{d}, \\ (\rho \theta)_t + \operatorname{div}(\rho \mathbf{u} \theta) + P \operatorname{div} \mathbf{u} - \kappa \Delta \theta = 2\mu |\mathfrak{D}(\mathbf{u})|^2 + \lambda (\operatorname{div} \mathbf{u})^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2, \\ \mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \\ |\mathbf{d}| = 1, \end{cases} \tag{1.14}$$

and the constant viscosity coefficients μ and λ satisfy

$$\mu > 0, \ \mu + \lambda \geq 0. \tag{1.15}$$

The present paper aims at giving a blow-up criterion of strong solutions to the system (1.14) with the initial condition

$$(\rho, \rho \mathbf{u}, \theta, \mathbf{d})(x, 0) = (\rho_0, \rho_0 \mathbf{u}_0, \theta_0, \mathbf{d}_0)(x), \quad x \in \Omega, \tag{1.16}$$

and the boundary condition

$$\mathbf{u} = \mathbf{0}, \ \frac{\partial \theta}{\partial \mathbf{n}} = \mathbf{0}, \ \nabla \mathbf{d} \cdot \mathbf{n} = \mathbf{0}, \quad x \in \partial \Omega, \ t > 0, \tag{1.17}$$

where $\mathbf{n}$ is the unit outer normal vector of $\partial \Omega$.

Before stating our main result, we first explain the notations and conventions used throughout this paper. Set

$$\int \cdot dx \triangleq \int_{\Omega} \cdot dx.$$

For $1 \le p \le \infty$ and integer $k \ge 0$, the standard Sobolev spaces are denoted by:

$$L^p = L^p(\Omega),\ W^{k,p} = W^{k,p}(\Omega),\ H^k = H^{k,2}(\Omega),\ H_0^1 = \{u \in H^1 : u|_{\partial\Omega} = 0\}.$$

Now we define precisely what we mean by strong solutions to the problem (1.14)–(1.17).

Definition 1.1 *(Strong solutions).* $(\rho \ge 0, \mathbf{u}, \theta \ge 0, \mathbf{d})$ is called a strong solution to (1.14)–(1.17) in $\Omega \times (0, T)$ if for some $q > 2$,

$$\begin{cases} \rho \in C([0,T]; W^{1,q}),\ \rho_t \in C([0,T]; L^q), \\ \nabla\mathbf{u} \in L^\infty(0,T; H^1) \cap L^2(0,T; W^{1,q}), \\ \sqrt{\rho}\dot{\mathbf{u}},\ \sqrt{\rho}\dot{\theta} \in L^\infty(0,T; L^2), \\ \nabla\mathbf{d} \in C([0,T]; H^2) \cap L^2(0,T; H^3), \\ \mathbf{d}_t \in C([0,T]; H^1) \cap L^2(0,T; H^2), \\ \theta \in C([0,T]; H^2) \cap L^2(0,T; W^{2,q}), \end{cases}$$

and $(\rho, \mathbf{u}, \theta, \mathbf{d})$ satisfies both (1.14) almost everywhere in $\Omega \times (0, T)$ and (1.16) almost everywhere in Ω. Here

$$\dot{v} \triangleq v_t + \mathbf{u} \cdot \nabla v$$

denotes the material derivative of v.

Our main result reads as follows:

Theorem 1.1. *In addition to* (1.10)*, let the initial data* $(\rho_0 \ge 0, \mathbf{u}_0, \theta_0 \ge 0, \mathbf{d}_0)$ *satisfy for any given* $q > 2$*,*

$$\begin{cases} \rho_0 \in W^{1,q},\ \mathbf{u}_0 \in H_0^1 \cap H^2,\ \theta_0 \in H^2,\ \frac{\partial\theta_0}{\partial\mathbf{n}}|_{\partial\Omega} = \mathbf{0}, \\ \nabla\mathbf{d}_0 \in H^2,\ \nabla\mathbf{d}_0 \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0},\quad |\mathbf{d}_0| = 1, \end{cases} \tag{1.18}$$

and the compatibility conditions

$$-\mu\Delta\mathbf{u}_0 - (\lambda+\mu)\nabla \operatorname{div}\mathbf{u}_0 + \nabla(\rho_0\theta_0) + \nabla\mathbf{d}_0 \cdot \Delta\mathbf{d}_0 = \sqrt{\rho_0}\mathbf{g}_1, \tag{1.19}$$

$$-\kappa\Delta\theta_0 - 2\mu|\mathfrak{D}(\mathbf{u}_0)|^2 - \lambda(\operatorname{div}\mathbf{u}_0)^2 - |\Delta\mathbf{d}_0 + |\nabla\mathbf{d}_0|^2\mathbf{d}_0|^2 = \sqrt{\rho_0}\mathbf{g}_2, \tag{1.20}$$

for some $\mathbf{g}_1, \mathbf{g}_2 \in L^2(\Omega)$*. Let* $(\rho, \mathbf{u}, \theta, \mathbf{d})$ *be a strong solution to the problem* (1.14)–(1.17)*. If* $T^* < \infty$ *is the maximal time of existence for that solution, then we have*

$$\lim_{T\to T^*} \|\operatorname{div}\mathbf{u}\|_{L^1(0,T;L^\infty)} = \infty. \tag{1.21}$$

Remark 1.1. The local existence of a strong solution with initial data as in Theorem 1.1 has been obtained in [4]. Hence, the maximal time T^* is well-defined. Moreover, it is worth mentioning that the condition (1.10) is not needed in establishing the local existence of strong solutions.

Remark 1.2. It should be noted that the blow up criterion (1.21) is somewhat surprising since it is independent of the orientation field $\mathbf{d}$ and is the same as that of the full compressible Navier-Stokes equations [33].

Remark 1.3. The geometric condition (1.10) is introduced by Lei-Li-Zhang in [18], and recently it is proved in [26] that the 2D incompressible nonhomogeneous nematic liquid crystal flows has a unique global strong solution under the condition (1.10), so the result in our paper is reasonable from this point. The blow up criterion here shows that $\operatorname{div}\mathbf{u}$ plays an important role in the fluid dynamics.

We now make some comments on the analysis of this paper. We mainly make use of continuation arguments to prove Theorem 1.1. That is, suppose that (1.21) were false, i.e.,

$$\lim_{T\to T^*} \|\operatorname{div}\mathbf{u}\|_{L^1(0,T;L^\infty)} \le M_0 < \infty.$$

We want to show that

$$\sup_{0\le t\le T^*} \left(\|\rho\|_{W^{1,q}} + \|\theta\|_{H^2} + \|\nabla\mathbf{u}\|_{H^1} + \|\nabla\mathbf{d}\|_{H^2}\right) \le C < +\infty.$$

We first obtain (see Lemma 3.3) a control on the $L_t^\infty L_x^2$-norm of $\nabla\mathbf{u}$ and $\nabla^2\mathbf{d}$. To this end, the key ingredient of the analysis is a logarithmic Sobolev inequality (see Lemma 2.5). The inequality implies the uniform estimate of $\|(\mathbf{u},\nabla\mathbf{d})\|_{L^2(0,T;L^\infty)}$ due to the a priori estimate of $\|(\mathbf{u},\nabla\mathbf{d})\|_{L^2(0,T;H^1)}$ from the energy estimate (3.2). Then we derive the key a priori estimates on $L^\infty(0,T;L^q)$-norm of $\nabla\rho$ by solving a logarithmic Gronwall inequality based on a Brézis-Wainger type inequality (see Lemma 2.6) and the a priori estimates we have just derived.

The rest of this paper is organized as follows. In Section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

We begin with the following Gagliardo-Nirenberg inequality (see [8, Theorem 10.1, p. 27]).

Lemma 2.1 *(Gagliardo-Nirenberg). Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Assume that $1 \le q, r \le \infty$, and j, m are arbitrary integers satisfying $0 \le j < m$. If $v \in W^{m,r}(\Omega) \cap L^q(\Omega)$, then we have*

$$\|D^j v\|_{L^p} \le C\|v\|_{L^q}^{1-a}\|v\|_{W^{m,r}}^a,$$

where

$$-j+\frac{2}{p}=(1-a)\frac{2}{q}+a\left(-m+\frac{2}{r}\right),$$

and

$$a \in \begin{cases} [\frac{j}{m},1), & \text{if } m-j-\frac{2}{r} \text{ is a nonnegative integer},\\ [\frac{j}{m},1], & \text{otherwise}. \end{cases}$$

The constant C depends only on m, j, q, r, a, and Ω. In particular, we have

$$\|v\|_{L^4}^4 \le C\|v\|_{L^2}^2\|v\|_{H^1}^2,$$

which will be used in the next section.

Next, the following useful result (see [18, Theorem 1.5]) will help us to get the estimate of $\|\nabla^2\mathbf{d}\|_{L^2(0,T;L^2)}$ in terms of the basic energy inequality.

Lemma 2.2. *Let $0<\varepsilon_1<\infty$ and $0<\varepsilon_2\le 1$ be any two given positive constants. Let $\Omega\subset\mathbb{R}^2$ be a bounded smooth domain. Assume that $\mathbf{d}:\Omega\to\mathbb{S}^2$ satisfies $\nabla\mathbf{d}\in H^1(\Omega)$ with $\|\nabla\mathbf{d}\|_{L^2}\le\varepsilon_1$ and $d_3\ge\varepsilon_2$. Then there exists a positive constant $\delta_0\in(0,1)$ such that*

$$\|\nabla\mathbf{d}\|_{L^4}^4 \le (1-\delta_0)\|\nabla^2\mathbf{d}\|_{L^2}^2, \tag{2.1}$$

which particularly implies that

$$\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2 \ge \frac{\delta_0}{2}\left(\|\Delta\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4\right). \tag{2.2}$$

Next, we give some regularity results of the following Lamé system with Dirichlet boundary condition, the proof can be found in [29, Proposition 2.1].

$$\begin{cases} -\mu\Delta\mathbf{U}-(\mu+\lambda)\nabla\operatorname{div}\mathbf{U}=\mathbf{F}, & x\in\Omega,\\ \mathbf{U}=\mathbf{0}, & x\in\partial\Omega. \end{cases} \tag{2.3}$$

Lemma 2.3. *Let $q\ge 2$ and $\mathbf{U}$ be a weak solution of (2.3). There exists a constant C depending only on q, μ, λ, and Ω such that the following estimates hold:*

- *If $\mathbf{F}\in L^q(\Omega)$, then*

$$\|\mathbf{U}\|_{W^{2,q}} \le C\|\mathbf{F}\|_{L^q};$$

- *If $\mathbf{F}\in W^{-1,q}(\Omega)$ (i.e., $\mathbf{F}=\operatorname{div}\mathbf{f}$ with $\mathbf{f}=(f_{ij})_{3\times 3}$, $f_{ij}\in L^q(\Omega)$), then*

$$\|\mathbf{U}\|_{W^{1,q}} \le C\|\mathbf{f}\|_{L^q};$$

- *If* $\mathbf{F} \in W^{-1,q}(\Omega)$ *(i.e.,* $\mathbf{F} = \operatorname{div} \mathbf{f}$ *with* $\mathbf{f} = (f_{ij})_{3\times 3}$, $f_{ij} \in L^\infty(\Omega)$*), then*

$$[\nabla \mathbf{U}]_{BMO} \le C\|\mathbf{f}\|_{L^\infty}.$$

Here $BMO(\Omega)$ stands for the John-Nirenberg's space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO} = \|f\|_{L^2} + [f]_{BMO}$$

with

$$[f]_{BMO} = \sup_{x\in\Omega, r\in(0,d)} \frac{1}{|\Omega_r(x)|} \int\limits_{\Omega_r(x)} |f(y) - f_{\Omega_r}(x)| dy,$$

and

$$f_{\Omega_r}(x) = \frac{1}{|\Omega_r(x)|} \int\limits_{\Omega_r(x)} f(y) dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r, and d is the diameter of Ω. $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

Next, we decompose the velocity field into two parts in order to overcome the difficulty caused by the boundary, namely $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v}$ is the solution to the Lamé system

$$\begin{cases} \mu\Delta\mathbf{v} + (\mu+\lambda)\nabla \operatorname{div} \mathbf{v} = \nabla P, & x \in \Omega, \\ \mathbf{v} = \mathbf{0}, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

and $\mathbf{w}$ satisfies the following boundary value problem

$$\begin{cases} \mu\Delta\mathbf{w} + (\mu+\lambda)\nabla \operatorname{div} \mathbf{w} = \rho\dot{\mathbf{u}} + \nabla\mathbf{d}\cdot\Delta\mathbf{d}, & x \in \Omega, \\ \mathbf{w} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{2.5}$$

By virtue of Lemma 2.3, one has the following key estimates for $\mathbf{v}$ and $\mathbf{w}$.

Lemma 2.4. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be a solution of* (2.4) *and* (2.5) *respectively. Then for any* $p \ge 2$*, there is a constant* $C > 0$ *depending only on* p, μ, λ*, and* Ω *such that*

$$\|\mathbf{v}\|_{W^{1,p}} \le C\|P\|_{L^p}, \tag{2.6}$$

and

$$\|\mathbf{w}\|_{W^{2,p}} \le C \left\|\rho\dot{\mathbf{u}} + \nabla\mathbf{d}\cdot\Delta\mathbf{d}\right\|_{L^p}. \tag{2.7}$$

Next, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brézis-Wainger [1]. The reader can refer to [13, Section 2] for the proof.

Lemma 2.5. *Assume Ω is a bounded smooth domain in $\mathbb{R}^2$ and $f \in L^2(s,t;W^{1,q}(\Omega))$ with some $q>2$ and $0 \le s < t \le \infty$, then there is a constant $C>0$ depending only on q and Ω such that*

$$\|f\|_{L^2(s,t;L^\infty)}^2 \le C\left(1+\|f\|_{L^2(s,t;H^1)}^2 \log(e+\|f\|_{L^2(s,t;W^{1,q})})\right). \tag{2.8}$$

Finally, the following variant of the Brézis-Wainger inequality plays a crucial role in obtaining the estimate of $\|(\nabla\rho, \nabla P)\|_{L^q}$. For its proof, please refer to [29, Lemma 2.3].

Lemma 2.6. *Assume Ω is a bounded smooth domain in $\mathbb{R}^2$ and $f \in W^{1,q}(\Omega)$ with some $q>2$, then there is a constant $C>0$ depending only on q and Ω such that*

$$\|f\|_{L^\infty} \le C\left(1+\|f\|_{BMO}\log(e+\|f\|_{W^{1,q}})\right). \tag{2.9}$$

3. Proof of Theorem 1.1

Let $(\rho, \mathbf{u}, \theta, \mathbf{d})$ be a strong solution described in Theorem 1.1. Suppose that (1.21) were false, that is, there exists a constant $M_0>0$ such that

$$\lim_{T\to T^*}\|\operatorname{div}\mathbf{u}\|_{L^1(0,T;L^\infty)} \le M_0 < \infty. \tag{3.1}$$

First of all, we have the following energy estimate.

Lemma 3.1. *Under the condition (3.1), it holds that for any $T\in[0,T^*)$,*

$$\sup_{0\le t\le T}\left(\|\rho\|_{L^\infty}+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\rho\theta\|_{L^1}+\|\nabla\mathbf{d}\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)dt \le C, \tag{3.2}$$

where and in what follows, C, C_1 stand for generic positive constant depending only on Ω, M_0, κ, λ, μ, T^*, and the initial data.

Proof. 1. We obtain from $(1.14)_1$, (3.1), and [3, pp. 340–341] that the density ρ is non-negative and

$$\sup_{0\le t\le T}\|\rho\|_{L^\infty} \le C. \tag{3.3}$$

2. Applying the standard maximum to $(1.14)_3$ together with [5, pp. 43–44] (see also [3, p. 341]) shows that

$$\inf_{x\in\Omega}\theta(x,t) \ge 0. \tag{3.4}$$

3. Multiplying $(1.14)_2$ by $\mathbf{u}$ and $(1.14)_4$ by $-(\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d})$, respectively, then adding the two resulting equations together, and integrating over Ω, we obtain after integrating by parts that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\rho|\mathbf{u}|^2+|\nabla\mathbf{d}|^2\right)dx+\int\left[\mu|\nabla\mathbf{u}|^2+(\lambda+\mu)(\operatorname{div}\mathbf{u})^2+|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2\right]dx\\
&=\int P\operatorname{div}\mathbf{u}dx.
\end{aligned}
\tag{3.5}
$$

Integrating (1.14)$_3$ with respect to x and then adding the resulting equality to (3.5) give rise to

$$
\frac{d}{dt}\int\left(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{2}|\nabla\mathbf{d}|^2+P\right)dx=0,
\tag{3.6}
$$

which combined with (1.18) leads to

$$
\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2+\|\rho\theta\|_{L^1}\right)\le C.
\tag{3.7}
$$

This together with (3.5) and Cauchy-Schwarz inequality yields

$$
\sup_{0\le t\le T}\left(\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^2}^2\right)+\int_0^T\left(\|\nabla\mathbf{u}\|_{L^2}^2+\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2\right)dt\le C.
\tag{3.8}
$$

4. Applying the maximum principle to the equation of d_3 (i.e., the third component of $\mathbf{d}$, see [16]) together with the geometric condition (1.10) (see [17]) yields that for any $t>0$,

$$
\inf_{x\in\Omega}d_3(x,t)\ge\inf_{x\in\Omega}d_{03}(x)\ge\varepsilon_0,
$$

which along with Lemma 2.2 implies that for some $\sigma_0>0$,

$$
\|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}\|_{L^2}^2\ge\sigma_0\left(\|\Delta\mathbf{d}\|_{L^2}^2+\|\nabla\mathbf{d}\|_{L^4}^4\right).
\tag{3.9}
$$

So the desired (3.2) follows from (3.3), (3.7), (3.8), (3.9), and the following fact (see [30])

$$
\|\nabla^2\mathbf{d}\|_{L^2}=\|\nabla(\nabla\mathbf{d})\|_{L^2}\le C\|\operatorname{div}(\nabla\mathbf{d})\|_{L^2}+C\|\operatorname{curl}(\nabla\mathbf{d})\|_{L^2}=C\|\Delta\mathbf{d}\|_{L^2}.
\tag{3.10}
$$

This completes the proof of Lemma 3.1. □

Based on (3.2) and (1.14)$_4$, we have the following $L_t^\infty L_x^q$-norm of $\nabla\mathbf{d}$ for any $q\in[2,\infty)$, which plays a crucial role for the higher order estimates of solutions.

Lemma 3.2. *Under the condition (3.1), it holds that for any $q\in[2,\infty)$ and $T\in[0,T^*)$,*

$$
\sup_{0\le t\le T}\|\nabla\mathbf{d}\|_{L^q}+\int_0^T\int|\nabla\mathbf{d}|^{q-2}|\nabla^2\mathbf{d}|^2dxdt\le C.
\tag{3.11}
$$

Proof. This estimate was proved by Huang et al. [15, Lemma 2.3] for three-dimensional case. For the convenience of the reader, we sketch it here. Applying ∇ to $(1.14)_4$, we have

$$\nabla \mathbf{d}_t - \Delta \nabla \mathbf{d} = -\nabla(\mathbf{u} \cdot \nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d}). \tag{3.12}$$

Multiplying (3.12) by $q|\nabla \mathbf{d}|^{q-2}\nabla \mathbf{d}$ $(q \geq 2)$ and integrating the resulting equation over Ω, we then obtain from Gagliardo-Nirenberg inequality and (3.2) that

$$\begin{aligned}
&\frac{d}{dt}\int |\nabla \mathbf{d}|^q dx + \int \left(q|\nabla \mathbf{d}|^{q-2}|\nabla^2 \mathbf{d}|^2 + q(q-2)|\nabla \mathbf{d}|^{q-2}|\nabla|\nabla \mathbf{d}||^2 \right) dx \\
&= -q\int \partial_i \mathbf{u} \cdot \nabla d_j \partial_i d_j |\nabla \mathbf{d}|^{q-2} dx + \int \operatorname{div} |\nabla \mathbf{d}|^q dx + q\int |\nabla \mathbf{d}|^{q+2} dx \\
&\leq C\left(\|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{d}\|_{L^4}^2 \right) \||\nabla \mathbf{d}|^{\frac{q}{2}}\|_{L^4}^2 \\
&\leq C\left(\|\nabla \mathbf{u}\|_{L^2} + \|\nabla \mathbf{d}\|_{L^2}\|\nabla \mathbf{d}\|_{H^1} \right) \||\nabla \mathbf{d}|^{\frac{q}{2}}\|_{L^2}\||\nabla \mathbf{d}|^{\frac{q}{2}}\|_{H^1} \\
&\leq \varepsilon \|\nabla|\nabla \mathbf{d}|^{\frac{q}{2}}\|_{L^2}^2 + C(\varepsilon)\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2 + 1 \right) \|\nabla \mathbf{d}\|_{L^q}^q.
\end{aligned} \tag{3.13}$$

Choosing ε suitably small in (3.13), we deduce (3.11) after using Gronwall's inequality and (3.2). □

Lemma 3.3. *Under the condition (3.1), it holds that for any $T \in [0, T^*)$,*

$$\sup_{0 \leq t \leq T} \left(\|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2 \right) + \int_0^T \left(\|\nabla \theta\|_{L^2}^2 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^3 \mathbf{d}\|_{L^2}^2 \right) dt \leq C. \tag{3.14}$$

Proof. 1. Multiplying $(1.14)_3$ by θ and integrating the resulting equation over Ω yields

$$\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2 + \kappa\|\nabla \theta\|_{L^2}^2 \leq C\|\operatorname{div} \mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2}^2 + \int \theta|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 dx + C\int \theta|\nabla \mathbf{u}|^2 dx. \tag{3.15}$$

Due to (3.1), we shall estimate the last two terms on the right hand side of (3.15). Integration by parts together with $(1.14)_5$, (3.11), and Gagliardo-Nirenberg inequality implies for any $\varepsilon \in (0, 1]$,

$$\begin{aligned}
\int \theta|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 dx &= \int \theta(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|) \cdot (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|) dx \\
&\leq C\int \theta|\nabla \mathbf{d}||\nabla^3 \mathbf{d}| dx + C\int |\nabla \theta||\nabla \mathbf{d}||\nabla^2 \mathbf{d}| dx \\
&\quad + C\int \theta|\nabla \mathbf{d}|^4 dx + C\int \theta|\nabla^2 \mathbf{d}||\nabla \mathbf{d}|^2 dx \\
&\leq C\|\theta\|_{L^4}\|\nabla \mathbf{d}\|_{L^4}\|\nabla^3 \mathbf{d}\|_{L^2} + C\|\nabla \theta\|_{L^2}\|\nabla \mathbf{d}\|_{L^4}\|\nabla^2 \mathbf{d}\|_{L^4}
\end{aligned}$$

$$
\begin{aligned}
&+ C\|\theta\|_{L^2}\|\nabla \mathbf{d}\|_{L^8}^2 + C\|\theta\|_{L^4}\|\nabla^2 \mathbf{d}\|_{L^2}\|\nabla \mathbf{d}\|_{L^8}^2 \\
\le{}& C\left(1+\|\nabla\theta\|_{L^2}\right)\|\nabla^3\mathbf{d}\|_{L^2} + C\|\nabla\theta\|_{L^2}\|\nabla^2\mathbf{d}\|_{L^2}^{\frac12}\|\nabla^2\mathbf{d}\|_{H^1}^{\frac12} \\
&+ C\left(1+\|\nabla\theta\|_{L^2}\right) + C\left(1+\|\nabla\theta\|_{L^2}\right)\|\nabla^2\mathbf{d}\|_{L^2} \\
\le{}& \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla^2\mathbf{d}\|_{L^2}^2 + C,
\end{aligned}
\tag{3.16}
$$

where in the third inequality we have used the following fact

$$
\|\theta\|_{L^r} \le C\left(1+\|\nabla\theta\|_{L^2}\right), \text{ for any } 1\le r<\infty. \tag{3.17}
$$

Indeed, if we denote the average of θ by $\bar\theta = \frac{1}{|\Omega|}\int \theta dx$, it follows from (3.2) that

$$
\bar\theta\int \rho dx \le \int \rho\theta dx + \int \rho|\theta-\bar\theta| dx \le C + C\|\nabla\theta\|_{L^2},
$$

which along with Poincaré's inequality leads to the desired (3.17). Next, in order to control the last term on the right hand side of (3.15), we borrow some idea from [10] (see also [33]). Multiplying $(1.14)_2$ by $\mathbf{u}\theta$ and integrating by parts yield

$$
\begin{aligned}
&\mu\int \theta|\nabla\mathbf{u}|^2 dx + (\mu+\lambda)\int \theta(\operatorname{div}\mathbf{u})^2 dx \\
&= -\int \rho\dot{\mathbf{u}}\cdot\mathbf{u}\theta dx - \mu\int \mathbf{u}\cdot\nabla\mathbf{u}\cdot\nabla\theta dx - (\mu+\lambda)\int(\operatorname{div}\mathbf{u})\mathbf{u}\cdot\nabla\theta dx \\
&\quad -\int \nabla P\cdot\mathbf{u}\theta dx - \int \nabla\mathbf{d}\cdot\Delta\mathbf{d}\cdot\mathbf{u}\theta dx \triangleq \sum_{i=1}^{5} J_i.
\end{aligned}
\tag{3.18}
$$

By Hölder's inequality and Young's inequality, it holds that for any $\varepsilon,\delta\in(0,1]$,

$$
\begin{aligned}
\sum_{i=1}^{3}|J_i| &\le \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2} + C\|\nabla\theta\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}\|_{L^2} \\
&\le \varepsilon\|\nabla\theta\|_{L^2}^2 + \delta\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\left(\|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2\right).
\end{aligned}
\tag{3.19}
$$

Integration by parts gives

$$
\begin{aligned}
J_4 &= \int \rho\theta^2 \operatorname{div}\mathbf{u} dx + \int \rho\theta\mathbf{u}\cdot\nabla\theta dx \\
&\le \|\operatorname{div}\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2}\|\rho\|_{L^\infty}^{\frac12} \\
&\le \varepsilon\|\nabla\theta\|_{L^2}^2 + C\left(\|\operatorname{div}\mathbf{u}\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}^2\right)\|\sqrt{\rho}\theta\|_{L^2}^2.
\end{aligned}
\tag{3.20}
$$

Moreover, we obtain from (3.11) and (3.17) that

$$\begin{aligned}
J_5 &\le \|\nabla \mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^2}\|\theta\|_{L^4}\|\mathbf{u}\|_{L^\infty}\\
&\le C\left(1+\|\nabla\theta\|_{L^2}\right)\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{d}\|_{L^2}\\
&\le \varepsilon\|\nabla\theta\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2\mathbf{d}\|_{L^2}^2 + C.
\end{aligned}\tag{3.21}$$

Substituting (3.19)–(3.21) into (3.18), we have

$$\begin{aligned}
\mu\int\theta|\nabla\mathbf{u}|^2dx &\le 3\varepsilon\|\nabla\theta\|_{L^2}^2+\delta\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+C\|\operatorname{div}\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\theta\|_{L^2}^2\\
&\quad + C\|\mathbf{u}\|_{L^\infty}^2\left(\|\sqrt{\rho}\theta\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)+C.
\end{aligned}\tag{3.22}$$

Thus, inserting (3.16) and (3.22) into (3.15), we derive after choosing ε suitably small that

$$\begin{aligned}
&\frac{d}{dt}\|\sqrt{\rho}\theta\|_{L^2}^2+\kappa\|\nabla\theta\|_{L^2}^2\\
&\le C\left(1+\|\operatorname{div}\mathbf{u}\|_{L^\infty}+\|\mathbf{u}\|_{L^\infty}^2\right)\left(1+\|\sqrt{\rho}\theta\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)\\
&\quad+\delta\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2.
\end{aligned}\tag{3.23}$$

2. Multiplying $(1.14)_2$ by $\mathbf{u}_t$ and integrating the resulting equation over Ω give rise to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\mu|\nabla\mathbf{u}|^2+(\mu+\lambda)(\operatorname{div}\mathbf{u})^2\right)dx+\int\rho|\dot{\mathbf{u}}|^2dx\\
&=\int\rho\dot{\mathbf{u}}\cdot\mathbf{u}\cdot\nabla\mathbf{u}dx+\int P\operatorname{div}\mathbf{u}_t dx-\int\mathbf{u}_t\cdot\operatorname{div}(\nabla\mathbf{d}\odot\nabla\mathbf{d})dx+\frac{1}{2}\int\mathbf{u}_t\cdot\nabla|\nabla\mathbf{d}|^2dx\\
&\triangleq\sum_{i=1}^4 K_i.
\end{aligned}\tag{3.24}$$

It follows from Cauchy-Schwarz inequality and (3.3) that

$$K_1\le\frac{1}{4}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+C\|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2.\tag{3.25}$$

To bound K_2, we decompose $\mathbf{u}$ into $\mathbf{u}=\mathbf{v}+\mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ satisfy (2.4) and (2.5), respectively. Then we have

$$\begin{aligned}
K_2&=\frac{d}{dt}\left(\int P\operatorname{div}\mathbf{u}dx\right)-\int P_t\operatorname{div}\mathbf{u}dx\\
&=\frac{d}{dt}\left(\int P\operatorname{div}\mathbf{u}dx\right)-\int P_t\operatorname{div}\mathbf{v}dx-\int P_t\operatorname{div}\mathbf{w}dx\\
&=\frac{d}{dt}\left(\int P\operatorname{div}\mathbf{u}dx\right)+\int\nabla P_t\cdot\mathbf{v}dx-\int P_t\operatorname{div}\mathbf{w}dx\\
&=\frac{d}{dt}\left(\int P\operatorname{div}\mathbf{u}dx\right)+\int(\mu\Delta\mathbf{v}+(\mu+\lambda)\nabla\mathbf{v})_t\cdot\mathbf{v}dx-\int P_t\operatorname{div}\mathbf{w}dx
\end{aligned}$$

$$= \frac{1}{2}\frac{d}{dt}\int \Big(2P\operatorname{div}\mathbf{u} - \mu|\nabla \mathbf{v}|^2 - (\mu+\lambda)(\operatorname{div}\mathbf{v})^2\Big)dx - \int P_t \operatorname{div}\mathbf{w}dx. \tag{3.26}$$

Denote

$$E \triangleq \theta + \frac{1}{2}|\mathbf{u}|^2,$$

then we infer from (1.14) that E satisfies

$$\begin{aligned}(\rho E)_t + \operatorname{div}(\rho\mathbf{u}E + P\mathbf{u}) = {} & \frac{1}{2}\mu\Delta|\mathbf{u}|^2 + \mu\operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{u}) + \lambda\operatorname{div}(\mathbf{u}\operatorname{div}\mathbf{u}) \\ & - \nabla\mathbf{d}\cdot\Delta\mathbf{d}\cdot\mathbf{u} + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2,\end{aligned}$$

which combined with (3.3), (3.11), and Gagliardo-Nirenberg inequality gives

$$\begin{aligned}
-\int P_t\operatorname{div}\mathbf{w}dx = {} & -\int(\rho E)_t\operatorname{div}\mathbf{w}dx + \frac{1}{2}\int(\rho|\mathbf{u}|^2)_t\operatorname{div}\mathbf{w}dx \\
= {} & -\int(\rho E\mathbf{u} + P\mathbf{u} - \mu\nabla\mathbf{u}\cdot\mathbf{u} - \mu\mathbf{u}\cdot\nabla\mathbf{u} - \lambda\mathbf{u}\operatorname{div}\mathbf{u})\cdot\nabla\operatorname{div}\mathbf{w}dx \\
& - \frac{1}{2}\int\operatorname{div}(\rho\mathbf{u})|\mathbf{u}|^2\operatorname{div}\mathbf{w}dx + \int\rho\mathbf{u}\cdot\mathbf{u}_t\operatorname{div}\mathbf{w}dx \\
& + \int\nabla\mathbf{d}\cdot\Delta\mathbf{d}\cdot\mathbf{u}\operatorname{div}\mathbf{w}dx - \int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\operatorname{div}\mathbf{w}dx \\
= {} & -\int\Big(2P\mathbf{u} + \frac{1}{2}\rho|\mathbf{u}|^2\mathbf{u} - \mu\nabla\mathbf{u}\cdot\mathbf{u} - \mu\mathbf{u}\cdot\nabla\mathbf{u} - \lambda\mathbf{u}\operatorname{div}\mathbf{u}\Big)\cdot\nabla\operatorname{div}\mathbf{w}dx \\
& + \frac{1}{2}\int\rho|\mathbf{u}|^2\mathbf{u}\cdot\nabla\operatorname{div}\mathbf{w}dx + \int\rho\dot{\mathbf{u}}\cdot\mathbf{u}\operatorname{div}\mathbf{w}dx \\
& + \int\nabla\mathbf{d}\cdot\Delta\mathbf{d}\cdot\mathbf{u}\operatorname{div}\mathbf{w}dx - \int|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\operatorname{div}\mathbf{w}dx \\
= {} & -\int(2P\mathbf{u} - \mu\nabla\mathbf{u}\cdot\mathbf{u} - \mu\mathbf{u}\cdot\nabla\mathbf{u} - \lambda\mathbf{u}\operatorname{div}\mathbf{u})\cdot\nabla\operatorname{div}\mathbf{w}dx \\
& + \int\rho\dot{\mathbf{u}}\cdot\mathbf{u}\operatorname{div}\mathbf{w}dx + \int\nabla\mathbf{d}\cdot\Delta\mathbf{d}\cdot\mathbf{u}\operatorname{div}\mathbf{w}dx \\
& - \int\Big(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\Big)\cdot\Big(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}\Big)\operatorname{div}\mathbf{w}dx \\
\le {} & C\|\mathbf{u}\|_{L^\infty}\Big(\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_{L^2} + \|\nabla\mathbf{u}\|_{L^2}\Big)\|\nabla^2\mathbf{w}\|_{L^2} \\
& + C\|\mathbf{u}\|_{L^\infty}\|\rho\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2} \\
& + C\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla\mathbf{w}\|_{L^2} + C\|\nabla\mathbf{d}\|_{L^\infty}\|\nabla^3\mathbf{d}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2} \\
& + C\|\nabla\mathbf{d}\|_{L^8}^2\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla\mathbf{w}\|_{L^2} + C\|\nabla\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{w}\|_{L^2} \\
& + C\|\nabla\mathbf{d}\|_{L^8}^4\|\nabla\mathbf{w}\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\mathbf{u}\|_{L^\infty}\left(\|\sqrt{\rho}\theta\|_{L^2}+\|\nabla\mathbf{u}\|_{L^2}\right)\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}+\|\nabla^2\mathbf{d}\|_{L^4}\right)\\
&\quad + C\|\mathbf{u}\|_{L^\infty}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2}+C\|\mathbf{u}\|_{L^\infty}\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla\mathbf{w}\|_{L^2}\\
&\quad + C\|\nabla\mathbf{d}\|_{L^\infty}\|\nabla^3\mathbf{d}\|_{L^2}\|\nabla\mathbf{w}\|_{L^2}+C\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla\mathbf{w}\|_{L^2}\\
&\quad + C\|\nabla^2\mathbf{d}\|_{L^4}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}+\|\nabla^2\mathbf{d}\|_{L^4}\right)+C\|\nabla\mathbf{w}\|_{L^2}\\
&\le \frac{1}{4}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2+C\left(1+\|\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{d}\|_{L^\infty}^2\right)\\
&\quad \times\left(1+\|\sqrt{\rho}\theta\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2\right)\\
&\quad + C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla^2\mathbf{d}\|_{L^2}^2,
\end{aligned}
\tag{3.27}
$$

where one has used the following

$$
\begin{aligned}
\|\nabla^2\mathbf{w}\|_{L^2} &\le C\|\rho\dot{\mathbf{u}}\|_{L^2}+C\||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^2}\\
&\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}+C\|\nabla\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}\\
&\le C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}+C\|\nabla^2\mathbf{d}\|_{L^4},
\end{aligned}
$$

due to (2.7), (3.3), and (3.11). Thus, we get from (3.26) and (3.27) that

$$
\begin{aligned}
K_2 &\le \frac{1}{2}\frac{d}{dt}\int\left(2P\operatorname{div}\mathbf{u}-\mu|\nabla\mathbf{v}|^2-(\mu+\lambda)(\operatorname{div}\mathbf{v})^2\right)dx\\
&\quad + C\left(1+\|\mathbf{u}\|_{L^\infty}^2+\|\nabla\mathbf{d}\|_{L^\infty}^2\right)\left(1+\|\sqrt{\rho}\theta\|_{L^2}^2+\|\nabla\mathbf{u}\|_{L^2}^2+\|\nabla\mathbf{v}\|_{L^2}^2+\|\nabla^2\mathbf{d}\|_{L^2}^2\right)\\
&\quad + C\|\nabla^3\mathbf{d}\|_{L^2}^2+\frac{1}{4}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2.
\end{aligned}
\tag{3.28}
$$

Integration by parts and Cauchy-Schwarz inequality imply that for any $\delta_1\in(0,1]$,

$$
\begin{aligned}
K_3 &= \int(\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}_t\,dx\\
&= \frac{d}{dt}\int(\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}dx-\int(\nabla\mathbf{d}_t\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}dx-\int(\nabla\mathbf{d}\odot\nabla\mathbf{d}_t)\cdot\nabla\mathbf{u}dx\\
&\le \frac{d}{dt}\int(\nabla\mathbf{d}\odot\nabla\mathbf{d})\cdot\nabla\mathbf{u}dx+\delta_1\|\nabla\mathbf{d}_t\|_{L^2}^2+C(\delta_1)\|\nabla\mathbf{d}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2.
\end{aligned}
\tag{3.29}
$$

Similarly, one has

$$
K_4 \le \frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{d}|^2\operatorname{div}\mathbf{u}dx+\delta_1\|\nabla\mathbf{d}_t\|_{L^2}^2+C(\delta_1)\|\nabla\mathbf{d}\|_{L^\infty}^2\|\nabla\mathbf{u}\|_{L^2}^2. \tag{3.30}
$$

Putting (3.25), (3.28), (3.29), and (3.30) into (3.24), we get

$$\begin{aligned}
&B'(t) + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \\
&\le C\left(1 + \|\mathbf{u}\|_{L^\infty}^2 + \|\nabla \mathbf{d}\|_{L^\infty}^2\right)\left(1 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2\right) \\
&\quad + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + 4\delta_1\|\nabla \mathbf{d}_t\|_{L^2}^2,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
B(t) \triangleq{}& \frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\lambda+\mu}{2}\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \frac{\mu}{2}\|\nabla \mathbf{v}\|_{L^2}^2 + \frac{\lambda+\mu}{2}\|\operatorname{div}\mathbf{v}\|_{L^2}^2 \\
&- \int P \operatorname{div}\mathbf{u}dx + \frac{1}{2}\int |\nabla \mathbf{d}|^2 \operatorname{div}\mathbf{u}dx - \int (\nabla \mathbf{d} \odot \nabla \mathbf{d}) \cdot \nabla \mathbf{u}dx
\end{aligned} \tag{3.32}$$

satisfies

$$\frac{\mu}{4}\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2\right) + \|\sqrt{\rho}\theta\|_{L^2}^2 - C \le B(t) \le C\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2\right) + C\|\sqrt{\rho}\theta\|_{L^2}^2 + C, \tag{3.33}$$

due to (3.3) and (3.11). Then, adding (3.23) to (3.31) and choosing δ suitably small give rise to

$$\begin{aligned}
&\frac{d}{dt}\left(B(t) + \|\sqrt{\rho}\theta\|_{L^2}^2\right) + \kappa\|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \\
&\le C\left(1 + \|\operatorname{div}\mathbf{u}\|_{L^\infty} + \|\mathbf{u}\|_{L^\infty}^2 + \|\nabla \mathbf{d}\|_{L^\infty}^2\right)\left(1 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2\right) \\
&\quad + C_1\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\delta_1\|\nabla \mathbf{d}_t\|_{L^2}^2.
\end{aligned} \tag{3.34}$$

3. From (3.12) and (3.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int |\Delta \mathbf{d}|^2 dx + \int \left(|\nabla \mathbf{d}_t|^2 + |\nabla \Delta \mathbf{d}|^2\right) dx \\
&= \int |\nabla \mathbf{d}_t - \nabla \Delta \mathbf{d}|^2 dx \\
&= \int |-\nabla(\mathbf{u}\cdot\nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d})|^2 dx \\
&\le C\||\nabla \mathbf{d}||\nabla^2 \mathbf{d}|\|_{L^2}^2 + C\|\nabla \mathbf{d}\|_{L^\infty}^2\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^\infty}^2\|\nabla^2 \mathbf{d}\|_{L^2}^2 + C \\
&\le C\left(\|\nabla \mathbf{d}\|_{L^\infty}^2 + \|\mathbf{u}\|_{L^\infty}^2\right)\left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2\right) + C.
\end{aligned} \tag{3.35}$$

Adding (3.35) multiplied by $\tilde{C}$ large enough to (3.34) and choosing δ_1 small enough, we derive that

$$\begin{aligned}
&\frac{d}{dt}\left(B(t) + \|\sqrt{\rho}\theta\|_{L^2}^2 + \tilde{C}\|\Delta \mathbf{d}\|_{L^2}^2\right) + \|\nabla\theta\|_{L^2}^2 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2 + \|\nabla^3 \mathbf{d}\|_{L^2}^2 \\
&\le C\left(1 + \|\operatorname{div}\mathbf{u}\|_{L^\infty} + \|\nabla \mathbf{d}\|_{L^\infty}^2 + \|\mathbf{u}\|_{L^\infty}^2\right)\left(1 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{v}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}\|_{L^2}^2\right).
\end{aligned} \tag{3.36}$$

4. Let

$$\Phi(t) \triangleq 2 + \sup_{0\le\tau\le t}\left(2 + \|\sqrt{\rho}\theta\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{v}\|_{L^2}^2 + \|\nabla^2\mathbf{d}\|_{L^2}^2\right) + \int_0^t \left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{d}_\tau\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\right) d\tau. \tag{3.37}$$

Then we obtain from (3.36), (3.33), (3.1), and Gronwall's inequality that for every $0 \le s \le T < T^*$,

$$\Phi(T) \le C\Phi(s)\exp\left\{C\int_s^T \left(\|\mathbf{u}\|_{L^\infty}^2 + \|\nabla\mathbf{d}\|_{L^\infty}^2\right) d\tau\right\}. \tag{3.38}$$

From Lemma 2.5, we get

$$\begin{aligned}
&\|\mathbf{u}\|_{L^2(s,T;L^\infty)}^2 + \|\nabla\mathbf{d}\|_{L^2(s,T;L^\infty)}^2 \\
&\le C\left[1 + \left(\|\mathbf{u}\|_{L^2(s,T;H^1)}^2 + \|\nabla\mathbf{d}\|_{L^2(s,T;H^1)}^2\right)\log\left(e + \|\mathbf{u}\|_{L^2(s,T;W^{1,3})} + \|\nabla\mathbf{d}\|_{L^2(s,T;W^{1,3})}\right)\right] \\
&\le C_2\left[1 + \left(\|\nabla\mathbf{u}\|_{L^2(s,T;L^2)}^2 + \|\nabla^2\mathbf{d}\|_{L^2(s,T;L^2)}^2\right)\log(C\Phi(T))\right],
\end{aligned} \tag{3.39}$$

where one has used the Poincaré inequality, (3.2), and the following facts

$$\begin{aligned}
\|\mathbf{u}\|_{W^{1,3}}^2 &\le \|\mathbf{w}\|_{W^{1,3}}^2 + \|\mathbf{v}\|_{W^{1,3}}^2 \\
&\le C\|\mathbf{w}\|_{W^{2,2}}^2 + C\|P\|_{L^3}^2 \\
&\le C\left(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}^2\right) \\
&\le C\left(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{d}\|_{L^4}^2\|\nabla^2\mathbf{d}\|_{L^4}^2\right) \\
&\le C\left(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2\mathbf{d}\|_{L^2}\|\nabla^2\mathbf{d}\|_{H^1}\right) \\
&\le C\left(1 + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2\mathbf{d}\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\right),
\end{aligned}$$

and

$$\|\nabla\mathbf{d}\|_{W^{1,3}}^2 \le C\|\nabla\mathbf{d}\|_{W^{2,2}}^2 \le C\left(1 + \|\nabla^2\mathbf{d}\|_{L^2}^2 + \|\nabla^3\mathbf{d}\|_{L^2}^2\right).$$

The combination (3.38) and (3.39) gives rise to

$$\Phi(T) \le C\Phi(s)(C\Phi(T))^{C_2\left(\|\nabla\mathbf{u}\|_{L^2(s,T;L^2)}^2 + \|\nabla^2\mathbf{d}\|_{L^2(s,T;L^2)}^2\right)}. \tag{3.40}$$

Recalling (3.2), one can choose s close enough to T^* such that

$$\lim_{T\to T^{*-}} C_2\left(\|\nabla\mathbf{u}\|^2_{L^2(s,T;L^2)} + \|\nabla^2\mathbf{d}\|^2_{L^2(s,T;L^2)}\right) \le \frac{1}{2}.$$

Hence, for $s < T < T^*$, we have

$$\Phi(T) \le C\Phi^2(s) < \infty.$$

This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Under the condition* (3.1)*, it holds that for any* $T \in [0, T^*)$*,*

$$\sup_{0\le t\le T}\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2} + \|\nabla\mathbf{d}_t\|^2_{L^2} + \|\nabla\theta\|^2_{L^2}\right) + \int_0^T\left(\|\nabla\dot{\mathbf{u}}\|^2_{L^2} + \|\nabla^2\mathbf{d}_t\|^2_{L^2} + \|\sqrt{\rho}\dot{\theta}\|^2_{L^2}\right)dt \le C, \tag{3.41}$$

and

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot{\theta}\|^2_{L^2} + \int_0^T\|\nabla\dot{\theta}\|^2_{L^2}dt \le C. \tag{3.42}$$

Proof. 1. We first show (3.41). Operating $\partial_t + \operatorname{div}(\mathbf{u}\cdot)$ to the j-th component of $(1.14)_2$ and multiplying the resulting equation by $\dot{u}^j$, one gets by some calculations that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int\rho|\dot{\mathbf{u}}|^2dx &= \mu\int\dot{u}^j(\partial_t\Delta u^j + \operatorname{div}(\mathbf{u}\Delta u^j))dx + (\lambda+\mu)\int\dot{u}^j(\partial_t\partial_j(\operatorname{div}\mathbf{u})\\ &\quad + \operatorname{div}(\mathbf{u}\partial_j(\operatorname{div}\mathbf{u})))dx - \int\dot{u}^j(\partial_j P_t + \operatorname{div}(\mathbf{u}\partial_j P))dx\\ &\quad - \int\dot{u}^j\left(\partial_t(\nabla\mathbf{d}\cdot\Delta d_j) + \operatorname{div}(\mathbf{u}\cdot\nabla\mathbf{d}\cdot\Delta d_j)\right)dx\\ &\triangleq \sum_{i=1}^4 J_i.\end{aligned} \tag{3.43}$$

Integration by parts leads to

$$\begin{aligned}J_1 &= -\mu\int(\partial_i\dot{u}^j\partial_t\partial_i u^j + \Delta u^j\mathbf{u}\cdot\nabla\dot{u}^j)dx\\ &= -\mu\int(|\nabla\dot{\mathbf{u}}|^2 - \partial_i\dot{u}^j u^k\partial_k\partial_i u^j - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j + \Delta u^j\mathbf{u}\cdot\nabla\dot{u}^j)dx\\ &= -\mu\int(|\nabla\dot{\mathbf{u}}|^2 + \partial_i\dot{u}^j\partial_k u^k\partial_i u^j - \partial_i\dot{u}^j\partial_i u^k\partial_k u^j - \partial_i u^j\partial_i u^k\partial_k\dot{u}^j)dx\\ &\le -\frac{3\mu}{4}\|\nabla\dot{\mathbf{u}}\|^2_{L^2} + C\|\nabla\mathbf{u}\|^4_{L^4}.\end{aligned} \tag{3.44}$$

Similarly, one has

$$J_2 \le -\frac{\lambda+\mu}{2}\|\operatorname{div}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4. \tag{3.45}$$

It follows from integration by parts, $(1.14)_1$, (3.3), (3.17), and Young's inequality that

$$\begin{aligned}
J_3 &= \int (\partial_j \dot{u}^j P_t + \partial_j P\mathbf{u}\cdot\nabla\dot{u}^j)dx \\
&= \int \partial_j\dot{u}^j P_t dx - \int P\partial_j(\mathbf{u}\cdot\nabla\dot{u}^j)dx \\
&= \int \partial_j\dot{u}^j\,[(\rho\theta)_t + \operatorname{div}(\rho\theta\mathbf{u})]\,dx - \int \rho\theta\partial_j\mathbf{u}\cdot\nabla\dot{u}^j dx \\
&= \int \partial_j\dot{u}^j\rho\dot{\theta}dx - \int \rho\theta\partial_j\mathbf{u}\cdot\nabla\dot{u}^j dx \\
&\le C\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\sqrt{\rho}\dot{\theta}\|_{L^2} + C\|\rho\|_{L^\infty}\|\theta\|_{L^4}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\dot{\mathbf{u}}\|_{L^2} \\
&\le \frac{\mu}{4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned} \tag{3.46}$$

Integrating by parts and applying (3.11), (3.14), Sobolev's inequality, and Gagliardo-Nirenberg inequality, we arrive at

$$\begin{aligned}
|J_4| &\le C\int |\nabla\dot{\mathbf{u}}|\Big(|\nabla\mathbf{d}||\nabla\mathbf{d}_t| + |\mathbf{u}||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\Big)dx \\
&\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}\|_{L^4}^2\|\nabla\mathbf{d}_t\|_{L^4}^2 + C\|\mathbf{u}\|_{L^6}^2\|\nabla\mathbf{d}\|_{L^6}^2\|\nabla^2\mathbf{d}\|_{L^6}^2 \\
&\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{H^1} + C\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla^2\mathbf{d}\|_{H^1}^2 \\
&\le \frac{\mu}{8}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{d}_t\|_{L^2}^2 + \varepsilon\|\nabla^2\mathbf{d}_t\|_{L^2}^2 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C.
\end{aligned} \tag{3.47}$$

Inserting (3.44)–(3.47) into (3.43) yields

$$\begin{aligned}
\frac{d}{dt}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \mu\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 &\le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 \\
&\quad + C\|\nabla\mathbf{d}_t\|_{L^2}^2 + \varepsilon\|\nabla^2\mathbf{d}_t\|_{L^2}^2 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C.
\end{aligned} \tag{3.48}$$

2. Differentiation (3.12) with respect to t leads to

$$\nabla\mathbf{d}_{tt} - \Delta\nabla\mathbf{d}_t = -\nabla(\mathbf{u}\cdot\nabla\mathbf{d})_t + \nabla(|\nabla\mathbf{d}|^2\mathbf{d})_t. \tag{3.49}$$

Multiplying (3.49) by $\nabla\mathbf{d}_t$ and integrating the resulting equation over Ω gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla\mathbf{d}_t|^2dx+\int|\nabla^2\mathbf{d}_t|^2dx\\
&\le C\int|\nabla\mathbf{d}||\nabla\mathbf{u}_t||\nabla\mathbf{d}_t|dx+C\int|\mathbf{u}_t||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx+C\int|\nabla\mathbf{d}_t|^2|\nabla\mathbf{u}|dx\\
&\quad+C\int|\nabla\mathbf{d}|^2|\mathbf{d}_t||\nabla^2\mathbf{d}_t|dx+C\int|\nabla\mathbf{d}||\nabla\mathbf{d}_t||\nabla^2\mathbf{d}_t|dx\triangleq\sum_{i=1}^{5}S_i.
\end{aligned}\tag{3.50}
$$

By Hölder's, Young's, Sobolev's, Gagliardo-Nirenberg inequalities, (3.11), and (3.14), one has

$$
\begin{aligned}
S_1+S_2\le&C\int|\nabla\mathbf{d}||\nabla\dot{\mathbf{u}}||\nabla\mathbf{d}_t|dx+C\int|\dot{\mathbf{u}}||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx\\
&+C\int|\nabla^2\mathbf{d}||\mathbf{u}\cdot\nabla\mathbf{u}||\nabla\mathbf{d}_t|dx+C\int|\mathbf{u}\cdot\nabla\mathbf{u}||\nabla^2\mathbf{d}_t||\nabla\mathbf{d}|dx\\
\le&C\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\nabla\mathbf{d}\|_{L^4}\|\nabla\mathbf{d}_t\|_{L^4}+C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\dot{\mathbf{u}}\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}\\
&+C\|\nabla^2\mathbf{d}\|_{L^6}\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{u}\|_{L^{12}}\|\nabla\mathbf{d}_t\|_{L^4}+C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{d}\|_{L^6}\|\mathbf{u}\|_{L^{12}}\\
\le&\frac{\delta}{2}\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C(\delta)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^2+C\|\nabla\mathbf{d}_t\|_{L^4}^2\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C(\delta)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2+C\|\nabla^3\mathbf{d}\|_{L^2}^2+C\|\nabla\mathbf{u}\|_{L^4}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2;\\
S_3\le&C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{d}_t\|_{L^4}^2\le C\|\nabla\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{H^1}\le\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2;\\
S_5\le&C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}\\
\le&\frac{\delta}{2}\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}_t\|_{H^1}\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2.
\end{aligned}
$$

To bound S_4, it follows from $(1.14)_4$, Sobolev's inequality, (3.11), and (3.14) that

$$
\begin{aligned}
\|\mathbf{d}_t\|_{L^2}&=\|-\mathbf{u}\cdot\nabla\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}+\Delta\mathbf{d}\|_{L^2}\\
&\le C\left(\|\mathbf{u}\|_{L^6}\|\nabla\mathbf{d}\|_{L^3}+\|\nabla\mathbf{d}\|_{L^4}^2+\|\nabla^2\mathbf{d}\|_{L^2}\right)\\
&\le C\|\nabla\mathbf{u}\|_{L^2}+C\\
&\le C.
\end{aligned}\tag{3.51}
$$

Hence we deduce from Hölder's, Young's, (3.11), and (3.51) that

$$
\begin{aligned}
S_4\le&C\|\nabla^2\mathbf{d}_t\|_{L^2}\|\nabla\mathbf{d}\|_{L^8}^2\|\mathbf{d}_t\|_{L^4}\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\mathbf{d}_t\|_{H^1}^2\\
\le&\delta\|\nabla^2\mathbf{d}_t\|_{L^2}^2+C\|\nabla\mathbf{d}_t\|_{L^2}^2+C.
\end{aligned}
$$

Substituting the above estimates on S_i $(i=1,\cdots,5)$ into (3.50), we obtain after choosing δ suitably small that

$$\frac{d}{dt}\|\nabla \mathbf{d}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{d}_t\|_{L^2}^2 \le C\|\nabla \mathbf{d}_t\|_{L^2}^2 + C_3\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^2 + C. \tag{3.52}$$

Adding (3.48) multiplied by $\frac{C_3+1}{\mu}$ to (3.52) and choosing ε suitably small, one has

$$\begin{aligned}&\frac{d}{dt}\left(\mu^{-1}(C_3+1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2\right) + \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}_t\|_{L^2}^2 \\ &\le C_4\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla \mathbf{d}_t\|_{L^2}^2 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C.\end{aligned} \tag{3.53}$$

It follows from Lemma 2.4, Gagliardo-Nirenberg inequality, (3.3), (3.17), (3.11), and (3.14) that

$$\begin{aligned}\|\nabla \mathbf{u}\|_{L^4}^4 &\le C\|\nabla \mathbf{v}\|_{L^4}^4 + C\|\nabla \mathbf{w}\|_{L^4}^4 \\ &\le C\|\rho\theta\|_{L^4}^4 + C\|\nabla \mathbf{w}\|_{L^2}^2\|\nabla \mathbf{w}\|_{H^1}^2 \\ &\le C\|\rho\|_{L^\infty}^4\|\theta\|_{L^4}^4 + C(\|\rho\dot{\mathbf{u}}\|_{L^2} + \||\nabla \mathbf{d}||\Delta \mathbf{d}|\|_{L^2})^2 \\ &\le C + C\|\nabla\theta\|_{L^2}^4 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla \mathbf{d}\|_{L^4}^2\|\nabla^2 \mathbf{d}\|_{L^4}^2 \\ &\le C + C\|\nabla\theta\|_{L^2}^4 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla^2 \mathbf{d}\|_{L^2}\|\nabla^2 \mathbf{d}\|_{H^1} \\ &\le C + C\|\nabla\theta\|_{L^2}^4 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2,\end{aligned} \tag{3.54}$$

which together with (3.53) implies that

$$\begin{aligned}&\frac{d}{dt}\left(\mu^{-1}(C_3+1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2\right) + \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla^2 \mathbf{d}_t\|_{L^2}^2 \\ &\le C_4\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla \mathbf{d}_t\|_{L^2}^2 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C.\end{aligned} \tag{3.55}$$

3. Multiplying $(1.14)_3$ by $\dot{\theta}$ and integrating the resulting equation by parts yield that

$$\begin{aligned}&\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2dx + \int\rho|\dot{\theta}|^2dx \\ &= \kappa\int \mathbf{u}\cdot\nabla\theta\Delta\theta dx - \int\rho\theta\,\mathrm{div}\,\mathbf{u}\dot{\theta}dx + \lambda\int(\mathrm{div}\,\mathbf{u})^2\dot{\theta}dx \\ &\quad + 2\mu\int|\mathfrak{D}(\mathbf{u})|^2\dot{\theta}dx + \int|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2\mathbf{d}|^2\dot{\theta}dx \triangleq \sum_{n=1}^{5} I_n.\end{aligned} \tag{3.56}$$

It follows from Sobolev's inequality, Gagliardo-Nirenberg inequality, and (3.17) that for any $\varepsilon \in (0, 1]$,

$$\begin{aligned}\|\theta\|_{L^\infty}^2 &\le C\|\theta\|_{W^{1,4}}^2 \\ &\le C\|\theta\|_{L^4}^2 + C\|\nabla\theta\|_{L^4}^2 \\ &\le C\|\nabla\theta\|_{L^2}^2 + C + C\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1} \\ &\le \varepsilon\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C,\end{aligned} \tag{3.57}$$

which together with (3.14), the standard $W^{2,2}$-estimate of $(1.14)_3$, (3.3), (3.11), (3.17), and Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
\|\theta\|_{H^2}^2 &\le C\int \rho\dot{\theta}^2 dx + C\int \rho^2\theta^2|\nabla \mathbf{u}|^2 dx + C\int |\nabla \mathbf{u}|^4 dx + C\int |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^4 dx + C\|\theta\|_{L^2}^2 \\
&\le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2\|\theta\|_{L^\infty}^2\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C \\
&\le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\varepsilon\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C \\
&\le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\varepsilon\|\nabla^2\theta\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C.
\end{aligned}$$

Hence, we obtain after choosing ε small enough that

$$\|\theta\|_{H^2}^2 \le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C. \tag{3.58}$$

Thus, we obtain from Sobolev's inequality, Gagliardo-Nirenberg inequality, (3.14), and (3.58) that

$$\begin{aligned}
|I_1| &\le C\|\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4}\|\nabla^2\theta\|_{L^2} \\
&\le C\|\nabla \mathbf{u}\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\theta\|_{H^1}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2} \\
&\le \delta\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla^3 \mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C.
\end{aligned} \tag{3.59}$$

From (3.3) and (3.17), we get

$$\begin{aligned}
|I_2| &\le C\|\rho\|_{L^\infty}^{\frac{1}{2}}\|\nabla \mathbf{u}\|_{L^4}\|\theta\|_{L^4}\|\sqrt{\rho}\dot{\theta}\|_{L^2} \\
&\le \delta\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C.
\end{aligned} \tag{3.60}$$

Integration by parts together with (3.17) leads to

$$\begin{aligned}
I_3 &= \lambda\int (\operatorname{div}\mathbf{u})^2\theta_t dx + \lambda\int (\operatorname{div}\mathbf{u})^2(\mathbf{u}\cdot\nabla\theta) dx \\
&= \lambda\frac{d}{dt}\int (\operatorname{div}\mathbf{u})^2\theta dx - 2\lambda\int \theta \operatorname{div}\mathbf{u} \operatorname{div}\dot{\mathbf{u}} dx + 2\lambda\int \theta \operatorname{div}\mathbf{u} \operatorname{div}(\mathbf{u}\cdot\nabla\theta) dx \\
&\quad + \lambda\int (\operatorname{div}\mathbf{u})^2(\mathbf{u}\cdot\nabla\theta) dx \\
&= \lambda\frac{d}{dt}\int (\operatorname{div}\mathbf{u})^2\theta dx - 2\lambda\int \theta \operatorname{div}\mathbf{u} \operatorname{div}\dot{\mathbf{u}} dx + 2\lambda\int \theta \operatorname{div}\mathbf{u}\partial_i u^j\partial_j u^i dx \\
&\quad + \lambda\int \mathbf{u}\cdot\nabla(\theta(\operatorname{div}\mathbf{u})^2) dx \\
&= \lambda\frac{d}{dt}\int (\operatorname{div}\mathbf{u})^2\theta dx - 2\lambda\int \theta \operatorname{div}\mathbf{u} \operatorname{div}\dot{\mathbf{u}} dx + 2\lambda\int \theta \operatorname{div}\mathbf{u}\partial_i u^j\partial_j u^i dx - \lambda\int \theta(\operatorname{div}\mathbf{u})^3 dx \\
&\le \lambda\frac{d}{dt}\int (\operatorname{div}\mathbf{u})^2\theta dx + C\|\theta\|_{L^4}\|\nabla \mathbf{u}\|_{L^4}\|\nabla\dot{\mathbf{u}}\|_{L^2} + C\|\theta\|_{L^4}\|\nabla \mathbf{u}\|_{L^4}^3
\end{aligned}$$

$$\leq \lambda \frac{d}{dt}\int (\operatorname{div}\mathbf{u})^2\theta dx + \eta\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C. \tag{3.61}$$

Similarly to (3.61), one has for any $\eta \in (0, 1]$,

$$I_4 \leq 2\mu\frac{d}{dt}\int |\mathfrak{D}(\mathbf{u})|^2\theta dx + \eta\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C. \tag{3.62}$$

For the term I_5, we deduce from Gagliardo-Nirenberg inequality, Sobolev's inequality, (3.11), (3.14), (3.58), and (3.51) that

$$\begin{aligned}
I_5 &= \int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta_t dx + \int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2(\mathbf{u}\cdot\nabla\theta)dx \\
&= \frac{d}{dt}\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx - \int (|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2)_t\theta dx + \int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2(\mathbf{u}\cdot\nabla\theta)dx \\
&\leq \frac{d}{dt}\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx + C\|\theta\|_{L^\infty}\left(\|\nabla^2\mathbf{d}\|_{L^2} + \|\nabla\mathbf{d}\|_{L^4}^2\right) \\
&\quad \times \left(\|\nabla^2\mathbf{d}_t\|_{L^2} + \|\nabla\mathbf{d}\|_{L^8}^2\|\mathbf{d}_t\|_{L^4} + \|\nabla\mathbf{d}\|_{L^4}\|\nabla\mathbf{d}_t\|_{L^4}\right) \\
&\quad + C(\|\nabla^2\mathbf{d}\|_{L^4}^2 + \|\nabla\mathbf{d}\|_{L^8}^4)\|\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4} \\
&\leq \frac{d}{dt}\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx + C\|\theta\|_{L^\infty}\left(\|\nabla^2\mathbf{d}_t\|_{L^2} + \|\mathbf{d}_t\|_{L^2}^{\frac12}\|\mathbf{d}_t\|_{H^1}^{\frac12} + \|\nabla\mathbf{d}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{d}_t\|_{H^1}^{\frac12}\right) \\
&\quad + C(\|\nabla^2\mathbf{d}\|_{L^2}\|\nabla^2\mathbf{d}\|_{H^1} + 1)\|\nabla\mathbf{u}\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\theta\|_{H^1}^{\frac12} \\
&\leq \frac{d}{dt}\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx + C\|\theta\|_{L^\infty}\left(\|\nabla^2\mathbf{d}_t\|_{L^2} + \|\nabla\mathbf{d}_t\|_{L^2}^{\frac12} + \|\nabla\mathbf{d}_t\|_{L^2}^{\frac12}\|\nabla\mathbf{d}_t\|_{H^1}^{\frac12}\right) \\
&\quad + C(\|\nabla^3\mathbf{d}\|_{L^2} + 1)\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\theta\|_{H^1}^{\frac12} \\
&\leq \frac{d}{dt}\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx + C\|\nabla^2\mathbf{d}_t\|_{L^2}^2 + C\|\nabla\mathbf{d}_t\|_{L^2}^2 \\
&\quad + \delta\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C.
\end{aligned} \tag{3.63}$$

Inserting (3.59)–(3.63) into (3.56) and choosing δ suitably small, we get

$$\begin{aligned}
&\frac{d}{dt}\left(\kappa\|\nabla\theta\|_{L^2}^2 - \Psi(t)\right) + \|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 \\
&\leq \eta\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla\mathbf{u}\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla^2\mathbf{d}_t\|_{L^2}^2 + C\|\nabla\mathbf{d}_t\|_{L^2}^2 \\
&\quad + C\|\nabla^3\mathbf{d}\|_{L^2}^2 + C,
\end{aligned} \tag{3.64}$$

where

$$\Psi(t) \triangleq 2\lambda\int (\operatorname{div}\mathbf{u})^2\theta dx + 4\mu\int |\mathfrak{D}(\mathbf{u})|^2\theta dx + 2\int |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\theta dx \tag{3.65}$$

satisfies

$$
\begin{aligned}
\Psi(t) &\le C\|\theta\|_{L^6}\left(\|\nabla\mathbf{u}\|^2_{L^{\frac{12}{5}}}+\|\nabla^2\mathbf{d}\|^2_{L^{\frac{12}{5}}}+\|\nabla\mathbf{d}\|^4_{L^{\frac{24}{5}}}\right)\\
&\le C\left(1+\|\nabla\theta\|_{L^2}\right)\left(\|\nabla\mathbf{u}\|^{\frac43}_{L^2}\|\nabla\mathbf{u}\|^{\frac23}_{L^4}+\|\nabla^2\mathbf{d}\|^{\frac43}_{L^2}\|\nabla^2\mathbf{d}\|^{\frac23}_{L^4}+1\right)\\
&\le \frac{\kappa}{4}\|\nabla\theta\|^2_{L^2}+C\|\nabla\mathbf{u}\|^{\frac43}_{L^4}+C\|\nabla^2\mathbf{d}\|^{\frac23}_{L^2}\|\nabla^2\mathbf{d}\|^{\frac23}_{H^1}+C\\
&\le \frac{\kappa}{4}\|\nabla\theta\|^2_{L^2}+C\|\nabla\theta\|^{\frac43}_{L^2}+C\|\sqrt{\rho}\dot{\mathbf{u}}\|^{\frac12}_{L^2}+C\|\nabla^3\mathbf{d}\|^{\frac12}_{L^2}+C\|\nabla^3\mathbf{d}\|^{\frac23}_{L^2}+C\\
&\le \frac{\kappa}{2}\|\nabla\theta\|^2_{L^2}+\frac{1}{2\mu(C_4+1)}\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2}+\frac{1}{2(C_4+1)}\|\nabla\mathbf{d}_t\|^2_{L^2}+C,
\end{aligned}
\tag{3.66}
$$

owing to (3.54), (3.14), and the following fact

$$
\begin{aligned}
\|\nabla^3\mathbf{d}\|_{L^2} &\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla(\mathbf{u}\cdot\nabla\mathbf{d})\|_{L^2}+C\|\nabla(|\nabla\mathbf{d}|^2\mathbf{d})\|_{L^2}\\
&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\mathbf{u}\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}+C\||\mathbf{u}||\nabla^2\mathbf{d}|\|_{L^2}+C\|\nabla\mathbf{d}\|^3_{L^6}+C\||\nabla\mathbf{d}||\nabla^2\mathbf{d}|\|_{L^2}\\
&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\theta\|_{L^2}+C\|\sqrt{\rho}\dot{\mathbf{u}}\|^{\frac12}_{L^2}+C\|\nabla^3\mathbf{d}\|^{\frac12}_{L^2}+C\\
&\quad+C\|\mathbf{u}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}+C\|\nabla\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}\|_{L^4}\\
&\le C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\theta\|_{L^2}+C\|\sqrt{\rho}\dot{\mathbf{u}}\|^{\frac12}_{L^2}+C\|\nabla^3\mathbf{d}\|^{\frac12}_{L^2}+C\\
&\quad+C\|\nabla\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{d}\|^{\frac14}_{L^2}\|\nabla^2\mathbf{d}\|^{\frac34}_{L^6}+C\|\nabla^2\mathbf{d}\|^{\frac14}_{L^2}\|\nabla^2\mathbf{d}\|^{\frac34}_{L^6}\\
&\le \frac12\|\nabla^3\mathbf{d}\|_{L^2}+C\|\nabla\mathbf{d}_t\|_{L^2}+C\|\nabla\theta\|_{L^2}+C\|\sqrt{\rho}\dot{\mathbf{u}}\|^{\frac12}_{L^2}+C.
\end{aligned}
\tag{3.67}
$$

Consequently, adding (3.64) multiplied by C_4+1 to (3.55) and choosing η small enough, we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\bar{\Psi}(t)+\|\nabla\dot{\theta}\|^2_{L^2}+\|\nabla\dot{\mathbf{u}}\|^2_{L^2}+\|\nabla^2\mathbf{d}_t\|^2_{L^2}\\
&\le C\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2}+C\|\nabla\theta\|^4_{L^2}+C\|\nabla\mathbf{d}_t\|^2_{L^2}+C,
\end{aligned}
\tag{3.68}
$$

where

$$
\bar{\Psi}(t)\triangleq\mu^{-1}(C_3+1)\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2}+\|\nabla\mathbf{d}_t\|^2_{L^2}+\kappa(C_4+1)\|\nabla\theta\|^2_{L^2}-(C_4+1)\Psi(t)
$$

satisfies

$$
\begin{aligned}
&\mu^{-1}\left(C_3+\frac12\right)\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2}+\frac12\|\nabla\mathbf{d}_t\|^2_{L^2}+\frac{\kappa(C_4+1)}{2}\|\nabla\theta\|^2_{L^2}-C\\
&\le\bar{\Psi}(t)\le C\|\sqrt{\rho}\dot{\mathbf{u}}\|^2_{L^2}+C\|\nabla\mathbf{d}_t\|^2_{L^2}+C\|\nabla\theta\|^2_{L^2}+C
\end{aligned}
\tag{3.69}
$$

due to (3.66). Then the desired (3.41) follows from (3.68), Gronwall's inequality, and (3.14).

4. We now turn to the proof of (3.42). Operating $\partial_t + \operatorname{div}(\mathbf{u}\cdot)$ to $(1.14)_3$ gives rise to

$$\begin{aligned}\rho(\dot\theta + \mathbf{u}\cdot\nabla\dot\theta) = {} & \kappa\Delta\dot\theta + \kappa[\operatorname{div}\mathbf{u}\Delta\theta - \partial_i(\partial_i\mathbf{u}\cdot\nabla\theta) - \partial_i\mathbf{u}\cdot\nabla\partial_i\theta] - \rho\dot\theta\operatorname{div}\mathbf{u} - \rho\theta\operatorname{div}\dot{\mathbf{u}}\\ & + \rho\theta\partial_k u^l\partial_l u^k + (\lambda(\operatorname{div}\mathbf{u})^2 + 2\mu|\mathfrak{D}(\mathbf{u})|^2)\operatorname{div}\mathbf{u} + 2\lambda(\operatorname{div}\dot{\mathbf{u}} - \partial_k u^l\partial_l u^k)\operatorname{div}\mathbf{u}\\ & + \mu(\partial_i u^j + \partial_j u^i)(\partial_i\dot u^j + \partial_j\dot u^i - \partial_i u^k\partial_k u^j - \partial_j u^k\partial_k u^i)\\ & + \partial_t|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2 + \operatorname{div}(|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\mathbf{u}).\end{aligned} \tag{3.70}$$

Then multiplying (3.70) by $\dot\theta$ and integration by parts lead to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int\rho|\dot\theta|^2dx + \kappa\int|\nabla\dot\theta|^2dx\\ &\le C\int|\nabla\mathbf{u}|(|\nabla^2\theta||\dot\theta| + |\nabla\theta||\nabla\dot\theta|)dx + C\int|\nabla\mathbf{u}|^2|\dot\theta|(|\nabla\mathbf{u}| + \theta)dx\\ &\quad + C\int\rho|\dot\theta|^2|\nabla\mathbf{u}|dx + C\int\rho\theta|\nabla\dot{\mathbf{u}}||\dot\theta|dx + C\int|\nabla\mathbf{u}||\nabla\dot{\mathbf{u}}||\dot\theta|dx\\ &\quad + \int\dot\theta\partial_t|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2dx + \int\dot\theta\operatorname{div}(|\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\mathbf{u})dx \triangleq \sum_{n=1}^{7}K_n.\end{aligned} \tag{3.71}$$

It follows from (3.58), (3.54), (3.67), and (3.41) that

$$\|\theta\|_{H^2} \le C\|\sqrt{\rho}\dot\theta\|_{L^2} + C. \tag{3.72}$$

Moreover, by the similar arguments as (3.17), we have

$$\|\dot\theta\|_{L^r} \le C\|\sqrt{\rho}\dot\theta\|_{L^2} + C\|\nabla\dot\theta\|_{L^2}, \text{ for any } 1 \le r < \infty. \tag{3.73}$$

Thus, by virtue of Hölder's, Young's, Sobolev's, Gagliardo-Nirenberg inequalities, (3.54), (3.67), (3.41), (3.72), (3.73), (3.11), (3.51), and (3.14), we can bound each term K_n as follows

$$\begin{aligned}K_1 \le{} & C\|\nabla\mathbf{u}\|_{L^4}\|\nabla^2\theta\|_{L^2}\|\dot\theta\|_{L^4} + C\|\nabla\mathbf{u}\|_{L^4}\|\nabla\theta\|_{L^4}\|\nabla\dot\theta\|_{L^2} \le \frac{\kappa}{12}\|\nabla\dot\theta\|_{L^2}^2\\ & + C\|\sqrt{\rho}\dot\theta\|_{L^2}^2 + C;\\ K_2 \le{} & C\|\nabla\mathbf{u}\|_{L^4}^3\|\dot\theta\|_{L^4} + C\|\nabla\mathbf{u}\|_{L^4}^2\|\theta\|_{L^4}\|\dot\theta\|_{L^4} \le \frac{\kappa}{12}\|\nabla\dot\theta\|_{L^2}^2 + C\|\sqrt{\rho}\dot\theta\|_{L^2}^2 + C;\\ K_3 \le{} & C\|\nabla\mathbf{u}\|_{L^4}\|\dot\theta\|_{L^4}\|\sqrt{\rho}\dot\theta\|_{L^2} \le \frac{\kappa}{12}\|\nabla\dot\theta\|_{L^2}^2 + C\|\sqrt{\rho}\dot\theta\|_{L^2}^2;\\ K_4 + K_5 \le{} & C\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\dot\theta\|_{L^4}\|\theta\|_{L^4} + C\|\nabla\dot{\mathbf{u}}\|_{L^2}\|\dot\theta\|_{L^4}\|\nabla\mathbf{u}\|_{L^4} \le \frac{\kappa}{12}\|\nabla\dot\theta\|_{L^2}^2\\ & + C\|\sqrt{\rho}\dot\theta\|_{L^2}^2 + \|\nabla\dot{\mathbf{u}}\|_{L^2}^2;\\ K_6 \le{} & C\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla^2\mathbf{d}_t\|_{L^2}\|\dot\theta\|_{L^4} + C\|\nabla^2\mathbf{d}\|_{L^4}\|\nabla\mathbf{d}\|_{L^4}\|\nabla\mathbf{d}_t\|_{L^4}\|\dot\theta\|_{L^4}\\ & + C\|\nabla\mathbf{d}\|_{L^8}^2\|\nabla^2\mathbf{d}_t\|_{L^2}\|\dot\theta\|_{L^4} + C\|\nabla\mathbf{d}\|_{L^{16}}^4\|\mathbf{d}_t\|_{L^2}\|\dot\theta\|_{L^4}\end{aligned}$$

$$
\begin{aligned}
&\quad + C\|\nabla \mathbf{d}\|_{L^{12}}^3 \|\nabla \mathbf{d}_t\|_{L^2} \|\dot{\theta}\|_{L^4} + C\|\nabla^2 \mathbf{d}\|_{L^6} \|\nabla \mathbf{d}\|_{L^{24}}^2 \|\mathbf{d}_t\|_{L^2} \|\dot{\theta}\|_{L^4} \\
&\le \frac{\kappa}{12}\|\nabla\dot{\theta}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla^2 \mathbf{d}_t\|_{L^2}^2; \\
K_7 &\le C\|\mathbf{u}\|_{L^6}\|\nabla\dot{\theta}\|_{L^2}\left(\|\nabla^2\mathbf{d}\|_{L^6}^2 + \|\nabla \mathbf{d}\|_{L^{12}}^4\right) \\
&\le C\|\nabla\mathbf{u}\|_{L^2}\|\nabla\dot{\theta}\|_{L^2}\left(\|\nabla^2\mathbf{d}\|_{H^1}^2 + 1\right) \\
&\le C\|\nabla\dot{\theta}\|_{L^2} \le \frac{\kappa}{12}\|\nabla\dot{\theta}\|_{L^2}^2 + C.
\end{aligned}
$$

Substituting the above estimates on K_n $(n = 1, 2, \cdots, 7)$ into (3.71) yields that

$$
\frac{d}{dt}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + \kappa\|\nabla\dot{\theta}\|_{L^2}^2 \le C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\nabla^2\mathbf{d}_t\|_{L^2}^2 + C,
$$

which together with Gronwall's inequality and (3.41) leads to the desired (3.42). This finishes the proof of Lemma 3.4. □

The following lemma will treat the higher order derivatives of the solutions which are needed to guarantee the extension of local strong solution to be a global one.

Lemma 3.5. *Under the condition* (3.1)*, and let $q > 2$ be as in Theorem 1.1, then it holds that for any $T \in [0, T^*)$,*

$$
\sup_{0\le t\le T}\left(\|\rho\|_{W^{1,q}} + \|\nabla\mathbf{u}\|_{H^1} + \|\nabla\mathbf{d}\|_{H^2} + \|\theta\|_{H^2}\right) \le C. \tag{3.74}
$$

Proof. 1. It follows from (3.67) and (3.41) that

$$
\sup_{0\le t\le T}\|\nabla^3\mathbf{d}\|_{L^2} \le C,
$$

which combined with (3.2) and (3.14) gives rise to

$$
\sup_{0\le t\le T}\|\nabla\mathbf{d}\|_{H^2} \le C. \tag{3.75}
$$

2. We derive from (3.72) and (3.42) that

$$
\sup_{0\le t\le T}\|\theta\|_{H^2} \le C. \tag{3.76}
$$

3. For $q > 2$, it follows from the mass equation $(1.14)_1$ that $\nabla\rho$ satisfies

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\rho\|_{L^q} &\le C(q)(1 + \|\nabla\mathbf{u}\|_{L^\infty})\|\nabla\rho\|_{L^q} + C(q)\|\nabla^2\mathbf{u}\|_{L^q} \\
&\le C(1 + \|\nabla\mathbf{w}\|_{L^\infty} + \|\nabla\mathbf{v}\|_{L^\infty})\|\nabla\rho\|_{L^q} + C\left(\|\nabla^2\mathbf{w}\|_{L^q} + \|\nabla^2\mathbf{v}\|_{L^q}\right) \\
&\le C(1 + \|\nabla\mathbf{w}\|_{L^\infty} + \|\nabla\mathbf{v}\|_{L^\infty})\|\nabla\rho\|_{L^q} + C\|\nabla^2\mathbf{w}\|_{L^q} + C
\end{aligned} \tag{3.77}
$$

due to the following fact

$$\|\nabla^2\mathbf{v}\|_{L^q} \le C\|\nabla P\|_{L^q} \le C\|\nabla\rho\|_{L^q}\|\theta\|_{L^\infty} + C\|\nabla\theta\|_{L^q}\|\rho\|_{L^\infty} \le C\|\nabla\rho\|_{L^q} + C,$$

which follows from the standard L^q-estimate for the following elliptic system

$$\begin{cases} \mu\Delta\mathbf{v} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{v} = \nabla P, \quad x \in \Omega, \\ \mathbf{v} = \mathbf{0}, \;\; x \in \partial\Omega, \end{cases}$$

(3.76), and (3.2). From Lemma 2.6 and (2.6), one gets

$$\begin{aligned} \|\nabla\mathbf{v}\|_{L^\infty} &\le C\left(1 + \|\nabla\mathbf{v}\|_{BMO}\log(e + \|\nabla\mathbf{v}\|_{W^{1,q}})\right) \\ &\le C\left(1 + (\|\rho\theta\|_{L^2} + \|\rho\theta\|_{L^\infty})\log(e + \|\nabla\mathbf{v}\|_{W^{1,q}})\right) \\ &\le C\left(1 + \log(e + \|\nabla\rho\|_{L^q})\right). \end{aligned} \tag{3.78}$$

By virtue of Sobolev's embedding theorem, (2.7), (3.2), and (3.75), one deduces that

$$\begin{aligned} \|\nabla\mathbf{w}\|_{L^\infty} &\le \|\mathbf{w}\|_{W^{2,q}} \\ &\le C\|\rho\dot{\mathbf{u}}\|_{L^q} + C\||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^q} \\ &\le C\|\dot{\mathbf{u}}\|_{L^q} + C\|\nabla\mathbf{d}\|_{H^2}^2 \\ &\le C\|\nabla\dot{\mathbf{u}}\|_{L^2} + C. \end{aligned} \tag{3.79}$$

Moreover, we have

$$\|\nabla^2\mathbf{w}\|_{L^q} \le \|\mathbf{w}\|_{W^{2,q}} \le C\|\nabla\dot{\mathbf{u}}\|_{L^2} + C. \tag{3.80}$$

Substituting (3.78)–(3.80) into (3.77), we derive that

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \le C\left(1 + \|\nabla\dot{\mathbf{u}}\|_{L^2} + \log(e + \|\nabla\rho\|_{L^q})\right)\|\nabla\rho\|_{L^q} + C\|\nabla\dot{\mathbf{u}}\|_{L^2} + C. \tag{3.81}$$

Thus, we get from Gronwall's inequality and (3.41) that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q} \le C,$$

which together with (3.2) and the boundedness of Ω yields

$$\sup_{0\le t\le T}\|\rho\|_{W^{1,q}} \le C. \tag{3.82}$$

4. We infer from (2.6), (2.7), and (3.1) that

$$\|\nabla^2\mathbf{u}\|_{L^2} \le \|\nabla^2\mathbf{v}\|_{L^2} + \|\nabla^2\mathbf{w}\|_{L^2} \le C\|\nabla(\rho\theta)\|_{L^2} + C\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + C\||\nabla\mathbf{d}||\Delta\mathbf{d}|\|_{L^2},$$

which combined with (3.82), Hölder's inequality, (3.41), (3.76), and (3.75) implies that

$$\sup_{0\le t\le T}\|\nabla^2\mathbf{u}\|_{L^2}\le C. \tag{3.83}$$

Thus the desired (3.74) follows from (3.75), (3.76), (3.82), (3.83), and (3.14). The proof of Lemma 3.5 is finished. □

With Lemmas 3.1–3.5 at hand, we are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We argue by contradiction. Suppose that (1.21) were false, that is, (3.1) holds. Note that the general constant C in Lemmas 3.1–3.5 is independent of $t<T^*$, that is, all the a priori estimates obtained in Lemmas 3.1–3.5 are uniformly bounded for any $t<T^*$. Hence, the function

$$(\rho,\mathbf{u},\theta,\mathbf{d})(x,T^*)\triangleq\lim_{t\to T^*}(\rho,\mathbf{u},\theta,\mathbf{d})(x,t)$$

satisfy the initial condition (1.18) at $t=T^*$.

Furthermore, standard arguments yield that $\rho\dot{\mathbf{u}},\rho\dot{\theta}\in C([0,T];L^2)$, which implies

$$(\rho\dot{\mathbf{u}},\rho\dot{\theta})(x,T^*)=\lim_{t\to T^*}(\rho\dot{\mathbf{u}},\rho\dot{\theta})\in L^2.$$

Hence,

$$\begin{aligned}-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}+\nabla(\rho\theta)+\nabla\mathbf{d}\cdot\Delta\mathbf{d}|_{t=T^*}&=\sqrt{\rho}(x,T^*)g_1(x),\\ -\kappa\Delta\theta-2\mu|\mathfrak{D}(\mathbf{u})|^2-\lambda(\operatorname{div}\mathbf{u})^2-|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2|_{t=T^*}&=\sqrt{\rho}(x,T^*)g_2(x),\end{aligned}$$

with

$$g_1(x)\triangleq\begin{cases}\rho^{-1/2}(x,T^*)(\rho\dot{\mathbf{u}})(x,T^*), & \text{for } x\in\{x|\rho(x,T^*)>0\},\\ 0, & \text{for } x\in\{x|\rho(x,T^*)=0\},\end{cases}$$

and

$$g_2(x)\triangleq\begin{cases}\rho^{-1/2}(x,T^*)(\rho\dot{\theta}+\rho\theta\operatorname{div}\mathbf{u})(x,T^*), & \text{for } x\in\{x|\rho(x,T^*)>0\},\\ 0, & \text{for } x\in\{x|\rho(x,T^*)=0\},\end{cases}$$

satisfying $g_1,g_2\in L^2$ due to (3.74). Therefore, one can take $(\rho,\mathbf{u},\theta,\mathbf{d})(x,T^*)$ as the initial data and extend the local strong solution beyond T^*. This contradicts the assumption on T^*. Thus we finish the proof of Theorem 1.1. □

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