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Periodic solutions for an N-dimensional cyclic feedback system with delay ☆

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Abstract

We study models for N cyclically coupled variables (e.g., neuron activities) with overall negative delayed feedback, and without symmetry or monotonicity properties. Our aim is to extract the common parts of similar approaches that are known in dimensions one, two and three so far, to exhibit how these parts work for general dimension N, and to show how this framework includes previous as well as new results. We provide a fixed point theorem and a related theorem on periodic orbits for semiflows on Banach spaces, which then yield periodic solutions of cyclic delayed negative feedback systems for general N. We also give criteria for the global asymptotic stability in the same systems, which are derived by relating the systems to interval maps.

MSC: 34K13; 34K11; 34K20; 37C25; 92B20

Keywords: N-dimensional delay systems with non-monotone cyclic coupling; Fixed point theorem; Periodic solutions; Global asymptotic stability

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1. Introduction

As in [20], we consider a cyclically coupled system of differential equations of the form

$$\begin{cases} \dot{y}_1(t) = -\mu_1 y_1(t) + f_1(y_2(t-\tau_2)) \\ \dot{y}_2(t) = -\mu_2 y_2(t) + f_2(y_3(t-\tau_3)) \\ \qquad \dots \\ \dot{y}_N(t) = -\mu_N y_N(t) + f_N(y_1(t-\tau_1)), \end{cases}$$

with delays $\tau_j \geq 0$ and with decay coefficients $\mu_j > 0,\ j = 1, \dots, N$, and C^1 feedback functions $f_j : \mathbb{R} \to \mathbb{R},\ j = 1, \dots, N$. Systems of this type appear in various biological applications, e.g., as models for protein synthesis, for neural networks with a cyclical structure, or for oscillators generating biological rhythms. See for example [12], [19], [26], [38], [43] for more details. Some systems may have the above form only after a transformation of the form $y = y^* + x$, where $y^* \in \mathbb{R}^N$ is an equilibrium of the original equation; compare [19], pp. 42-43.

The theory of such systems in the case of monotone coupling is established in [31], the main result being that a Poincaré-Bendixson-type theorem holds. In particular, if the ω-limit set of a solution does not contain equilibria, then it must be a nonconstant periodic solution.

We study the existence of periodic solutions to the above system without monotonicity conditions, but with the assumption that each f_j has either negative or positive feedback with respect to zero, and that the overall feedback is negative. That is, for $x \in \mathbb{R} \setminus \{0\}$ and $j = 1, \dots, N$ one has

$$\operatorname{sign}[f_j(x) \cdot x] = \sigma_j \in \{-1, +1\}, \text{ and } \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_N = -1. \tag{H1}$$

The latter implies that $f_j(0) = 0,\ j = 1, ..., N$, holds so the system has the only constant solution $y_1 = y_2 = \ldots = y_N \equiv 0$.

In the non-monotone case with negative delayed feedback, even for $N = 1$, the dynamics can be complicated (see [24], [25], and the related results for ordinary differential equations from [11]).

For the cases $N = 1$ or $N = 2$, periodic solutions were obtained from versions of the Browder ejective fixed point theorem in, e.g., [16], [33], and [1]. A similar approach yielded periodic solutions of two-dimensional systems with two or even four delays in papers [39], [36], [44], and for arbitrary N in papers [17], [26].

In the present paper, we try to do as much as possible for general N, under the assumption of cyclic negative coupling (which is not assumed in all of the quoted references).

We now outline how the above system is transformed to a more convenient standard form (compare [1]). Setting $\tau := \tau_1 + \tau_2 + ... + \tau_N$ and $z_1 := y_1$, $z_2 := y_2(t - \tau_2)$, ..., $z_N := y_N(t - \tau_2 - \tau_3 - ... - \tau_N)$, the system transforms into a system with the single delay τ that appears only in the last equation.

By further transformations of the form $x_j(t) = -y_j(t)$ and $g_j(x) := f_j(-x)$, we can achieve that for the transformed system

$$(\mathrm{S})\begin{cases}\dot{x}_1(t) = -\mu_1 x_1(t) + g_1(x_2(t))\\ \dot{x}_2(t) = -\mu_2 x_2(t) + g_2(x_3(t))\\ \quad \cdots\cdots\\ \dot{x}_N(t) = -\mu_N x_N(t) + g_N(x_1(t-\tau))\end{cases}$$

one has

$$x g_j(x) > 0 \text{ for } x \neq 0 \text{ and } j = 1, 2, ..., N-1 \text{ and } x g_N(x) < 0 \text{ for } x \neq 0. \tag{H2}$$

We consider system (S) (where the g_j are C^1 functions) from now on. We assume that $N \geq 2$ and that

$$a_j := g_j'(0) \neq 0, \ (j = 1, 2, ..., N), \tag{H3}$$

which together with (H2) implies $a_1, a_2, ..., a_{N-1} > 0$ and $a_N < 0$, and we set

$$a := -a_1 \cdot a_2 \cdot \ldots \cdot a_N > 0.$$

By a solution of system (S) we mean an N-tuple of functions $(x_1, x_2, ..., x_N)$, where $x_1 : [-\tau, \infty) \to \mathbb{R}$ is continuous and has a differentiable restriction to $[0, \infty)$, and $x_2, x_3, ..., x_N : [0, \infty) \to \mathbb{R}$ are differentiable, and the equations in system (S) hold for all $t \geq 0$. Set $C := C^0([-\tau, 0], \mathbb{R})$. As the state space for system (S), we shall mostly use the set $\mathbb{X} := C \times \mathbb{R}^{N-1}$, since only the past of the x_1-variable plays a role in the system.

It is seen by the method of steps that any initial value $\psi = (\varphi, x_2^0, x_3^0, ...x_N^0) \in \mathbb{X}$ defines a unique corresponding solution $(x_1, x_2, ..., x_N)$ of (S). For this solution we have $x_1{}_{|[-1,0]} = \varphi$ and $x_2(0) = x_2^0, ..., x_N(0) = x_N^0$. (In section 2, we partially consider more general linear equations, which require a state space including a past of all variables.)

With the a_j from above, the linearization of system (S) at the zero solution is

$$(\mathrm{L})\begin{cases}\dot{x}_1(t) = -\mu_1 x_1(t) + a_1 x_2(t)\\ \dot{x}_2(t) = -\mu_2 x_2(t) + a_2 x_3(t))\\ \quad \cdots\cdots\\ \dot{x}_N(t) = -\mu_N x_N(t) + a_N x_1(t-\tau)\end{cases}$$

The general approach that we take is well-known, but so far not in arbitrary dimension: the oscillation properties of solutions define a return map on an appropriate cone, and such map can be shown to satisfy the conditions of some version of the ejective fixed point theorem.

We exhibit the connections between terms appearing in the Laplace transform of the equation, spectral projections, and the bilinear form named after Jack Hale in Section 2. There we also provide a general strategy to obtain lower estimates for these terms, and show how these considerations give a unified view on several previous results.

With view on the nonlinear system (S), we provide a suitable fixed point theorem and a corresponding theorem on periodic orbits in section 3. This periodicity result is stated for semiflows on Banach spaces, and thus not dependent on a particular type of equation.

One main condition of that theorem is a lower bound for a spectral projection on a cone within which one searches for fixed points. We comment on the circumstances under which this condition may or may not be satisfied in Section 4; there and in other places we use the detailed analysis of system (L) from [4].

In Section 5 we construct a return map for system (S), based on a condition which guarantees oscillatory behavior for the solutions starting in a suitable cone. This condition is in general weaker than absence of real eigenvalues.

Applying the main result of Section 3 to this map, we obtain a theorem on existence of periodic solutions in Section 6.

We take another point of view in Section 7, where we give a result on global stability of the zero solution of system (S) under different assumptions.

The final Section 8 briefly comments on possible alternative approaches, and corrects some minor errata from [20].

2. Characteristic equation, eigenspace projection, and Laplace transform

Consider a general N-dimensional linear retarded non-homogeneous differential equation of the form

$$(L,h) \qquad \dot{x}(t) = Lx_t + h(t)$$

with $x(t) \in \mathbb{R}^N$, $x_t(\theta) := x(t+\theta)$ $(\theta \in [-\tau, 0])$ as usual, and with a continuous and linear functional $L : C^0([-\tau,0], \mathbb{R}^N) \to \mathbb{R}^N$, and $h : \mathbb{R} \to \mathbb{R}^N$ continuous. The Riesz representation theorem implies that L can be written as a Riemann-Stieltjes-Integral in the form $L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta)$, with a matrix-valued function $\eta : [-\tau, 0] \to \mathbb{R}^{N\times N}$ of bounded variation. The space $C^N := C^0([-\tau,0], \mathbb{R}^N)$ and the functional L can be complexified in the obvious way by defining $C^N_{\mathbb{C}} := C^0([-\tau,0], \mathbb{C}^N)$ and $L_{\mathbb{C}}(u+iv) := Lu + iLv$ for $u, v \in C^N$. We use the max-norm on C^N and $C^N_{\mathbb{C}}$.

The exponential Ansatz $\phi(t) = e^{\lambda t}w$ with $\lambda \in \mathbb{C}$, $w \in \mathbb{C}^N$ for complex-valued solutions of the homogeneous equation $(L, h = 0)$ leads to the equation

$$\Delta(\lambda)w = 0,$$

with the characteristic matrix

$$\Delta(\lambda) := \lambda I_N - \int_{-\tau}^{0} e^{\lambda\theta} d\eta(\theta)$$

(where I_N is the $N \times N$ identity matrix), see [18], p. 200.

ϕ as above is a nonzero solution if and only if λ is a zero of the characteristic function, i.e.,

$$\chi(\lambda) := \det(\Delta(\lambda)) = 0,$$

and $0 \neq w$ is an eigenvector for the eigenvalue zero of $\Delta(\lambda)$.

For $\lambda \in \mathbb{C}$, we also define a linear functional $K_{\lambda,L} : C^N_{\mathbb{C}} \to \mathbb{C}^N$ associated to λ and the functional L by

$$K_{\lambda,L}(\varphi) := \varphi(0) + \int_{-\tau}^{0} d\eta(\theta) \cdot \left(\int_{\theta}^{0} e^{-\lambda(s-\theta)} \varphi(s)\, ds \right).$$

This corresponds to K from formula (4.7), p. 206 in [18].

Recall the notion of the *adjoint matrix* $\hat{M}$ to a given matrix $M = (m_{ij}) \in \mathbb{C}^{N\times N}$, which is the transpose of the matrix of cofactors: If $\hat{m}_{ij}$ is the algebraic complement or cofactor to the coefficient m_{ij} in M (that is, $(-1)^{i+j}$ times the determinant of dimension $N-1$ obtained by canceling the $i-$th row and the $j-$th column in M) then $(\hat{M})_{ij} = \hat{m}_{ji}$, $i, j = 1, ..., N$. One has $\hat{M} \cdot M = M \cdot \hat{M} = \det(M) \cdot I_N$, so $M^{-1} = \frac{1}{\det(M)} \hat{M}$ in case M is invertible. (The adjoint matrix is not to be confused with the transposed matrix.)

We focus on the consideration of the situation when $\lambda \in \mathbb{C}$ is a simple zero of the characteristic function χ, i.e.,

$$\chi(\lambda) = \det(\Delta(\lambda)) = 0, \quad \chi'(\lambda) \neq 0,$$

and that $w \in \mathbb{C}^N \setminus \{0\}$ is a corresponding vector in the (one-dimensional) kernel of $\Delta(\lambda)$. Then the (complex) one-dimensional subspace of $C_{\mathbb{C}}$ spanned by the function $\phi_{\lambda,w} : [-\tau, 0] \ni \theta \mapsto e^{\lambda\theta} w \in \mathbb{C}^N$ is invariant under the semigroup generated by the homogeneous equation $(L, 0)$, and λ is an isolated eigenvalue of its infinitesimal generator A with eigenfunction $\phi_{\lambda,w}$. (See [18], section 7.3, in particular, Theorem 4.1 on p. 205.)

We state some simple properties of $\Delta(\lambda)$:

Remark 2.1. Assume that $\lambda \in \mathbb{C}$ is a simple zero of the characteristic function χ. Then

a) The adjoint matrix $\hat{\Delta}(\lambda)$ is of rank one and has the form

$$\hat{\Delta}(\lambda) = w \cdot v$$

with a column vector w and a row vector v (dyadic product), which are right and left eigenvectors (for the eigenvalue zero) of $\Delta(\lambda)$ in the sense that

$$\Delta(\lambda)w = 0, \ v\Delta(\lambda) = 0.$$

b) The function $\phi(t) := e^{\lambda t} w$ satisfies $\phi'(t) = L\phi_t$, and the function $\psi(t) := e^{-\lambda t} v$ satisfies the adjoint equation

$$\dot{\psi}(t) = -\int_{-\tau}^{0} \psi(t-\theta)\, d\eta(\theta)$$

as defined in formula (21.4), p. 105 in [14].

Proof. Ad a): 1. Differentiating $\hat{\Delta}(\mu) \cdot \Delta(\mu) = \det(\Delta(\mu)) \cdot I_N = \chi(\mu) \cdot I_N$ at $\mu = \lambda$, we obtain that $(\hat{\Delta})'(\lambda) \cdot \Delta(\lambda) + \hat{\Delta}(\lambda) \cdot (\Delta)'(\lambda) = \chi'(\lambda) \cdot I_N$. If we had $\hat{\Delta}(\lambda) = 0$, it would follow from $\chi'(\lambda) \neq 0$ that $\Delta(\lambda)$ is invertible. Hence $\hat{\Delta}(\lambda) \neq 0$.

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2. Since $\hat{\Delta}(\lambda) \cdot \Delta(\lambda) = \det(\Delta(\lambda)) \cdot I_N = 0$, every row of $\hat{\Delta}(\lambda)$ is 'orthogonal' to the image space (column space) of $\Delta(\lambda)$. Since 0 is a simple eigenvalue of $\Delta(\lambda)$, that column space has codimension 1, so the rank of $\hat{\Delta}(\lambda)$ is at most one. In view of the first part, the rank is one.

3. It follows that $\hat{\Delta}(\lambda)$ is of the form $w \cdot v$, with both vectors nonzero. $w_i \neq 0$ for some $i \in \{1, ..., N\}$ and $\hat{\Delta}(\lambda)\Delta(\lambda) = 0$ together imply $v\Delta(\lambda) = 0$. Similarly, $\Delta(\lambda)\hat{\Delta}(\lambda) = 0$ and $v_j \neq 0$ for some j together imply $\Delta(\lambda)w = 0$.

Ad b): These facts are known from [14]; we include a short proof for completeness.

$$\dot{\phi}(t) - L\phi_t = \lambda e^{\lambda t} w - L([-\tau, 0] \ni \theta \mapsto e^{\lambda(t+\theta)} w) = e^{\lambda t} \underbrace{[\lambda I_N - \int_{-1}^{0} e^{\lambda\theta} d\eta(\theta)]}_{=\Delta(\lambda)} w = 0.$$

$$\begin{aligned}\dot{\psi}(t) + \int_{-\tau}^{0} \psi(t-\theta)\, d\eta(\theta) &= v \cdot [-\lambda e^{-\lambda t} I_N + \int_{-\tau}^{0} e^{-\lambda(t-\theta)}\, d\eta(\theta)] \\ &= -e^{-\lambda t} v \cdot \underbrace{[\lambda I_N - \int_{-\tau}^{0} e^{\lambda\theta}\, d\eta(\theta)]}_{=\Delta(\lambda)} = 0. \quad \square\end{aligned}$$

We still assume that λ is a simple zero of χ. With w as above, every $\varphi \in C^N_{\mathbb{C}}$ has a spectral projection $\pi_\lambda \varphi$ to the eigenspace $\mathbb{C} \cdot [\theta \mapsto e^{\lambda\theta} w]$, obviously of the form $(\pi_\lambda\varphi)(\theta) = c_\lambda(\varphi) \cdot e^{\lambda\theta} \quad (\theta \in [-\tau, 0])$ with a complex linear functional $c_\lambda : C^N_{\mathbb{C}} \to \mathbb{C}^N$ satisfying

$$c_\lambda([\theta \mapsto e^{\lambda\theta} w]) = w.$$

In [14], formula (20.5), p. 100, a bilinear form $C^0([0, \tau], \mathbb{C}^{N*}) \times C^0([-\tau, 0], \mathbb{C}^N) \to \mathbb{C}$ associated to η (and thereby to L) is introduced. We denote it by $(\alpha, \beta) \mapsto [\alpha, \beta]_H$, namely,

$$[\alpha, \beta]_H = \alpha(0)\beta(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \alpha(s-\theta) d\eta(\theta)\beta(s)\, ds.$$

Here $\mathbb{C}^{N*}$ stands for row vectors, and the ds-integration is the 'inner' integration while the variable θ runs from $-\tau$ to 0, and the order of terms in the integrand is chosen to suitably express the matrix multiplication (row vector times matrix times column vector). It is known that the spectral projection of any $\varphi \in C^N_{\mathbb{C}}$ can be expressed with the help of this bilinear form.

One has the following relation between spectral projection, the functional $K_{\lambda,L}$ from above, the solution ψ of the adjoint equation and the bilinear form defined by Hale.

Proposition 2.2. *Assume that $\lambda \in \mathbb{C}$ is a simple zero of the characteristic function χ, and that v, w and ψ are as in Remark 2.1, so $\hat{\Delta}(\lambda) = w \cdot v$. Then*

$$\forall \varphi \in C^N_{\mathbb{C}} : \ (\pi_\lambda\varphi)(\theta) = c_\lambda(\varphi) \cdot e^{\lambda\theta} \quad (\theta \in [-\tau, 0]), \ \textit{with}$$

$$c_\lambda(\varphi) = \frac{1}{\chi'(\lambda)} w \cdot v \cdot K_{\lambda,L}(\varphi) = \frac{1}{\chi'(\lambda)} w \cdot [\psi\big|_{[0,\,\tau]}, \varphi]_H. \tag{2.1}$$

Proof. 1. Assume $\varphi \in C_{\mathbb{C}}^N$. With the infinitesimal generator A as introduced above, the spectral projection is given by the Dunford integral of the resolvent $R(z; A)$:

$$\pi_\lambda = \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} R(z; A)\, dz,$$

with $\varepsilon > 0$ small enough so that λ is the only spectral value of A in the closed ball $\overline{B(\lambda, \varepsilon)}$. (See, e.g., [40], p. 321, and formula (8-10) on p. 315 for the case when $f = 1$ on $\overline{B(\lambda, \varepsilon)}$ and zero otherwise.)

We now interchange the evaluation at $\varphi \in C_{\mathbb{C}}^N$ with the integral (which is possible since the Riemann sums for the integral converge even in $L_c(C_{\mathbb{C}}^N, C_{\mathbb{C}}^N)$), and then also exchange the evaluation at $\theta \in [-\tau, 0]$ with the integral (which is possible since this evaluation is continuous from $C_{\mathbb{C}}$ to $\mathbb{C}^n$). Thus we obtain

$$(\pi_\lambda \varphi)(\theta) = \frac{1}{2\pi i} \oint_{|z-\lambda|=\varepsilon} [R(z; A)\varphi](\theta)\, dz = \mathrm{Res}_\lambda[z \mapsto (R(z; A)\varphi)(\theta)]$$

(where the last integral is $\mathbb{C}^N$−valued, and the residue is to be understood component-wise). We now employ formula (4.6) from Corollary 4.1, p. 206 of [18] to obtain

$$(\pi_\lambda \varphi)(\theta) = \mathrm{Res}_\lambda[z \mapsto e^{z\theta} \cdot ([\Delta(z)]^{-1} K_{z,L}(\varphi) + \int_\theta^0 e^{-zs} \varphi(s)\, ds)].$$

The second term in the above sum (together with the factor $e^{z\theta}$) represents a holomorphic function of z and thus does not contribute to the residue, and we have $[\Delta(z)]^{-1} = \dfrac{1}{\det \Delta(z)} \hat{\Delta}(z) = \dfrac{1}{\chi(z)} \hat{\Delta}(z)$, if $|z - \lambda| = \varepsilon$. Further, if f, g are holomorphic in a neighborhood of λ and g has a simple zero at λ, then $\mathrm{Res}_\lambda \frac{f}{g} = \frac{f(\lambda)}{g'(\lambda)}$. Thus we arrive at

$$(\pi_\lambda \varphi)(\theta) = \frac{e^{\lambda\theta}}{\chi'(\lambda)} \hat{\Delta}(\lambda) K_{\lambda,L}(\varphi)$$

which in view of $\hat{\Delta}(\lambda) = w \cdot v$ proves the first equation in (2.1). For the second equation, we show that the numbers $[\psi\big|_{[0,\,\tau]}, \varphi]_H$ and $v \cdot K_{\lambda,L}(\varphi)$ are equal.

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$$
\begin{aligned}
[\psi\big|_{[0,\,\tau]}, \varphi]_H &= \psi(0)\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} \psi(s-\theta)\,d\eta(\theta)\varphi(s)\,ds \\
&= v\cdot\varphi(0) - \int_{-\tau}^{0}\int_{0}^{\theta} e^{-\lambda(s-\theta)} v\,d\eta(\theta)\varphi(s)\,ds \\
&= v\cdot[\varphi(0) + \int_{-\tau}^{0} d\eta(\theta)\left(\int_{\theta}^{0} e^{-\lambda(s-\theta)}\varphi(s)\,ds\right) \\
&= v\cdot K_{\lambda,L}(\varphi). \quad \square
\end{aligned}
$$

Recall now that the Laplace transform is defined by $(\mathcal{L}x)(\lambda) := \int_0^\infty e^{-\lambda t}x(t)\,dt$, for $x : [0,\infty) \to \mathbb{C}^N$ and $\lambda \in \mathbb{C}$ such that the integral converges (as improper Riemann integral in each of the N components). $\mathcal{L}$ has the following properties (assuming all indicated derivatives and integrals exist, and x is defined at least on $[-\tau,\infty)$ for the last property):

$$
\begin{aligned}
&\text{For } M \in \mathbb{C}^{n\times n}, \mathcal{L}(Mx)(\lambda) = M(\mathcal{L}x)(\lambda) \\
&(\mathcal{L}\dot{x})(\lambda) = -x(0) + \lambda(\mathcal{L}x)(\lambda) \\
&[\mathcal{L}x(\cdot-\tau)](\lambda) = e^{-\lambda\tau}\left[\int_{-\tau}^{0} e^{-\lambda t}x(t)\,dt + (\mathcal{L}x)(\lambda)\right].
\end{aligned}
\tag{2.2}
$$

Proposition 2.3. *Let $x : [-\tau,\infty) \to \mathbb{C}^n$ be a solution of equation (L,h) with initial function $\varphi \in (C^0[-\tau,0],\mathbb{R}^N)$, and that $\lambda \in \mathbb{C}$ is such that the Laplace transforms appearing below all converge absolutely in the sense of Lebesgue integrals. (This is, e.g., the case when x and h are both bounded and* Re $(\lambda) < 0$*). Then*

$$
\Delta(\lambda)(\mathcal{L}x)(\lambda) = K_{\lambda,L}(\varphi) + (\mathcal{L}h)(\lambda).
$$

Proof. Applying the Laplace transform to equation (L,h) and using (2.2) we obtain

$$
-x(0) + \lambda(\mathcal{L}x)(\lambda) = \mathcal{L}(t \mapsto Lx_t)(\lambda) + (\mathcal{L}h)(\lambda). \tag{2.3}
$$

For the first term on the right-hand side we calculate, using a suitable version of Fubini's theorem to reverse the order of integration (e.g., Theorem 8.8 in Section 8 of [37]):

$$\begin{aligned}
\mathcal{L}(t \mapsto Lx_t)(\lambda) &= \int_0^\infty e^{-\lambda t} \int_{-\tau}^0 d\eta(\theta) x(t+\theta))dt \\
&= \int_{-\tau}^0 d\eta(\theta) \int_0^\infty e^{-\lambda t} x(t+\theta)\, dt = \int_{-\tau}^0 d\eta(\theta) \int_\theta^\infty e^{-\lambda(s-\theta)} x(s)\, ds \\
&= \int_{-\tau}^0 d\eta(\theta) \left[\int_\theta^0 e^{-\lambda(s-\theta)} x(s)\, ds + \underbrace{\int_0^\infty e^{-\lambda(s-\theta)} x(s)\, ds}_{=e^{\lambda\theta}(\mathcal{L}x)(\lambda)} \right] \\
&= \int_{-\tau}^0 e^{\lambda\theta} d\eta(\theta) \cdot (\mathcal{L}x)(\lambda) + \int_{-\tau}^0 d\eta(\theta) \int_\theta^0 e^{-\lambda(s-\theta)} x(s)\, ds.
\end{aligned}$$

Inserting this result in (2.3), collecting the terms which are multiples of $(\mathcal{L}x)(\lambda)$ on the left hand side, and recalling that $x_0 = \varphi$ gives

$$\underbrace{[\lambda I_N - \int_{-\tau}^0 e^{\lambda\theta} d\eta(\theta)]}_{=\Delta(\lambda)}(\mathcal{L}x)(\lambda) = \underbrace{\varphi(0) + \int_{-\tau}^0 d\eta(\theta) \int_\theta^0 e^{-\lambda(s-\theta)} \varphi(s)\, ds}_{=K_{\lambda,L}(\varphi)} + (\mathcal{L}h)(\lambda),$$

which is the assertion. □

Corollary 2.4. *In the case of Proposition 2.3, assume in addition that λ is a simple zero of the characteristic function χ, and that $\hat{\Delta}(\lambda) = w \cdot v$ as in Remark 2.1. Then*

$$0 = v \cdot K_{\lambda,L}(\varphi) + v \cdot (\mathcal{L}h)(\lambda).$$

Proof. This follows directly from Proposition 2.3, using $v \cdot \Delta(\lambda) = 0$. □

We now consider special cases of the above formulas:

Corollary 2.5. *If the functional L has the form $L\varphi = A\varphi(0) + B\varphi(-\tau)$ with $A, B \in \mathbb{R}^{N\times N}$ then for $\lambda \in \mathbb{C}$*

$$K_{\lambda,L}(\varphi) = \varphi(0) + e^{-\lambda\tau} B \int_{-\tau}^0 e^{-\lambda s} \varphi(s)\, ds,$$

and

$$\Delta(\lambda) = \lambda I_N - e^{-\lambda\tau} B - A.$$

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Proof. The matrix-valued function η with $L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta)$ can be chosen as

$$\eta(\theta) := \begin{cases} 0, & \theta = -\tau \\ B, & -\tau < \theta < 0 \\ B + A, & \theta = 0. \end{cases}$$

From the definitions of $K_{\lambda,L}$ and $\Delta(\lambda)$ one obtains for $\varphi \in C$:

$$K_{\lambda,L}(\varphi) = \varphi(0) + B \int_{-\tau}^{0} e^{-\lambda(s-(-\tau))}\varphi(s)\,ds + A \int_{0}^{0} \ldots = \varphi(0) + e^{-\lambda\tau} B \int_{-\tau}^{0} e^{-\lambda s}\varphi(s)\,ds,$$

and

$$\Delta(\lambda) = \lambda I_N - e^{-\lambda\tau} B - e^{\lambda\cdot 0} A = \lambda I_N - e^{-\lambda\tau} B - A. \quad \square$$

In [20], $N-$dimensional cyclic systems of the form (S) were considered for the special case $N = 3$ and $\tau = 1$. The linear functional L corresponding to the linearization of such a system at the zero solution (namely, system (L)) takes the form $L\varphi = A\varphi(0) + B\varphi(-\tau)$ of Corollary 2.5, with $a_i := g_i'(0)$ $(i = 1, \ldots, n)$ and the $N \times N$-matrices

$$A := \begin{pmatrix} -\mu_1 & a_1 & 0 & \ldots & 0 \\ 0 & -\mu_2 & a_2 & \ldots & 0 \\ & & \ldots & & \\ 0 & & \ldots & -\mu_{N-1} & a_{N-1} \\ 0 & & \ldots. & & -\mu_N \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & \ldots & 0 \\ & \ldots & & \\ 0 & 0 & \ldots & 0 \\ a_N & 0 & \ldots & 0 \end{pmatrix}.$$

Corollary 2.6. *For the case of the cyclic feedback system* (S)*, and with A, B as above, one has for* $\varphi = \begin{pmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_N \end{pmatrix}$ *that* $K_{\lambda,L}(\varphi) = \varphi(0) + e^{-\lambda\tau} a_N \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \int_{-\tau}^{0} e^{-\lambda s}\varphi_1(s)\,ds \end{pmatrix}$,

$$\Delta(\lambda) = \begin{pmatrix} \lambda + \mu_1 & -a_1 & 0 & \ldots & 0 \\ 0 & \lambda + \mu_2 & -a_2 & \ldots & 0 \\ & & \ldots & & \\ 0 & & \ldots & \lambda + \mu_{N-1} & -a_{N-1} \\ -e^{-\lambda\tau} a_N & & \ldots & & \lambda + \mu_N \end{pmatrix},$$

and

$$\chi(\lambda) = (\lambda + \mu_1) \cdot \ldots \cdot (\lambda + \mu_N) - a_1 \cdot \ldots \cdot a_N e^{-\lambda\tau}. \tag{2.4}$$

Further, if λ is a simple zero for χ then $\Delta(\lambda)$ has the left eigenvector (with the eigenvalue zero)

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$$v = [(\lambda+\mu_2)\cdot \ldots \cdot(\lambda+\mu_N), a_1(\lambda+\mu_3)\cdot\ldots\cdot(\lambda+\mu_N), a_1a_2(\lambda+\mu_4)\cdot\ldots\cdot(\lambda+\mu_N),$$
$$\ldots, (\lambda+\mu_N)a_1a_2\ldots a_{N-2}, a_1a_2\ldots a_{N-1}],\ \textit{i.e.}\ ,$$

$$v_k = \prod_{j=1}^{k-1} a_j \prod_{j=k+1}^{N} (\lambda+\mu_j), \quad k = 1, \ldots, N,$$

where empty products are to be read as 1. Finally, for the linear part in the equation from Corollary 2.4 one has

$$v\cdot K_{\lambda,L}(\varphi) = \sum_{k=1}^{N} \Big[\prod_{j=1}^{k-1} a_j \prod_{j=k+1}^{N} (\lambda+\mu_j)\Big]\varphi_k(0) \quad + \quad \prod_{j=1}^{N} a_j \int_{-\tau}^{0} e^{-\lambda(s+\tau)}\varphi_1(s)\,ds.$$

Proof. 1. The formulas for $K_{\lambda,L}(\varphi)$ and $\Delta(\lambda)$ follow directly from Corollary 2.5 by inserting the special form of A and B.

2. Expanding the determinant of $\Delta(\lambda)$ along the first column, one gets

$$\begin{aligned}\chi(\lambda) &= (\lambda+\mu_1)\cdot\ldots\cdot(\lambda+\mu_N) + (-1)^{N+1}(-e^{-\lambda\tau}a_N)\cdot(-1)^{N-1}a_1\cdot\ldots\cdot a_{N-1}\\ &= (\lambda+\mu_1)\cdot\ldots\cdot(\lambda+\mu_N) - a_1\cdot\ldots\cdot a_N e^{-\lambda\tau}.\end{aligned}$$

3. If now λ is a simple zero for χ then v as above multiplied by the first column of $\Delta(\lambda)$ gives $(\lambda+\mu_1)\cdot\ldots\cdot(\lambda+\mu_N) - a_1\cdot\ldots\cdot a_N e^{-\lambda} = \chi(\lambda) = 0$. For $k \in \{2, \ldots, N\}$, multiplying v by the k-th column of $\Delta(\lambda)$ results in

$$\begin{aligned}-a_{k-1}v_{k-1} + (\lambda+\mu_k)v_k &= -a_{k-1}\prod_{j=1}^{k-2} a_j \prod_{j=k}^{N}(\lambda+\mu_j) + (\lambda+\mu_k)\prod_{j=1}^{k-1} a_j \prod_{j=k+1}^{N}(\lambda+\mu_j)\\ &= -\prod_{j=1}^{k-1} a_j \prod_{j=k}^{N}(\lambda+\mu_j) + \prod_{j=1}^{k-1} a_j \prod_{j=k}^{N}(\lambda+\mu_j) = 0.\end{aligned}$$

The formula for $v\cdot K_{\lambda,L}(\varphi)$ follows immediately, noting that $v_N = \prod_{j=1}^{N-1} a_j$. □

With the characteristic function from (2.4), we turn now to the analysis of the characteristic equation $\chi(\lambda) = 0$. Let p_j denote the coefficients of the polynomial part $p(\lambda)$ of the characteristic function, i.e., $p(\lambda) := (\lambda+\mu_1)\cdot\ldots\cdot(\lambda+\mu_N) = \sum_{j=0}^{N} p_j\lambda^j$ (the p_j are the elementary symmetric functions of $(-\mu_1, \ldots, -\mu_N)$).

Lemma 2.7. *(Solutions of the characteristic equation). Consider the cyclic feedback system* (S)*, its linearization* (L)*, and the characteristic function* χ*, as in Corollary 2.6. Set* $\mathcal{Z} := \Big\{\lambda \in \mathbb{C} \;\Big|\; \chi(\lambda) = 0\Big\}$*, and (as in the introduction)* $a := |a_1 \ldots a_N|$.

(i) For fixed delay $\tau > 0$*, there exists* $A_0 > 0$ *(dependent on* τ*) such that if* $a > A_0$ *then the corresponding function* χ *has no real zero. The number* A_0 *can be chosen uniformly for all* $\tau \geq 1$*; a possible choice is* $A_0 := \max\limits_{j=0,\ldots,N} |p_j|j!$.

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(ii) If $\lambda = \rho + i\omega \in \mathcal{Z}$ and $\omega \neq 0$ then $\chi'(\lambda) \neq 0$.
(iii) For every $\rho \in \mathbb{R}$ there is at most one $\omega \in [0, \infty)$ such that $\rho \pm i\omega \in \mathcal{Z}$.
(iv) For fixed delay $\tau > 0$, there exists $A_1 > 0$ such that for $a \geq A_1$ there exists a leading eigenvalue $\lambda^* = \rho + i\omega \in Z$ (i.e., an eigenvalue with maximal real part), with $\rho = \rho(a) > 0$ and $\omega = \omega(a) \in (0, \pi/\tau)$. ρ and ω are both strictly increasing as functions of the parameter a on the interval $[A_1, \infty)$, and $\rho(a) \to \infty$ and $\omega(a) \to \pi/\tau$ as $a \to \infty$. A possible choice for A_1 is $A_1 := \sqrt{\prod_{j=1}^{N}(\mu_j^2 + \frac{\pi^2}{\tau^2})}$.

Proof. Most of these results are proved in Section 2 of [4], namely: (ii) in Lemma 2, (iv) in Lemma 3, and (iii) in Claim 1 within the proof of Lemma 3. In the proof of Lemma 3 from [4] (in the passage preceding formula (11)) it is shown that the value α_1 of the parameter a where the first eigenvalue pair crosses the imaginary axis at $\pm\omega_1$, with $\omega_1 \in (0, \pi/\tau)$, is given by $\alpha_1 = \sqrt{\prod_{j=1}^{N}(\mu_j^2 + \omega_1^2)}$. For all larger a, the assertions of part (iv) of the present lemma are true. This value satisfies $\alpha_1 < \sqrt{\prod_{j=1}^{N}(\mu_j^2 + \frac{\pi^2}{\tau^2})}$.

Assertion (i) is proved in Lemma 1 of [4], except for the statement that A_0 can be chosen uniformly for all $\tau \geq 1$, which we prove now: Note that nonnegative real solutions of the characteristic equation

$$\chi(\lambda) = (\lambda + \mu_1) \cdot \ldots \cdot (\lambda + \mu_N) + ae^{-\lambda\tau} = 0$$

do not exist. For $\tau > 0$, define $A_0(\tau) := \max_{j=0,\ldots,N} \frac{|p_j| j!}{\tau^j}$. For negative real λ and $a \geq A_0(\tau)$ one then has

$$|\sum_{j=0}^{N} p_j \lambda^j| \leq \sum_{j=0}^{N} \frac{|p_j| j!}{\tau^j} \frac{(|\lambda|\tau)^j}{j!} \leq A_0(\tau) \sum_{j=0}^{N} \frac{(|\lambda|\tau)^j}{j!} < ae^{|\lambda|\tau} = ae^{-\lambda\tau},$$

and hence $\chi(\lambda) \neq 0$ for all negative λ, if $a \geq A_0(\tau)$. Now for $\tau \geq 1$ one has

$$A_0(\tau) \leq A_0(1) = \max_{j=0,\ldots,N} |p_j| j!,$$

which shows that the assertion of (i) is true for all $\tau \geq 1$ if $a \geq A_0(1)$. □

Remark: In several previous papers, e.g. [33], [16], [1], [20], calculations using the Laplace transform (not always mentioning that) were carried out until one arrived at an equation analogous to the one in Corollary 2.4, and for the case $h(t) = g(x_t)$ with $g(\psi) = o(||\psi||)$ in the sense that $|g(\psi)| \leq c||\psi||$ with a constant $c > 0$ that can be made arbitrarily small (either by taking $||\psi||$ small enough, or, as in [33] (proof of Lemma 2.2, p. 367), by taking $||\psi||$ large enough. (In dimensions larger than one, the norm $|| \cdot ||$ here usually involved the max-norm of the one component appearing with delay in the equation, such as the first component in system (S), and only the evaluation at zero of the remaining components.) Further, initial values φ from a cone $\mathfrak{K}$ were considered, with the property that for some $c_1 > 0$ one has

$$\forall \varphi \in \mathfrak{K}: \quad |v \cdot K_{\lambda,L}(\varphi)| \geq c_1 ||\varphi||. \tag{2.5}$$

(The previous papers worked with concrete representations of v and $K_{\lambda,L}$ for the case of specific equations.) In view of formula (2.1), it is obvious that the last lower estimate is equivalent to the existence of a lower bound for the spectral projection of the type

$$\forall \varphi \in \mathfrak{K}: ||\pi_\lambda(\varphi)|| \geq c||\varphi||. \tag{2.6}$$

Then, using the concrete instance of the equation from Corollary 2.4, one could conclude that for sufficiently small $\delta > 0$ there exists no nonzero solution x starting with $x_0 = \varphi \in \mathfrak{K}$ and $||\varphi|| = \delta$ such that, for example, $||x_t|| \leq 2\delta$ for all $t \geq 0$; the argument is based on the non-balance between the $O(\delta)$ linear and the $O(\delta^2)$ quadratic term in that equation. This result was essential in proving ejectivity of the fixed point $0 \in \mathfrak{K}$ under a return map defined by the oscillatory character of solutions. Note that an analogue of condition (2.5) is an assumption of the ejectivity criterion stated in Theorem 3.1 of [7]. A similar estimate along the whole trajectory, until it returns to the cone $\mathfrak{K}$, and not only on $\mathfrak{K}$ itself, is provided, e.g., in Lemma 5.15 of [3]. In [26] (for general dimension N, but with a nonlinearity in only one equation), a lower bound of the type (2.5) was proved working with the Hale bilinear form and using the structure of solutions of the adjoint equation (compare (2.1)), in the spirit of the proof of Lemma 5.6, p. 266 from [15].

Lower estimates of the type (2.5) (for φ in some cone) were obtained in several papers by splitting the functional on the left-hand side into real and imaginary part, possibly splitting these into sub-parts, and then distinguishing cases where one or the other part is dominant. Our next goal is to give a unified and general view of such arguments that extends to any dimension N. The following simple general result is helpful for this purpose. The functionals h_1, h_2 below are not necessarily linear or continuous, although that is the case in typical applications. Note that in the case when h_2 is continuous the upper bound in condition (ii) is not a consequence of the continuity of h_2.

Proposition 2.8. *Let $(Y, || \, ||_Y)$ and $(Z, || \, ||_Z)$ be normed spaces over $\mathbb{R}$, and let X be defined as*

$$X := Y \times Z \text{ with the norm } ||(y,z)|| := \max\{||y||_Y,\ ||z||_Z\}.$$

Let $\mathcal{K}$ be a subset of X. Assume that the real valued functional $h_1 : X \to \mathbb{R}$ and the complex-valued functional $h_2 : X \to \mathbb{C}$ satisfy the following conditions with some constants $a_1, a_2, b_2 > 0$:

$\forall (y,z) \in \mathcal{K}$:

(i) $a_1||y||_Y \leq |h_1(y,z)|$;
(ii) $|h_2(y,z)| \leq b_2||z||$;
(iii) $a_2||z||_Z \leq |\operatorname{Im} h_2(y,z)|$.

Then, defining the functional $h : X \to \mathbb{C}$ by $h(y,z) := h_1(y,z) + h_2(y,z)$, there exists $c > 0$ such that

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$$\forall x = (y, z) \in \mathcal{K}: \quad |h(x)| \geq c||x||.$$

Proof. In the proof, we write $||\cdot||$ for all appearing norms.

1. If $(y, z) \in \mathcal{K}$ and $||y|| \geq \frac{2b_2}{a_1}||z||$ then $||h_2(y, z)|| \leq b_2||z|| \leq \frac{a_1}{2}||y||$ and

$$|h(y, z)| \geq |h_1(y, z)| - |h_2(y, z)| \geq a_1||y|| - \frac{a_1}{2}||y|| = \frac{a_1}{2}||y|| \geq b_2||z||,$$

so

$$|h(y, z)| \geq \min\{\frac{a_1}{2}, b_2\} \cdot \max\{||y||, ||z||\}. \tag{2.7}$$

2. If $(y, z) \in \mathcal{K}$ and $||y|| \leq \frac{2b_2}{a_1}||z||$ then, using the fact that h_1 is real-valued and condition (iii), we see that

$$|h(y, z)| \geq |\operatorname{Im} h(y, z)| = |\operatorname{Im} h_2(y, z)| \geq a_2||z|| \geq \frac{a_2 a_1}{2b_2}||y||,$$

so

$$|h(y, z)| \geq \min\{a_2, \frac{a_2 a_1}{2b_2}\} \cdot \max\{||y||, ||z||\}. \tag{2.8}$$

3. Since for $x = (y, z)$ we have $||x|| = \max\{||y||, ||z||\}$, the assertion now follows from (2.7) and (2.8), setting $c := \min\left\{\min\{\frac{a_1}{2}, b_2\}, \min\{a_2, \frac{a_2 a_1}{2b_2}\}\right\}$. □

Consider now again the situation of Lemma 2.7, with $a > A_1$. Let $\lambda = \lambda^*(a) = \rho + i\omega$ be the leading eigenvalue of system (L), so $\rho > 0$ and $\omega \in (0, \pi/\tau)$. Recall the functional $v \cdot K_{\lambda,L}$ as in Corollary 2.6, which corresponds to the spectral projection onto the eigenspace of λ, as described in Proposition 2.2. As mentioned, a lower estimate for this functional as in Proposition 2.8 was essential in several previous papers (for the proof of ejectivity) and is also an essential assumption of our theorem on periodic orbits (Theorem 3.4) in Section 3. Next we provide a criterion in terms of the parameters of system (L) for such a lower estimate to hold.

It is now convenient to use the state space $\mathbb{X} = C^0([-\tau, 0], \mathbb{R}) \times \mathbb{R}^{N-1}$ (compare the introduction), as opposed to the space $C^0([-\tau, 0], \mathbb{R}^N)$ which was used for the more general considerations at the beginning of this section. The norm on $\mathbb{X}$ is given by $||(\varphi, x_2(0), \ldots, x_N(0)||_{\mathbb{X}} := \max\{|\varphi|_\infty, |x_2(0)|, ..., |x_N(0)|\}$.

For $X_0 = (\varphi, x_2(0), \ldots, x_N(0)) \in \mathbb{X}$, the general expression $v \cdot K_{\lambda,L}(\varphi)$ in the sense of Corollary 2.6 and condition (2.5) has to be replaced by the functional

$$h(X_0) := \sum_{k=1}^{N} \left\{ (\prod_{j=1}^{k-1} a_j) \cdot x_k(0) \cdot \prod_{j=k+1}^{N} (\lambda + \mu_j) \right\} + \prod_{j=1}^{N} a_j \int_{-\tau}^{0} e^{-\lambda(s+\tau)}\varphi(s)\, ds, \tag{2.9}$$

see Corollary 2.6. (Here an empty product is to be read as 1, and $x_1(0) = \varphi(0)$.)

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We define the cone

$$\mathfrak{K} := \{X_0 = (\varphi, x_2(0), \ldots, x_N(0)) \in \mathbb{X} | \varphi(-\tau) = 0, \tag{2.10}$$
$$[-\tau, 0] \ni s \mapsto \varphi(s)\exp\{\mu_1 s\} \text{ is increasing, and } x_k(0) \geq 0, 2 \leq k \leq N\}.$$

From this definition, for $X_0 = (\varphi, x_2(0), \ldots, x_N(0)) \in \mathfrak{K}$ one has

$$\forall s \in [-\tau, 0]: \ \varphi(s) \leq e^{-\mu_1 s}\varphi(0) \leq e^{\mu_1 \tau}\varphi(0),$$

so that

$$0 \leq \varphi(0) \leq ||\varphi||_\infty \leq e^{\mu_1 \tau}\varphi(0). \tag{2.11}$$

Lemma 2.9. *If*

$$\forall k \in \{1, ..., N-3\}: \ \operatorname{Im} \prod_{j=k+1}^{N} (\lambda + \mu_j) > 0 \tag{2.12}$$

then the functional h from (2.9) *satisfies a lower estimate of the form*

$$\forall X_0 \in \mathfrak{K}: \ |h(X_0)| \geq c||X_0||_{\mathbb{X}}$$

(with some $c > 0$), which corresponds to an estimate of type (2.5) *and* (2.6).

Remark. 1) Condition (2.12) is to be read as automatically satisfied if $N \in \{1, 2, 3\}$. It is equivalent to the seemingly stronger condition

$$\forall k \in \{1, ..., N-1\}: \ \operatorname{Im} \prod_{j=k+1}^{N} (\lambda + \mu_j) > 0, \tag{2.13}$$

since for $k = N-2$ the product to consider equals $(\rho + \mu_{N-1} + i\omega)(\rho + \mu_N + i\omega)$, which has imaginary part $\omega[2\rho + \mu_{N-1} + \mu_N] > 0$, and for $k = N-1$ the product equals $\rho + \mu_n + i\omega$.

2) The numbers $\lambda + \mu_j = \rho + \mu_j + i\omega$ are all of the form $r_j \exp(i\varphi_j)$ with $\varphi_j := \arctan[\omega/(\rho + \mu_j)] \in (0, \pi/2)$. Hence conditions (2.12) and (2.13) are both equivalent to

$$\forall k \in \{1, ..., N-1\}: \sum_{j=k+1}^{N} \arctan\left[\frac{\omega}{\rho + \mu_j}\right] < \pi.$$

Note that the difference between two of the above sums for different k lies in $(0, \pi/2)$, and for $k = N-1$ the sum equals $\arctan[\omega/(\rho + \mu_N)] \in (0, \pi/2)$, so all such sums have to be in $(0, \pi)$ (and cannot reach, for instance, the interval $(2\pi, 3\pi)$), if all the imaginary parts in (2.13) are positive.

The last condition again is equivalent to

$$\sum_{j=2}^{N} \arctan\left[\frac{\omega}{\rho+\mu_j}\right] < \pi. \tag{2.14}$$

Proof of Lemma 2.9. We have $h(X_0) = h_1(X_0) + h_2(X_0)$, where

$$h_1(X_0) := \left(\prod_{j=1}^{N-1} a_j\right) \cdot x_N(0)$$

is real-valued, and (of course)

$$\begin{aligned} h_2(X_0) &= h(X_0) - h_1(X_0) \\ &= \sum_{k=1}^{N-1} \left\{ \left(\prod_{j=1}^{k-1} a_j\right) \cdot x_k(0) \cdot \prod_{j=k+1}^{N} (\lambda+\mu_j) \right\} + \prod_{j=1}^{N} a_j \int_{-\tau}^{0} e^{-\lambda(s+\tau)} \varphi(s)\, ds. \end{aligned}$$

In the sense of the notation of Proposition 2.8, we use the splitting

$$X_0 = [\underbrace{\varphi, x_2(0), \ldots, x_{N-1}(0)}_{=z}, \underbrace{x_N(0)}_{=y}],$$

which corresponds to the choice of Y and Z as

$$Y := (\mathbb{R}, |\ |) \text{ and } Z := (C^0([-\tau, 0), \mathbb{R}), ||\ ||_\infty) \times \mathbb{R}^{N-2}$$

with the norm $||(\varphi, x_2(0), \ldots, x_{N-1}(0))||_Z := \max\{||\varphi||_\infty, \max\{|x_2(0)|, ..., |x_{N-1}(0)|\}\}$. Obviously h_1 satisfies the estimate

$$|h_1(X_0)| \geq \left(\prod_{j=1}^{N-1} a_j\right) \cdot |x_N(0)|,$$

corresponding to condition (i) of Proposition 2.8.

Further, using $e^{-\lambda(s+\tau)} \leq 1$ for $s \in [-\tau, 0]$ we get the estimate

$$|h_2(X_0)| \leq \left\{ \sum_{k=1}^{N-1} \left(\prod_{j=1}^{k-1} a_j\right) \prod_{j=k+1}^{N} |\lambda+\mu_j| \right\} \max_{k=1,\ldots,N-1} |x_k(0)| + |\prod_{j=1}^{N} a_j| \tau \cdot ||\varphi||_\infty.$$

Since $x_1(0) = \varphi(0)$, we obtain (with the meaning of 'z' and 'Z' indicated above:

$$|h_2(X_0)| \le \left[\left\{ \sum_{k=1}^{N-1} \dots \right\} + |\prod_{j=1}^{N} a_j|\tau \right] \cdot \max\{|x_1(0)|, \max\{\max_{k=2,\dots,N-1} |x_k(0)|, ||\varphi||_\infty\}\}$$
$$\le \underbrace{[\dots]}_{=:b_2} \max\{||\varphi||_\infty, \max_{k=2,\dots,N-1} |x_k(0)|\} = b_2 ||z||_Z,$$

and hence condition (ii) of Proposition 2.8 is also satisfied. We provide a lower estimate for $|\operatorname{Im} h_2|$ now (assuming that $X_0 \in \mathfrak{K}$, which was not necessary so far). Assumption (2.12) in the form (2.13) shows that

$$\nu := \min_{k \in 1,\dots,N-1} \operatorname{Im} \prod_{j=k+1}^{N} (\lambda + \mu_j) > 0.$$

Further, $\omega \in (0, \pi/\tau)$ implies that $\sin(-\omega(s+\tau)) \le 0$ for $s \in [-\tau, 0]$, so for the integral term in h_2 one has

$$\operatorname{Im} \left[\int_{-\tau}^{0} e^{-\lambda(s+\tau)} \varphi(s)\, ds \right] = \int_{-\tau}^{0} e^{-\rho(s+\tau)} \sin(-\omega(s+\tau)) \varphi(s)\, ds \le 0.$$

Since $\prod_{j=1}^{N} a_j < 0$, we conclude that

$$\operatorname{Im} h_2(X_0) \ge \sum_{k=1}^{N-1} \left\{ \left(\prod_{j=1}^{k-1} a_j \right) \cdot x_k(0) \cdot \operatorname{Im} \prod_{j=k+1}^{N} (\lambda + \mu_j) \right\} \ge \sum_{k=1}^{N-1} \left(\prod_{j=1}^{k-1} a_j \right) \cdot \nu \cdot x_k(0)$$
$$\ge \underbrace{\min_{k=1,\dots,N-1} \left(\prod_{j=1}^{k-1} a_j \right) \cdot \nu}_{=:\tilde{\nu}} \cdot \max_{k=1,\dots,N-1} x_k(0).$$

Using (2.11) we obtain with the indicated definition of $\tilde{\nu}$:

$$\operatorname{Im} h_2(X_0) \ge \tilde{\nu} \max\{e^{-\mu_1 \tau} ||\varphi||_\infty, \max_{k=2,\dots,N-1} x_k(0)\} \ge \tilde{\nu} e^{-\mu_1 \tau} \max\{||\varphi||_\infty,$$
$$\max_{k=2,\dots,N-1} x_k(0)\} = \tilde{\nu} e^{-\mu_1 \tau} ||z||_Z.$$

We see that condition (iii) of Proposition 2.8 also holds, and the assertion of the present lemma follows now from that proposition. □

We show now how the calculations from some earlier results, providing lower bounds of type (2.5) or (2.6), fit into the framework of Lemma 2.9.

1) Hadeler and Tomiuk [16] considered the scalar equation $\dot{x}(t) = -\nu x(t) - f(x(t-1))$ with continuous f satisfying $x \cdot f(x) > 0$ for $x \neq 0$, $\nu > 0$ and $f'(0) =: \alpha > 0$. The characteristic

function χ and, for a zero λ of χ, the functional $K_{\lambda,L}$ are in this one-dimensional case given by $\chi(\lambda) = \lambda + \nu + \alpha e^{-\lambda}$ (compare formula (9) in [16]), and by

$$K_{\lambda,L}(\varphi) = \varphi(0) - \alpha \int_{-1}^{0} e^{-\lambda(s+1)}\varphi(s)\,ds = \varphi(0) + (\lambda+\nu)\int_{-1}^{0} e^{-\lambda s}\varphi(s)\,ds.$$

(Note that the eigenvector v in this case can just be replaced by the number 1.) The cone considered in [16] is $\mathfrak{K} := \left\{\varphi \in C \;\middle|\; \varphi(-1) = 0, t \mapsto e^{\nu t}\varphi(t) \text{ is increasing}\right\}$, which for $N = 1$ corresponds to the cone $\mathfrak{K}$ as in (2.10). For α large enough relative to ν, χ has a zero $\lambda = \mu + i\gamma$ with $\gamma \in (0, \pi)$ and $\mu > 0$ (see Corollary to Lemma 3 in [16]).

With such α and λ, the existence of a lower bound for $K_{\lambda,L}$ as in Lemma 2.9 is clear from that lemma (see the remark after it). The reader may compare this to the lower bound derived by a different argument on p. 93 of [16], after formula (21).

In the earlier paper [33], Nussbaum considered the scalar delay equation

$$y'(t) = -f(y(t-1))$$

under the assumption that $f(x) = \beta x + \tilde{f}(x)$ with $\beta > \pi/2$ and f asymptotically linear in the sense that $|\tilde{f}(x)|/|x| \to 0$ for $|x| \to \infty$. Thus the equation can be written as

$$y'(t) = -\beta y(t-1) - \tilde{f}(y(t-1)).$$

Apart from the different notation in [33] compared to [16] (β instead of α, and $\lambda = \mu + i\nu$ instead of $\mu + i\gamma$ for the leading eigenvalue), and the asymptotics for 'large' instead of small solutions, the considerations involving Laplace transform and comparison of linear and nonlinear part are similar to the ones from [16], for the special case $\nu = 0$. The lower bound can in this case also be obtained from Lemma 2.9 for the case $N = 1$. It corresponds to the lower bounds derived on p. 369 of [33] in equations (2.5) and (2.8).

2) An der Heiden [1] considered (with x, y instead of x_1, x_2)) the two-dimensional system

$$\dot{x}_1(t) = -ax_1(t) + x_2(t), \quad \dot{x}_2(t) = -bx_2(t) - f(x_1(t-1)).$$

As in the above examples, there was a leading eigenvalue $\lambda = \mu + i\nu$ with $\mu > 0$ and $\nu \in (0, \pi)$, and negative feedback was assumed for the cyclic coupling. As mentioned in the remark following Lemma 2.9, the existence of the lower bound is again automatically guaranteed in this case, where $N = 2$. The lower bound for the estimate $|\operatorname{Im} h_2|$ in Proposition 2.8 corresponds to the fact that in paper [1], after application of the Laplace transform, also only the imaginary part was considered (compare formula (23) on p. 607 of [1], and the last estimate on the same page).

3) In [20], cyclic systems of the form as in Corollary 2.6 were considered in the case $N = 3, \tau = 1$, and the cone $\mathfrak{K}$ was defined as in (2.10). As noted in the remark after Lemma 2.9, condition (2.12) is also automatically satisfied in this case. The expression $\omega[2\rho + \mu_3 + \mu_2]$ from part 1) of that remark appeared in the definition of the constant m_1 preceding Proposition 4.3 in [20], and the proof of that proposition in [20] can be seen as a special case of the proof of Lemma 2.9.

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We now proceed towards proving that an estimate of the type (2.6) holds on the cone $\mathfrak{K}$, if the number a as in Lemma 2.7 (a measure of the total feedback strength at zero) or the delay τ are sufficiently large. The characteristic equation for $\lambda = \rho + i\omega$ with $\rho \geq 0, \omega \geq 0$ is equivalent to the following two equations (with $a = |a_1....a_N|$), see [4], equation (14) in Section 2):

$$e^{\rho\tau} \prod_{j=1}^{N} \left[(\rho + \mu_j)^2 + \omega^2\right]^{1/2} = a, \tag{2.15}$$

$$\Theta(\omega, \rho) := \sum_{j=1}^{N} \arctan\left[\frac{\omega}{\rho + \mu_j}\right] = (2m-1)\pi - \omega\tau \text{ for some } m \in \mathbb{N}. \tag{2.16}$$

Proposition 2.10. *With A_1 as in Lemma 2.7, there exists $A_2 \geq A_1$ such that if $a \geq A_2$ then the lower bound condition* (2.6) *is satisfied with $\mathfrak{K}$ from* (2.10) *for the leading eigenvalue $\lambda = \rho + i\omega$ of system* (L).

A possible choice for A_2 is as follows: Set $K := \max\{\max_{j=1,\dots,N} \mu_j, \dfrac{\max\{\pi, N-1\}}{\tau}\}$, define $\alpha(x) := e^{\tau x} \cdot [\sqrt{5}x]^N$ for $x > 0$, and set $A_2 := \max\{\alpha(K), A_1\}$.

Proof. Define the constant K, the function α and $A_2 := \max\{\alpha(K), A_1\}$ as in the statement of the proposition, with A_1 as in Lemma 2.7. Then for $a \geq A_2$ there is a unique leading eigenvalue $\lambda = \rho + i\omega$ (depending on a), with $\rho > 0$ and $\omega \in (0, \pi/\tau)$. For such a, one has necessarily $\rho \geq K$, since $\rho < K$ and equation (2.15) would imply that with $g(\rho) := e^{\rho\tau} \prod_{j=1}^{N}[(\rho+\mu_j)^2+\omega^2]^{1/2}$

$$\begin{aligned} a = g(\rho) &< e^{K\tau} \prod_{j=1}^{N} [(K+K)^2 + (\pi/\tau)^2]^{1/2} \leq e^{K\tau} \prod_{j=1}^{N} [5K^2]^{1/2} \\ &= e^{K\tau}[\sqrt{5}K]^N = \alpha(K) \leq A_2 \leq a, \end{aligned}$$

a contradiction. Thus, for $a \geq A_2$ we have (from the choice of K) that $\rho \geq \frac{N-1}{\tau}$, and hence

$$\sum_{j=2}^{N} \arctan\left[\frac{\omega}{\rho + \mu_j}\right] < \sum_{j=2}^{N} \frac{\omega}{\rho + \mu_j} < \sum_{j=2}^{N} \frac{(\pi/\tau)}{\rho} = \frac{(N-1)\pi}{\tau\rho} \leq \pi.$$

We see that condition (2.14) is satisfied if $a \geq A_2$, and the result follows from Lemma 2.9. □

We want to prove an analogue of Proposition 2.10 for the case of large delay τ rather than large a, and for this purpose we need the following analogue of Lemma 2.7. The analysis is similar to the one given in [4], where most of Lemma 2.7 is proved, but slightly different.

Lemma 2.11. *Assume that $a = |a_1....a_N|$ and the parameters $\mu_1, ..., \mu_N$ satisfy*

$$\prod_{j=1}^{N} \mu_j < a. \tag{2.17}$$

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Then for $\tau > 0$ *so large that*

$$a) \quad \sqrt{\prod_{j=1}^{N}(\mu_j^2 + \frac{\pi}{\tau}^2)} \le a, \qquad b) \quad \tau > \sum_{j=2}^{N} \frac{1}{\mu_j}, \tag{2.18}$$

the characteristic function χ *has a unique zero* $\lambda = \rho + i\omega$ *with the properties* $\rho > 0$, $\omega \in (0, \pi/\tau)$, *and this* λ *is the leading eigenvalue (i.e., has maximal real part). Further, the lower bound condition* (2.6) *for the spectral projection on the cone* $\mathfrak{K}$ *is then satisfied.*

Proof. Equation (2.16) for $m = 1$, namely, $\Theta(\omega, \rho) = \pi - \omega\tau$ has a unique solution $\omega^*(\rho, \tau) \in (0, \pi/\tau)$ for every $\tau > 0$ and $\rho \ge 0$, since the difference between the left hand and the right hand side equals $-\pi$ for $\omega = 0$ and equals $\Theta(\pi/\tau, \rho) > 0$ for $\omega = \pi/\tau$, and since this difference is strictly increasing with ω. Note that $\omega^*(\rho, \tau)$ is strictly increasing with ρ. Further, since all μ_j are positive, we have $\omega^*(\rho, \tau) \to 0$ $(\tau \to \infty)$, uniformly w.r. to $\rho \ge 0$. Inserting $\omega^*(\rho, \tau)$ into equation (2.15) we obtain

$$e^{\rho\tau} \prod_{j=1}^{N} \left[(\rho + \mu_j)^2 + \omega^*(\rho, \tau)^2\right]^{1/2} = a. \tag{2.19}$$

For τ large enough so that (2.18), a) holds (condition (2.17) ensures that such τ exist), there exists a unique number $\rho^*(\tau) > 0$ such that (2.19) holds with $\rho = \rho^*(\tau)$, since (2.18), a) and $\omega^*(\rho, \tau) < \pi/\tau$ show that the left-hand side of (2.19) is then less than a for $\rho = 0$. This left hand side is strictly increasing in ρ, and goes to infinity as $\rho \to \infty$. It follows that $\lambda^*(\tau) := \rho^*(\tau) + i\omega^*(\rho^*(\tau), \tau)$ is a zero of the characteristic function.

We prove that $\rho^*(\tau)$ is the maximal real part of all eigenvalues: Equation (2.15) shows that the real part of all eigenvalues is bounded above, so there is a maximal real part.

Claim: If τ satisfies (2.18), b) above then for $\rho \ge 0$ there is no second solution of equation (2.16) in $(0, \pi/\tau)$ besides $\omega^*(\rho, \tau)$.

Proof. Any other solution ω would have to correspond to a number $m \in \mathbb{N}$, $m \ge 2$ (since $\omega^*(\rho, \tau)$ is unique for $m = 1$), but for $\omega \in (0, \pi/\tau)$ the right hand side of equation (2.16) is larger than $(4 - 1)\pi - \pi = 2\pi$ if $m \ge 2$, while the left hand side satisfies

$$\Theta(\omega, \rho) \le \Theta(\omega, 0) = \sum_{j=1}^{N} \arctan\left[\frac{\omega}{\mu_j}\right] \le \pi/2 + \sum_{j=2}^{N} \frac{\omega}{\mu_j} \le \pi/2 + \frac{\pi}{\tau}\sum_{j=2}^{N}\frac{1}{\mu_j}, \tag{2.20}$$

and the last expression is less than $\pi/2 + \pi = 3\pi/2$ if τ satisfies (2.18), b).

The above claim shows that if τ satisfies both (2.18) a) and b) then any eigenvalue $\rho + i\omega$ with $\rho, \omega \ge 0$ different from $\lambda^*(\tau)$ has to satisfy $\omega > \frac{\pi}{\tau} > \omega^*(\rho^*(\tau), \tau)$ (note that positive real eigenvalues are impossible). Now $\omega > \omega^*(\rho^*(\tau), \tau)$ and equation (2.15) show that necessarily $\rho < \rho^*(\tau)$, so $\rho^*(\tau)$ is maximal.

Finally, with an estimate analogous to (2.20), applied to $\rho := \rho^*(\tau)$ and $\omega := \omega^*(\rho^*(\tau), \tau)$, we see from part b) of assumption (2.18) that for such values of τ one has

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$$\sum_{j=2}^{N} \arctan[\omega/(\rho+\mu_j)] < \sum_{j=2}^{N} \frac{\pi}{\tau\mu_j} < \pi.$$

Hence condition (2.14) holds, and the assertion on the lower bound condition follows from Lemma 2.9. □

3. A fixed point theorem and periodic orbits for semiflows

The approach of deriving periodic solutions from fixed points of a return map P, which is defined by zeroes of one component of the system, is well known. The difficulty is always that zero is a trivial fixed point of P, so one needs to find a nontrivial one. In many papers this was achieved by applications of some version of the ejective fixed point theorem of Browder, which means that one has to show ejectivity of the trivial fixed point zero. Alternatively, one can 'cut out' a neighborhood of zero from the domain of P and then get periodic solutions from Schauder's fixed point theorem – this was done in [20], but there only a fixed point of an iterate of P was obtained (i.e., a periodic solution with possibly a larger number of zeroes in the relevant component within one period, while one application of P corresponds to the second zero). The approach of using the Schauder fixed point theorem to derive periodic solutions of differential delay equations was also used in papers [35,41].

In this section we provide a fixed point theorem which possibly does not have maximal generality from the topological point of view, but is tailored to be easily applicable to the equations that we have in mind, and also gives a fixed point for the first iterate of P. Our result is based on the following theorem (Satz 8.2.4, p. 174 in [10]). Among a number of related results elsewhere, this statement appears to be most convenient for our purposes.

Theorem 3.1. *Let E be a Banach space, $\Omega \subset E$ open, and $f : \Omega \to E$ compact. Further assume that $C \subset E$ is closed and convex, Ω_1 is open in E with $C \subset \Omega_1 \subset \Omega$, $m \in \mathbb{N}$, that the iterate f^m is defined on Ω_1, and the following conditions hold:*

$$\text{(i)} \quad \bigcup_{j=0}^{m} f^j(C) \subset \Omega_1 \qquad\qquad \text{(ii)} \quad f^m(\Omega_1) \subset C.$$

Then f has a fixed point in C.

The above theorem is very close to Theorem 2 on p. 298 of [5] except that Ω_1 is not assumed to be convex. The proof of this theorem, as well as the proof of related results such as Browder's ejective fixed point theorem ([6], p. 575), or Lemma 1 in [5], p. 292, or Theorem 1 by Nussbaum in [32], p. 187), uses ultimately the relation of the Lefschetz number $\Lambda(f^m)$ of the m-th iterate of some map f to the Lefschetz number of f itself, as explained in the proof of Lemma 1 in [5]. This relation is such that $\Lambda(f^m) \neq 0$ implies $\Lambda(f) \neq 0$ if the underlying space is topologically simple, in the sense that it has the same homology (cohomology) as a one-point space. This is, in particular, the case for convex subsets of a Banach space. These ideas go back to work of Leray [23] and Deleanu [8]. The statement given in formula (4.2) on p. 449 of [21] is very similar to Theorem 3.1, with slightly more restrictive assumptions.

On the basis of the above theorem we want to prove a theorem that directly links to the semiflows generated by the equations that we have in mind. We first obtain the following result, which can be seen as a version of the ejective fixed point theorem from [6]. We include the proof for completeness.

Theorem 3.2. *Let $\mathfrak{B}$ be a compact convex subset of the Banach space E with $0 \in \mathfrak{B}$, and $\mathfrak{B} \neq \{0\}$. Assume that $P : \mathfrak{B} \to \mathfrak{B}$ and $\eta : \mathfrak{B} \to \mathbb{R}$ are continuous, and that the following conditions hold:*

(i) $\eta(0) = 0,\ \eta(x) > 0$ *for* $x \in \mathfrak{B} \setminus \{0\}$.
(ii) *for every* $\delta > 0$, *the set* $\mathfrak{B}_\delta := \left\{ x \in \mathfrak{B} \mid \eta(x) \geq \delta \right\}$ *is convex.*
(iii) $\exists \delta_0 > 0 : \forall x \in \mathfrak{B} \setminus \{0\}\ \exists j(x) \in \mathbb{N} :\ \eta[P^{j(x)}(x)] > \delta_0$.

Then P *has a fixed point in* $\mathfrak{B}_{\delta_0}$ *(which, in particular, is not zero).*

Proof. $\mathfrak{B}_{\delta_0}$ is compact (and not empty). For $\delta > 0$ we define the set $\mathfrak{B}^+_\delta := \left\{ x \in \mathfrak{B} \mid \eta(x) > \delta \right\}$. For every $x \in \mathfrak{B}_{\delta_0}$ there exists a neighborhood U_x of x in $\mathfrak{B}$ with $P^{j(x)} U_x \subset \mathfrak{B}^+_{\delta_0}$. Taking a finite subcovering $U_{x_1}, ..., U_{x_n}$ and setting $N_0 := \max\{j(x_1), ..., j(x_n)\}$, we obtain

$$\forall x \in \mathfrak{B}_{\delta_0}\ \exists j(x) \in \{0, ..., N_0\} : P^{j(x)}(x) \in \mathfrak{B}^+_{\delta_0}.$$

(Note that this does not imply $P^{N_0}(\mathfrak{B}_{\delta_0}) \subset \mathfrak{B}_{\delta_0}$). Set $\mathcal{O} := \bigcup_{j=0}^{N_0} P^j(\mathfrak{B}_{\delta_0})$. The set $\mathcal{O}$ is compact.

Claim: $P(\mathcal{O}) \subset \mathcal{O}$.

Proof. For $x \in \mathcal{O}$ there exist $y \in \mathfrak{B}_{\delta_0}$ and $k \in \{0, ..., N_0\}$ with $x = P^k(y)$. Then

$$P(x) = P^{k+1}(y) \in \bigcup_{j=1}^{N_0} P^j(\mathfrak{B}_{\delta_0}) \cup \{P^{N_0+1}(y)\} \subset \mathcal{O} \cup \{P^{N_0+1}(y)\},$$

and $j(y) \in \{1, ..., N_0\}$ implies

$$P^{N_0+1}(y) = P^{N_0+1-j(y)} P^{j(y)}(y) \in P^{N_0+1-j(y)}(\mathfrak{B}^+_{\delta_0}) \subset \bigcup_{j=1}^{N_0} P^j(\mathfrak{B}^+_{\delta_0}) \subset \mathcal{O},$$

so $P(x) \in \mathcal{O}$.

It follows that

$$\mathcal{O} = \bigcup_{j=0}^{\infty} P^j(\mathfrak{B}_{\delta_0}).$$

Since P maps only zero to zero, we have $0 \notin \mathcal{O}$, and the compactness of $\mathcal{O}$, the continuity of η and property (i) give that
$\delta_1 := \frac{1}{2} \min_{x \in \mathcal{O}} \eta(x) > 0$. Obviously then that the inclusion holds

$$\mathcal{O} \subset \mathfrak{B}^+_{\delta_1}. \tag{3.1}$$

The set $\mathfrak{B}_{\delta_1}$ is compact, convex, and as in the analogous argument for $\mathfrak{B}_{\delta_0}$ above, there exists $N_1 \in \mathbb{N}$ such that

$$\forall\, x \in \mathfrak{B}_{\delta_1} \exists k(x) \in \{1, ..., N_1\} : \; P^{k(x)}(x) \in \mathfrak{B}_{\delta_0^+}.$$

For $x \in \mathfrak{B}_{\delta_1}$ and $j \geq N_1$ we have $P^j(x) = P^{j-k(x)}P^{k(x)}(x) \in P^{j-k(x)}(\mathfrak{B}_{\delta_0^+}) \subset \mathcal{O}$, and hence we get from (3.1)

$$\forall\, j \geq N_1 : \; P^j(\mathfrak{B}_{\delta_1}) \subset \mathcal{O} \subset \mathfrak{B}^+_{\delta_1}. \tag{3.2}$$

The set $\mathcal{O}_1 := \bigcup_{j=0}^{N_1} P^j(\mathfrak{B}_{\delta_1})$ is compact, and using (3.2) we obtain

$$P^{N_1}(\mathcal{O}_1) \subset \bigcup_{j=0}^{N_1} P^j(P^{N_1}(\mathfrak{B}_{\delta_1})) \subset \mathfrak{B}^+_{\delta_1}. \tag{3.3}$$

The Tietze-Dugundji theorem ([9], Theorem 6.1, p. 188) shows that there exists a continuous extension $\bar{P}$ of P to E with $\bar{P}(E) \subset \mathrm{conv}(P(\mathfrak{B}))$, and from $P(\mathfrak{B}) \subset \mathfrak{B}$ and convexity of $\mathfrak{B}$ we conclude

$$\bar{P}(E) \subset \mathfrak{B}.$$

The continuity of $\bar{P}$, $\bar{P}(E) \subset \mathfrak{B}$, the fact that $\mathfrak{B}^+_{\delta_1}$ is open in $\mathfrak{B}$, and inclusion (3.3) imply that there exists an open neighborhood Ω_1 of $\mathcal{O}_1$ in E such that

$$\bar{P}^{N_1}(\Omega_1) \subset \mathfrak{B}^+_{\delta_1} \subset \mathfrak{B}_{\delta_1}. \tag{3.4}$$

We want to apply Theorem 3.1 above with $\Omega := E$, $f := \bar{P}$, $C := \mathfrak{B}_{\delta_1}$, N_1 in place of m, and with the above set Ω_1. We see from (3.4) that condition (ii) of Theorem 3.1 holds, and from

$$\bigcup_{j=0}^{N_1} \bar{P}^j(\mathfrak{B}_{\delta_1}) = \bigcup_{j=0}^{N_1} P^j(\mathfrak{B}_{\delta_1}) = \mathcal{O}_1 \subset \Omega_1$$

we see that condition (i) holds as well. From Theorem 3.1 we obtain a fixed point of $\bar{P}$ in $\mathfrak{B}_{\delta_1}$, which obviously is a nonzero fixed point of P. $\square$

Remarks: 1) Condition (ii) above is satisfied if η is given by a linear functional. Condition (iii) is closely related to the notion of ejectivity from [6], p. 575.

2) The construction of the set $\mathfrak{B}_{\delta_1}$ in the above proof, which is invariant under all iterates P^j, $j \geq N_1$, is analogous to the construction of the corresponding set in the proof of Theorem 1.1, p. 691 of our earlier paper [20]. As mentioned there, the method is inspired by the proof of Lemma 1 in the paper of Browder [6], p. 576.

3) Theorem 3.2 is stated with a view on convenient application. A number of similar, more general results, together with a wealth of historical remarks on related work, can be found in Kapitel 8 of the book of Eisenack and Fenske [10].

In the remainder of this section we link the above results for maps to a general semiflow, without reference to a particular equation, under the following assumptions:

(A1) $(E, |\,|_E)$ is a Banach space, and $\Phi : \mathbb{R}_0^+ \times E \to E$ is a continuous semiflow, $\Phi(t, 0) = 0$ for all t, $D_2\Phi$ exists on $\mathbb{R}_0^+ \times E$, and $D_2\Phi(t, x) \to D_2\Phi(t, 0)$ as $x \to 0$, uniformly for t in compact intervals.

(A2) The operators $T(t) := D_2\Phi(t, 0) \in L_c(E, E)$ form a C^0-semigroup of linear operators. There exist real numbers $\alpha < \beta$ with $\beta > 0$ and a decomposition $E = U \oplus S$ into $T(t)$-invariant closed subspaces, where $U \neq \{0\}$, and a constant $K > 0$ such that

$$\forall t \geq 0 : \ |T(t)u|_E \geq K^{-1} e^{\beta t} |u|_E \ (u \in U), \quad |T(t)s|_E \leq K e^{\alpha t} |s|_E \ (s \in S).$$

Under these assumptions, there is an adapted equivalent norm $||\cdot||$ on E with $||u + s|| = \max\{||u||, ||s||\}$ for $u \in U, s \in S$ and such that with respect to this norm, the constant K can be chosen equal to one, i.e.:

$$\forall t \geq 0 : \ ||T(t)u|| \geq e^{\beta t} ||u|| \ (u \in U), \quad ||T(t)s|| \leq e^{\alpha t} ||s|| \ (s \in S). \tag{3.5}$$

(see, e.g., Lemma 2.1, p. 10 in [2]; note that there the resulting norm on the space E was defined by $||x|| := ||\pi_U x|| + ||\pi_S x||$). Let $\pi_U, \pi_S \in L_c(E, E)$ denote the projections onto U and S, respectively, defined by $E = U \oplus S$. With regard to the norm induced by the adapted norm one has $||\pi_U|| = 1$, and $||\pi_S|| = 1$ if $S \neq \{0\}$. For $c \in (0, 1]$ we now define the cone

$$K_c := \Big\{ x \in E \ \Big| \ ||\pi_U x|| \geq c||x|| \Big\}$$

(note that $c \leq 1$ is necessary to have $K_c \neq \{0\}$). With these assumptions and notations, we have the following lemma. The methods here are very much inspired by the paper by Bates and Jones [2].

Lemma 3.3 *(Local cone invariance and expansion). Let $c \in (0, 1]$ and $t_1, T_1 \in \mathbb{R}$ be given, $0 < t_1 < T_1$. Set $q := \dfrac{e^{\beta t_1} + 1}{2}$ (then $q > 1$, since $\beta > 0$). There exists $\delta > 0$ such that*

$$\tau \in [t_1, T_1], \ x \in K_c, ||x|| < \delta \Longrightarrow \Phi(\tau, x) \in K_c, \ ||\pi_U \Phi(\tau, x)|| \geq q||\pi_U x||.$$

Proof. We have $\Phi(\tau, x) = T(\tau)x + r(\tau, x)$, with

$$r(t, x) := \Phi(t, x) - T(t)x = \left[\int_0^1 (D_2\Phi(t, sx) - D_2\Phi(t, 0))\, ds \right] x \quad (t \geq 0, x \in E).$$

Set $\lambda := \min\{c(e^{\beta t_1} - 1)/2,\ c(e^{\beta t_1} - e^{\alpha t_1})/(c+1)\}$. From assumption (A1), we can choose $\delta > 0$ such that $|| \int_0^1 (D_2\Phi(t, sx) - D_2\Phi(t, 0))\, ds \,||_{L_c(E,E)} \leq \lambda,$ if $||x|| < \delta$ and $t \in [t_1, T_1]$. It follows that

$$||r(\tau, x)|| \leq \lambda ||x|| \text{ if } ||x|| < \delta,\ \tau \in [t_1, T_1].$$

Assume now $\tau \in [t_1, T_1],\ x \in K_c,\ ||x|| < \delta$. Then $||x|| \leq \frac{1}{c}||\pi_U x||$. Using (3.5) and $||\pi_U|| = 1$ we obtain

$$\begin{aligned} ||\pi_U \Phi(\tau, x)|| &= ||\pi_U T(\tau)x + \pi_U r(\tau, x)|| = ||T(\tau)\pi_U x + \pi_U r(\tau, x)|| \\ &\geq e^{\beta t_1}||\pi_U x|| - \lambda ||x|| \geq (e^{\beta t_1} c - \lambda)||x||, \end{aligned} \tag{3.6}$$

and, analogously,

$$||\pi_S \Phi(\tau, x)|| \leq e^{\alpha t_1}||\pi_S x|| + \lambda ||x|| \leq (e^{\alpha t_1} + \lambda)||x||.$$

We have

$$||\Phi(\tau, x)|| = \max\{||\pi_S \Phi(\tau, x)||,\ ||\pi_U \Phi(\tau, x)||\} \leq \max\{(e^{\alpha t_1} + \lambda)||x||,\ ||\pi_U \Phi(\tau, x)||\}.$$

If the last max equals $||\pi_U \Phi(\tau, x)||$ then $c \leq 1$ shows $||\pi_U \Phi(\tau, x)|| \geq ||\Phi(\tau, x)|| \geq c||\Phi(\tau, x)||$, so $\Phi(\tau, x) \in K_c$. Otherwise, the choice of λ and the above estimate for $||\pi_U \Phi(\tau, x)||$ give

$$\begin{aligned} c||\Phi(\tau, x)|| &\leq c(e^{\alpha t_1} + \lambda)||x|| = (ce^{\alpha t_1} + (c+1)\lambda)||x|| - \lambda||x|| \leq ce^{\beta t_1}||x|| - \lambda||x|| \\ &\leq ||\pi_U \Phi(\tau, x)||, \end{aligned}$$

so again $\Phi(\tau, x) \in K_c$.

In addition, the choice of λ, the property $x \in K_c$, and (3.6) show that

$$\begin{aligned} ||\pi_U \Phi(\tau, x)|| &\geq (e^{\beta t_1} - \frac{\lambda}{c})||\pi_U x|| \geq [e^{\beta t_1} - (e^{\beta t_1} - 1)/2] \cdot ||\pi_U x|| = \frac{e^{\beta t_1} + 1}{2}||\pi_U x|| \\ &= q||\pi_U x||. \quad \square \end{aligned}$$

Remark. Note that the lower bound t_1 for the times τ in the above proof allows to conclude the invariance of K_c without use of Gronwall's Lemma, and therefore also without an underlying differential equation for the semiflow and without a variation-of-constants-formula.

We can now prove the main result of this section. The approach of using a lower bound condition on the projection to the unstable direction (assumption 3) below) is familiar from, e.g., the periodicity results Theorem 2.1 on p. 93 of [13], Theorem 28.1 on p. 152 of [14], or Theorem 3.1 on p. 497 of [7]. Our version below does not use the notion of ejectivity, or any connection to a particular type of equation behind.

Theorem 3.4. *Assume that the semiflow $\Phi : \mathbb{R}_0^+ \times E \to E$ on the Banach space $(E, |\,|_E)$ satisfies assumptions* (A1) *and* (A2) *above. Let $\{0\} \neq \mathfrak{K} \subset E$ be closed and convex with $0 \in \mathfrak{K}$. Assume that $0 < t_1 < T_1$, that map $\tau : \mathfrak{K} \setminus \{0\} \to [t_1, T_1]$ is continuous, and $\Phi(\tau(\psi), \psi) \in \mathfrak{K}$ for $\psi \in \mathfrak{K} \setminus \{0\}$. Define $P : \mathfrak{K} \to \mathfrak{K}$ by*

$$P(0) := 0, \qquad P(\psi) := \Phi(\tau(\psi), \psi) \text{ for } \psi \neq 0.$$

We further assume:

1) *P is compact;*
2) *$P(\psi) \neq 0$ if $\psi \neq 0$;*
3) *$\exists c > 0 : \mathfrak{K} \subset K_c$ (i.e., $\forall \varphi \in \mathfrak{K} : ||\pi_U \varphi|| \geq c||\varphi||$);*
4) *There exist a continuous linear functional $\eta : E \to \mathbb{R}$ and $c_1 > 0$ such that*

$$\forall \varphi \in \mathfrak{K} : c_1 |\varphi|_E \leq \eta(\varphi).$$

Then P has a fixed point φ^* in $\mathfrak{K} \setminus \{0\}$, corresponding to a periodic trajectory $\Phi(\cdot, \varphi^*)$ of the semiflow with period $\tau(\varphi^*)$.

Proof. 1. Uniform continuity of Φ on the compact set $[t_1, T_1] \times \{0\}$ and $\Phi(t, 0) = 0$ for $t \geq 0$ together with $\tau(\mathfrak{K} \setminus \{0\}) \subset [t_1, T_1]$ imply $P(\varphi) \to 0$ as $\varphi \to 0$, so P is continuous.

2. Let $|| \cdot ||$ be the adapted, equivalent norm on E, as described before Lemma 3.3 above. We can assume that the functional η satisfies the estimate from condition 4) with respect to this norm. Let $q > 1$ and $\delta > 0$ be as in Lemma 3.3. It follows from $\mathfrak{K} \subset K_c$, from the fact that τ takes values in $[t_1, T_1]$, the definition of P and from Lemma 3.3 that

$$\forall \varphi \in \mathfrak{K}, ||\varphi|| < \delta : \ ||\pi_U P(\varphi)|| \geq q||\pi_U \varphi||.$$

Inductively (by using $P(\mathfrak{K}) \subset \mathfrak{K}$) one obtains $||\pi_U P^n(\varphi)|| \geq q^n ||\pi_U \varphi||$, as long as $||P^j(\varphi)|| < \delta$ for all $j \in \{0, ..., n-1\}$. Using condition 2) we see that

$$\forall \varphi \in \mathfrak{K} \setminus \{0\} \ \exists j(\varphi) \in \mathbb{N} : \ ||\pi_U P^{j(\varphi)}(\varphi)|| \geq \delta. \tag{3.7}$$

(Note that condition 2) is used to obtain $j(\varphi) \geq 1$ in case when $||\pi_U \varphi|| \geq \delta$ is already satisfied.)

3. Set $\mathfrak{B} := \overline{\text{conv}}(P(\mathfrak{K}))$, where $\overline{\text{conv}}$ denotes the closure of the convex hull (note $\mathfrak{B}$ is convex). Since $P(\mathfrak{K}) \subset \mathfrak{K}$ and $\mathfrak{K}$ is closed and convex, we have $\mathfrak{B} \subset \mathfrak{K}$. Further, the compactness of P and the Mazur theorem ([27]) imply that $\mathfrak{B}$ is compact. Clearly $P(\mathfrak{K}) \subset \mathfrak{B}$, in particular, $P(\mathfrak{B}) \subset \mathfrak{B}$, as required in Theorem 3.2. It follows from assumption 4) and linearity of η that η satisfies conditions (i) and (ii) of that theorem.

4. With c_1 from assumption 4) (for the new norm), choose $\delta_0 \in (0, \dfrac{c_1 \delta}{c_1 + 1})$. For $\varphi \in \mathfrak{B} \setminus \{0\}$ and $j(\varphi) \in \mathbb{N}$ as in (3.7), one has $||\pi_U P^{j(\varphi)}(\varphi)|| > \dfrac{\delta}{c_1 + 1}$, and hence

$$\eta(P^{j(\varphi)}(\varphi)) \geq c_1 ||P^{j(\varphi)}(\varphi)|| \geq c_1 ||\pi_U P^{j(\varphi)}(\varphi)|| > \frac{c_1 \delta}{(c_1 + 1)} > \delta_0,$$

so condition (iii) of Theorem 3.2 is also satisfied with this δ_0, and the assertion follows from Theorem 3.2. □

4. The method of projection to a single eigenvalue

Condition 3) of Theorem 3.4 requires a lower estimate for the values $||\pi_U\varphi||$ for φ in the set $\mathfrak{K}$. In the papers dealing with lower dimensions 1, 2 and 3 [34,1,20], the set $\mathfrak{K}$ was an appropriate cone, and the method to provide such a lower estimate was always to consider only the projection π_λ to the eigenspace associated with the single leading eigenvalue λ. Of course, if this projection already satisfies an estimate of the required form, this is even more true for the projection π_U to the whole unstable space U of the equilibrium zero, which may be of much higher dimension.

On the other hand, if the unstable space U is of higher dimension, and if there exists more than one conjugate pair of eigenvalues with positive real part and imaginary part in $(0, \pi/\tau)$, then it is well feasible that $\mathfrak{K}$ may contain an eigenfunction φ associated to some eigenvalue μ (not from the leading pair), in which case one would have $\pi_\lambda\varphi = 0$ and $\varphi \in \mathfrak{K}$, so that π_λ would not be bounded below on $\mathfrak{K}$.

In such a situation, the 'classical' method used in papers [34,1,20], which employs only one eigenvalue (and the corresponding Laplace transform) would not work. Consider an eigenvalue $\mu = \rho + i\omega$ with $\omega \in (0, \pi/\tau)$. Then a corresponding function of the form $\varphi : t \mapsto e^{\rho(t+\tau)}\sin(\omega(t+\tau))$, which is non-negative on $[-\tau, 0]$ and zero at $-\tau$, has no more than one zero in every interval of length τ. Recall the state space $\mathbb{X}$ and the cone $\mathfrak{K}$ from Section 2 (see (2.10)), and consider an initial state $X_0 = (\varphi, x_2(0), ..., x_N(0)) \in \mathbb{X}$ which has φ in the first component. Then $X_0 \in \mathfrak{K}$ if and only if the function $[-\tau, 0] \ni t \mapsto e^{\mu_1 t}\varphi(t) = e^{(\mu_1+\rho)(t+\tau)}\sin(\omega(t+\tau))$ is non-decreasing. Although the exponential factor is increasing, it is not clear that the full function will be non-decreasing. However, when ω satisfies $\omega \in (0, \pi/2\tau]$ then it is obviously increasing, since then both the exponential factor and the sin-term are increasing on $[-\tau, 0]$.

We did a numerical evaluation of the zeroes of the characteristic function (eigenvalues), using a Newton procedure. We observed, for example, the following: for dimension $N = 6$ and all μ_j equal to 0.1 (j = 1,..., 6), delay $\tau = 1$ and $a = |a_1 \cdot \ldots \cdot a_6| = 8.0$, there exist two different eigenvalues $\lambda_1 = \rho_1 + i\omega_1$, $\lambda_2 = \rho_2 + i\omega_2$ with $0 < \rho_1 < \rho_2$, and with ω_1, ω_2 both contained in $(0, \pi/2)$, and λ_2 has maximal real part among all eigenvalues. The particular approximate values are $\lambda_1 = 0.20 + 1.33i,\ \lambda_2 = 0.99 + 0.51i$.

In this case both functions $\varphi_j : t \mapsto e^{(\mu_1+\rho_j)(t+\tau)}\sin(\omega_j(t+\tau))$, $j = 1, 2$, define elements $X_j := (\varphi_j, 0, ..., 0)$ of the cone $\mathfrak{K}$, and $\pi_{\lambda_1}X_2 = \pi_{\lambda_2}X_1 = 0$. In particular, the projection π_{λ_2} is zero on X_1, so not bounded below on the cone $\mathfrak{K}$. Thus, the functional $v \cdot K_{\lambda_2,L}(\cdot)$ (compare Proposition 2.2 and Corollary 2.4) which arises (also) through the Laplace transform and the analogue of which was employed in papers [34,1,20], is not bounded below on $\mathfrak{K}$ here, so that the method from these papers would not be directly applicable.

In such a situation, there may still exist a lower bound for the 'whole' projection π_U onto the unstable space, or (even better), to some proper subspace of U larger than the one corresponding to λ_2. Of course, one could also consider working with a different cone than the one which was used so far.

Shortly after we made our numerical observations in the example with $N = 6$ described above, it was proved in [4] (Proposition 6) by a thorough analysis of the characteristic equation that when $N \leq 4$ there cannot be more than one pair of complex conjugate eigenvalues with imaginary part in $(-\pi/\tau, \pi/\tau)$; however, when $N \geq 5$ a case with at least two such pairs of eigenvalues can

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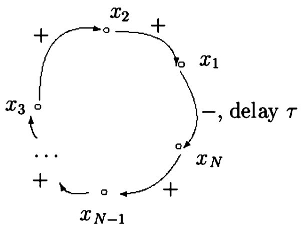

Fig. 1. The feedback structure of system (S).

always occur (for suitable values of a and the μ_j). It is also shown in [4] that this phenomenon of multiple pairs disappears when a becomes sufficiently large.

5. The return map

Recall the nonlinear system (S), the linearized system (L), and the real values $a_j := g_j'(0)$, $j = 1, ..., N$ with $a_j > 0$, $j = 1, ..., N-1$, and $a_N < 0$ from the introduction.

For the purposes of this section, we use the phase space $\mathbb{X} = C^0([-\tau, 0], \mathbb{R}) \times \mathbb{R}^{N-1}$ for system (S) (which is possible because only component x_1 contains the delay). The norm on $\mathbb{X}$ is given by $||(\varphi, x_2^0, \ldots, x_N^0)||_{\mathbb{X}} := \max\{|\varphi|_\infty, |x_2^0|, ..., |x_N^0|\}$.

By a solution $x = (x_1, ..., x_N)$ of system (S) on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$, we mean that x_1 is defined on $[t_0 - \tau, \infty)$ and $x_2, ..., x_n$ are defined on $[t_0, \infty)$. System (S) induces a continuous semiflow $\Phi : [0, \infty) \times \mathbb{X} \to \mathbb{X}$.

The feedback structure is visualized in Fig. 1. We shall often use it in arguments combined with the variation of constants formula, e.g., of the following form: if $u(t_0) \geq 0$, the function g has positive feedback, $v \geq 0$ on $[t_0, t_1]$ and $\dot{u}(t) = -\mu u(t) + g(v(t))$ on $[t_0, t_1]$, then $u \geq 0$ on $[t_0, t_1]$. This follows from the solution's representation as

$$u(t) = \underbrace{\exp[-\mu(t-t_0)]u(t_0)}_{\geq 0} + \int_{t_0}^{t} \underbrace{\exp[-\mu(t-s)]g(v(s))}_{\geq 0}\, ds.$$

A solution $(x_1, ..., x_N)$ of system (S) is also a solution of the system

$$(\tilde{\mathrm{S}}) \begin{cases} \varepsilon_1 \dot{x}_1(t) = -x_1(t) + G_1(x_2(t)) \\ \varepsilon_2 \dot{x}_2(t) = -x_2(t) + G_2(x_3(t)) \\ \quad \cdots \qquad\qquad \cdots \\ \varepsilon_{N-1} \dot{x}_{N-1}(t) = -x_{N-1}(t) + G_{N-1}(x_N(t)) \\ \varepsilon_N \dot{x}_N(t) = -x_N(t) + G_N(x_1(t-\tau)), \end{cases}$$

where $\varepsilon_j = 1/\mu_j$ and $G_j(\cdot) = (1/\mu_j) g_j(\cdot)$, $1 \leq j \leq N$. We say that a function $u : [0, \infty) \to \mathbb{R}$ has a property eventually (e.g., '$u \geq 0$ ev.', or '$u \in J$ ev.' with some set J) if there exists $T \geq 0$ such that the property holds for all $u(t)$ with $t \geq T$.

We assume that all nonlinearities g_j are C^1, and that g_N (and hence also G_N) is bounded either from above or from below. We shall use the convention $\tau_j := 0$ for $j \in \{1, ..., N-1\}$ and

$\tau_N := \tau$. Our first aim is to construct an invariant, bounded and attracting set for the semiflow induced by system (S). Define the composite function G as $G = G_1 \circ G_2 \circ \cdots \circ G_N$.

Proposition 5.1. *Assume that* $(x_1, ..., x_n)$ *is a solution of system* $(\tilde{\mathrm{S}})$ *(then also of* (S)*), and that* $A \subset \mathbb{R}$ *is a (not necessarily bounded) closed interval with* $0 \in \mathrm{int}(A)$.

a) For $j \in \{1, ..., N\}$ *and* $t \geq 0$ *one has the following implications:*

(i) $x_{(j+1)\mathrm{mod}\, N} \in A$ *ev.* $\Longrightarrow$ $x_j \in G_j(A)$ *ev.*

(ii) $x_{(j+1)\bmod N}([t - \tau_j, \infty)) \subset A,\ x_j(t) \in G_j(A) \Longrightarrow x_j([t, \infty)) \subset G_j(A)$.

b) If $x_1 \in A$ *ev. then also* $x_1 \in G(A)$ *ev.*

Proof. Assertion (i) of part a) is contained in [4] as claim (18) within the proof of Lemma 6 there, but note that the property $0 \in \mathrm{int}(A)$ is implicitly assumed in [4], in order to have that a limit x^* of x_j is nonzero, if x_j does not enter the set $G_j(A)$ eventually. Also, the proof in [4] actually is given for the case of a compact interval (uses 'endpoints'), but also works for unbounded closed intervals. Assertion (ii) of part a) follows from Proposition 1 of [4].

Proof of b): Assume $x_1 \in A$ ev. Applying part a) with $j := N$ we see that $x_N \in G_N(A)$ ev., then applying part a) again with $j := N - 1$ we obtain $x_{N-1} \in G_{N-1}(G_N(A))$ ev., and proceeding inductively we see that $x_1 \in G_1(G_2(...G_N(A)))...) = G(A)$ ev. □

The following result is closely related to Lemma 5, the passage after Proposition 1, and Lemma 6 in [4].

Theorem 5.2. *1) Assume that a compact interval* I_1 *contains* 0 *in its interior and is invariant under* G *(i.e.,* $G(I_1) \subset I_1$*), and set* $I_N = G_N(I_1), I_j = G_j(I_{j+1}),\ j = 2, ..., N - 1$, $\vec{I} := (I_1, ..., I_N)$ *and*

$$\mathbb{X}_{\vec{I}} := \left\{(\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{X} \;\middle|\; \varphi(t) \in I_1\ (t \in [-\tau, 0]),\ x_j^0 \in I_j,\ j = 2, ..., N\right\}.$$

Then for every initial state in the set $\mathbb{X}_{\vec{I}}$ *the corresponding solution* $(x_1, ..., x_N)$ *satisfies* $\forall t \geq 0 :$ $x_j(t) \in I_j,\ j = 1, ..., N$*, so that* $\mathbb{X}_{\vec{I}}$ *is forward invariant under the semiflow* Φ.

2) $I_1 := \overline{G^2(\mathbb{R})}$ *is a possible choice for* I_1 *as in 1), and with this choice the set* $\mathbb{X}_{\vec{I}}$ *also is attracting within finite time in the following sense: For every* $X_0 \in \mathbb{X}$ *there exists a finite* $T(X_0) \geq 0$ *such that* $\forall t \geq T(X_0) :\ \Phi(t, X_0) \in \mathbb{X}_{\vec{I}}$.

Proof. Part 1) is proved in Lemma 5 of [4].

Proof of 2): Assume that G_N is bounded from above (the case when it is bounded from below is analogous), so $0 < \sup_{x\in\mathbb{R}} G_N(x) < \infty$. Then positive feedback of $G_1, ..., G_{N-1}$ implies that $G = G_1 \circ ... \circ G_N$ is bounded from above. Negative feedback of G then shows that the second iterate G^2 is bounded below by $G(\sup_{x\in\mathbb{R}} G(x))$, and (of course) G^2 is also bounded from above, since G is. It follows that $I_1 := \overline{G^2(\mathbb{R})}$ is a compact interval, and negative feedback of G implies $0 \in \mathrm{int}(I_1)$. Now the continuity of G implies

$$G(I_1) = G(\overline{G^2(\mathbb{R})}) \subset \overline{G(G^2(\mathbb{R}))} = \overline{G^3(\mathbb{R})} \subset \overline{G^2(\mathbb{R})} = I_1,$$

which shows the invariance of I_1 required in 1). It remains to prove the attractivity property of the set $\mathbb{X}_{\vec{I}}$ corresponding to I_1. Applying Proposition 5.1 b) with $A := \mathbb{R}$ first, we see that for every solution one has $x_1 \in G(\mathbb{R})$ ev. Applying Proposition 5.1 b) again, with $A := \overline{G(\mathbb{R})}$ we see that $x_1 \in G(\overline{G(\mathbb{R})}) \subset \overline{G^2(\mathbb{R})} = I_1$ ev. From Proposition 5.1 a), (i) with $j := N$ we then obtain recursively $x_N \in G_N(I_1) = I_N$ ev., $x_{N-1} \in G_{N-1}(I_N) = I_{N-1}$ ev., etc., so $x_j \in I_j$ ev., $j = 2, ..., N$. Hence we have $\Phi(t, X_0) \in \mathbb{X}_{\bar{I}}$ ev. for every initial state $X_0 \in \mathbb{X}$. □

Under an additional assumption on α-Hölder-continuity of the derivatives g_j' in a neighborhood of zero, it is also proved in [4] (see Theorem 1 there) that if the characteristic equation associated to system (L) has no real roots, then every non-zero solution of system (S) oscillates (in the sense that all components x_j do not satisfy $x_j \geq 0$ or $x_j \leq 0$ on any interval of the form $[T, \infty)$).

Our approach in this section is to provide a criterion for oscillation of all solutions starting in a suitable subset of $\mathbb{X}$. This criterion can be stated in an explicit form more transparent than conditions for the absence of real eigenvalues; also, it does not require the smoothness higher than C^1. From the oscillation properties we then construct a return map.

We introduce the cone segment

$$\mathfrak{K}_{\bar{I}} = \{\Psi = (\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{X}_{\bar{I}} \mid \varphi(-\tau) = 0, s \mapsto \varphi(s)\exp\{\mu_1 s\} \text{ is increasing on } [-\tau, 0], \text{ and } x_k^0 \geq 0, 2 \leq k \leq N\},$$

and the sets

$$\mathbb{X}_{\bar{I}}^+ := \big\{X_0 = (\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{X}_{\bar{I}} \,\big|\, \varphi \geq 0, x_k^0 \geq 0, 2 \leq k \leq N, \text{ and } \max\{\varphi(0), x_2^0, ... x_N^0\} > 0\big\},$$

$$\mathbb{X}_{\bar{I}}^- := \big\{X_0 = (\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{X}_{\bar{I}} \,\big|\, \varphi \leq 0, x_j^0 \leq 0, 2 \leq j \leq N, \text{ and } \min\{\varphi(0), x_2^0, ... x_N^0\} < 0\big\},$$

$$\mathbb{O}_{\bar{I}} := \big\{X_0 = (\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{X}_{\bar{I}} \,\big|\, \varphi \geq 0, \text{ and } \exists k \in \{1, ..., N\} : x_k^0 > 0, x_j^0 \geq 0\,(j = 1, ..., k), \text{ and } x_j^0 \leq 0\,(j = k+1, ..., N) \text{ (if } k < N)\big\}.$$

Note that then $\mathfrak{K}_{\bar{I}} \setminus \{0\} \subset \mathbb{X}_{\bar{I}}^+$, since for $(\varphi, x_2^0, \ldots, x_N^0) \in \mathfrak{K}_{\bar{I}}$, the property $\varphi(0) = 0$ implies $\varphi = 0$. Further, $\mathbb{X}_{\bar{I}}^+ \subset \mathbb{O}_{\bar{I}}$, so that we have

$$\mathfrak{K}_{\bar{I}} \setminus \{0\} \subset \mathbb{X}_{\bar{I}}^+ \subset \mathbb{O}_{\bar{I}}.$$

The essential property of initial states X_0 from $\mathbb{O}_{\bar{I}}$ is that they may have positive and negative components, but in an ordered way, as described in the above definition.

Notation: For a solution $x = (x_1, ..., x_n)$ of (S) and $t \geq 0$, $(x_1)_t$ denotes the segment of x_1 at time $t : (x_1)_t(\theta) := x_1(t + \theta)$, $-\tau \leq \theta \leq 0$. By $x_j(t+) > 0\ (< 0)$ we mean that there exists $\delta > 0$ such that $x_j > 0\ (< 0)$ on $(t, t + \delta]$.

Proposition 5.3. *Assume* $Y = (\psi, y_2, ..., y_N) \in \overline{\mathbb{O}_{\bar{I}}} \setminus (\mathbb{O}_{\bar{I}} \cup \{0\})$*, and consider the corresponding solution* $t \mapsto (y_1(t), ..., y_N(t))$ *of system* (S)*. There exists* $\delta \in (0, \tau)$ *with* $y_1(\delta) < 0$.

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Proof. Y is limit of a sequence $(Y^{(n)})$ of elements of $\mathbb{O}_{\vec{I}}$, and for every n there exists $k(n) \in \{1, ..., N\}$ corresponding to $Y^{(n)}$ as in the definition of $\mathbb{O}_{\vec{I}}$. Choosing a subsequence (and calling it $(Y^{(n)})$ again) we can assume that $k(n) = k \in \{1, ..., N\}$ for all $n \in \mathbb{N}$. Passing to the limit in the definition of $\mathbb{O}_{\vec{I}}$ and using that $Y \notin \mathbb{O}_{\vec{I}}$ then shows

$$\psi \geq 0, \; y_j = 0 \, (j = 1, ..., k), \quad y_j \leq 0, \; j \in \{k+1, ..., N\} \text{ (if } k < N). \tag{5.1}$$

Case a): $y_j < 0$ for some $j \in \{k+1, ..., N\}$. Then one sees from system (S) and (5.1) that $y_l(0+) < 0, \; l = 1, ..., j$, in particular, $y_1(0+) < 0$, so there exists $\delta > 0$ as asserted.

Case b): $y_j = 0$ for all $j \in \{k+1, ..., N\}$, which means $y_j = 0, \; j \in \{1, ..., N\}$. The condition $Y \neq 0$ then implies $\psi \neq 0$ (but we have $\psi(0) = y_1 = 0$). Set $t_0 := 0$ if $\psi(-\tau) > 0$, and $t_0 := \max\left\{t \in [0, \tau] \;\middle|\; \psi = 0 \text{ on } [-\tau, -\tau + t]\right\}$ otherwise. Then $\psi \neq 0$ implies $t_0 < \tau$, and $\int_0^t e^{-\mu_N(t-r)} g_N(y_1(r-\tau))\,dr = \int_0^t e^{-\mu_N(t-r)} g_N(\psi(r-\tau))\,dr < 0$ for every t in $(t_0, \tau]$. The differential equation for y_N now gives that $y_N = 0$ on $[0, t_0]$ and $y_N(t_0+) < 0$. Similar to case a), system (S) shows that the same is true with $y_l, l = 1, ..., N-1$ instead of y_N. In particular, $y_1(t_0+) < 0$, and the assertion follows. □

We now study the behavior of solutions starting in $\mathbb{X}^+_{\vec{I}}$ and with the property that the x_1 component has a first positive zero: Define

$$\mathbb{Z}_{\vec{I}} := \Big\{X_0 \in \mathbb{O}_{\vec{I}} \;\Big|\; \text{With the corresponding solution } (x_1, ..., x_N), \; x_1 \text{ has a first zero } z_1^{(1)} \text{ in } (0, \infty)\Big\}.$$

We will later provide a condition under which $\mathbb{Z}_{\vec{I}} = \mathbb{O}_{\vec{I}}$. Defining $-\vec{I} := (-I_1, ..., -I_N)$, we have sets $\mathfrak{K}_{-\vec{I}}$, $\mathbb{X}^{\pm}_{-\vec{I}}$ and $\mathbb{O}_{-\vec{I}}$ corresponding to the intervals $-I_1, ..., -I_N$ analogous to the ones defined with $\vec{I}$ above. Then

$$-\mathbb{X}^-_{\vec{I}} = \mathbb{X}^+_{-\vec{I}}, \quad -\mathbb{X}^+_{\vec{I}} = \mathbb{X}^-_{-\vec{I}}.$$

Lemma 5.4. *Let $X_0 = (\varphi, x_2^0, \ldots, x_N^0) \in \mathbb{O}_{\vec{I}}$ be given, and let $x = (x_1, x_2, ..., x_N)$ be the corresponding solution of system* (S). *For this X_0, let k be as in the definition of $\mathbb{O}_{\vec{I}}$.*

1) Then

$$x_j(0+) > 0 \text{ for } j \in \{1, ..., k\}, \tag{5.2}$$

$$x_j(0+) < 0 \text{ for } j \in \{k+1, ..., N\}, \text{ if } k < N. \tag{5.3}$$

2) If $X_0 \in \mathbb{Z}_{\vec{I}}$ then the numbers $z_1^{(j)} := \min\left\{t > 0 \;\middle|\; x_j(t) = 0\right\}$ for $j = 1, ..., k$ are well-defined. We set $z_1^{(j)} := 0$ for $j = k+1, ..., N$, if $k < N$. Then

$$x_j < 0 \text{ on } (z_1^{(j)}, z_1^{(1)} + \tau], \; j = 1, ..., N, \text{ and} \tag{5.4}$$

$$z_1^{(k)} < z_1^{(k-1)} < \ldots < z_1^{(1)}. \tag{5.5}$$

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(Note that for $j \in \{1, ..., k\}$, the number $z_1^{(j)}$ is the first zero of x_j in $(0, \infty)$, even if possibly $x_j(0) = 0$ for some $j \in 1, ..., k-1$.)

3) If $X_0 \in \mathbb{Z}_{\vec{I}}$ and $z_1^{(1)} > \tau$ then $Y := \Phi(\tau, X_0)$ satisfies $Y \in \mathbb{O}_{\vec{I}}$.

4) If $X_0 \in \mathbb{Z}_{\vec{I}}$ then $\dot{x}_1(z_1^{(1)}) < 0$, and $z_1^{(1)}$ is continuous as a function of the initial value $X_0 \in \mathbb{Z}_{\vec{I}}$, and the map

$$Q : \mathbb{Z}_{\vec{I}} \ni X_0 = (\varphi, x_2^0, ..., x_N^0) \mapsto [(x_1)_{z_1^{(1)}+\tau}, x_2(z_1^{(1)} + \tau), ..., x_N(z_1^{(1)} + \tau)]$$
$$= \Phi(z_1^{(1)} + \tau, X_0) \in \mathbb{X}_{\vec{I}}^-$$

is well-defined and continuous, and takes values in $-\mathfrak{K}_{-\vec{I}} \setminus \{0\}$.

Proof. From the definition of $\mathbb{O}_{\vec{I}}$ we have $X_0 \neq 0$, and

$$x_j(0) \leq 0 \text{ for } j \in \{k+1, ..., N\}, \text{ if } k < N. \tag{5.6}$$

Property (5.2) follows from the continuity for $j = k$, since $x_k(0) > 0$, and then (recursively) from $x_j(0) \geq 0$ for $j \in \{1, ..., k\}$, using the positive feedback from x_k to x_{k-1} etc.

In case $k < N$, the negative feedback from x_1 to x_N, property (5.2) for $j = 1$, $x_N(0) \leq 0$ and the variation of constants formula show that $x_N(0+) < 0$, and the positive feedback from x_N to x_{N-1} etc. together with (5.6) shows that (5.3) is true. Part 1) is proved.

Now pick $\delta > 0$ such that $x_j > 0$ on $(0, \delta]$ for $j = 1, ..., k$. The positive feedback from x_j to x_{j-1} shows the following implications for $j \in \{2, ..., k\}$ (in case $k \geq 2$):

$$T \geq \delta \text{ and } x_j \geq 0 \text{ on } [\delta, T] \Longrightarrow x_{j-1} > 0 \text{ on } [\delta, T]. \tag{5.7}$$

Since from the definition of $\mathbb{Z}_{\vec{I}}$ we know that x_1 has a first zero $z_1^{(1)}$ in $(0, \infty)$, it follows from (5.7) that all components x_j, $2 \leq j \leq k$ must have a first zero $z_1^{(j)}$ in $(0, \infty)$, which lies in (δ, ∞). The inequalities in (5.5) follow from (5.7).

Proof of (5.4): From (5.2) for $j = 1$, we have $x_1 > 0$ on $(0, z_1^{(1)})$.

(i) In case $k < N$, it follows from $x_N(0) \leq 0$, the variation of constants formula, and the negative delayed feedback from x_1 to x_N that $x_N < 0$ on $(0, z_1^{(1)} + \tau]$. With the positive feedback from x_N to x_{N-1} etc. and $x_j(0) \leq 0$ for $j \in \{k+1, .., N\}$, this implies

$$x_j < 0 \text{ on } (0, z_1^{(1)} + \tau] \text{ for } j \in \{k+1, ..., N\}, \tag{5.8}$$

which coincides with (5.4) for $j \in \{k+1, ..., N\}$, since $z_1^{(j)} = 0$ for these j. From system (S), (5.5) and (5.8) we see (recursively) that

$$\dot{x}_j(z_1^{(j)}) < 0, \; x_j < 0 \text{ on } (z_1^{(j)}, z_1^{(1)} + \tau], \; j = 1, ..., k, \tag{5.9}$$

since for these j one has $x_{j+1} < 0$ on $[z_1^{(j)}, z_1^{(1)} + \tau]$. In particular, property (5.4) holds also for $j = 1, ..., k$ in case $k < N$.

(ii) In case $k = N$ we have from (5.2) that $x_j(0+) > 0$, $j = 1, ..., N$, and (from the already proved statement (5.5) for $k = N$) that $0 < z_1^{(N)} < z_1^{(N-1)} < ... < z_1^{(1)}$. The feedback properties show $x_N < 0$ on $(z_1^{(N)}, z_1^{(1)} + \tau]$, and we conclude

$$\dot{x}_j(z_1^{(j)}) = g_j[x_{(j+1)\mathrm{mod} N}(z_1^{(j)} - \tau_j)] < 0, \ x_j < 0 \text{ on } (z_1^{(j)}, z_1^{(1)} + \tau], \ j = 1, ..., N, \tag{5.10}$$

with $\tau_N = \tau$ and $\tau_j = 0$, $j = 1, ..., N - 1$. This proves (5.4) for $j = 1, ..., k = N$ also in this case. Part 2) is proved.

Proof of 3): Denote $Y = \Phi(\tau, X_0)$ by $Y =: (\psi, y_2, ..., y_N)$. First, invariance of $\mathbb{X}_{\vec{I}}$ under the semiflow implies that certainly $Y \in \mathbb{X}_{\vec{I}}$. Next, (5.2) and the assumption $z_1^{(1)} > \tau$ imply $x_1 > 0$ on $(0, \tau]$, so that one has

$$\psi \geq 0, \text{ and } \psi(0) > 0. \tag{5.11}$$

First case: $z_1^{(j)} \geq \tau$ for all $j \in \{1, ..., N\}$. Then the definition of $z_1^{(j)}$ implies $k = N$ (for the number k corresponding to $X_0 \in \mathbb{O}_{\vec{I}}$), and $x_j > 0$ on $(0, \tau)$ $(j = 1, ..., N)$, which shows $y_2, ..., y_N \geq 0$, and thus $Y \in \mathbb{O}_{\vec{I}}$.

Second case: $z_1^{(j)} < \tau$ for some $j \in \{1, ..., N\}$. Then we can define

$$\tilde{k} := \min\left\{ j \in \{1, ..., N\} \,\middle|\, z_1^{(j)} < \tau \right\},$$

and the assumption $z_1^{(1)} > \tau$ implies $\tilde{k} > 1$. We see from (5.5) that $z_1^{(j)} < \tau$ for $j \in \{\tilde{k}, ..., N\}$, and from (5.4) we conclude

$$y_j = x_j(\tau) < 0 \text{ for } j \in \{\tilde{k}, ..., N\}. \tag{5.12}$$

For $j \in \{1, ..., \tilde{k} - 1\}$ we have $z_1^{(j)} \geq \tau$ and thus $x_j > 0$ on $(0, \tau)$. It follows that

$$y_j \geq 0, \quad j = 2, ..., \tilde{k} - 1 \text{ (in case } \tilde{k} \geq 3). \tag{5.13}$$

From (5.11), (5.12) and (5.13) we see that $Y \in \mathbb{O}_{\vec{I}}$.

Proof of 4): It is clear from (5.9) and (5.10) that $\dot{x}_1(z_1^{(1)}) < 0$, which implies (via the implicit function theorem) that $z_1^{(1)}$ is continuous as a function of $X_0 \in \mathbb{Z}_{\vec{I}}$ (even can be extended continuously to a neighborhood of $\mathbb{Z}_{\vec{I}}$ in $\mathbb{X}$). Since $Q(X_0) = \Phi(z_1^{(1)}(X_0) + \tau, X_0)$, it follows that Q is continuous on $\mathbb{Z}_{\vec{I}}$. One also sees from (5.9) and (5.10) and from invariance of $\mathbb{X}_{\vec{I}}$ that Q takes nonzero values in $\mathbb{X}^-_{\vec{I}} = -\mathbb{X}^+_{-\vec{I}}$. In order to prove that these values lie in $-\mathfrak{K}_{-\vec{I}} \setminus \{0\}$, it remains to show that with $\zeta := x_1(z_1^{(1)} + \tau + \cdot)\big|_{[-\tau, 0]}$, the function $[-\tau, 0] \ni s \mapsto e^{\mu_1 s}\zeta(s)$ is decreasing. Now for $s \in [-\tau, 0]$, one has

$$\begin{aligned}\frac{d}{dt}\Big|_{t=s}[e^{\mu_1 t}x_1(z_1^{(1)}+\tau+t)] &= e^{\mu_1 s}\{\mu_1\zeta(s)+\dot{x}_1(z_1^{(1)}+\tau+s)\}\\ &= e^{\mu_1 s}\{\mu_1\zeta(s)-\mu_1\zeta(s)+g_1(x_2(z_1^{(1)}+\tau+s))\}\le 0,\end{aligned}$$

since (from (5.4)) $x_2 < 0$ on $[z_1^{(1)}, z_1^{(1)}+\tau]$. Part 4) is proved. □

So far we have the map Q not necessarily defined on all of $\mathfrak{K}_{\vec{I}}\setminus\{0\}$, but on $\mathbb{Z}_{\vec{I}}$. Next, we give a condition which implies $\mathbb{Z}_{\vec{I}}=\mathbb{O}_{\vec{I}}$, so that then Q is defined on the subset $\mathfrak{K}_{\vec{I}}\setminus\{0\}$ of $\mathbb{O}_{\vec{I}}$.

Lemma 5.5. *Recall the 'total feedback strength' $a=|a_1\cdot\ldots\cdot a_N|$ from Lemma 2.7, and define $M^*:=\left(\prod_{k=1}^{N}\mu_k\right)/\min\{\mu_1,\ldots,\mu_N\}$. If*

$$a\tau > M^* \tag{5.14}$$

then for every solution $(x_1,\ldots,x_N)$ of system (S) *with an initial value $X_0\in\mathbb{O}_{\vec{I}}$, the component x_1 has a first zero $z_1^{(1)}(X_0)$ in $(0,\infty)$, so $\mathbb{Z}_{\vec{I}}=\mathbb{O}_{\vec{I}}$. Also, $x_1>0$ on $(0,z_1^{(1)}(X_0))$.*

Proof. Assume that condition (5.14) is satisfied, consider $X_0\in\mathbb{O}_{\vec{I}}$, and the corresponding solution $(x_1,\ldots,x_N)$. Assume that x_1 has no zero in $(0,\infty)$. Recall that $x_1(0+)>0$ (from (5.2)), so that

$$x_1\ge 0 \text{ on } [-\tau,0],\ x_1>0 \text{ on } (0,\infty), \tag{5.15}$$

and we can define $l:=\max\Big\{j\in\{1,\ldots,N\}\ \Big|\ x_j>0 \text{ on } (0,\infty)\Big\}$. Let the number $k\in\{1,..,N\}$ associated to X_0 be as in the definition of $\mathbb{O}_{\vec{I}}$. Negative feedback from x_1 to x_N, positive feedback from x_N to x_{N-1} etc., and $x_j(0)\le 0$ for $j\in\{k+1,\ldots,N\}$ in case $k<N$ imply

$$x_j<0 \text{ on } (0,\infty) \text{ for } j\in\{k+1,\ldots,N\}\ (\text{ if } k<N).$$

Obviously we have $1\le l\le k$. Positive feedback from x_j to x_{j-1} for $j\in\{2,\ldots,l\}$ and $x_j(0)\ge 0$ for these j show

$$\forall\, j\in\{1,\ldots,l\}:\ x_j>0 \text{ on } (0,\infty). \tag{5.16}$$

From (5.2) we know that $x_j(0+)>0$ for $j\in\{1,\ldots,k\}$. Hence the definition of l implies in case $l<k$ that for $j\in\{l+1,\ldots,k\}$, the function x_j has a first zero $z_1^{(j)}$ in $(0,\infty)$. Positive feedback from x_j to x_{j-1} and $x_j>0$ on $(0,z_1^{(j)})$ for these j imply

$$0<z_1^{(k)}\le z_1^{(k-1)}\le\ldots\le z_1^{l+1} \text{ in case } l<k.$$

If $k<N$ we set $z_1^{(j)}:=0$ for $j\in\{k+1,\ldots,N\}$. With this convention, the numbers $z_1^{(j)}$ are defined for $j\in\{l+1,\ldots,N\}$ if $l<N$, satisfy $x_j(z_1^{(j)})\le 0$, and we have

$$z_1^{(N)}\le z_1^{(N-1)}\ldots\le\ldots\le z_1^{(l+1)}\quad (\text{if } l<N). \tag{5.17}$$

If $l < N$ then $x_N(z_1^{(N)}) \le 0$, and the negative delayed feedback from x_1 to x_N and (5.15) show that $x_N \le 0$ on $[z_1^{(N)}, \infty)$. Inductively we see from (5.17) and system (S) that in case $l < N$

$$x_j \le 0 \text{ on } [z_1^{(j)}, \infty) \text{ for } j \in \{l+1, ..., N\}. \tag{5.18}$$

It is shown in [4], Cor. 3 and Cor. 4, that for a nonoscillatory solution as we consider now, one has

$$|\dot{x}_j(t)| > 0 \text{ for all large enough } t, \text{ and } x_j(t) \to 0 \ (t \to \infty), \ j = 1, ..., N. \tag{5.19}$$

(Note that this result does not depend on Hölder conditions which are assumed later in Theorem 1 of [4].) In view of (5.19), we conclude from(5.16) and (5.18) that for all large enough t:

$$\begin{aligned} &x_j(t) > 0 > \dot{x}_j(t), \quad j \in \{1, ..., l\}, \\ &\dot{x}_j(t) > 0 > x_j(t), \quad j \in \{l+1, ..., N\} \text{ (if } l < N). \end{aligned} \tag{5.20}$$

From system (S) we see that with $G_j := \frac{1}{\mu_j} g_j \ (j = 1, ..., N)$ and $\tau_j := 0, \ j = 1, ..., N - 1, \ \tau_N := \tau$ one has for large enough t

$$\begin{aligned} &x_j(t) > G_j(x_{(j+1) \bmod N}(t - \tau_j)) > 0, \quad j \in \{1, ..., l-1\} \text{ (if } l > 1), \\ &x_j(t) < G_j(x_{(j+1) \bmod N}(t - \tau_j)) < 0, \quad j \in \{l+1, ..., N\} \text{ (if } l < N). \end{aligned} \tag{5.21}$$

Using (5.21) around 'one loop' of the cyclic feedback system, we conclude that for all large enough t (taking indices $\bmod N$)

$$|x_{l+1}(t)| > |G_{l+1}(G_{l+2}(...G_N(G_1(...G_{l-1}(x_l(t - \tau_l^*))...)|, \tag{5.22}$$

where $\tau_l^* := \sum_{\substack{j=1,...,N \\ j \ne l}} \tau_j = \begin{cases} 0, \ l = N \\ \tau, \ l < N. \end{cases}$

Note that (5.22) is to be read as

$$x_1(t) > |G_1(G_2(...G_{N-1}(x_N(t)))...)| \text{ if } l = N, \text{ and}$$
$$|x_2(t)| > |G_2(G_3(...G_N(x_1(t - \tau)))...)| \text{ if } l = 1.$$

Condition (5.14) implies that with $M_l := \mu_1 \cdot ... \not{\mu}_l ... \mu_N$ we have $a\tau > M_l$, and thus permits to choose $\varepsilon > 0$ such that

$$(a - \varepsilon)\tau > M_l.$$

We have

$$\lim_{x \to 0, x \neq 0} \left| \frac{[G_{l+1}(G_{l+2}(... G_N(G_1(... G_{l-1}(x)))...)]}{x} \right| = \frac{1}{\mu_1 \cdot ... \not{\mu}_l ... \mu_N} \prod_{j=1, j \ne l}^{N} |g_j'(0)|.$$

There exists a neighborhood U of zero in $\mathbb{R}$ such that (note the lower case letter in g_l below)

$$
\begin{aligned}
\forall x \in U : \ & |g_l[G_{l+1}(G_{l+2}(\dots G_N(G_1(\dots G_{l-1}(x)))\dots)]| \\
& \geq \frac{1}{\mu_1 \cdot \dots \not{\mu}_l \dots \mu_N} \cdot [\prod_{j=1}^{N} |g_j'(0)| - \varepsilon)] \cdot |x| \\
& = \frac{a-\varepsilon}{M_l} \cdot |x|.
\end{aligned}
\tag{5.23}
$$

The monotonicity of g_l in a neighborhood of zero, the convergence of x_l to zero and (5.22), (5.23) combined show that for all large enough t

$$
|g_l(x_{l+1}(t))| \geq \frac{a-\varepsilon}{M_l} \cdot |x_l(t-\tau_l^*|
\tag{5.24}
$$

(to be read as $|g_N(x_1(t))| \geq \dfrac{a-\varepsilon}{M_N}|x_N(t)|$, if $l = N$). Now, from the l-th equation of system (S), we have for $t \geq \tau$

$$
x_l(t) - x_l(t-\tau) = \int_{t-\tau}^{t} [-\mu_l x_l(s) + g_l(x_{(l+1)\mathrm{mod}N}(s-\tau_l)] \, ds.
\tag{5.25}
$$

From (5.20) for $j = l$ we get for all large enough t

$$
\int_{t-\tau}^{t} [-\mu_l x_l(s)] \, ds \leq -\mu_l \tau x_l(t).
\tag{5.26}
$$

From (5.18), $x_{(l+1)\mathrm{mod}N}$ converges to zero and is monotone for large times. Further, for large s we see from (5.20) that $x_{(l+1)\mathrm{mod}N}(s) < 0$ if $l < N$ and $x_{(l+1)\mathrm{mod}N}(s) > 0$ if $l = N$. In both cases, positive feedback of g_l (if $l < N$) and negative feedback of g_N (if $l = N$) give $g_l(x_{(l+1)\mathrm{mod}N}(s-\tau_l)) < 0$ for large enough s. Using (5.24), positivity and monotonicity of x_l, and $\tau_l^* + \tau_l = \tau$, we see that

$$
\begin{aligned}
\forall s \in [t-\tau, t] : \ g_l(x_{(l+1)\mathrm{mod}N}(s-\tau_l)) & \leq -\frac{a-\varepsilon}{M_l} \cdot x_l(s - \tau_l^* - \tau_l) = -\frac{a-\varepsilon}{M_l} \cdot x_l(s-\tau) \\
& \leq -\frac{a-\varepsilon}{M_l} \cdot x_l(t-\tau),
\end{aligned}
$$

hence

$$
\int_{t-\tau}^{t} g_l(x_{(l+1)\mathrm{mod}N}(s-\tau_l)) \, ds \leq -\frac{(a-\varepsilon)\tau}{M_l} \cdot x_l(t-\tau).
\tag{5.27}
$$

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Combining (5.25), (5.26) and (5.27) yields (for large enough t)

$$x_l(t) - x_l(t-\tau) \le -\mu_l \tau x_l(t) - \frac{(a-\varepsilon)\tau}{M_l} \cdot x_l(t-\tau),$$

which implies $(1+\mu_l\tau)x_l(t) \le [1 - \underbrace{\frac{(a-\varepsilon)\tau}{M_l}}_{>1}]x_l(t-\tau) < 0$, contradicting the positivity of $x_l(t)$. Thus the assumption that x_1 has no zero in $(0,\infty)$ is contradictory, so under condition (5.14) x_1 has a first zero $z_1^{(1)}(X_0)$ in $(0,\infty)$, when $X_0 \in \mathbb{O}_{\bar{l}}$. It is clear from $x_1(0+) > 0$ that $x_1 > 0$ on $(0, z_1^{(1)}(X_0))$. □

Remarks: 1) The special property of (hypothetical) solutions starting in $\mathbb{O}_{\bar{l}}$ and such that the x_1 component has no zero in $(0,\infty)$ is that for such solutions (as opposed to general non-oscillatory solutions) the eventually positive and negative components are ordered – in the sense that $x_j(t) > 0,\ j = 1, ..., l$, and $x_j(t) < 0$ for $j \in \{l+1, ..., N\}$ and all large t (see (5.20)). This property allowed us to derive the estimates (5.22) and (5.24), and the contradiction that x_l should have a zero. It is the order of positive and negative components that makes the proof of oscillation for nonzero solutions starting in $\mathbb{O}_{\bar{l}}$ possible under condition (5.14), although, in general, this condition does not exclude real (negative) eigenvalues of the linearized system. In fact, consider the cases $N \in \{1,2\}$: Assuming $\mu_2 \ge \mu_1$ in case $N = 2$, condition (5.14) reads as $a\tau > 1$ in case $N = 1$ and as $a\tau > \mu_2$ in case $N = 2$. The characteristic function χ satisfies $\chi > 0$ on $[-\mu_1, \infty)$, and condition (5.14) shows in both cases that $\chi'(-\mu_1) < 0$. It follows then from $\chi'' > 0$ (even on all of $\mathbb{R}$ in both cases) that χ has no zero in $(-\infty, -\mu_1]$ and hence is positive on all of $\mathbb{R}$.

For $N = 3$ (and larger N), condition (5.14) does not exclude negative real eigenvalues. For example, setting $\mu_1 = \mu_2 = \mu_3 := 1/4$, $\tau := 1$ and $a := e^{-2} > 1/9$, one has $a\tau > \mu_1^2$, so (5.14) holds. But with $\lambda := -5/4$, one has

$$\chi(\lambda) = (\lambda + \mu_1)^3 + ae^{-\lambda\tau} = -1 + e^{-2}e^{5/4} < 0,$$

so χ has a zero in $(-5/4, -1/4)$.

2) Lemma 5.5 can be seen as an extended version of Lemma 2.3 in the paper [34] of Nussbaum.

3) As Lemma 5.5 shows, under condition (5.14) the property $X_0 \in \mathfrak{K}_{\bar{l}}$ is not necessary for x_1 to have a first positive zero, but $X_0 \in \mathbb{O}_{\bar{l}}$ is sufficient.

However, in the derivation of the lower bound for the spectral projection (property (2.13) which is used in the proof of Lemma 2.9), it was important that for the initial function φ of the x_1-component, $||\varphi||_\infty$ can be estimated by $\varphi(0)$. For this purpose, the properties that $\varphi \ge 0$ and is increasing would also be sufficient – but the latter are not reproduced under the return map P, while the property defining $\mathfrak{K}_{\bar{l}}$ is reproduced. This is the reason why the return map will be defined on the cone section $\mathfrak{K}_{\bar{l}}$ in the present paper, and it was defined on analogous cones in earlier papers (e.g., [1,16,20,26]).

The following result will be used to show that the return times to $\mathfrak{K}_{\bar{l}}$ are bounded from above.

Proposition 5.6. *There exists a constant $\gamma > 0$ with the following property: If $(x_1, ..., x_N)$ is a solution of system* (S) *with initial state $X_0 = [x_1|_{[-\tau,0]}, x_2(0), ..., x_N(0)] \in \mathfrak{K}_{\bar{l}}$, then*

$$
\begin{aligned}
||\dot{x}_1|_{[0,\,\tau]}||_\infty &\le \gamma \cdot \max\{[||x_1|_{[0,\,\tau]}||_\infty, |x_2(\tau)|, \dots, |x_N(\tau)|\} \\
&= \gamma \cdot ||\Phi(\tau, X_0)||_{\mathbb{X}}.
\end{aligned}
$$

Proof. From the variation of constants formula and system (S) we have

$$
\forall\, t \in [0, \tau]: \; x_j(t) = e^{-\mu_j(t-\tau)} x_j(\tau) + \int_\tau^t e^{-\mu_j(t-s)} g_j(x_{j+1}(s - \tau_j))\, ds,
$$

where x_{N+1} is to be read as x_1, and $\tau_j = 0$ for $j = 1, \dots, N-1,\; \tau_N = \tau$. The Lipschitz continuity of g_j on I_{j+1} (with a Lipschitz constant γ_j) shows that

$$
||x_j|_{[0,\,\tau]}||_\infty \le e^{\mu_j \tau} |x_j(\tau)| + \tau e^{\mu_j \tau} \gamma_j ||x_{j+1}|_{[0,\,\tau] - \tau_j}||_\infty.
$$

Consequently, with $c_j := \max\{e^{\mu_j\tau}, \tau e^{\mu_j\tau}\gamma_j\}$, we have

$$
\begin{cases}
||x_j|_{[0,\,\tau]}||_\infty \le c_j[|x_j(\tau)| + ||x_{j+1}|_{[0,\,\tau]}||_\infty], \;\; j = 1, \dots, N-1, \\
||x_N|_{[0,\,\tau]}||_\infty \le c_N[|x_N(\tau)| + ||x_1|_{[-\tau,\,0]}||_\infty].
\end{cases}
\tag{5.28}
$$

We set $\varphi := x_1|_{[-\tau,\,0]}$. Using (5.28) recursively, we obtain

$$
\begin{aligned}
||x_{N-1}|_{[0,\,\tau]}||_\infty &\le c_{N-1}[|x_{N-1}(\tau)| + c_N(|x_N(\tau)| + ||\varphi||_\infty)] \\
&\le \max\{c_{N-1}, c_{N-1}c_N\}[|x_{N-1}(\tau)| + |x_N(\tau)| + ||\varphi||_\infty], \\
||x_{N-2}|_{[0,\,\tau]}||_\infty &\le c_{N-2}[|x_{N-2}(\tau)| + ||x_{N-1}|_{[0,\,\tau]}||_\infty] \\
&\le \;\text{const} \cdot [|x_{N-2}(\tau)| + |x_{N-1}(\tau)| + |x_N(\tau)| + ||\varphi||_\infty],
\end{aligned}
$$

and finally there exists a constant $K_2 > 0$ such that

$$
||x_2|_{[0,\,\tau]}||_\infty \le K_2[|x_2(\tau)| + \dots + |x_N(\tau)| + ||\varphi||_\infty].
$$

It follows that

$$
\begin{aligned}
||\dot{x}_1|_{[0,\,\tau]}||_\infty &\le \mu_1 ||x_1|_{[0,\,\tau]}||_\infty + \gamma_2 ||x_2|_{[0,\,\tau]}||_\infty \\
&\le \mu_1 ||x_1|_{[0,\,\tau]}||_\infty + \gamma_2 K_2[|x_2(\tau)| + \dots + |x_N(\tau)| + ||\varphi||_\infty].
\end{aligned}
$$

Now the definition of $\mathfrak{K}_{\bar{I}}$ implies

$$
||\varphi||_\infty \le e^{\mu_1\tau}|x_1(0)| \le e^{\mu_1\tau} ||x_1|_{[0,\,\tau]}||_\infty,
$$

and hence

$$||\dot{x}_1|_{[0,\,\tau]}||_\infty \leq \underbrace{\{\gamma_2 K_2(N-1) + (\mu_1 + \gamma_2 K_2)e^{\mu_1 \tau}\}}_{=:\,\gamma} \max\{||x_1|_{[0,\,\tau]}||_\infty, |x_2(\tau)|, ..., |x_N(\tau)|\},$$

which proves the inequality in the proposition. The subsequent equality comes from the definition of $|| \cdot ||_{\mathbb{X}}$. □

Lemma 5.7. *Assume condition (5.14), so that $\mathbb{Z}_{\vec{I}} = \mathbb{O}_{\vec{I}}$. (Then $\mathfrak{K}_{\vec{I}} \setminus \{0\} \subset \mathbb{X}^+_{\vec{I}} \subset \mathbb{O}_{\vec{I}} = \mathbb{Z}_{\vec{I}}$, so the first zero $z_1^{(1)}$ and the map Q are defined, in particular, on $\mathfrak{K}_{\vec{I}} \setminus \{0\}$.)*

1) There exists $T_1 > 0$ such that for $X_0 \in \mathfrak{K}_{\vec{I}} \setminus \{0\}$ one has $z_1^{(1)}(X_0) \leq T_1$.

2) The extension of $Q|_{\mathfrak{K}_{\vec{I}} \setminus \{0\}}$ to $\mathfrak{K}_{\vec{I}}$ by $Q(0) := 0$ is continuous, compact, and takes values in $-\mathfrak{K}_{-\vec{I}}$.

Proof. Proof of 1): Assume there exists a sequence of initial states

$$X_0^{(n)} = [\varphi^{(n)}, x_2^{(n)}(0), ..., x_N^{(n)}(0)] \in \mathfrak{K}_{\vec{I}} \setminus \{0\}$$

and of corresponding solutions $t \mapsto [x_1^{(n)}(t), x_2^{(n)}(t), ..., x_N^{(n)}(t)]$ such that

$$z_1^{(1)}(X_0^{(n)}) \to \infty \text{ as } n \to \infty. \tag{5.29}$$

We can then assume

$$z_1^{(1)}(X_0^{(n)}) > \tau \text{ for all } n \in \mathbb{N}, \tag{5.30}$$

and consider $Y^{(n)} := \Phi(\tau, X_0^{(n)}) =: [\psi^{(n)}, y_2^{(n)}, ..., y_N^{(n)}]$. Since $X_0^{(n)} \in \mathfrak{K}_{\vec{I}} \setminus \{0\} \subset \mathbb{O}_{\vec{I}} = \mathbb{Z}_{\vec{I}}$, we know from part 3) of Lemma 5.4 that $Y^{(n)} \in \mathbb{O}_{\vec{I}}$ $(n \in \mathbb{N})$. Together with (5.30) we see that

$$\psi^{(n)}(0) \neq 0, \text{ and } Y^{(n)} \in \mathbb{O}_{\vec{I}} \setminus \{0\} \text{ for all } n \in \mathbb{N}.$$

$\psi^{(n)}$, $\dot{\psi}^{(n)}$, and $y_2^{(n)}, ..., y_N^{(n)}$ are bounded uniformly with respect to $n \in \mathbb{N}$, and the theorems of Arzelà-Ascoli and Bolzano-Weierstraß give a subsequence $\{Y^{\alpha(n)}\} \subset \{Y^{(n)}\}$ converging to $Y^* = (\psi^*, y_2^*, ..., y_N^*) \in \mathbb{X}_{\vec{I}}$ (in the norm on $\mathbb{X}$). We can assume that the whole sequence $(Y^{(n)})$ converges to Y^*, and obviously $Y^* \in \overline{\mathbb{O}_{\vec{I}}}$. Consider the solution $(y_1, ..., y_N)$ of system (S) with initial value Y^*. Convergence of $Y^{(n)}$ to Y^* implies the convergence

$$[x_1^{(n)}(t+\tau), ..., x_N^{(n)}(t+\tau)] \to [y_1(t), ..., y_N(t)] \tag{5.31}$$

uniformly on compact subintervals of $[0, \infty)$.

Case 1: $Y^* \in \mathbb{O}_{\vec{I}}$. Then, due to $\mathbb{O}_{\vec{I}} = \mathbb{Z}_{\vec{I}}$, Lemma 5.4 (in particular, part 4)) shows that y_1 has a first zero $z^{(1)}(Y^*)$ in $(0, \infty)$, with $\dot{y}_1(z^{(1)}(Y^*)) < 0$. In view of (5.31), this implies that for all large enough n, $x_1^{(n)}$ has a first zero in $(0, z^{(1)}(Y^*) + \tau + 1)$, contradicting the assumption $z_1^{(1)}(X_0^{(n)}) \to \infty$.

Case 2: $Y^* \in \overline{\mathbb{O}_{\vec{I}}} \setminus (\mathbb{O}_{\vec{I}} \cup \{0\})$. Then Proposition 5.3 shows that there exists $\delta > 0$ with $y_1(\delta) < 0$. From (5.31) we see that $x_1^{(n)}(\tau + \delta) < 0$ for large n, and a contradiction is obtained as in case 1.

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Case 3: $Y^* = 0$. (Since $Y^* \in \overline{\mathbb{O}_{\bar{I}}}$, this is the remaining case.) There exists $r > 0$ such that with the intervals I_j from Theorem 5.2 one has $[-r, r] \subset \bigcap_{j=1}^{N} I_j$. Then the rescaled sequence

$$Z^{(n)} := r \cdot \frac{Y^{(n)}}{||Y^{(n)}||_{\mathbb{X}}}$$

is contained in $\mathbb{O}_{\bar{I}}$ and satisfies $||Z^{(n)}||_{\mathbb{X}} = r$ for all $n \in \mathbb{N}$. From Proposition 5.6 we obtain $||\dot{x}_1^{(n)}|_{[0,\tau]}||_\infty \le \gamma ||\Phi(\tau, X_0^{(n)}||_{\mathbb{X}}$, so the inequality $||\dot{\psi}^{(n)}||_\infty \le \gamma ||Y^{(n)}||_{\mathbb{X}}$ $(n \in \mathbb{N})$ holds. If we write $Z^{(n)} = (\zeta^{(n)}, w_2^{(n)}, ..., w_N^{(n)})$, we conclude that for all $n \in \mathbb{N}$

$$||\zeta^{(n)}||_\infty \le r, \ |w_j^{(n)}| \le r, \ j = 2, ..., N, \quad \text{and} \quad ||\dot{\zeta}^{(n)}||_\infty \le r \cdot \frac{||\dot{\psi}^{(n)}||_\infty}{||Y^{(n)}||_{\mathbb{X}}} \le r\gamma.$$

Again, applying the theorems of Arzelà-Ascoli and Bolzano-Weierstraß and choosing a subsequence, we can assume that $\{Z^{(n)}\}$ converges to a limit Z^* in $\overline{\mathbb{O}_{\bar{I}}}$. Then $||Z^*||_{\mathbb{X}} = r$, in particular, $Z^* \ne 0$. Now consider the solution $t \mapsto [w_1(t), ..., w_N(t)]$ of the linearized system (L) with initial state Z^*, and the solutions $[y_1^{(n)}, .., y_N^{(n)}]$ of the nonlinear system (S) with initial state $Y^{(n)}$. Setting

$$\alpha_j^{(n)}(t) := \int_0^1 g_j'(s \cdot y_{j+1}^{(n)}(t))\, ds, \ j = 1, ..., N-1, \quad \alpha_N^{(n)}(t) := \int_0^1 g_N'(s \cdot y_1^{(n)}(t-\tau))\, ds,$$

we have

$$\begin{cases} \dot{y}_j^{(n)}(t) = -\mu_j y_j(t) + \alpha_j^{(n)}(t) y_{j+1}^{(n)}(t), \ j = 1, ..., N-1, \\ \dot{y}_N^{(n)}(t) = -\mu_N y_N(t) + \alpha_N^{(n)}(t) y_1^{(n)}(t-\tau). \end{cases} \tag{5.32}$$

(The technique here is similar to the proof of Lemma 3.10 in [20].) The convergence of $Y^{(n)}$ to $Y^* = 0$ implies that for $j = 1, ..., N$ one has the convergence $\alpha_j^{(n)}(t) \to a_j = g_j'(0)$ uniformly on every compact subinterval of $[0, \infty)$, as $n \to \infty$. The functions $\tilde{y}_j^{(n)}(t) := r \cdot \frac{y_j^{(n)}(t)}{||Y^{(n)}||_{\mathbb{X}}}$ also satisfy the nonautonomous linear system (5.32), and, in addition, their initial states converge to Z^*. Thus we obtain

$$\tilde{y}_j^{(n)}(t) \to w_j(t) \ (n \to \infty), \ j = 1, ..., N, \ \text{uniformly on compact subintervals of } [0, \infty). \tag{5.33}$$

If now $Z^* \in \mathbb{O}_{\bar{I}}$ then Lemma 5.5 shows that w_1 has a first zero $z^{(1)}(Z^*)$ in $(0, \infty)$, with $\dot{w}_1(z^{(1)}(Y^*)) < 0$. It follows from (5.33) that then for all large enough n, $\tilde{y}_1^{(n)}$ and therefore also $y_1^{(n)}$ has a first zero in $(0, z^{(1)}(Z^*) + 1)$. Hence $x_1^{(n)}$ has a first zero in $(0, z^{(1)}(Z^*) + \tau + 1)$, contradicting assumption (5.29) (similarly to case 1).

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If $Z^* \notin \mathbb{O}_{\vec{I}}$ then $Z^* \in \overline{\mathbb{O}_{\vec{I}}} \setminus (\mathbb{O}_{\vec{I}} \cup \{0\})$, and Proposition 5.3 shows that there exists $\delta > 0$ with $w_1(\delta) < 0$. For large n, it follows that $\tilde{y}_1^{(n)}(\delta) < 0$ and $y_1^{(n)}(\delta) = x_1^{(n)}(\delta + \tau) < 0$, contradicting (5.29).

We have seen in all cases that assumption (5.29) is contradictory, and hence $z_1^{(1)}$ is bounded above by some T_1 on the set $\mathfrak{K}_{\vec{I}} \setminus \{0\}$.

Proof of 2): The continuity of the extended map Q follows from the fact that $\Phi(t, \psi) \to 0$ as $\psi \to 0$ uniformly on $[0, T_1 + \tau]$ (since the semiflow Φ is uniformly continuous on the compact set $[0, T_1 + \tau] \times \{0\}$).

The image of Q is bounded (since it is contained in $\mathbb{X}_{\vec{I}}$), and if $(\varphi_1, y_2, ..., y_N) = Q((\varphi, x_2^0, ..., x_N^0))$ then φ_1 is C^1, and $\dot{\varphi}_1 = \dot{x}_1(z_1^{(1)} + \tau \cdot)|_{[-\tau, 0]}$ takes only values in the compact set $g_2(I_2)$. The compactness of the closure of the image of Q now follows from the Arzelà-Ascoli theorem and the Bolzano-Weierstraß theorem. Finally, we see from part 4) of Lemma 5.4 that Q maps into $-\mathfrak{K}_{-\vec{I}}$. $\square$

The subsequent construction of a return map P from the above map Q is analogous to the corresponding passage in [20]; see e.g. Remark 3.5 there. The system (S^-) with the nonlinearities h_j defined by $h_j(x) := -g_j(-x)$ has the same feedback properties as system (S), and one sees from the construction of the intervals I_j in [4] that one can take the intervals $-I_j$ for system (S^-) to obtain a corresponding invariant set $\mathbb{X}_{-\vec{I}}$ for system (S^-). Thus one obtains a semiflow Φ^- and a map Q^- for system (S^-) analogous to Q above, with a 'time function' $z_{1,-}^{(1)}$ analogous to $z_1^{(1)}$ above. Note that then for an initial state $X_0 \in \mathbb{X}$ and $t \geq 0$ one has $\Phi^-(t, -X_0) = -\Phi(t, X_0)$. Further, the nonlinearities h_j have the same derivatives at zero as the g_j, so condition (5.14) is the same for system (S^-) as for system (S).

Assume now that condition (5.14) is satisfied, and consider an initial state $X_0 \in X_{\vec{I}} \setminus \{0\}$. Then, in view of part 2) of Lemma 5.4, we have $Q(X_0) \in -\mathfrak{K}_{-\vec{I}}$. The solution $(y_1, ..., y_N)$ of system (S) with initial state $Y_0 = Q(X_0)$ (which is the time translate of the solution with initial state X_0 by the time $z_1^{(1)}(X_0) + \tau$) satisfies that $-(y_1, ..., y_N)$ is a solution of system (S^-), with initial state $-Q(X_0) \in \mathfrak{K}_{-\vec{I}}$. It follows that

$$\begin{aligned} Q^-(-Q(X_0)) &= Q^-(-Y_0) = \Phi^-(z_{1,-}^{(1)}(-Y_0) + \tau, -Y_0) = -\Phi(z_{1,-}^{(1)}(-Y_0) + \tau, Y_0) \\ &= -\Phi[(z_{1,-}^{(1)}(-Y_0) + \tau, \Phi(z_1^{(1)}(X_0) + \tau, X_0)] = -\Phi[(z_{1,-}^{(1)}(-Y_0) + z_1^{(1)}(X_0) + 2\tau, X_0]. \end{aligned} \tag{5.34}$$

Note that in case $X_0 \neq 0$ the time $z_{1,-}^{(1)}(-Y_0) + z_1^{(1)}(X_0) + \tau$ coincides with the second zero $z_2^{(1)}(X_0)$ in $(0, \infty)$ of the component x_1 of the solution of system (S) with initial state X_0. We obtain the following result:

Corollary 5.8. *Assume condition* (5.14).

1) The map $P : \mathfrak{K}_{\vec{I}} \to \mathfrak{K}_{\vec{I}}$ defined by $P(X_0) := -Q^-[-Q(X_0)]$ satisfies $P(0) = 0$ and is of the form $P(X_0) = \Phi(\theta(X_0), X_0)$ for $X_0 \neq 0$, where the return time θ is a continuous function on $\mathfrak{K}_{\vec{I}} \setminus \{0\}$ given by $\theta(X_0) = z_2^{(1)}(X_0) + \tau$, and there exists $T^{**} > 0$ such that

$$\forall X_0 \in \mathfrak{K}_{\vec{I}} \setminus \{0\} : 2\tau \leq \theta(X_0) \leq T^{**}. \tag{5.35}$$

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2) P is continuous and compact.
3) $P(X_0) \neq 0$ if $X_0 \neq 0$.

Proof. Ad 1): $P(0) = 0$ follows from $Q(0) = Q^-(0) = 0$. It was shown above that $P(X_0)$ has the asserted form in case $X_0 \neq 0$. From part 2) of Lemma 5.7 we see that map Q^- maps into $-\mathfrak{K}_{-(-\vec{I})} = -\mathfrak{K}_{\vec{I}}$, and hence P maps into $\mathfrak{K}_{\vec{I}}$.

The time maps involved in the construction of Q and Q^- are both bounded above, which implies the existence of $T^{**} > 0$ as in (5.35). One sees from (5.34) that $\theta \geq 2\tau$.

Ad 2): The continuity and compactness of the map P extended to zero follow from part 2) of Lemma 5.7.

Finally, Assertion 3) follows from part 4 of Lemma 5.4. □

6. Periodic solutions

We can now put together the previous results on the linearized system (L), the return map for system (S), and Theorem 3.4 on periodic orbits, to prove our main result.

Remark 6.1. Assume that there exists a leading eigenvalue $\lambda = \rho + i\omega$, $\rho > 0$, $\omega > 0$ of system (L), as described in part (iv) of Lemma 2.7 and Lemma 2.11. Then the semiflow induced by system (S) on the Banach space $\mathbb{X}$ satisfies assumptions (A1) and (A2) from Section 3.

Proof. For N-dimensional systems with more general coupling structure than considered here, the state space is usually taken as $\mathcal{C} := (C^0([-\tau, 0], \mathbb{C}^N), ||\cdot||_\infty)$ instead of $\mathbb{X}$. The space $\mathbb{X}$ can be isometrically embedded into $\mathcal{C}$ by constant extension of $x_2(0), ..., x_n(0))$ into the past:

$$\iota : \mathbb{X} \ni [\varphi, x_2(0), ..., x_N(0)] \mapsto [\varphi, \hat{x}_2(0), ..., \hat{x}_N(0)] \in \mathcal{C},$$

where the hats denote the corresponding constant functions on $[-\tau, 0]$. With the semiflows $\Phi_{\mathbb{X}}$ and $\Phi_{\mathcal{C}}$ induced on the indicated spaces, the maps ι and

$$\pi_0 : \mathcal{C} \ni [\varphi_1, ..., \varphi_N] \mapsto [\varphi_1, \varphi_2(0), ..., \varphi_N(0)],$$

one has for $X_0 \in \mathbb{X}$ and $t \geq 0$: $\Phi_{\mathbb{X}}(t, X_0) = \pi_0[\Phi_{\mathcal{C}}(t, \iota(X_0))]$. Thus, results for $\mathbb{X}$ follow easily from those for $\mathcal{C}$. We write Φ for $\Phi_{\mathbb{X}}$ from now on. With the semigroup $T(t)_{t\geq 0}$ induced by system (L) on $\mathbb{X}$, we have $D_2\Phi(t, 0) = T(t)$ $(t \geq 0)$, since the derivative of $D_2\Phi(t, \cdot)$ with respect to the initial state at zero is given by the solutions of the variational equation along the constant zero solution, i.e., by solutions of system (L). For $R > 0$, the uniform continuity of Φ on the compact set $[0, R] \times \{0\}$ implies that if $||X_0||_{\mathbb{X}} \to 0$ then $\Phi(t, X_0) \to 0$ uniformly on $[0, T]$. We conclude that the coefficients $g_j'(x_{j+1}(t - \tau_j))$ in the variational equation of system (S) along a solution with initial state X_0 converge to the coefficients $a_j = g_j'(0)$ of system (L) uniformly on compact intervals in $[0, \infty)$, as $X_0 \to 0$. This implies that $D_2\Phi(t, X_0) \to D_2\Phi(t, 0) = T(t)$ as $X_0 \to 0$, uniformly on compact t-intervals, i.e., property (A1) holds.

With the (real) spectral subspaces U and S corresponding to the spectral set $\{\lambda, \bar{\lambda}\}$ (and its complement in the spectrum), one has $\dim_{\mathbb{R}} U = 2$ and $\mathbb{X} = U \oplus S$. The fact that $\mathrm{Re}\,(\mu) < \rho$ for all eigenvalues $\mu \neq \lambda$ implies that $\max \left\{ \mathrm{Re}\,(\mu) \;\middle|\; \mu \text{ eigenvalue of (L)}, \mu \neq \lambda \right\} < \rho$. The exponential estimates in (A2) then follow from Theorem 4.1, p. 181 in [15], or the analogous

Theorem 6.1 on p. 214 of [18] (one has to observe that the constant $\gamma > 0$ in these theorems can be chosen arbitrarily small). □

In the main theorem of this section, we consider the cyclic feedback system (S) with delay $\tau > 0$, with decay rates $\mu_1, \ldots, \mu_N > 0$, and with C^1 functions g_j, where g_N is bounded from above or from below. Recall the numbers $a_j = g_j'(0)$, which satisfy $a_1, ..., a_{N-1} > 0, a_N < 0$, and the number $a = |a_1 \cdot ... \cdot a_N| > 0$. For the statement of the theorem, we repeat the definitions of the following constants from Sections 2 and 5:

$$M^* := \frac{\prod_{k=1}^{N} \mu_k}{\min\{\mu_1, ..., \mu_N\}}; \; A_1 := \sqrt{\prod_{j=1}^{N}(\mu_j^2 + \frac{\pi^2}{\tau^2})}; \; K := \max\{\max_{j=1,...,N} \mu_j, \; \frac{\max\{\pi, N-1\}}{\tau}\}.$$

We define $\alpha(x) := e^{\tau x} \cdot [\sqrt{5}x]^N$ for $x > 0$, and set $A_2 := \max\{\alpha(K), A_1\}$.

The theorem below gives two sufficient conditions for the existence of periodic solutions. The first one expresses the case of strong feedback ('large a'), while the second one concerns the case of 'large delay τ'.

Theorem 6.2. *Under the above assumptions on system* (S) *and with the constants defined above, consider the following conditions:*

(1) a) $a \geq A_2$ and b) $a\tau > M^*$;
(2) c) $a \geq A_1$ and d) $\tau \geq \sum_{j=1}^{N} \frac{1}{\mu_j}$.

If the inequalities a) and b) from (1) hold, or if the inequalities c) and d) from (2) hold, then system (S) *has a non-constant periodic solution with initial value in the cone $\mathfrak{K}$. The minimal period is given by $z_2^{(1)} + \tau$, where $z_2^{(1)}$ is the second positive zero of the x_1-component of the periodic solution.*

Proof. We verify the hypotheses of Theorem 3.4 for the semiflow Φ on the Banach space $\mathbb{X}$ induced by system (S).

Assume first that condition (1) holds (i.e., inequalities a) and b) are true). Then, since $A_2 \geq A_1$, Lemma 2.7 shows the existence of a leading eigenvalue as required in Remark 6.1. Further, condition a) and Proposition 2.10 give that the lower estimate (2.6) holds. Finally, condition b) coincides with condition (5.14) of Lemma 5.5, so Corollary 5.8 gives a continuous and compact return map P, with a return time θ as required in Theorem 3.4, and satisfying conditions 1) and 2) of that theorem.

Assume next that condition (2) is satisfied (i.e., c) and d) hold). Then we obtain the existence of a leading eigenvalue and the validity of the lower bound (2.6) from Lemma 2.11, since, in particular, inequality d) implies $\tau > \sum_{j=2}^{N} \frac{1}{\mu_j}$. Note now that the definition of A_1 together with inequalities c) and d) give

$$a\tau \geq A_1 \sum_{j=1}^{N} \frac{1}{\mu_j} > \prod_{k=1}^{N} \mu_k \cdot \sum_{j=1}^{N} \frac{1}{\mu_j} = \sum_{j=1}^{N} M_j > \max_{j=1,...,N} M_j,$$

i.e., condition b) holds also, and again Lemma 5.5 and Corollary 5.8 give the return map P as above.

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Remark 6.1 shows in both cases that assumptions (A1) and (A2) from Section 3 hold. We have seen that in both cases we obtain a compact continuous return map P with a return time θ as required in Theorem 3.4, and satisfying conditions 1) and 2) of that theorem, and that the lower bound property (2.6) (assumption 3) of Theorem 3.4 is true in both cases.

Finally, we define the continuous linear functional η on $\mathbb{X}$ by $\eta(X_0) := \varphi(0) + x_2 + ... + x_N$ for $X_0 = (\varphi, x_2, ..., x_N) \in \mathbb{X}$. If $X_0 \in \mathfrak{K}$ then the definition of $\mathfrak{K}$ implies $\varphi(0) \geq \exp(-\mu_1\tau)||\varphi||_\infty$, and

$$\eta(X_0) = \varphi(0) + |x_2| + ... + |x_N| \geq \exp(-\mu_1\tau)\max\{||\varphi||_\infty, |x_2|, ..., |x_N|\} = \exp(-\mu_1\tau)||X_0||_{\mathbb{X}},$$

showing that assumption 4) of Theorem 3.4 holds. The statement about the minimal period is clear from Corollary 5.8. □

7. Global asymptotic stability of zero

Recall the functions G_j $(j = 1, ..., N)$ and $G = G_1 \circ .... \circ G_N$ from section 5. We show here how the invariance and attractivity results from that section can be employed to obtain the convergence of all solutions to the zero solution. This is a complementary result to those on the existence of periodic solutions in the previous section,

Proposition 7.1. *Assume that G has a compact invariant interval I_1 with 0 in its interior and such that $G^n(x) \to 0$ as $n \to \infty$ $\forall x \in I_1$. Then* $\bigcap_{n\in\mathbb{N}} G^n(I_1) = \{0\}$.

Proof. Set $[a_n, b_n] := G^n(I_1)$ for $n \in \mathbb{N}$. Each function G^n has either positive or negative feedback (depending on the parity of n), which implies that $a_n < 0 < b_n$ for all n. Further it follows from $G^{n+1}(I_1) \subset G^n(I_1)$ that

$$a_n \leq a_{n+1} < 0 = G(0) < b_{n+1} \leq b_n \quad (n \in \mathbb{N}).$$

The monotone sequences (a_n) and (b_n) have limits $a^* \leq 0$ and $b^* \geq 0$, and we have to show $a^* = b^* = 0$. We first prove

$$G([a^*, b^*]) = [a^*, b^*]. \tag{7.1}$$

Since $G([a^*, b^*]) \subset G([a_n, b_n]) = [a_{n+1}, b_{n+1}]$ for all n, we have $G([a^*, b^*]) \subset [a^*, b^*]$. If we had $\min_{x\in[a^*,b^*]} G(x) > a^*$ then $a_n \to a^*$, $b_n \to b^*$ and continuity of G would imply $a_{n+1} = \min_{x\in[a_n,b_n]} G(x) > a^*$ for large n, a contradiction. Hence $\min_{x\in[a^*,b^*]} G(x) = a^*$, and with the analogous argument for b^* we obtain (7.1).

Assume now $a^* < 0$. Since G^2 has positive feedback, we see from $G(0) = 0$ and (7.1) that

$$G^2([a^*, 0]) = [a^*, 0].$$

If $G^2(a^*) > a^*$ then there exists $x \in (a^*, 0)$ with $G^2(x) = a^* < x$, and hence we obtain from the intermediate value theorem an $x^* \in (a^*, x)$ with $G^2(x^*) = x^*$. If $G^2(a^*) = a^*$ then $x^* := a^*$ is a non-zero fixed point of G^2. Both cases contradict the property $G^n(x^*) \to 0$ for $n \to \infty$. Thus the assumption is contradictory and we have $a^* = 0$; analogously one shows that $b^* = 0$. □

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Theorem 7.2 *(Global asymptotic stability).*

a) Assume there exists a compact interval I_1 with $0 \in int(I_1)$, $G(I_1) \subset I_1$, and such that the fixed point 0 of G is globally attracting on I_1, as in Proposition 7.1, i.e., $\lim_{n\to\infty} G^n(x) = 0$ *for every $x \in I_1$. Then the zero solution is globally asymptotically stable within the set $\mathbb{X}_{\bar{I}}$ as defined in Theorem 5.2: for every initial state $X_0 \in \mathbb{X}_{\bar{I}}$ one has $\Phi(t, X_0) \to 0$ as $t \to \infty$.*

b) If 0 is globally attractive for G within the G-invariant interval $I_1 := \overline{G^2(\mathbb{R})}$ from part 2) of Theorem 5.2, then $0 \in \mathbb{X}$ is the global attractor for the whole semiflow induced by system (S).

Proof. Proof of a): Let I_1 be as in a), and consider an initial state X_0 in the associated set $\mathbb{X}_{\bar{I}}$. Part 1) of Theorem 5.2 shows that the corresponding solution satisfies $x_1([-\tau, \infty)) \subset I_1$. Applying part b) of Proposition 5.1 iteratively, we see that $x_1 \in G^n(I_1)$ ev. for all $n \in \mathbb{N}$. In view of Proposition 7.1 this implies $x_1(t) \to 0$ as $t \to \infty$. With Corollary 2 from [4], one sees that $x_N(t) \to 0$ as $t \to \infty$, and using the same argument repeatedly, that all components x_j converge to zero.

Proof of b): We know from Theorem 5.2 that I_1 is as required in part a), and that every solution enters the corresponding set $\mathbb{X}_{\bar{I}}$ in finite time. Part b) follows hence from part a). □

8. Final remarks

We mention three possible alternative approaches to obtain periodic solutions for our system (S), which are not yet realized:

1) The Morse decomposition result for scalar delay equations from [28] includes the statement that nonconstant periodic solutions exist in each level set of a zero-counting Liapunov functional. The corresponding result for systems of more than one equation would include the result on periodic solutions from the present paper, but is to our knowledge presently not proved.

2) In the spirit of the geometric description of subsets of the global attractor in [42] (for negative monotone feedback) and [22] (for positive monotone feedback), one might conjecture that if the linearization at zero has a conjugate pair of eigenvalues in the right half plane then the global continuation of the local unstable manifold at zero contains a nonconstant periodic orbit in its closure. Again, such a result would include ours, but does not exist presently.

3) An extensive theory of cyclic delay systems with *monotone* feedback was developed in [30] and [31]. Assuming that these results grant the existence of periodic solutions, it could be possible to define an index-preserving homotopy from a non-monotone system to one that falls in the class described in [31]. (This was noted by John Mallet-Paret, [29].)

As a final minor addition here we take the opportunity to correct some errata that occurred in our earlier paper [20]:

1) On page 668 in the introduction, preceding the system with only one delay, it should be 'Then, setting $q_j(t) := z_j(\tau t)$, ...'

2) In part a) of Definition 3.2 on p. 676, the last sentence should be 'We say that z is oscillatory or oscillates, if z takes positive and negative values on every interval $[T, \infty)$ if $T > a$.'

3) The definition of the cone $\mathfrak{K}$ on p. 681 should come before Lemma 3.8 on p. 680, and in the assumptions of Lemma 3.8 it should be '... with initial value $X_0 = (\varphi, x_2^0, x_3^0) \in \mathfrak{K}$, and such that ...'

4) In part c) of Corollary 3.1 on p. 686 it should be $z_2^1(X_0)$ instead of $z_1^2(X_0)$.

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