Journal of Differential Equations **177**, 49–76 (2001)
doi:10.1006/jdeq.2001.3998, available online at http://www.idealibrary.com on IDEAL®

Isolated Singularities for Fully Nonlinear Elliptic Equations

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Received November 8, 1999; revised October 30, 2000

We obtain Serrin type characterization of isolated singularities for solutions of fully nonlinear uniformly elliptic equations $F(D^2u)=0$. The main result states that any solution to the equation in the punctured ball bounded from one side is either extendable to the solution in the entire ball or can be controlled near the centre of the ball by means of special fundamental solutions. In comparison with semi- and quasilinear results the proofs use the viscosity notion of generalised solution rather than distributional or Sobolev weak solutions. We also discuss one way of defining the expression $-\mathscr{P}^+_{\lambda,\Lambda}(D^2u)$, $(\mathscr{P}^-_{\lambda,\Lambda}(D^2u))$ as a measure for viscosity supersolutions (subsolutions) of the corresponding equation. Here $\mathscr{P}^\pm_{\lambda,\Lambda}$ are the Pucci extremal operators. © 2001 Elsevier Science

Key Words: isolated singularities; viscosity solutions to fully nonlinear elliptic equations.

1. INTRODUCTION

The classical result due to Serrin [35] describes isolated singularities of solutions to the equation

$$\operatorname{div} \mathbf{A}(x,u,Du)+B(x,u,Du)=0. \tag{1.1}$$

Let Ω be an open set in $\mathbf{R}^n$, $n\geqslant 2$, containing 0. Serrin's theorem asserts that under special growth assumptions on the structure of the vector function $\mathbf{A}$ and the scalar function B every weak solution u of (1.1) in $\Omega\setminus\{0\}$, $u\in W^{1,2}_{\text{loc}}(\Omega\setminus\{0\})$, $u\geqslant$ const, either can be defined at 0 as a solution of (1.1) in Ω (that is, *the singularity at* 0 *is removable*) or

$$c\leqslant u/E\leqslant 1/c, \qquad c>0 \tag{1.2}$$

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0022-0396/01 $35.00
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in some neighbourhood of 0. Here E is a special fundamental singular solution of Eq. (1.1). In particular for the p-Laplace equation $\Delta_p u = \operatorname{div}(|Du|^{p-2}\, Du) = 0$, $1 < p \leqslant n$,

$$E(x) = E_p(x) = \begin{cases} \dfrac{1}{|x|^{(n-p)/(p-1)}}, & 1 < p < n \\ -\log |x|, & p = n; \end{cases}$$

see also [44, Chap. 1]. The main goal of the present paper is to establish a Serrin type characterisation of isolated singularities for solutions of *fully nonlinear* (nonlinear on the second derivatives) uniformly elliptic equations

$$F(D^2 u) = 0. \tag{1.3}$$

Before stating our results let us make some bibliographical remarks. Isolated singularities for *semi*- and *quasilinear* elliptic and parabolic equations were investigated by many authors. The number of papers on this subject is very substantial and it is hard to review the literature in a short article. We refer to Véron's monograph [44] for results for these classes of equations and a rich bibliography up to 1996. Among the recent results we mention [8, 24]. In [8] singularities of solutions quasilinear *subelliptic* equations of type (1.1) on stratified Lie groups were carefully investigated. In particular the Serrin characterisation (1.2) was proved. In [24] a simpler proof of the characterisation (1.2) was obtained for positive solutions of the semilinear Yamabe type equation. Originally this equation was treated in [7, 16]. The important tools in [8, 24, 44] are the flexible notions of generalised solution: weak solution for quasilinear equations or, sometimes, distributional solution for semilinear equations. Such notions of generalised solutions are not applicable to fully nonlinear elliptic equations (1.3). In the present paper we use the recently introduced *viscosity* generalised solutions to investigate isolated singularities for fully nonlinear equations.

Our main results are the three theorems 1.1, 1.2, and 1.4 below. The *plan* of this paper is as follows. In Section 2 we formulate preliminaries concerning viscosity solutions of elliptic equations. In Section 3 the main theorems are proved. In Section 4 we study superharmonic functions connected with some fully nonlinear operators, see the end of this Section for a summary. Now we recall some notions and finish this Section stating the main results.

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product in $\mathbf{R}^n$, $n \geqslant 2$. $B(x, R)$ denotes an open ball in $\mathbf{R}^n$ with centre x and radius R, $B_R = B(0, R)$. By $\mathbf{S}^n$, $n \geqslant 2$, we denote the space of real $n \times n$ symmetric matrices equipped with its usual order; that is for $N \in \mathbf{S}^n$ $N \geqslant 0$ means $\langle Nx, x \rangle \geqslant 0$ for all $x \in \mathbf{R}^n$; I stands for the identity matrix. In Eq. (1.3) $F\colon \mathbf{S}^n \to \mathbf{R}^1$. We will assume that

F is a *uniformly elliptic operator*. That is, there are two constants $\Lambda \geqslant \lambda > 0$ (which are called the *ellipticity constants*) such that for any $M \in \mathbf{S}^n$

$$\lambda \operatorname{trace}(N) \leqslant F(M+N) - F(M) \leqslant \Lambda \operatorname{trace}(N) \qquad \forall N \geqslant 0,$$

or equivalently $\lambda I \leqslant [\partial F(M)/\partial M_{ij}] \leqslant \Lambda I$. Examples of fully nonlinear uniformly elliptic equations arising in applications are the Bellman and Isaacs equations; see [5, 17]. Important operators for the viscosity theory (and for our work) are the *Pucci extremal operators* $\mathscr{P}^{\pm}_{\lambda,\Lambda}$, see Section 2 for definitions. By a *solution* of (1.3) we always mean the viscosity solution (Sect. 2) unless otherwise indicated.

Our first result concerns the Pucci extremal operators $\mathscr{P}^{\pm}_{\lambda,\Lambda}$ and their fundamental solutions $E^{\pm} = E^{\pm}_{\Lambda/\lambda}$, $e^{\pm} = e^{\pm}_{\Lambda/\lambda}$, see (2.5)–(2.9).

THEOREM 1.1. *Let $u \in C^2_{\mathrm{loc}}(B_R \setminus \{0\})$, $u \geqslant 0$, satisfy*

$$\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u) = 0 \qquad \text{in } B_R \setminus \{0\}, \tag{1.4}$$

where $B_R \subset \mathbf{R}^n$, $n \geqslant 2$, $0 < \lambda < \Lambda$, $1 < \Lambda/\lambda \leqslant n-1$. Then either the singularity at 0 is removable and u is a classical solution of (1.4) *in the entire ball B_R, or there exists a real number $\gamma > 0$ such that*

$$u(x) = \gamma E^{+}_{\Lambda/\lambda}(x) + O(1), \qquad x \to 0, \tag{1.5}$$

and

$$D^\alpha u(x) = \gamma D^\alpha E^{+}_{\Lambda/\lambda}(x) + o\left(\frac{1}{|x|^{\lambda(n-1)/\Lambda - 1 + |\alpha|}}\right), \qquad x \to 0, \tag{1.6}$$

for all multi-indices α with $1 \leqslant |\alpha| \leqslant 2$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

According to the Evans–Krylov estimates, any viscosity solution to (1.4) enjoys $C^{2,\alpha}_{\mathrm{loc}}$ regularity [29] and, consequently, is a classical solution. Because of the lack of differentiability of the matrix function (2.5) we cannot in general expect the existence of derivatives of order 3 and higher for solutions of (1.4). Theorem 1.1 and its proof are also valid for $\lambda = \Lambda$, $\mathscr{P}^{+}_{\Lambda,\Lambda}(D^2u) = \Lambda\, \Delta u$. To exclude the classical case of harmonic functions we will always assume that $0 < \lambda < \Lambda$. The proof of Theorem 1.1 is based on the scale invariance of (1.4) and the classical maximum principle. It uses a construction of Kichenassamy and Véron [22]. Concerning the condition $\Lambda/\lambda \leqslant n-1$ in Theorems 1.1, 1.2, see Remark 1.3. Of course, statements completely analogous to Theorem 1.1 hold for $\mathscr{P}^{+}_{\lambda,\Lambda}$ and e^{+}, or for $\mathscr{P}^{-}_{\lambda,\Lambda}$ and E^{-}, e^{-}, see (2.5)–(2.9).

Due to the Evans–Krylov $C^{2,\alpha}_{\mathrm{loc}}$ regularity results and the convexity of $\mathscr{P}^{+}_{\lambda,\Lambda}$ we could avoid the use of the viscosity solutions in the proof of

Theorem 1.1. For example, in the proof of the removability part of Theorem 1.1 we argue utilising a simple linearisation. For general equations (1.3) the best local regularity known is the Trudinger $C^{1,\alpha}$ regularity [31, 38, 39] (see also [5, Chap. 5]), so the linearisation cannot be employed. However, for the general nonlinear equation (1.3) we have the following generalisation of the Liouville theorem [9, 30] on removable singularities.

THEOREM 1.2. *Let $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$ be a solution to*

$$F(D^2 u) = 0 \qquad in \quad B_R \setminus \{0\}, \tag{1.7}$$

where $B_R \subset \mathbf{R}^n$, $n \geqslant 2$, and F is a uniformly elliptic operator with the ellipticity constants $0 < \lambda < \Lambda$, $1 < \Lambda/\lambda \leqslant n-1$. If

$$u(x) = o(E^+_{\Lambda/\lambda}(x)), \qquad x \to 0, \tag{1.8}$$

then the singularity at 0 *is removable and u is a solution of* (1.7) *in the entire ball B_R.*

The example of fundamental solutions $E^\pm$ of the operators $\mathscr{P}^\pm_{\lambda,\Lambda}$ shows that condition (1.8) is *sharp*. The main idea in the proof of Theorem 1.2 is purely viscosity in nature.

Remark 1.3. The condition $\Lambda/\lambda \leqslant n-1$ for operators $\mathscr{P}^\pm_{\lambda,\Lambda}(D^2u)$ (or for $F(D^2u)$) is analogous to the condition $p \leqslant n$ for the p-Laplacian (or to the well known growth restriction for general quasilinear operators in (1.1) [35]). For $\Lambda/\lambda > n-1$, the fundamental solution $E^+_{\Lambda/\lambda}$ for the operator $\mathscr{P}^\pm_{\lambda,\Lambda}$ is Hölder continuous at the nonremovable singularity, (2.8), as in the case for the fundamental solution for the p-Laplacian for $p > n$. Consequently, in removability statements like our Theorem 1.2, the absolute value restrictions are no longer sufficient for $\Lambda/\lambda > n-1$. Nevertheless, the ideas behind Theorems 1.1, 1.2 work for any Λ/λ. For example, using "tilting" arguments as in the proof of Theorem 1.2 and (2.7) it is not hard to show that if $|u(x) - u(0)| \leqslant C\,|x|^\beta$ for some $\beta > 1 - (\lambda(n-1)/\Lambda)$ then 0 is a removable singularity for (1.7), even when the ellipticity constants for F satisfy $\Lambda/\lambda > n-1$. The example of $\mathscr{P}^\pm_{\lambda,\Lambda}$ and $E^\pm$ shows that this condition on β is sharp. The proof of the characterisation (1.4)–(1.6) moreover can be easily adapted to embrace the case $\Lambda/\lambda > n-1$, see [22] for the case of the p-Laplacian with $p > n$.

In our Theorem 1.4 below we establish a characterisation of isolated singularities for nonlinear equations more general than (1.4). The proof of Theorem 1.4 relies on an idea different from that in the proof of Theorem 1.1. Namely we use an observation from [43], based on the moving plane method. We will also need the following conditions on the fully nonlinear uniformly elliptic operator F:

I. F is rotationally invariant, that is

$$F(TXT^t) = F(X) \qquad \text{for all} \quad X, T \in \mathbf{S}^n, \quad TT^t = I;$$

II. $F(X) = 0 \Rightarrow F(tX) = 0$ for all $t > 0$;

III. There exist two radial solutions E, e to $F(D^2u) = 0$ in $\mathbf{R}^n \setminus \{0\}$ such that

$$\lim_{x \to 0} E = -\lim_{x \to 0} e = +\infty.$$

THEOREM 1.4. *Let $u \in C_{\mathrm{loc}}(B_R \setminus \{0\})$, $u \geqslant 0$, be a solution to*

$$F(D^2u) = 0 \qquad \text{in } B_R \setminus \{0\}, \tag{1.9}$$

where the uniformly elliptic operator F satisfies **I–III**. *Then either the singularity at* 0 *is removable and u is a solution of* (1.9) *in the entire ball B_R, or there exists a real number $\gamma > 0$ such that*

$$u(x) = \gamma E(x) + O(1), \qquad x \to 0. \tag{1.10}$$

We remark that the asymptotic conditions (1.5), (1.10) are stronger than (1.2). Using the blow-up construction in the proof of Theorem 1.1, the estimate (1.10) can be refined further as in (1.6), if more information on the structure of E is given.

In Section 4 we discuss some properties of viscosity supersolutions u, that is

$$\mathscr{P}^+_{\lambda, \Lambda}(D^2u) \leqslant 0. \tag{1.11}$$

This material is related to the question whether it is possible to define $-\mathscr{P}^+_{\lambda, \Lambda}(D^2E^+_{\Lambda/\lambda})$ as the Dirac mass δ. The set of all supersolutions u of (1.11) is a convex cone [5, Chap. 5]. In the case $\lambda = \Lambda$ this cone is exactly the cone of the classical superharmonic functions [19, Chap. 3]. For superharmonic functions the expression $-\mathscr{P}^+_{1,1}(D^2u) = -\Delta u$ can be defined as a Radon measure, weakly* continuous with respect to L^1_{loc} convergence, see e.g. [19, Chap. 3].

Recently Trudinger and Wang [41–43] generalised this classical result as follows. They defined fully nonlinear expressions $F_k[u]$ as Radon measures for viscosity subsolutions u, $F_k[u] \geqslant 0$, $k = 1, \ldots, n$. Here the k-Hessian operator $F_k[u]$ is the kth elementary symmetric function of the eigenvalues of the Hessian matrix $[D^2u]$, $F_1[u] = \Delta u$, $F_n[u] = \det D^2u$. Trudinger and Wang proved weak* continuity of $F_k[u]$ with respect to L^1_{loc} convergence, and existence and uniqueness theorems for the Dirichlet problem. In Section 4, using an idea from [42] we prove that if u is a viscosity supersolution of (1.11) with $0<\lambda<\Lambda$, then $D_{ij}u$ are signed Radon measures for all $i, j=1, \ldots, n$. Then we define the expression $-\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u)$ as a Radon measure and prove an upper semicontinuity result for this measure with respect to L^1_{loc} convergence. Properties of $-\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u)$ on the cone of supersolutions are analogous to properties of the operator

$$\Phi[u]=\left(\det\left[\frac{\partial^2 u}{\partial z_j\,\partial \bar{z}_k}\right]\right)^{1/n} \tag{1.12}$$

on the cone of plurisubharmonic functions in $\mathbf{C}^n$, $n\geqslant 2$, discussed by Bedford and Taylor in [2, Sect. 5].

2. PRELIMINARIES

First we recall some well known facts from viscosity theory. The viscosity notion of generalised solution to fully nonlinear equations was introduced and investigated by Crandall, Lions, Evans, Jensen, Ishii, and others. See the surveys [11, 13] for exposition, history and bibliography. For fully nonlinear uniformly elliptic equations of the form

$$F(D^2u)=0 \tag{2.1}$$

(and for more general equations) their existence and uniqueness results, together with the regularity results of Trudinger [38, 39], Caffarelli [4], and others, form a complete theory; also see [11, 13], and the monograph [5]. For (2.1) with concave or convex F (in particular for Eq. (2.10)), the questions of existence, uniqueness and regularity were first investigated by Lions utilising a connection with stochastic control theory, see [27–29].

Consider Eq. (2.1), where the real function u is defined in a bounded domain $\Omega\subset\mathbf{R}^n$, $n\geqslant 2$, and F is a uniformly elliptic operator. We say that a polynomial P of degree 2 touches the function $f\colon\Omega\to\mathbf{R}^1$ above at $x_0\in\Omega$ if there is a neighbourhood $B(x_0,\varepsilon)$, $\varepsilon>0$ such that

$$f\leqslant P \qquad \text{in } B(x_0,\varepsilon) \qquad \text{and} \qquad f(x_0)=P(x_0). \tag{2.2}$$

Similarly P touches f below if $P\leqslant f$ in $B(x_0,\varepsilon)$, $f(x_0)=P(x_0)$. An upper semicontinuous function $u\colon\Omega\to\mathbf{R}^1\cup\{-\infty\}$ (resp. lower semicontinuous function $u\colon\Omega\to\mathbf{R}^1\cup\{+\infty\}$) is a *viscosity subsolution* (resp. *viscosity supersolution*) of (2.1) in Ω when the following condition holds:

if $x_0\in\Omega$, and P is any polynomial of degree 2 touching u above at x_0 then $F(D^2P(x_0))\geqslant 0$

(respectively if P touches u below at x_0 then $F(D^2P(x_0))\leqslant 0$).

We say that u is a *viscosity solution* of (2.1) when it is simultaneously a subsolution and a supersolution (in particular u is continuous). We say that $F(D^2u) \geqslant 0$ (resp. $\leqslant 0$, $=0$) in the viscosity sense in Ω whenever u is a viscosity subsolution (resp. supersolution, solution) of (2.1) in Ω. In what follows, by a solution we always mean a viscosity solution.

This definition is taken from [5, Chap. 2]. Definitions in [13, Sect. 2], [11, Sect. 2] are slightly different and use arbitrary C^2_{loc} functions instead of polynomials of degree 2. However, all the definitions are equivalent, see [5, Chap. 2].

We refer to [20] for the theory of viscosity solutions and to [6, 37] for L^p-viscosity solutions of the general equations $F(x, u, Du, D^2u) = 0$, with respectively continuous and measurable dependence on x. A complete treatment of the parabolic counterpart can be found in [12].

The comparison principle holds for viscosity solutions of (2.1), [5, Chap. 5; 21]. That is, for $u, v \in C(\bar{\Omega})$, $F(D^2u) \geqslant 0$, $F(D^2v) \leqslant 0$, the inequality $u|_{\partial\Omega} \leqslant v|_{\partial\Omega}$ implies $u \leqslant v$ in Ω.

Let F be a uniformly elliptic operator. If $u \in C_{\mathrm{loc}}(B_{3\rho/2})$, $u \geqslant 0$ in $B_{3\rho/2}$, is a viscosity solution to (2.1) in $B_{3\rho/2}$, then the the Krylov–Safonov Harnack inequality [5, Chap. 6], holds:

$$\sup_{B_\rho} u \leqslant C(\lambda, \Lambda, n)(\inf_{B_\rho} u + \rho \, \|F(0)\|_{L^n(B_{3\rho/2})}). \tag{2.3}$$

Inequality (2.3) for classical and strong solutions of (2.1) was proved in [33]; see also [26, Chap. 4; 17, Chap. 17]. For quasilinear equations (1.1), the Harnack inequality was established by Serrin earlier in [34].

If F is additionally convex or concave on $\mathbf{S}^n$ then solutions to (2.1) enjoy $C^{2,\alpha}_{\mathrm{loc}}$ regularity. Moreover, the following Evans–Krylov estimate [14, 25], holds. If $u \in C_{\mathrm{loc}}(B_1)$ is a solution to (2.1) in B_1, then

$$\|u\|_{C^{2,\alpha}(\bar{B}_{1/2})} \leqslant C \, \|u\|_{L^\infty(B_1)}, \tag{2.4}$$

where $0 < \alpha < 1$ and $C = C(\lambda, \Lambda, n) > 0$. The proof of (2.4) can be found for classical solutions in [15; 17, Chap. 17; 40], and for viscosity solutions in [4, 5].

Important examples of fully nonlinear uniformly elliptic operators are the *Pucci extremal operators* $\mathscr{P}^{\pm}_{\lambda,\Lambda}(M)$, $M \in \mathbf{S}^n$. If μ_j, $j = 1, \ldots, n$ are the eigenvalues of M, and $0 < \lambda \leqslant \Lambda$ then (see [5, Chap. 2])

$$\mathscr{P}^{+}_{\lambda,\Lambda}(M) = \sup_{\lambda I \leqslant A \leqslant \Lambda I} \left(\sum_{i,j=1}^{n} A_{ij} M_{ij} \right) = \Lambda \sum_{\mu_j > 0} \mu_j + \lambda \sum_{\mu_j < 0} \mu_j, \tag{2.5}$$

$$\mathscr{P}^{-}_{\lambda,\Lambda}(M) = \inf_{\lambda I \leqslant A \leqslant \Lambda I} \left(\sum_{i,j=1}^{n} A_{ij} M_{ij} \right) = \lambda \sum_{\mu_j > 0} \mu_j + \Lambda \sum_{\mu_j < 0} \mu_j. \tag{2.6}$$

For arbitrary uniformly elliptic operator F with the ellipticity constants $0<\lambda\leqslant\Lambda$, the following property holds for viscosity sub- and supersolutions, [5, Chap. 2]:

$$\begin{aligned} F(D^2u)\geqslant 0 &\Rightarrow \mathcal{P}^+_{\lambda,\Lambda}(D^2u)\geqslant -F(0)\\ F(D^2u)\leqslant 0 &\Rightarrow \mathcal{P}^-_{\lambda,\Lambda}(D^2u)\leqslant -F(0). \end{aligned} \tag{2.7}$$

The *fundamental solutions* E^+, e^+ to the operator $\mathcal{P}^+_{\lambda,\Lambda}$ are defined by

$$E^+(x)=E^+_{\Lambda/\lambda}(x)=\begin{cases} \dfrac{1}{|x|^{(n-1)\lambda/\Lambda-1}} & \text{if} \quad 1\leqslant\Lambda/\lambda<n-1\\ -\log|x| & \text{if}\quad \Lambda/\lambda=n-1\\ -|x|^{1-(n-1)\lambda/\Lambda} & \text{if}\quad n-1<\Lambda/\lambda, \end{cases} \tag{2.8}$$

$$e^+(x)=e^+_{\Lambda/\lambda}(x)=\begin{cases} \dfrac{-1}{|x|^{(n-1)\Lambda/\lambda-1}} & \text{if}\quad \Lambda/\lambda\geqslant 1 \quad\text{and}\quad n\geqslant 3\\ \dfrac{-1}{|x|^{\Lambda/\lambda-1}} & \text{if}\quad \Lambda/\lambda>1\quad\text{and}\quad n=2\\ \log|x| & \text{if}\quad \Lambda=\lambda\quad\text{and}\quad n=2. \end{cases} \tag{2.9}$$

Note that $E^+_{\Lambda/\lambda}\neq -e^+_{\Lambda/\lambda}$ if $\Lambda/\lambda>1$. Using the rotational invariance of the Pucci extremal operators, it is easy to check that E^+, e^+ satisfy the equation

$$\mathcal{P}^+_{\lambda,\Lambda}(D^2u)=0 \tag{2.10}$$

in $\mathbf{R}^n\setminus\{0\}$. As a direct consequence of the comparison principle in spherical shells any radial solution to (2.10) in $\mathbf{R}^n\setminus\{0\}$ has either the form aE^++b, or ae^++b, where $a\geqslant 0$, $b\in\mathbf{R}^1$. We define the fundamental solutions E^-, e^-, to the operator $\mathcal{P}^-_{\lambda,\Lambda}$ by

$$E^-=E^-_{\Lambda/\lambda}=-E^+_{\Lambda/\lambda},\qquad e^-=e^-_{\Lambda/\lambda}=-e^+_{\Lambda/\lambda}. \tag{2.11}$$

In this paper we will consider only the operator $\mathcal{P}^+_{\lambda,\Lambda}$. Using the equality

$$\mathcal{P}^+_{\lambda,\Lambda}(M)=-\mathcal{P}^-_{\lambda,\Lambda}(-M) \tag{2.12}$$

it is easy to formulate and prove results for $\mathcal{P}^-_{\lambda,\Lambda}$ parallel to the results for $\mathcal{P}^+_{\lambda,\Lambda}$.

In [32] Pucci introduced the slightly different extremal operators

$$M_\alpha(m_\alpha)(D^2u)=\sup(\inf)\left(\sum_{i,j=1}^n A_{ij}D_{ij}u\right),$$

where sup (inf) is taken over all $A \in \mathbf{S}^n$ such that $\operatorname{Tr} A = 1$, $A \geqslant \alpha I$, $0 < \alpha \leqslant 1/n$. See also [17, Chap. 17]. The operators M_α, m_α have fundamental solutions of type (2.8), (2.9). The results of this paper can be obtained by the same methods for M_α, m_α, and similar operators.

3. PROOFS OF THE RESULTS

Proof of Theorem 1.1. We employ only classical solutions to (1.4). The plan of the proof is as follows. First, using linearisation and properties of the classical solutions, we show that either the singularity at 0 is removable or

$$E^+/C \leqslant u \leqslant CE^+ \qquad \text{in } B_{R/2} \setminus \{0\}. \tag{3.1}$$

Then we will refine (3.1) and prove (1.5), (1.6).

We claim that for u satisfying (1.4) there exists

$$\lim_{x \to 0} u(x) = u_0, \qquad 0 \leqslant u_0 \leqslant +\infty. \tag{3.2}$$

In fact, let

$$u_0 = \liminf_{x \to 0} u(x).$$

If $u_0 = +\infty$, then from the definition $u(x) \to +\infty$, $x \to 0$. If $u_0 < +\infty$, consider the function

$$u_\varepsilon = u - u_0 + \varepsilon, \qquad \varepsilon > 0.$$

It satisfies (1.4) in $B_R \setminus \{0\}$ and is positive in some neighbourhood of 0. By the Harnack inequality (2.3)

$$\max_{\partial B_{r_j}} u_\varepsilon \leqslant C(n)\, \varepsilon$$

for a monotone sequence $\{r_j\}$, $r_j \to 0$. Due to the comparison principle for classical solutions in shells $B_{r_j} \setminus B_{r_{(j+1)}}$ we conclude by letting $\varepsilon \to 0$ that

$$\limsup_{x \to 0} u(x) = u_0.$$

We show that (3.1) holds in the case $u_0 = +\infty$. Let

$$m_j = \min_{\partial B_{R4^{-j}}} u, \qquad M_j = \max_{\partial B_{R4^{-j}}} u, \qquad j = 1, 2 \ldots.$$

We define α_j, $\alpha_j \geqslant 0$, such that

$$m_j = \alpha_j (E^+(x) - E^+(R/2)), \qquad |x| = R4^{-j}.$$

If the sequence $\{\alpha_j\}$ is unbounded then the comparison principle applied to u and $\alpha_j(E^+ - E^+(R/2))$ in the shells $B_{R/2} \setminus B_{R4^{-j}}$ implies $u \equiv +\infty$, a contradiction. Thus $\alpha_j \leqslant C_1$ for some constant $C_1 > 0$. Combining this with the Harnack inequality (2.3) for classical solutions, we obtain

$$M_j \leqslant C(\lambda, \Lambda, n)\, m_j \leqslant C_2 (E^+(x) - E^+(R/2)) \qquad \text{for all} \quad x \in \partial B_{R4^{-j}}.$$

Using the comparison principle, we conclude that

$$u \leqslant CE^+ \qquad \text{in} \quad B_{R/2} \setminus \{0\}.$$

The proof of the lower bound in (3.1) is the same.

To finish the first part of the proof we now show that in the case $u_0 < +\infty$ in (3.2) the singularity is removable. The proof is similar to the linear case [44, Chap. 1]. Consider the function $v \in C^{2,\alpha}(\bar{B}_{R/2})$ such that

$$\begin{cases} \mathcal{P}^+_{\lambda, \Lambda}(D^2 v) = 0 & \text{in } B_{R/2} \\ v = u & \text{on } \partial B_{R/2}. \end{cases}$$

The function $(u - v) \in C^2_{\text{loc}}(B_{R/2} \setminus \{0\}) \cap L^\infty(B_{R/2})$ satisfies the equation

$$\sum_{i,j=1}^n A_{ij}(x)\, D_{ij}(u - v) = 0 \qquad \text{in } B_{R/2} \setminus \{0\},$$

where the measurable coefficients A_{ij}, $\lambda I \leqslant [A_{ij}] \leqslant \Lambda I$, are obtained by linearisation, that is

$$A_{ij}(x) = \int_0^1 \frac{\partial}{\partial r_{ij}} \mathcal{P}^+_{\lambda, \Lambda}(t D^2 u(x) + (1 - t)\, D^2 v(x))\, dt. \tag{3.3}$$

By definition, the function $E^+ = E^+_{\Lambda/\lambda}$ satisfies

$$\sum_{i,j=1}^n A_{ij}(x)\, D_{ij} E^+ \leqslant 0 \qquad \text{in } B_{R/2} \setminus \{0\}.$$

Using comparison principle for $(u - v)$ and εE^+, $\varepsilon \to 0$, we obtain $u \leqslant v$ in $B_{R/2} \setminus \{0\}$. The opposite inequality is proved in the same way. We conclude that

$$u = v \qquad \text{in } B_{R/2} \setminus \{0\}$$

and the singularity at 0 is removable.

We have proved that either the singularity at 0 is removable or (3.1) holds. In the remaining part of the proof we show that (3.1) can be refined and (1.5), (1.6) hold.

The scale invariance of the equation in (1.4) and the Evans–Krylov estimate (2.4) give the following: let y, z satisfy $0<|y|<|z|<1$, and let the function f satisfy (1.4) in $B_3\setminus\{0\}$. Then

$$|Df(y)|\leqslant C\,\frac{\|f\|_{L^\infty(B_2\setminus B_{|y|/2})}}{|y|},\tag{3.4}$$

$$|D^2f(y)|\leqslant C\,\frac{\|f\|_{L^\infty(B_2\setminus B_{|y|/2})}}{|y|^2},\tag{3.5}$$

$$|D^2f(y)-D^2f(z)|\leqslant C\,\frac{\|f\|_{L^\infty(B_2\setminus B_{|y|/2})}|y-z|^\alpha}{|y|^{2+\alpha}},\tag{3.6}$$

where $0<\alpha<1$, $C=C(\lambda,\Lambda,n)$.

In the proof of (1.5) and (1.6) we treat the cases $1<\Lambda/\lambda<n-1$ and $1<\Lambda/\lambda=n-1$ separately.

Case (i). $1<\Lambda/\lambda<n-1$. Due to the invariance of the equation we can assume that $R=1/2$ and $\max_{\partial B_{1/4}}u=0$. Let

$$\Gamma(r)=\max_{r\leqslant|x|\leqslant 1/4}(u(x)/E^+(x)),\qquad 0<r<1/4.\tag{3.7}$$

From the comparison principle there exists x_r, $|x_r|=r$, such that

$$\Gamma(r)=\max_{\partial B_r}(u/E^+)=u(x_r)/E^+(x_r).\tag{3.8}$$

Consequently $\Gamma(r)\uparrow\gamma$ as $r\downarrow 0$, where

$$\gamma=\overline{\lim_{x\to 0}}\,(u(x)/E^+(x)),\qquad \gamma>0.\tag{3.9}$$

We introduce the function $v_r\colon B_{1/(4r)}\setminus\{0\}\to\mathbf{R}$, by

$$v_r(x)=u(rx)/E^+(a_r),\qquad |a_r|=r.\tag{3.10}$$

The function v_r satisfies equation in (1.4) in $B_{1/(4r)}\setminus\{0\}$, and from (3.1), (3.7), (3.9)

$$0\leqslant v_r\leqslant\gamma E^+\qquad \text{in } B_2\setminus\{0\}\tag{3.11}$$

for all $r>0$ small enough. Let $K\subset B_2$ be a compact set, $0\notin K$. From (3.11) and (3.4)–(3.6)

$$\|v_r\|_{C^{2,\alpha}(K)}\leqslant\gamma C_K\qquad\text{for all}\quad r\in(0,1/4),\tag{3.12}$$

where the constant $C_K = C_K(\lambda, \Lambda, n, \operatorname{dist}(0, K))$ is independent of r. From (3.12) we can find a sequence $r_j \to 0$ such that $v_{r_j} \to v$ in $C^2_{\mathrm{loc}}(B_2 \setminus \{0\})$. The function v satisfies equation (1.4) in $B_2 \setminus \{0\}$. Moreover, from (3.11)

$$0 \leqslant v(x) \leqslant \gamma E^+(x) \qquad \text{for all} \quad x \in B_2 \setminus \{0\}. \tag{3.13}$$

Let $\xi_j = x_{r_j}/r_j$, $|\xi_j| = 1$, where x_r is given in (3.8). We can assume $\xi_j \to \xi$, $|\xi| = 1$. From (3.8)–(3.10) we have

$$\frac{v_{r_j}(\xi_j)}{E^+(\xi_j)} = \Gamma(r_j) \to \gamma, \qquad j \to \infty.$$

Thus

$$\frac{v(\xi)}{E^+(\xi)} = \gamma, \qquad |\xi| = 1. \tag{3.14}$$

From (3.13) and (3.14), due to the strong maximum principle, [17, Chap. 2],

$$v(x) = \gamma E^+(x) \qquad \text{for all} \quad x \in B_2 \setminus \{0\}.$$

Thus in particular the limit function v is independent of the sequence $\{r_j\}$. Consequently $v_r \to \gamma E^+$ as $r \to 0$ in $C^2_{\mathrm{loc}}(B_2 \setminus \{0\})$. Hence if we put $D^0 u = u$, then

$$D^\beta v_r \to \gamma D^\beta E^+ \qquad \text{as} \quad r \to 0, \quad \text{uniformly on } \partial B_1,$$

for $0 \leqslant |\beta| \leqslant 2$. Thus

$$\lim_{x \to 0} \left(\frac{D^\beta u(x)}{E^+(x)\, |x|^{-|\beta|}} \right) = \gamma (D^\beta E^+) \left(\frac{x}{|x|} \right). \tag{3.15}$$

For $1 \leqslant |\beta| \leqslant 2$, (1.6) then follows from (3.15).

In case $\beta = 0$ it is possible to strengthen (3.15). Consider the functions

$$V_\varepsilon^+(x) = (\gamma + \varepsilon)\, E^+(x) - (\gamma + \varepsilon)\, E^+(z_0) + \max_{\partial B_{1/2}} u,$$

$$V_\varepsilon^-(x) = (\gamma - \varepsilon)\, E^+(x) - (\gamma - \varepsilon)\, E^+(z_0) + \min_{\partial B_{1/2}} u,$$

where $|z_0| = 1/2$. From the comparison principle we get $V_\varepsilon^- \leqslant u \leqslant V_\varepsilon^+$, and letting ε go to 0 then give (1.5).

Case (ii). $1 < \Lambda/\lambda = n-1$. First we proceed as in the case $\Lambda/\lambda < n-1$, again assuming that

$$R = 1/2, \qquad \max_{\partial B_{1/4}} u = 0.$$

Repeating the arguments (3.7)–(3.9), we conclude that

$$u(x) \leqslant \gamma E^+_{n-1}(x) \qquad \text{for all} \quad x \in B_{1/4} \setminus \{0\},$$

where γ is defined by (3.9). Combining this with (3.1), we conclude that for some $R_0 > 0$

$$0 \leqslant u(x) \leqslant \gamma E^+_{n-1}(x) \qquad \text{for all} \quad x \in B_{R_0} \setminus \{0\}. \tag{3.16}$$

Consider the function v_r defined by (3.10) with $E^+ = E^+_{n-1}$. Using (3.16) and (3.4)–(3.6), we obtain (3.12) for any compact set $K \subset \mathbf{R}^n \setminus \{0\}$. Consequently we can find a sequence $\{r_j\}$, $r_j \to 0$ when $j \to \infty$, such that

$$v_{r_j} \to \tilde{v} \qquad \text{in } C^2_{\text{loc}}(\mathbf{R}^n \setminus \{0\}).$$

The function $\tilde{v}$ satisfies equation (1.4) in $\mathbf{R}^n \setminus \{0\}$. Moreover, (3.16) gives

$$\begin{aligned} 0 \leqslant \tilde{v}(x) &= \lim_{j\to\infty} v_{r_j}(x) = \lim_{j\to\infty} \frac{u(r_j x)}{-\log r_j} \\ &\leqslant \lim_{j\to\infty} \frac{\gamma(-\log r_j - \log|x|)}{-\log r_j} = \gamma \qquad \text{for all} \quad x \in \mathbf{R}^n \setminus \{0\}. \end{aligned}$$

Thus $\tilde{v}$ is a bounded solution to (1.4) in $\mathbf{R}^n \setminus \{0\}$. It does not satisfy (3.1). Thus, as we have already proved, the singularity at 0 is removable. The Harnack inequality (2.3) applied to the solution $\tilde{v}$ gives $\tilde{v}(x) \equiv b \leqslant \gamma$. (For this Liouville theorem see [5, Chap. 4].) For the sequence $\{\xi_j\}$, $\xi_j = x_{r_j}/r_j$, $|\xi_j| = 1$, where x_r is defined in (3.8), we have

$$v_{r_j}(\xi_j) \to \gamma, \qquad j \to \infty.$$

We conclude that $b = \gamma$. Consequently

$$v_r \to \gamma \qquad \text{in} \quad C^2_{\text{loc}}(\mathbf{R}^n \setminus \{0\}) \quad \text{as} \quad r \to 0. \tag{3.17}$$

As in the case $\Lambda/\lambda < n-1$, this gives

$$u(x) = -\gamma \log|x| + O(1), \qquad x \to 0. \tag{3.18}$$

Thus (1.5) proved. However, the estimates for v_{r_j} contained in (3.17) do not give (1.6).

To prove the asymptotic estimates (1.6) for the derivatives it is enough to show that there exists a constant $N \in \mathbf{R}$ such that

$$\|u(r\,\cdot\,)-\gamma E^{+}(r\,\cdot\,)-N\|_{C^2(\overline{B_2\setminus B_1})} \to 0 \qquad \text{when} \quad r\to 0.$$

We establish this convergence by showing that:

- For any sequence $\{\rho_j\}$, $\rho_j \to 0$ when $j\to\infty$, there exists a subsequence $\{r_j\}$, $r_k=\rho_{j_k}$, and a function $\varphi\in C^2(\overline{B_2\setminus B_1})$ such that

$$\|u(r_j\,\cdot\,)-\gamma E^{+}(r_j\,\cdot\,)-\varphi\|_{C^2(\overline{B_2\setminus B_1})} \to 0, \qquad j\to\infty. \tag{3.19}$$

- There exists a constant $N\in\mathbf{R}$ such that

$$\|u(r\,\cdot\,)-\gamma E^{+}(r\,\cdot\,)-N\|_{L^\infty(B_2\setminus B_1)} \to 0, \qquad r\to 0. \tag{3.20}$$

Of course, it follows from (3.19), (3.20) that $\varphi\equiv N$. In the remaining part of the proof we establish assertions (3.19), (3.20).

We introduce for $0<r<1/4$ the function

$$w_r(x)=u(rx)-\gamma(-\log r), \tag{3.21}$$

defined in $B_{1/4r}\setminus\{0\}$. Let us first obtain some estimates for the function w_r and its derivatives. We have from (3.18)

$$-C+\gamma E^{+}(x)\leqslant w_r(x)\leqslant C+\gamma E^{+}(x), \tag{3.22}$$

where

$$C=\sup_{0<|x|<1/4}|u(x)-\gamma E^{+}(x)|.$$

Using estimates (3.22) and (2.4) for the function w_r we obtain

$$\|w_r\|_{C^{2,\alpha}(\{x:\,1/2\leqslant|x|\leqslant 2\})}\leqslant C\,\|w_r\|_{L^\infty(\{x:\,1/4\leqslant|x|\leqslant 3\})}, \tag{3.23}$$

C is independent of r. Estimate (3.23) leads us to the estimate for $C^{2,\alpha}$-norm of u, and then definition (3.21) leads us back to the estimates for w_r. The latter are:

$$|D^\beta w_r(x)|\leqslant C\,\frac{1}{|x|^{|\beta|}}, \qquad 1\leqslant|\beta|\leqslant 2, \tag{3.24}$$

$$|D^2 w_r(x)-D^2 w_r(y)|\leqslant C\,\frac{|x-y|^\alpha}{|x|^{2+\alpha}}, \qquad 0<\alpha<1, \tag{3.25}$$

where $0<|x|<|y|<1/(8r)$ and $C=C(u)$ is independent of r.

From (3.22), (3.24), (3.25) for any sequence $\{\rho_j\}$, $\rho_j \to 0$, there is a subsequence $\{r_j\}$, $r_k = \rho_{j_k}$, $0 < r_k < 1/4$, such that

$$w_{r_j} \to w \qquad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\}), \qquad j \to \infty, \tag{3.26}$$

or, equivalently,

$$u(r_j \cdot) - \gamma E^+(r_j \cdot) \to w(\cdot) - \gamma E^+(\cdot) \qquad \text{in } C^2_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\}), \quad j \to \infty, \tag{3.27}$$

for a function $w \in C^2_{\mathrm{loc}}(\mathbf{R}^n \setminus \{0\})$. Thus (3.19) is proved with $\varphi = w - \gamma E^+$.

Now we pass to the proof of (3.20). The function w (chosen for a fixed $\{r_j\}$ in (3.26)) is a solution to (1.4) in $\mathbf{R}^n \setminus \{0\}$ due to the Evans–Krylov estimate (2.4). We also have

$$|w(x) - \gamma E^+(x)| \leqslant C, \qquad x \neq 0,$$

with C independent on x. The bounded, smooth function $w - \gamma E^+$ satisfies in $\mathbf{R}^n \setminus \{0\}$ the linear homogeneous uniformly elliptic equation obtained by linearisation (3.3) of (1.4). Thus by the Krylov–Safonov Harnack inequality for the strong solutions [26, Chap. 4; 33] we obtain

$$\lim_{x \to 0} (w - \gamma E^+) = M.$$

The assertion (3.20) will be proved with $N = M$ if we establish that

$$\lim_{x \to 0} (u(x) - \gamma E^+(x)) = M. \tag{3.28}$$

In particular (3.28) implies that M does not depend on $\{r_j\}$. We remark that (3.28) is a refinement of (3.18). The same refinement holds for (1.5) with $1 < \Lambda/\lambda < n - 1$.

For the proof of (3.28) we fix an arbitrary $\varepsilon > 0$ and choose $\rho_0 > 0$ so small that

$$|w - \gamma E^+ - M| \leqslant \varepsilon \qquad \text{on } \partial B_{\rho_0}.$$

Combining this with (3.27), we obtain

$$\gamma E^+ + M - 2\varepsilon \leqslant u \leqslant \gamma E^+ + M + 2\varepsilon \qquad \text{on } \partial B_{r_j \rho_0}$$

for all j large enough. At the same time, both u and $\gamma E^+ + M + \text{const}$ are solutions to

$$\mathscr{P}^+_{\lambda, \Lambda}(D^2 U) = 0 \qquad \text{in } B_{1/2} \setminus \{0\}.$$

From the comparison principle in spherical shells, we conclude that

$$\limsup_{x \to 0} |u(x) - \gamma E^+(x) - M| \leqslant 2\varepsilon.$$

Thus (3.20) holds. ∎

Proof of Theorem 1.2. The best known regularity results for solutions of the general equation (1.7) are the Trudinger $C^{1,\alpha}$ estimates, [31, 39]. Thus we cannot use linearisation as in the proof of Theorem 1.1. The proof of Theorem 1.2 consists of two steps. The first step is to show that any function u satisfying (1.7), (1.8) can be defined at 0 by continuity. The second step is to verify that this continuous function is the viscosity solution to equation (1.7) in the entire ball B_R.

To show the existence of

$$\lim_{x \to 0} u(x) = u_0, \qquad -\infty < u_0 < +\infty, \tag{3.29}$$

we first prove that $u \in L^\infty(B_{R/2})$. In fact, (1.7) (2.7) imply

$$\mathscr{P}^+_{\lambda, \Lambda}(D^2 u) \geqslant -F(0) \qquad \text{in } B_R \setminus \{0\}.$$

The uniform ellipticity of F implies that for any $A, a > 0$

$$\mathscr{P}^+_{\lambda, \Lambda}(D^2(aE^+(x) - A\,|x|^2)) \leqslant -\lambda 2An \qquad \text{in } B_R \setminus \{0\}.$$

Consequently, from (1.8) and the comparison principle for viscosity solutions, we conclude that for any $x \in B_{R/2} \setminus \{0\}$ and all sufficiently small $a > 0$

$$u(x) \leqslant aE^+_{\Lambda/\lambda}(x) - \frac{|F(0)|}{\lambda}\,|x|^2 + \frac{|F(0)|}{\lambda}\,(R/2)^2 + \max_{\partial B_{R/2}} u.$$

Letting $a \to 0$ we see that u is bounded from above in $B_{R/2}$. A bound from below is obtained by using the same arguments with $\mathscr{P}^-_{\lambda, \Lambda}$ and $E^-_{\Lambda/\lambda}$, $|E^-| = |E^+|$. We conclude the proof of (3.29) by using the Harnack inequality (2.3) and the comparison principle in the same way as in the proof of (3.2). But this time we need (2.3) and the comparison principle for viscosity solutions.

From (3.29) we can assume that $u \in C_{\mathrm{loc}}(B_R)$. We show that this function satisfies

$$F(D^2 u) = 0 \qquad \text{in } B_R$$

in the viscosity sense. Let us show that u is a viscosity *subsolution*. (The proof that u is a supersolution is the same.) We need to establish that

$$F(D^2 P) \geqslant 0 \tag{3.30}$$

for any quadratic polynomial P touching u above at 0. Let P be a quadratic polynomial such that (2.9) holds for $f=u$, $x_0=0$. Due to the invariance of the equation, we can assume that $u(0)=P(0)=0$, $DP(0)=0$.

First we note that u cannot have a strict local maximum at 0. To see this, suppose that there is a ball of radius $r<\min\{R/2, 1\}$ such that

$$\max_{\partial B_r} u = M < 0.$$

Consider the sequence of functions $M+(E^+/j)$, $j \geqslant 1$. Let $0<|x_0|<r$. Due to the comparison principle in the shell $B_r \setminus B_{\rho_j}$ for sufficiently small $\rho_j>0$ one has

$$u(x_0) \leqslant M + E^+(x_0)/j \qquad \text{for all} \quad j \geqslant 1.$$

When $j\to\infty$ we obtain $u \leqslant M<0$ in $B_r\setminus\{0\}$, which contradicts the continuity of u at 0. Consequently we can assume that there is a sequence $\{z_j\}$, $z_j\to 0$, $j\to\infty$, such that all coordinates of every z_j are positive and

$$u(z_j) \geqslant 0, \qquad j=1,2,\ldots. \tag{3.31}$$

Now we establish (3.30). Fix any $\delta>0$. There exists $r_0=r_0(\delta)>0$ such that the polynomial $P_\delta(x)=P(x)+\delta\,|x|^2/2$ satisfies

$$P_\delta(0)=u(0), \qquad P_\delta(x)>u(x) \quad \text{for any} \ x\in B_{r_0}.$$

Consequently there exists $\varepsilon=\varepsilon(\delta)>0$ such that, for the polynomial

$$\begin{aligned} P_{\delta,\varepsilon}(x) &= P_\delta(x)-\varepsilon(x_1+\cdots+x_d) \\ &= P(x)+\delta\,|x|^2/2-\varepsilon(x_1+\cdots+x_d), \end{aligned}$$

we have

$$u(0)-P_{\delta,\varepsilon}(0)=0, \ (u-P_{\delta,\varepsilon})<0 \qquad \text{on } \partial B_{r_0}. \tag{3.32}$$

From (3.31) we obtain

$$u(z_j)-P_{\delta,\varepsilon}(z_j)>0$$

for z_j and sufficiently large j. Thus from (3.32) we can find a point $x^\varepsilon\in B_{r_0}$, $x^\varepsilon\neq 0$, such that

$$u(x^\varepsilon)-P_{\delta,\varepsilon}(x^\varepsilon)=\max_{B_{r_0}}(u-P_{\delta,\varepsilon})>0. \tag{3.33}$$

From (3.32), (3.33), the polynomial

$$Q_{\delta,\varepsilon}(x) = P_{\delta,\varepsilon}(x) + u(x^\varepsilon) - P_{\delta,\varepsilon}(x^\varepsilon)$$

touches u above at x^ε, $x^\varepsilon \neq 0$. The function u satisfies (1.7) in the viscosity sense. Consequently

$$0 \leqslant F(D^2 Q_{\delta,\varepsilon}) = F(D^2 P + \delta I).$$

Letting $\delta \to 0$ we obtain (3.30). ∎

Proof of Theorem 1.4. Following the beginning of the proof of Theorem 1.1, we see that either

$$\lim_{x\to 0} u(x) = u_0, \qquad -\infty < u_0 < +\infty,$$

or

$$E/C \leqslant u \leqslant CE \qquad \text{in } B_{R/2} \setminus \{0\} \tag{3.34}$$

for some constant $C > 0$. (Note that in order to repeat the arguments of the corresponding part of the proof of Theorem 1.1 it is necessary use conditions **II** and **III**.)

In case $-\infty < u_0 < +\infty$, we can follow the proof of Theorem 1.2 and establish that

$$F(D^2 u) = 0 \qquad \text{in } B_R$$

in the viscosity sense. (Note that repeating the arguments of the corresponding part of the proof of Theorem 1.2 we also use conditions **II** and **III**.)

In the remaining part of the proof we will refine (3.34) and prove the asymptotic estimate (1.10). We can assume that $\min_{\partial B_{R/2}} u = 0$, and set $M = \max_{\partial B_{R/2}} u$. Consider the sequence $\{v_j\}$, where $v_j \in C(\bar{B}_{R/2} \setminus B_{2^{-j}R})$ is defined as follows:

$$\begin{aligned} F(D^2 v_j) &= 0 && \text{in } B_{R/2} \setminus \bar{B}_{2^{-j}R} \\ v_j &= M && \text{on } \partial B_{R/2} \\ v_j &= u && \text{on } \partial B_{2^{-j}R}. \end{aligned} \tag{3.35}$$

From the comparison principle

$$v_j - M \leqslant u \leqslant v_j \qquad \text{in } B_{R/2} \setminus B_{2^{-j}R}. \tag{3.36}$$

We claim that there exists a subsequence $\{j_k\}$ and a function $U \in C_{\text{loc}}(B_{R/2} \setminus \{0\})$ such that

$$v_{j_k} \to U \qquad \text{in } C_{\text{loc}}(B_{R/2} \setminus \{0\}). \tag{3.37}$$

Accepting this assertion we now finish the proof of the theorem.

From (3.36), (3.37)

$$U - M \leqslant u \leqslant U \qquad \text{in } B_{R/2} \setminus \{0\}. \tag{3.38}$$

In particular $U(x) \to +\infty$ when $x \to 0$. From (3.35), (3.37), and the stability of the viscosity notions with respect to uniform convergence, we obtain

$$\begin{cases} F(D^2U) = 0 & \text{in } B_{R/2} \setminus \{0\} \\ U = M & \text{on } \partial B_{R/2} \\ U(x) \to +\infty & \text{if } x \to 0. \end{cases} \tag{3.39}$$

The rotational invariance condition **I** and the comparison principle for lower semicontinuous viscosity supersolutions allows us to apply the moving plane arguments of [36] to solutions of (3.39). We thus see that U is radially symmetric. Next, by the comparison principle we conclude from **II** and (3.34) that

$$U = \gamma E + a$$

for some $\gamma > 0$, $a \in \mathbf{R}$. Combining this with (3.38) we derive (1.10) and the theorem is proved.

It is left to establish (3.37) for the sequence $\{v_j\}$ given by (3.35). From (3.34), (3.36) and the comparison principle, we obtain

$$0 \leqslant v_j \leqslant CE + M \qquad \text{in } B_{R/2} \setminus B_{2^{-j}R}$$

with C and M independent of j. The existence of a subsequence $\{v_{j_k}\}$ with property (3.37) now follows directly from the Arzela–Ascoli lemma and the local $C^{0,\alpha}$ estimates for solutions of uniformly elliptic equations, e.g. [5, Chap. 4]. ∎

4. PROPERTIES OF VISCOSITY SUPERSOLUTIONS

For a domain $\Omega \subset \mathbf{R}^n$, let $\Psi_{\lambda/\Lambda}(\Omega)$ be the set of all viscosity supersolutions of

$$\mathscr{P}^+_{\lambda,\Lambda}(D^2u) \leqslant 0 \qquad \text{in } \Omega$$

which are not identically equal to $+\infty$. This set is well-defined since $\mathscr{P}^+_{\lambda,\Lambda}(D^2u) \leqslant 0$ implies that $\mathscr{P}^+_{t\lambda,t\Lambda}(D^2u) \leqslant 0$, $t > 0$. For example, if by continuity we define $E^+_{\Lambda/\lambda}(0) = +\infty$, then $E^+_{\Lambda/\lambda}$ is lower semicontinuous in $\mathbf{R}^n$ and consequently $E^+_{\Lambda/\lambda} \in \Psi_{\lambda/\Lambda}(\mathbf{R}^n)$.

The set $\Psi_{\lambda/\Lambda}(\Omega)$ is a convex cone, [5, Chap. 5], and obviously

$$\Psi_s(\Omega) \subset \Psi_t(\Omega), \qquad 0 < s \leqslant t \leqslant 1.$$

The cone $\Psi_1(\Omega)$ is in fact the cone of classical superharmonic functions, see e.g. [19, Chap. 3]. In this Section we will investigate properties of functions in $\Psi_t(\Omega)$, $0 < t < 1$. Corollary 4.3 states that for any $u \in \Psi_t(\Omega)$, $0 < t < 1$, the second derivatives $D_{ij}u$, i, $j = 1, \ldots, n$, are signed Radon measures in Ω. Of course, this is not true for classical superharmonic functions. Properties of functions whose Hessian matrices are Radon measures have been investigated in the literature, see e.g. [1] and references therein. Thus Corollary 4.3 implies that the results of [1] hold for functions in $\Psi_t(\Omega)$, $0 < t < 1$.

Remark 4.1. Originally we were motivated by the problem of which properties of the Laplace operator on $\Psi_1(\Omega)$ could be extended to the operator $\mathscr{P}^+_{\lambda,\Lambda}$ acting on $\Psi_{\lambda/\Lambda}(\Omega)$. More precisely, is it possible to define the expression $-\mathscr{P}^+_{\lambda,\Lambda}(D^2u)$, $u \in \Psi_{\lambda/\Lambda}(\Omega)$, as a measure with some sequential continuity properties so as to construct a nontrivial existence-uniqueness theory for the Dirichlet problem? It is tempting to ask whether it is possible to define $\mathscr{P}^+_{\lambda,\Lambda}$ such that

$$-\mathscr{P}^+_{\lambda,\Lambda}(D^2E^+_{\Lambda/\lambda}) = C\delta, \qquad C > 0,$$

where δ is the Dirac mass at 0. If $u \in \Psi_1(\Omega)$, then $-\Delta u$ is a Radon measure, weakly* continuous with respect to L^1_{loc} convergence. Recently Trudinger and Wang [41, 42] obtained a similar result (see the introduction) for viscosity subsolutions of fully nonlinear Hessian equations. For k-Hessian operators F_k and for quasilinear equations involving the p-Laplacian Δ_p (and even for more general operators) such definitions exist, and the corresponding fundamental solutions satisfy the equation with a Dirac mass δ. In contrast with $\mathscr{P}^+_{\lambda,\Lambda}$, the operators F_k and Δ_p have a variational structure. Concerning Δ_p-superharmonic functions and measures related to them, see [3; 23; 44, Chap. 5]. For our results for $\mathscr{P}^+_{\lambda,\Lambda}$, see Remark 4.4 and Proposition 4.5 below.

It follows immediately from (2.5) that $u \in C^2_{\text{loc}}(\Omega)$ belongs to $\Psi_t(\Omega)$, $0 < t \leqslant 1$, if and only if for any $x \in \Omega$ the eigenvalues of the Hessian matrix $[D^2u(x)]$ lie in the cone

$$\Gamma_t = \left\{ \xi \in \mathbf{R}^n : \sum_{\xi_j \geqslant 0} \xi_j + t \sum_{\xi_j \leqslant 0} \xi_j \leqslant 0 \right\}. \tag{4.1}$$

The function

$$\sum_{\xi_j \geqslant 0} \xi_j + t \sum_{\xi_j \leqslant 0} \xi_j = t \sum_{j=1}^{n} \xi_j + (1-t) \sum_{j=1}^{n} \max\{\xi_j, 0\}$$

is convex on $\mathbf{R}^n$. Thus Γ_t, $0 < t \leqslant 1$, is a convex cone in $\mathbf{R}^n$. The cone Γ_t^* dual to Γ_t is defined by

$$\Gamma_t^* = \{\eta \in \mathbf{R}^n : \langle \eta, \xi \rangle \geqslant 0 \ \ \forall \xi \in \Gamma_t\}. \tag{4.2}$$

From the definition, $\Gamma_s^* \supset \Gamma_t^*$, $0 < s \leqslant t \leqslant 1$, and $\Gamma_1^* = (-1, \cdots, -1)$. A radial test function

$$\varphi \geqslant 0, \qquad \int \varphi = 1, \qquad \operatorname{supp}(\varphi) \subset B_1,$$

is called a smoothing kernel, $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. We recall that for a distribution $f \in \mathscr{D}'(\Omega)$, the condition $f \geqslant 0$ means that

$$(f, \psi) \geqslant 0 \qquad \text{for every} \quad \psi \in \mathscr{D}(\Omega), \qquad \psi \geqslant 0.$$

Every nonnegative distribution is a Radon measure, see e.g. [45, Chap. 1]. The following lemma was inspired by [42].

Lemma 4.2. *Let Ω be a domain in $\mathbf{R}^n$, $0 < t \leqslant 1$. If $u \in \Psi_t(\Omega)$, then*

$$\sum_{i,j=1}^{n} A_{ij} D_{ij} u \geqslant 0 \qquad \text{in } \mathscr{D}'(\Omega) \tag{4.3}$$

for all $A \in \mathbf{S}^n$ with eigenvalues in Γ_t^*. Conversely, if $U \in \mathscr{D}'(\Omega)$ satisfies (4.3) *for all $A \in \mathbf{S}^n$ with eigenvalues in Γ_t^*, then U is equivalent to a unique $u \in \Psi_t(\Omega)$.*

Proof. We remark that the cone dual to Γ_t^* is Γ_t. Thus the lemma holds for $u \in C^2_{\text{loc}}(\Omega)$.

Let $u \in \Psi_t(\Omega) \subset \Psi_1(\Omega)$. So in particular $u \in L^1_{\text{loc}}(\Omega)$, see [19, Chap. 3]. Consider the inf-convolution (another name is the lower ε-envelope) of u, defined by

$$u_\varepsilon^-(x) = \inf_{y \in \Omega'} \{u(y) - \varepsilon + |x-y|^2/\varepsilon\},$$

where $\varepsilon > 0$, $x \in \Omega' \subset\subset \Omega$. One has

$$u^-_{1/k} \in \Psi_t(\Omega') \cap C(\overline{\Omega'}), \qquad u^-_{1/k} \uparrow u \quad \text{in} \quad \Omega' \quad \text{when} \quad k \to \infty.$$

The proof of these facts for continuous u can be found in [5, Chap. 5]. For lower semicontinuous u the proof is essentially the same. Thus

$$u^-_{1/k} \to u \qquad \text{in } L^1(\Omega') \qquad \text{when} \quad k \to \infty.$$

From the continuity of $u^-_{1/k}$ and the convexity of the matrix function $\mathscr{P}^+_{\lambda, \Lambda}$, it follows that

$$u^-_{1/k} * \varphi_\varepsilon \in \Psi_t(\Omega''), \qquad \Omega'' \subset\subset \Omega',$$

see [5, Chap. 6]. Now choose $\varepsilon_k \to 0$ such that $u^-_{1/k} * \varphi_{\varepsilon_k} \to u$ in $L^1(\Omega'')$ and consequently in $\mathscr{D}'(\Omega'')$. For smooth functions $u^-_{1/k} * \varphi_{\varepsilon_k}$ it is clear that (4.3) holds. Thus (4.3) holds for the limit distribution u as well.

Let now $U \in \mathscr{D}'(\Omega)$ be such that (4.3) holds. In particular $-\Delta U \geqslant 0$. By a classical result for superharmonic functions, [19, Chap. 3], we have $U \in L^1_{\mathrm{loc}}(\Omega)$. Thus if φ is a smoothing kernel, there is a unique superharmonic function $u\colon \Omega \to \mathbf{R}^1 \cup \{+\infty\}$ such that $U = u$ and

$$u * \varphi_\varepsilon(x) \uparrow u(x), \qquad \varepsilon \to 0,$$

for every $x \in \Omega$. The function $u * \varphi_\varepsilon$ defined in $\Omega' \subset\subset \Omega$ is smooth, and (4.3) holds for it because

$$\left(\sum_{i,j=1}^{n} A_{ij} D_{ij}(u * \varphi_\varepsilon), \psi \right) = \left(\sum_{i,j=1}^{n} A_{ij} D_{ij} u, \psi * \varphi_\varepsilon \right) \geqslant 0,$$

for all $\psi \in \mathscr{D}(\Omega')$, $\psi \geqslant 0$. Thus $u * \varphi_\varepsilon \in \Psi_t(\Omega')$, $\Omega' \subset\subset \Omega$. Finally since u is the limit of an increasing sequence of viscosity supersolutions, it is also a supersolution, [11, Sect. 8]. ∎

COROLLARY 4.3. *If $u \in \Psi_t(\Omega)$, $0 < t < 1$, then the distributional derivatives $D_{ij}u$ are signed Radon measures for all $i, j = 1, \ldots, n$.*

Proof. Due to the invariance of $\mathscr{P}^+_{\lambda, \Lambda}$ with respect to orthogonal transformations, it is sufficient to prove that $D_{11}u$ is a signed Radon measure. From (4.3)

$$\sum_{i,j=1}^{n} A_{ij}\, D_{ij} u$$

is a Radon measure for any corresponding A. Moreover, for $0 < t \leqslant 1$

$$A^{(1)} = \operatorname{diag}\{-1, \ldots, -1\} \in \Gamma_t^*.$$

Elementary calculations using (4.1), (4.2) also show that

$$A^{(2)} = \operatorname{diag}\{-1/t, -1, \ldots, -1\} \in \Gamma_t^*.$$

Thus

$$\begin{aligned} -D_{11}u - \cdots - D_{nn}u &= \mu_1, \\ -D_{11}u/t - D_{22}u - \cdots - D_{nn}u &= \mu_2, \end{aligned}$$

and for $0 < t < 1$,

$$D_{11}u = t(\mu_1 - \mu_2)/(1-t)$$

is a signed Radon measure. ∎

Now following the abstract construction of Goffman and Serrin [18] we use Corollary 4.3 to define the Radon measure $-\mathscr{P}^+_{\lambda,\Lambda}(D^2u)$ as a function of measures $D_{ij}u$, $i, j = 1, \ldots, n$, $u \in \Psi_{\lambda/\Lambda}(\Omega)$. This was done in [2, Sect. 5] for the operator $\Phi[\cdot]$ in (1.12) on the cone of plurisubharmonic functions. Our operator $-\mathscr{P}^+_{\lambda,\Lambda}$ enjoys the properties needed to make the construction of [18] applicable:

- The set

$$\{M \in \mathbf{S}^n : \mathscr{P}^+_{\lambda,\Lambda}(M) \leqslant 0\}$$

is a convex matrix cone.

- The function $-\mathscr{P}^+_{\lambda,\Lambda}$ is concave and homogeneous of degree 1 on this cone.

- For any $M \in \mathbf{S}^n$

$$|\mathscr{P}^+_{\lambda,\Lambda}(M)| \leqslant C(n) \sum_{i,j=1}^{n} |M_{ij}|.$$

In what follows we will refer to the proofs in the carefully written Sect. 5 in [2] whenever proofs for $-\mathscr{P}^+_{\lambda,\Lambda}(D^2u)$ and for $\Phi[v] = (\det[v_{z_j\bar{z}_k}])^{1/n}$ are the same.

Let us pass to the definition. Let $u \in \Psi_{\lambda/\Lambda}(\Omega)$. From Corollary 4.3 and Lemma 4.2 the matrix $[D^2u(E)] = [D_{ij}u(E)]$ has the eigenvalues in $\Gamma_{\lambda/\Lambda}$ for every Borel set $E \subset\subset \Omega$. We define Borel measure $\mu_{\lambda,\Lambda}[u] = -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u)$ by prescribing

$$\mu_{\lambda,\Lambda}[u] = -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u) = \inf\left\{ \sum_{k=1}^{\infty} -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u(E_k)) : \right.$$
$$\left. E = \bigcup_{k=1}^{\infty} E_k,\ E_k \text{ disjoint Borel subsets of } \Omega \right\}, \tag{4.4}$$

for arbitrary Borel subset $E \subset \Omega$. It is clear that $\mu_{\lambda,\Lambda}[u](K) < +\infty$ for any compact $K \subset\subset \Omega$. Thus $\mu_{\lambda,\Lambda}[u]$ is a Radon measure for $u \in \Psi_{\lambda/\Lambda}(\Omega)$.

Let us give another definition equivalent to [2; 18, (4.4)]. We fix any measure ν such that all $D_{ij}u$ are absolutely continuous with respect to ν. For example $\nu = \sum |D_{ij}u|$, where $|D_{ij}u|$ is the total variation of $D_{ij}u$. By Radon–Nikodym theorem $D_{ij}u = h_{ij}\,d\nu$, where h_{ij} are Borel measurable functions in Ω. Then

$$\mu_{\lambda,\Lambda}[u] = -\mathscr{P}^{+}_{\lambda,\Lambda}([h_{ij}])\,d\nu, \qquad u \in \Psi_{\lambda/\Lambda}(\Omega).$$

In particular if $u \in \Psi_{\lambda/\Lambda}(\Omega)$, $D_{ij}u \in L^1_{\text{loc}}(\Omega)$, $i, j = 1, \dots, n$, then $\mu_{\lambda,\Lambda}[u]$ is absolutely continuous and

$$\mu_{\lambda,\Lambda}[u] = -\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u(x))\,dx. \tag{4.5}$$

We can also define $\mu_{\lambda,\Lambda}[u]$ using the supremum of the family of signed measures $\sum A_{ij}D_{ij}u$, $\lambda I \leqslant A \leqslant \Lambda I$, see [10, Chap. 3]. Such definition for *solutions* to $\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u) = 0$ and more general equations was discussed in [26, Chap. 2], and [27, Sect. 1]. Using (4.5) it is not hard to show that this definition is equivalent to the definitions above.

In Proposition 4.5 below we indicate some properties of $\mu_{\lambda,\Lambda}[u]$.

Remark 4.4. From (4.5) it follows that

$$\mu_{\lambda,\Lambda}[E^{+}_{\Lambda/\lambda}] = 0 \quad \text{in } \mathbf{R}^n, \qquad \lambda < \Lambda. \tag{4.6}$$

Thus there can be no comparison principle for $\mu_{\lambda,\Lambda}[u]$, $u \in \Psi_{\lambda/\Lambda}(\Omega)$ if $\lambda < \Lambda$. However part 3 of Proposition 4.5 shows that this effect is due to the structure of $\mathscr{P}^{+}_{\lambda,\Lambda}$ rather than a particular way of defining $\mu_{\lambda,\Lambda}[u]$. From (4.6) and part 3 of Proposition 4.5 it follows that if a measure $\nu_{\lambda,\Lambda}[u]$ defined on $\Psi_{\lambda/\Lambda}(\Omega)$ coincides with $-\mathscr{P}^{+}_{\lambda,\Lambda}(D^2u)\,dx$ for smooth $u \in \Psi_{\lambda/\Lambda}(\Omega)$ and $\nu_{\lambda,\Lambda}[u_j] \to \nu_{\lambda,\Lambda}[u]$ weakly* as $u_j \to u$ in L^1_{loc}, then necessarily $\nu_{\lambda,\Lambda}[E^+] = 0$ rather than $\nu_{\lambda,\Lambda}[E^+] = C\delta$, $C > 0$.

Proposition 4.5. *Let* $0 < \lambda < \Lambda$.

1. *If* $u_j \to u$ *in* $L^1_{\mathrm{loc}}(\Omega)$, $u_j \in \Psi_{\lambda/\Lambda}(\Omega)$, $j \geqslant 1$, *then after a possible modification on a set of the Lebesgue measure* 0 $u \in \Psi_{\lambda/\Lambda}(\Omega)$, *and*

$$\mu_{\lambda,\Lambda}[u](K) \geqslant \limsup_{j\to\infty} \mu_{\lambda,\Lambda}[u_j](K)$$

for every compact set $K \subset \Omega$.

2. *If* $u \in \Psi_{\lambda/\Lambda}(\Omega)$, φ *is a smoothing kernel*, $\varepsilon > 0$, $\Omega' + \operatorname{supp} \varphi_\varepsilon \subset \Omega$, *then* $u * \varphi_\varepsilon \in \Psi_{\lambda/\Lambda}(\Omega')$.

3. *If* $u \in \Psi_{\lambda/\Lambda}(\Omega)$, *then*

$$\lim_{\varepsilon\to 0} \mu_{\lambda,\Lambda}[u * \varphi_\varepsilon] = \mu_{\lambda,\Lambda}[u]$$

in the weak sense.*

4. *If* $\Lambda/\lambda > n-1$, *then* $\Psi_{\lambda/\Lambda}(\Omega) \subset C^\alpha_{\mathrm{loc}}(\Omega)$ *for* $\alpha = 1-(n-1)\,\lambda/\Lambda$.

Proof. 1. Passing to the limit when $j \to \infty$ we see by Lemma 4.2 that (4.3) holds for u. Applying Lemma 4.2 again we see that u is equivalent to a function from $\Psi_{\lambda/\Lambda}(\Omega)$. Proof of the last part of the assertion is the same as in [2, Sect. 5; 18, Sect. 2].

2, 3. Proofs are the same as in [2, Sect. 5].

4. Let first $u \in \Psi_{\lambda/\Lambda}(\Omega) \cap C^\infty_{\mathrm{loc}}(\Omega)$. Fixing a ball $B = B(y, R) \subset\subset \Omega$ we consider the fundamental solution $E^+_{\Lambda/\lambda}(x-y)$ in $B\backslash\{y\}$. By the classical comparison principle

$$u(y)-u(x) \leqslant (\operatorname*{osc}_B u) \left(\frac{|x-y|}{R}\right)^\alpha, \qquad \alpha = 1 - \frac{(n-1)\,\lambda}{\Lambda}, \tag{4.7}$$

for all $x \in B$. Let $\sigma \geqslant 0$, $d_x = \operatorname{dist}(x, \partial\Omega)$,

$$|u|^{(\sigma)}_{0;\Omega} = \sup_{x\in\Omega} (d^\sigma_x\, |u(x)|),$$

$$[u]^{(\sigma)}_{\alpha;\Omega} = \sup_{x,\, y\in B,\, x\neq y} (\min\{d_x, d_y\})^{\sigma+\alpha} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Then the direct consequence of (4.7) is the estimate

$$[u]^{(\sigma)}_{\alpha;\Omega} \leqslant C_1(n)\, |u|^{(\sigma)}_{0;\Omega} \tag{4.8}$$

valid for any $\sigma \geqslant 0$. Using (4.8) with $\sigma = n$ and the interpolation inequality [42]

$$|u|^{(n)}_{0;\Omega} \leqslant \varepsilon^\alpha [u]^{(n)}_{\alpha;\Omega} + \frac{C_2(n)}{\varepsilon^n} \int_\Omega |u|$$

with ε small enough we obtain

$$[u]^{(n)}_{\alpha;\Omega} \leqslant C(n) \int_{\Omega} |u|. \tag{4.9}$$

For arbitrary $u \in \Psi_{\lambda/\Lambda}(\Omega)$ applying (4.9) to $u * \varphi_\varepsilon$ and using parts 1, 2 we complete the proof of 4. The example of the function $E^{+}_{\Lambda/\lambda} \in \Psi_{\lambda/\Lambda}(\mathbf{R}^n)$ shows that the Hölder exponent $\alpha = 1-(n-1)\,\lambda/\Lambda$ is the best possible. ∎

ACKNOWLEDGMENT

I thank my advisors N. S. Trudinger and X. J. Wang for discussions of their works [41]–[43], sharing of ideas, and useful discussions during the preparation of this paper. I also thank M. Kocan and A. Święch for their remarks about the results.

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