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ScienceDirect
J. Differential Equations 239 (2007) 213–250
Journal of Differential Equations
www.elsevier.com/locate/jde

Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations ☆

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Received 26 January 2007; revised 12 April 2007

Available online 3 May 2007

Abstract

Sufficient and necessary conditions for the embeddings between Besov spaces $B^{s_1}_{p,q}$ and modulation spaces $M^{s_2}_{p,q}$ are obtained. Moreover, using the frequency-uniform decomposition method, we study the Cauchy problem for the generalized BO, KdV and NLS equations, for which the global well-posedness of solutions with the small rough data in certain modulation spaces $M^{s}_{2,1}$ is shown.

MSC: 35Q55; 35Q53; 46E35

Keywords: Korteweg–de Vries equation; Benjamin–Ono equation; Nonlinear Schrödinger equation; Rough data; Embedding between Besov and modulation spaces

1. Introduction

In this paper we study the Cauchy problem for the generalized Korteweg–de Vries (KdV), Benjamin–Ono (BO) and nonlinear Schrödinger equations

$$\partial_t u + \partial_x^3 u + u^\kappa \partial_x u = 0, \quad u(0,x) = u_0(x), \tag{1.1}$$

$$\partial_t u + \mathcal{H}\big(\partial_x^2 u\big) + u^\kappa \partial_x u = 0, \quad u(0,x) = u_0(x), \tag{1.2}$$

☆ This work is supported in part by the National Science Foundation of China, grant 10571004; and the 973 Project Foundation of China, grant 2006CB805902.

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.04.009

$$\mathrm{i}\partial_t u + \partial_x^2 u + \mu\partial_x\big(|u|^\kappa u\big) = 0, \quad u(0,x) = u_0(x), \tag{1.3}$$

where $\mathcal{H} := \mathrm{i}\,\mathscr{F}^{-1}\,\mathrm{sign}(\xi)\mathscr{F}$ denotes the Hilbert transform; $u(t,x)$ is a complex-valued function of $(t,x)\in\mathbb{R}^{1+1}$, $\mathrm{i}=\sqrt{-1}$, $\mu\in\mathbb{C}$, $\partial_t=\partial/\partial t$, $\partial_x=\partial/\partial x$ and $\partial_x^m=\partial^m/\partial^m x$, $m=2,3$, $\kappa\geqslant 4$ is an integer, u_0 is a complex-valued function of $x\in\mathbb{R}$. We will study the global well-posedness of (1.1)–(1.3) with small rough data in a class of modulation spaces $M^s_{2,1}$.

In order to state our main results precisely, we now recall the definition of modulation spaces. Let $Q_0=\{\xi\colon -1/2\leqslant\xi_i<1/2,\ i=1,\dots,n\}$ be the unit cube and $Q_\mu=\mu+Q_0$, $\mu\in\mathbb{R}^n$. Let $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ be a function sequence satisfying

$$\begin{cases} \big|\sigma_k(\xi)\big|\geqslant c, & \forall\xi\in Q_k,\\ \operatorname{supp}\sigma_k\subset\big\{\xi\colon |\xi-k|\leqslant\sqrt{n}\big\}, & \\ \displaystyle\sum_{k\in\mathbb{Z}^n}\sigma_k(\xi)\equiv 1, & \forall\xi\in\mathbb{R}^n,\\ \big|D^\alpha\sigma_k(\xi)\big|\leqslant C_m, & \forall\xi\in\mathbb{R}^n,\ |\alpha|\leqslant m\in\mathbb{N}. \end{cases} \tag{1.4}$$

Denote

$$\Upsilon=\big\{\{\sigma_k\}_{k\in\mathbb{Z}^n}\colon \{\sigma_k\}_{k\in\mathbb{Z}^n} \text{ satisfies (1.4)}\big\}. \tag{1.5}$$

Let $\{\sigma_k\}_{k\in\mathbb{Z}^n}\in\Upsilon$ be a function sequence and

$$\Box_k := \mathscr{F}^{-1}\sigma_k\mathscr{F}, \quad k\in\mathbb{Z}^n, \tag{1.6}$$

which are said to be the frequency-uniform decomposition operators. For any $k\in\mathbb{Z}^n$, we write $|k|=|k_1|+\cdots+|k_n|$, $\langle k\rangle=1+|k|$. For any $s\in\mathbb{R}$, $0<p,q\leqslant\infty$, we denote $\|\cdot\|_p:=\|\cdot\|_{L^p(\mathbb{R}^n)}$ and

$$M^s_{p,q}\big(\mathbb{R}^n\big)=\left\{f\in\mathscr{S}'\big(\mathbb{R}^n\big)\colon \|f\|_{M^s_{p,q}}=\left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\|\Box_k f\|_p^q\right)^{1/q}<\infty\right\}, \tag{1.7}$$

$M^s_{p,q}:=M^s_{p,q}(\mathbb{R}^n)$ is said to be a modulation space, which was first introduced by Feichtinger [9] in the cases $1\leqslant p,q\leqslant\infty$. The norm $\|\cdot\|_{M^s_{p,q}}$ adopted here is an equivalent norm on modulation spaces; cf. [9,12,23,24,38,40]. For simplicity, we will write $M^0_{p,q}=M_{p,q}$.

Besov spaces $B^s_{p,q}:=B^s_{p,q}(\mathbb{R}^n)$ are defined as follows (cf. [1,35]). Let $\psi:\mathbb{R}^n\to[0,1]$ be a smooth radial bump function adapted to the ball $B(0,2)$: $\psi(\xi)=1$ as $|\xi|\leqslant 1$ and $\psi(\xi)=0$ as $|\xi|\geqslant 2$. We write $\delta(\cdot):=\psi(\cdot)-\psi(2\,\cdot)$ and $\delta_k:=\delta(2^{-k}\cdot)$ for $k\geqslant 1$; $\delta_0:=1-\sum_{k\geqslant 1}\delta_k$. We say that $\Delta_k:=\mathscr{F}^{-1}\delta_k\mathscr{F}$, $k\in\mathbb{Z}_+$, are the dyadic decomposition operators. Besov spaces $B^s_{p,q}$ are defined in the following way:

$$B^s_{p,q}=\left\{f\in\mathscr{S}'\big(\mathbb{R}^n\big)\colon \|f\|_{B^s_{p,q}}=\left(\sum_{j=0}^{\infty}2^{sjq}\|\Delta_j f\|_p^q\right)^{1/q}<\infty\right\}. \tag{1.8}$$

First, we will give some sufficient and necessary conditions for the embeddings between Besov and modulation spaces, which cover the corresponding results as in [11,28,34,40].

Theorem 1.1. *Let* $0 < p, q \leqslant \infty$, $s_1, s_2 \in \mathbb{R}$. *Then we have*

(1) $B^{s_1}_{p,q} \subset M^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \tau(p,q)$, *where*

$$\tau(p,q) = \max\left\{0,\ n\left(\frac{1}{q} - \frac{1}{p}\right), n\left(\frac{1}{q} + \frac{1}{p} - 1\right)\right\};$$

(2) $M^{s_1}_{p,q} \subset B^{s_2}_{p,q}$ *if and only if* $s_1 \geqslant s_2 + \sigma(p,q)$, *where*

$$\sigma(p,q) = \max\left\{0,\ n\left(\frac{1}{p} - \frac{1}{q}\right), n\left(1 - \frac{1}{p} - \frac{1}{q}\right)\right\}.$$

In this paper, we are particularly interested in the space $M^s_{2,1}(\mathbb{R})$, for which the norm can be rewritten as

$$\|f\|_{M^s_{2,1}(\mathbb{R})} \sim \sum_{k\in\mathbb{Z}} \langle k\rangle^s \|\chi_{[k,\,k+1)}\mathscr{F} f\|_2,$$

where $\chi_{[k,\,k+1)}$ denotes the characteristic function on $[k,\ k+1)$. In view of Theorem 1.1 we see that $B^{1/2}_{2,1}(\mathbb{R}) \subset M_{2,1}(\mathbb{R}) \subset B^0_{2,1}(\mathbb{R})$ are both sharp embeddings. So, one can regard $M_{2,1}(\mathbb{R})$ as the lower regularity version of $B^{1/2}_{2,1}(\mathbb{R})$. Moreover, in [40] we have shown that $M_{2,1} \not\subset \dot B^{\varepsilon}_{2,\infty} \cup \dot B^{\varepsilon}_{\infty,\infty}$ $(\forall \varepsilon > 0)$,[1] from which we see that $M_{2,1}(\mathbb{R})$ has no derivative regularity. We have the following

Theorem 1.2. *Let* $\kappa \geqslant 4$, $u_0 \in M_{2,1}(\mathbb{R})$. *Then there exists* $T > 0$ *such that* (1.1) *has a unique solution* $u \in C([-T,T]; M_{2,1}) \cap X^T_{\mathrm{KdV}}$, *where*

$$X^T_{\mathrm{KdV}} = \left\{u\colon \sum_{k\in\mathbb{Z}} \|\Box_k u\|_{L^{\kappa+1}_x L^{2(\kappa+1)}_{t\in(-T,T)}} < \infty\right\}. \tag{1.9}$$

Moreover, if there exists a small $\delta > 0$ *such that* $\|u_0\|_{M_{2,1}} \leqslant \delta$, *then* (1.1) *has a unique global solution* $u \in C(\mathbb{R}, M_{2,1}) \cap X^\infty_{\mathrm{KdV}}$.

Recall that Kenig, Ponce and Vega [20] showed the global well-posedness of Eq. (1.1) with small data in the Sobolev spaces $\dot H^{s_\kappa}$ for $\kappa \geqslant 4$, $s_\kappa = 1/2 - 2/\kappa$. Molinet and Ribaud [26] generalized their work to the case $u_0 \in \dot B^{s_\kappa}_{2,\infty}$. Due to $M_{2,1} \not\subset \dot B^{s_\kappa}_{2,\infty}$ if $\kappa > 4$, our Theorem 1.2 obtains new local and global well-posedness result for a class of rough Cauchy data. However, our result cannot cover the cases $\kappa = 2, 3$, the sharp global well-posedness in H^s for the cases $k = 2, 3$ has been shown in [6,33].

Theorem 1.3. *Let* $\kappa \geqslant 4$, $u_0 \in M^{1/\kappa}_{2,1}(\mathbb{R})$. *Assume that there exists a small* $\delta > 0$ *such that* $\|u_0\|_{M^{1/\kappa}_{2,1}} \leqslant \delta$. *Then* (1.2) *has a unique solution* $u \in C(\mathbb{R}, M^{1/\kappa}_{2,1}) \cap X_{\mathrm{BO}}$, *where*

[1] In [40], the result was stated as $M_{2,1} \not\subset B^{\varepsilon}_{2,\infty} \cup B^{\varepsilon}_{\infty,\infty}$, however, it was shown in fact that $P_{\geqslant 1} M_{2,1} \not\subset \dot B^{\varepsilon}_{2,\infty} \cup \dot B^{\varepsilon}_{\infty,\infty}$, $P_{\geqslant 1} = \mathscr{F}^{-1}\chi_{(|\xi|\geqslant 1)}\mathscr{F}$.

$$X_{\mathrm{BO}} = \left\{u \in \mathscr{S}'\left(\mathbb{R}^{1+1}\right):\ \|u\|_{X_{\mathrm{BO}}} \lesssim \delta\right\}, \tag{1.10}$$

$$\|u\|_{X_{\mathrm{BO}}} = \sum_{k\in\mathbb{Z}}\left(\|\Box_k u\|_{L^\kappa_x L^\infty_t} + \langle k\rangle^{1/\kappa}\|\Box_k u\|_{L^{2+\kappa}_{x,t}}\right) + \sum_{|k|\gg 1}\langle k\rangle^{1/2+1/\kappa}\|\Box_k u\|_{L^\infty_x L^2_t}. \tag{1.11}$$

We remark that Molinet and Ribaud [26] obtained the global well-posedness of Eq. (1.2) with small data $u_0 \in \dot{B}^{s(\kappa)}_{2,1}$, $s(\kappa) = 1/2 - 1/\kappa$, $\kappa \geqslant 4$. In view of Theorem 1.1, one sees that $M^{1/\kappa}_{2,1} \not\subset \dot{B}^{s(\kappa)}_{2,1}$ if $\kappa > 4$. So, our Theorem 1.3 obtains new global well-posedness result for a class of rough initial data. When $\kappa = 2, 3$, our method is invalid for Eq. (1.2) (see [22,27]).

The derivative nonlinear Schrödinger equation (1.3) has been studied by many authors in the case $\kappa = 2$; cf. [13,14,29,32]. Applying the gauge transform technique, Hayashi [13] was able to show the global well-posedness of Eq. (1.3) in the energy space H^1; cf. also [29]; and the global well-posedness of Eq. (1.3) in H^s $(s > 1/2)$ was obtained in [7]. For the higher power cases $\kappa \geqslant 4$, Wang [39] introduced a generalized gauge transform $\varphi := \exp(\mathrm{i}\int_{-\infty}^{x}|u(t,y)|^\kappa\,dy)u$ and obtained some sufficient conditions for the well-posedness results of Eq. (1.3) in the energy space H^1. Molinet and Ribaud's techniques on the generalized BO equation can be developed to Eq. (1.3) and the global well-posedness with small data $u_0 \in \dot{B}^{s(\kappa)}_{2,1}$, $s(\kappa) = 1/2 - 1/\kappa$, $\kappa \geqslant 4$, was essentially obtained in [26]. The following is a global well-posedness result with small rough data in $M^{1/\kappa}_{2,1}$:

Theorem 1.4. *Let $\kappa \geqslant 4$ be an even integer, $\mu \in \mathbb{C}$, $u_0 \in M^{1/\kappa}_{2,1}(\mathbb{R})$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{M^{1/\kappa}_{2,1}} \leqslant \delta$. Then* (1.3) *has a unique solution u satisfying the same conclusions as in Theorem* 1.3.

One can easily generalize the above results to the nonlinearity with an exponential growth, say, we consider the following problem:

$$\mathrm{i}\partial_t u + \partial_x^2 u + \mu\partial_x\left(\left(e^{|u|^\kappa} - 1\right)u\right) = 0, \quad u(0,x) = u_0(x), \tag{1.12}$$

$$\partial_t u + \partial_x^3 u + \partial_x\left(\left(e^{u^\kappa} - 1\right)u\right) = 0, \quad u(0,x) = u_0(x). \tag{1.13}$$

Theorem 1.5. *Let $\kappa \geqslant 4$ be an even integer, $\mu \in \mathbb{C}$, $u_0 \in M^{1/\kappa}_{2,1}(\mathbb{R})$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{M^{1/\kappa}_{2,1}} \leqslant \delta$. Then the same results as in Theorem* 1.4 *hold for Eq.* (1.12).

Theorem 1.6. *Let $\kappa \geqslant 4$, $u_0 \in M_{2,1}(\mathbb{R})$. Then the same results as in Theorem* 1.2 *hold for Eq.* (1.13).

Finally, we consider the following NLS equation:

$$\mathrm{i}\partial_t u + \partial_x^2 u = F(u, \bar{u}, \partial_x u, \partial_x \bar{u}), \quad u(0,x) = u_0(x), \tag{1.14}$$

where

$$F(u, \bar{u}, \partial_x u, \partial_x \bar{u}) = \sum_{m+1\leqslant \kappa_1+\kappa_2+\nu_1+\nu_2\leqslant \tilde{m}+1} \lambda_{\kappa_1\kappa_2\nu_1\nu_2} u^{\kappa_1}\bar{u}^{\kappa_2}(\partial_x u)^{\nu_1}(\partial_x \bar{u})^{\nu_2} \tag{1.15}$$

is a multi-polynomial of $u, \bar{u}, \partial_x u$ and $\partial_x \bar{u}$ with $\tilde{m} \geqslant m \geqslant 4$, $\lambda_{\kappa_1\kappa_2\nu_1\nu_2} \in \mathbb{C}$. Equation (1.14) has been studied in [5,15,18,21]. Using the energy method, together with the gauge transformation technique, Hayashi and Ozawa [15] showed the well-posedness in H^3, $m \geqslant 3$. On the basis of pseudo-differential calculus, Chihara [5] constructed the gauge transformation in higher spatial dimensions and obtained the local well-posedness of Eq. (1.14) in higher spatial dimensions. Kenig, Ponce and Vega [18,21] developed the smooth effect estimates of Kato's type for the Schrödinger semigroups and established the local well-posedness for the smooth data, the solutions are almost global if the smooth initial data are small enough. In this paper we consider the small initial data in the space $M^{1+1/m}_{2,1}$, which has the lower regularity. We have the following

Theorem 1.7. *Let $\tilde{m} \geqslant m \geqslant 4$, $u_0 \in M^{1+1/m}_{2,1}(\mathbb{R})$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{M^{1+1/m}_{2,1}} \leqslant \delta$. Then* (1.14) *has a unique solution $u \in C(\mathbb{R}, M^{1+1/m}_{2,1}) \cap X_{\mathrm{dNLS}}$, where*

$$X_{\mathrm{dNLS}} = \left\{u \in \mathscr{S}'\left(\mathbb{R}^{1+1}\right):\ \|u\|_{X_{\mathrm{dNLS}}} \lesssim \delta\right\}, \tag{1.16}$$

$$\begin{aligned} \|u\|_{X_{\mathrm{dNLS}}} := \sum_{i=0,1}\sum_{k\in\mathbb{Z}}\left(\left\|\Box_k \partial_x^i u\right\|_{L^m_x L^\infty_t} + \langle k\rangle^{1/m}\left\|\Box_k \partial_x^i u\right\|_{(L^\infty_t L^2_x)\cap L^{2+m}_{x,t}}\right) \\ + \sum_{i=0,1}\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/m}\left\|\Box_k \partial_x^i u\right\|_{L^\infty_x L^2_t}. \end{aligned} \tag{1.17}$$

Theorem 1.7 can be developed to the case $u_0 \in H^s$ with $s > 3/2 - 1/\tilde{m}$, see [41]. If $F(u, \bar{u}, u_x, \bar{u}_x) = |u|^\kappa u + \partial_x(|u|^\nu u)$, combining Theorems 1.4 and 1.7, we have

Corollary 1.8. *Let $F(u, \bar{u}, u_x, \bar{u}_x) = \lambda_1|u|^{\kappa_1}u + \lambda_2|u|^{\kappa_2}u + \partial_x(\mu_1|u|^{\nu_1}u + \mu_2|u|^{\nu_2}u)$ with $4 \leqslant m \leqslant \kappa_1, \kappa_2, \nu_1, \nu_2 \leqslant \tilde{m}$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$, $u_0 \in M^{1/m}_{2,1}(\mathbb{R})$. Assume that there exists a small $\delta > 0$ such that $\|u_0\|_{M^{1/m}_{2,1}} \leqslant \delta$. Then* (1.14) *has a unique solution $u \in C(\mathbb{R}, M^{1/m}_{2,1}) \cap X_{\mathrm{NLS}}$, where*

$$X_{\mathrm{NLS}} = \left\{u \in \mathscr{S}'\left(\mathbb{R}^{1+1}\right):\ \|u\|_{X_{\mathrm{NLS}}} \lesssim \delta\right\}, \tag{1.18}$$

$$\begin{aligned} \|u\|_{X_{\mathrm{NLS}}} := \sum_{k\in\mathbb{Z}}\left(\|\Box_k u\|_{L^m_x L^\infty_t} + \langle k\rangle^{1/m}\|\Box_k u\|_{(L^\infty_t L^2_x)\cap L^{2+m}_{x,t}}\right) \\ + \sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/m}\|\Box_k u\|_{L^\infty_x L^2_t}. \end{aligned} \tag{1.19}$$

This paper is organized as follows. In Section 2 we will state a complex interpolation theorem on modulation spaces $M^s_{p,q}$, which is useful for us establishing the inclusions between Besov spaces and modulation spaces and the details of the proof will be given in Appendix A of this paper. In Section 3 we will prove Theorem 1.1. Some dispersive smooth effects for the Schrödinger semigroup will be given in Section 4 and Theorems 1.3–1.5 will be proved in Section 5. Section 6 is devoted to considering the smooth effects for the KdV semigroup and we show our Theorem 1.2 in Section 7. Finally, Theorem 1.7 will be shown in Section 8.

The following are some notations which will be frequently used in this paper: $\mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the sets of reals, positive integers and integers, respectively. $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. $c < 1$, $C > 1$ will denote positive universal constants, which can be different at different places. $a \lesssim b$ stands for $a \leqslant Cb$ for some constant $C > 1$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. We denote by p' the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$, Sobolev spaces $H^s = (I - \Delta)^{-s/2} L^2$, homogeneous Sobolev spaces $\dot{H}^s = (-\Delta)^{-s/2} L^2$. Some properties of these function spaces can be found in [1,35]. We will use the function spaces $L^q_{t \in I} L^p_x$ and $L^p_x L^q_{t \in I}$ for which the norms are defined by

$$\|f\|_{L^q_{t\in I} L^p_x} = \left(\int_I \left(\int_{\mathbb{R}^n} |f(t,x)|^p \, dx \right)^{q/p} dt \right)^{1/q},$$

$$\|f\|_{L^p_x L^q_{t\in I}} = \left(\int_{\mathbb{R}^n} \left(\int_I |f(t,x)|^q \, dt \right)^{p/q} dx \right)^{1/p},$$

and I will be omitted if $I = \mathbb{R}$, i.e., we simply write $L^q_t L^p_x := L^q_{t\in\mathbb{R}} L^p_x$, $L^p_x L^q_t := L^p_x L^q_{t\in\mathbb{R}}$. We denote $L^p_{x,t\in I} := L^p_x L^p_{t\in I}$ and I will be omitted if $I = \mathbb{R}$. We denote by $\mathscr{S} := \mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n)$ the Schwartz space and its dual space, respectively. $B(x, R)$ stands for the ball in $\mathbb{R}^n$ with center x and radius R. $\mathscr{F}$ or $\widehat{\ }$ denotes the Fourier transform; $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. We will frequently use the Bernstein multiplier estimate; cf. [1,16,35]: For any $r \in [1, \infty]$,

$$\left\| \mathscr{F}^{-1} \varphi \mathscr{F} f \right\|_r \leqslant C \|\varphi\|_{H^s} \|f\|_r, \quad s > n/2. \tag{1.20}$$

Let $f(t,x) \in \mathscr{S}(\mathbb{R}^{1+n})$ and $\mathscr{F}^{-1}\varphi \in L^1(\mathbb{R}^n)$. In view of Minkowski's and Young's inequalities, we easily see the following Bernstein's estimate:

$$\left\| \mathscr{F}^{-1} \varphi \mathscr{F} f \right\|_{L^p_x L^q_t} \leqslant \left\| \mathscr{F}^{-1} \varphi \right\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p_x L^q_t}, \quad p, q \geqslant 1. \tag{1.21}$$

2. Complex interpolation for $M^s_{p,q}$

In this section we consider the complex interpolation for the modulation spaces $M^s_{p,q}$ with $s \in \mathbb{R}$, $0 < p, q \leqslant \infty$. Recall that the complex interpolation for $M^s_{p,q}$ in the cases $1 \leqslant p, q \leqslant \infty$ has been studied by Feichtinger [9,10].

We now recall the complex interpolation space $(A_0, A_1)_\theta$; cf. Calderón [2], Calderón and Torchinsky [3,4] and Triebel [35]. Let $A = \{z\colon 0 < \operatorname{Re} z < 1\}$ be a strip in the complex plane. Its closure $\{z\colon 0 \leqslant \operatorname{Re} z \leqslant 1\}$ is denoted by $\overline{A}$. We say that $f(z)$ is an $\mathscr{S}'(\mathbb{R}^n)$-analytic function in A if the following properties are satisfied:

(1) For every fixed $z \in \overline{A}$, $f(z) \in \mathscr{S}'(\mathbb{R}^n)$.
(2) For any compact subset $\Omega \subset \mathbb{R}^n$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset \Omega$, $(\mathscr{F}^{-1} \varphi \mathscr{F} f)(x, z)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \overline{A}$.
(3) For any compact subset $\Omega \subset \mathbb{R}^n$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset \Omega$, $(\mathscr{F}^{-1} \varphi \mathscr{F} f)(x, z)$ is an analytic function in A for every fixed $x \in \mathbb{R}^n$.

We will denote by $\mathcal{A}(\mathscr{S}')$ the set of all $\mathscr{S}'(\mathbb{R}^n)$-analytic functions in A.

Definition 2.1. (Cf. Triebel [35].) Let $0 < p_i, q_i \leqslant \infty$, $s_i \in \mathbb{R}$ with $i = 0, 1$. We define

$$F\big(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\big) = \big\{f(z) \in \mathcal{A}(\mathscr{S}')\colon\ f(\ell + \mathrm{i}t) \in M^{s_\ell}_{p_\ell,q_\ell},\ \ell = 0, 1\big\},$$
$$\big\|f(z)\big\|_{F(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})} = \max_{\ell=0,1} \sup_{t\in\mathbb{R}} \big\|f(\ell + \mathrm{i}t)\big\|_{M^{s_\ell}_{p_\ell,q_\ell}};$$

and

$$\big(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\big)_\theta = \big\{g\colon\ \exists f(z) \in F\big(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\big) \text{ such that } f(\theta) = g\big\}, \tag{2.1}$$
$$\|g\|_{(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})_\theta} = \inf\big\|f(z)\big\|_{F(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})},$$

where the infimum is taken over all admissible functions $f(z)$ in the sense of (2.1).

The next proposition is also essentially known; cf. Triebel [35]:

Proposition 2.2. *We have*

$$\|g\|_{(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})_\theta} = \inf \sup_{t\in\mathbb{R}} \big\|f(\mathrm{i}t)\big\|^{1-\theta}_{M^{s_0}_{p_0,q_0}} \big\|f(1 + \mathrm{i}t)\big\|^{\theta}_{M^{s_1}_{p_1,q_1}},$$

where the infimum is taken over all $f(z)$ with $f(z) \in F(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})$ such that $f(\theta) = g$.

The following is the interpolation theorem for the modulation spaces $M^s_{p,q}$ in the cases $0 < p, q \leqslant \infty$.

Theorem 2.3. *Let $0 < p, q, p_i, q_i \leqslant \infty$, $s, s_i \in \mathbb{R}$ with $i = 0, 1$ and*

$$s = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.2}$$

Then we have

$$\big(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\big)_\theta = M^s_{p,q}.$$

Recall that the result of Theorem 2.3 is quite similar to Besov spaces, indeed, if (2.2) holds, then we have

$$\big(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\big)_\theta = B^s_{p,q}. \tag{2.3}$$

Theorem 2.3 seems essentially known, cf. [36,37]. The proof proceeds in a similar way as that of (2.3) and we leave it into Appendix A of this paper.

3. Sharp embeddings between $B^s_{p,q}$ and $M^s_{p,q}$

The first result on the inclusions between $B^s_{p,q}$ and $M^s_{p,q}$ in the cases $1 \leqslant p, q \leqslant \infty$ is due to Gröbner's unpublished thesis [11], where he obtained a sufficient embedding condition, and by our Theorem 1.1, his result is optimal at the vertices of the square $\{(1/p, 1/q)\colon\ 0 \leqslant 1/p, 1/q \leqslant 1\}$ in the $(1/p, 1/q)$-plane. Toft [34] used a different way showing the sharp sufficient conditions for the inclusions between $B^s_{p,q}$ and $M^s_{p,q}$ in the cases $1 \leqslant p, q \leqslant \infty$, and our sufficient conditions in Theorem 1.1 in the case $1 \leqslant p, q \leqslant \infty$ is identical with Toft's result. Wang and Hudzik [40] generalized their embeddings to the cases $0 < p, q \leqslant \infty$. Theorem 1.1 covers the corresponding results as in [11,28,34,40]. In this section we give the details of the proof of Theorem 1.1 and our technique is the frequency-uniform decomposition method applied in [11, 40], which is different from those in [28,34].

In order to prove Theorem 1.1, we need the following multiplier estimate; cf. Peetre [30], Triebel [35].

Proposition 3.1 *(Bernstein's multiplier estimate). Let $\Omega \subset \mathbb{R}^n$ be a compact subset. Let $0 < p \leqslant 1$. Then we have*

$$\big\|\mathscr{F}^{-1} M \mathscr{F} f\big\|_p \lesssim \|M\|_{B^{n(1/p-1/2)}_{2,p}} \|f\|_p$$

for all $f \in L^p_\Omega = \{f \in L^p\colon\ \operatorname{supp} \hat{f} \subset \Omega\}$, $M \in B^{n(1/p-1/2)}_{2,p}$.

In view of Proposition 3.1, we have

Corollary 3.2. *Let $b > 0$, $0 < p \leqslant 1$. Then we have*

$$\big\|\mathscr{F}^{-1} M \mathscr{F} f\big\|_p \leqslant C \big\|M(b\,\cdot)\big\|_{B^{n(1/p-1/2)}_{2,p}} \|f\|_p$$

for all $f \in L^p_{B(0,b)}$, $M \in B^{n(1/p-1/2)}_{2,p}$, where the constant $C > 0$ is independent of $b > 0$.

It is easy to see that if $f \in L^p_{B(0,b)}$, then $f(b^{-1}\cdot) \in L^p_{B(0,1)}$. Since

$$\big(\mathscr{F}^{-1} M \mathscr{F} f\big)(x) = \big[\mathscr{F}^{-1} M(b\,\cdot)\widehat{f(b^{-1}\cdot)}\big](bx),$$

one can use Proposition 3.1 to get the result of Corollary 3.2.

Lemma 3.3. *Let $0 < p \leqslant \infty$. Then we have*[2]

$$B^{n(1/(p\wedge 1)-1)}_{p,\infty} \subset M_{p,\infty}. \tag{3.1}$$

[2] Here we have corrected a mistake in [40, Lemma 2.13], where $B^0_{p,\infty} \subset M_{p,\infty}$ in the case $0 < p < 1$ is not true.

Proof. If $p \geqslant 1$, (3.1) was shown in [11,34,40]. We now consider the case $0 < p < 1$. By Corollary 3.2, we have for $|k| \gg 1$, $|k| \in [2^{j-1}, 2^j)$,

$$\|\Box_k f\|_p = \left\| \mathscr{F}^{-1} \sigma_k \sum_{\ell=-4}^{4} \delta_{j+\ell} \mathscr{F} f \right\|_p \lesssim \left\| \sigma_k\left(2^{j+5}\cdot\right) \right\|_{B^{n(1/p-1/2)}_{2,p}} \sum_{\ell=-4}^{4} \|\Delta_{j+\ell} f\|_p. \tag{3.2}$$

Applying the scaling inequality (cf. [35])

$$\left\| g(\lambda\cdot) \right\|_{B^s_{p,q}} \leqslant \lambda^{s-n/p} \|g\|_{B^s_{p,q}}, \quad \lambda \gtrsim 1,$$

we have[3]

$$\left\| \sigma_k\left(2^{j+5}\cdot\right) \right\|_{B^{n(1/p-1/2)}_{2,p}} \lesssim 2^{jn(1/p-1)} \|\sigma_k\|_{B^{n(1/p-1/2)}_{2,p}} \lesssim 2^{jn(1/p-1)}. \tag{3.3}$$

Inserting (3.3) into (3.2), we immediately obtain that

$$\|\Box_k f\|_p \lesssim 2^{jn(1/p-1)} \sum_{\ell=-4}^{4} \|\Delta_{j+\ell} f\|_p. \tag{3.4}$$

It follows from (3.4) that (3.1) holds. □

Proof of Theorem 1.1. First, we prove (1). Denote $\mathbb{R}^2_+ = \{(1/p, 1/q)\colon\ 1/p, 1/q \geqslant 0\}$ and as in Fig. 1

$$\begin{aligned} S_1 &= \left\{(1/p, 1/q) \in \mathbb{R}^2_+\colon\ 1/q \geqslant 1/p,\ 1/p \leqslant 1/2\right\}; \\ S_2 &= \left\{(1/p, 1/q) \in \mathbb{R}^2_+\colon\ 1/q \leqslant 1/p,\ 1/p + 1/q \leqslant 1\right\}; \\ S_3 &= \mathbb{R}^2_+ \setminus (S_1 \cup S_2). \end{aligned}$$

Step 1 (Sufficiency). We divide our proof into the following three cases.

Case 1.1. $(1/p, 1/q) \in S_3$. It is easy to see that $\tau(p,q) = n(1/p + 1/q - 1)$. Taking (p_0, q_0) and (p_1, q_1) such that

$$\begin{gathered} \frac{1}{p_0} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q_0} = 0; \\ \frac{1}{p_1} = \frac{1}{2}, \quad \frac{1}{q_1} = \frac{1}{p} + \frac{1}{q} - \frac{1}{2}, \end{gathered}$$

one has that for $\theta = \frac{1}{q}(\frac{1}{p} + \frac{1}{q} - \frac{1}{2})^{-1}$,

[3] Since $\{\sigma_k\}_{k\in\mathbb{Z}^n}$, $\{\tau_k\}_{k\in\mathbb{Z}^n} \in \Upsilon$ generate equivalent norms on $M^s_{p,q}$ (cf. [40]), we can always assume that $\sigma_k = \sigma_0(\cdot - k)$ in the definition of $M^s_{p,q}$.

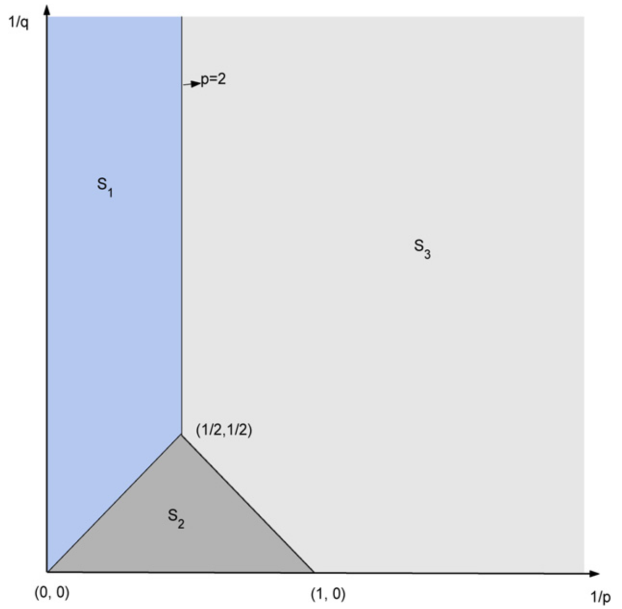

Fig. 1. The distribution of $\tau(p,q)$ in $\mathbb{R}^2_+$: $\tau(p,q)=n(\frac{1}{q}-\frac{1}{p})$ in S_1, $\tau(p,q)=0$ in S_2, and $\tau(p,q)=n(\frac{1}{p}+\frac{1}{q}-1)$ in S_3.

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\qquad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1};$$
$$\frac{1}{p}+\frac{1}{q}-1=(1-\theta)\left(\frac{1}{p_0}-1\right)+\left(\frac{1}{q_1}-\frac{1}{2}\right)\theta.$$

In view of Lemma 2.10 in [40] and Lemma 3.3, we have

$$B^{n(1/q_1-1/2)}_{2,q_1}\subset M_{2,q_1},\qquad B^{n(1/p_0-1)}_{p_0,\infty}\subset M_{p_0,\infty}.$$

A complex interpolation yields

$$B^{n(1/p+1/q-1)}_{p,q}\subset M_{p,q},$$

which implies the result, as desired.

Case 1.2. $(1/p,1/q)\in S_1$. For any $(1/p,1/q)\in \dot{S}_1$ ($\dot{S}_1$ denotes the set of all inner points of S_1), we can interpolate $(1/p,1/q)$ between $(1/\infty,1/\infty)$ and a point $(1/p_1,1/q_1)$ in the line $\{(1/p,1/q)\colon\ p=2, q<2\}$ and finally obtain that

$$B^{n(1/q-1/p)}_{p,q}\subset M_{p,q}. \tag{3.5}$$

Noticing that $\tau(p,q)=n(1/q-1/p)$ if $(1/p,1/q)\in\dot{S}_1$, we see that (3.5) implies the result. If $(1/p,1/q)=(0,1/q)$, the result was shown in [40].

Case 1.3. $(1/p,1/q)\in S_2$. This case has been discussed in [34,40].

Step 2 (Necessity). We need to show that for any $0<\eta\ll 1$,

$$B^{\tau(p,q)-\eta}_{p,q} \not\subset M_{p,q}. \tag{3.6}$$

Case 2.1. $(1/p,1/q)\in S_3$. Let $f=\mathscr{F}^{-1}\delta_j$, $j\gg 1$. It is easy to see that

$$\|f\|^q_{B^{\tau(p,q)-\eta}_{p,q}} = \sum_{\ell=-1}^{1} 2^{q(\tau(p,q)-\eta)(j+\ell)} \big\|\mathscr{F}^{-1}\delta_{j+\ell}\delta_j\big\|^q_p \lesssim 2^{q(n/q-\eta)j}. \tag{3.7}$$

On the other hand, we may assume, without loss of generality that the dyadic decomposition function

$$\delta_j(\xi)=1, \quad \text{if } \xi\in D_j := \left\{\xi\colon\ \frac{5}{4}\cdot 2^{j-1}\leqslant|\xi|\leqslant\frac{3}{4}\cdot 2^{j+1}\right\}.$$

Noticing that the set

$$\Lambda_j=\big\{k\in\mathbb{Z}^n\colon\ B\big(k,\sqrt{n}\big)\subset D_j\big\}$$

contains at least $O(2^{jn})$ many lattice points, we have

$$\|f\|^q_{M_{p,q}} = \sum_{k\in\mathbb{Z}^n}\big\|\Box_k\mathscr{F}^{-1}\delta_j\big\|^q_p \geqslant \sum_{k\in\Lambda_j}\big\|\mathscr{F}^{-1}\sigma_k\delta_j\big\|^q_p \gtrsim 2^{nj}. \tag{3.8}$$

Hence, it follows from (3.7) and (3.8) that

$$\|f\|_{M_{p,q}} \gtrsim 2^{\eta j}\|f\|_{B^{\tau(p,q)-\eta}_{p,q}},$$

which implies (3.6), as desired.

Case 2.2. $(1/p,1/q)\in S_2$. If $q<\infty$, this case has been discussed in [40, Proposition 2.8]. It suffices to consider the case $q=\infty$. Taking $k(j)=(2^j,0,\dots,0)$ and $f=\mathscr{F}^{-1}\sigma_{k(j)}$, we easily see that

$$\|f\|_{M_{p,\infty}} \gtrsim 1 \gtrsim 2^{\eta j}\|f\|_{B^{-\eta}_{p,\infty}}.$$

Case 2.3. $(1/p,1/q)\in S_1$. We can assume, without loss of generality that the frequency-uniform decomposition function $\sigma_k(\xi)=0$ if $\xi\notin\tilde{Q}_k := \{\xi\colon |\xi_i-k_i|\leqslant 5/8,\ 1\leqslant i\leqslant n\}$, and the dyadic decomposition function $\delta_j(\xi)=1$ if $\xi\in D_j (=\{\xi\colon \frac{5}{4}\cdot 2^{j-1}\leqslant|\xi|\leqslant\frac{3}{4}\cdot 2^{j+1}\})$. Let

$$A_j=\big\{k\in\mathbb{Z}^n\colon\ \tilde{Q}_k\subset D_j\big\}, \quad j\gg 1. \tag{3.9}$$

It is easy to see that A_j has at least $O(2^{nj})$ elements. Let $f\in\mathscr{S}(\mathbb{R}^n)$ be a radial function satisfying $\operatorname{supp}\hat{f}\subset B(0,1/8)$ and

$$g(x)=\sum_{k\in A_j} e^{\mathrm{i}xk}(\tau_k f)(x), \quad \tau_k f=f(\cdot-k). \tag{3.10}$$

Noticing that $\operatorname{supp}\widehat{\tau_k f}\subset B(0,1/8)$, we see that $\operatorname{supp}\tau_k(\widehat{\tau_k f})\cap\operatorname{supp}\sigma_\ell=\emptyset$ if $k\neq\ell$. It follows that

$$\begin{aligned}\|g\|_{M_{p,q}} &\geqslant \Big(\sum_{k\in A_j}\big\|\mathscr{F}^{-1}\sigma_k\mathscr{F}g\big\|_p^q\Big)^{1/q}=\Big(\sum_{k\in A_j}\big\|\mathscr{F}^{-1}\sigma_k\tau_k(\widehat{\tau_k f})\big\|_p^q\Big)^{1/q}\\ &=\Big(\sum_{k\in A_j}\big\|\mathscr{F}^{-1}\sigma_0(\widehat{\tau_k f})\big\|_p^q\Big)^{1/q}\gtrsim 2^{jn/q}. \end{aligned}\tag{3.11}$$

On the other hand, we have $\operatorname{supp}\hat g\subset\{\xi\colon 2^{j-1}\leqslant|\xi|\leqslant 2^{j+1}\}$. Hence,

$$\|g\|_{B^{n(1/q-1/p)}_{p,q}}\leqslant\Big(\sum_{\ell=-1}^{1}2^{nj(1-q/p)}\big\|\mathscr{F}^{-1}\delta_{j+\ell}\mathscr{F}g\big\|_p^q\Big)^{1/q}. \tag{3.12}$$

In view of Bernstein's multiplier estimate and Hölder's inequality, we have

$$\big\|\mathscr{F}^{-1}\delta_{j+\ell}\mathscr{F}g\big\|_p\lesssim\|g\|_p\leqslant\|g\|_\infty^{1-2/p}\|g\|_2^{2/p}. \tag{3.13}$$

By Plancherel's identity,

$$\|g\|_2=\|\hat g\|_2=\Big(\int_{\mathbb{R}^n}\sum_{k\in A_j}\big|\tau_k\big(e^{-\mathrm{i}k\xi}\hat f(\xi)\big)\big|^2\,d\xi\Big)^{1/2}\lesssim 2^{nj/2}. \tag{3.14}$$

We can further assume that $f(x)=f(|x|)$ is a monotone function of $|x|$. Since $f\in\mathscr{S}(\mathbb{R}^n)$, we see that

$$\big|f(x-k)\big|\lesssim\big(1+|x-k|\big)^{-N}, \quad N\gg 1. \tag{3.15}$$

We denote

$$B_0=\big\{k\in A_j\colon |x-k|\leqslant 2\big\}, \qquad B_i=\big\{k\in A_j\colon 2^i<|x-k|\leqslant 2^{i+1}\big\}.$$

We easily see that B_i contains at most $O(2^{ni})$ elements. It follows from (3.15) that

$$\big|g(x)\big|\leqslant\sum_{i\geqslant 0}\sum_{k\in B_i}\big|f(x-k)\big|\lesssim f(0)+\sum_{i\geqslant 1}2^{ni}\big|f\big(2^i\big)\big|\lesssim\sum_{i\geqslant 0}2^{(n-N)i}\lesssim 1. \tag{3.16}$$

Collecting (3.13), (3.14) and (3.16), we have

$$\big\|\mathscr{F}^{-1}\delta_{j+\ell}\mathscr{F}g\big\|_p\lesssim 2^{nj/p}. \tag{3.17}$$

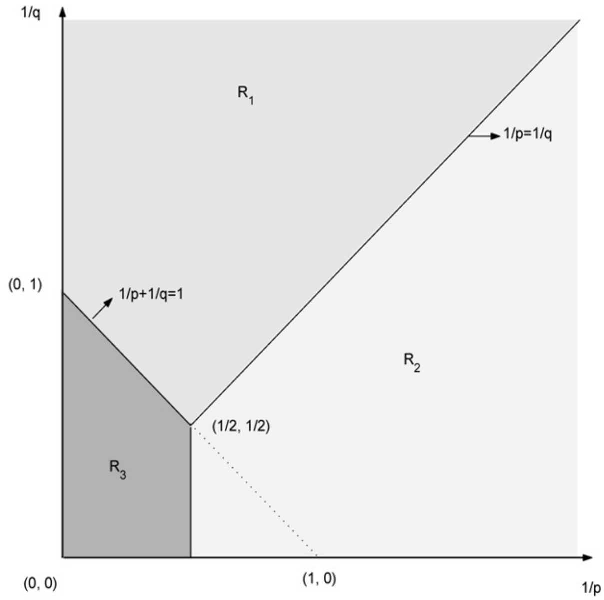

Fig. 2. The distribution of $\sigma(p,q)$ in $\mathbb{R}^2_+$: $\sigma(p,q)=0$ in R_1, $\sigma(p,q)=n(\frac{1}{p}-\frac{1}{q})$ in R_2, and $\sigma(p,q)=n(1-\frac{1}{p}-\frac{1}{q})$ in R_3.

Inserting (3.17) into (3.12), and then using (3.11), we immediately obtain that

$$\|g\|_{B^{n(1/q-1/p)}_{p,q}} \lesssim 2^{nj/q} \lesssim \|g\|_{M_{p,q}}, \tag{3.18}$$

which implies (3.6). This finishes the proof of (1).

Next, we prove (2). The sufficiency has been shown by Toft [34] in the case $1 \leqslant p,q \leqslant \infty$ and by Wang and Hudzik [40] in the case $0 < p,q \leqslant \infty$. Put (see Fig. 2)

$$R_1 = \{(1/p, 1/q) \in \mathbb{R}^2_+:\ 1/q \geqslant 1/p,\ 1/p + 1/q \geqslant 1\};$$
$$R_2 = \{(1/p, 1/q) \in \mathbb{R}^2_+:\ 1/q \leqslant 1/p,\ 1/p \geqslant 1/2\};$$
$$R_3 = \mathbb{R}^2_+ \setminus (R_1 \cup R_2).$$

We now prove the necessity. It suffices to show that

$$M^{\sigma(p,q)-\eta}_{p,q} \not\subset B^0_{p,q}, \quad \forall \eta > 0. \tag{3.19}$$

We will consider the following three separate cases.

Case (i). $(1/p, 1/q) \in R_1$. This case has been discussed in our earlier work [40, Proposition 2.8].

Case (ii). $(1/p, 1/q) \in R_3$. Assume for a contrary that (3.19) does not hold, i.e., there is an $\eta > 0$ such that $M^{n(1-1/p-1/q)-\eta}_{p,q} \subset B^0_{p,q}$. If $1 \leqslant p,q < \infty$, by the duality we have $B^0_{p',q'} \subset$

$M_{p',q'}^{-n(1/p'+1/q'-1)+\eta}$, which contradicts (3.6). If $p=\infty$ or $q=\infty$, we can use the same way as in Case 2.1 to get the result. Indeed, putting $f=\mathscr{F}^{-1}\delta_j$, we have

$$\|f\|_{B^0_{p,q}} \geqslant \left\|\mathscr{F}^{-1}\delta_j\delta_j\right\|_p \gtrsim 2^{nj(1-1/p)}. \tag{3.20}$$

On the other hand,

$$\|f\|_{M^{n/p'}_{p,\infty}} \leqslant \sup_k \langle k\rangle^{n(1-1/p)} \left\|\mathscr{F}^{-1}\sigma_k\delta_j\right\|_p \lesssim 2^{nj(1-1/p)}, \tag{3.21}$$

$$\|f\|_{M^{n/q'}_{\infty,q}} \leqslant \left(\sum_{|k|\in[2^{j-1},2^{j+1}]} 2^{nj(q-1)}\left\|\mathscr{F}^{-1}\sigma_k\delta_j\right\|_\infty^q\right)^{1/q} \lesssim 2^{nj}. \tag{3.22}$$

In view of (3.20)–(3.22) we also have (3.19) in the case $p=\infty$ or $q=\infty$.

Case (iii). $(1/p,1/q)\in R_2$. We use the idea as in Case 2.3. Let $f\in\mathscr{S}(\mathbb{R}^n)$ satisfy $f(0)=1$ and $\operatorname{supp}\hat{f}\subset Q_0$. Taking $0<a\ll 1$ which will be fixed in (3.28), we denote $f_a(x)=f(x/a)$. One easily sees that $\operatorname{supp}\hat{f}_a\subset Q_{0,a}:=\{\xi\colon |\xi_i|\leqslant 1/2a,\ 1\leqslant i\leqslant n\}$. Recall that $D_j=\{\xi\colon \frac{5}{4}\cdot 2^{j-1}\leqslant|\xi|\leqslant\frac{3}{4}\cdot 2^{j+1}\}$ contains at least $O(a^n2^{jn})$ many pairwise disjoint cubes $Q_{k(i),a}:=k(i)+Q_{0,a}$, $i=1,\dots,O(a^n2^{jn})$. Denote $A_j=\{k(i)\colon i=1,\dots,O(a^n2^{jn})\}$ and

$$g(x)=\sum_{k\in A_j} e^{\mathrm{i}xk}(\tau_k f_a)(x). \tag{3.23}$$

Since for any $N\gg 1$,

$$\left|f(x)\right|\leqslant C_N\left(1+|x|\right)^{-N}, \tag{3.24}$$

we see that

$$\left|f_a(x)\right|\leqslant C_N a^N|x|^{-N}. \tag{3.25}$$

By the continuity of $f(x)$ and $f(0)=1$, one sees that there exists $\varrho>0$ such that[4]

$$\left|f_a(x)\right|>1/2, \quad x\in B(0,\varrho). \tag{3.26}$$

It follows from (3.23) and (3.26) that for any $x\in B(k(i),\varrho)$,

$$\begin{aligned}\left|g(x)\right| &\geqslant \left|f_a\bigl(x-k(i)\bigr)\right| - \sum_{k\in A_j\setminus\{k(i)\}}\left|f_a(x-k)\right| \\ &\geqslant \frac{1}{2} - \sum_{k\in A_j\setminus\{k(i)\}}\left|f_a(x-k)\right|. \end{aligned}\tag{3.27}$$

[4] It is easy to see that ϱ can be chosen as $\varrho=a\varrho_0$, $\varrho_0>0$ depends only on f and is independent of a.

We write $A_{j,\ell} := \{k \in A_j\colon\ 2^\ell \leqslant |k - k(i)| < 2^{\ell+1}\}$. One can further assume that $f(x)$ is a monotone function of $|x|$. Since $A_{j,\ell}$ has at most $O(a^n 2^{\ell n})$ elements, we have for any $x \in B(k(i), \varrho)$,

$$\sum_{k\in A_j\setminus\{k(i)\}} \left|f_a(x-k)\right| \leqslant \sum_{\ell\geqslant 1}\sum_{k\in A_{j,\ell}} \left|f_a(x-k)\right| \leqslant C\sum_{\ell\geqslant 1} a^n 2^{n\ell}\left|f_a\big(2^\ell-\varrho\big)\right|$$
$$\lesssim \sum_{\ell\geqslant 1} C_N a^{n+N} 2^{(n-N)\ell} \leqslant 1/4, \tag{3.28}$$

where we have chosen $N \geqslant n+1$ and $CC_N a^{n+N} \leqslant 1/4$. Hence, it follows from (3.27) and (3.28) that

$$\left|g(x)\right| \geqslant 1/4, \quad x \in B\big(k(i), \varrho\big). \tag{3.29}$$

Thus, in view of (3.29), we have

$$\left\|g(x)\right\|_p \geqslant \left\| \sum_{i=1}^{O(a^n 2^{nj})} \frac{1}{4}\chi_{B(k(i),\varrho)} \right\|_p \gtrsim (a\varrho)^{n/p} 2^{nj/p}, \tag{3.30}$$

where ϱ and a are independent of $j \gg 1$. We can assume, without loss of generality that $\delta_j(\xi) = 1$ if $\xi \in D_j$. Due to $\operatorname{supp}\hat{g} \subset D_j$, we have $\mathscr{F}^{-1}\delta_j\mathscr{F}g = g$. So, (3.30) implies that

$$\|g\|_{B^0_{p,q}} \geqslant \left\|\mathscr{F}^{-1}\delta_j\mathscr{F}g\right\|_p \gtrsim (a\varrho)^{n/p} 2^{nj/p}. \tag{3.31}$$

On the other hand,

$$\|g\|_{M^{n(1/p-1/q)}_{p,q}} = \left(\sum_{|k|\in[2^{j-1},2^{j+1}]} 2^{nj(q/p-1)}\left\|\mathscr{F}^{-1}\sigma_k\mathscr{F}g\right\|_p^q\right)^{1/q}$$
$$\leqslant 2^{nj/p} \sup_{|k|\in[2^{j-1},2^{j+1}]} \left\|\mathscr{F}^{-1}\sigma_k\mathscr{F}g\right\|_p. \tag{3.32}$$

Since $\operatorname{supp}\sigma_k$ overlaps at most finite many $\operatorname{supp}\tau_\ell(\widehat{\tau_\ell f_a})$, in view of Bernstein's multiplier estimate, one easily sees that

$$\left\|\mathscr{F}^{-1}\sigma_k\mathscr{F}g\right\|_p \lesssim \|f\|_p. \tag{3.33}$$

So, we have from (3.32) and (3.33) that

$$\|g\|_{M^{n(1/p-1/q)}_{p,q}} \lesssim 2^{nj/p}. \tag{3.34}$$

By (3.31) and (3.34) we immediately have (3.19). We have finished the proof of Theorem 1.1. $\square$

In view of Theorem 1.1 and the sharp embeddings between Besov spaces:

$$B^{s_0}_{p_0,q} \subset B^s_{p,q}, \quad s_0 - \frac{n}{p_0} \geqslant s - \frac{n}{p},\ s_0 \geqslant s,\ p_0 \leqslant p, \tag{3.35}$$

we get the following

Corollary 3.4. *Let* $0 < p_0, p, q_0, q \leqslant \infty$, $s_0, s \in \mathbb{R}$ *and let* $\tau(p,q)$, $\sigma(p,q)$ *be as in Theorem* 1.1.

(1) *Let* $p_0 \leqslant p$. *Then* $B^{s_0}_{p_0,q_0} \subset M^s_{p,q}$ *if and only if* $s_0 - n/p_0 \geqslant s - n/p + \tau(p,q)$, $s_0 \geqslant s$, $q_0 \leqslant q$.
(2) *Let* $p \leqslant p_0$. *Then* $M^s_{p,q} \subset B^{s_0}_{p_0,q_0}$ *if and only if* $s - n/p \geqslant s_0 - n/p_0 + \sigma(p,q)$, $s_0 \leqslant s$, $q_0 \geqslant q$.

4. Smooth effects of the Schrödinger semigroup

In the sequel, we will always assume that the spatial dimension $n = 1$. Since the Schrödinger semigroup and the BO semigroup have the same basic estimates (see below, Lemmas 4.1 and 4.3), it follows that the Schrödinger semigroup and the BO semigroup enjoy the same smooth effects and so, it suffices to consider the case of the Schrödinger semigroup. We will mainly use the sharp version of Kato's smoothing effects described by the following lemma which is due to Kenig, Ponce and Vega; cf. [18,19].

Lemma 4.1. *Let* $W(t) = \mathscr{F}^{-1} e^{\mathrm{i}t\xi^2} \mathscr{F}$. *Then we have*

$$\big\| W(t) f \big\|_{L^\infty_x L^2_t} \lesssim \|f\|_{\dot H^{-1/2}}, \tag{4.1}$$

$$\big\| W(t) f \big\|_{L^4_x L^\infty_t} \lesssim \|f\|_{\dot H^{1/4}}, \tag{4.2}$$

$$\big\| W(t) f \big\|_{L^\infty_t L^2_x} \lesssim \|f\|_2. \tag{4.3}$$

We will use Lemma 4.1 deriving some smoothing effect estimates in local frequency spaces. The smooth effect estimates for the dyadic localization to the frequency spaces were obtained by Molinet and Ribaud [25,26]. Our idea is to substitute the dyadic decomposition by the uniform partition to the frequency spaces, which enable us to deal with a class of initial data with lower regularity indices (one can compare (4.2) with (4.5) below). We have

Lemma 4.2. *Let* $W(t) = \mathscr{F}^{-1} e^{\mathrm{i}t\xi^2} \mathscr{F}$. *Then we have for any* $p \geqslant 4$ *and* $k \in \mathbb{Z}$,

$$\big\| \Box_k W(t) f \big\|_{L^\infty_x L^2_t} \lesssim \|\Box_k f\|_{\dot H^{-1/2}}, \tag{4.4}$$

$$\big\| \Box_k W(t) f \big\|_{L^p_x L^\infty_t} \lesssim \|\Box_k f\|_{\dot H^{1/p}}, \tag{4.5}$$

$$\big\| \Box_k W(t) f \big\|_{L^\infty_t L^2_x} \lesssim \|\Box_k f\|_2, \tag{4.6}$$

where the omitted constants in (4.4)–(4.6) *are independent of* $k \in \mathbb{Z}$.

Proof. By Lemma 4.1, it suffices to show that (4.5) holds. In fact, in view of (4.2) and Plancherel's equality, we have

$$\left\|\Box_k W(t)f\right\|_{L_x^4 L_t^\infty} \lesssim \|f\|_{\dot H^{1/4}}. \tag{4.7}$$

By Hausdorff–Young's, Hölder's inequalities and Plancherel's identity,

$$\begin{aligned}\left\|\Box_k W(t)f\right\|_{L_t^\infty L_x^\infty} &\lesssim \left\|e^{\mathrm{i}t\xi^2}\sigma_k \mathscr{F} f\right\|_{L_t^\infty L_x^1}\\ &\lesssim \|\sigma_k\|_2 \|\mathscr{F} f\|_{L_x^2} \lesssim \|f\|_2.\end{aligned} \tag{4.8}$$

We emphasize that the omitted constants in (4.7) and (4.8) are independent of $k \in \mathbb{Z}$. A complex interpolation between (4.7) and (4.8) yields

$$\left\|\Box_k W(t)f\right\|_{L_x^p L_t^\infty} \lesssim \|f\|_{\dot H^{1/p}}. \tag{4.9}$$

We can assume that $\operatorname{supp}\sigma_0 \subset (-1,1)$, it follows that $\sigma_k \sum_{|\ell|\leqslant 1} \sigma_{k+\ell} = \sigma_k$. In view of (4.9), we have

$$\left\|\Box_k W(t)f\right\|_{L_x^p L_t^\infty} \lesssim \sum_{|\ell|\leqslant 1} \left\|\Box_{k+\ell} W(t)\Box_k f\right\|_{L_x^p L_t^\infty} \lesssim \|\Box_k f\|_{\dot H^{1/p}}, \tag{4.10}$$

which implies the result, as desired. □

Next, we consider the estimates for the integral operator $\int_0^t W(t-\tau)f(\tau)\,d\tau$. The following result is also due to Kenig, Ponce and Vega [18]:

Lemma 4.3. *Let $W(t) = \mathscr{F}^{-1} e^{\mathrm{i}t\xi^2}\mathscr{F}$. Then we have*

$$\left\|\int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^1 L_t^2}, \tag{4.11}$$

$$\left\|\int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L_t^\infty L_x^2} \lesssim \left\|D_x^{1/2} f\right\|_{L_x^1 L_t^2}. \tag{4.12}$$

On the basis of Lemma 4.3, we have

Lemma 4.4. *Let $W(t) = \mathscr{F}^{-1} e^{\mathrm{i}t\xi^2}\mathscr{F}$, $D_x^s = (-\partial_x^2)^{s/2}$. Then we have for any $p \geqslant 4$ and $k \in \mathbb{Z}$,*

$$\left\|\Box_k \int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L_x^\infty L_t^2} \lesssim \|\Box_k f\|_{L_x^1 L_t^2}, \tag{4.13}$$

$$\left\|\Box_k \int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L_t^\infty L_x^2} \lesssim \left\|D_x^{1/2}\Box_k f\right\|_{L_x^1 L_t^2}, \tag{4.14}$$

$$\left\|\Box_k \int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L_x^p L_t^\infty} \lesssim \left\|D_x^{1/2+1/p}\Box_k f\right\|_{L_x^1 L_t^2}, \tag{4.15}$$

where the omitted constants in (4.13)–(4.15) *are independent of $k \in \mathbb{Z}$.*

Proof. We will use the dual estimate method. By Lemma 4.3, we see that (4.13) and (4.14) uniformly hold for all $k \in \mathbb{Z}$, i.e., the omitted constants are independent of $k \in \mathbb{Z}$. So, it suffices to show that (4.15) holds. First, we prove that

$$\left\| \Box_k \int_{\mathbb{R}} W(t-\tau) f(\tau)\, d\tau \right\|_{L_t^\infty L_x^2} \lesssim \left\| D_x^{1/p} \Box_k f \right\|_{L_x^{p'} L_t^1}. \tag{4.16}$$

Indeed, for any $f, g \in \dot{\mathscr{S}}(\mathbb{R}^2)^5$, we have from Hölder's inequality and Lemma 4.2 that,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} W(t-\tau) f(\tau)\, d\tau, g(t) \right) dt \right| \\
&\quad = \left| \int_{\mathbb{R}} \left(\Box_k f(\tau), \int_{\mathbb{R}} W(\tau - t) g(t)\, dt \right) d\tau \right| \\
&\quad \lesssim \left\| D_x^{1/p} \Box_k f \right\|_{L_x^{p'} L_t^1} \sum_{|\ell| \leqslant 1} \left\| D_x^{-1/p} \Box_{k+\ell} \int_{\mathbb{R}} W(\cdot - t) g(t)\, dt \right\|_{L_x^p L_\tau^\infty} \\
&\quad \lesssim \left\| D_x^{1/p} \Box_k f \right\|_{L_x^{p'} L_t^1} \sum_{|\ell| \leqslant 1} \| \Box_{k+\ell} g \|_{L_t^1 L_x^2} \\
&\quad \lesssim \left\| D_x^{1/p} \Box_k f \right\|_{L_x^{p'} L_t^1} \| g \|_{L_t^1 L_x^2},
\end{aligned} \tag{4.17}$$

whence, (4.16) uniformly hold for all $k \in \mathbb{Z}$. In view of (4.12) and (4.16), we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} W(t-\tau) f(\tau)\, d\tau, g(t) \right) dt \right| \\
&\quad \lesssim \left\| \Box_k D_x^{1/p} \int_{\mathbb{R}} W(-\tau) f(\tau)\, d\tau \right\|_2 \sum_{|\ell| \leqslant 1} \left\| \Box_{k+\ell} D_x^{-1/p} \int_{\mathbb{R}} W(-t) g(t)\, dt \right\|_2 \\
&\quad \lesssim \left\| D_x^{-1/2+1/p} \Box_k f \right\|_{L_x^1 L_t^2} \sum_{|\ell| \leqslant 1} \| \Box_{k+\ell} g \|_{L_x^{p'} L_t^1} \\
&\quad \lesssim \left\| D_x^{-1/2+1/p} \Box_k f \right\|_{L_x^1 L_t^2} \| g \|_{L_x^{p'} L_t^1},
\end{aligned} \tag{4.18}$$

which implies that

$$\left\| \Box_k \int_{\mathbb{R}} W(t-\tau) \partial_x f(\tau)\, d\tau \right\|_{L_x^p L_t^\infty} \lesssim \left\| D_x^{1/2+1/p} \Box_k f \right\|_{L_x^1 L_t^2} \tag{4.19}$$

uniformly hold for all $k \in \mathbb{Z}$. By Christ–Kiselev's lemma (cf. [25]) and (4.19), we immediately have (4.15). □

[5] $\dot{\mathscr{S}}(\mathbb{R}^2) = \{ f \in \mathscr{S}(\mathbb{R}^2)\colon D^\alpha \mathscr{F} f(0) = 0,\ \forall \alpha \}$.

Lemma 4.5. *Let $W(t)=\mathscr{F}^{-1}e^{\mathrm{i}t\xi^2}\mathscr{F}$. Then we have for any $r\geqslant 6$, $p\geqslant 4$ and $k\in\mathbb{Z}$,*

$$\big\|\Box_k W(t)f\big\|_{L^r_{x,t}}\lesssim\|\Box_k f\|_2, \tag{4.20}$$

$$\left\|\Box_k\int_0^t W(t-\tau)f(\tau)\,d\tau\right\|_{L^r_{x,t}\cap(L^\infty_t L^2_x)}\lesssim\|\Box_k f\|_{L^{r'}_{x,t}}, \tag{4.21}$$

$$\left\|\Box_k\int_0^t W(t-\tau)f(\tau)\,d\tau\right\|_{L^\infty_x L^2_t}\lesssim\big\|D_x^{-1/2}\Box_k f\big\|_{L^{r'}_{x,t}}, \tag{4.22}$$

$$\left\|\Box_k\int_0^t W(t-\tau)f(\tau)\,d\tau\right\|_{L^p_x L^\infty_t}\lesssim\big\|D_x^{1/p}\Box_k f\big\|_{L^{r'}_{x,t}}, \tag{4.23}$$

$$\left\|\Box_k\int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau\right\|_{L^r_{x,t}}\lesssim\big\|D_x^{1/2}\Box_k f\big\|_{L^1_x L^2_t}, \tag{4.24}$$

where the omitted constants in (4.20)–(4.24) *are independent of $k\in\mathbb{Z}$.*

Proof. By the Strichartz estimate, we have (cf. [17])

$$\big\|W(t)f\big\|_{L^6_{x,t}}\lesssim\|f\|_2, \tag{4.25}$$

which can also be derived by an interpolation between the inequalities in (4.1) and (4.2). It follows from (4.25) and (4.8) that

$$\big\|\Box_k W(t)f\big\|_{L^6_{x,t}}\lesssim\|f\|_2, \tag{4.26}$$

$$\big\|\Box_k W(t)f\big\|_{L^\infty_{x,t}}\lesssim\|f\|_2. \tag{4.27}$$

Interpolating between (4.26) and (4.27), then using the almost orthogonal properties of $\{\Box_k\}$, we have (4.20). By a standard dual estimate method, we have (4.21), see [40] for details.

Analogous to (4.18), we can use (4.12) and (4.21) to get that

$$\left|\int_{\mathbb{R}}\left(\Box_k\int_{\mathbb{R}}W(t-\tau)f(\tau)\,d\tau,\ g(t)\right)dt\right|\lesssim\big\|D_x^{-1/2}\Box_k f\big\|_{L^{r'}_{x,t}}\|g\|_{L^1_x L^2_t}, \tag{4.28}$$

$$\left|\int_{\mathbb{R}}\left(\Box_k\int_{\mathbb{R}}W(t-\tau)f(\tau)\,d\tau,\ g(t)\right)dt\right|\lesssim\big\|D_x^{-1/2}\Box_k f\big\|_{L^1_x L^2_t}\|g\|_{L^{r'}_{x,t}}, \tag{4.29}$$

which implies (4.22) and (4.24), as desired.

Analogous to (4.18), by (4.16) and (4.21) we have

$$\left|\int_{\mathbb{R}}\left(\Box_k\int_{\mathbb{R}} W(t-\tau)f(\tau)\,d\tau,\ g(t)\right)dt\right| \lesssim \left\|D_x^{1/p}\Box_k f\right\|_{L^{r'}_{x,t}}\|g\|_{L_x^{p'}L_t^1}, \tag{4.30}$$

which implies (4.23). □

5. Proofs of Theorems 1.3–1.5

In this section we prove our Theorem 1.3. Our main idea is to apply the smooth effect estimates obtained in Section 4 and we consider the following resolution space:

$$X=\left\{u\in\mathscr{S}'\big(\mathbb{R}^{1+1}\big):\ \|u\|_X=\sum_{k\in\mathbb{Z}}|||\Box_k u|||+\sum_{|k|\geqslant K}\langle k\rangle^{1/2+1/\kappa}\|\Box_k u\|_{L_x^\infty L_t^2}\leqslant\rho\right\},$$

where

$$|||\Box_k u||| = \|\Box_k u\|_{L_x^\kappa L_t^\infty}+\langle k\rangle^{1/\kappa}\|\Box_k u\|_{L_{x,t}^{2+\kappa}}, \tag{5.1}$$

and $K\geqslant 2$ which will be fixed as in the following. The construction of the above resolution space follows some ideas as in [31], [25,26] and [40]. However, Planchon [31], Molinet and Ribaud [25,26] mainly used the dyadic decomposition to the frequency space; the frequency-uniform decomposition was applied in [38,40], but the dispersive smooth effects were not involved in [38,40]. We consider the mapping

$$\mathscr{T}: u(t)\to W(t)u_0-\mathscr{A}\big(u^\kappa\partial_x u\big), \tag{5.2}$$

where we denote

$$W(t)=\mathscr{F}^{-1}e^{\mathrm{i}t|\xi|\xi}\mathscr{F},\qquad \mathscr{A}=\int_0^t W(t-\tau)\cdot d\tau. \tag{5.3}$$

We now give a brief explanation to the resolution space X. Recall that (4.13) is the only known estimate which can be applied for controlling the 1-order derivative in the nonlinearity, so $L_x^\infty L_t^2$ is introduced in the working space X. However, taking $k=0$ in (4.13), one has that

$$\|\Box_0\mathscr{A}f\|_{L_x^\infty L_t^2}\lesssim\left\|D_x^{-1}\Box_0 f\right\|_{L_x^1L_t^2},$$

which is an extremely bad estimate for the low frequency case. So, (4.13) has no use to the low frequency case. To overcome this difficulty, our idea is to use the space $L_{x,t}^{2+\kappa}$ and the Strichartz estimates treating the low frequency part of the solutions, see Lemma 4.5. By (4.13), in order to estimate $\|\Box_k(u^{\kappa+1})\|_{L_x^1L_t^2}$, we need to introduce $L_x^\kappa L_t^\infty$ in X. In view of (4.5), the initial data $u_0\in M_{2,1}^{1/\kappa}$ seems necessary.

Proof of Theorem 1.3. For $|k| \geqslant K \geqslant 2$, in view of Lemmas 4.2, 4.5 and Bernstein's multiplier estimates,

$$\langle k\rangle^{1/2+1/\kappa}\big\|\Box_k W(t)u_0\big\|_{L_x^\infty L_t^2} \lesssim \langle k\rangle^{1/2+1/\kappa}\|\Box_k u_0\|_{\dot H^{-1/2}} \lesssim \langle k\rangle^{1/\kappa}\|\Box_k u_0\|_2, \tag{5.4}$$

$$\big\|\Box_k W(t)u_0\big\|_{L_x^\kappa L_t^\infty} \lesssim \|\Box_k u_0\|_{\dot H^{1/\kappa}} \lesssim \langle k\rangle^{1/\kappa}\|\Box_k u_0\|_2, \tag{5.5}$$

$$\langle k\rangle^{1/\kappa}\big\|\Box_k W(t)u_0\big\|_{L_{x,t}^{2+\kappa}} \lesssim \langle k\rangle^{1/\kappa}\|\Box_k u_0\|_2. \tag{5.6}$$

Notice that (5.5) and (5.6) hold for all $k \in \mathbb{Z}$. It follows from (5.4)–(5.6) that

$$\big\|\Box_k W(t)u_0\big\|_X \lesssim \|u_0\|_{M_{2,1}^{1/\kappa}}. \tag{5.7}$$

Now we consider the estimates of $\|\mathscr{A}(u^\kappa \partial_x u)\|_X$. Using the frequency-uniform decomposition, one has that

$$\begin{aligned}
&\sum_{|k|\geqslant K}\langle k\rangle^{1/2+1/\kappa}\big\|\Box_k\mathscr{A}\big(u^\kappa\partial_x u\big)\big\|_{L_x^\infty L_t^2}\\
&\quad\lesssim \sum_{|k|\geqslant K}\ \sum_{k_1,\dots,k_{\kappa+1}\in\mathbb{Z}}\langle k\rangle^{1/2+1/\kappa}\big\|\partial_x\mathscr{A}\Box_k(\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u)\big\|_{L_x^\infty L_t^2}\\
&\quad= \sum_{|k|\geqslant K}\ \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|\geqslant K}\langle k\rangle^{1/2+1/\kappa}\big\|\partial_x\mathscr{A}\Box_k(\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u)\big\|_{L_x^\infty L_t^2}\\
&\qquad+ \sum_{|k|\geqslant K}\ \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|< K}\langle k\rangle^{1/2+1/\kappa}\big\|\partial_x\mathscr{A}\Box_k(\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u)\big\|_{L_x^\infty L_t^2}\\
&\quad:= \mathrm{I}+\mathrm{II}.
\end{aligned}\tag{5.8}$$

Since $\operatorname{supp}\sigma_k \subset (k-1,k+1)$ and $\operatorname{supp}(\sigma_{k_1}\hat u)*\cdots*(\sigma_{k_{\kappa+1}}\hat u) \subset B(k_1+\cdots+k_{\kappa+1},\kappa+1)$, we have from Lemma 4.4 and Bernstein's estimate (1.21) that

$$\begin{aligned}
\mathrm{I} &= \sum_{|k|\geqslant K}\ \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|\geqslant K}\langle k\rangle^{1/2+1/\kappa}\big\|\partial_x\mathscr{A}\Box_k(\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u)\big\|_{L_x^\infty L_t^2}\,\chi_{(|k-k_1-\cdots-k_{\kappa+1}|\leqslant\kappa+2)}\\
&\lesssim \sum_{|k|\geqslant K}\ \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|\geqslant K}\langle k\rangle^{1/2+1/\kappa}\|\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u\|_{L_x^1 L_t^2}\,\chi_{(|k-k_1-\cdots-k_{\kappa+1}|\leqslant\kappa+2)}.
\end{aligned}\tag{5.9}$$

We may assume, without loss of generality that in (5.9), $|k_1| = |k_1|\vee\cdots\vee|k_{\kappa+1}|$. It follows from (5.9), Young's and Hölder's inequalities that

$$\mathrm{I} \lesssim \Big(\sum_{|k_1|\geqslant K}\langle k_1\rangle^{1/2+1/\kappa}\|\Box_{k_1}u\|_{L_x^\infty L_t^2}\Big)\prod_{i=2}^{\kappa+1}\sum_{k_i\in\mathbb{Z}}\|\Box_{k_i}u\|_{L_x^\kappa L_t^\infty} \lesssim \|u\|_X^{\kappa+1}. \tag{5.10}$$

We estimate II. By Lemma 4.5,

$$\begin{aligned}\mathrm{II} &\leqslant C_K \sum_{|k|\geqslant K}\ \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|<K} \big\|\Box_k(\Box_{k_1}u\ldots\Box_{k_{\kappa+1}}u)\big\|_{L_{x,t}^{\frac{2+\kappa}{1+\kappa}}}\ \chi_{(|k-k_1-\cdots-k_{\kappa+1}|\leqslant\kappa+2)}\\ &\leqslant C_K \sum_{|k_1|\vee\cdots\vee|k_{\kappa+1}|<K} \|\Box_{k_1}u\|_{L_{x,t}^{2+\kappa}}\ldots\|\Box_{k_{\kappa+1}}u\|_{L_{x,t}^{2+\kappa}} \leqslant C_K\|u\|_X^{\kappa+1}. \qquad (5.11)\end{aligned}$$

In view of (5.8), (5.10) and (5.11), we obtain that

$$\sum_{|k|\geqslant K} \langle k\rangle^{1/2+1/\kappa}\big\|\Box_k\mathscr{A}\big(u^\kappa\partial_x u\big)\big\|_{L_x^\infty L_t^2} \lesssim \|u\|_X^{\kappa+1}. \qquad (5.12)$$

We consider the estimate of $\sum_{k\in\mathbb{Z}} |||\Box_k\mathscr{A}(u^\kappa\partial_x u)|||$. By (4.21) and (4.24) in Lemma 4.5 and Bernstein's estimate (1.21),

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/\kappa}\big\|\Box_k\mathscr{A}\big(u^\kappa\partial_x u\big)\big\|_{L_{x,t}^{2+\kappa}} &\lesssim \sum_{|k|<(\kappa+2)K}\langle k\rangle^{1/\kappa}\big\|\partial_x\Box_k\big(u^{\kappa+1}\big)\big\|_{L_{x,t}^{\frac{2+\kappa}{1+\kappa}}}\\ &\quad+\sum_{|k|\geqslant(\kappa+2)K}\langle k\rangle^{1/\kappa}\big\|D_x^{1/2}\Box_k\big(u^{\kappa+1}\big)\big\|_{L_x^1L_t^2}\\ &\leqslant C_K\sum_{|k|<(\kappa+2)K}\big\|\Box_k\big(u^{\kappa+1}\big)\big\|_{L_{x,t}^{\frac{2+\kappa}{1+\kappa}}}\\ &\quad+\sum_{|k|\geqslant(\kappa+2)K}\langle k\rangle^{1/\kappa}\big\|D_x^{1/2}\Box_k\big(u^{\kappa+1}\big)\big\|_{L_x^1L_t^2}\\ &:=\mathrm{III}+\mathrm{IV}. \qquad (5.13)\end{aligned}$$

Analogous to the estimates of II, we see that

$$\mathrm{III}\leqslant C_K\|u\|_X^{\kappa+1}. \qquad (5.14)$$

Now we consider the estimate of IV. It follows from Minkowski's and Young's inequalities that

$$\begin{aligned}\big\|D_x^{1/2}\Box_k u^{\kappa+1}\big\|_{L_x^1L_t^2} &\lesssim \sum_{|\ell|\leqslant 1}\big\|\mathscr{F}^{-1}\big(\sigma_{k+\ell}|\xi|^{1/2}\big)*\Box_k u^{\kappa+1}\big\|_{L_x^1L_t^2}\\ &\lesssim \sum_{|\ell|\leqslant 1}\big\|\mathscr{F}^{-1}\big(\sigma_{k+\ell}|\xi|^{1/2}\big)\big\|_{L_x^1}\big\|\Box_k u^{\kappa+1}\big\|_{L_x^1L_t^2}. \qquad (5.15)\end{aligned}$$

By Bernstein's multiplier estimates,

$$\big\|\mathscr{F}^{-1}\big(\sigma_{k+\ell}|\xi|^{1/2}\big)\big\|_{L_x^1}\lesssim\langle k\rangle^{1/2}. \qquad (5.16)$$

So,

$$\mathrm{IV}\lesssim\sum_{|k|\geqslant(\kappa+2)K}\langle k\rangle^{1/2+1/\kappa}\big\|\Box_k\big(u^{\kappa+1}\big)\big\|_{L_x^1L_t^2}. \qquad (5.17)$$

Similar to (5.9),

$$\mathrm{IV} \lesssim \sum_{|k|\geqslant(\kappa+2)K} \sum_{k_1,\dots,k_{\kappa+1}\in\mathbb{Z}} \langle k\rangle^{1/2+1/\kappa} \|\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u\|_{L^1_xL^2_t}\, \chi_{(|k-k_1-\dots-k_{\kappa+1}|\leqslant\kappa+2)}. \tag{5.18}$$

It is easy to see that in the summation of (5.18), $|k_1|\vee\cdots\vee|k_{\kappa+1}|\geqslant K$ if $K\geqslant\kappa+1$. Hence, we can repeat the procedure as in the estimates of I to obtain that

$$\mathrm{IV} \leqslant C_K\|u\|_X^{\kappa+1}. \tag{5.19}$$

So, collecting (5.13), (5.14) and (5.19), we obtain that

$$\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/\kappa}\big\|\Box_k\mathscr{A}\big(u^\kappa\partial_xu\big)\big\|_{L^{2+\kappa}_{x,t}} \lesssim \|u\|_X^{\kappa+1}. \tag{5.20}$$

By Lemmas 4.4 and 4.5,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\big\|\Box_k\mathscr{A}\big(u^\kappa\partial_xu\big)\big\|_{L^\kappa_xL^\infty_t} &\lesssim \sum_{|k|<(\kappa+2)K}\big\|\Box_k\big(u^{\kappa+1}\big)\big\|_{L^{\frac{2+\kappa}{1+\kappa}}_{x,t}}\\ &\quad+\sum_{|k|\geqslant(\kappa+2)K}\big\|D_x^{1/2+1/\kappa}\Box_ku^{\kappa+1}\big\|_{L^1_xL^2_t},\end{aligned} \tag{5.21}$$

which reduces to the estimates of III and IV as in (5.13). Hence,

$$\sum_{k\in\mathbb{Z}}\big|\!\big|\!\big|\Box_k\mathscr{A}\big(u^\kappa\partial_xu\big)\big|\!\big|\!\big| \lesssim \|u\|_X^{\kappa+1}. \tag{5.22}$$

Now, collecting (5.7) and (5.22), we have

$$\|\mathscr{T}u\|_X \lesssim \|u_0\|_{M^{1/\kappa}_{2,1}} + \|u\|_X^{\kappa+1}. \tag{5.23}$$

Similarly,

$$\|\mathscr{T}u-\mathscr{T}v\|_X \lesssim \big(\|u\|_X^\kappa+\|v\|_X^\kappa\big)\|u-v\|_X. \tag{5.24}$$

So, by the standard contraction mapping argument, we see that (1.2) has a solution $u\in X$. Moreover, this solution is unique in X. By (4.6) and (4.14), in an analogous way as above, we have $u\in C(\mathbb{R}, M^{1/\kappa}_{2,1})$. This finishes the proof of Theorem 1.3. □

Since all of the estimates obtained in Section 4 adapt to both the BO and the Schrödinger semigroups, we see that Theorem 1.4 can be shown in the same way as in the proof of Theorem 1.3 and the details will be omitted.

Proof of Theorem 1.5. We follow the proof of Theorem 1.3 and the proof will be outlined. Put

$$\begin{aligned} X &= \Big\{u\in\mathscr{S}'\big(\mathbb{R}^{1+1}\big):\ \|u\|_X=\sum_{k\in\mathbb{Z}}|\!|\!|\Box_ku|\!|\!| + \sum_{|k|\geqslant2}\langle k\rangle^{1/2+1/\kappa}\|\Box_ku\|_{L^\infty_xL^2_t}\leqslant\rho\Big\},\\ &\qquad |\!|\!|\Box_ku|\!|\!| = \|\Box_ku\|_{L^\kappa_xL^\infty_t}+\langle k\rangle^{1/\kappa}\|\Box_ku\|_{L^{2+\kappa}_{x,t}\cap(L^\infty_tL^2_x)}.\end{aligned} \tag{5.25}$$

We consider the mapping

$$\mathscr{T}: u(t) \to S(t)u_0 + \mathrm{i}\mu\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa} - 1\big)u\big), \tag{5.26}$$

where

$$S(t) = \mathscr{F}^{-1}e^{-\mathrm{i}t\xi^2}\mathscr{F}, \qquad \mathscr{A} = \int_0^t S(t-\tau)\cdot d\tau. \tag{5.27}$$

Following (5.7) and (4.14),

$$\big\|S(t)u_0\big\|_X \lesssim \|u_0\|_{M^{1/\kappa}_{2,1}}. \tag{5.28}$$

It follows from Taylor's expansion of $e^{|u|^\kappa}$ that

$$\begin{aligned}
&\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/\kappa}\big\|\Box_k\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa}-1\big)u\big)\big\|_{L^\infty_x L^2_t}\\
&\quad\lesssim \sum_{j\geqslant 1}\frac{1}{j!}\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/\kappa}\sum_{k_1,\dots,k_{\kappa j+1}\in\mathbb{Z}}\big\|\Box_k\mathscr{A}\partial_x(\Box_{k_1}\tilde{u}\dots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^\infty_x L^2_t}\\
&\quad\lesssim \sum_{j\geqslant 1}\frac{1}{j!}\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/\kappa}\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\big\|\Box_k(\Box_{k_1}\tilde{u}\dots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^1_x L^2_t}\\
&\qquad+\sum_{j\geqslant 1}\frac{1}{j!}\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/\kappa}\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|<2}\big\|\Box_k(\Box_{k_1}\tilde{u}\dots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^{\frac{2+\kappa}{1+\kappa}}_{x,t}}\\
&\quad:= \mathrm{I}+\mathrm{II},
\end{aligned}\tag{5.29}$$

where $\tilde{u}$ denotes u or $\bar{u}$. Using the same way as in the proof of Theorem 1.3, one has that for $|k_1| = |k_1|\vee\cdots\vee|k_{\kappa j+1}|$,

$$\begin{aligned}
&\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\big\|\Box_k(\Box_{k_1}\tilde{u}\dots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^1_x L^2_t}\\
&\quad\leqslant \sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\|\Box_{k_1}\tilde{u}\dots\Box_{k_{\kappa j+1}}\tilde{u}\|_{L^1_x L^2_t}\chi_{(|k-k_1-\cdots-k_{\kappa j+1}|\leqslant\kappa j+2)}\\
&\quad\leqslant \sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\|\Box_{k_1}\tilde{u}\|_{L^\infty_x L^2_t}\prod_{i=2}^{\kappa+1}\|\Box_{k_i}\tilde{u}\|_{L^\kappa_x L^\infty_t}\\
&\qquad\times\prod_{i=\kappa+2}^{\kappa j+1}\|\Box_{k_i}\tilde{u}\|_{L^\infty_{x,t}}\chi_{(|k-k_1-\cdots-k_{\kappa j+1}|\leqslant\kappa j+2)}.
\end{aligned}\tag{5.30}$$

Hence, we have from (5.30) and $\|\Box_k f\|_\infty \lesssim \|\Box_k f\|_2$ that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/2+1/\kappa}\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\big\|\Box_k(\Box_{k_1}\tilde{u}\ldots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^1_xL^2_t}\\
&\quad\lesssim(\kappa j)^C\sum_{|k_1|\geqslant 2,k_2,\ldots,k_{\kappa j+1}\in\mathbb{Z}}\langle k_1\rangle^{1/2+1/\kappa}\|\Box_{k_1}\tilde{u}\|_{L^\infty_xL^2_t}\\
&\qquad\times\prod_{i=2}^{\kappa+1}\|\Box_{k_i}\tilde{u}\|_{L^\kappa_xL^\infty_t}\prod_{i=\kappa+2}^{\kappa j+1}\|\Box_{k_i}\tilde{u}\|_{L^\infty_{x,t}}\\
&\quad\lesssim(\kappa j)^C\big(\|u\|_X\big)^{\kappa j+1}.
\end{aligned}\tag{5.31}
$$

By (5.29) and (5.31), we have for any $u\in X$,

$$
\mathrm{I}\lesssim\sum_{j\in\mathbb{N}}\frac{1}{j!}(\kappa j)^C\delta^{\kappa j+1}.\tag{5.32}
$$

Using (5.32) and following the same way as in the proof of Theorem 1.3, we can get that II has the same upper bound as I. Hence,

$$
\sum_{|k|\geqslant 2}\langle k\rangle^{1/2+1/\kappa}\big\|\Box_k\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa}-1\big)u\big)\big\|_{L^\infty_xL^2_t}\lesssim\sum_{j\in\mathbb{N}}\frac{1}{j!}(\kappa j)^C\delta^{\kappa j+1}.\tag{5.33}
$$

Similarly,

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\big\|\Box_k\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa}-1\big)u\big)\big\|_{L^\kappa_xL^\infty_t}\\
&\quad\lesssim\sum_{j\geqslant 1}\frac{1}{j!}\sum_{k\in\mathbb{Z}}\sum_{k_1,\ldots,k_{\kappa j+1}\in\mathbb{Z}}\big\|\Box_k\mathscr{A}\partial_x(\Box_{k_1}\tilde{u}\ldots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^\kappa_xL^\infty_t}\\
&\quad\lesssim\sum_{j\geqslant 1}\frac{1}{j!}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/2+1/\kappa}\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|\geqslant 2}\big\|\Box_k(\Box_{k_1}\tilde{u}\ldots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^1_xL^2_t}\\
&\qquad+\sum_{j\geqslant 1}\frac{1}{j!}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/2+1/\kappa}\sum_{|k_1|\vee\cdots\vee|k_{\kappa j+1}|<2}\big\|\Box_k(\Box_{k_1}\tilde{u}\ldots\Box_{k_{\kappa j+1}}\tilde{u})\big\|_{L^{\frac{2+\kappa}{1+\kappa}}_{x,t}}\\
&\quad:=\mathrm{III}+\mathrm{IV}.
\end{aligned}\tag{5.34}
$$

One can use the same way as in I and II to get the estimate of III and IV. Hence,

$$
\sum_{k\in\mathbb{Z}}\big\|\Box_k\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa}-1\big)u\big)\big\|_{L^\kappa_xL^\infty_t}\lesssim\sum_{j\in\mathbb{N}}\frac{1}{j!}(\kappa j)^C\delta^{\kappa j+1}.\tag{5.35}
$$

Similarly,

$$
\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/\kappa}\big\|\Box_k\mathscr{A}\partial_x\big(\big(e^{|u|^\kappa}-1\big)u\big)\big\|_{(L^\infty_tL^2_x)\cap L^{2+\kappa}_{x,t}}\lesssim\sum_{j\in\mathbb{N}}\frac{1}{j!}(\kappa j)^C\delta^{\kappa j+1}.\tag{5.36}
$$

Finally,

$$\|\mathscr{T}u\|_X \lesssim \|u_0\|_{M^{1/\kappa}_{2,1}} + \sum_{j\in\mathbb{N}} \frac{1}{j!}(\kappa j)^C \delta^{\kappa j+1}, \tag{5.37}$$

which implies the result, as desired. □

6. Smooth effects of KdV semigroup

In this section, we will consider the smooth effect estimates for the solutions of the linear KdV equation. The following lemma is also due to Kenig, Ponce and Vega; cf. [20].

Lemma 6.1. *Let $W(t) = \mathscr{F}^{-1}e^{-\mathrm{i}t\xi^3}\mathscr{F}$. Then we have*

$$\big\|W(t)f\big\|_{L^\infty_x L^2_t} \lesssim \|f\|_{\dot H^{-1}}, \tag{6.1}$$

$$\big\|W(t)f\big\|_{L^4_x L^\infty_t} \lesssim \|f\|_{\dot H^{1/4}}, \tag{6.2}$$

$$\big\|W(t)f\big\|_{L^\infty_t L^2_x} = \|f\|_2. \tag{6.3}$$

By Lemma 6.1, we have

Lemma 6.2. *Let $W(t) = \mathscr{F}^{-1}e^{-\mathrm{i}t\xi^3}\mathscr{F}$. Then we have for any $p \geqslant 4$ and $k \in \mathbb{Z}$,*

$$\big\|\Box_k W(t)f\big\|_{L^\infty_x L^2_t} \lesssim \|\Box_k f\|_{\dot H^{-1}}, \tag{6.4}$$

$$\big\|\Box_k W(t)f\big\|_{L^p_x L^\infty_t} \lesssim \|\Box_k f\|_{\dot H^{1/p}}, \tag{6.5}$$

$$\big\|\Box_k W(t)f\big\|_{L^\infty_t L^2_x} \lesssim \|\Box_k f\|_2, \tag{6.6}$$

where the omitted constants in (6.4)–(6.6) *are independent of $k \in \mathbb{Z}$.*

Proof. Using the same way as in the proof of Lemma 4.2, we can get the result, as desired. □

Corollary 6.3. *We have for any $p \geqslant 4$ and $k \in \mathbb{Z}$,*

$$\big\|\Box_k W(t)f\big\|_{L^{p+1}_x L^{2(p+1)}_t} \lesssim \|\Box_k f\|_2, \tag{6.7}$$

where the omitted constant in (6.7) *is independent of $k \in \mathbb{Z}$.*

Proof. By (6.4) and (6.5),

$$\big\|\Box_k W(t)f\big\|_{L^\infty_x L^2_t} \lesssim \|f\|_{\dot H^{-1}}, \tag{6.8}$$

$$\big\|\Box_k W(t)f\big\|_{L^p_x L^\infty_t} \lesssim \|f\|_{\dot H^{1/p}}. \tag{6.9}$$

One can interpolate L^2 between $\dot H^{1/p}$ and $\dot H^{-1}$, and interpolate $L_x^{p+1}L_t^{2(p+1)}$ between $L_x^pL_t^\infty$ and $L_x^\infty L_t^2$ to obtain the result. $\square$

The next lemma is the smooth effect estimates for the nonhomogeneous part of the solution to the linear KdV equation; cf. Kenig, Ponce and Vega [20] (cf. also [8]):

Lemma 6.4. *Let $W(t)=\mathscr{F}^{-1}e^{-\mathrm{i}t\xi^3}\mathscr{F}$. Then we have*

$$\left\| \int_0^t W(t-\tau)\partial_x^2 f(\tau)\,d\tau \right\|_{L_x^\infty L_t^2} \lesssim \|f\|_{L_x^1L_t^2}, \tag{6.10}$$

$$\left\| \int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau \right\|_{L_t^\infty L_x^2} \lesssim \|f\|_{L_x^1L_t^2}. \tag{6.11}$$

On the basis of Lemma 6.4, we have

Lemma 6.5. *Let $W(t)=\mathscr{F}^{-1}e^{-\mathrm{i}t\xi^3}\mathscr{F}$. Then we have for any $p\geqslant 4$ and $k\in\mathbb{Z}$,*

$$\left\| \Box_k\int_0^t W(t-\tau)\partial_x^2 f(\tau)\,d\tau \right\|_{L_x^\infty L_t^2} \lesssim \|\Box_k f\|_{L_x^1L_t^2}, \tag{6.12}$$

$$\left\| \Box_k\int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau \right\|_{L_t^\infty L_x^2} \lesssim \|\Box_k f\|_{L_x^1L_t^2}, \tag{6.13}$$

$$\left\| \Box_k\int_0^t W(t-\tau)\partial_x f(\tau)\,d\tau \right\|_{L_x^{p+1}L_t^{2(p+1)}} \lesssim \|\Box_k f\|_{L_x^1L_t^2}, \tag{6.14}$$

where the omitted constants in (6.12)–(6.14) *are independent of $k\in\mathbb{Z}$.*

Proof. By Lemma 6.4, it suffices to show that (6.14) holds. First, we prove that

$$\left\| \Box_k\int_{\mathbb{R}} W(t-\tau) f(\tau)\,d\tau \right\|_{L_t^\infty L_x^2} \lesssim \|\Box_k f\|_{L_x^{(p+1)/p}L_t^{2(1+p)/(2p+1)}}. \tag{6.15}$$

Indeed, for any $f,g\in\dot{\mathscr{S}}(\mathbb{R}^2)$, we have from Hölder's inequality and Corollary 6.3 that,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} W(t-\tau) f(\tau)\,d\tau,\ g(t)\right) dt \right| \\
&\quad\lesssim \|\Box_k f\|_{L_x^{(p+1)/p}L_t^{2(p+1)/(2p+1)}} \sum_{|\ell|\leqslant 1} \left\| \Box_{k+\ell}\int_{\mathbb{R}} W(\cdot-t)g(t)\,dt \right\|_{L_x^{p+1}L_\tau^{2(p+1)}}
\end{aligned}$$

$$\lesssim \|\Box_k f\|_{L_x^{(p+1)/p} L_t^{2(p+1)/(2p+1)}} \|g\|_{L_t^1 L_x^2}, \tag{6.16}$$

which implies (6.15). In view of (6.11) and (6.15), we have

$$\begin{aligned} &\left| \int_{\mathbb{R}} \left(\Box_k \int_{\mathbb{R}} W(t-\tau) f(\tau)\, d\tau,\ g(t) \right) dt \right| \\ &\quad \lesssim \left\| \Box_k \int_{\mathbb{R}} W(-\tau) f(\tau)\, d\tau \right\|_2 \sum_{|\ell| \leqslant 1} \left\| \Box_{k+\ell} \int_{\mathbb{R}} W(-t) g(t)\, dt \right\|_2 \\ &\quad \lesssim \left\| D_x^{-1} \Box_k f \right\|_{L_x^1 L_t^2} \|g\|_{L_x^{(p+1)/p} L_t^{2(p+1)/(2p+1)}}, \end{aligned} \tag{6.17}$$

which implies

$$\left\| \Box_k \int_{\mathbb{R}} W(t-\tau) \partial_x f(\tau)\, d\tau \right\|_{L_x^{p+1} L_t^{2(p+1)}} \lesssim \|\Box_k f\|_{L_x^1 L_t^2}. \tag{6.18}$$

Using the same way as in Lemma 4.4, we immediately have (6.14). □

7. Proof of Theorem 1.2

In this section we prove our Theorem 1.2. The main idea is similar to that of the generalized BO equation and we consider the following resolution space:

$$X = \left\{ u \in \mathscr{S}'(\mathbb{R}^{1+1}):\ \|u\|_X = \sum_{k \in \mathbb{Z}} \|\Box_k u\|_{L_x^{\kappa+1} L_t^{2(\kappa+1)}} \leqslant \rho \right\}. \tag{7.1}$$

We consider the mapping

$$\mathscr{T} : u(t) \to W(t) u_0 - \mathscr{B}\big(u^\kappa \partial_x u\big), \tag{7.2}$$

where we denote,

$$W(t) = e^{-t\partial_x^3}, \qquad \mathscr{B} = \int_0^t W(t-\tau) \cdot d\tau. \tag{7.3}$$

Proof of Theorem 1.2. *Step* 1. We show the global well-posedness for the small data in $M_{2,1}$. In view of Corollary 6.3,

$$\left\| \Box_k W(t) u_0 \right\|_{L_x^{\kappa+1} L_t^{2(\kappa+1)}} \lesssim \|\Box_k u_0\|_2. \tag{7.4}$$

By Lemma 6.5,

$$\left\| \Box_k \mathscr{B}\big(u^\kappa \partial_x u\big) \right\|_{L_x^{\kappa+1} L_t^{2(\kappa+1)}} \lesssim \left\| \Box_k u^{\kappa+1} \right\|_{L_x^1 L_t^2}. \tag{7.5}$$

Combining (7.4) with (7.5), we get

$$\|\mathscr{T}u\|_X \lesssim \|u_0\|_{M_{2,1}} + \sum_{k\in\mathbb{Z}} \big\|\Box_k u^{\kappa+1}\big\|_{L^1_x L^2_t}. \tag{7.6}$$

Using the frequency-uniform decomposition, one has that

$$\big\|\Box_k u^{\kappa+1}\big\|_{L^1_x L^2_t} \lesssim \sum_{k_1,\dots,k_{\kappa+1}\in\mathbb{Z}} \big\|\Box_k(\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u)\big\|_{L^1_x L^2_t}. \tag{7.7}$$

Analogous to the proof of Theorem 1.3, we see that

$$\begin{aligned}\big\|\Box_k u^{\kappa+1}\big\|_{L^1_x L^2_t} &\lesssim \sum_{k_1,\dots,k_{\kappa+1}\in\mathbb{Z}} \|\Box_{k_1}u\dots\Box_{k_{\kappa+1}}u\|_{L^1_x L^2_t}\chi_{(|k-k_1-\dots-k_{\kappa+1}|\leqslant\kappa+2)}\\ &\lesssim \sum_{k_1,\dots,k_{\kappa+1}\in\mathbb{Z}}\prod_{i=1}^{\kappa+1}\|\Box_{k_i}u\|_{L^{\kappa+1}_x L^{2(\kappa+1)}_t}\chi_{(|k-k_1-\dots-k_{\kappa+1}|\leqslant\kappa+2)}.\end{aligned} \tag{7.8}$$

In view of (7.6) and (7.8), we have for any $u\in X$,

$$\|\mathscr{T}u\|_X \lesssim \|u_0\|_{M_{2,1}} + \|u\|_X^{\kappa+1}. \tag{7.9}$$

Similarly, for any $u, v\in X$,

$$\|\mathscr{T}u-\mathscr{T}v\|_X \lesssim \big(\|u\|_X^{\kappa}+\|v\|_X^{\kappa}\big)\|u-v\|_X. \tag{7.10}$$

So, by the standard contraction mapping argument, we see that (1.1) has a solution $u\in X$. Moreover, this solution is unique in X. By (6.6) and (6.13), we have $u\in C(\mathbb{R}, M_{2,1})$.

Step 2. We show the local well-posedness for any Cauchy data in $M_{2,1}$. Put

$$X^T = \Big\{u\in\mathscr{S}'\big(\mathbb{R}^{1+1}\big):\ \|u\|_{X^T} = \sum_{k\in\mathbb{Z}}\|\Box_k u\|_{L^{\kappa+1}_x L^{2(\kappa+1)}_{t\in[-T,T]}} \leqslant \rho\Big\}. \tag{7.11}$$

Since $u_0\in M_{2,1}$, there exists $K\in\mathbb{N}$ verifying

$$C\sum_{|k|>K}\|\Box_k u_0\|_2 \leqslant \rho/4. \tag{7.12}$$

It follows from (7.4) that

$$\sum_{|k|>K}\big\|\Box_k W(t)u_0\big\|_{L^{\kappa+1}_x L^{2(\kappa+1)}_{t\in\mathbb{R}}} \leqslant C\sum_{|k|>K}\|\Box_k u_0\|_2 \leqslant \rho/4. \tag{7.13}$$

For $|k|\leqslant K$, in view of $\lim_{T\to 0}\|\Box_k W(t)u_0\|_{L^{\kappa+1}_x L^{2(\kappa+1)}_{t\in[-T,T]}} = 0$, we see that there exists $T\ll 1$ such that

$$\sum_{|k|\leqslant K} \left\| \Box_k W(t)u_0 \right\|_{L_x^{\kappa+1} L_{t\in[-T,T]}^{2(\kappa+1)}} \leqslant \rho/4, \tag{7.14}$$

whence, we have

$$\sum_{k\in\mathbb{Z}} \left\| \Box_k W(t)u_0 \right\|_{L_x^{\kappa+1} L_{t\in[-T,T]}^{2(\kappa+1)}} \leqslant \rho/2. \tag{7.15}$$

Then, we can repeat the procedure above as in Step 1 to get the local well-posedness of Eq. (1.1). This finishes the proof of Theorem 1.2. □

8. Proof of Theorem 1.7

We may assume, without loss of generality that

$$F(u, \bar{u}, \partial_x u, \partial_x \bar{u}) := F(u, \partial_x u) = \sum_{m+1\leqslant \kappa+\nu\leqslant \tilde{m}+1} u^\kappa (\partial_x u)^\nu \tag{8.1}$$

and the general cases can be handled in the same way. We consider the mapping

$$\mathscr{T} : u(t) \to S(t)u_0 - \mathrm{i}\mathscr{A} F(u, \partial_x u), \tag{8.2}$$

where $S(t)$ and $\mathscr{A}$ are as in (5.27). For convenience, we write for $i = 0, 1$,

$$N_i(u) := \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/m} \left\| \Box_k \partial_x^i u \right\|_{(L_t^\infty L_x^2)\cap L_{x,t}^{2+m}}, \tag{8.3}$$

$$M_i(u) := \sum_{k\in\mathbb{Z}} \left\| \Box_k \partial_x^i u \right\|_{L_x^m L_t^\infty}, \tag{8.4}$$

$$T_i(u) := \sum_{|k|\geqslant 2} \langle k\rangle^{1/2+1/m} \left\| \Box_k \partial_x^i u \right\|_{L_x^\infty L_t^2}. \tag{8.5}$$

Proof of Theorem 1.7. Let X_{dNLS} be as in (1.16). It follows from Lemmas 4.1 and 4.5 that

$$N_i\big(S(t)u_0\big) + M_i\big(S(t)u_0\big) + T_i\big(S(t)u_0\big) \lesssim \|u_0\|_{M_{2,1}^{i+1/m}}.$$

Hence,

$$\left\| S(t)u_0 \right\|_{X_{\mathrm{dNLS}}} \lesssim \|u_0\|_{M_{2,1}^{1+1/m}}. \tag{8.6}$$

By (8.2) and (8.6),

$$\|\mathscr{T} u\|_{X_{\mathrm{dNLS}}} \lesssim \|u_0\|_{M_{2,1}^{1+1/m}} + \left\| \mathscr{A} F(u, u_x) \right\|_{X_{\mathrm{dNLS}}}. \tag{8.7}$$

So, we need to estimate $\|\mathscr{A} F(u, u_x)\|_{X_{\mathrm{dNLS}}}$ and we divide our discussions into the following four steps.

Step 1. We consider the estimates of $N_0(\mathscr{A}F(u,u_x))$. By the Strichartz estimate (4.21),

$$N_0\big(\mathscr{A}F(u,u_x)\big) \leqslant \sum_{m+1\leqslant\kappa+\nu\leqslant\tilde m+1}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/m}\big\|\Box_k\mathscr{A}\big(u^\kappa u_x^\nu\big)\big\|_{(L_t^\infty L_x^2)\cap L_{x,t}^{2+m}}$$
$$\lesssim \sum_{m+1\leqslant\kappa+\nu\leqslant\tilde m+1}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/m}\big\|\Box_k\big(u^\kappa u_x^\nu\big)\big\|_{L_{x,t}^{\frac{2+m}{1+m}}}. \tag{8.8}$$

For simplicity, we will use the following notations as in [40]:

$$\|u\|_{\ell_\Box^{1,s}(X)} = \sum_{k\in\mathbb{Z}}\langle k\rangle^s\|\Box_k u\|_X, \qquad \|u\|_{\ell_\Box^{1}(X)} = \sum_{k\in\mathbb{Z}}\|\Box_k u\|_X. \tag{8.9}$$

Considering the nonlinear estimates, we have

Lemma 8.1. *Let $s\geqslant 0$, $1\leqslant p, p_i,\gamma,\gamma_i\leqslant\infty$ satisfy*

$$\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_N}, \qquad \frac{1}{\gamma}=\frac{1}{\gamma_1}+\cdots+\frac{1}{\gamma_N}. \tag{8.10}$$

Then

$$\|u_1\dots u_N\|_{\ell_\Box^{1,s}(L_t^\gamma L_x^p)} \lesssim \|u_1\|_{\ell_\Box^{1,s}(L_t^{\gamma_1}L_x^{p_1})}\prod_{i=2}^N\|u_i\|_{\ell_\Box^{1}(L_t^{\gamma_i}L_x^{p_i})}$$
$$+\|u_2\|_{\ell_\Box^{1,s}(L_t^{\gamma_2}L_x^{p_2})}\prod_{i\neq 2,\, i=1,\dots,N}\|u_i\|_{\ell_\Box^{1}(L_t^{\gamma_i}L_x^{p_i})}+\cdots$$
$$+\prod_{i=1}^{N-1}\|u_i\|_{\ell_\Box^{1}(L_t^{\gamma_i}L_x^{p_i})}\|u_N\|_{\ell_\Box^{1,s}(L_t^{\gamma_N}L_x^{p_N})}, \tag{8.11}$$

and in particular, if $u_1=\cdots=u_N=u$, then

$$\big\|u^N\big\|_{\ell_\Box^{1,s}(L_t^\gamma L_x^p)} \lesssim \|u\|_{\ell_\Box^{1,s}(L_t^{\gamma_1}L_x^{p_1})}\prod_{i=2}^N\|u\|_{\ell_\Box^{1}(L_t^{\gamma_i}L_x^{p_i})}. \tag{8.12}$$

Substituting the spaces $L_t^\gamma L_x^p$ and $L_t^{\gamma_i}L_x^{p_i}$ by $L_x^pL_t^\gamma$ and $L_x^{p_i}L_t^{\gamma_i}$, respectively, (8.11) *and* (8.12) *also holds.*

Proof. See [40, Lemma 7.1]. □

We now use Lemma 8.1 to estimate the nonlinearity in (8.8) and we divide our discussion into the following three cases.

Case 1. $\nu=0$. Taking $s=1/m$, $N=\kappa$ in Lemma 8.1, from (8.12) we easily see that

$$\big\|u^\kappa\big\|_{\ell_\Box^{1,1/m}(L_{x,t}^{\frac{2+m}{1+m}})} \lesssim \|u\|_{\ell_\Box^{1,1/m}(L_{x,t}^{2+m})}\|u\|^m_{\ell_\Box^{1}(L_{x,t}^{2+m})}\|u\|^{\kappa-m-1}_{\ell_\Box^{1}(L_{x,t}^{\infty})}. \tag{8.13}$$

Noticing the fact that

$$\|u\|_{\ell^1_\square(L^\infty_{x,t})} \lesssim \|u\|_{\ell^1_\square(L^\infty_t L^2_x)}, \tag{8.14}$$

and from the definition of $N_0(u)$, we immediately have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle k\rangle^{1/m} \big\|\square_k\big(u^\kappa\big)\big\|_{L^{\frac{2+m}{1+m}}_{x,t}} &\lesssim \|u\|_{\ell^{1,1/m}_\square(L^{2+m}_{x,t})} \|u\|^{\kappa-1}_{\ell^1_\square(L^{2+m}_{x,t})\cap\ell^1_\square(L^\infty_t L^2_x)} \\ &\lesssim \|u\|^\kappa_{\ell^{1,1/m}_\square(L^{2+m}_{x,t})\cap\ell^{1,1/m}_\square(L^\infty_t L^2_x)} \leqslant N_0(u)^\kappa. \end{aligned}\tag{8.15}$$

Case 2. $\kappa = 0$. Substituting κ by ν, and u by u_x in Case 1, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle k\rangle^{1/m} \big\|\square_k\big(u_x^\nu\big)\big\|_{L^{\frac{2+m}{1+m}}_{x,t}} &\lesssim \|\partial_x u\|_{\ell^{1,1/m}_\square(L^{2+m}_{x,t})} \|\partial_x u\|^{\nu-1}_{\ell^1_\square(L^{2+m}_{x,t})\cap\ell^1_\square(L^\infty_t L^2_x)} \\ &\lesssim N_1(u)^\nu. \end{aligned}\tag{8.16}$$

Case 3. $\kappa, \nu \geqslant 1$. It suffices to consider the case $\kappa < m+1$, the case $\kappa \geqslant m+1$ is easier than this case. Taking $s = 1/m$, $N = \kappa+\nu$, $u_1 = \cdots = u_\kappa = u$ and $u_{\kappa+1} = \cdots = u_{\kappa+\nu} = u_x$ in Lemma 8.1, and noticing (8.14), it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \langle k\rangle^{1/m} \big\|\square_k\big(u^\kappa u_x^\nu\big)\big\|_{L^{\frac{2+m}{1+m}}_{x,t}} &\lesssim \|u\|_{\ell^{1,1/m}_\square(L^{2+m}_{x,t})} \|u\|^{\kappa-1}_{\ell^1_\square(L^{2+m}_{x,t})} \|u_x\|^{m+1-\kappa}_{\ell^1_\square(L^{2+m}_{x,t})} \|u_x\|^{\kappa+\nu-m-1}_{\ell^1_\square(L^\infty_{x,t})} \\ &\lesssim \|u\|^\kappa_{\ell^{1,1/m}_\square(L^{2+m}_{x,t})} \|u_x\|^\nu_{\ell^1_\square(L^{2+m}_{x,t})\cap\ell^1_\square(L^\infty_t L^2_x)} \\ &\lesssim N_0(u)^\kappa N_1(u)^\nu. \end{aligned}\tag{8.17}$$

Collecting (8.8), (8.15)–(8.17), we have

$$N_0\big(\mathscr{A} F(u,u_x)\big) \lesssim \sum_{m+1\leqslant\kappa+\nu\leqslant\tilde m+1} N_0(u)^\kappa N_1(u)^\nu. \tag{8.18}$$

Step 2. We consider the estimates of $N_1(\mathscr{A} F(u,u_x))$. Similar to (8.8), we have

$$N_1\big(\mathscr{A} F(u,u_x)\big) \leqslant \sum_{m+1\leqslant\kappa+\nu\leqslant\tilde m+1} \big\|\mathscr{A}\partial_x\big(u^\kappa u_x^\nu\big)\big\|_{\ell^{1,1/m}_\square((L^\infty_t L^2_x)\cap L^{2+m}_{x,t})}. \tag{8.19}$$

By Lemmas 4.4 and 4.5,

$$\begin{aligned}&\big\|\mathscr{A}\partial_x\big(u^\kappa u_x^\nu\big)\big\|_{\ell^{1,1/m}_\square((L^\infty_t L^2_x)\cap L^{2+m}_{x,t})} \\ &\quad\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/2+1/m} \sum_{|k_1|\vee\cdots\vee|k_{\kappa+\nu}|\geqslant 2} \big\|\square_k(\square_{k_1}u\ldots\square_{k_\kappa}u\square_{k_{\kappa+1}}u_x\ldots\square_{k_{\kappa+\nu}}u_x)\big\|_{L^1_x L^2_t} \\ &\qquad+ \sum_{k\in\mathbb{Z}} \langle k\rangle^{1/m} \sum_{|k_1|\vee\cdots\vee|k_{\kappa+\nu}|<2} \big\|\partial_x\square_k(\square_{k_1}u\ldots\square_{k_\kappa}u\square_{k_{\kappa+1}}u_x\ldots\square_{k_{\kappa+\nu}}u_x)\big\|_{L^{\frac{2+m}{1+m}}_{x,t}} \\ &\quad:= \mathrm{I}_{\kappa,\nu} + \mathrm{II}_{\kappa,\nu}. \end{aligned}\tag{8.20}$$

Noticing that $|k-k_1-\dots-k_{\kappa+\nu}|\leqslant \kappa+\nu+1$ in $\mathrm{II}_{\kappa,\nu}$, one sees that in $\mathrm{II}_{\kappa,\nu}$, the summation on $k\in\mathbb{Z}$ is in lower frequency and has at most finite nonzero terms. Hence, in view of the Bernstein's multiplier estimate (1.20), the estimate of $\mathrm{II}_{\kappa,\nu}$ reduces to the case of $N_0(\mathscr{A}F(u,u_x))$:

$$\mathrm{II}_{\kappa,\nu}\lesssim \sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1} N_0(u)^\kappa N_1(u)^\nu. \tag{8.21}$$

The estimates of $\mathrm{I}_{\kappa,\nu}$ proceeds in a similar way as in Section 5. Noticing that $|k-k_1-\dots-k_{\kappa+\nu}|\leqslant \kappa+\nu+1$ in $\mathrm{I}_{\kappa,\nu}$, one has that

$$\mathrm{I}_{\kappa,\nu}\lesssim \sum_{|k_1|\vee\dots\vee|k_{\kappa+\nu}|\geqslant 2}\Big\langle \max_{1\leqslant i\leqslant\kappa+\nu}|k_i|\Big\rangle^{1/2+1/m}\|\Box_{k_1}u\dots\Box_{k_\kappa}u\Box_{k_{\kappa+1}}u_x\dots\Box_{k_{\kappa+\nu}}u_x\|_{L^1_xL^2_t}. \tag{8.22}$$

Using Hölder's inequality, we have

$$\begin{aligned}\|v_1\dots v_{\kappa+\nu}\|_{L^1_xL^2_t}&\leqslant\|v_1\|_{L^\infty_xL^2_t}\prod_{i=2}^{\kappa+\nu}\|v_i\|_{L^{\kappa+\nu-1}_xL^\infty_t}\\&\leqslant\|v_1\|_{L^\infty_xL^2_t}\prod_{i=2}^{\kappa+\nu}\Big(\|v_i\|^{\frac{m}{\kappa+\nu-1}}_{L^m_xL^\infty_t}\|v_i\|^{\frac{\kappa+\nu-1-m}{\kappa+\nu-1}}_{L^\infty_{x,t}}\Big).\end{aligned} \tag{8.23}$$

In (8.22), it suffices to consider the cases $|k_1|=|k_1|\vee\dots\vee|k_{\kappa+\nu}|$ and $|k_{\kappa+1}|=|k_1|\vee\dots\vee|k_{\kappa+\nu}|$, the other cases can be handled in a similar way. If $|k_1|=|k_1|\vee\dots\vee|k_{\kappa+\nu}|$, by (8.14), (8.22) and (8.23) we have

$$\begin{aligned}\mathrm{I}_{\kappa,\nu}\lesssim{}&\|u\|_{\ell^{1,1/2+1/m}_\Box(L^\infty_xL^2_t)}\Big(\|u\|^{\frac{m}{\kappa+\nu-1}}_{\ell^1_\Box(L^m_xL^\infty_t)}\|u\|^{\frac{\kappa+\nu-1-m}{\kappa+\nu-1}}_{\ell^1_\Box(L^\infty_tL^2_x)}\Big)^{\kappa-1}\\&\times\Big(\|u_x\|^{\frac{m}{\kappa+\nu-1}}_{\ell^1_\Box(L^m_xL^\infty_t)}\|u_x\|^{\frac{\kappa+\nu-1-m}{\kappa+\nu-1}}_{\ell^1_\Box(L^\infty_tL^2_x)}\Big)^{\nu}.\end{aligned} \tag{8.24}$$

If $|k_{\kappa+1}|=|k_1|\vee\dots\vee|k_{\kappa+\nu}|$, similar to (8.24), we have

$$\begin{aligned}\mathrm{I}_{\kappa,\nu}\lesssim{}&\|u_x\|_{\ell^{1,1/2+1/m}_\Box(L^\infty_xL^2_t)}\Big(\|u\|^{\frac{m}{\kappa+\nu-1}}_{\ell^1_\Box(L^m_xL^\infty_t)}\|u\|^{\frac{\kappa+\nu-1-m}{\kappa+\nu-1}}_{\ell^1_\Box(L^\infty_tL^2_x)}\Big)^{\kappa}\\&\times\Big(\|u_x\|^{\frac{m}{\kappa+\nu-1}}_{\ell^1_\Box(L^m_xL^\infty_t)}\|u_x\|^{\frac{\kappa+\nu-1-m}{\kappa+\nu-1}}_{\ell^1_\Box(L^\infty_tL^2_x)}\Big)^{\nu-1}.\end{aligned} \tag{8.25}$$

Collecting (8.24) and (8.25), we have from the definition of $M_i(u)$, $N_i(u)$, $T_i(u)$ that

$$\mathrm{I}_{\kappa,\nu}\lesssim\big(T_0(u)+T_1(u)\big)\big(M_0(u)+M_1(u)+N_0(u)+N_1(u)\big)^{\kappa+\nu-1}. \tag{8.26}$$

Hence, in view of (8.19)–(8.21) and (8.26) we have

$$N_1\big(\mathscr{A}F(u,u_x)\big)\lesssim\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\|u\|^{\kappa+\nu}_{X_{\mathrm{dNLS}}}. \tag{8.27}$$

Step 3. By Lemmas 4.4 and 4.5, using a similar way as that of $N_1(\mathscr{A}F(u,u_x))$, we see that the estimates of $T_1(\mathscr{A}F(u,u_x))$ and $M_1(\mathscr{A}F(u,u_x))$ can be reduced to the similar versions as in the right-hand side of (8.20), it follows that

$$M_1\big(\mathscr{A}F(u,u_x)\big)+T_1\big(\mathscr{A}F(u,u_x)\big)\lesssim\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\|u\|_{X_{\mathrm{dNLS}}}^{\kappa+\nu},\tag{8.28}$$

the details are omitted.

Step 4. Finally, we consider the estimates of $M_0(\mathscr{A}F(u,u_x))$ and $T_0(\mathscr{A}F(u,u_x))$, which are analogous to $M_0(\mathscr{A}F(u,u_x))$. Indeed, by (4.22) and (4.23),

$$M_0\big(\mathscr{A}F(u,u_x)\big)\lesssim\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1/m}\big\|\Box_k\big(u^\kappa u_x^\nu\big)\big\|_{L_{x,t}^{\frac{2+m}{1+m}}},\tag{8.29}$$

$$T_0\big(\mathscr{A}F(u,u_x)\big)\lesssim\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\sum_{|k|\geqslant2}\langle k\rangle^{1/m}\big\|\Box_k\big(u^\kappa u_x^\nu\big)\big\|_{L_{x,t}^{\frac{2+m}{1+m}}}.\tag{8.30}$$

So, in view of the estimates of (8.8) and (8.18), we have

$$M_0\big(\mathscr{A}F(u,u_x)\big)+T_0\big(\mathscr{A}F(u,u_x)\big)\lesssim\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\|u\|_{X_{\mathrm{dNLS}}}^{\kappa+\nu}.\tag{8.31}$$

Summarizing (8.7), (8.18), (8.27), (8.28) and (8.31), we have

$$\|\mathscr{T}u\|_{X_{\mathrm{dNLS}}}\lesssim\|u_0\|_{M_{2,1}^{1+1/m}}+\sum_{m+1\leqslant\kappa+\nu\leqslant\tilde{m}+1}\|u\|_{X_{\mathrm{dNLS}}}^{\kappa+\nu}.\tag{8.32}$$

Following the same way as in Section 5, we can get the results. □

Acknowledgments

The authors are grateful to Professors H. Feichtinger and J. Toft for their enlightening conservations on the interpolation of modulation spaces, and on the inclusions between Besov and modulation spaces, respectively. J. Toft kindly informed us that M. Sugimoto and N. Tomita independently obtained partial results of Theorem 1.1, where they mainly used the dilation property and Feichtinger's norm on modulation spaces, which is different from our frequency-uniform decomposition method. The authors are also indebted to the referee for his valuable comments and pointing out the reference [23].

Appendix A. Proof of Theorem 2.3

The proof of Theorem 2.3 is similar to that of the complex interpolation of Besov spaces (cf. Triebel [36,37]) and we will give the details of the proof for completeness. We need the following (cf. Triebel [35,36])

Lemma A.1. *Let A be the strip, $0 < r < \infty$. Then there exist two positive function $\mu_0(\theta, t)$ and $\mu_1(\theta, t)$ in $(0, 1) \times \mathbb{R}$ such that*

$$\left|g(z)\right|^r \leqslant \left((1-\theta)^{-1} \int_{\mathbb{R}} \left|g(\mathrm{i}t)\right|^r \mu_0(\theta, t)\, dt\right)^{1-\theta} \left(\theta^{-1} \int_{\mathbb{R}} \left|g(1+\mathrm{i}t)\right|^r \mu_1(\theta, t)\, dt\right)^{\theta}$$

with $\theta = \operatorname{Re} z$ for any analytic function $f(z)$ in A which is uniformly continuous and bounded in $\overline{A}$. Moreover,

$$(1-\theta)^{-1} \int_{\mathbb{R}} \mu_0(\theta, t)\, dt = \theta^{-1} \int_{\mathbb{R}} \mu_1(\theta, t)\, dt = 1.$$

Lemma A.2. *Assume that the conditions of Theorem 2.3 are satisfied. Then we have*

$$\left(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\right)_\theta \subset M^s_{p,q}.$$

Proof. Since $(I-\Delta)^{\sigma/2} : M^s_{p,q} \to M^{s-\sigma}_{p,q}$ and $F(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}) \to F(M^{s_0-\sigma}_{p_0,q_0}, M^{s_1-\sigma}_{p_1,q_1})$ are isomorphic mappings, it suffices to consider the case $s = 0$, i.e., we need to show that

$$\left(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1}\right)_\theta \subset M_{p,q}, \quad 0 = (1-\theta)s_0 + \theta s_1.$$

Taking $0 < r < \min_{i=0,1}(p_i, q_i)$, $f \in F(M^{s_0}_{p_0,q_0}, M^{s_1}_{p_1,q_1})$ with $f(\theta) = g$, we have

$$\|g\|_{M_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \left\| \left|\Box_k f(\theta, \cdot)\right|^r \right\|^{q/r}_{p/r}\right)^{1/q}. \tag{A.1}$$

Denote

$$a_k(x) = (1-\theta)^{-1} \int_{\mathbb{R}} \left|\Box_k f(\mathrm{i}t, \cdot)\right|^r \mu_0(\theta, t)\, dt,$$

$$b_k(x) = \theta^{-1} \int_{\mathbb{R}} \left|\Box_k f(1+\mathrm{i}t, \cdot)\right|^r \mu_1(\theta, t)\, dt.$$

In view of Lemma A.1, we have from Hölder's and Minkowski's inequalities that

$$\begin{aligned} \left\| \left|\Box_k f(\theta, \cdot)\right|^r \right\|_{p/r} &\leqslant \|a_k\|^{1-\theta}_{p_0/r} \|b_k\|^{\theta}_{p_1/r} \\ &\leqslant \left((1-\theta)^{-1} \int_{\mathbb{R}} \mu_0(t) \left\|\Box_k f(\mathrm{i}t)\right\|^r_{p_0} dt\right)^{1-\theta} \\ &\quad \times \left(\theta^{-1} \int_{\mathbb{R}} \mu_1(t) \left\|\Box_k f(1+\mathrm{i}t)\right\|^r_{p_1} dt\right)^{\theta}. \end{aligned} \tag{A.2}$$

Noticing that $(1-\theta)s_0+\theta s_1=0$, inserting (A.2) into (A.1) and using Hölder's and Minkowski's inequalities, we have

$$\begin{aligned}\|g\|_{M_{p,q}} &\leqslant \Bigg\{\sum_{k\in\mathbb{Z}^n}\Bigg[\Bigg((1-\theta)^{-1}\int_{\mathbb{R}}\mu_0(t)\langle k\rangle^{s_0 r}\big\|\Box_k f(\mathrm{i}t)\big\|_{p_0}^r\,dt\Bigg)^{1-\theta}\\&\quad\times\Bigg(\theta^{-1}\int_{\mathbb{R}}\mu_1(t)\langle k\rangle^{s_1 r}\big\|\Box_k f(1+\mathrm{i}t)\big\|_{p_1}^r\,dt\Bigg)^{\theta}\Bigg]^{q/r}\Bigg\}^{1/q}\\&\leqslant \Bigg\{\sum_{k\in\mathbb{Z}^n}\Bigg((1-\theta)^{-1}\int_{\mathbb{R}}\mu_0(t)\langle k\rangle^{s_0 r}\big\|\Box_k f(\mathrm{i}t)\big\|_{p_0}^r\,dt\Bigg)^{q_0/r}\Bigg\}^{(1-\theta)/q_0}\\&\quad\times\Bigg\{\sum_{k\in\mathbb{Z}^n}\Bigg(\theta^{-1}\int_{\mathbb{R}}\mu_1(t)\langle k\rangle^{s_1 r}\big\|\Box_k f(1+\mathrm{i}t)\big\|_{p_1}^r\,dt\Bigg)^{q_1/r}\Bigg\}^{\theta/q_1}\\&\leqslant \sup_t\big\|f(\mathrm{i}t)\big\|_{M^{s_0}_{p_0,q_0}}^{1-\theta}\big\|f(1+\mathrm{i}t)\big\|_{M^{s_1}_{p_1,q_1}}^{\theta},\end{aligned}$$

which implies the result, as desired. □

Proof of Theorem 2.3. The idea follows from Triebel's [36]. The difference is that the frequency-uniform decomposition is different from the dyadic decomposition, we need to modify the maximal function used in [36] by the following one in (A.3).

By Lemma A.2, it suffices to show that

$$M_{p,q}\subset\big(M^{s_0}_{p_0,q_0},M^{s_1}_{p_1,q_1}\big)_\theta,\quad 0=(1-\theta)s_0+\theta s_1.$$

Put $\lambda_k=\sum_{\ell\in\Lambda}\sigma_{k+\ell}$, $\Lambda=\{\ell\in\mathbb{Z}^n\colon\ \operatorname{supp}\sigma_0\cap\operatorname{supp}\sigma_\ell\neq\emptyset\}$. Let $g\in M_{p,q}$ and let g_k^* be the maximal function of $\Box_k g$, i.e.,

$$g_k^*(x)=\sup_{y\in\mathbb{R}^n}\frac{|(\Box_k g)(x-y)|}{1+|y|^a},\quad a>\frac{n}{p}. \tag{A.3}$$

Let

$$f(z,\cdot)=\sum_{k\in\mathbb{Z}^n}\mathscr{F}^{-1}\lambda_k\mathscr{F}\big[\langle k\rangle^{\varrho_1(z)}(\Box_k g)^{\varrho_2(z)}\big]\big\|g_k^*\big\|_{L^p}^{\varrho_3(z)}\big\|\big(g_k^*\big)\big\|_{\ell^q(L^p)}^{\varrho_4(z)},$$

where

$$\varrho_1(z)=-(1-z)s_0-zs_1,\quad \varrho_2(z)=p\bigg(\frac{1-z}{p_0}+\frac{z}{p_1}\bigg),$$
$$\varrho_3(z)=q\bigg(\frac{1-z}{q_0}+\frac{z}{q_1}\bigg)-\varrho_2(z),\quad \varrho_4(z)=1-\varrho_2(z)-\varrho_3(z).$$

It is easy to see that $f(\theta)=g$. For any $\ell\in\mathbb{Z}^n$,

$$\Box_\ell f(z) = \sum_{k\in\Lambda_1} \mathscr{F}^{-1}\sigma_\ell\lambda_{k+\ell}\mathscr{F}\big[\langle k+\ell\rangle^{\varrho_1(z)}(\Box_{k+\ell}g)^{\varrho_2(z)}\big]\big\|g^*_{k+\ell}\big\|_{L^p}^{\varrho_3(z)}\big\|\big(g^*_k\big)\big\|_{\ell^q(L^p)}^{\varrho_4(z)}, \tag{A.4}$$

where $\Lambda_1 = \{k\colon \operatorname{supp}\sigma_0 \cap \operatorname{supp}\lambda_k \neq \emptyset\}$ has at most $O(\sqrt{n})$ many elements. Noticing that

$$\begin{aligned}\big|\mathscr{F}^{-1}\sigma_\ell\lambda_{k+\ell}\mathscr{F}\big[\langle k+\ell\rangle^{\varrho_1(z)}(\Box_{k+\ell}g)^{\varrho_2(z)}\big]\big| &\lesssim \langle k+\ell\rangle^{\operatorname{Re}\varrho_1(z)}\big(\big|\mathscr{F}^{-1}\sigma_0\lambda_k\big| * \big|g_{k+\ell}^{\operatorname{Re}\varrho_2(z)}\big|\big)\\ &\lesssim \langle k+\ell\rangle^{\operatorname{Re}\varrho_1(z)}\big|g^*_{k+\ell}\big|^{\operatorname{Re}\varrho_2(z)},\end{aligned} \tag{A.5}$$

it follows from (A.4) and (A.5) that

$$\big|\Box_\ell f(\mathrm{i}t)\big| \lesssim \sum_{k\in\Lambda_1}\langle k+\ell\rangle^{-s_0}\big|g^*_{k+\ell}\big|^{p/p_0}\big\|g^*_{k+\ell}\big\|_{L^p}^{\operatorname{Re}\varrho_3(\mathrm{i}t)}\big\|\big(g^*_k\big)\big\|_{\ell^q(L^p)}^{1-q/q_0}. \tag{A.6}$$

Hence, in view of (A.6) and the maximal function estimates (cf. [35, Theorem 1.6.2])

$$\big\|f(\mathrm{i}t)\big\|_{M^{s_0}_{p_0,q_0}} \lesssim \big\|\big(g^*_k\big)\big\|_{\ell^q(L^p)} \lesssim \|g\|_{M_{p,q}}. \tag{A.7}$$

Similarly, $\|f(1+\mathrm{i}t)\|_{M^{s_1}_{p_1,q_1}}$ has the same upper bound as in (A.7). By Proposition 2.2, we get the result, as desired. □

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