J. Differential Equations 253 (2012) 1376–1398

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Solutions of Schrödinger equations with inverse square potential and critical nonlinearity

Yinbin Deng [a], Lingyu Jin [b], Shuangjie Peng [a,*]

[a] Department of Mathematics, Huazhong Normal University, Wuhan 430079, PR China
[b] College of Science, South China Agriculture University, Guangzhou 510642, PR China

ARTICLE INFO

Article history:
Received 10 November 2011
Revised 14 March 2012
Available online 16 May 2012

MSC:
35J10
35J20
35J60

Keywords:
Compactness
Critical Sobolev exponent
Nonlinear Schrödinger equation
Positive solutions
Palais–Smale sequence

ABSTRACT

In this paper, we are concerned with the following nonlinear Schrödinger equations with inverse square potential and critical Sobolev exponent

$$\begin{aligned} &-\Delta u-\mu\frac{u}{|x|^2}+a(x)u=|u|^{2^*-2}u+f(x,u),\\ &u\in H^1\big(\mathbb{R}^N\big), \end{aligned} \tag{P}$$

where $2^*=2N/(N-2)$ is the critical Sobolev exponent, $0\leqslant\mu<\bar\mu:=\frac{(N-2)^2}{4}$, $a(x)\in C(\mathbb{R}^N)$. We first give a representation to the Palais–Smale sequence related to (P) and then obtain an existence result of positive solutions of (P). Our assumptions on $a(x)$ and $f(x,u)$ are weaker than the known cases even if $\mu=0$.

1. Introduction

We consider the following nonlinear Schrödinger equations with inverse square potential and critical Sobolev exponent

$$-\Delta u-\mu\frac{u}{|x|^2}+a(x)u=|u|^{2^*-2}u+f(x,u),\quad u\in H^1\big(\mathbb{R}^N\big), \tag{1.1}$$

where $2^*:=2N/(N-2)$ is the critical Sobolev exponent, $0\leqslant\mu<\bar\mu:=\frac{(N-2)^2}{4}$.

* Corresponding author.
E-mail address: sjpeng@mail.ccnu.edu.cn (S. Peng).

0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2012.05.009

The main reason of interest in inverse square potentials (Hardy term) relies in their criticality: indeed they have the same homogeneity as the Laplacian and the critical Sobolev exponent and do not belong to the Kato class, hence they cannot be regarded as a lower order perturbation term. Besides, potentials with this rate of decay are critical also in nonrelativistic quantum mechanics, as they represent an intermediate threshold between regular potentials (for which there are ordinary stationary states) and singular potentials (for which the energy is not lower-bounded and the particle falls to the center), for more details see [17]. We also mention that inverse square singular potentials arise in many other physical contexts: molecular physics, see e.g. [20], quantum cosmology [5], linearization of combustion models [1,3]. Moreover, we emphasize the correspondence between nonrelativistic Schrödinger operators with inverse square potentials and relativistic Schrödinger operators with Coulomb potentials, see [13]. When $f(x,u)$ has the form $K(x)f(u)$, we point out that problem (1.1) arises in nonlinear optics, in plasma physics and in condensed matter physics, where the presence of many particles leads one to consider nonlinear terms which simulate the interaction effect among them. The function $a(x)$ represents the potential acting on the particle and $K(x)$ a particle-interaction term, which avoids spreading of the wave packets in the time-dependent version of the above equation.

Another reason why we investigate (1.1), in addition to the inverse square potential, is the presence of the critical Sobolev exponent and the unbounded domain $\mathbb{R}^N$, which cause the loss of compactness of embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$. Hence, including the noncompactness of the imbedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N, |x|^{-2}\,dx)$, we face a type of triple loss of compactness whose interacting each other will result in some new difficulties. In last two decades, loss of compactness leads to many interesting existence and nonexistence phenomena for elliptic equations (see, for example, [1,2,7,9,10,16,19,21,22,25,26,28] and the references therein). The main purpose of this paper is to exhibit some new existence phenomena for problem (1.1).

The first goal of this paper is to provide a careful analysis of the features of a Palais–Smale sequence for the functional related to (1.1). To this end, following the point of view adopted by Struwe [25] for the Dirichlet problem, we employ the blow-up technique to characterize all energy values where the Palais–Smale condition fails. More precisely, we represent any diverging Palais–Smale sequence as the sum of critical points of a family of limiting functionals which are invariant under scaling. Because of the inverse square potential, the critical Sobolev exponent, the nonlinear term $f(x,u)$ and the unboundedness of the domain, there are three types of critical points of a new family of limiting functionals in our problem. As a by-product, we find the lowest level at which Palais–Smale condition may fail. Thus we are able to determine safe sublevels where standard critical point theorems can be applied. For the readers' convenience, we mentioned some literatures in which characterization of Palais–Smale sequences were obtained. The well-known result was on the Brezis–Nirenberg problem in bounded domains and was given by Struwe in [25], where the noncompactness is completely described by a single blow-up profile. Zhu and Cao in [28] represented Palais–Smale sequences for (1.1) without the inverse square potential and critical Sobolev term by translating the ground state of the limiting equation corresponding to (1.1) to infinity. Recently, for (1.1) with $a(x) = 0$ and $f(x,u) = 0$, Smets in [24] and Cao and Peng in [9] obtained a representation of Palais–Smale sequences on unbounded and bounded domains respectively. Their results show that blowing up Palais–Smale sequences can bear exactly two types of bubbles. For more results, we refer the readers to [4,15,23,26,27] and the references therein.

Our second purpose in this paper is to obtain positive solutions for (1.1) under weaker conditions on $a(x)$ and the nonlinear term $f(x,u)$ by applying the previous compactness analysis. First, following a well-known strategy developed for the Dirichlet problem, we show that the functional related to (1.1) satisfies the geometrical assumptions of the Mountain Pass Theorem. Then we again use the blow-up technique to show that the Mountain Pass level is actually below a compactness threshold. Usually, to ensure that the Mountain Pass level is actually below the compactness threshold, we should impose the conditions that $a(x)$ is positive and $a(x) < \lim_{|x|\to\infty} a(x)$ or $f(x,u) > \lim_{|x|\to\infty} f(x,u)$ (see e.g. [8]). However, in this paper, we may weaken this kind of conditions.

To mention our main results, we need to introduce some notations and assumptions.

Let $H^1(\mathbb{R}^N)$ be the standard Sobolev space with the usual norm

$$\|u\|_{H^1(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx\right)^{1/2},$$

and

$$D^{1,2}\left(\mathbb{R}^N\right) = \left\{u \in L^2_{loc}\left(\mathbb{R}^N\right):\ |\nabla u| \in L^2\left(\mathbb{R}^N\right)\right\}$$

with the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} |\nabla u|^2\, dx\right)^{1/2}.$$

We assume that

(a1) $a(x) \in C(\mathbb{R}^N)$, $\lim_{|x|\to+\infty} a(x) = \bar{a} > 0$ and there exists a constant $\lambda_1 > 0$ such that

$$\int\limits_{\mathbb{R}^N} \left(1 - \left(\frac{2}{N-2}\right)^2 \mu\right)|\nabla u|^2 + a(x)u^2\, dx \geqslant \lambda_1 \int\limits_{\mathbb{R}^N} \left(\bar{a} - a(x)\right)u^2, \quad \text{for all } u \in H^1\left(\mathbb{R}^N\right);$$

(Without loss of generality, we may assume $\bar{a} = 1$.)

(a2) $f(x,t)$ is differentiable with respect to $t \in [0,+\infty)$ for all $x \in \mathbb{R}^N$ and continuous with respect to $x \in \mathbb{R}^N$ for all $t \in [0,+\infty)$. Moreover, we extend $f(x,t) \equiv 0$ for all $t \in (-\infty,0)$, $x \in \mathbb{R}^N$;

(a3) there exists a constant $p \in (1, \frac{N+2}{N-2})$ such that $\lim_{t\to+\infty} \frac{f(x,t)}{t^p} = 0$ and $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^N$;

(a4) there exists a constant $\theta \in (\frac{N-2}{N+2}, 1)$ such that $\theta t \frac{\partial}{\partial t} f(x,t) \geqslant f(x,t) > 0$, for all $x \in \mathbb{R}^N$, $t > 0$;

(a5) $\lim_{|x|\to+\infty} f(x,t) = \bar{f}(t)$ uniformly on any compact subset of $[0,\infty)$ and there exists a constant $\nu > 2$ such that for any $\varepsilon > 0$ we can find $C_\varepsilon > 0$ satisfying

$$f(x,t) - \bar{f}(t) \geqslant -e^{-\nu|x|}\left(\varepsilon t + C_\varepsilon t^p\right) \quad \text{for all } x \in \mathbb{R}^N,\ t \geqslant 0,$$

where $p \in (1, \frac{N+2}{N-2})$ is given by (a3).

Remark 1.1. Assumption (a1) is applied to prove Lemma 2.3 which gives that

$$\left(\int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2 - \mu \frac{u^2}{|x|^2}\right) dx\right)^{\frac{1}{2}}$$

is an equivalence norm of $H^1(\mathbb{R}^N)$. We provide an interesting example in Section 5 to show that $a(x)$ may be negative in some bounded domain in $\mathbb{R}^N$.

Remark 1.2. We can easily verify that the function

$$f(x,t) = \begin{cases} (1 - e^{-\nu|x|})t^q & \text{for } t \geqslant 0,\ x \in \mathbb{R}^N, \\ 0 & \text{for } t < 0,\ x \in \mathbb{R}^N, \end{cases}$$

satisfying our assumptions (a2)–(a5). Here $\nu > 2$, $1 < q < p$ and p is given by (a3). This example shows that $f(x,u) < \bar{f}(u)$ may be permitted in our case.

The energy functional associated with problem (1.1) is defined by

$$I(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u|^2 + a(x)u^2 - \mu\frac{u^2}{|x|^2}\right)dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\left(u^+\right)^{2^*}dx - \int\limits_{\mathbb{R}^N}F(x,u)\,dx, \quad u \in H^1\left(\mathbb{R}^N\right),$$

where $F(x,u) = \int_0^u f(x,t)\,dt$.

Now, we list three limiting equations related to problem (1.1).

The first limiting equation of (1.1) at infinity is

$$-\Delta u + u = |u|^{2^*-2}u + \bar{f}(u), \quad u \in H^1\left(\mathbb{R}^N\right), \tag{1.2}$$

and the corresponding variational functional is

$$I^\infty = \frac{1}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u|^2 + u^2\right)dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\left(u^+\right)^{2^*}dx - \int\limits_{\mathbb{R}^N}\bar{F}(u)\,dx, \quad u \in H^1\left(\mathbb{R}^N\right),$$

where $\bar{F}(u) = \int_0^u \bar{f}(t)\,dt$.

The limiting equation of (1.1) relating to both Hardy term and critical nonlinear term is

$$-\Delta u - \mu\frac{u}{|x|^2} = |u|^{2^*-2}u, \quad u \in D^{1,2}\left(\mathbb{R}^N\right), \tag{1.3}$$

and the corresponding variational functional is

$$I_\mu = \frac{1}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\left(u^+\right)^{2^*}dx, \quad u \in D^{1,2}\left(\mathbb{R}^N\right).$$

The last limiting equation of (1.1) relating to critical nonlinear term is

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}\left(\mathbb{R}^N\right), \tag{1.4}$$

and the corresponding variational functional is

$$I_0 = \frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla u|^2\,dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\left(u^+\right)^{2^*}dx, \quad u \in D^{1,2}\left(\mathbb{R}^N\right).$$

The set of positive solutions of (1.4) is the well-known $(N+1)$-parameter family of

$$U_0^{\varepsilon,y}(x) := \varepsilon^{(2-N)/2}U_0\left(\frac{x-y}{\varepsilon}\right),$$

where

$$U_0(x) := C(N)\left(1+|x|^2\right)^{\frac{2-N}{2}},$$

for an appropriate constant $C(N) > 0$. These solutions are also known to minimize the Sobolev quotient

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \int_{\mathbb{R}^N} |\nabla u|^2\,dx \Big/ \left(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}.$$

For $0 < \mu < \bar{\mu}$, it was shown in [26] that all positive solutions of (1.3) are of the form $U_\mu^\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U_\mu(x/\varepsilon)$, where

$$U_\mu(x) := C_\mu(N) \frac{1}{|x|^{\sqrt{\bar{\mu}}-\beta}(1+|x|^{\frac{2\beta}{\sqrt{\bar{\mu}}}})^{\frac{N-2}{2}}}$$

for an appropriate constant $C_\mu(N) > 0$, and $\beta := \sqrt{\bar{\mu}-\mu}$. These solutions minimize the quotient

$$S_\mu = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right) dx \Big/ \left(\int_{\mathbb{R}^N} |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}.$$

Clearly, S_μ is decreasing in μ, and a simple computation shows that $\lim_{\mu\to\bar{\mu}} S_\mu = 0$.

Define

$$J^\infty = \inf_{u\in\mathcal{N}} I^\infty(u),$$

where

$$\mathcal{N} = \left\{u \in H^1(\mathbb{R}^N)\setminus\{0\} \;\Big|\; \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 - |u|^{2^*} - \bar{f}(u)u\right) dx = 0\right\}.$$

It is known that $\mathcal{N} \neq \emptyset$ since problem (1.2) has at least one positive solution if $N \geqslant 4$ (see [14]).

Gidas, Ni and Nirenberg [18] showed that J^∞ can be achieved by a function $w \in \mathcal{N}$, moreover there exist $a_1, a_2 > 0$ such that for all $x \in \mathbb{R}^N$,

$$a_1\left(|x|+1\right)^{-\frac{N-1}{2}} e^{-|x|} \leqslant w(x) \leqslant a_2\left(|x|+1\right)^{-\frac{N-1}{2}} e^{-|x|}. \tag{1.5}$$

For convenience, we also define the following quantities, which will represent the amount of I carried over by blowing-up bubbles:

$$D_0 := \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla U_0|^2 - \frac{1}{2^*}U_0^{2^*}\right) dx = \frac{1}{N} S^{N/2},$$

$$D_\mu := \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla U_\mu|^2 - \frac{\mu}{2}\frac{U_\mu^2}{|x|^2} - \frac{1}{2^*}U_\mu^{2^*}\right) dx = \frac{1}{N} S_\mu^{N/2}.$$

We prove that diverging Palais–Smale sequences can be represented as sums of scaled critical points of the functional I^∞, I_0 or I_μ by exploiting suitable blow-up arguments. The main result of our paper is as follows:

Theorem 1.1. *Suppose $\beta > 1$, $N \geqslant 5$, $a(x)$ and $f(x,u)$ satisfy* (a1)–(a5)*, $\{u_n\}$ is a nonnegative Palais–Smale sequence of I at level $d \geqslant 0$, then there exist three nonnegative integers l_1, l_2 and l_3, l_1 sequences $\{R_n^j\} \subset \mathbb{R}^+$ $(1 \leqslant j \leqslant l_1)$, $2l_2$ sequences $\{r_n^j\} \subset \mathbb{R}^+$, $\{x_n^j\} \subset \mathbb{R}^N \backslash \{0\}$ $(1 \leqslant j \leqslant l_2)$, l_3 sequences $\{y_n^j\} \subset \mathbb{R}^N$ $(1 \leqslant j \leqslant l_3)$, $0 \leqslant u \in H^1(\mathbb{R}^N)$, $0 < u^j \in H^1(\mathbb{R}^N)$ $(1 \leqslant j \leqslant l_3)$ such that up to a subsequence*:

- $I'(u) = 0$, $I^{\infty\prime}(u^j) = 0$ $(1 \leqslant j \leqslant l_3)$;
- $R_n^j \to 0$ $(1 \leqslant j \leqslant l_1)$ *as* $n \to \infty$;
- $x_n^j \to x^j \in \mathbb{R}^N \cup \{\infty\}$, $r_n^j \to 0$ *and* $\frac{r_n^j}{|x_n^j|} \to 0$ $(1 \leqslant j \leqslant l_2)$, *as* $n \to \infty$;
- $|y_n^j| \to \infty$ $(1 \leqslant j \leqslant l_3)$, *as* $n \to \infty$;
- $d = I(u) + l_1 D_\mu + l_2 D_0 + \sum_{i=1}^{l_3} I^\infty(u^i)$;
- $$\left\| u_n - u - \sum_{j=1}^{l_1} U_\mu^{R_n^j} - \sum_{j=1}^{l_2} U_0^{r_n^j, x_n^j} - \sum_{j=1}^{l_3} u^j\big(x - y_n^j\big) \right\|_{H^1(\mathbb{R}^N)} = o_n(1) \quad \text{as } n \to \infty. \tag{1.6}$$

In particular, if $u \not\equiv 0$, then u is a positive weak solution of (1.1)*. Note that the corresponding sum in* (1.6) *will be treated as zero if $l_i = 0$ $(i = 1, 2, 3)$.*

Using above representation result of Palais–Smale sequences and Mountain Pass Theorem, we can obtain the following existence result of positive solutions.

Theorem 1.2. *Assume that $a(x)$, $f(x,u)$ satisfy* (a1)–(a5)*, $\beta > 1$, $N \geqslant 5$, $1 - a(x) + \frac{\mu}{|x|^2} > 0$. Then problem* (1.1) *has a positive solution $u \in H^1(\mathbb{R}^N)$ which satisfies*

$$I(u) < \min\left\{ \frac{1}{N} S_\mu^{N/2}, J^\infty \right\}.$$

Compared with the global compactness results proved in [24], where the Palais–Smale sequence was dealt in space $D^{1,2}(\mathbb{R}^N)$, we investigate the Palais–Smale sequence in space $H^1(\mathbb{R}^N)$ by using the compactness-concentration principle of Lions in [21,22]. More precisely, we will analyze carefully the behavior of the L^2-norm of the Palais–Smale sequence to determine what kind of blow-up occurs. We also point out here that it is the boundedness of L^2-norm of the Palais–Smale sequence that results in the phenomena that $R_n^j \to 0$ and $r_n^j \to 0$ as $n \to \infty$ in Theorem 1.1. This is different from the result in [24] where R_n^j and r_n^j may tend to infinity as $n \to \infty$. It is also the L^2-integrability of U_0 and U_μ that needs us to impose the conditions $N \geqslant 5$ and $\beta > 1$. Concerning the existence result, we emphasize that, as mentioned before, compared with the known assumption $0 < a(x) < \bar{a}$ or $f(x,u) > \bar{f}(u)$, $a(x) \geqslant \bar{a}$ or $f(x,u) < \bar{f}(u)$ may be permitted in our case. Moreover, our $a(x)$ may be negative in some bounded domain in $\mathbb{R}^N$.

In the sequel, we will denote by $B(x,r)$ a ball centered at x with radius r and B_r a ball centered at 0 with radius r. For simplicity, we will use the same C or c to denote various generic positive constants. $o_n(1)$ denotes a datum which tends to 0 as $n \to \infty$.

2. Some preliminary lemmas

In this section, we list some lemmas. The proofs of some lemmas can be found in the corresponding references.

Lemma 2.1. *(See Lemma 2.1 in [28].) Let $\{\rho_n\}_{n \geqslant 1}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geqslant 0 \quad \text{on } \mathbb{R}^N, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} \rho_n \, dx = \lambda > 0, \tag{2.1}$$

where $\lambda > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}$ satisfying one of the following two possibilities:

(i) (*Vanishing*):

$$\lim_{k\to+\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{y+B_R}\rho_{n_k}(x)\,dx = 0, \quad \textit{for all } 0 < R < +\infty. \tag{2.2}$$

(ii) (*Nonvanishing*): $\exists\alpha > 0$, $0 < R < +\infty$ *and* $\{y_k\} \subset \mathbb{R}^N$ *such that*

$$\lim_{k\to+\infty}\int\limits_{y_k+B_R}\rho_{n_k}(x)\,dx \geqslant \alpha > 0.$$

Lemma 2.2. *(See Lemma 2.3 in [28].) Let $1 < p \leqslant \infty$, $1 \leqslant q < \infty$, with $q \neq \frac{Np}{N-p}$ if $p < N$. Assume that u_n is bounded in $L^q(\mathbb{R}^N)$, $|\nabla u_n|$ is bounded in $L^p(\mathbb{R}^N)$ and*

$$\sup_{y\in\mathbb{R}^N}\int\limits_{y+B_R}|u_n|^q\,dx \to 0 \quad \textit{for some } R > 0 \textit{ as } n \to \infty.$$

Then $u_n \to 0$ in $L^\alpha(\mathbb{R}^N)$, for $\alpha \in (q, \frac{Np}{N-p})$.

Lemma 2.3. *Assume that $a(x)$ satisfies* (a1)*. Then there exist two positive constants C and c such that*

$$c\|u\|^2_{H^1(\mathbb{R}^N)} \leqslant \int\limits_{\mathbb{R}^n}\left(|\nabla u|^2 + a(x)u^2 - \frac{\mu}{|x|^2}u^2\right)dx \leqslant C\|u\|^2_{H^1(\mathbb{R}^N)}.$$

Proof. By Hardy inequality [1] and (a1), we find that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N}\left(|\nabla u|^2 + a(x)u^2 - \frac{\mu}{|x|^2}u^2\right)dx\\
&\quad\geqslant \int\limits_{\mathbb{R}^N}\left(\left(1-\left(\frac{2}{N-2}\right)^2\mu\right)|\nabla u|^2 + u^2 - \big(1-a(x)\big)u^2\right)dx\\
&\quad\geqslant \int\limits_{\mathbb{R}^N}\left(\left(1-\left(\frac{2}{N-2}\right)^2\mu\right)|\nabla u|^2 + u^2\right)dx - \frac{1}{\lambda_1+1}\int\limits_{\mathbb{R}^N}\left(\left(1-\left(\frac{2}{N-2}\right)^2\mu\right)|\nabla u|^2 + u^2\right)dx\\
&\quad= \left(1-\frac{1}{\lambda_1+1}\right)\int\limits_{\mathbb{R}^N}\left(\left(1-\left(\frac{2}{N-2}\right)^2\mu\right)|\nabla u|^2 + u^2\right)dx\\
&\quad\geqslant C\|u\|^2_{H^1(\mathbb{R}^N)}
\end{aligned}$$

since $\lambda_1 > 0$.

On the other hand, from (a1) we obtain that

$$\int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + a(x)u^2 - \mu \frac{u^2}{|x|^2} \right) dx \leqslant \int\limits_{\mathbb{R}^N} \left(|\nabla u|^2 + \|a(x)\|_{L^\infty} u^2 \right) dx \leqslant C\|u\|_{H^1(\mathbb{R}^N)}$$

for all $u \in H^1(\mathbb{R}^N)$. This completes the proof. □

Lemma 2.4. *Assume* (a1)–(a5)*. Let $\{u_n\}$ be a Palais–Smale sequence of I at level $d \in \mathbb{R}$. Then $d \geqslant 0$ and $\{u_n\} \subset H^1(\mathbb{R}^N)$ is bounded. Moreover, every Palais–Smale sequence for I at a level zero converges strongly to zero.*

Proof. It follows from (a4) that

$$\frac{\theta}{1+\theta} uf(x,u) \geqslant F(x,u) \quad \text{and} \quad \frac{\theta}{1+\theta} \geqslant \frac{1}{2^*}.$$

Hence, from Lemma 2.3, we see

$$\begin{aligned} d + o(\|u_n\|) &= I(u_n) - \frac{\theta}{1+\theta}\langle I'(u_n), u_n\rangle \\ &\geqslant \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right) \int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^2 + a(x)|u_n|^2 - \mu \frac{|u_n|^2}{|x|^2} \right) dx \\ &\quad - \int\limits_{\mathbb{R}^N} F(x,u_n)\,dx + \frac{\theta}{1+\theta} \int\limits_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\ &\geqslant \left(\frac{1}{2} - \frac{\theta}{1+\theta}\right) \int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^2 + a(x)|u_n|^2 - \mu \frac{|u_n|^2}{|x|^2} \right) dx \geqslant c\|u_n\|^2_{H^1(\mathbb{R}^N)}. \end{aligned} \tag{2.3}$$

Thus $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $d \geqslant 0$. Moreover, if $d = 0$, then

$$\lim_{n\to\infty} \|u_n\|_{H^1(\mathbb{R}^N)} = 0. \qquad \square$$

Lemma 2.5. *Assume* (a1)–(a5)*. Let $\{u_n\}$ be a Palais–Smale sequence of I at level $d \in \mathbb{R}$. Then $\{u_n^+\}$ is also a Palais–Smale sequence of I at level d, where $u_n^+ = \max\{u_n, 0\}$.*

Proof. Let $\{u_n\}$ be a Palais–Smale sequence of I at level $d \in \mathbb{R}$. By the definition of I we have that as $n \to +\infty$

$$I(u_n) = \frac{1}{2} \int\limits_{\mathbb{R}^N} \left(|\nabla u_n|^2 + a(x)u_n^2 - \mu \frac{u_n^2}{|x|^2} \right) dx - \frac{1}{2^*} \int\limits_{\mathbb{R}^N} \left|u_n^+\right|^{2^*} dx - \int\limits_{\mathbb{R}^N} F(x,u_n)\,dx = d + o_n(1)$$

and

$$\begin{aligned} \langle I'(u_n), \psi\rangle &= \int\limits_{\mathbb{R}^N} \left(\nabla u_n \psi + a(x)u_n\psi - \mu \frac{u_n\psi}{|x|^2} \right) dx - \int\limits_{\mathbb{R}^N} \left|u_n^+\right|^{2^*-1} \psi\,dx - \int\limits_{\mathbb{R}^N} f(x,u_n)\psi\,dx \\ &= o_n(1)\|\psi\|_{H^1(\mathbb{R}^N)} \end{aligned}$$

for all $\psi \in H^1(\mathbb{R}^N)$. Taking $\psi = u_n^-$ we have, from (a2) and Lemma 2.3, that

$$\begin{aligned}
o_n(1)\|u_n^-\|_{H^1(\mathbb{R}^N)} &= \langle I'(u_n), u_n^-\rangle \\
&= \int_{\mathbb{R}^N}\left(\nabla u_n \nabla u_n^- + a(x)u_n u_n^- - \mu\frac{u_n u_n^-}{|x|^2}\right) - \int_{\mathbb{R}^N}|u_n^+|^{2^*-1}u_n^-\,dx - \int_{\mathbb{R}^N} f(x,u_n)u_n^-\,dx \\
&= \int_{\mathbb{R}^N}\left(|\nabla u_n^-|^2 + a(x)|u_n^-|^2 - \mu\frac{|u_n^-|^2}{|x|^2}\right)dx \\
&\geqslant C\|u_n^-\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

Hence, we obtain

$$\|u_n^-\|_{H^1(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

It follows from Lemma 2.3 and (2.4) that

$$\int_{\mathbb{R}^N}\left(|\nabla u_n^-|^2 + a(x)(u_n^-)^2 - \mu\frac{(u_n^-)^2}{|x|^2}\right)dx \to 0$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} I(u_n^+) = \lim_{n\to\infty} I(u_n) = d$$

and

$$\langle I'(u_n^+), \psi\rangle = \langle I'(u_n), \psi\rangle + o_n(1)\|\psi\|_{H^1(\mathbb{R}^N)}$$

for all $\psi \in H^1(\mathbb{R}^N)$ as $n \to \infty$. This completes the proof. $\square$

Remark 2.1. All nontrivial critical points of I are the positive solutions. In fact, let $u \in H^1(\mathbb{R}^N)$ be a nontrivial critical point of I. Arguing as in the proof of Lemma 2.5, we can obtain that $\|u^-\|_{H^1(\mathbb{R}^N)} = 0$ which gives that $u \geqslant 0$ a.e. in $\mathbb{R}^N$. Standard regularity argument show that $u \in C^2(\mathbb{R}^N \setminus \{0\})$. Therefore, we know from the strong maximum principle that u is positive.

Remark 2.2. Lemma 2.5 and Remark 2.1 still hold for the functionals I_μ, I_0 and I^∞.

Let $\{u_n\}$ be a nonnegative Palais–Smale sequence of I. Up to a subsequence, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Obviously, $I'(u_0) = 0$. Set $v_n = u_n - u_0$, then

$$v_n \rightharpoonup 0 \quad \text{weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty.$$

Moreover, we have

Lemma 2.6. *(See Lemma 2.2 in [28].) Suppose that* $f(x,u)$ *satisfies* (a3) *then*

$$\lim_{n\to\infty}\left[\int_{\mathbb{R}^N} F(x,u_n)\,dx-\int_{\mathbb{R}^N} F(x,u_0)\,dx-\int_{\mathbb{R}^N} F(x,v_n)\,dx\right]=0.$$

Lemma 2.7. $\{v_n\}$ *is a Palais–Smale sequence for* I *at level* $d_0=d-I(u_0)$*.*

Proof. For any test function $v\in C_0^\infty(\mathbb{R}^N)$, we can easily prove that

$$\int_{\mathbb{R}^N} f(x,v_n)v\,dx=\int_{\mathbb{R}^N} f(x,u_n)v\,dx-\int_{\mathbb{R}^N} f(x,u_0)v\,dx+o_n(1),\quad \text{as } n\to\infty. \tag{2.5}$$

Now, using Brezis–Lieb Lemma in [6] and Lemma 2.6, we have that, as $n\to\infty$,

$$I(v_n)=I(u_n)-I(u_0)+o_n(1)=d-I(u_0)+o_n(1),$$
$$\langle I'(v_n),v\rangle=\langle I'(u_n),v\rangle-\langle I'(u_0),v\rangle=o_n(1),$$

which means that v_n is a Palais–Smale sequence for I at level $d_0=d-I(u_0)$. □

Lemma 2.8. *Let* υ *be a unit vector of* $\mathbb{R}^N$ *and* w *be that in* (1.5)*. There exist some constants* $C_1>0$ *and* $C_2>0$ *independent of* $R\geqslant 1$ *such that*

(1) $\int_{\{x\in\mathbb{R}^N\,||x|\leqslant 1\}}(w(x-R\upsilon))^2\,dx\geqslant C_1R^{-(N-1)}e^{-2R}$*, for* $R\geqslant 1$,
(2) $\int_{\mathbb{R}^N}e^{-\nu|x|}(w(x-R\upsilon))^{p+1}\,dx\leqslant C_2e^{-\min\{\nu,p+1\}R}$*, for* $R\geqslant 1$.

Proof. This lemma can be proved by the similar arguments as that of Lemma 3.6 in [12]. □

3. Noncompactness analysis

In this section, we prove Theorem 1.1 by a delicate analysis of the nonnegative Palais–Smale sequences of I. The main idea of our proof is similar to that of [10,25] except needing to deal with the difficulty caused by the L^2-norm and the nonlinear term.

Proof of Theorem 1.1. Let $\{u_n\}$ be a nonnegative (PS) sequence. It follows from Lemma 2.4 that $\{u_n\}$ is bounded. Hence, we assume that, up to a subsequence, as $n\to\infty$,

$$u_n\rightharpoonup u\quad \text{weakly in } H^1(\mathbb{R}^N),$$
$$u_n\to u\quad \text{strongly in } L^p_{loc}(\mathbb{R}^N) \text{ for } 1\leqslant p<2^*,$$
$$u_n\to u\quad \text{a.e. in } \mathbb{R}^N.$$

Denote $v_n=u_n-u$. Lemma 2.7 implies that $\{v_n\}$ is a Palais–Smale sequence of I and $v_n\rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ satisfying

$$I(v_n)=I(u_n)-I(u)+o_n(1), \tag{3.1}$$
$$I'(v_n)=I'(u_n)-I'(u)+o_n(1), \tag{3.2}$$
$$\|v_n\|_{H^1(\mathbb{R}^N)}=\|u_n\|_{H^1(\mathbb{R}^N)}-\|u\|_{H^1(\mathbb{R}^N)}+o_n(1). \tag{3.3}$$

If $\|v_n\|^2_{H^1(\mathbb{R}^N)} \to 0$ $(n \to \infty)$, we see from (3.1)–(3.3) that Theorem 1.1 holds true with $l_1 = l_2 = l_3 = 0$.

Now we may assume that

$$\|v_n\|^2_{H^1(\mathbb{R}^N)} \to l > 0 \quad \text{as } n \to \infty.$$

Case a: $\|v_n\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$.

We first claim that $\|v_n\|_{L^q(\mathbb{R}^N)} \to 0$ for all $q \in [2, 2^*)$ as $n \to \infty$. In fact, since $2 < q < 2^*$, there exists $\theta_q \in (0, 1)$ such that

$$\frac{1}{q} = \frac{\theta_q}{2} + \frac{1-\theta_q}{2^*}. \tag{3.4}$$

Now, using the fact that $\{v_n\}$ is bounded in $L^{2^*}(\mathbb{R}^N)$ and interpolation inequality, we have

$$\|v_n\|_{L^q(\mathbb{R}^N)} \leqslant \|v_n\|^{\theta_q}_{L^2(\mathbb{R}^N)} \|v_n\|^{1-\theta_q}_{L^{2^*}(\mathbb{R}^N)} \leqslant c\|v_n\|^{\theta_q}_{L^2(\mathbb{R}^N)} = o_n(1).$$

Hence, the claim follows.

Now, using the claim, we can easily prove that, as $n \to \infty$

$$\left|\int_{\mathbb{R}^N} F(x, v_n)\,dx\right| \leqslant \varepsilon \int_{\mathbb{R}^N} v_n^2\,dx + C_\varepsilon \int_{\mathbb{R}^N} |v_n|^{p+1}\,dx = o_n(1),$$

$$\int_{\mathbb{R}^N} f(x, v_n)\varphi\,dx = o_n(1), \quad \forall \varphi \in C_0^\infty\big(\mathbb{R}^N\big).$$

Thus,

$$I_\mu(v_n) = I(v_n) + o_n(1), \qquad I'_\mu(v_n) = I'(v_n) + o_n(1), \quad \text{as } n \to \infty,$$

which implies that $\{v_n\}$ is a nonnegative Palais–Smale sequence of I_μ. Hence, employing Theorem 3.1 in [24], we can find two nonnegative integers k, l and l sequences $\{R_n^j\} \subset \mathbb{R}^+$ $(1 \leqslant j \leqslant l)$, $2k$ sequences $\{r_n^j\} \subset \mathbb{R}^+$ and $\{x_n^j\} \subset \mathbb{R}^N \backslash \{0\}$ $(1 \leqslant j \leqslant k)$ such that, as $n \to \infty$,

$$d = I(v_n) + I(u) + o_n(1) = I_\mu(v_n) + I(u) + o_n(1) = lD_\mu + kD_0 + I(u) + o_n(1), \tag{3.5}$$

$$\left\| u_n - u - \sum_{j=1}^{k} U_0^{r_n^j, x_n^j} - \sum_{j=1}^{l} U_\mu^{R_n^j} \right\|_{D^{1,2}(\mathbb{R}^N)} \to 0, \tag{3.6}$$

where $R_n^j \to 0$ or ∞ $(1 \leqslant j \leqslant l)$, $x_n^j \to x^j \in \mathbb{R}^N \cup \{\infty\}$ and $r_n^j/|x_n^j| \to 0$ $(1 \leqslant j \leqslant k)$.

In this case, we only need to prove (1.6) with $l_3 = 0$. For this, we claim that $R_n^j \to 0$ $(1 \leqslant j \leqslant l)$ and $r_n^j \to 0$ $(1 \leqslant j \leqslant k)$ as $n \to \infty$. Indeed, if this claim is not true, then, without loss of generality, we can choose $R_n^h \to \infty$ for some $h \in \{1, \ldots, l\}$ such that, for n sufficiently large,

$$0 < c < \frac{R_n^h}{R_n^j}, \quad j \neq h, \qquad 0 < c < \frac{R_n^h}{r_n^j}, \quad j = 1, \ldots, k,$$

where c is a positive constant. Solving (3.4), we have

$$\theta_p = 1 + \frac{N}{p} - \frac{N}{2}.$$

Since U_0 and U_μ are positive and $\beta > 1$, we see from interpolation inequality and (3.6) that

$$\begin{aligned}
&\left(R_n^h\right)^{1+\frac{N}{p}-\frac{N}{2}} \|U_\mu\|_{L^p(\mathbb{R}^N)} - o_n(1) \\
&= \left\| U_\mu^{R_n^h} \right\|_{L^p(\mathbb{R}^N)} - \|v_n\|_{L^p(\mathbb{R}^N)} \\
&\leqslant \left\| v_n - \sum_{j=1}^{k} U_0^{r_n^j, x_n^j} - \sum_{j=1}^{l} U_\mu^{R_n^j} \right\|_{L^p(\mathbb{R}^N)} \\
&\leqslant \left\| v_n - \sum_{j=1}^{k} U_0^{r_n^j, x_n^j} - \sum_{j=1}^{l} U_\mu^{R_n^j} \right\|_{L^2(\mathbb{R}^N)}^{\theta_p} \left\| v_n - \sum_{j=1}^{k} U_0^{r_n^j, x_n^j} - \sum_{j=1}^{l} U_\mu^{R_n^j} \right\|_{L^{2^*}(\mathbb{R}^N)}^{1-\theta_p} \\
&\leqslant o_n(1) \left\| U_\mu^{R_n^j} \right\|_{L^2(\mathbb{R}^N)}^{\theta_p} \\
&= o_n(1) \left(R_n^j\right)^{1+\frac{N}{p}-\frac{N}{2}},
\end{aligned}$$

which is impossible and our claim follows. Therefore,

$$\begin{aligned}
\left\| \sum_{j=1}^{k} U_0^{r_n^j, x_n^j} \right\|_{L^2(\mathbb{R}^N)} &= \sum_{j=1}^{k} r_n^j \|U_0\|_{L^2(\mathbb{R}^N)} = o_n(1), \\
\left\| \sum_{j=1}^{l} U_\mu^{R_n^j} \right\|_{L^2(\mathbb{R}^N)} &= \sum_{j=1}^{l} R_n^j \|U_\mu\|_{L^2(\mathbb{R}^N)} = o_n(1).
\end{aligned} \tag{3.7}$$

Combining the fact $\|v_n\|_{L^2(\mathbb{R}^N)} = o_n(1)$ and (3.6), (3.7), we conclude that (1.6) with $l_3 = 0$ is true.

Case b:

$$\|v_n\|_{L^2(\mathbb{R}^N)} \to a > 0, \quad \text{as } n \to \infty. \tag{3.8}$$

By Lemma 2.1, there exists a subsequence still denoted by $\{v_n\}$ such that one of the following two cases occurs.

i) Vanishing occurs.

By Lemma 2.1 we have

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |v_n|^2 \, dx \to 0 \quad \text{as } n \to \infty, \ \forall 0 < R < \infty.$$

By Sobolev's inequality and Lemma 2.2 we have

$$\int_{\mathbb{R}^N} |v_n|^p \, dx \to 0 \quad \text{as } n \to \infty, \ \forall 2 < p < 2^*.$$

So, similar to **Case a**, we see

$$\int_{\mathbb{R}^N} F(x, v_n)\,dx = o_n(1), \qquad \int_{\mathbb{R}^N} f(x, v_n)\varphi\,dx = o_n(1), \quad \forall \varphi \in C_0^\infty\big(\mathbb{R}^N\big),$$

and v_n is a positive Palais–Smale sequence of the functional corresponding to

$$-\Delta u - \mu \frac{u}{|x|^2} + a(x)u = |u|^{2^*-2}u, \quad u \in H^1\big(\mathbb{R}^N\big). \tag{3.9}$$

Proceeding as done in **Case a**, we can also find two nonnegative integers m, h and m sequences $\{R_n'^j\} \subset \mathbb{R}^+$ $(1 \leqslant j \leqslant m)$, $2h$ sequences $\{r_n'^j\} \subset \mathbb{R}^+$ and $\{x_n'^j\} \subset \mathbb{R}^N \backslash \{0\}$ $(1 \leqslant j \leqslant h)$ such that, as $n \to \infty$,

$$d = I(v_n) + I(u) + o_n(1) = mD_\mu + hD_0 + I(u) + o_n(1),$$
$$\left\| v_n - \sum_{j=1}^{m} U_0^{r_n'^j, x_n'^j} - \sum_{j=1}^{h} U_\mu^{R_n'^j} \right\|_{D^{1,2}(\mathbb{R}^N)} \to 0,$$

where $R_n'^j \to 0$ $(1 \leqslant j \leqslant m)$, $x_n'^j \to x'^j \in \mathbb{R}^N \cup \{\infty\}$ and $r_n'^j \to 0$ $(1 \leqslant j \leqslant h)$.

However, in this case, similar to (3.7),

$$\left\| v_n - \sum_{j=1}^{m} U_0^{r_n'^j, x_n'^j} - \sum_{j=1}^{h} U_\mu^{R_n'^j} \right\|_{L^2(\mathbb{R}^N)} = \|v_n\|_{L^2(\mathbb{R}^N)} + o_n(1) \to a > 0.$$

As a consequence, the following nonvanishing case must occur.

ii) Nonvanishing occurs.

By Lemma 2.1, there exist $\alpha > 0$, $0 < \bar{R} < +\infty$, $\{y_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{y_n + B_{\bar{R}}} |v_n|^2\,dx \geqslant \alpha > 0. \tag{3.10}$$

Without loss of generality we choose $|y_n| \to \infty$ as $n \to \infty$. Otherwise, $\{v_n\}$ is tight, and thus $\|v_n\|_{L^2(\mathbb{R}^N)} \to 0$ as $n \to \infty$. This contradicts (3.8). Denote $\bar{v}_n = v_n(x + y_n)$. Since $\|\bar{v}_n\|_{H^1(\mathbb{R}^N)} = \|v_n\|_{H^1(\mathbb{R}^N)} \leqslant c$, without loss of generality, we assume that as $n \to \infty$,

$$\bar{v}_n \rightharpoonup v_0 \quad \text{in } H^1\big(\mathbb{R}^N\big),$$
$$\bar{v}_n \to v_0 \quad \text{in } L^p_{loc}\big(\mathbb{R}^N\big), \text{ for any } 1 \leqslant p < 2^*.$$

For any $\phi \in C_0^\infty(\mathbb{R}^N)$, we see, as $n \to \infty$,

$$\int_{\mathbb{R}^N} \frac{\bar{v}_n \phi}{|x + y_n|^2}\,dx = \int_{\mathbb{R}^N} \frac{v_n \phi_n}{|x|^2}\,dx$$
$$= \int_{|x|>R} \frac{v_n \phi_n}{|x|^2}\,dx + o_n(1)$$

$$\leqslant \frac{1}{R^2}\left(\int\limits_{\mathbb{R}^N} v_n^2\,dx\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^N} \phi_n^2\,dx\right)^{\frac{1}{2}} + o_n(1),$$

where $\phi_n = \phi(x - y_n)$. Let $R \to \infty$, then we have

$$\int\limits_{\mathbb{R}^N} \frac{\bar{v}_n\phi}{|x+y_n|^2}\,dx = o_n(1). \tag{3.11}$$

Similarly we have

$$\int\limits_{\mathbb{R}^N} \frac{\bar{v}_n^2}{|x+y_n|^2}\,dx = o_n(1). \tag{3.12}$$

Since $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and $\lim_{n\to\infty} a(x+y_n) = 1$, we have as $n \to \infty$,

$$\int\limits_{\mathbb{R}^N} a(x)v_n\phi_n\,dx = \int\limits_{\mathbb{R}^N} \bar{v}_n\phi\,dx + \int\limits_{\mathbb{R}^N} \big[a(x+y_n) - 1\big]\bar{v}_n\phi\,dx$$

and

$$\left|\int\limits_{\mathbb{R}^N} \big[a(x+y_n) - 1\big]\bar{v}_n\phi\,dx\right| \leqslant c\left(\int\limits_{\mathbb{R}^N} \big|a(x+y_n) - 1\big|^2\phi^2\,dx\right)^{1/2} = o_n(1),$$

that is,

$$\int\limits_{\mathbb{R}^N} \bar{v}_n\phi\,dx = \int\limits_{\mathbb{R}^N} a(x)v_n\phi_n\,dx + o_n(1) \quad \text{as } n \to \infty. \tag{3.13}$$

From (a2), (a5) and Lebesgue convergence theorem,

$$\begin{aligned}
\int\limits_{\mathbb{R}_N} f(x, v_n)\phi_n\,dx &= \int\limits_{\mathbb{R}^N} f(x+y_n, \bar{v}_n)\phi\,dx \\
&= \int\limits_{\mathbb{R}^N} \bar{f}(\bar{v}_n)\phi\,dx + \int\limits_{\mathbb{R}^N} \big[f(x+y_n, \bar{v}_n) - \bar{f}(\bar{v}_n)\big]\phi\,dx \\
&= \int\limits_{\mathbb{R}^N} \bar{f}(\bar{v}_n)\phi\,dx + o_n(1) \quad \text{as } n \to \infty.
\end{aligned} \tag{3.14}$$

From (3.11)–(3.14), and the fact that v_n is a Palais–Smale sequence of I, we have

$$\big\langle I^{\infty\prime}(\bar{v}_n), \phi\big\rangle = \big\langle I'(v_n), \phi_n\big\rangle + o_n(1) = o_n(1). \tag{3.15}$$

Hence $\bar{v}_n$ is a nonnegative Palais–Smale sequence of $I^\infty(u)$, and v_0 is a weak solution of (1.2).

Now we claim that $v_0 \not\equiv 0$.

In fact, from (3.8), we may assume there exists a sequence $\{y_n\}$ satisfying (3.10) and

$$\exists R > 0, \qquad \int\limits_{B(y_n,R)} |v_n|^2\,dx = c + o_n(1) > 0, \tag{3.16}$$

where $c \in (0, a]$ is a constant. If $v_0 = 0$, we have

$$\int\limits_{B_R} |\bar{v}_n|^2\,dx = \int\limits_{B(y_n,R)} |v_n|^2\,dx = o_n(1),$$

which contradicts (3.16).

Denote $z_n = \bar{v}_n - v_0$. From (a2), (a5) and Lebesgue convergence theorem,

$$\begin{aligned}\left|\int\limits_{\mathbb{R}^N} F(x+y_n,\bar{v}_n) - \bar{F}(\bar{v}_n)\,dx\right| &= \left|\int\limits_{\mathbb{R}^N}\int\limits_0^{\bar{v}_n} f(x+y_n,t) - \bar{f}(t)\,dt\,dx\right| \\ &\leqslant \left|\int\limits_{\mathbb{R}^N}\big[f(x+y_n,m_n\bar{v}_n) - \bar{f}(m_n\bar{v}_n)\big]\bar{v}_n\,dx\right| \\ &= o_n(1),\end{aligned} \tag{3.17}$$

where $0 \leqslant m_n(x) \leqslant 1$. Thus we have

$$\begin{aligned} I(v_n) &= \frac{1}{2}\int\limits_{\mathbb{R}^N}\big(|\nabla v_n|^2 + a(x)|v_n|^2\big)\,dx - \frac{\mu}{2}\int\limits_{\mathbb{R}^N}\frac{v_n^2}{|x|^2}\,dx \\ &\quad - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\big|v_n^+\big|^{2^*}\,dx - \int\limits_{\mathbb{R}^N} F(x,v_n)\,dx \\ &= \frac{1}{2}\int\limits_{\mathbb{R}^N}\big(|\nabla\bar{v}_n|^2 + a(x+y_n)|\bar{v}_n|^2\big)\,dx - \frac{\mu}{2}\int\limits_{\mathbb{R}^N}\frac{|\bar{v}_n|^2}{|x+y_n|^2}\,dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\big|\bar{v}_n^+\big|^{2^*}\,dx \\ &\quad - \int\limits_{\mathbb{R}^N}\big(F(x+y_n,\bar{v}_n) - \bar{F}(\bar{v}_n)\big)\,dx - \int\limits_{\mathbb{R}^N}\bar{F}(\bar{v}_n)\,dx \\ &= \frac{1}{2}\int\limits_{\mathbb{R}^N}\big(|\nabla\bar{v}_n|^2 + |\bar{v}_n|^2\big)\,dx - \frac{1}{2^*}\int\limits_{\mathbb{R}^N}\big|\bar{v}_n^+\big|^{2^*}\,dx - \int\limits_{\mathbb{R}^N}\bar{F}(\bar{v}_n)\,dx + o_n(1),\end{aligned}$$

where the last equality holds from (3.12) and (3.13).

As a result,

$$\begin{gathered}\|z_n\|_{H^1(\mathbb{R}^N)} = \|\bar{v}_n\|^2_{H^1(\mathbb{R}^N)} - \|v_0\|^2_{H^1(\mathbb{R}^N)} + o_n(1), \\ I(z_n) = I^\infty(\bar{v}_n) - I^\infty(v_0) + o_n(1) = I(v_n) - I^\infty(v_0) + o_n(1), \\ I'(z_n) = I'(v_n) - I'^\infty(v_0) + o_n(1) = o_n(1).\end{gathered}$$

Hence, $z_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ as $n \to \infty$, and z_n is a new Palais–Smale sequence of I. If $\|z_n\|_{L^2(\mathbb{R}^N)} \to c > 0$ as $n \to \infty$, by applying the above procedure recursively, the iteration must stop after finite steps. Moreover, the last Palais–Smale sequence denoted still by $\{v_n\}$ must satisfy that

$$\|v_n\|_{L^2(\mathbb{R}^N)} \to 0 \quad \text{as } n \to \infty.$$

Then the proof goes back again to **Case a** and $\{v_n\}$ is a Palais–Smale sequence of I_μ. So we can complete the proof of Theorem 1.1 by iteration. □

4. Existence of the solution for problem (1.1)

In this section, we prove Theorem 1.2 by applying Theorem 1.1 and Mountain Pass Theorem. Since

$$I(tu) = \frac{t^2}{2}\left[\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\frac{u^2}{|x|^2}\right)dx + \int_{\mathbb{R}^N} a(x)u^2\,dx\right] - \frac{|t|^{2^*}}{2^*}\int_{\mathbb{R}^N}\left|u^+\right|^{2^*}dx - \int_{\mathbb{R}^N} F(x,tu)\,dx,$$

we deduce that for fixed $u \not\equiv 0$ in $H^1(\mathbb{R}^N)$, $I(tu) \to -\infty$ if $t \to +\infty$.

Since for $p \in (1, \frac{N+2}{N-2})$ and $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} F(x,u)\,dx \leqslant \int_{\mathbb{R}^N}\left(\varepsilon u^2 + C_\varepsilon|u|^{p+1}\right)dx \leqslant C_\varepsilon\|u\|^{p+1}_{H^1(\mathbb{R}^N)} + \varepsilon\|u\|^2_{H^1(\mathbb{R}^N)},$$

choosing ε small enough, we have

$$I(u) \geqslant C\|u\|^2_{H^1(\mathbb{R}^N)} - C\left(\|u\|^{p+1}_{H^1(\mathbb{R}^N)} + \|u\|^{2^*}_{H^1(\mathbb{R}^N)}\right), \quad 1 < p < 2^* - 1.$$

Hence, there exists $r_0 > 0$ small such that $I(u)|_{\partial B(0,r_0)} \geqslant \rho > 0$.

As a consequence, $I(u)$ satisfies the geometry structure of Mountain Pass Theorem. Define

$$c_\mu =: \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I\left(\gamma(t)\right),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N))\colon\ \gamma(0) = 0,\ \gamma(1) = \psi_0 \in H^1(\mathbb{R}^N)\}$. The ψ_0 is chosen such that $I(t\psi_0) \leqslant 0$ for all $t \geqslant 1$. From the Mountain Pass lemma without (PS) condition and Lemma 2.5 we deduce that there exists a nonnegative (PS) sequence of $\{u_n\}$ of I in $H^1(\mathbb{R}^N)$ at the level c_μ, that is,

$$I(u_n) \to c_\mu \quad \text{and} \quad I'(u_n) \to 0, \quad \text{in } H^{-1}\left(\mathbb{R}^N\right) \text{ as } n \to \infty.$$

The following proposition can be obtained by applying Theorem 1.1 and Mountain Pass lemma directly.

Proposition 4.1. *Assume $a(x)$, $f(x,u)$ satisfy* (a1)–(a5). *If*

$$c_\mu < \min\left\{\frac{1}{N}S^{N/2}, \frac{1}{N}S_\mu^{N/2}, J^\infty\right\},$$

then c_μ is a critical value of I.

Remark 4.1. From [11], we know that $S_\mu < S$ if $\mu > 0$.

In the following, we will verify that the level value c_μ is in an interval where the (PS) condition holds. By Proposition 4.1 and Remark 4.1, we only need to verify that

$$c_\mu < \min\left\{\frac{1}{N}S^{N/2}, \frac{1}{N}S_\mu^{N/2}, J^\infty\right\} = \min\left\{\frac{1}{N}S_\mu^{N/2}, J^\infty\right\} \quad \text{for } \mu \geqslant 0. \tag{4.1}$$

To this end, let $\varphi(x) \in C_0^\infty(B_{2R})$, $\varphi(x) = 1$ for $|x| \leqslant R$, $\varphi(x) = 0$ for $|x| \geqslant 2R$. Set $u_\varepsilon(x) = \varphi(x)U_\mu^\varepsilon(x)$. Denote

$$v_\varepsilon(x) = \frac{u_\varepsilon}{(\int_{\mathbb{R}^N} |u_\varepsilon|^{2^*}\,dx)^{\frac{1}{2^*}}}.$$

Then we have the following estimates (see [10, Lemma 2.4]):

$$\int_{\mathbb{R}^N} \left(|\nabla v_\varepsilon|^2 - \mu \frac{v_\varepsilon^2}{|x|^2}\right) dx = S_\mu + O\left(\varepsilon^{2\beta}\right);$$

$$\int_{\mathbb{R}^N} |v_\varepsilon|^2\,dx = \begin{cases} O(\varepsilon^2), & \beta > 1, \\ O(\varepsilon^{2\beta}|\log\varepsilon|), & \beta = 1, \\ O(\varepsilon^{2\beta}), & \beta < 1. \end{cases}$$

Now we can prove the following lemma:

Lemma 4.1. *Under the assumptions* (a1)–(a5)*, we have*

$$\max_{t>0} I(tv_\varepsilon) < \frac{1}{N}S_\mu^{N/2}, \tag{4.2}$$

for $\varepsilon > 0$ small enough,

Proof. Let t_ε achieve $\max_{t>0} I(tv_\varepsilon)$, then t_ε is uniformly bounded. Hence, for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned} \max_{t>0} I(tv_\varepsilon) &= I(t_\varepsilon v_\varepsilon) \\ &\leqslant \max_{t>0}\left\{\frac{t^2}{2}\int_{\mathbb{R}^N}\left(|\nabla v_\varepsilon|^2 - \mu\frac{v_\varepsilon^2}{|x|^2}\right)dx - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|v_\varepsilon|^{2^*}\,dx\right\} \\ &\quad + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N} a(x){v_\varepsilon}^2\,dx - \int_{\mathbb{R}^N} F(x, t_\varepsilon u_\varepsilon)\,dx \\ &= \frac{1}{N}S_\mu^{\frac{N}{2}} + O\left(\varepsilon^{2\beta}\right) - \int_{\mathbb{R}^N} F(x, t_\varepsilon v_\varepsilon)\,dx + \begin{cases} O(\varepsilon^2), & \beta > 1, \\ O(\varepsilon^2|\log\varepsilon|), & \beta = 1, \\ O(\varepsilon^{2\beta}), & \beta < 1. \end{cases} \end{aligned} \tag{4.3}$$

Now we verify that

$$\lim_{\varepsilon\to 0^+}\varepsilon^{-2}\int_{\mathbb{R}^N} F(x,tv_\varepsilon)\,dx=+\infty,\quad \text{if } \beta>1. \tag{4.4}$$

From (a4), there exist some constants $C>0$ and $\theta'>0$ such that $F(x,t)\geqslant Ct^{2+\theta'}$, $\forall x\in\mathbb{R}^N$. Then

$$\begin{aligned}\varepsilon^{-2}\int_{B(0,2R)} F(x,t_\varepsilon v_\varepsilon)\,dx &\geqslant C\varepsilon^{-2}\int_{B(0,R)} |u_\varepsilon|^{2+\theta'}\,dx\\ &\geqslant C\varepsilon^{-\theta'\sqrt{\bar\mu}}\int_0^{R\varepsilon^{-1}}\frac{r^{N-1+(2+\theta')(\beta-\sqrt{\bar\mu})}}{(1+r^{\frac{2\beta}{\sqrt{\bar\mu}}})^{\frac{N-2}{2}(2+\theta')}}\,dr.\end{aligned} \tag{4.5}$$

If $R\geqslant 1$, obviously (4.4) holds for ε small enough. If $R<1$, we have

$$\begin{aligned}&\varepsilon^{-2}\int_R^1 r^{N-1}F(x,t_\varepsilon v_\varepsilon)\,dr\\ &\leqslant C\varepsilon^{-2}\int_R^1 r^{N-1}\big(|u_\varepsilon|^2+|u_\varepsilon|^p\big)\,dr\\ &\leqslant C\varepsilon^{N-2}\int_{\frac{R}{\varepsilon}}^{\frac{1}{\varepsilon}}\left(\frac{r^{N-1+2(\beta-\sqrt{\bar\mu})}}{(1+r^{\frac{2\beta}{\sqrt{\bar\mu}}})^{N-2}}\varepsilon^{(2-N)}+\frac{r^{N-1+p(\beta-\sqrt{\bar\mu})}}{(1+r^{\frac{2\beta}{\sqrt{\bar\mu}}})^{\frac{N-2}{2}p}}\varepsilon^{(\frac{2-N}{2})p}\right)dr\\ &\leqslant C\varepsilon^{N-2}\left(\frac{(\bar\tau\varepsilon^{-1})^{N-1+2(\beta-\sqrt{\bar\mu})}}{(1+(\bar\tau\varepsilon^{-1})^{\frac{2\beta}{\sqrt{\bar\mu}}})^{N-2}}\varepsilon^{2-N}+\frac{(\bar\tau\varepsilon^{-1})^{N-1+p(\beta-\sqrt{\bar\mu})}}{(1+(\bar\tau\varepsilon^{-1})^{\frac{2\beta}{\sqrt{\bar\mu}}})^{\frac{N-2}{2}p}}\varepsilon^{(\frac{2-N}{2})p}\right)(1-R)\varepsilon^{-1}\\ &\leqslant C\varepsilon^{-2}O\big(\varepsilon^{2\beta}\big)\leqslant C\quad \text{as } \beta>1,\end{aligned} \tag{4.6}$$

where $\bar\tau\in(R,1)$, C is a positive constant.

Hence, (4.4) follows from (4.5) and (4.6). Combining (4.3) and (4.4), we get (4.2) for sufficiently small ε. □

Lemma 4.2. *Under the assumptions* (a1)–(a5), *we have*

$$\sup_{t\geqslant 0} I(tw_R)<J^\infty, \tag{4.7}$$

for R sufficiently large, where $w_R=w(x-R\upsilon)$, w and υ are defined in Lemma 2.8.

Proof. Since $I(tw_R)\to-\infty$ as $t\to+\infty$ uniformly in $R\geqslant 1$, there exists $\bar t>0$ such that

$$\sup_{t\geqslant 0} I(tw_R)=\sup_{0\leqslant t\leqslant\bar t} I(tw_R).$$

To verify (4.7), we only need to show

$$\sup_{0\leqslant t\leqslant \bar{t}} I(tw_R) < J^\infty \quad \text{for } R \text{ large enough.}$$

Since $a(x)\in C(\mathbb{R}^N)$, we can choose a small $\tau\in(0,1)$ such that

$$1-a(x)+\frac{\mu}{|x|^2}\geqslant\frac{\mu}{2|x|^2},\quad \forall|x|\leqslant\tau.$$

Then, we find

$$\begin{aligned}
\int_{\mathbb{R}^N}\left(1-a(x)+\frac{\mu}{|x|^2}\right)w_R^2\,dx &\geqslant \int_{|x|\leqslant\tau}\frac{\mu a_1^2}{2\tau^2}\big(|x-R\upsilon|+1\big)^{-(N-1)}e^{-2|x-R\upsilon|}\,dx\\
&\geqslant a_1^2\frac{\mu}{2\tau^2}(R+2)^{-(N-1)}e^{-2(R-2\tau)}\int_{|x|\leqslant\tau}dx\\
&\geqslant C_1\tau^{N-2}R^{-(N-1)}e^{-2(R-2\tau)}=\bar{C}R^{-(N-1)}e^{-2R}, \qquad (4.8)
\end{aligned}$$

where $\bar{C}, C_1$ are positive constants.

On the other hand, it follows from (a5) and Lemma 2.8 that

$$\begin{aligned}
\int_{\mathbb{R}^N}\big(\bar{F}(tw_R)-F(x,tw_R)\big)\,dx &= \int_{\mathbb{R}^N}\int_0^{tw_R}\big(\bar{f}(s)-f(x,s)\big)\,ds\,dx\\
&\leqslant\int_{\mathbb{R}^N}\int_0^{tw_R}e^{-\nu|x|}\big(\varepsilon s+C_\varepsilon s^p\big)\,ds\,dx\\
&\leqslant\varepsilon\frac{\bar{t}^2}{2}\int_{\mathbb{R}^N}e^{-\nu|x|}w_R^2\,dx+C_\varepsilon\frac{\bar{t}^{p+1}}{p+1}\int_{\mathbb{R}^N}e^{-\nu|x|}w_R^{p+1}\,dx\\
&\leqslant\varepsilon B_1R^{-(N-1)}e^{-2R}+C_\varepsilon B_2e^{-\min\{\nu,p+1\}R},
\end{aligned}$$

where $B_1=C_2\bar{t}^2/2$, $B_2=C_3\bar{t}^{p+1}/(p+1)$ are positive constants.

Hence, noting $\nu>2$, we see that for R large enough,

$$\begin{aligned}
I(tw_R)&\leqslant I^\infty(tw_R)-\frac{t^2}{2}\int_{\mathbb{R}^N}\left(1-a(x)+\frac{\mu}{|x|^2}\right)w_R^2\,dx+\int_{\mathbb{R}^N}\big(\bar{F}(tw_R)-F(x,tw_R)\big)\,dx\\
&\leqslant J^\infty-cR^{-N+1}e^{-2R}+\varepsilon B_1R^{-N+1}e^{-2R}+C_\varepsilon B_2e^{-\min\{\nu,p+1\}R}\\
&< J^\infty. \qquad \square \qquad (4.9)
\end{aligned}$$

Complement of the proof of Theorem 1.2. Choose

$$v_0=\begin{cases} v_\varepsilon, & \text{if } \frac{1}{N}S_\mu^{N/2}<J^\infty,\\ w_R, & \text{if } \frac{1}{N}S_\mu^{N/2}\geqslant J^\infty.\end{cases}$$

It follows from Lemma 4.1 and Lemma 4.2 that

$$c_\mu \leqslant \sup_{t \geqslant 0} I(tv_0) < \min\left\{\frac{1}{N} S_\mu^{N/2}, J^\infty\right\}.$$

Consequently, by using Proposition 4.1 and Remark 2.1 we can find a positive critical point u of I at the level c_μ which must be a positive solution for problem (1.1).

As a result, we complete the proof. □

5. An example

In this section, we provide an example to show that the functions $a(x)$ satisfying (a1) may be negative in some bounded domain in $\mathbb{R}^N$.

Theorem 5.1. *The assumption* (a1) *holds naturally if* $a(x) \in C(\mathbb{R}^N)$ *satisfying*

i) $a(x) \to \bar{a} > 0$ *as* $|x| \to +\infty$;
ii) $-h \leqslant a(x)$ *and the set* $\{x \in \mathbb{R}^N\colon -h \leqslant a(x) \leqslant 0\}$ *is nonempty and bounded, where* $h \in (0, h^*)$ *and* h^* *is a small positive constant.*

Proof. Set $d = 1 - (\frac{2}{N-2})^2\mu$. We first claim that there exist $h^* > 0$ and $\lambda^* > 0$ such that

$$\int_{\mathbb{R}^N} \left(d|\nabla u|^2 + a(x)u^2\right) \geqslant \lambda^* \int_{\mathbb{R}^N} |u|^2 \quad \text{for all } u \in H^1\left(\mathbb{R}^N\right) \text{ and } h \in \left(0, h^*\right). \tag{5.1}$$

In fact, for any bounded smooth domain $\Lambda \subset \mathbb{R}^N$, denote by $\lambda_1(\Lambda)$ the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Lambda)$. Then we can see from page 420 in [19] that $\lambda_1(B_1(0)) \geqslant \pi^2$ and hence

$$\lambda_1\left(B_r(0)\right) \geqslant \frac{\lambda_1(B_1(0))}{r^2} \geqslant \frac{\pi^2}{r^2}.$$

By assumptions i) and ii), we can find $\rho \geqslant 0$ such that

$$\left\{x \in \mathbb{R}^N\colon a(x) \leqslant 0\right\} \subset B_\rho(0)$$

and

$$\inf_{\mathbb{R}^N \setminus B_\rho(0)} a(x) > \frac{\bar{a}}{2}.$$

For $R > \rho$, choose $\psi(x) \in C_0^\infty(B_R(0))$ satisfying $0 \leqslant \psi(x) \leqslant 1$, $\psi(x) = 1$ in $B_\rho(0)$, $\psi(x) = 0$ in $\mathbb{R}^N \setminus B_R(0)$ and $|\nabla\psi| \leqslant \frac{2}{R-\rho}$. Then for any $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 &= \int_{\mathbb{R}^N} \left|\nabla\left(\psi u + (1-\psi)u\right)\right|^2 \\ &= \int_{B_R(0)} \left|\nabla(\psi u)\right|^2 + 2\int_{\mathbb{R}^N} u(1-\psi)\nabla u \nabla\psi + 2\int_{\mathbb{R}^N} u^2 \nabla(1-\psi)\nabla\psi\end{aligned}$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}^N} \psi(1-\psi)|\nabla u|^2 + 2\int_{\mathbb{R}^N} u\psi\nabla(1-\psi)\nabla u + \int_{\mathbb{R}^N} u^2\big|\nabla(1-\psi)\big|^2 \\
&+2\int_{\mathbb{R}^N} u\psi\nabla u\nabla(1-\psi) + \int_{\mathbb{R}^N}(1-\psi)^2|\nabla u|^2 \\
\geqslant\ & \lambda_1\big(B_R(0)\big)\int_{B_R(0)} |\psi u|^2 - \frac{6}{R-\rho}\int_{\mathbb{R}^N}|u||\nabla u| - \frac{4}{(R-\rho)^2}\int_{\mathbb{R}^N} u^2 \\
\geqslant\ & \frac{\lambda_1(B_1(0))}{R^2}\int_{B_\rho(0)}|u|^2 - \frac{5}{(R-\rho)^2}\int_{\mathbb{R}^N} u^2 - \frac{1}{9}\int_{\mathbb{R}^N}|\nabla u|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{\mathbb{R}^N}\big(d|\nabla u|^2 + a(x)u^2\big) &\geqslant \frac{d\pi^2}{R^2}\int_{B_\rho(0)}|u|^2 + \int_{\mathbb{R}^N} a(x)u^2 - \frac{5d}{(R-\rho)^2}\int_{\mathbb{R}^N}u^2 - \frac{d}{9}\int_{\mathbb{R}^N}|\nabla u|^2 \\
&= \int_{B_\rho(0)}\left(\frac{d\pi^2}{R^2} - \frac{5d}{(R-\rho)^2} + a(x)\right)|u|^2 + \int_{\mathbb{R}^N\setminus B_\rho(0)}\left(a(x) - \frac{5d}{(R-\rho)^2}\right)|u|^2 \\
&\quad - \frac{d}{9}\int_{\mathbb{R}^N}|\nabla u|^2.
\end{aligned}
$$

Choose R large such that

$$
\frac{\pi^2}{R^2} - \frac{5}{(R-\rho)^2} \geqslant \frac{\pi^2-5}{2R^2}, \quad \text{and} \quad \frac{5d}{(R-\rho)^2} < \frac{\bar{a}}{4}.
$$

Then

$$
\frac{10}{9}\int_{\mathbb{R}^N}\big(d|\nabla u|^2 + a(x)u^2\big) \geqslant \int_{B_\rho(0)}\left(\frac{(\pi^2-5)d}{2R^2} + \frac{10}{9}a(x)\right)u^2 + \frac{\bar{a}}{4}\int_{\mathbb{R}^N\setminus B_\rho(0)} u^2.
$$

Setting $h^* = \frac{9}{10}\frac{(\pi^2-5)d}{4R^2}$, we find that for $0 \leqslant h \leqslant h^*$,

$$
\begin{aligned}
\frac{10}{9}\int_{\mathbb{R}^N}\big(d|\nabla u|^2 + a(x)u^2\big) &\geqslant \frac{(\pi^2-5)d}{4R^2}\int_{B_\rho(0)} u^2 + \frac{\bar{a}}{4}\int_{\mathbb{R}^N\setminus B_\rho(0)} u^2 \\
&\geqslant \min\left\{\frac{(\pi^2-5)d}{4R^2}, \frac{\bar{a}}{4}\right\}\int_{\mathbb{R}^N} u^2.
\end{aligned}
$$

The inequality (5.1) follows if we set $\lambda^* = \frac{9}{10}\min\{\frac{(\pi^2-5)d}{4R^2}, \frac{\bar{a}}{4}\}$ and hence our claim holds true.

Now we are ready to verify the inequality in assumption (a1). From (5.1), we have

$$\begin{aligned}\int\limits_{\mathbb{R}^N}\left(\left(1-\left(\frac{2}{N-2}\right)^2\mu\right)|\nabla u|^2+a(x)u^2\right)&=\int\limits_{\mathbb{R}^N}\left(d|\nabla u|^2+a(x)u^2\right)\\&\geqslant\lambda^*\int\limits_{\mathbb{R}^N}u^2=\frac{\lambda^*}{\bar a+h^*}\int\limits_{\mathbb{R}^N}\left(\bar a+h^*\right)u^2\\&\geqslant\frac{\lambda^*}{\bar a+h^*}\int\limits_{\mathbb{R}^N}\left(\bar a-a(x)\right)u^2.\end{aligned}$$

We can complete the proof by taking $\lambda_1=\frac{\lambda^*}{\bar a+h^*}$. $\square$

Acknowledgments

The paper was supported by the funds from NSFC (11071093, 11071094, 11125101, 11101160). The authors would like to thank the anonymous referees for carefully reading this paper and suggesting many useful comments.

References

[1] J.P. Garcia Azorero, I. Peral Alonso, Hardy inequalities and some critical elliptic and parabolic problems, J. Differential Equations 144 (1998) 441–476.
[2] A. Bahri, J.M. Coron, On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain, Comm. Pure Appl. Math. 41 (1988) 253–294.
[3] P. Baras, J.A. Goldstein, The heat equation with a singular potential, Trans. Amer. Math. Soc. 284 (1984) 121–139.
[4] V. Benci, G. Cerami, Positive solutions of semilinear elliptic problem in exterior domains, Arch. Ration. Mech. Anal. 99 (1987) 283–300.
[5] H. Berestycki, M.J. Esteban, Existence and bifurcation of solutions for an elliptic degenerate problem, J. Differential Equations 134 (1997) 1–25.
[6] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[7] H. Brezis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.
[8] D. Cao, Positive solution and bifurcation from the essential spectrum of a semilinear elliptic equation on $\mathbb{R}^N$, Nonlinear Anal. 15 (1990) 1045–1052.
[9] D. Cao, S. Peng, A global compactness result for singular elliptic problems involving critical Sobolev exponent, Proc. Amer. Math. Soc. 131 (2003) 1857–1866.
[10] D. Cao, S. Peng, A note on the sign-changing solutions to elliptic problem with critical Sobolev and Hardy terms, J. Differential Equations 193 (2003) 424–434.
[11] F. Catrina, Z.Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001) 229–258.
[12] K. Chen, C. Peng, Multiplicity and bifurcation of positive solutions for nonhomogeneous semilinear elliptic problems, J. Differential Equations 240 (2007) 58–91.
[13] I. Daubechies, E. Lieb, One-electron relativistic molecules with Coulomb interaction, Comm. Math. Phys. 90 (1983) 497–510.
[14] Y. Deng, Z. Guo, G. Wang, Nodal solutions for p-Laplace equations with critical growth, Nonlinear Anal. 54 (2003) 1121–1151.
[15] Y. Deng, L. Jin, S. Peng, A Robin boundary problem with Hardy potential and critical nonlinearities, J. Anal. Math. 104 (2008) 125–154.
[16] V. Felli, E.M. Marchini, S. Terracini, On Schrödinger operators with multipolar inverse-square potentials, J. Funct. Anal. 250 (2007) 265–316.
[17] W.M. Frank, D.J. Land, R.M. Spector, Singular potentials, Rev. Modern Phys. 43 (1971) 36–98.
[18] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, Adv. Math. Suppl. Stud. A 7 (1981) 209–243.
[19] E. Jannelli, The role played by space dimension in elliptic critical problems, J. Differential Equations 156 (1999) 407–426.
[20] J.M. Lévy-Leblond, Electron capture by polar molecules, Phys. Rev. 153 (1967) 1–4.
[21] P.L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact cases. Part I, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145.

[22] P.L. Lions, The concentration-compactness principle in the calculus of variations, The locally compact cases. Part II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.
[23] D. Pierrotti, S. Terracini, On a Neumann problem with critical exponent and critical nonlinearity on the boundary, Comm. Partial Differential Equations 20 (1995) 1155–1187.
[24] D. Smets, Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, Trans. Amer. Math. Soc. 357 (2005) 2909–2938.
[25] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187 (1984) 511–517.
[26] S. Terracini, On positive solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations 2 (1996) 241–264.
[27] M. Willem, Minimax Theorems, Birkhäuser Verlag, Basel, 1996.
[28] X. Zhu, D. Cao, The concentration-compactness principle in nonlinear elliptic equations, Acta Math. Sci. 9 (1989) 307–323.