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Realization problems for limit cycles of planar polynomial vector fields

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Abstract

We show that for any finite configuration of closed curves $\Gamma \subset \mathbb{R}^2$, one can construct an explicit planar polynomial vector field that realizes Γ, up to homeomorphism, as the set of its limit cycles with prescribed periods, multiplicities and stabilities. The only obstruction given on this data is the obvious compatibility relation between the stabilities and the parity of the multiplicities. The constructed vector fields are Darboux integrable and admit a polynomial inverse integrating factor.

Keywords: Limit cycle; Polynomial vector field; Integrating factor; Realization problem

1. Introduction and statement of the main theorem

We consider the planar vector field

$$X = P(x, y)\partial_x + Q(x, y)\partial_y, \tag{1.1}$$

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where P and Q are polynomials. The degree of X is defined as the maximum of the degrees of P and Q, and will be denoted by $\deg(X)$. The study of the limit cycles of planar polynomial vector fields has a long tradition, starting with the celebrated second part of Hilbert's 16th problem, cf. [12], which consists in finding the maximum number of limit cycles for the vector field (1.1) in terms of the degree $\deg(X)$, and studying the relative positions of these cycles. We recall that a limit cycle of X is a periodic trajectory that is isolated. Despite having been formulated more than a century ago, Hilbert's 16th problem is still wide open, even for quadratic vector fields, i.e. $\deg(X) = 2$.

A related problem that has attracted considerable attention in the last years is the realization problem for limit cycles. To state it in a precise way let us introduce some basic definitions. Let C be a closed curve embedded in $\mathbb{R}^2$. A configuration of cycles is a finite set $\Gamma = \{C_1, \dots, C_n\}$ of closed curves.

Definition 1.1. We say that two configurations of cycles Γ, Γ' are *equivalent* if there exists a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi(\Gamma) = \Gamma'$. A vector field X is said to *realize* a configuration of cycles Γ if its set of limit cycles is equivalent to Γ.

The realization (or inverse) problem asks if, for any configuration of cycles Γ, there exists a planar vector field X that realizes Γ. In this problem it is usual to prescribe other dynamical properties of the limit cycles (e.g. stability) as well as some additional conditions on the vector field X (e.g. regularity).

The realization problem for limit cycles was first addressed by Al'mukhamedov [1] for C^k vector fields using the theory of Lyapunov functions. The same techniques allowed Sverdlove [17] to solve the C^k realization problem completely (also prescribing the stability of the cycles), fixing a gap in Al'mukhamedov's construction that had been noticed by other authors, see [17] for details.

Regarding polynomial vector fields, the first realization result was obtained by Bautin in [3] (later corrected in [7]), where explicit expressions for polynomial planar vector fields with a prescribed set of algebraic limit cycles were derived. Alternative constructions appear in the works of Winkel [18], Christopher [5] and Korchagin [13]. The idea of using inverse integrating factors to construct polynomial vector fields with prescribed algebraic limit cycles was first given in [11,4]. This idea culminated in the complete solution to the realization problem in the polynomial setting obtained by Llibre and Rodríguez in [14] using inverse integrating factors and the Darboux theory of integrability. An alternative proof using Lyapunov functions was given in [16], with a generalization to higher dimensions. A remarkable recent solution of the realization problem was obtained by Coll, Dumortier and Prohens [6] using polynomial Liénard equations and the theory of slow–fast systems. All these works provide an upper bound for $\deg(X)$ which depends on the configuration of cycles. Notice that since the degree of X is not fixed a priori, the realization problem is much easier than Hilbert's 16th problem.

The main result of this paper is a realization theorem for planar polynomial vector fields prescribing not only the configuration of limit cycles, but also their periods, multiplicities and stabilities. These are the three most basic invariants under smooth conjugacy, so it is reasonable to consider them as a part of the realization problem. The limit cycles of the systems constructed in [14] and [6] are all hyperbolic and their stabilities depend on the configuration that is realized (of course, there are no semistable limit cycles due to the hyperbolicity).

To state our main theorem let us introduce some notation. We call D_C the compact set bounded in $\mathbb{R}^2$ by the closed curve C. We say that a limit cycle C is stable (unstable) in the interior

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if all the trajectories of X in D_C that are sufficiently close to C approach the limit cycle as $t \to \infty$ ($t \to -\infty$). Analogously, if all the trajectories of X in $\mathbb{R}^2 \setminus D_C$ that are close enough to C approach the limit cycle as $t \to \infty$ ($t \to -\infty$), we say that C is stable (unstable) in the exterior. It is obvious that the exterior stability of a cycle is determined by its interior stability and multiplicity.

Theorem 1.2. *Consider sets $\{T_1, \ldots, T_n\}$ of positive constants and $\{m_1, \ldots, m_n\}$ of positive integers. Let $\Gamma = \{C_1, \ldots, C_n\}$ be a configuration of cycles with fixed interior stabilities. Then Γ is realized as the set of limit cycles of a polynomial vector field X, where each C_k has multiplicity m_k, period T_k and the required stability. Moreover, there exists an explicit upper bound for the degree of X in terms of the prescribed quantities (see Theorem 5.2).*

The proof of Theorem 1.2 is based on the construction of Llibre and Rodríguez [14]. As in [14], the realized limit cycles are algebraic and the vector fields are Darboux integrable and admit a polynomial inverse integrating factor. We would like to remark that the limit cycles of a polynomial vector field do not need to be algebraic, cf. [15], but we are not aware of any construction of polynomial vector fields with given limit cycles that are not algebraic. It thus remains an interesting open problem to characterize those analytic curves that can be limit cycles of a polynomial vector field.

The structure of the paper is as follows. In Section 2 we construct a vector field X_T realizing the configuration of cycles Γ with prescribed periods. The construction of a vector field X_m with prescribed limit cycles and multiplicities is presented in Section 3, where we also combine both constructions to obtain a vector field X_{Tm} prescribing periods and multiplicities at the same time. In Section 4 we construct a vector field X_{Ts} realizing the configuration of cycles Γ with prescribed periods and stabilities. Finally, using the constructions in the previous sections, the main theorem is proved in Section 5.

2. A planar polynomial vector field with prescribed limit cycles and periods

In the following theorem we show that any configuration of cycles Γ can be realized by a planar polynomial vector field, where the period of each limit cycle is also prescribed. The method of the proof is a variation of the construction introduced in [14]. In the statement of the theorem, a curve $C_k \in \Gamma$ is called primary if no other curve $C_j \in \Gamma$, $j \neq k$, is contained in the domain D_{C_k}.

We recall that a *Darboux first integral* is a (possibly multivalued) function G that can be written as:

$$G = e^{g/h} \prod_{l=1}^{L} f_l^{\lambda_l},$$

where f_l, g and h are complex polynomials and $\lambda_l \in \mathbb{C}$ are complex constants, and a smooth function V is an *inverse integrating factor* of a vector field X if

$$X \cdot \nabla V = V \operatorname{div}(X),$$

where div denotes the divergence operator.

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Theorem 2.1. *Let $\Gamma = \{C_1, \dots, C_n\}$ be a configuration of cycles and $\{T_1, \dots, T_n\}$ a set of positive constants, then Γ is realized by a planar polynomial vector field X_T with $\deg(X_T) < 2(n+r)$, where each periodic orbit C_k has period T_k. Here r is the number of primary cycles in Γ. Moreover, X_T admits a polynomial inverse integrating factor, it is Darboux integrable and all its limit cycles are algebraic and hyperbolic.*

Proof. It is well known [14] that any configuration of cycles is homeomorphic to a set of circles. Accordingly, we can take that Γ consists of n disjoint circles of radii $\{r_k\}_{k=1}^n$ centered at the points $\{p_k\}_{k=1}^n$, i.e.

$$C_k = \{f_k(x, y) = 0\} \quad \text{with} \quad f_k(x, y) := (x - x_k)^2 + (y - y_k)^2 - r_k^2, \tag{2.1}$$

where $p_k = (x_k, y_k)$. We can safely assume that the first r cycles $\{C_1, \dots, C_r\}$ are primary and that no point p_k is contained in any circle C_j.

Let us introduce the following auxiliary functions:

$$\begin{aligned}
g_k(x, y) &:= (x - x_k)^2 + (y - y_k)^2, \\
A &:= \prod_{k=1}^{n} f_k, \\
A_T &:= \prod_{k=1}^{n} f_k^{\tau_k}, \\
B &:= \prod_{k=1}^{r} g_k, \\
C &:= \exp\left(-2\sum_{k=1}^{r} \theta_k\right), \\
D_T &:= A_T B C, \\
H_T &:= \ln D_T,
\end{aligned}$$

where τ_k are positive constants that will be fixed later in order to prescribe the desired periods T_k, and θ_k is an angular (multivalued) function defined as

$$\theta_k := \arctan\left(\frac{y - y_k}{x - x_k}\right). \tag{2.2}$$

We claim that the vector field X_T in the statement of the theorem can be defined as

$$X_T := P_T \partial_x + Q_T \partial_y, \qquad \text{with} \qquad \begin{aligned} P_T &:= -AB\frac{\partial H_T}{\partial y}, \\ Q_T &:= AB\frac{\partial H_T}{\partial x}. \end{aligned}$$

We observe that in the case that $\tau_k = 1$ for all k, then $A_T = A$ and the vector field X_T is the same as in the construction by Llibre and Rodríguez [14]. Throughout the paper, we shall use the notation $X_{LR} := X_T|_{\tau_k=1}$.

To prove that X_T satisfies all the claims in the statement of the theorem, let us first show that it is a polynomial vector field with $\deg(X_T) < 2(n+r)$, that D_T is a Darboux first integral and that $V_T := AB$ is an inverse integrating factor. Indeed, noticing that

$$B\frac{\partial C}{\partial x} = C\frac{\partial B}{\partial y}, \qquad B\frac{\partial C}{\partial y} = -C\frac{\partial B}{\partial x}, \tag{2.3}$$

a straightforward computation shows that P_T and Q_T can be written as

$$P_T = A\left(\frac{\partial B}{\partial x} - \frac{\partial B}{\partial y}\right) - B\sum_{k=1}^{n}\tau_k\mu_k\frac{\partial f_k}{\partial y}, \tag{2.4}$$

$$Q_T = A\left(\frac{\partial B}{\partial x} + \frac{\partial B}{\partial y}\right) + B\sum_{k=1}^{n}\tau_k\mu_k\frac{\partial f_k}{\partial x}, \tag{2.5}$$

where

$$\mu_k := \prod_{j\neq k}^{n} f_j.$$

It is clear from these expressions that P_T and Q_T are polynomials of degree at most $2(n+r)-1$. Moreover, it is easy to check that the Darboux function D_T satisfies $X_T \cdot \nabla D_T = 0$, thus implying that D_T is a Darboux first integral of X_T. Another easy computation shows that the vector field $\frac{X_T}{V_T}$ is divergence-free in $\mathbb{R}^2 \setminus V_T^{-1}(0)$, and hence V_T is an inverse integrating factor of X_T.

Accordingly, all the limit cycles of X_T are contained in the zero set $V_T^{-1}(0)$ of its inverse integrating factor [10]. Therefore, since

$$V_T^{-1}(0) = \Gamma \cup \{p_1, \dots, p_r\},$$

we conclude that if X_T has a limit cycle, it has to be precisely one of the circles $\{C_1, \dots, C_n\}$. Let us now prove that indeed all of them are realized as limit cycles.

We first show that each C_k is a periodic trajectory of X_T. Since $V_T^{-1}(0)$ is invariant under the flow of X_T, it is enough to prove that X_T does not vanish on each C_k. Using Eqs. (2.4) and (2.5) we can evaluate the components P_T and Q_T of the vector field on each circle $C_k = \{f_k = 0\}$, thus obtaining

$$X_T|_{C_k} = \tau_k B|_{C_k}\mu_k|_{C_k}\left(-\frac{\partial f_k}{\partial y}\bigg|_{C_k}\partial_x + \frac{\partial f_k}{\partial x}\bigg|_{C_k}\partial_y\right) = \tau_k X_{LR}|_{C_k}. \tag{2.6}$$

Since the gradient of f_k only vanishes at the point p_k, and the functions B and μ_k do not vanish on C_k (because all the circles in Γ are disjoint and no point p_j is contained in any circle C_k), we infer that X_T has no zeros on C_k, which is then a periodic orbit.

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Let us assume that the periodic orbit C_k is not a limit cycle. Since X_T is polynomial, it then follows that C_k must belong to a period annulus, i.e. it is surrounded by a family of periodic orbits. Consider a periodic orbit γ_k close enough to C_k so that it is disjoint from the set Γ and all the points p_j. In particular, we have that $V_T|_{\gamma_k}$ does not vanish. Defining the 1-form $\omega_T := -Q_T dx + P_T dy$, we have that

$$\int_{\gamma_k} \frac{\omega_T}{V_T} = 0 \tag{2.7}$$

because γ_k is a periodic orbit of X_T. On the other hand, using the definitions of P_T, Q_T and V_T, we can also write

$$\begin{aligned} \int_{\gamma_k} \frac{\omega_T}{V_T} &= -\int_{\gamma_k} dH_T = -\int_{\gamma_k} \Big[d\,(\ln A_T) + d\,(\ln B) + d\,(\ln C) \Big] \\ &= 2\sum_{j=1}^{r} \int_{\gamma_k} d\theta_j = \pm 4\pi s_k \neq 0, \end{aligned}$$

where we have used that the functions $\ln A_T$ and $\ln B$ are smooth on γ_k because A_T and B do not vanish there, and hence $\int_{\gamma_k} d\,(\ln A_T) = 0 = \int_{\gamma_k} d\,(\ln B)$. The number s_k in the last equality is the number of primary cycles contained in D_{C_k}, which is at least 1 (it is 1 if and only if C_k is primary). The sign of the last expression depends on how we orient the circle C_k. Since this formula contradicts Eq. (2.7), we conclude that all the periodic orbits C_k are limit cycles of X_T. Moreover, since the vanishing order of the inverse integrating factor V_T on each cycle C_k is 1, it follows that all the limit cycles are hyperbolic [8,9].

To conclude the proof of the theorem, let us show that each constant τ_k can be chosen in such a way that the period of the limit cycle C_k is T_k. Indeed, since $X_T|_{C_k} = \tau_k X_{LR}|_{C_k}$ by Eq. (2.6), denoting by T_k^{LR} the period of the limit cycle C_k for the vector field X_{LR}, we have that the orbit C_k has period T_k if we choose

$$\tau_k = \frac{T_k^{LR}}{T_k}.$$

The theorem then follows. □

3. A planar polynomial vector field with prescribed limit cycles, multiplicities and periods

We first prove that, for any configuration of cycles Γ, we can construct a planar polynomial vector field that realizes it as a set of limit cycles with prescribed multiplicities. As in Section 2, the method of the proof is modeled upon the construction of Llibre and Rodríguez [14].

Theorem 3.1. *Let $\Gamma = \{C_1, \ldots, C_n\}$ be a configuration of cycles and $\{m_1, \ldots, m_n\}$ a set of positive integers, then Γ is realized by a planar polynomial vector field X_m with $\deg(X_m) < 2(r + \sum m_k)$, where each periodic orbit C_k has multiplicity m_k. Here r is the number of primary cycles in Γ. Moreover, X_m admits a polynomial inverse integrating factor, it is Darboux integrable and all its limit cycles are algebraic.*

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Proof. Arguing as in the proof of Theorem 2.1, we can assume that Γ is a set of n circles $C_k = \{f_k(x, y) = 0\}$, see Eq. (2.1) for the definition of f_k, where the first r ones are the primary cycles and no point p_k lies on any C_j.

We consider the auxiliary functions A, B, C defined in Section 2, as well as the functions:

$$
\begin{aligned}
A_m &:= \prod_{k=1}^{n} f_k^{m_k}, \\
D &:= ABC, \\
\mathcal{D} &:= D \exp\left(\Lambda \sum_{k=1}^{n} h_k\right), \\
H &:= \ln D, \\
\lambda_k &:= \prod_{j \neq k}^{n} f_j^{m_j}, \\
F &:= \Lambda \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial y}, \\
G &:= \Lambda \sum_{k=1}^{n} \lambda_k \frac{\partial f_k}{\partial x},
\end{aligned}
$$

where Λ is a constant defined as

$$
\Lambda := \sum_{k=1}^{n} (m_k - 1),
$$

which vanishes if and only if $m_k = 1$ for every $k = 1, \cdots, n$, and

$$
h_k := \begin{cases} \dfrac{f_k^{1-m_k}}{(1-m_k)} & \text{if } m_k \geq 2, \\ \ln f_k & \text{if } m_k = 1. \end{cases}
$$

We claim that the vector field X_m in the statement of the theorem can be defined using these functions as:

$$
X_m := P_m \partial_x + Q_m \partial_y \qquad \text{with} \qquad
\begin{aligned}
P_m &:= -A_m B \frac{\partial H}{\partial y} - BF, \\
Q_m &:= A_m B \frac{\partial H}{\partial x} + BG.
\end{aligned}
$$

Observe that if $m_k = 1$ for all k, then $A_m = A$ and $F = 0 = G$, thus we have $X_m|_{m_k=1} = X_{LR}$, where X_{LR} is the vector field constructed by Llibre and Rodríguez in [14].

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To prove that X_m satisfies the desired properties, let us show that it is a polynomial vector field with $\deg(X_m) < 2(r + \sum m_k)$, that $\mathcal{D}$ is a Darboux first integral and that $V_m := A_m B$ is an inverse integrating factor. Indeed, by direct computations we obtain that P_m and Q_m can be written as

$$P_m = P_{LR} \prod_{k=1}^{n} f_k^{m_k-1} - BF, \tag{3.1}$$

$$Q_m = Q_{LR} \prod_{k=1}^{n} f_k^{m_k-1} + BG. \tag{3.2}$$

From these expressions it follows that P_m and Q_m are polynomials of degree at most $2(n + \sum m_k) - 1$. Moreover, using the identities:

$$\frac{\partial \mathcal{D}}{\partial x} = \mathcal{D}\left(\frac{1}{D}\frac{\partial D}{\partial x} + \frac{G}{A_m}\right), \qquad \frac{\partial \mathcal{D}}{\partial y} = \mathcal{D}\left(\frac{1}{D}\frac{\partial D}{\partial y} + \frac{F}{A_m}\right),$$

one can easily check that $\mathcal{D}$ is a Darboux first integral of X_m. It is also straightforward to show that the vector field $\frac{X_m}{V_m}$ is divergence-free in $\mathbb{R}^2 \setminus V_m^{-1}(0)$, thus implying that V_m is an inverse integrating factor of X_m.

Arguing as in the proof of Theorem 2.1, we conclude that the circles $\{C_1, \cdots, C_n\}$ are invariant under the flow of X_m, and that if X_m has a limit cycle, it has to be precisely one of these circles. Let us now prove that all of them are limit cycles.

First we check that X_m does not vanish on each C_k. Using Eqs. (3.1) and (3.2) we can write

$$X_m|_{C_k} = B|_{C_k}\lambda_k|_{C_k}\left(\Lambda + f_k^{m_k-1}|_{C_k}\right)\left(-\frac{\partial f_k}{\partial y}\Big|_{C_k}\partial_x + \frac{\partial f_k}{\partial x}\Big|_{C_k}\partial_y\right). \tag{3.3}$$

Notice that the functions B, λ_k and $\Lambda + f_k^{m_k-1}$ do not vanish on C_k because all the circles in Γ are disjoint, no point p_j is contained in a circle C_k and $\Lambda > 0$ unless $m_j = 1$ for all j. In the case that all the limit cycles have multiplicity 1, it follows that $\Lambda + f_k^{m_k-1} = 1$. Moreover, the gradient of f_k only vanishes at the point p_k, so we infer that X_m has no zeros on C_k, which is then a periodic orbit.

Since the vector field X_m is polynomial, if C_k is not a limit cycle then it must belong to a period annulus. Let us assume that this is the case, and take a periodic orbit γ_k close enough to C_k so that it is disjoint from the set Γ and all the points p_j. In particular, we have that the function $V_m|_{\gamma_k}$ does not vanish. Defining the 1-form $\omega_m := -Q_m dx + P_m dy$, we have that

$$\int_{\gamma_k} \frac{\omega_m}{V_m} = 0 \tag{3.4}$$

because γ_k is a periodic orbit of X_m. Using the definitions of P_m, Q_m and V_m, and proceeding as in the proof of Theorem 2.1, we can also write

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$$
\begin{aligned}
\int_{\gamma_k} \frac{\omega_m}{V_m} &= -\int_{\gamma_k} \left[dH + \Lambda \sum_{j=1}^{n} dh_j \right] \\
&= -\int_{\gamma_k} \Big[d(\ln A) + d(\ln B) + d(\ln C) \Big] = \pm 4\pi s_k \neq 0.
\end{aligned}
$$

To obtain this formula we have used that the functions $\ln A$, $\ln B$ and h_j are smooth on γ_k because A, B and f_j do not vanish there. The number s_k in the last equality is the number of primary cycles contained in D_{C_k}, which is at least 1 by definition. Since this formula contradicts Eq. (3.4), we deduce that all the periodic orbits C_k are limit cycles of X_m.

To conclude the proof of the theorem, we notice that, by construction, the vanishing order of the inverse integrating factor V_m on the limit cycle C_k is m_k. Since the multiplicity of a limit cycle is equal to the vanishing order of the inverse integrating factor [8,9], it follows that the multiplicity of C_k is m_k, in particular C_k is hyperbolic if and only if $m_k = 1$. $\square$

Combining the constructions in Theorems 2.1 and 3.1, and arguing exactly as in their proofs, it is easy to check that the vector field X_{Tm} defined as

$$
X_{Tm} := P_{Tm}\partial_x + Q_{Tm}\partial_y, \qquad \text{with} \qquad
\begin{aligned}
P_{Tm} &:= -A_m B \frac{\partial H_T}{\partial y} - B F_T, \\
Q_{Tm} &:= A_m B \frac{\partial H_T}{\partial x} + B G_T,
\end{aligned}
$$

realizes the set of cycles Γ with prescribed periods and multiplicities. Here the functions B and H_T were defined in the proof of Theorem 2.1, A_m in the proof of Theorem 3.1 and we set

$$
\begin{aligned}
F_T &:= \Lambda \sum_{k=1}^{n} \tau_k \lambda_k \frac{\partial f_k}{\partial y}, \\
G_T &:= \Lambda \sum_{k=1}^{n} \tau_k \lambda_k \frac{\partial f_k}{\partial x}.
\end{aligned}
$$

The constants τ_k are chosen so that the limit cycle C_k has period T_k. Indeed, noticing that

$$
X_{Tm}|_{C_k} = \tau_k \Big(\Lambda + f_k^{m_k - 1}|_{C_k} \Big) \prod_{j \neq k} f_j^{m_j - 1}|_{C_k} X_{LR}|_{C_k}, \tag{3.5}
$$

if we denote by $\gamma_k^{LR}(t)$ the integral curve parametrizing C_k of the Llibre–Rodríguez vector field X_{LR} realizing Γ, the constant τ_k is chosen as

$$
\tau_k = \frac{1}{T_k} \int_0^{T_k^{LR}} \frac{dt}{[\Lambda + f_k^{m_k-1}(\gamma_k^{LR}(t))] \prod_{j \neq k}^{N} f_j^{m_j-1}(\gamma_k^{LR}(t))},
$$

thus implying that the period of C_k for the vector field X_{Tm} is T_k. Here T_k^{LR} is the period of the integral curve $\gamma_k^{LR}(t)$. The theorem can then be stated as follows:

Theorem 3.2. *Let $\Gamma = \{C_1, \ldots, C_n\}$ be a configuration of cycles, $\{m_1, \ldots, m_n\}$ a set of positive integers and $\{T_1, \ldots, T_n\}$ a set of positive constants, then Γ is realized by a planar polynomial vector field X_{Tm} with $\deg(X_{Tm}) < 2(r + \sum m_k)$, where each periodic orbit C_k has multiplicity m_k and period T_k. Here r is the number of primary cycles in Γ. Moreover, X_{Tm} admits a polynomial inverse integrating factor, it is Darboux integrable and all its limit cycles are algebraic.*

4. A planar polynomial vector field with prescribed limit cycles, periods and stabilities

The goal of this section is to prove that, for any configuration of cycles Γ, we can construct a planar polynomial vector field that realizes it as a set of hyperbolic limit cycles with prescribed stabilities and periods. To this end, we first show that the stability of each limit cycle C_k of the polynomial vector field X_T constructed in Section 2 can be characterized in terms of the relative position of C_k with respect to the other cycles. In what follows, we will say that the limit cycle C_k has stability -1 if it is stable and 1 if it is unstable. Since all the limit cycles considered in this section are hyperbolic, the interior and exterior stabilities are the same; the case of semistable limit cycles will be addressed in Section 5.

Lemma 4.1. *The limit cycle C_k of the vector field X_T introduced in Section 2 has stability $(-1)^{N_k}$, where $N_k := card\{C_j : C_k \subset D_{C_j},\ j \neq k\}$.*

Proof. Denoting by $\gamma_k(t)$, $t \in \mathbb{R}$, the integral curve of X_T whose image is the limit cycle C_k, we first compute the derivative of the angular variable θ_j defined in Eq. (2.2) for each j:

$$\begin{aligned}\frac{d\theta_j(\gamma_k(t))}{dt} &= X_T(\gamma_k(t)) \cdot \nabla\theta_j(\gamma_k(t)) \\ &= 2\tau_k B(\gamma_k(t))\mu_k(\gamma_k(t)) \left.\frac{(x-x_k)(x-x_j)+(y-y_k)(y-y_j)}{(x-x_j)^2+(y-y_j)^2}\right|_{\gamma_k(t)}, \end{aligned} \tag{4.1}$$

which is negative (positive) if the orientation of C_k induced by the integral curve $\gamma_k(t)$ is clockwise (counterclockwise). The constants τ_k and the functions B and μ_k were defined in the proof of Theorem 2.1. Assuming that the point (x_j, y_j) is contained in the interior of the compact set D_{C_k}, it easily follows that

$$\left.\frac{(x-x_k)(x-x_j)+(y-y_k)(y-y_j)}{(x-x_j)^2+(y-y_j)^2}\right|_{\gamma_k(t)} > 0. \tag{4.2}$$

Accordingly, since B is always positive over C_k, the sign of $\frac{d\theta_j(\gamma_k(t))}{dt}$ is given by the sign of μ_k over C_k. Noticing that the function $f_j = (x-x_j)^2 + (y-y_j)^2 - r_j^2$ is positive over C_k if and only if C_k is not contained in D_{C_j}, the definition of μ_k implies that its sign is precisely $(-1)^{N_k}$.

The stability of the limit cycle C_k is given by the sign of the following integral [2]

$$\mathcal{L} := \int_0^{T_k} \operatorname{div} X_T(\gamma_k(t))dt.$$

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Using the identity $X_T \cdot \nabla V_T = V_T \operatorname{div} X_T$ we obtain

$$\begin{aligned}
\mathcal{L} &= \int_0^{T_k} X_T(\gamma_k(t)) \cdot \nabla \ln A(\gamma_k(t)) dt \\
&\quad + \int_0^{T_k} X_T(\gamma_k(t)) \cdot \nabla \ln B(\gamma_k(t)) dt \\
&= \int_0^{T_k} X_T(\gamma_k(t)) \cdot \nabla \ln f_k(\gamma_k(t)) dt,
\end{aligned}$$

where to pass to the second equality we have used that the functions $\ln B$ and $\ln f_j$ with $j \neq k$ are smooth on C_k, and therefore the corresponding integrals vanish. Using the definition of X_T, the identities (2.3) and the value of X_T on C_k computed in (2.6), after a few straightforward computations we can write

$$\begin{aligned}
\mathcal{L} &= \frac{-1}{\tau_k} \int_0^{T_k} X_T(\gamma_k(t)) \cdot \nabla \ln C(\gamma_k(t)) \\
&\quad + \frac{-1}{\tau_k} \int_0^{T_k} X_T(\gamma_k(t)) \cdot \nabla \ln \Big(B \prod_{j \neq k} f_j^{\tau_j} \Big)(\gamma_k(t)).
\end{aligned}$$

As before, the second integral in this expression vanishes because the function $B \prod_{j \neq k} f_j^{\tau_j}$ is smooth on C_k. Finally, from the definition of the function C we obtain

$$\mathcal{L} = \frac{2}{\tau_k} \sum_{j=1}^{r} \int_0^{T_k} \frac{d\theta_j(\gamma_k(t))}{dt} dt = \frac{(-1)^{N_k} 4\pi s_k}{\tau_k},$$

where $s_k \in \{1, \dots, r\}$ is the number of primary cycles contained in D_{C_k}, and we have used the sign $(-1)^{N_k}$ computed before (observe that there is a non-vanishing contribution to the integral above if and only if θ_j is the angle whose center (x_j, y_j) is contained in D_{C_k}, so that Eq. (4.2) holds). We then conclude that the stability of C_k is $(-1)^{N_k}$, as we wanted to show. □

This lemma proves that the stability of the limit cycle C_k of X_T is fixed by the configuration of cycles that we want to realize. In the following theorem, which is the main result of this section, we show how to modify the vector field X_T in order to prescribe the stabilities of its limit cycles. The idea is to add additional cycles to the configuration Γ to obtain a new configuration $\tilde{\Gamma}$ so that $\tilde{N}_k$ for the new configuration has the desired sign. Since we do not want to realize the extra cycles $\tilde{\Gamma} \backslash \Gamma$, we can remove them by adding a singular (zero) point over each extra limit cycle of the vector field $\tilde{X}_T$ realizing $\tilde{\Gamma}$.

Theorem 4.2. *Let $\Gamma = \{C_1, \dots, C_n\}$ be a configuration of cycles, $\{T_1, \dots, T_n\}$ a set of positive constants and $\{\nu_1, \dots, \nu_n\}$ a set of ± 1. Then Γ is realized by a planar polynomial vector field X_{Ts} with $\deg(X_{Ts}) < 2(3n + r)$, where each periodic orbit C_k is hyperbolic, has period T_k and stability ν_k. Here r is the number of primary cycles in Γ. Moreover, X_{Ts} admits a polynomial inverse integrating factor, it is Darboux integrable and all its limit cycles are algebraic.*

Proof. As in previous sections, we can assume that Γ consists of circles. Take a circle C_{n+k} centered at p_k of radius $r_k - \varepsilon\nu_k$. Recall that r_k is the radius of the circle C_k and p_k is its center. It is clear that we can take $\varepsilon > 0$ small enough such that all the circles are disjoint and no p_j lies on any C_k and C_{n+k}. We denote the whole configuration by $\tilde{\Gamma} := \{C_1, \cdots, C_{2n}\}$. Observe that the number of primary cycles and their centers remain unchanged (a primary cycle C_k in Γ with $\nu_k > 0$ is no longer a primary cycle in $\tilde{\Gamma}$, instead C_{n+k} will be primary, but it has the same center), and therefore the function B will be the same for Γ and $\tilde{\Gamma}$.

Now we construct a vector field $\tilde{X}_T$ as in Theorem 2.1 realizing the $2n$ cycles of $\tilde{\Gamma}$ where each limit cycle C_k has an associated constant τ_k that will be fixed later (cf. the proof of Theorem 2.1 for the definition of such a constant) and $\tau_{n+k} = 1$ so that the limit cycle C_{n+k} has period T^{LR}_{n+k}, for $k \in \{1, \dots, n\}$. It is obvious from the definition of $\tilde{\Gamma}$ that the parity of the number $\tilde{N}_k$ defined in Lemma 4.1 only depends on the relative positions of C_{n+k} and C_k, and that $(-1)^{\tilde{N}_k} = \nu_k$ for $k = 1, \dots, n$. The lemma then implies that the cycles C_k have the desired stability. In order to remove the additional cycles C_{n+k}, we consider the functions:

$$l_k(x, y) := (x - a_k)^2 + (y - b_k)^2$$

$$L_{Ts} := \prod_{k=1}^{n} l_k$$

where each $q_k := (a_k, b_k) \in C_{n+k}$ is a point at the extra cycle C_{n+k}. Then the vector field:

$$X_{Ts} := L_{Ts}\tilde{X}_T$$

satisfies the statements of the theorem. Indeed, since the factor L_{Ts} is positive on Γ, this set is realized by X_{Ts} as algebraic limit cycles, while the cycles C_{n+k} contain a singular point of X_{Ts}, thus becoming homoclinic connections. The factor L_{Ts} does not change the stability of each cycle C_k of $\tilde{X}_T$, which is ν_k. Moreover, if $\gamma^{LR}_k(t)$ is the integral curve parametrizing C_k of the Llibre–Rodríguez vector field X_{LR} that realizes $\tilde{\Gamma}$, see Section 2 for the definition, the constant τ_k must be chosen such that

$$\tau_k = \frac{1}{T_k} \int_0^{T^{LR}_k} \frac{dt}{L_{Ts}(\gamma^{LR}_k(t))},$$

which implies that the period of C_k for the vector field X_{Ts} is T_k. Here T^{LR}_k is the period of the integral curve $\gamma^{LR}_k(t)$. Finally, observe that the degree of X_{Ts} is $2(3n + r)$ because the vector field $\tilde{X}_T$ has degree $2(2n + r)$ and the factor L_{Te} has degree $2n$. Additionally, $V_{Ts} := ABL_{Ts}$ is an inverse integrating factor of X_{Ts}, and the Darboux first integral $\tilde{H}_T$ of $\tilde{X}_T$ is a first integral of X_{Ts} as well. This completes the proof of the theorem. □

5. The main theorem

In this section we construct a vector field X that realizes a configuration of cycles Γ with prescribed periods, stabilities and multiplicities, thus establishing the main theorem of the paper. As in Section 4 we shall use $\nu_k \in \{-1, 1\}$ to denote the *interior stability* that we want to prescribe for the limit cycle C_k (negative means stable in the interior and positive is unstable). Observe that the *exterior stability* of C_k is determined by ν_k and the multiplicity m_k as $(-1)^{m_k+1}\nu_k$. In the following lemma we show that the interior stability of each limit cycle C_k of the polynomial vector field X_{Tm} constructed in Section 3 can be characterized in terms of the relative position of C_k with respect to the other limit cycles and the set of multiplicities $\{m_1, \dots, m_n\}$. In the case that $m_k = 1$ for all k we recover Lemma 4.1.

Lemma 5.1. *The limit cycle C_k of the vector field X_{Tm} constructed in Section 3 has interior stability $\nu_k = (-1)^{m_k+M_k+1}$, where*

$$M_k := \sum_{j\in\mathcal{M}_k} m_j$$

and $\mathcal{M}_k := \{j \in \{1, \dots, n\} \ / \ C_k \subset D_{C_j},\ j \neq k\}$.

Proof. Consider a closed curve $\hat{C}_k$ in the region bounded by C_k and close enough to it so that $\hat{C}_k$ is disjoint from all C_j and p_j. Since C_k is a limit cycle of X_{Tm} we can assume that $\hat{C}_k$ is transverse to the integral curves of X_{Tm} at each point. The interior stability of C_k is determined by the sign of the flux through $\hat{C}_k$ of X_{Tm}, which is given by

$$Flux_k := \int_{\hat{C}_k} X_{Tm} \cdot n ds, \tag{5.1}$$

where n is the unit normal vector on $\hat{C}_k$ pointing outwards and s parametrizes the curve $\hat{C}_k$ in the positive direction (i.e. counterclockwise).

The sign of the flux (5.1) can be easily computed using the 1-form $\omega_{Tm} := -Q_{Tm}dx + P_{Tm}dy$ and the inverse integrating factor $V_{Tm} := A_m B$ of X_{Tm}. Indeed, noticing that V_{Tm} does not vanish at any point of $\hat{C}_k$, and observing that

$$\frac{\omega_{Tm}}{V_{Tm}} = -dH_T - \Lambda \sum_{j=1}^{n} \frac{df_j}{f_j^{m_j}},$$

see the definitions of all the involved functions in Sections 3 and 4, it is obvious that the sign of $Flux_k$ is the same as the sign of

$$\widehat{Flux_k} := V_{Tm}(p_0) \int_{\hat{C}_k} \frac{\omega_{Tm}}{V_{Tm}} = -V_{Tm}(p_0) \int_{\hat{C}_k} dH_T = 4\pi s_k V_{Tm}(p_0),$$

where p_0 is any fixed point on $\hat{C}_k$. The second equality follows from the fact that each term $\frac{df_j}{f_j^{m_j}}$ does not contribute to the integral because f_j does not vanish at any point of $\hat{C}_k$. For the last equality we have used the expression of $\int dH_T$ obtained in the proof of Theorem 2.1, taking into account that the curve $\hat{C}_k$ is positively oriented.

Since the interior stability of C_k is given by $-\operatorname{sign}(Flux_k)$, the formula above implies that $\nu_k = -\operatorname{sign}(V_{Tm}(\hat{C}_k)) = -\operatorname{sign}(A_m(\hat{C}_k))$. Then, arguing as in the proof of Lemma 4.1, but taking into account the multiplicity, we easily obtain the desired expression for ν_k. □

In the following theorem we show how to modify the vector field X_{Tm} in order to prescribe the interior stabilities of its limit cycles. The idea is the same as in the proof of Theorem 4.2: we add additional cycles to the configuration Γ to obtain a new configuration $\tilde{\Gamma}$ so that $\tilde{\nu}_k$ for the new configuration has the desired sign. To remove the extra cycles $\tilde{\Gamma}\backslash\Gamma$, we add a singular point over each extra limit cycle of the vector field $\tilde{X}_{Tm}$ realizing $\tilde{\Gamma}$.

Theorem 5.2. *Let $\Gamma = \{C_1, \ldots, C_n\}$ be a configuration of cycles, $\{m_1, \ldots, m_n\}$ a set of positive integers, $\{T_1, \ldots, T_n\}$ a set of positive constants and $\{\nu_1, \ldots, \nu_n\}$ a set of ± 1. Then Γ is realized by a planar polynomial vector field X with*

$$\deg(X) < 2\left(2(N-n) + r + \sum_{k=1}^{n} m_k\right)$$

where each periodic orbit C_k has multiplicity m_k, period T_k and interior stability ν_k. As usual, r is the number of primary cycles in Γ, and $N := n + n_1 + 2n_2$ where $n_1 := card\{k : m_k \text{ is odd}\}$ and $n_2 := card\{k : m_k \text{ is even and } M_k \text{ is odd}\}$. Moreover, X admits a polynomial inverse integrating factor, it is Darboux integrable and all its limit cycles are algebraic.

Proof. As usual, we assume that Γ consists of circles. Let us define a new configuration of circles $\tilde{\Gamma} := \Gamma \cup \{C_{n+1}, \ldots, C_N\}$ following these rules:

- If m_k is odd, we add a concentric circle of radius $r_k - \nu_k\varepsilon$.
- If m_k is even and $\nu_k = -1$, we add nothing.
- If m_k is even and $\nu_k = 1$, we add two concentric circles of radii $r_k \pm \varepsilon$.

Here ε is a small enough constant so that all the circles in $\tilde{\Gamma}$ are disjoint, and disjoint from p_k. Using Lemma 5.1 it is easy to check that $N = n + n_1 + 2n_2$.

Now, we construct a vector field $\tilde{X}_{Tm}$ as in Section 3 that realizes the configuration $\tilde{\Gamma}$, with m_k, $k \in \{1, \ldots, n\}$, the multiplicity we want to prescribe and $m_{n+j} = 1$, $j \in \{1, \ldots, N-n\}$. Moreover, the constant τ_k corresponding to each limit cycle C_k will be fixed later, while we take $\tau_{n+j} = 1$. A simple argument using Lemma 5.1 implies that the interior stability of each limit cycle C_k of $\tilde{X}_{Tm}$ is precisely ν_k.

Taking an arbitrary point $q_j := (a_j, b_j) \in C_{n+j}$ for every extra circle, we define the function L as in the proof of Theorem 4.2, i.e.

$$L := \prod_{j=1}^{N-n} l_j,$$

where $l_j := (x - a_j)^2 + (y - b_j)^2$, and the vector field

$$X := L\tilde{X}_{Tm}.$$

Since the factor L is positive on Γ, this set is realized by X as algebraic limit cycles, while the remaining cycles $C_{n+j} \subset \tilde{\Gamma}$ contain a singular point, thus becoming homoclinic connections. The factor L does not change the interior stability of each cycle C_k, which is then ν_k by construction, nor the multiplicity m_k. The constants $\{\tau_k\}_{k=1}^n$ can also be chosen such that each limit cycle C_k of X has period T_k. More precisely, since the vector field $\tilde{X}_{Tm}$ on each limit cycle C_k can be written as in Eq. (3.5), we conclude that

$$\tau_k = \frac{1}{T_k} \int_0^{T_k^{LR}} \frac{dt}{L(\gamma_k^{LR}(t))[\tilde{\Lambda} + f_k^{m_k-1}(\gamma_k^{LR}(t))] \prod_{j\neq k}^N f_j^{m_j-1}(\gamma_k^{LR}(t))},$$

where we are using the notation introduced in Section 3.

Finally, an easy computation shows that X is a polynomial vector field of degree as in the statement of the theorem, and $V := A_m BL$ is an inverse integrating factor. Using the functions defined in Sections 2 and 3, it is ready to check that the function

$$A_T BC \exp\left(\Lambda \sum_{j=1}^N \tau_j h_j\right)$$

is a Darboux first integral of the vector field X. This completes the proof of the theorem. □

Remark 5.3. We observe that the vector field X in the proof of Theorem 5.2 can be constructed without including the functions f_{n+j}, $j \in \{1, \dots, N - n\}$, in the quantities H_T, F_T and G_T appearing in the definition of $\tilde{X}_{Tm}$.

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