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J. Differential Equations 269 (2020) 5802–5831

Journal of Differential Equations

www.elsevier.com/locate/jde

On a length-preserving inverse curvature flow of convex closed plane curves

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Received 17 September 2019; revised 6 April 2020; accepted 7 April 2020
Available online 24 April 2020

Abstract

This paper deals with a $1/\kappa^{\alpha}$-type length-preserving nonlocal flow of convex closed plane curves for all $\alpha > 0$. Under this flow, the convexity of the evolving curve is preserved. For a global flow, it is shown that the evolving curve converges smoothly to a circle as $t \to \infty$. Some numerical blow-up examples and a sufficient condition leading to the global existence of the flow are also constructed.

MSC: primary 53C44; secondary 35B40, 35K15, 35K55

Keywords: Convex curve; Inverse curvature flow; Length-preserving; Nonlocal flow

1. Introduction

Let $\alpha > 0$ be a constant, which can be arbitrary, and let $\gamma_0 \subset \mathbb{R}^2$ be a given smooth convex closed curve parametrized by $X_0(\varphi) : S^1 \to \mathbb{R}^2$. We study $1/\kappa^{\alpha}$-type nonlocal length-preserving flow of the form

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https://doi.org/10.1016/j.jde.2020.04.028

$$\begin{cases} \dfrac{\partial X(\varphi,t)}{\partial t} = \left(\dfrac{1}{2\pi} \displaystyle\int_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot,t)ds - \kappa^{-\alpha}(\varphi,t) \right) \mathbf{N}_{in}(\varphi,t), \quad t>0 \\ X(\varphi,0) = X_0(\varphi), \quad \varphi \in S^1, \end{cases} \tag{1}$$

where, in (1), $\kappa(\varphi,t)$ is the curvature of the evolving curve $X(\varphi,t)$; $L(t)$ is the length of $X(\varphi,t)$; $\mathbf{N}_{in}(\varphi,t)$ is the inward unit normal of $X(\varphi,t)$; and the integral $\int ds$ is with respect to arc length parameter s. For time $T>0$, a family of smooth convex closed curves $X(\varphi,t): S^1 \times [0,T) \to \mathbb{R}^2$ is said to evolve by the flow if it satisfies the initial value problem (1) on the domain $S^1 \times [0,T)$. For simplicity, we also say that $X(\varphi,t): S^1 \times [0,T) \to \mathbb{R}^2$ is a convex solution of (1).

Note that the flow (1) is parabolic in the sense that the curvature term $F(\kappa) := -\kappa^{-\alpha}$ in the speed function of (1) is a strictly increasing function of $\kappa \in (0,\infty)$. Similar to the discussions in [7] (using Leray-Schauder's fixed point theory), or in [6] (using the linearization method), or in many other nonlocal flow papers, there is a unique smooth convex solution of (1) defined on $S^1 \times [0,T)$ for some short time $T>0$. Therefore, we have short-time existence of a convex solution to (1).

Our goal in this paper is to understand the long-time behavior of the flow (1). There are two aspects:

The noteworthy feature of the $1/\kappa^\alpha$-type nonlocal length-preserving flow (1) is that a **singularity** (curvature blow-up to $+\infty$) can happen in finite time even if the enclosed area $A(t)$ is increasing and the isoperimetric ratio $L^2(t)/4\pi A(t)$ of the evolving curve is decreasing during the evolution (so the curve $X(\cdot,t)$ is getting circular in the isoperimetric sense!). For the case $\alpha = 1$ in (1), the formation of a singularity has been shown to occur for some initial convex closed curves; see [11]. In Section 5, we shall give an intuitive example (using ellipse as the initial curve) to demonstrate the formation of a singularity in finite time and also provide some numerical blow-up examples.

Due to these blow-up examples, the optimal result we can prove is that, as long as the curvature $\kappa(\cdot,t)$ does not blow up in any finite time, the solution $X(\cdot,t)$ of the flow (1) converges smoothly to a fixed circle with radius $L(0)/2\pi$ as $t \to \infty$. See Theorem 2.4. For convenience, we also call (1) a **global flow** if it exists in time interval $[0,\infty)$. Assume the flow (1) is global, the key step in the proof of Theorem 2.4 is to show that the function $v(\varphi,t) := 1/\kappa^\alpha(\varphi,t)$ (the solution to the quasilinear equation (10)) tends to the constant $(L(0)/2\pi)^\alpha$ as $t \to \infty$. It is first shown that the $W^{2,2}(S^1)$-norm of v is uniformly bounded (see the proof of Lemma 3.1). For the case $\alpha \geq 1$, one can use the fact that the area functional $A(t)$ is increasing to prove the convergence of $v(\cdot,t)$ (see Lemmas 2.1, 3.9 and 3.11). For the case $0<\alpha<1$, we adopt an interesting geometric method which relies on the **Green-Osher's inequality** for convex closed plane curves and the classical **Blaschke Selection Theorem** in the theory of convex geometry (see Lemma 3.12).

The work on the flow (1) is a continuation of our previous works in [5,10–12]. For a brief survey on other related nonlocal flows of convex closed curves, please see the introduction section in [5,12,13].

2. Properties of the flow (1)

It is well-known that if $X(\cdot,t): S^1 \times [0,T) \to \mathbb{R}^2$ is a family of evolving simple closed curves (not necessarily convex), its length $L(t)$ and enclosed area $A(t)$ satisfy the following equations:

$$\frac{dL}{dt}(t) = -\int\limits_{X(\cdot,t)} \langle \mathbf{W}, \kappa \mathbf{N}_{in} \rangle \, ds, \qquad \frac{dA}{dt}(t) = -\int\limits_{X(\cdot,t)} \langle \mathbf{W}, \mathbf{N}_{in} \rangle \, ds, \quad t \in [0, T), \tag{2}$$

where $\mathbf{W} = \partial X/\partial t$ is the velocity vector of X and $\langle , \rangle$ is the inner product in $\mathbb{R}^2$. By (2), we have:

Lemma 2.1. *Assume $X(\varphi, t) : S^1 \times [0, T) \to \mathbb{R}^2$ is a **convex solution** of (1). Then we have*

$$\frac{dL}{dt}(t) \equiv 0, \quad \forall\, t \in [0, T) \tag{3}$$

and

$$\frac{dA}{dt}(t) = \int\limits_{X(\cdot,t)} \kappa^{-\alpha}(\cdot, t) ds - \frac{L(t)}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds \geq 0, \quad \forall\, t \in [0, T). \tag{4}$$

Moreover, we have $dA(t)/dt = 0$ if and only if the curve $X(\cdot, t)$ is a circle.

Proof. By (1) and (2), we have

$$\begin{aligned}
\frac{dL}{dt}(t) &= -\int\limits_{X(\cdot,t)} \left(\frac{1}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds - \kappa^{-\alpha} \right) \kappa ds \\
&= \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds - \left(\frac{1}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds \right) \cdot 2\pi = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{dA}{dt}(t) &= -\int\limits_{X(\cdot,t)} \left(\frac{1}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds - \kappa^{-\alpha} \right) ds \\
&= \int\limits_{X(\cdot,t)} \kappa^{-\alpha}(\cdot, t) ds - \frac{L(t)}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds.
\end{aligned} \tag{5}$$

To see that we have $dA(t)/dt \geq 0$, we use the **outward normal angle** $\theta \in [0, 2\pi]$ to express the right hand side of (5) as (note that we have $ds = \kappa^{-1} d\theta$ and $L(t) = \int_0^{2\pi} \kappa^{-1}(\theta, t) \, d\theta$)

$$\begin{aligned}
\frac{dA}{dt}(t) &= \int\limits_{X(\cdot,t)} \kappa^{-\alpha}(\cdot, t) ds - \frac{L(t)}{2\pi} \int\limits_{X(\cdot,t)} \kappa^{1-\alpha}(\cdot, t) ds \\
&= \frac{1}{2\pi} \left(\int_0^{2\pi} d\theta \int_0^{2\pi} \frac{1}{\kappa^{\alpha+1}(\theta, t)} d\theta - \int_0^{2\pi} \frac{1}{\kappa(\theta, t)} d\theta \int_0^{2\pi} \frac{1}{\kappa^{\alpha}(\theta, t)} d\theta \right)
\end{aligned} \tag{6}$$

and by the Hölder inequality, we have

$$\int_0^{2\pi} \frac{1}{\kappa(\theta,t)} d\theta \leq \left(\int_0^{2\pi} \left(\frac{1}{\kappa(\theta,t)} \right)^{\alpha+1} d\theta \right)^{\frac{1}{\alpha+1}} (2\pi)^{\frac{\alpha}{\alpha+1}}$$

and

$$\int_0^{2\pi} \frac{1}{\kappa^{\alpha}(\theta,t)} d\theta \leq \left(\int_0^{2\pi} \left(\frac{1}{\kappa^{\alpha}(\theta,t)} \right)^{\frac{\alpha+1}{\alpha}} d\theta \right)^{\frac{\alpha}{\alpha+1}} (2\pi)^{\frac{1}{\alpha+1}}.$$

The above two inequalities together imply $dA(t)/dt \geq 0$. Moreover, we see that $dA(t)/dt = 0$ if and only if both inequalities are equalities, which implies $\kappa(\theta,t)$ is a constant, i.e. a circle. □

As an immediate consequence of Lemma 2.1, we have:

Lemma 2.2. *Assume $X(\varphi,t) : S^1 \times [0,T) \to \mathbb{R}^2$ is a convex solution of (1). Then*

$$L(t) \equiv L(0) \quad \text{and} \quad A(0) \leq A(t) \leq \frac{L^2(0)}{4\pi}, \quad \forall\, t \in [0,T) \tag{7}$$

and the isoperimetric ratio $L^2(t)/4\pi A(t)$ of $X(\cdot,t)$ is decreasing in $t \in [0,T)$. Moreover, it is strictly decreasing unless the initial curve X_0 is a circle (which is an equilibrium solution of the flow (1)).

According to the explanation in Section 1, for given initial convex curve X_0, the flow (1) has a convex solution $X(\varphi,t) : S^1 \times [0,T) \to \mathbb{R}^2$ defined on some short time interval $[0,T)$. The curvature $\kappa(\varphi,t)$, in terms of the outward normal angle $\theta \in S^1$ of the convex curve $X(\varphi,t)$, satisfies the evolution equation (see [3,2] for computational details)

$$\frac{\partial \kappa}{\partial t}(\theta,t) = \kappa^2(\theta,t) \left[\left(-\kappa^{-\alpha}(\theta,t) \right)_{\theta\theta} + \left(\frac{1}{2\pi} \int_0^{2\pi} \kappa^{-\alpha}(\theta,t) d\theta - \kappa^{-\alpha}(\theta,t) \right) \right],$$
$$(\theta,t) \in S^1 \times [0,T), \tag{8}$$

where $\kappa(\theta,0) = \kappa_0(\theta) > 0,\ \theta \in S^1$, is the curvature of X_0. To obtain a better-looking evolution equation, we look at the radius of curvature $\rho(\theta,t) := 1/\kappa(\theta,t)$ and get

$$\frac{\partial \rho}{\partial t}(\theta,t) = \left(\rho^{\alpha} \right)_{\theta\theta}(\theta,t) + \rho^{\alpha}(\theta,t) - \frac{1}{2\pi} \int_0^{2\pi} \rho^{\alpha}(\theta,t) d\theta, \quad (\theta,t) \in S^1 \times [0,T) \tag{9}$$

with $\rho(\theta,0) = \rho_0(\theta) = 1/\kappa_0(\theta)$. In particular, if we let $v(\theta,t) = \rho^{\alpha}(\theta,t),\ \alpha > 0$, we get

$$\frac{\partial v}{\partial t} = \alpha v^p \left(v_{\theta\theta} + v - \lambda(t)\right), \quad v = v(\theta, t), \quad p = 1 - \frac{1}{\alpha}, \quad \lambda(t) = \frac{1}{2\pi}\int_0^{2\pi} v d\theta. \tag{10}$$

At this moment, we may use equation (9) to give an intuitive explanation why the curvature $\kappa(\theta, t)$ can blow up in finite time. Roughly speaking, when the curvature $\kappa(\theta, t)$ is large on some interval of θ_* (where $\kappa(\theta_*, t) = \kappa_{\max}(t)$), the function $\rho(\theta, t)$ is small on that interval. Hence the diffusion term $(\rho^\alpha)_{\theta\theta}(\theta_*, t)$ is also small (positively) and it does not help here. The nonlocal term $-\lambda(t)$ will make $\rho(\theta_*, t)$ to become even smaller, causing $\rho(\theta_*, t)$ to drop to zero (i.e. curvature blow-up) in finite time.

Remark 2.3. Instead, if we consider the κ^α-type nonlocal length-preserving flow, the evolution of the curvature $\kappa(\theta, t)$ is given by

$$\frac{\partial \kappa}{\partial t}(\theta, t) = \kappa^2(\theta, t)\left[\left(\kappa^\alpha\right)_{\theta\theta}(\theta, t) + \kappa^\alpha(\theta, t) - \lambda(t)\right], \quad \alpha > 0, \quad \lambda(t) = \frac{1}{2\pi}\int_0^{2\pi} \kappa^\alpha(\theta, t)\, d\theta.$$

When the curvature $\kappa(\theta, t)$ is large on some interval of θ_* (where $\kappa(\theta_*, t) = \kappa_{\max}(t)$), the diffusion term $(\kappa^\alpha)_{\theta\theta}(\theta_*, t)$, $\alpha > 0$, is also large (negatively), which will prevent the blow-up of the curvature and eventually we have convergence to circle. See [12].

As explained in the above and in Section 1, a singularity (curvature blow-up to $+\infty$) can happen in finite time under the flow (1). Therefore, the optimal result we can obtain is the following:

Theorem 2.4. *Assume $\alpha > 0$ and $X_0(\varphi)$, $\varphi \in S^1$, is a smooth convex closed curve. Consider the length-preserving flow (1) and assume that the curvature κ* ***will not blow up to $+\infty$ in any finite time*** *during the evolution. Then the flow exists for all time $t \in [0, \infty)$ and is length-preserving. Each $X(\cdot, t)$ remains smooth, convex, and it converges to a fixed round circle with radius $L(0)/2\pi$ in C^∞ norm as $t \to \infty$.*

Remark 2.5. For the case $\alpha = 1$, see [10].

3. Proof of Theorem 2.4

We shall decompose the proof of Theorem 2.4 into several lemmas. The first one is to show that if the curvature κ will not blow up to $+\infty$ in finite time, then it will not drop down to 0 in finite time either.

3.1. Uniform convexity of the evolving curve; lower bound of the curvature

Lemma 3.1 (***Lower bound of the curvature***). *Let $X(\varphi, t) : S^1 \times [0, T) \to \mathbb{R}^2$ be a convex solution of the flow (1). There is a positive constant $C > 0$, which depends only on the initial curve X_0 and is independent of time T, so that we have*

$$\kappa(\varphi, t) \geq C > 0, \quad \forall\ (\varphi, t) \in S^1 \times [0, T). \tag{11}$$

Proof. The idea is to use the fact that the curve $X(\cdot,t)$ has fixed length and to derive an integral estimate analogous to Lemma 4.3.5 in [3]. In the following computation, we shall express the curvature κ and the functions ρ and v in terms of the outward normal angle $\theta\in S^1$.

By (10), we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^{2\pi}\left(\left(\frac{\partial v}{\partial\theta}\right)^2-v^2\right)d\theta=2\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\frac{\partial^2 v}{\partial\theta\partial t}-v\frac{\partial v}{\partial t}\right)d\theta=-2\int_0^{2\pi}\left(\frac{\partial^2 v}{\partial\theta^2}+v\right)\frac{\partial v}{\partial t}d\theta\\
&=-2\int_0^{2\pi}\left(\frac{1}{\alpha}v^{-p}\frac{\partial v}{\partial t}+\lambda(t)\right)\frac{\partial v}{\partial t}d\theta=-\frac{2}{\alpha}\int_0^{2\pi}v^{-p}\left(\frac{\partial v}{\partial t}\right)^2d\theta-\frac{1}{2\pi}\frac{d}{dt}\left(\int_0^{2\pi}vd\theta\right)^2\\
&\leq-\frac{1}{2\pi}\frac{d}{dt}\left(\int_0^{2\pi}vd\theta\right)^2.
\end{aligned}\tag{12}$$

Integrating the above inequality with respect to time on the interval $[0,t]$ gives

$$\begin{aligned}
&\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2d\theta-\int_0^{2\pi}v^2d\theta\\
&\leq\left[\int_0^{2\pi}\left(\frac{\partial v_0}{\partial\theta}\right)^2d\theta-\int_0^{2\pi}(v_0)^2d\theta\right]-\frac{1}{2\pi}\left[\left(\int_0^{2\pi}vd\theta\right)^2-\left(\int_0^{2\pi}v_0d\theta\right)^2\right]\\
&\leq\int_0^{2\pi}\left(\frac{\partial v_0}{\partial\theta}\right)^2d\theta+\frac{1}{2\pi}\left(\int_0^{2\pi}v_0d\theta\right)^2:=C_0
\end{aligned}\tag{13}$$

where $v_0(\theta)=v(\theta,0)$. Therefore, there exists a constant C_0 given by (13), which depends only on X_0 and is independent of time, such that

$$\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2d\theta\leq\int_0^{2\pi}v^2(\theta,t)\,d\theta+C_0,\quad\forall\, t\in[0,T).\tag{14}$$

For each $t\in[0,T)$ and each $\delta\in(0,2\pi)$, define the number

$$\rho_\delta^*(t)=\sup\{b|\rho(\theta,t)>b\text{ on some interval of }\theta\text{ with length }\delta\}.\tag{15}$$

Intuitively, we see that for δ close to 0, $\rho_\delta^*(t)$ is close to $\rho_{\max}(t)=\max_{\theta\in S^1}\rho(\theta,t)$ and for δ close to 2π, $\rho_\delta^*(t)$ is close to $\rho_{\min}(t)=\min_{\theta\in S^1}\rho(\theta,t)$.

For given $\delta\in(0,2\pi)$, one can choose a subinterval $(\theta_1,\theta_2)\subset[0,2\pi]$ with length δ so that $\rho(\theta,t)\geq\rho_\delta^*(t)$ in (θ_1,θ_2) and $\rho(\theta_1,t)=\rho(\theta_2,t)=\rho_\delta^*(t)$. By

$$L(t)=\int_0^{2\pi}\rho(\theta,t)\,d\theta=L(0)\geq\int_{\theta_1}^{\theta_2}\rho(\theta,t)\,d\theta\geq\delta\rho_\delta^*(t),\quad\forall\, t\in[0,T),$$

one obtains the inequality

$$\rho_\delta^*(t)\leq\frac{L(0)}{\delta},\quad\forall\, t\in[0,T),\quad\forall\,\delta\in(0,2\pi),\tag{16}$$

and thus $\rho(\theta,t)\leq\frac{L(0)}{\delta}$ except on intervals (θ_1,θ_2) with length less than or equal to δ. In other words, if there is some interval $I=(\theta_1,\theta_2)\subset[0,2\pi]$ such that

$$\rho(\theta,t)>\frac{L(0)}{\delta},\quad\forall\,\theta\in I\quad\text{and}\quad\rho(\theta_1,t)=\rho(\theta_2,t)=\frac{L(0)}{\delta},$$

then by (16) and the definition of $\rho_\delta^*(t)$, we must have $|I|\leq\delta$. On such a small interval I, we have

$$\begin{aligned}\rho^\alpha(\theta,t)=v(\theta,t)=v(\theta_1,t)+\int_{\theta_1}^{\theta}\frac{\partial v}{\partial\theta}(\theta,t)\,d\theta&\leq\left(\frac{L(0)}{\delta}\right)^\alpha+\sqrt{\delta}\left[\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2d\theta\right]^{\frac{1}{2}}\\&\leq\left(\frac{L(0)}{\delta}\right)^\alpha+\sqrt{\delta}\left(\int_0^{2\pi}v^2(\theta,t)\,d\theta+C_0\right)^{\frac{1}{2}},\quad\forall\,\theta\in I=(\theta_1,\theta_2),\end{aligned}\tag{17}$$

where the inequality (14) is used in the last step. By (17), the function $v_{\max}(t)$, $t\in[0,T)$, satisfies

$$v_{\max}(t)\leq\left(\frac{L(0)}{\delta}\right)^\alpha+\sqrt{\delta}\left[2\pi(v_{\max}(t))^2+C_0\right]^{\frac{1}{2}}\leq\left(\frac{L(0)}{\delta}\right)^\alpha+\sqrt{\delta}\left(\sqrt{2\pi}\,v_{\max}(t)+\sqrt{C_0}\right).\tag{18}$$

Choosing $\delta=1/(8\pi)$, we get

$$v_{\max}(t)=\left(\frac{1}{\kappa_{\min}(t)}\right)^\alpha\leq 2(8\pi L(0))^\alpha+\sqrt{\frac{C_0}{2\pi}},\quad\forall\, t\in[0,T).\tag{19}$$

Therefore, the function $v(\theta,t)$ has a time-independent positive upper bound, so does $\rho(\theta,t)=1/\kappa(\theta,t)$. Consequently, the curvature $\kappa(\theta,t)$ has a time-independent positive lower bound. The proof of (11) is finished. $\square$

3.2. Long-time existence of the flow (1)

Lemma 3.2. *Let $\alpha>0$ and $X_0(\varphi)$, $\varphi\in S^1$, be a smooth convex closed curve. Assume $X(\varphi,t): S^1\times[0,T)\to\mathbb{R}^2$ is a convex solution of the flow (1) such that its curvature $\kappa(\varphi,t)$ will not blow up to $+\infty$ as $t\to T$. Then the evolving curve $X(\cdot,t)$ converges to a smooth convex closed curve $X(\cdot,T)$ as $t\to T$ and the flow (1) can continue beyond time T.*

Proof. On the domain $S^1 \times [0, T)$, by the assumption and Lemma 3.1, the curvature $\kappa(\theta, t)$ has uniform positive upper and lower bounds, which means that the equation (10) is uniformly parabolic on $S^1 \times [0, T)$ with the nonlocal term $\lambda(t) = (2\pi)^{-1} \int_0^{2\pi} v(\theta, t)\, d\theta > 0$ being bounded on $[0, T)$. The usual parabolic regularity theory can still be applied to a nonlocal equation of the form (10) and guarantee that all space-time derivatives of $v(\theta, t)$ are bounded on $S^1 \times [0, T)$, which in turn implies that all space-time derivatives of $\kappa(\theta, t)$ are uniformly bounded on the domain $S^1 \times [0, T)$ (presumably, the bound on each derivative of $\kappa(\theta, t)$ may depend on the order of differentiation and on T). By the well-known Arzela-Ascoli theorem, one can conclude that there exists a smooth positive function $\kappa_T(\theta)$ on S^1 such that

$$\lim_{t \to T} \|\kappa(\theta, t) - \kappa_T(\theta)\|_{C^m(S^1)} = 0, \quad \forall\, m \in \{0\} \bigcup \mathbb{N}. \tag{20}$$

More precisely, for any sequence $t_k \to T$ there exists a subsequence, which we still denote it as t_k, where $t_k \to T$, such that $\kappa(\theta, t_k)$ converges uniformly to some function $\kappa_T(\theta)$, which is at least continuous. Next, we note that if there are two sequences $t_k \to T$ and $t_m \to T$ such that $\kappa(\theta, t_k) \to \kappa_T(\theta)$ and $\kappa(\theta, t_m) \to \tilde{\kappa}_T(\theta)$, then by the mean value theorem

$$|\kappa(\theta, t_k) - \kappa(\theta, t_m)| = \left| \frac{\partial \kappa}{\partial t}(\theta, t_*) \cdot (t_k - t_m) \right|, \quad t_* \text{ lies between } t_k \text{ and } t_m,$$

we have $\lim_{t_k \to T} \kappa(\theta, t_k) = \lim_{t_m \to T} \kappa(\theta, t_m)$ and so $\kappa_T(\theta) = \tilde{\kappa}_T(\theta)$. This observation guarantees that we have **full-time uniform convergence** to the same limit $\kappa_T(\theta)$, i.e. $\lim_{t \to T} \kappa(\theta, t) = \kappa_T(\theta)$. If we apply the same argument to the function $\kappa_\theta(\theta, t)$, we can obtain the full-time uniform convergence of $\kappa_\theta(\theta, t)$ to some function $g(\theta)$, which is at least continuous. Standard theorem in advanced calculus tells us that $g(\theta) = \kappa_T'(\theta)$ (i.e. $\kappa_T(\theta)$ must be differentiable). Therefore, we have the convergence result in (20) for $m = 0$ and 1. Repeating the process to $\kappa_{\theta\theta}(\theta, t)$, $\kappa_{\theta\theta\theta}(\theta, t)$, ..., etc. will give us the convergence result in (20) for $m = 2, 3, 4, \ldots$. Finally, since $X(\cdot, T)$ is a smooth convex closed curve, one can use it as the new initial data for the flow (1). By the short-time existence property as explained in Section 1, the flow (1) can continue beyond time T. The proof is done. □

As an immediate consequence of Lemma 3.2, we obtain:

Lemma 3.3 (***Long-time existence of the flow (1)***). *Assume $\alpha > 0$ and $X_0(\varphi)$, $\varphi \in S^1$, is a smooth convex closed curve. Consider the length-preserving flow (1) and assume that the curvature κ of $X(\cdot, t)$ will not blow up to $+\infty$ in any finite time during the evolution. Then the flow has a unique convex solution $X(\varphi, t) : S^1 \times [0, \infty) \to \mathbb{R}^2$ defined for all time.*

Remark 3.4. However, at this moment, Lemma 3.3 cannot exclude the possibility that the curvature may blow up to $+\infty$ as $t \to \infty$. We will show that this cannot happen in Section 3.3.

Proof. This is a straightforward consequence of Lemma 3.1 and Lemma 3.2. □

3.3. Convergence of the flow (1) under the global existence assumption

In this section, we will show that the flow (1) converges to a circle if it exists on the time interval $[0, \infty)$ (which is so if the curvature κ of $X(\cdot, t)$ will not blow up to $+\infty$ in finite time).

The proof is divided into two cases: the case $\alpha \geq 1$ and the case $0 < \alpha < 1$. We first show that the radius of curvature $\rho(\theta, t)$ converges to a positive constant $\rho_\infty = L(0)/2\pi$. Using this fact, one can prove that the norm $\|X_t(\cdot, t)\|_{C^0(S^1)}$ of the flow decays exponentially in time. The convergence of the evolving curve $X(\cdot, t)$ is obtained by integrating the flow equation (1) with respect to time.

The following two useful properties will be needed later on. We leave their proofs to the readers.

Lemma 3.5. *Let $\{h_n(x)\}_{n=1}^{\infty}$ be a sequence of differentiable functions defined on the interval $[a, b] \subset \mathbb{R}$. If there exist constants $C > 0$ and $p > 1$, such that*

$$\int_a^b \left|h_n'(x)\right|^p dx \leq C, \quad \forall\, n = 1,\ 2,\ 3,\ \ldots, \tag{21}$$

then $\{h_n(x)\}_{n=1}^{\infty}$ is equicontinuous on $[a, b]$.

Remark 3.6. The condition $p > 1$ in Lemma 3.5 is necessary.

Lemma 3.7. *Let $g(t) \geq 0$ be a differentiable function on $[0, \infty)$ with $\int_0^\infty g(t)\,dt < \infty$. If there exists a constant $C < 0$ such that $g'(t) \geq C$ on $[0, \infty)$ or there exists a constant $C > 0$ such that $g'(t) \leq C$ on $[0, \infty)$, then we must have*

$$g(t) \to 0 \quad as \quad t \to \infty. \tag{22}$$

Remark 3.8. The condition $g'(t) \geq C$ or $g'(t) \leq C$ in Lemma 3.7 is necessary.

3.3.1. The case $\alpha \geq 1$

Lemma 3.9. *Assume $\alpha \geq 1$ and $X(\varphi, t) : S^1 \times [0, \infty) \to \mathbb{R}^2$ is a convex solution of the length-preserving flow (1). Then we have*

$$\frac{dA}{dt}(t) \to 0 \quad as \quad t \to \infty. \tag{23}$$

Proof. Keep in mind that here we assume the flow solution $X(\varphi, t)$ is defined for all time $t \in [0, \infty)$. By (19), there is a constant $C > 0$ independent of time so that

$$0 \leq v(\theta, t) \leq C, \quad \forall\ (\theta, t) \in S^1 \times [0, \infty), \tag{24}$$

where, at this moment, we cannot exclude the bad scenario that $\lim_{t\to\infty} v_{\min}(t) = 0$ (which will be shown to be impossible to happen later on).

Let $g(t) = dA/dt$. By (4), we know that $g(t) \geq 0$ on $[0, \infty)$ with

$$\int_0^\infty g(t)\,dt = A(\infty) - A(0) < \frac{L^2(0)}{4\pi} < \infty.$$

Compute

$$g'(t)=\frac{d^2A}{dt^2}(t)=\frac{d}{dt}\left(-\frac{L(t)}{2\pi}\int_0^{2\pi}v\,d\theta+\int_0^{2\pi}v^{1+\frac{1}{\alpha}}d\theta\right)$$
$$=-\frac{L(0)}{2\pi}\int_0^{2\pi}\alpha v^p\left(v_{\theta\theta}+v-\lambda(t)\right)d\theta+(\alpha+1)\int_0^{2\pi}v\left(v_{\theta\theta}+v-\lambda(t)\right)d\theta,\quad p=1-\frac{1}{\alpha}$$
$$=\begin{cases}\frac{L(0)}{2\pi}\left[(\alpha-1)\int_0^{2\pi}v^{-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial\theta}\right)^2d\theta-\alpha\int_0^{2\pi}v^{2-\frac{1}{\alpha}}d\theta+\alpha\lambda(t)\int_0^{2\pi}v^{1-\frac{1}{\alpha}}d\theta\right]\\ +(\alpha+1)\left[-\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2d\theta+\int_0^{2\pi}v^2d\theta-\lambda(t)\int_0^{2\pi}v\,d\theta\right].\end{cases}\tag{25}$$

For $\alpha\geq 1$, by (25), (14), and (24), we have

$$g'(t)\geq-\alpha\frac{L(0)}{2\pi}\int_0^{2\pi}v^{2-\frac{1}{\alpha}}d\theta-(\alpha+1)\left(\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2d\theta+\lambda(t)\int_0^{2\pi}v\,d\theta\right)\geq -C\tag{26}$$

for some constant $C>0$ independent of time. Hence Lemma 3.7 implies $g(t)=dA/dt\to 0$ as $t\to\infty$. □

Remark 3.10. For the case $0<\alpha<1$, the first integral in (25) may approach to $-\infty$ as $t\to\infty$. Also note that we have

$$-\alpha\int_0^{2\pi}v^{2-\frac{1}{\alpha}}d\theta+\alpha\lambda(t)\int_0^{2\pi}v^{1-\frac{1}{\alpha}}d\theta$$
$$=-\frac{\alpha}{2\pi}\left[\int_0^{2\pi}d\theta\int_0^{2\pi}v^{2-\frac{1}{\alpha}}d\theta-\int_0^{2\pi}v\,d\theta\int_0^{2\pi}v^{1-\frac{1}{\alpha}}d\theta\right]$$
$$=-\frac{\alpha}{4\pi}\int_0^{2\pi}\left(\int_0^{2\pi}\left(v^{1-\frac{1}{\alpha}}(x,t)-v^{1-\frac{1}{\alpha}}(y,t)\right)\left(v(x,t)-v(y,t)\right)dx\right)dy\geq 0\tag{27}$$

due to $0<\alpha<1$. However, one cannot exclude the possibility that (27) tends to $+\infty$ as $t\to\infty$. Hence Lemma 3.7 is not applicable here.

As a consequence of Lemma 3.9, we have:

Lemma 3.11. *Assume $\alpha\geq 1$ and $X(\varphi,t):S^1\times[0,\infty)\to\mathbb{R}^2$ is a convex solution of the length-preserving flow (1). Then we have*

$$\lim_{t\to\infty}\left\|v(\theta,t)-\left(\frac{L(0)}{2\pi}\right)^{\alpha}\right\|_{C^0(S^1)}=0,\tag{28}$$

where $L(0)$ is the length of X_0 and $v(\theta,t)=\rho^{\alpha}(\theta,t)=1/\kappa^{\alpha}(\theta,t)$.

Proof. It follows from (14) and (19) that the L^2-norm of $\partial v/\partial\theta$ is uniformly bounded:

$$\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2 d\theta \le 2\pi\left(2(8\pi L)^{\alpha}+\sqrt{\frac{C_0}{2\pi}}\right)^2+C_0, \quad \forall\, t\in[0,\infty), \tag{29}$$

where $v(\theta,t)>0$ is defined on $S^1\times[0,\infty)$. By Lemma 3.5 and the Arzela-Ascoli theorem, for any sequence $t_k\to\infty$ there exists a subsequence (which we still denote it as t_k) such that $\lim_{k\to\infty} v(\theta,t_k)=v_{\infty}(\theta)$ uniformly on S^1, where $v_{\infty}(\theta)\ge 0$ is some nonnegative continuous bounded function on S^1.

Next, by Lemma 3.9, we know that

$$\frac{dA}{dt}(t)=-\frac{L(t)}{2\pi}\int_0^{2\pi} v\,d\theta+\int_0^{2\pi} v^{1+\frac{1}{\alpha}}d\theta\to 0 \quad \text{as} \quad t\to\infty,$$

which gives the identity

$$-\frac{L(\infty)}{2\pi}\int_0^{2\pi} v_{\infty}(\theta)\,d\theta+\int_0^{2\pi} v_{\infty}^{1+\frac{1}{\alpha}}(\theta)\,d\theta=0. \tag{30}$$

Due to the uniform convergence of $v^{1/\alpha}(\theta,t_k)$ to $v_{\infty}^{1/\alpha}(\theta)$ as $k\to\infty$ and the fact that the flow is length-preserving, the length $L(0)=L(\infty)$ can be expressed as

$$L(0)=\lim_{k\to\infty} L(t_k)=\lim_{k\to\infty}\int_0^{2\pi} v^{\frac{1}{\alpha}}(\theta,t_k)\,d\theta=\int_0^{2\pi} v_{\infty}^{\frac{1}{\alpha}}(\theta)\,d\theta. \tag{31}$$

By (31), one can rewrite (30) as

$$\int_0^{2\pi} v_{\infty}^{\frac{1}{\alpha}}(\theta)\,d\theta\int_0^{2\pi} v_{\infty}(\theta)\,d\theta=\int_0^{2\pi} d\theta\int_0^{2\pi} v_{\infty}^{1+\frac{1}{\alpha}}(\theta)\,d\theta, \tag{32}$$

which can be rewritten further as

$$\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left[\left(v_{\infty}^{\frac{1}{\alpha}}(x)-v_{\infty}^{\frac{1}{\alpha}}(y)\right)\left(v_{\infty}(x)-v_{\infty}(y)\right)\right]dxdy=0. \tag{33}$$

As the integrand in (33) is nonnegative everywhere on $S^1\times S^1$ and $v_{\infty}(\theta)\ge 0$ is a continuous function, $v_{\infty}(\theta)$ must be a constant function on S^1 and by (31) the constant is given by $(L(0)/2\pi)^{\alpha}$. Because every convergent subsequence tends to the same limit, we must have the convergence to $(L(0)/2\pi)^{\alpha}$ for all $t\to\infty$. The proof is done. □

3.3.2. The case $0<\alpha<1$

For $0<\alpha<1$, Lemma 3.11 is still correct, but due to Remark 3.10 we need to use a different proof.

Lemma 3.12. *Assume* $0<\alpha<1$ *and* $X(\varphi,t): S^1\times[0,\infty)\to\mathbb{R}^2$ *is a convex solution of the length-preserving flow (1). Then we have the same limit as in (28).*

Proof. We shall use a different approach with the help of Green-Osher's inequality. Note that for $0<\alpha<1$ the two functions $F_1(x)=x^{1+\alpha}$ and $F_2(x)=-x^\alpha$ are convex on $x\in(0,\infty)$. It follows from Green-Osher's inequality (see [4], Theorem 0.1) that for each $t\in[0,\infty)$ we have the inequalities

$$\int_0^{2\pi}\rho^{1+\alpha}d\theta\geq\pi\left[\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right)^{1+\alpha}+\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)^{1+\alpha}\right],\tag{34}$$

and

$$\int_0^{2\pi}-\rho^\alpha d\theta\geq\pi\left[-\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}-\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}\right],\tag{35}$$

where in the above $\rho=\rho(\theta,t)=1/\kappa(\theta,t)$, $L=L(t)$ (which is a constant), $A=A(t)$.

Using the evolution equation of $A(t)$ and (34), (35), one can conclude

$$\begin{aligned}\frac{dA}{dt}&=-\frac{L}{2\pi}\int_0^{2\pi}v\,d\theta+\int_0^{2\pi}v^{1+\frac{1}{\alpha}}d\theta,\quad v=\rho^\alpha\\&\geq\begin{cases}\frac{L}{2}\left[-\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}-\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}\right]\\+\pi\left[\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right)^{1+\alpha}+\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)^{1+\alpha}\right]\end{cases}\\&=\frac{\sqrt{L^2-4\pi A}}{2}\left[\left(\frac{L+\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}-\left(\frac{L-\sqrt{L^2-4\pi A}}{2\pi}\right)^{\alpha}\right].\end{aligned}$$

By the mean value theorem there exists a number $\xi(t)$ in the interval $(L-\sqrt{L^2-4\pi A},\ L+\sqrt{L^2-4\pi A})$ such that

$$\left(L+\sqrt{L^2-4\pi A}\right)^\alpha-\left(L-\sqrt{L^2-4\pi A}\right)^\alpha=2\alpha\xi^{\alpha-1}\sqrt{L^2-4\pi A}.$$

Substituting this identity into the above inequality and note that $0<\alpha<1$ we obtain

$$\frac{dA}{dt}\geq\frac{\alpha}{(2\pi)^\alpha}\xi^{\alpha-1}\left(L^2-4\pi A\right)\geq\frac{\alpha}{(2\pi)^\alpha}\left(L+\sqrt{L^2-4\pi A}\right)^{\alpha-1}\left(L^2-4\pi A\right),\quad\forall\,t\in[0,\infty).$$

So the isoperimetric difference satisfies

$$\begin{aligned}\frac{d}{dt}(L^2-4\pi A) &\le -\frac{4\pi\alpha}{(2\pi)^\alpha}\left(L+\sqrt{L^2-4\pi A}\right)^{\alpha-1}\left(L^2-4\pi A\right)\\ &\le -\frac{4\pi\alpha}{(2\pi)^\alpha}\left(L+\sqrt{L^2-4\pi A(0)}\right)^{\alpha-1}\left(L^2-4\pi A\right),\quad \forall\, t\in[0,\infty), \end{aligned}\tag{36}$$

where we have used the property that $A(t)$ is increasing in time. Note that now the coefficient in front of $L^2-4\pi A$ in (36) is a constant and integrating the above inequality can give us the decay

$$0\le L^2(t)-4\pi A(t)\le (L^2(0)-4\pi A(0))e^{-c_0 t},\quad \forall\, t\in[0,\infty),\tag{37}$$

where c_0 is the constant $c_0=\frac{4\pi\alpha}{(2\pi)^\alpha}\left(L(0)+\sqrt{L^2(0)-4\pi A(0)}\right)^{\alpha-1}>0$.

By (19), (14), and Lemma 3.5, the Arzela-Ascoli theorem can be applied to the function $v(\theta,t)$. For any sequence of time $t_k\to\infty$, there exists a subsequence (which, for convenience, we still denote it as t_k) such that $\lim_{k\to\infty} v(\theta,t_k)=v_\infty(\theta)$ uniformly on S^1, where $v_\infty(\theta)\ge 0$ is some nonnegative continuous bounded function on S^1. The corresponding convex closed curve $X(\cdot,t_k)$ has its isoperimetric difference decay to 0 as $k\to\infty$. Since $v(\theta,t_k)$ remains unchanged under a translation of $X(\cdot,t_k)$ in the plane, without loss of generality, we may assume that each curve $X(\cdot,t_k)$ encloses the origin $O=(0,0)$ of $\mathbb{R}^2$. As the length $L(t_k)$ of $X(\cdot,t_k)$ is independent of time, all $X(\cdot,t_k)$ must lie in some bounded set of $\mathbb{R}^2$. The classical Blaschke Selection Theorem (see Theorem 3.1 of [1]) implies the existence of a subsequence $X\left(\cdot,t_{k_j}\right)$ of $X(\cdot,t_k)$ such that $X\left(\cdot,t_{k_j}\right)$ converges to some convex closed curve $X_\infty(\cdot)$ in the Hausdorff metric. Continuity implies that the isoperimetric difference $L^2(\infty)-4\pi A(\infty)$ of $X_\infty(\cdot)$ is given by the limit of $L^2\left(t_{k_j}\right)-4\pi A\left(t_{k_j}\right)$, which is 0 due to the exponential decay (37). Therefore, the convex closed curve $X_\infty(\cdot)$ must be a circle with radius $L(0)/2\pi$. Since we have $\lim_{j\to\infty} v\left(\theta,t_{k_j}\right)=v_\infty(\theta)$ uniformly on S^1, we obtain $v_\infty(\theta)=(L(0)/2\pi)^\alpha$.

By the above discussion and a simple contradiction argument, we actually have the full-time convergence, i.e. $\lim_{t\to\infty} v(\theta,t)=(L(0)/2\pi)^\alpha$ uniformly on S^1. The proof is done. □

Remark 3.13. At this point, by (28), we only have the convergence of curvature $\kappa(\theta,t)$. See Lemma 3.16 for the convergence of $X(\varphi,t)$.

3.3.3. Higher derivatives estimate and convergence for any $\alpha>0$

If the flow (1) exists globally in time $t\in[0,\infty)$, then by Lemma 3.11 and Lemma 3.12, the evolution equation

$$\frac{\partial v}{\partial t}=\alpha v^p\left(v_{\theta\theta}+v-\lambda(t)\right),\quad v=v(\theta,t),\quad p=1-\frac{1}{\alpha},\quad \lambda(t)=\frac{1}{2\pi}\int_0^{2\pi} v\,d\theta\tag{38}$$

is uniformly parabolic on $S^1\times[0,\infty)$. Moreover, the quantity $\lambda(t)=(2\pi)^{-1}\int_0^{2\pi} v(\theta,t)\,d\theta>0$ is bounded on $[0,\infty)$, which means that it will not cause serious trouble. The usual parabolic regularity theory can still be applied to a nonlocal equation of the form (38) and can guarantee that all space-time derivatives of $v(\theta,t)$ are uniformly bounded on $S^1\times[0,\infty)$. For the readers' convenience, we provide a simple proof using the argument in [8].

Lemma 3.14. *For any $\alpha > 0$, if the length-preserving flow (1) has a convex solution $X(\varphi, t) : S^1 \times [0, \infty) \to \mathbb{R}^2$, then there exist constants C_k $(k = 1, 2, ...)$ independent of time such that*

$$\left| \frac{\partial^k v}{\partial \theta^k}(\theta, t) \right| \le C_k, \quad \forall (\theta, t) \in S^1 \times [0, \infty). \tag{39}$$

Proof. We shall prove the case $k = 1$ only. The proof for $k > 1$ is similar and can be argued by mathematical induction (see [8]). Let $u = v_\theta$ (the partial derivative of v with respect to θ), $Q = v^2$, and let $w = u + \beta Q$, where β is a constant to be chosen later on. We first compute

$$\frac{\partial u}{\partial t} = \left(\frac{\partial v}{\partial t} \right)_\theta = \alpha v^p u_{\theta\theta} + \alpha p v^{p-1} u u_\theta + \alpha v^{p-1} \left(p \left(v - \lambda(t) \right) + v \right) u, \quad p = 1 - \frac{1}{\alpha}, \tag{40}$$

and by $Q_\theta = 2vu,\ Q_{\theta\theta} = 2vu_\theta + 2u^2$, we get

$$\frac{\partial Q}{\partial t} = 2\alpha v^{p+1} \left(u_\theta + v - \lambda(t) \right) = \alpha v^p \left(Q_{\theta\theta} - 2u^2 \right) + 2\alpha v^{p+1} \left(v - \lambda(t) \right), \tag{41}$$

which together with the identity

$$w_\theta = u_\theta + \beta Q_\theta = u_\theta + 2\beta v u, \quad w_{\theta\theta} = u_{\theta\theta} + \beta Q_{\theta\theta},$$

gives

$$\begin{aligned} \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} + \beta \frac{\partial Q}{\partial t} &= \left\{ \begin{array}{l} \alpha v^p u_{\theta\theta} + \alpha p v^{p-1} u \left(w_\theta - 2\beta v u \right) + \alpha v^{p-1} \left(p \left(v - \lambda(t) \right) + v \right) u \\ + \alpha v^p \left(\beta Q_{\theta\theta} - 2\beta u^2 \right) + 2\alpha\beta v^{p+1} \left(v - \lambda(t) \right) \end{array} \right. \\ &= \left\{ \begin{array}{l} \alpha v^p \cdot w_{\theta\theta} + \alpha p v^{p-1} u \cdot w_\theta - 2\alpha\beta v^p (1 + p) u^2 \\ + \alpha v^{p-1} \left(p \left(v - \lambda(t) \right) + v \right) u + 2\alpha\beta v^{p+1} \left(v - \lambda(t) \right). \end{array} \right. \end{aligned} \tag{42}$$

Since $Q = v^2$ and $\lambda(t)$ are both bounded quantities, if $w = u + \beta Q$ becomes sufficiently large (either positively or negatively), it must be due to the term $u = v_\theta$. In (42), the coefficient of u^2 is

$$-2\alpha\beta v^p (1 + p) = -2\alpha\beta v^p \left(2 - \frac{1}{\alpha} \right) = -\beta v^p (4\alpha - 2), \quad p = 1 - \frac{1}{\alpha} \tag{43}$$

and we know that v^p has positive upper bound and positive lower bound due to (28). By (43) and the maximum principle, we can conclude the following:

(1). If $\alpha > 1/2$, choose $\beta = 1$, then $w = u + Q$ and $-\beta v^p (4\alpha - 2) = -v^p (4\alpha - 2)$ is strictly negative and if $w(\theta_*, t) = w_{\max}(t)$ becomes sufficiently large (positively) at the maximum point $\theta = \theta_*$, we get

$$\frac{\partial w}{\partial t} \le -v^p (4\alpha - 2) u^2 + \alpha v^{p-1} \left(p \left(v - \lambda(t) \right) + v \right) u + 2\alpha v^{p+1} \left(v - \lambda(t) \right) < 0 \quad \text{at} \quad (\theta_*, t). \tag{44}$$

The maximum principle implies that the function $w = u + Q$ has a positive time-independent upper bound and so is the function $u = v_\theta$. Similarly, if we choose $\beta = -1$, then $w = u - Q$ and $-\beta v^p (4\alpha - 2) = v^p (4\alpha - 2)$ is strictly positive and if $w(\theta_*, t) = w_{\min}(t)$ becomes sufficiently large (negatively) at the minimum point $\theta = \theta_*$, we get

$$\frac{\partial w}{\partial t} \geq v^p (4\alpha - 2) u^2 + \alpha v^{p-1} (p (v - \lambda(t)) + v) u - 2\alpha v^{p+1} (v - \lambda(t)) > 0 \quad \text{at} \quad (\theta_*, t). \tag{45}$$

The minimum principle implies that the function $w = u - Q$ has a negative time-independent lower bound and so is the function $u = v_\theta$.

(2). If $0 < \alpha < 1/2$, choose $\beta = 1$, then $w = u + Q$ and $-\beta v^p (4\alpha - 2) = -v^p (4\alpha - 2)$ is strictly positive and if $w(\theta_*, t) = w_{\min}(t)$ becomes sufficiently large (negatively) at the minimum point $\theta = \theta_*$, we get (45) and the minimum principle implies that the function $w = u + Q$ has a negative time-independent lower bound and so is the function $u = v_\theta$. Similarly, if we choose $\beta = -1$, then $w = u - Q$ and $-\beta v^p (4\alpha - 2) = v^p (4\alpha - 2)$ is strictly negative and if $w(\theta_*, t) = w_{\max}(t)$ becomes sufficiently large (positively) at the maximum point $\theta = \theta_*$, we get (44) and the maximum principle implies that the function $w = u - Q$ has a positive time-independent upper bound and so is the function $u = v_\theta$.

(3). When $\alpha = 1/2$, the coefficient $-\beta v^p (4\alpha - 2) = 0$ and the above maximum/minimum principle argument is not applicable. In such a case, we look at the evolution equation of $w = u + \beta v$ and by

$$w_\theta = u_\theta + \beta v_\theta = u_\theta + \beta u, \quad w_{\theta\theta} = u_{\theta\theta} + \beta v_{\theta\theta},$$

we get

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \begin{cases} \alpha v^p w_{\theta\theta} + \alpha p v^{p-1} u (w_\theta - \beta u) + \alpha v^{p-1} (p (v - \lambda(t)) + v) u \\ + \alpha v^p (\beta v - \beta \lambda(t)), \quad \text{where} \quad p = 1 - \frac{1}{\alpha} = -1 \end{cases} \\
&= \frac{1}{2} v^{-1} w_{\theta\theta} - \frac{1}{2} v^{-2} u w_\theta + \frac{1}{2} \beta v^{-2} u^2 + \frac{1}{2} v^{-2} \lambda(t) u + \frac{1}{2} v^{-1} (\beta v - \beta \lambda(t)).
\end{aligned}$$

Now similar to (1) and (2) (by choosing $\beta = 1$ or $\beta = -1$), we can get time-independent upper bound and lower bound of $u = v_\theta$.

By (1), (2), (3), the proof of (39) for the case $k = 1$ is finished. □

As a consequence of Arzela-Ascoli theorem, Lemma 3.11, Lemma 3.12 and Lemma 3.14, we immediately have the following smooth convergence:

Lemma 3.15. *Let $\alpha > 0$ and assume $X(\varphi, t) : S^1 \times [0, \infty) \to \mathbb{R}^2$ is a convex solution of the length-preserving flow (1). Then we have*

$$\lim_{t \to \infty} \left\| v(\theta, t) - \left(\frac{L(0)}{2\pi} \right)^\alpha \right\|_{C^m(S^1)} = 0, \quad \forall\, m \in \{0\} \bigcup \mathbb{N}, \tag{46}$$

where $L(0)$ is the length of X_0 and $v(\theta, t) = \rho^\alpha(\theta, t) = 1/\kappa^\alpha(\theta, t)$. In particular, we obtain

$$\lim_{t\to\infty}\left\|\rho(\theta,t)-\frac{L(0)}{2\pi}\right\|_{C^m(S^1)}=0,\quad \forall\, m\in\{0\}\bigcup\mathbb{N}. \tag{47}$$

Lemma 3.15 says that the evolving curve $X(\cdot,t)$ converges to a circle smoothly in the sense that its curvature $\kappa(\cdot,t)$ converges smooth to the constant $2\pi/L(0)$. Conceivably, although it seems very unlikely, the evolving curve $X(\cdot,t)$ may escape to infinity or oscillate indefinitely. This is because we only look at the convergence of the radius of curvature (or the curvature). To see that this will not happen, we will prove that $X(\cdot,t)$ has a limit as $t\to\infty$.

3.3.4. Exponential decay of the speed of the flow (1)

Lemma 3.16. *Let $\alpha>0$ and assume $X(\varphi,t):S^1\times[0,\infty)\to\mathbb{R}^2$ is a convex solution of the length-preserving flow (1). Then there exist constants $C_1,\ C_2>0$, both are independent of time, so that*

$$\left|\frac{\partial X}{\partial t}(\varphi,t)\right|\le C_1e^{-C_2t},\quad \forall\ (\varphi,t)\in S^1\times[0,\infty). \tag{48}$$

In particular, we have

$$\lim_{t\to\infty}X(\varphi,t)=X_0(\varphi)+\int_0^\infty X_t(\varphi,t)\,dt,\quad \forall\,\varphi\in S^1, \tag{49}$$

where, for each $\varphi\in S^1$, the integral $\int_0^\infty X_t(\varphi,t)\,dt$ converges and $X(\varphi,t)$ has a limit $X_\infty(\varphi)$ as $t\to\infty$.

Remark 3.17. Now Lemma 3.15 tells us that the limiting curve $X_\infty(\cdot)$ is a fixed circle with radius $L(0)/2\pi$. The center $(a,b)\in\mathbb{R}^2$ of $X_\infty(\cdot)$ is given by

$$(a,b)=\lim_{t\to\infty}\frac{1}{2\pi}\int_0^{2\pi}X(\theta,t)\,d\theta=\lim_{t\to\infty}\frac{1}{\pi}\int_0^{2\pi}p(\theta,t)(\cos\theta,\sin\theta)\,d\theta, \tag{50}$$

where $p(\theta,t)$ is the support function of the curve $X(\cdot,t)$.

Proof. Using the evolution of $u=v_\theta$ in (40) and $p=1-1/\alpha$, we can obtain

$$\frac{d}{dt}\int_0^{2\pi}\frac{1}{2}\left(\frac{\partial v}{\partial\theta}\right)^2d\theta=\int_0^{2\pi}uu_t\,d\theta=\begin{cases}\int_0^{2\pi}\alpha v^p uu_{\theta\theta}d\theta+\int_0^{2\pi}\alpha pv^{p-1}u^2u_\theta d\theta\\+\int_0^{2\pi}\alpha v^{p-1}\left(p\left(v-\lambda(t)\right)+v\right)u^2d\theta\end{cases} \tag{51}$$

and by

$$\int_0^{2\pi}\alpha v^p uu_{\theta\theta}d\theta=-\int_0^{2\pi}\left(\alpha v^p u\right)_\theta u_\theta d\theta=-\int_0^{2\pi}\left(\alpha pv^{p-1}u^2+\alpha v^p u_\theta\right)u_\theta d\theta$$

$$= -\int_0^{2\pi} \alpha v^p (u_\theta)^2 d\theta - \int_0^{2\pi} \alpha p v^{p-1} u^2 u_\theta d\theta,$$

(51) can be simplified as

$$\begin{aligned}
&\frac{d}{dt}\int_0^{2\pi} \frac{1}{2}\left(\frac{\partial v}{\partial \theta}\right)^2 d\theta = -\int_0^{2\pi} \alpha v^p (u_\theta)^2 d\theta + \int_0^{2\pi} \alpha v^{p-1}\left(p\left(v-\lambda(t)\right)+v\right)u^2 d\theta \\
&= -\int_0^{2\pi} \alpha v^{1-\frac{1}{\alpha}}\left(\frac{\partial^2 v}{\partial \theta^2}\right)^2 d\theta + (2\alpha-1)\int_0^{2\pi} v^{1-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial \theta}\right)^2 d\theta - (\alpha-1)\lambda(t)\int_0^{2\pi} v^{-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial \theta}\right)^2 d\theta \\
&= -\int_0^{2\pi} \alpha v^{1-\frac{1}{\alpha}}\left(\frac{\partial^2 v}{\partial \theta^2}\right)^2 d\theta + \alpha\int_0^{2\pi} v^{1-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial \theta}\right)^2 d\theta + (\alpha-1)\int_0^{2\pi} \left(v-\lambda(t)\right)v^{-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial \theta}\right)^2 d\theta.
\end{aligned} \tag{52}$$

Since $v = \rho^\alpha$, where $\rho = \rho(\theta, t) = 1/\kappa(\theta, t)$, we have

$$\begin{cases} \frac{\partial v}{\partial \theta} = \alpha\rho^{\alpha-1}\rho_\theta, \quad \frac{\partial^2 v}{\partial \theta^2} = \alpha\rho^{\alpha-1}\rho_{\theta\theta} + \alpha(\alpha-1)\rho^{\alpha-2}(\rho_\theta)^2 \\ \left(\frac{\partial^2 v}{\partial \theta^2}\right)^2 = \alpha^2\rho^{2\alpha-2}(\rho_{\theta\theta})^2 + 2\alpha^2(\alpha-1)\rho^{2\alpha-3}(\rho_\theta)^2\rho_{\theta\theta} + \alpha^2(\alpha-1)^2\rho^{2\alpha-4}(\rho_\theta)^4 \end{cases} \tag{53}$$

and by the identities

$$\begin{cases} \int_0^{2\pi} \rho\cos\theta d\theta = \int_0^{2\pi} \rho\sin\theta d\theta = 0, \quad \forall\, t \in [0, \infty) \\ \int_0^{2\pi} \rho_\theta d\theta = \int_0^{2\pi} \rho_\theta \cos\theta d\theta = \int_0^{2\pi} \rho_\theta \sin\theta d\theta = 0, \end{cases} \tag{54}$$

we have the following form of **Wirtinger inequality** (see [3], p. 92, or use Fourier series expansion to verify it)

$$\int_0^{2\pi} \left(\frac{\partial^2 \rho}{\partial \theta^2}\right)^2 d\theta \geq 4\int_0^{2\pi} \left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta. \tag{55}$$

Now we look at the integral $\int_0^{2\pi} \left(\partial^2 v/\partial\theta^2\right)^2 d\theta$ and by (53) we get

$$\begin{aligned}
&\int_0^{2\pi} \left(\frac{\partial^2 v}{\partial \theta^2}\right)^2 d\theta \\
&= \alpha^2\int_0^{2\pi} \rho^{2\alpha-2}\left(\frac{\partial^2 \rho}{\partial \theta^2}\right)^2 d\theta + \int_0^{2\pi} \left[2\alpha^2(\alpha-1)\rho^{2\alpha-3}\rho_{\theta\theta} + \alpha^2(\alpha-1)^2\rho^{2\alpha-4}\rho_\theta^2\right]\left(\frac{\partial \rho}{\partial \theta}\right)^2 d\theta
\end{aligned}$$

and since we have the convergence (47), for any small $\varepsilon > 0$, if t is large enough, we can conclude

$$\begin{aligned}\int_0^{2\pi}\left(\frac{\partial^2 v}{\partial\theta^2}\right)^2 d\theta &\geq \alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}-\varepsilon\right]\int_0^{2\pi}\left(\frac{\partial^2 \rho}{\partial\theta^2}\right)^2 d\theta - \varepsilon\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta \\ &\geq 4\alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}-\varepsilon\right]\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta - \varepsilon\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta, \end{aligned} \tag{56}$$

where in (56) we have used the Wirtinger inequality (55). On the other hand, we also have for large t the estimate

$$\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta = \alpha^2\int_0^{2\pi}\rho^{2\alpha-2}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta \leq \alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}+\varepsilon\right]\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta, \tag{57}$$

which, together with (56), implies

$$\begin{aligned}\int_0^{2\pi}\left(\frac{\partial^2 v}{\partial\theta^2}\right)^2 d\theta &\geq \left(4\alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}-\varepsilon\right]-\varepsilon\right)\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta \\ &\geq \frac{4\alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}-\varepsilon\right]-\varepsilon}{\alpha^2\left[\left(\frac{L(0)}{2\pi}\right)^{2\alpha-2}+\varepsilon\right]}\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta \geq (4-\delta)\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta, \end{aligned} \tag{58}$$

for some number $\delta > 0$ satisfying $\delta \to 0$ as $\varepsilon \to 0$. As a consequence of (58), we have for large t the following

$$\begin{aligned}&\int_0^{2\pi}\alpha v^{1-\frac{1}{\alpha}}\left(\frac{\partial^2 v}{\partial\theta^2}\right)^2 d\theta \\ &\geq \alpha\left[\left(\frac{L(0)}{2\pi}\right)^{1-\frac{1}{\alpha}}-\varepsilon\right]\int_0^{2\pi}\left(\frac{\partial^2 v}{\partial\theta^2}\right)^2 d\theta \geq \alpha\left[\left(\frac{L(0)}{2\pi}\right)^{1-\frac{1}{\alpha}}-\varepsilon\right](4-\delta)\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta. \end{aligned} \tag{59}$$

By (59) and (52), we conclude for large t the estimate

$$\begin{aligned}&\frac{d}{dt}\int_0^{2\pi}\frac{1}{2}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta \\ &= -\int_0^{2\pi}\alpha v^{1-\frac{1}{\alpha}}\left(\frac{\partial^2 v}{\partial\theta^2}\right)^2 d\theta + \alpha\int_0^{2\pi}v^{1-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta + (\alpha-1)\int_0^{2\pi}(v-\lambda(t))\,v^{-\frac{1}{\alpha}}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta\end{aligned}$$

$$
\leq \begin{cases} -\alpha\left[\left(\frac{L(0)}{2\pi}\right)^{1-\frac{1}{\alpha}}-\varepsilon\right](4-\delta)\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta \\ +\alpha\left[\left(\frac{L(0)}{2\pi}\right)^{1-\frac{1}{\alpha}}+\varepsilon\right]\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta+\varepsilon\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta \end{cases} \leq -C\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}\right)^2 d\theta, \quad \forall\, t\in[0,\infty) \tag{60}
$$

for some constant $C>0$, which is independent of $t,\ \varepsilon,\ \delta$ as long as t is large enough and $\varepsilon,\ \delta>0$ are small enough. Therefore, $\int_0^{2\pi}(\partial v/\partial\theta)^2\,d\theta$ decays exponentially:

$$
\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2 d\theta \leq Ke^{-2Ct}, \quad \forall\, t\in[0,\infty), \tag{61}
$$

where K is a positive constant depending only on the initial data X_0. By Sobolev's inequality, one can estimate the speed of the flow (1):

$$
\begin{aligned}
&\|X_t(\varphi,t)\|_{C^0(S^1)}=\left\|v(\theta,t)-\frac{1}{2\pi}\int_0^{2\pi}v(\theta,t)\,d\theta\right\|_{C^0(S^1)} \\
&\leq\sqrt{2\pi}\left(\int_0^{2\pi}(v(\theta,t)-\lambda(t))^2\,d\theta\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2\pi}}\left(\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2 d\theta\right)^{\frac{1}{2}} \\
&\leq\left(\sqrt{2\pi}+\frac{1}{\sqrt{2\pi}}\right)\left[\int_0^{2\pi}\left(\frac{\partial v}{\partial\theta}(\theta,t)\right)^2 d\theta\right]^{\frac{1}{2}}\leq\left(\sqrt{2\pi}+\frac{1}{\sqrt{2\pi}}\right)\sqrt{K}e^{-Ct}, \quad \forall\, t\in[0,\infty).
\end{aligned} \tag{62}
$$

It follows from (62) that the speed of the flow (1) decays exponentially. The evolving curve $X(\cdot,t)$ has a limit $X_\infty(\cdot)$ as $t\to\infty$ due to the convergence of the integral:

$$
\lim_{t\to\infty}X(\varphi,t)=X_0(\varphi)+\int_0^{\infty}X_t(\varphi,t)\,dt, \quad \forall\,\varphi\in S^1. \tag{63}
$$

The proof is done. □

3.3.5. Proof of Theorem 2.4

By Lemma 3.2, Lemma 3.3, Lemma 3.15, and Lemma 3.16, the proof of Theorem 2.4 is now complete. □

4. Some flows with global existence

As we shall see in Section 5, for some initial curve X_0, the curvature of $X(\cdot, t)$ can blow up to $+\infty$ in finite time. In this section, we give certain sufficient condition on X_0 so that under the flow (1), the convex solution $X(\cdot, t)$ exists for all time $t \in [0, \infty)$.

For a convex closed curve $X_0 \subset \mathbb{R}^2$ with radius of curvature $\rho(\theta) = 1/\kappa(\theta)$, where $\theta \in S^1$ is its outward normal angle, by Green-Osher's inequality [4], we have

$$\int_0^{2\pi} \rho^2(\theta)\, d\theta \geq \frac{L^2 - 2\pi A}{\pi}, \tag{64}$$

where the equality holds if and only if X_0 is a circle. By the classical isoperimetric inequality $L^2 \geq 4\pi A$, the above also gives

$$\int_0^{2\pi} \rho^2(\theta)\, d\theta - 2A \geq \frac{L^2 - 2\pi A}{\pi} - 2A \geq 0. \tag{65}$$

In order to construct a sufficient condition for the global existence of the flow, we need to have an upper bound of $\int_0^{2\pi} \rho^2(\theta)\, d\theta - 2A$. With the help of Fourier series expansion, we can obtain an upper bound, together with an improved lower bound.

Lemma 4.1. *For any convex closed curve $X \subset \mathbb{R}^2$ with radius of curvature $\rho(\theta)$, enclosed area A, and length L, we have the inequality*

$$\frac{2}{\pi}\left(L^2 - 4\pi A\right) \leq \int_0^{2\pi} \rho^2(\theta)\, d\theta - 2A \leq \frac{1}{3}\int_0^{2\pi} \left(\frac{d\rho}{d\theta}\right)^2 d\theta. \tag{66}$$

Moreover, the equality holds (either in the first place or in the second one) if and only if the ***support function*** *$p(\theta)$ of X has the form*

$$p(\theta) = \frac{L}{2\pi} + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta, \quad \forall\, \theta \in S^1 \tag{67}$$

for some constants $a_1,\ b_1,\ a_2,\ b_2$ satisfying

$$p''(\theta) + p(\theta) = \frac{L}{2\pi} - 3a_2 \cos 2\theta - 3b_2 \sin 2\theta > 0, \quad \forall\, \theta \in S^1. \tag{68}$$

Proof. The first inequality in (66) has been proved in [9] (see Lemma 1.7). Hence it suffices to prove the second one. It is known that the length L, area A, and radius of curvature $\rho(\theta)$ of X_0 can be expressed as in terms of $p(\theta)$ as

$$L = \int_0^{2\pi} p(\theta) d\theta, \quad A = \frac{1}{2}\int_0^{2\pi} p(\theta)\rho(\theta)\, d\theta, \quad \rho(\theta) = p''(\theta) + p(\theta) > 0. \tag{69}$$

Hence the Fourier expansion of $p(\theta)$ has the form

$$p(\theta) = \frac{L}{2\pi} + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right], \quad \theta \in S^1,$$

where $a_n,\ b_n$ are the Fourier coefficients of $p(\theta)$. By (69), we can easily derive the following:

$$\begin{cases} A = \frac{L^2}{4\pi} + \frac{\pi}{2} \sum_{n=2}^{\infty} \left(1 - n^2\right)\left(a_n^2 + b_n^2\right), \\ \int_0^{2\pi} \rho^2(\theta) d\theta = \frac{L^2}{2\pi} + \pi \sum_{n=2}^{\infty} \left(1 - n^2\right)^2 \left(a_n^2 + b_n^2\right), \\ \int_0^{2\pi} \left(\frac{d\rho}{d\theta}\right)^2 d\theta = \pi \sum_{n=2}^{\infty} n^2 \left(n^2 - 1\right)^2 \left(a_n^2 + b_n^2\right), \end{cases}$$

which gives

$$\begin{aligned} &\int_0^{2\pi} \rho^2(\theta) d\theta - 2A \\ &= \pi \sum_{n=2}^{\infty} n^2 \left(n^2 - 1\right)\left(a_n^2 + b_n^2\right) \le \frac{\pi}{3} \sum_{n=2}^{\infty} n^2 \left(n^2 - 1\right)^2 \left(a_n^2 + b_n^2\right) = \frac{1}{3} \int_0^{2\pi} \left(\frac{d\rho}{d\theta}\right)^2 d\theta, \end{aligned} \tag{70}$$

where the equality in (70) holds if and only if $a_n = b_n = 0$ for all $n \ge 3$. The proof is done. □

Now assume $X(\varphi, t)$ is a convex solution of the flow (1) defined on $S^1 \times [0, T)$ for some $T > 0$. Using the evolution equation (9) of ρ, one can compute

$$\frac{d}{dt} \int_0^{2\pi} \rho^2 d\theta = -2\alpha \int_0^{2\pi} \rho^{\alpha - 1} \left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta + 2 \int_0^{2\pi} \rho^{\alpha+1} d\theta - 2\lambda(t) L(t), \tag{71}$$

which, together with (6), gives

$$\frac{d}{dt} \left(\int_0^{2\pi} \rho^2 d\theta - 2A(t) \right) = -2\alpha \int_0^{2\pi} \rho^{\alpha - 1} \left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta. \tag{72}$$

Moreover, by equation (9) we have

$$\begin{aligned} &\frac{d}{dt} \int_0^{2\pi} \left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta \\ &= 2 \int_0^{2\pi} \frac{\partial\rho}{\partial\theta} \frac{\partial}{\partial\theta} \left(\frac{\partial^2 \rho^\alpha}{\partial\theta^2} + \rho^\alpha\right) d\theta = -2 \int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2} \frac{\partial^2 \rho^\alpha}{\partial\theta^2} d\theta + 2 \int_0^{2\pi} \alpha\rho^{\alpha - 1} \left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta \end{aligned}$$

$$=-2\int_0^{2\pi}\frac{\partial^2\rho}{\partial\theta^2}\left(\alpha\rho^{\alpha-1}\frac{\partial^2\rho}{\partial\theta^2}+\alpha\left(\alpha-1\right)\rho^{\alpha-2}\left(\frac{\partial\rho}{\partial\theta}\right)^2\right)d\theta+2\alpha\int_0^{2\pi}\rho^{\alpha-1}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta, \quad (73)$$

where by the integration by parts, we can rewrite the middle term in (73) and obtain

$$\begin{aligned}&\frac{d}{dt}\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta\\&=-2\alpha\int_0^{2\pi}\rho^{\alpha-1}\left(\frac{\partial^2\rho}{\partial\theta^2}\right)^2 d\theta+\frac{2}{3}\alpha\left(\alpha-1\right)\left(\alpha-2\right)\int_0^{2\pi}\rho^{\alpha-3}\left(\frac{\partial\rho}{\partial\theta}\right)^4 d\theta+2\alpha\int_0^{2\pi}\rho^{\alpha-1}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta.\end{aligned} \quad (74)$$

By (74) and (72), we get

$$\begin{aligned}&\frac{d}{dt}\left[\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta+\int_0^{2\pi}\rho^2 d\theta-2A\left(t\right)\right]\\&=-2\alpha\int_0^{2\pi}\rho^{\alpha-1}\left(\frac{\partial^2\rho}{\partial\theta^2}\right)^2 d\theta+\frac{2}{3}\alpha\left(\alpha-1\right)\left(\alpha-2\right)\int_0^{2\pi}\rho^{\alpha-3}\left(\frac{\partial\rho}{\partial\theta}\right)^4 d\theta\\&\leq\frac{2}{3}\alpha\left(\alpha-1\right)\left(\alpha-2\right)\int_0^{2\pi}\rho^{\alpha-3}\left(\frac{\partial\rho}{\partial\theta}\right)^4 d\theta.\end{aligned} \quad (75)$$

Now we assume $\alpha\in[1,2]$. It follows from inequality (75) that $\int_0^{2\pi}\rho_\theta^2 d\theta+\int_0^{2\pi}\rho^2 d\theta-2A\left(t\right)$ is decreasing in time under the flow (1). So we have

$$\begin{aligned}\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta&\leq-\int_0^{2\pi}\rho^2 d\theta+2A+\int_0^{2\pi}\left(\frac{d\rho_0}{d\theta}\right)^2 d\theta+\int_0^{2\pi}\rho_0^2 d\theta-2A\left(0\right)\\&\leq\int_0^{2\pi}\left(\frac{d\rho_0}{d\theta}\right)^2 d\theta+\int_0^{2\pi}\rho_0^2 d\theta-2A\left(0\right).\end{aligned}$$

Using the inequality (66), one gets

$$\int_0^{2\pi}\left(\frac{\partial\rho}{\partial\theta}\right)^2 d\theta\leq\frac{4}{3}\int_0^{2\pi}\left(\frac{d\rho_0}{d\theta}\right)^2 d\theta,\quad\forall\, t\in[0,T). \quad (76)$$

By the estimate (76), we can conclude the following:

Lemma 4.2. *Assume $\alpha \in [1,2]$ and the initial smooth convex closed curve X_0 satisfies the condition*

$$\frac{L(0)}{2\pi} \geq \left(\frac{8\pi}{3} \int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 d\theta \right)^{1/2} + \varepsilon \tag{77}$$

for some $\varepsilon > 0$. Then the solution $X(\varphi, t)$ of the flow (1) is defined on $S^1 \times [0, \infty)$.

Proof. By the short-time existence theorem, the flow (1) with initial data X_0 has a smooth convex solution $X(\varphi, t)$ defined on $S^1 \times [0, T)$ for some $T > 0$. For fixed $t \in [0, T)$, choose θ_1 so that $\rho(\theta_1, t) = L(t)/2\pi = L(0)/2\pi$ and choose θ_2 so that $\rho(\theta_2, t) = \rho_{\min}(t)$. By Hölder inequality, we have

$$\rho_{\min}(t) - \frac{L(0)}{2\pi} = \int_{\theta_1}^{\theta_2} \frac{\partial \rho}{\partial \theta} d\theta \geq -\sqrt{2\pi} \left(\int_0^{2\pi} \left(\frac{\partial \rho}{\partial \theta} \right)^2 d\theta \right)^{1/2}, \quad \forall \quad t \in [0, T),$$

which, together with the assumption (77) and estimate (76), gives

$$\begin{aligned} \rho_{\min}(t) &\geq \left(\frac{8\pi}{3} \int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 d\theta \right)^{1/2} + \varepsilon - \sqrt{2\pi} \left(\int_0^{2\pi} \left(\frac{\partial \rho}{\partial \theta} \right)^2 d\theta \right)^{1/2} \\ &\geq \left(\frac{8\pi}{3} \int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 d\theta \right)^{1/2} + \varepsilon - \sqrt{2\pi} \left(\frac{4}{3} \int_0^{2\pi} \left(\frac{d\rho_0}{d\theta} \right)^2 d\theta \right)^{1/2} = \varepsilon > 0, \quad \forall\, t \in [0, T). \end{aligned} \tag{78}$$

By Lemma 3.2, the flow solution $X(\varphi, t)$ can be continued beyond time T and is defined on a larger time interval $[0, \tilde{T})$. By the same argument, we still have $\rho_{\min}(t) \geq \varepsilon > 0$ for all $t \in [0, \tilde{T})$. Hence the solution $X(\varphi, t)$ must be defined on $S^1 \times [0, \infty)$ due to Lemma 3.3. □

Remark 4.3. There are lots of convex curves satisfying the condition (77). One can construct abundant examples by choosing large L and small $\max_{\theta \in S^1} |d\rho_0/d\theta|$. One can also construct examples by expanding convex curves according to the unit outward normal flow

$$\begin{cases} X_t(\varphi, t) = \mathbf{N}_{out}(\varphi, t), & t > 0 \\ X(\varphi, 0) = X_0(\varphi), & \varphi \in S^1. \end{cases} \tag{79}$$

After a long time, one has a convex curve $X(\cdot, T)$ satisfying (77), because under the flow (79) we have $L(t) = L(0) + 2\pi t$, but the integral quantity $\int_0^{2\pi} \left(\frac{\partial \rho}{\partial \theta} \right)^2 d\theta$ is independent of time.

In fact, the condition (77) is sufficient but not necessary to guarantee the global existence of the flow (1).

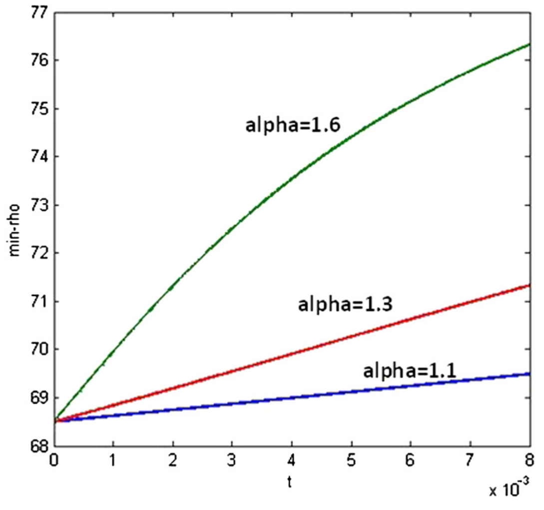


Fig. 1. $\rho_{\min}(t)$ of $\alpha = 1.1, 1.3$ and 1.6 in Example 4.4.

Example 4.4. Let $X_0(\theta)$ be a convex curve with the support function $p(\theta) = 80 + 2\cos(2\theta) + 0.5\cos(3\theta) + 0.1\cos(4\theta)$. Its radius of curvature is

$$\rho_0(\theta) = 80 - 6\cos(2\theta) - 4\cos(3\theta) - 1.5\cos(4\theta).$$

With the help of MATLAB, one can check that the minimum of ρ_0 is 68.5. It does not satisfy (77), because

$$\frac{L(0)}{2\pi} - \left(\frac{8\pi}{3}\int_0^{2\pi}\left(\frac{d\rho_0}{d\theta}\right)^2 d\theta\right)^{1/2} = -12.3436\cdots.$$

But the flow (1) still exists globally if $\alpha = 1.1,\ 1.3$ and 1.6. Fig. 1 presents the graph of the function $\rho_{\min}(t)$ under the flows.

5. Some blow-up examples

In this section, we first take a long narrow ellipse γ_0 as the initial curve and use it to demonstrate intuitively the formation of a singularity in finite time. Then we look at the evolution behavior (with the help of MATLAB) of several initial convex curves for the flow (1) and observe that some flows can blow up in finite time, while the others exist globally on the time interval $[0, \infty)$.

5.1. Curvature blow-up for an ellipse

Let γ_0 be an ellipse given by

$$\gamma_0 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \gg b > 0, \tag{80}$$

where $a \gg 0$ is sufficiently larger than $b > 0$ (here we fix b and in the limit we will let $a \to \infty$). A straightforward computation shows that the curvature $\kappa_0(\theta)$ of γ_0 is, in terms of its outward normal angle $\theta \in [0, 2\pi]$, given by

$$\kappa_0(\theta) = \frac{1}{a^2b^2}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{3}{2}}, \quad \max_{\theta\in[0,2\pi]}\kappa_0(\theta) = \kappa_0(0) = \frac{a}{b^2}, \quad \theta \in S^1. \tag{81}$$

Since $\kappa_0(\theta)$ is uniformly large on some fixed open interval $I \subset S^1$ centered at $\theta = 0$, the radius of curvature $\rho_0(\theta) = 1/\kappa_0(\theta)$ is uniformly small on I. One can also check that $\rho_0''(\theta)$ is also uniformly small on I due to the identities

$$\rho_0'(\theta) = -\frac{3}{2}a^2b^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-\frac{5}{2}}\left(b^2 - a^2\right)\sin 2\theta, \quad \rho_0'(0) = 0$$

and

$$\begin{aligned}\rho_0''(\theta) &= \frac{15}{4}a^2b^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-7/2}\left(b^2 - a^2\right)^2\sin^2 2\theta \\ &\quad + 3a^2b^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-5/2}\left(a^2 - b^2\right)\cos 2\theta, \quad \rho_0''(0) = 3\frac{b^2}{a}\left(1 - \frac{b^2}{a^2}\right)\end{aligned}$$

where $a \gg b > 0$. We now have $\min_{\theta\in[0,2\pi]}\rho_0(\theta) = \rho_0(0) = b^2/a > 0$, and at the minimum point $\theta = 0$ we have $\rho_0''(0) = 3a^{-1}b^2\left(1 - b^2a^{-2}\right) > 0$.

Recall that under the flow (1), the function $v(\theta, t) = \rho^\alpha(\theta, t)$, $v(\theta, 0) = \rho_0^\alpha(\theta)$, satisfies the equation

$$\frac{\partial v}{\partial t} = \alpha v^p\left(v_{\theta\theta} + v - \lambda(t)\right), \quad v = v(\theta, t), \quad p = 1 - \frac{1}{\alpha}, \quad \lambda(t) = \frac{1}{2\pi}\int_0^{2\pi} v\, d\theta. \tag{82}$$

We shall see what happens to the equation (82) initially if we use the ellipse γ_0 as initial curve with $a \gg b > 0$. We have

$$\begin{cases} v_0(\theta) = \rho_0^\alpha(\theta) = a^{2\alpha}b^{2\alpha}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-\frac{3\alpha}{2}}, \quad v_0(0) = \rho_0^\alpha(0) = \left(\frac{b^2}{a}\right)^\alpha \\ \lambda(0) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} v_0(\theta)\, d\theta = \dfrac{a^{2\alpha}b^{2\alpha}}{2\pi}\displaystyle\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-\frac{3\alpha}{2}} d\theta, \end{cases} \tag{83}$$

and by

$$v_0'(\theta) = \alpha\rho_0^{\alpha-1}(\theta)\,\rho_0'(\theta), \quad v_0''(\theta) = \alpha(\alpha - 1)\,\rho_0^{\alpha-2}(\theta)\left(\rho_0'(\theta)\right)^2 + \alpha\rho_0^{\alpha-1}(\theta)\,\rho_0''(\theta)$$

we get (note that $\rho_0'(0) = 0$)

$$v_0''(0) = \alpha\rho_0^{\alpha-1}(0)\,\rho_0''(0) = 3\alpha\left(\frac{b^2}{a}\right)^\alpha\left(1 - \frac{b^2}{a^2}\right). \tag{84}$$

By (83) and (84), the equation (82) at time $t = 0$ at $\theta = 0$ is

$$\begin{aligned}\frac{\partial v}{\partial t}(0,0) &= \alpha v^{p}(0,0)\left(v_{\theta\theta}(0,0) + v(0,0) - \lambda(0)\right), \quad p = 1 - \frac{1}{\alpha} \\ &= \alpha\left(\frac{b^2}{a}\right)^{\alpha-1}\left(\begin{array}{c} 3\alpha\left(\frac{b^2}{a}\right)^{\alpha}\left(1 - \frac{b^2}{a^2}\right) + \left(\frac{b^2}{a}\right)^{\alpha} \\ -\frac{a^{2\alpha}b^{2\alpha}}{2\pi}\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-\frac{3\alpha}{2}} d\theta \end{array}\right).\end{aligned} \tag{85}$$

Now it suffices to know the behavior of the integral in (85) when $a \gg 0$ is sufficiently large. It has been shown in [5] that we have the following asymptotic behavior as $a \to \infty$:

Lemma 5.1. *Let $a \gg b > 0$ be two constants. If we fix b and let $a \to \infty$, then we have the following asymptotic behavior:*

$$\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-p} d\theta = \begin{cases} O\left(a^{-2p}\right), & \forall\, p \in \left(0, \frac{1}{2}\right), \\ O\left(a^{-1}\log a\right), & p = \frac{1}{2}, \\ O\left(a^{-1}\right), & \forall\, p \in \left(\frac{1}{2}, \infty\right). \end{cases} \tag{86}$$

By (86), we have

$$\frac{a^{2\alpha}b^{2\alpha}}{2\pi}\int_0^{2\pi}\left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{-\frac{3\alpha}{2}} d\theta = \begin{cases} O\left(a^{-\alpha}\right), & \forall\, \alpha \in \left(0, \frac{1}{3}\right), \\ O\left(a^{-\frac{1}{3}}\log a\right), & \alpha = \frac{1}{3}, \\ O\left(a^{2\alpha-1}\right), & \forall\, \alpha \in \left(\frac{1}{3}, \infty\right), \end{cases}$$

which, together with (85), implies

$$\frac{\partial v}{\partial t}(0,0) = O\left(a^{1-2\alpha}\right)\ (\forall\, \alpha > 0) - \begin{cases} O\left(a^{1-2\alpha}\right), & \forall\, \alpha \in \left(0, \frac{1}{3}\right), \\ O\left(a^{\frac{1}{3}}\log a\right), & \alpha = \frac{1}{3}, \\ O\left(a^{\alpha}\right), & \forall\, \alpha \in \left(\frac{1}{3}, \infty\right). \end{cases} \tag{87}$$

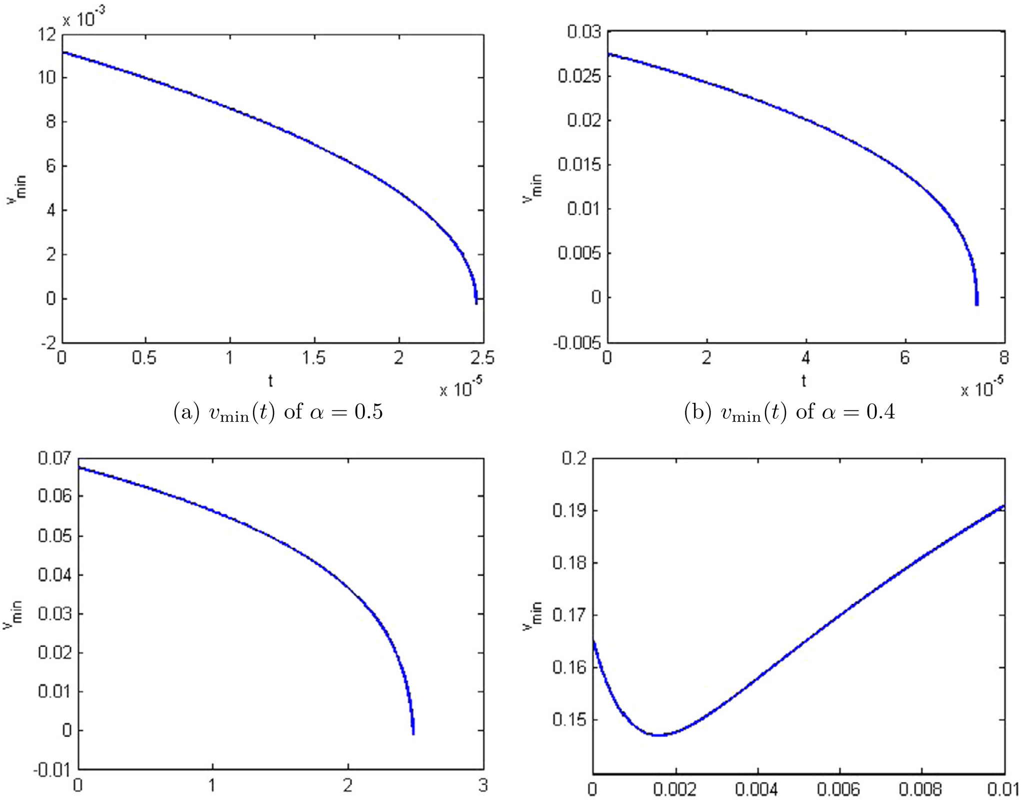
By (87), for $\alpha \geq 1/3$, if $a \gg 0$ is sufficiently large, we will have $(\partial v/\partial t)(0,0) \ll 0$, which will force $v_{\min}(t)$ (with $v_{\min}(0) = \left(b^2/a\right)^{\alpha}$) to drop to 0 in short time and a singularity can occur. In view of this, at least for $\alpha \geq 1/3$, we can make the following conjecture:

Conjecture 5.2. *For $\alpha \geq 1/3$, there exists an ellipse γ_0 such that under the flow (1) it develops a singularity in finite time. Moreover, the larger a and α are, the faster the blow-up will occur.*

Remark 5.3. However, for $0 < \alpha < 1/3$, we are not sure if $(\partial v/\partial t)(0,0)$ will become sufficiently large negatively for $a \gg 0$. We have no conclusion in this situation. See Conjecture 5.8 also.

Table 1
Some simulation results of the flow (1) with an ellipse γ_0.

α	simulation time ω	min v	status
0.5	2.46×10^{-5}	-2.5225×10^{-4}	blows up
0.4	7.438×10^{-5}	-8.7495×10^{-4}	blows up
0.3	2.48312×10^{-4}	-1.2×10^{-3}	blows up
0.2	0.01	0.1471	not blow up



(a) $v_{\min}(t)$ of $\alpha = 0.5$ (b) $v_{\min}(t)$ of $\alpha = 0.4$

(c) $v_{\min}(t)$ of $\alpha = 0.3$ (d) $v_{\min}(t)$ of $\alpha = 0.2$

Fig. 2. The functions $v_{\min}(t)$ with initial ellipse γ_0.

There some numerical simulations which support the Conjecture 5.2 and the Remark 5.3. Let γ_0 be an ellipse (80) with $a = 80$ and $b = 0.1$. The simulation results of the flow (1) with initial γ_0 are shown in Table 1 and Fig. 2, where $v_{\min}(t) := \min\{v(\theta, t)|\theta \in S^1\}$.

Remark 5.4. The above solutions of the PDEs are calculated by discretization method, so they may perform a little different from the continuous solutions. If $v_{\min}(t) > 0$ then the computation has no problem. Usually, if the flow blows up at time t_0, there is a very short time interval (t_0, t_1)

in which $v_{\min}(t) < 0$. After the time t_1, the solution in some cases of α becomes complex with imaginary part.

5.2. More numerical examples

Let $X_0(\theta)$ be a convex curve parameterized by the outward normal angle θ with the support function given by $p(\theta) = 10 + \cos(2\theta) + \cos(3\theta) - 0.25\cos(4\theta)$. Its radius of curvature is

$$\rho_0(\theta) = p(\theta) + \frac{d^2 p}{d\theta^2} = 10 - 3\cos(2\theta) - 8\cos(3\theta) + 3.75\cos(4\theta).$$

With the help of MATLAB, one can compute the minimum of $\rho_0(\theta)$, which is 0.065. So X_0 is a smooth convex closed curve in the plane. Let $X_0(\theta)$ evolve according to the flow (1). In the following we shall use the above $\rho_0(\theta)$ as the initial data in equation (9). We know that the flow (1) blows up if the function $\rho_{\min}(t) = \min_{\theta\in S^1} \rho_0(\theta, t)$ drops to 0 in finite time.

Example 5.5. Let α be $2,\ 3$, or 4. With the help of MATLAB, one can compute the solution $\rho(\theta, t)$ to (9), respectively. Then the function $\rho_{\min}(t)$ can be determined (see Fig. 3 (a), (b), (c)). These three flows all blow up in finite time.

The blow-up time for $\alpha = 2,\ 3$, and 4 is $t \approx 5.5 \times 10^{-4},\ 3.2 \times 10^{-5},\ 2.2 \times 10^{-6}$ respectively. It follows from the numerical computation that the bigger α is, the faster the flow blows up.

Example 5.6. Let α be $0.5,\ 1$, or 1.73. It follows from computations that these three flows all exist globally, because $\rho_{\min}(t) > 0$. The functions $\rho_{\min}(t)$ are presented in Fig. 4. For $\alpha = 1.73$, $\rho_{\min}(t)$ is decreasing initially and then becomes increasing as time proceeds. More examples show that $\rho_{\min}(t)$ is increasing if $\alpha < 1$.

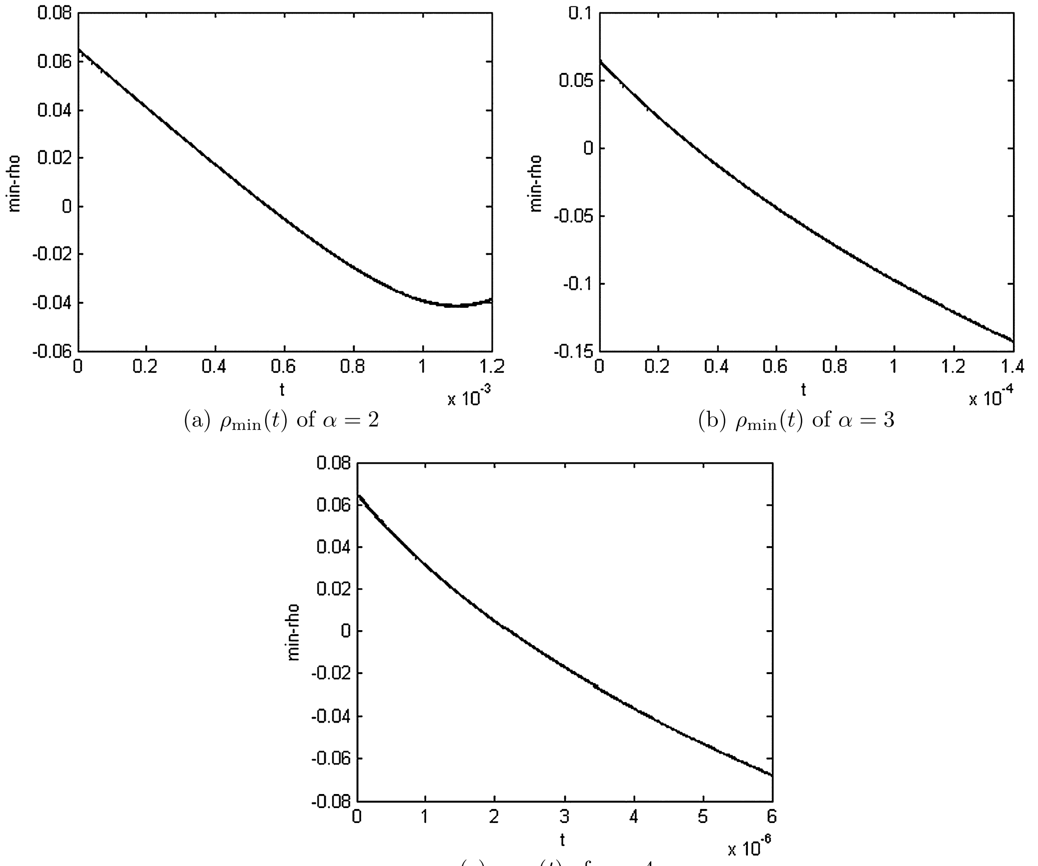
Remark 5.7. The flow (1) with $\alpha = 1$ has been studied by [10]. The authors proved that $X(\cdot, t)$ converges to a finite circle if the flow exists globally on the time interval $[0, \infty)$. In [11], using the idea of parallel curves, the author proved that for some initial curve X_0, the curvature κ does blow up in finite time.

The above blow-up examples indicate that a $1/\kappa^\alpha$-type non-local flow does not always exist globally for all initial convex closed curves. Given an initial convex curve, it seems that the smaller $\alpha > 0$ is, the longer the flow (1) lasts. In view of this, we make the following conjecture:

Conjecture 5.8. *Let $X_0 \subset \mathbb{R}^2$ be a smooth convex closed curve. If $\alpha > 0$ is small enough, then the length-preserving flow (1) with initial curve X_0 will exist on the time interval* $[0, \infty)$.

Acknowledgments

Laiyuan Gao is supported by National Natural Science Foundation of China (No. 11801230). Shengliang Pan is supported by National Natural Science Foundation of China (No. 11671298). Dong-Ho Tsai is supported by MOST of Taiwan (No. 105-2115-M-007-007-MY3). The three authors thank the referee for his/her comments.



(a) $\rho_{\min}(t)$ of $\alpha = 2$ (b) $\rho_{\min}(t)$ of $\alpha = 3$

(c) $\rho_{\min}(t)$ of $\alpha = 4$

Fig. 3. The functions $\rho_{\min}(t)$ in Example 5.5.

References

[1] K.S. Chou, X.P. Zhu, The Curve Shortening Problem, Chapman & Hall/CRC, 2001.
[2] M. Gage, On an area-preserving evolution equation for plane curves, in: Nonlinear Problems in Geometry, in: Contemp. Math., vol. 51, Amer. Math. Soc., 1985, pp. 51–62.
[3] M.E. Gage, R. Hamilton, The heat equation shrinking convex plane curves, J. Differ. Geom. 23 (1986) 69–96.
[4] M. Green, S. Osher, Steiner polynomials, Wulff flows, and some new isoperimetric inequalities for convex plane curves, Asian J. Math. 3 (1999) 659–676.
[5] L.Y. Gao, S.L. Pan, D.H. Tsai, Nonlocal flow driven by the radius of curvature with fixed curvature integral, J. Geom. Anal. (2019), https://doi.org/10.1007/s12220-019-00185-4, 27 pages.
[6] L.Y. Gao, Y.T. Zhang, On Yau's problem of evolving one curve to another: convex case, J. Differ. Equ. 266 (1) (2019) 179–201.
[7] L.S. Jiang, S.L. Pan, On a non-local curve evolution problem in the plane, Commun. Anal. Geom. 16 (2008) 1–26.
[8] Y.C. Lin, D.H. Tsai, On a simple maximum principle technique applied to equations on the circle, J. Differ. Equ. 245 (2008) 377–391.
[9] Y.C. Lin, D.H. Tsai, Application of Andrews and Green-Osher inequalities to nonlocal flow of convex plane curves, J. Evol. Equ. 12 (4) (2012) 833–854.
[10] S.L. Pan, J.N. Yang, On a non-local perimeter-preserving curve evolution problem for convex plane curves, Manuscr. Math. 127 (2008) 469–484.

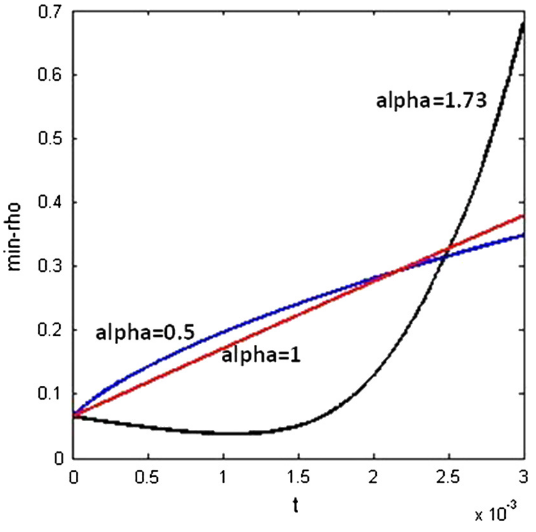


Fig. 4. $\rho_{\min}(t)$ of $\alpha = 0.5$, 1 and 1.73 in Example 5.6.

[11] D.H. Tsai, On flows that preserve parallel curves and their formation of singularities, J. Evol. Equ. 18 (2) (2018) 303–321.

[12] D.H. Tsai, X.L. Wang, On length-preserving and area-preserving nonlocal flow of convex closed plane curves, Calc. Var. Partial Differ. Equ. 54 (2015) 3603–3622.

[13] D.H. Tsai, X.L. Wang, The evolution of nonlocal curvature flow arising in a Hele-Shaw problem, SIAM J. Math. Anal. 50 (1) (2018) 1396–1431.