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Large time behavior in a three-dimensional degenerate chemotaxis-Stokes system modeling coral fertilization

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Abstract

This paper is concerned with the effects of nonlinear diffusion on global solvability and stabilization in a variety of models which generalizes the following prototype

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (n^{m-1} \nabla n) - \nabla \cdot (n \nabla c) - \rho n, \\ c_t + u \cdot \nabla c = \Delta c - c + \rho, \\ \rho_t + u \cdot \nabla \rho = \Delta \rho - \rho n, \\ u_t + \nabla P = \Delta u + (n + \rho) \nabla \phi, \\ \nabla \cdot u = 0 \end{cases}$$

with a given function $\phi \in W^{2,\infty}(\Omega)$, where $\Omega \subset \mathbb{R}^3$ is a general bounded domain with smooth boundary. Based on an energy-type argument combined with maximal Sobolev regularity theory, we conclude that if

$$m > \frac{37}{33},$$

an associated initial-boundary value problem admits at least one globally bounded weak solution which stabilizes toward the spatial homogeneous equilibrium $(n_\infty, \rho_\infty, \rho_\infty, 0)$ in the sense that

$$\|n(\cdot,t) - n_\infty\|_{L^p(\Omega)} + \|c(\cdot,t) - \rho_\infty\|_{W^{1,\infty}(\Omega)} + \|\rho(\cdot,t) - \rho_\infty\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty$$

for any $p > 1$, where $n_\infty := \frac{1}{|\Omega|}\left\{\int_\Omega n_0 - \int_\Omega \rho_0\right\}_+$ and $\rho_\infty := \frac{1}{|\Omega|}\left\{\int_\Omega \rho_0 - \int_\Omega n_0\right\}_+$.

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1. Introduction

We consider the following chemotaxis-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n = \nabla\cdot(D(n)\nabla n) - \nabla\cdot(n\nabla c) - \rho n, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + \rho, & x\in\Omega,\ t>0,\\ \rho_t + u\cdot\nabla\rho = \Delta\rho - \rho n, & x\in\Omega,\ t>0,\\ u_t + \nabla P = \Delta u + (n+\rho)\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

which describes the process of coral fertilization occurring in ocean flow, where the unknown functions n, c, ρ and u represent the density of sperm, the concentration of the chemical signal expelled by egg gametes, the density of eggs, and the velocity field of the ambient ocean flow, respectively. As shown in the experiments [4,5,21,22], the chemical signal can induce certain oriented motion of spermatozoid toward egg gametes, which is referred as chemotaxis and plays an enhanced role during the period of fertilization.

In the case when both sperms and eggs, which are transported by a known fluid velocity field u, enjoy the same density i.e. $n\equiv\rho$, and when the evolution of c is approximated by an elliptic equation under the assumption that the diffusion of the signal is much faster than that of the gametes, system (1.1) with $D\equiv 1$ is reduced to the original model as follows

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c) - n^q, & x\in\Omega,\ t>0,\\ 0 = \Delta c + n, & x\in\Omega,\ t>0. \end{cases} \tag{1.2}$$

Mathematical analysis has shown the temporal decay of the total mass $\int_{\mathbb{R}^2} n$ for both the super-critical case $q>2$ and the critical case $q=2$ ([13,14]).

If the transport of both the gametes and the chemical signal is also under the effects of an unknown solenoidal velocity field u whose evolution is governed by a (Navier–)Stokes equation, then system (1.2) becomes

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c) - \mu n^2, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u - \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0. \end{cases} \tag{1.3}$$

System (1.3) with $\kappa=0$ was proposed and studied in [7], where the authors constructed a global weak solution without any smallness restriction on initial data in the spatially two-dimensional context.

In more realistic case when the densities of sperms and eggs are not identical all the time, i.e., $n\not\equiv\rho$, system (1.3) turns into

$$
\begin{cases}
n_t + u \cdot \nabla n = \Delta n - \chi \nabla \cdot (n \nabla c) - \rho n, & x \in \Omega,\ t > 0, \\
\rho_t + u \cdot \nabla \rho = \Delta \rho - \rho n, & x \in \Omega,\ t > 0, \\
0 = \Delta c + \rho, & x \in \Omega,\ t > 0,
\end{cases}
\tag{1.4}
$$

where the reaction $-\rho n$ shows the consumption of the gametes caused by the contact of spermatozoid and egg in the process of fertilization. For system (1.4), similar conclusions as that derived for system (1.2) are proved in the three dimensional setting, and the achieved temporal decay properties therein are measured with higher regularity, i.e., $L^p(\mathbb{R}^N)$ with any $p > 1$ for $N = 2, 3$ [3]. With ρ replaced by $\rho - \frac{1}{|\Omega|} \int_\Omega \rho$ on the right-hand side of the c-equation, Espejo et al. [6] have showed the influence of the increasing chemical signal on the dynamic behavior of the gametes in the spatially two-dimensional case.

If one takes into account the transport of the chemical signal through the fluid flow whose velocity field is modeled by an incompressible (Navier–)Stokes equation, then system (1.4) can be extended to the following version, that is

$$
\begin{cases}
n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) - \rho n, & x \in \Omega,\ t > 0, \\
c_t + u \cdot \nabla c = \Delta c - c + \rho, & x \in \Omega,\ t > 0, \\
\rho_t + u \cdot \nabla \rho = \Delta \rho - \rho n, & x \in \Omega,\ t > 0, \\
u_t + \kappa (u \cdot \nabla) u = \Delta u - \nabla P + (n + \rho) \nabla \phi, & x \in \Omega,\ t > 0, \\
\nabla \cdot u = 0, & x \in \Omega,\ t > 0,
\end{cases}
\tag{1.5}
$$

which with $\kappa = 1$ was firstly investigated in [8] for global classical solvability and stability in two-dimensional bounded domain. In the spatially three dimensional context, whether for initial value problems or for initial-boundary value problems associated with (1.5), the construction of global solutions and even the detection of spatially homogeneous equilibria need to rely on appropriate smallness restrictions on initial data [3,9,17,20] or on certain nonlinear dampening mechanism, such as saturation effects of gametes [17,20], signal-dependent sensitivity [16] or slow p-Laplacian diffusion [18].

In this paper, our intentions are to explore how strong the dampening effects exerted by the porous medium type nonlinear diffusion of spermatozoid can prevent the occurrence of collapse to system (1.1) and to detect the convergence of the global solutions as time goes to infinity. In order to state our main results precisely, let us specify the evolution problem by considering (1.1) together with the initial data conditions

$$
n(x,0) = n_0(x),\ c(x,0) = c_0(x),\ \rho(x,0) = \rho_0(x) \ \text{ and } \ u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.6}
$$

as well as the boundary conditions

$$
D(n) \frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = \frac{\partial \rho}{\partial \nu} = 0 \ \text{ and } \ u = 0, \quad x \in \partial \Omega,\ t > 0, \tag{1.7}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, and moreover, the initial data in (1.6) fulfill

$$
\begin{cases}
n_0(x) \in C^\iota(\bar{\Omega}) \ \text{for certain } \iota > 0 \ \text{ with } n_0 > 0 \ \text{ in } \Omega, \\
c_0(x) \in W^{1,\infty}(\Omega) \ \text{ with } c_0 > 0 \ \text{ in } \Omega, \\
\rho_0(x) \in W^{1,\infty}(\Omega) \ \text{ with } \rho_0 > 0 \ \text{ in } \Omega, \ \text{ and} \\
u_0 \in D(A^\alpha) \ \text{ for some } \alpha \in (\frac{3}{4}, 1),
\end{cases}
\tag{1.8}
$$

where A stands for the Stokes operator whose domain is defined as $D(A) := W^{2,2}(\Omega;\mathbb{R}^3)\bigcap W_0^{1,2}(\Omega;\mathbb{R}^3)\bigcap L^2_\sigma(\Omega;\mathbb{R}^3)$ with $L^2_\sigma(\Omega;\mathbb{R}^3) := \{\omega \in L^2(\Omega;\mathbb{R}^3) | \nabla\cdot\omega = 0\}$ [23]. Apart from that, the function D in (1.1) is assumed to satisfy

$$D \in C^{\varsigma}_{loc}([0,\infty))\bigcap C^2_{loc}((0,\infty)) \text{ and } D(s) \geq C_D s^{m-1} \text{ for each } s \geq 0 \tag{1.9}$$

with certain $\varsigma \in (0,1)$, $C_D > 0$ and $m > 1$. As for the given function ϕ, we suppose that it fulfills

$$\phi \in W^{2,\infty}(\Omega). \tag{1.10}$$

Here and in the sequel, we abbreviate $\bar{\omega} := \frac{1}{|\Omega|}\int_\Omega \omega$ for $\omega \in L^1(\Omega)$.

Within the context of these hypothesis, our main results with regard to global solvability and stabilization can be formulated as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let (1.9) be valid with some*

$$m > \frac{37}{33}. \tag{1.11}$$

Then for given ϕ satisfying (1.10) and each (n_0, c_0, ρ_0, u_0) complying with (1.8), one can find a quadruple of functions fulfilling

$$\begin{cases} n \in L^\infty(\Omega\times(0,\infty))\bigcap C^0([0,\infty);(W_0^{2,2}(\Omega))^*), \\ c \in \bigcap_{q>1} L^\infty((0,\infty);W^{1,q}(\Omega))\bigcap C^0(\bar{\Omega}\times[0,\infty))\bigcap C^{1,0}(\bar{\Omega}\times(0,\infty)), \\ \rho \in \bigcap_{q>1} L^\infty((0,\infty);W^{1,q}(\Omega))\bigcap C^0(\bar{\Omega}\times[0,\infty))\bigcap C^{1,0}(\bar{\Omega}\times(0,\infty)), \\ u \in L^\infty(\Omega\times(0,\infty);\mathbb{R}^3)\bigcap L^2_{loc}([0,\infty);W_0^{1,2}(\Omega;\mathbb{R}^3)\bigcap L^2_\sigma(\Omega;\mathbb{R}^3))\bigcap C^0(\bar{\Omega}\times[0,\infty);\mathbb{R}^3), \end{cases} \tag{1.12}$$

such that (n, c, ρ, u) solves problem (1.1), (1.6), (1.7) in the sense of Definition 7.1. Furthermore, for each $p \geq 1$ the solution (n, c, ρ, u) stabilizes to the spatially homogeneous equilibrium $(n_\infty, \rho_\infty, \rho_\infty, 0)$ in accordance with

$$\|n(\cdot,t)-n_\infty\|_{L^p(\Omega)} + \|c(\cdot,t)-\rho_\infty\|_{W^{1,\infty}(\Omega)} + \|\rho(\cdot,t)-\rho_\infty\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t\to\infty \tag{1.13}$$

with $n_\infty := \frac{1}{|\Omega|}\left\{\int_\Omega n_0 - \int_\Omega \rho_0\right\}_+$ and $\rho_\infty := \frac{1}{|\Omega|}\left\{\int_\Omega \rho_0 - \int_\Omega n_0\right\}_+$.

To the best of our knowledge, for chemotaxis-fluid systems modeling coral fertilization, there is few rigorous mathematical results on large time behavior of the solutions under the effects of porous medium type diffusion. From this point of view, our results can be referred as an enrichment in this respect.

The core of our analysis lies in combining an energy-based reasoning with maximal Sobolev regularity theory appropriately. More precisely, we rely on a Lyapunov quasi-energy functional to establish some basic a priori estimates (see Sect. 3), which makes it possible to improve the regularity of the component n properly by means of a first bootstrap iteration (see Sect. 4). With this as a starting point, a second recursive reasoning based on maximal Sobolev regularity arguments enables us to derive the desired estimates, so as to achieve the further regularity of the solutions by virtue of some well-established reasonings (see Sects. 4–6). The compactness provided by the further regularity underlies both the verification of global solvability by a standard extraction procedure and the detection of the spatially homogeneous equilibria that the solutions eventually approach by an Ehrling-type argument (see Sects. 7–8).

2. Preliminaries

Due to the presence of nonlinear diffusion D, it is essential to investigate the regularized problems of (1.1), (1.6) and (1.7) at first. In accordance with the regularization procedures used in [29,30], we can introduce the following approximate variants of (1.1), (1.6) and (1.7)

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \nabla \cdot (D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla \cdot (n_\varepsilon F'_\varepsilon(n_\varepsilon)\nabla c_\varepsilon) - \rho_\varepsilon F_\varepsilon(n_\varepsilon), & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + \rho_\varepsilon, & x \in \Omega,\ t > 0, \\ \rho_{\varepsilon t} + u_\varepsilon \cdot \nabla \rho_\varepsilon = \Delta \rho_\varepsilon - \rho_\varepsilon F_\varepsilon(n_\varepsilon), & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} + \nabla P_\varepsilon = \Delta u_\varepsilon + (n_\varepsilon + \rho_\varepsilon)\nabla \phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = \frac{\partial \rho_\varepsilon}{\partial \nu} = 0,\ u_\varepsilon = 0, & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x),\ \ c_\varepsilon(x,0) = c_0(x),\ \ \rho_\varepsilon(x,0) = \rho_0(x),\ \ u_\varepsilon(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}$$

Here, the family of functions $(D_\varepsilon)_{\varepsilon \in (0,1)}$ satisfies

$$\begin{aligned} &D_\varepsilon \in C^2([0,\infty)) \ \text{ such that } \ D_\varepsilon(s) \ge \varepsilon \ \text{ for each } \ s \ge 0 \ \text{ and all } \ \varepsilon \in (0,1), \ \text{ and that} \\ &D(s) \le D_\varepsilon(s) \le D(s) + 2\varepsilon \ \text{ for each } \ s \ge 0 \ \text{ and all } \ \varepsilon \in (0,1). \end{aligned} \tag{2.2}$$

Moreover, for each $\varepsilon \in (0,1)$, $(F_\varepsilon)_{\varepsilon \in (0,1)}$ is defined as

$$F_\varepsilon(s) := \int_0^s \vartheta_\varepsilon(\sigma)d\sigma, \quad s \ge 0 \tag{2.3}$$

with $(\vartheta_\varepsilon)_{\varepsilon \in (0,1)} \subset C_0^\infty([0,\infty))$ fulfilling

$$0 \le \vartheta_\varepsilon \le 1 \ \text{ in } \ [0,\infty), \ \ \vartheta_\varepsilon \equiv 1 \ \text{ in } \ [0, \frac{1}{\varepsilon}] \ \text{ and } \ \vartheta_\varepsilon \equiv 0 \ \text{ in } \ [\frac{2}{\varepsilon}, \infty), \tag{2.4}$$

and thus (2.3) implies

$$F_\varepsilon \in C^\infty([0,\infty)), \ \ 0 \le F_\varepsilon(s) \le s \ \text{ and } \ 0 \le F'_\varepsilon(s) \le 1 \ \text{ for all } \ s \ge 0 \ \text{ and each } \ \varepsilon \in (0,1) \tag{2.5}$$

as well as

$$F_\varepsilon(s) \to s \ \text{ for any } \ s \ge 0 \ \text{ and } \ F'_\varepsilon(s) \to 1 \ \text{ for any } \ s > 0 \ \text{ as } \ \varepsilon \to 0. \tag{2.6}$$

In the framework of the reasoning in [31, Lemma 2.1], which is based on a combination of the maximal principle and the fixed point arguments, the local solvability of (2.1) as well as an extensible criterion can be established by a suitable adaption.

Lemma 2.1. *Let $\varepsilon \in (0,1)$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain, and let (1.10) be valid. Then for (n_0, c_0, ρ_0, u_0) satisfying (1.8), there exist a maximal existence time $T_{\max,\varepsilon} \in (0, +\infty]$ as well as functions $n_\varepsilon, c_\varepsilon, \rho_\varepsilon, u_\varepsilon, P_\varepsilon$ fulfilling $n_\varepsilon > 0, c_\varepsilon > 0, \rho_\varepsilon > 0$ and*

$$\begin{cases} n_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon})), \\ c_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon})) \bigcap L^\infty([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)), \\ \rho_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon})) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon})) \bigcap L^\infty([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)), \\ u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{\max,\varepsilon}); \mathbb{R}^3) \bigcap C^{2,1}(\bar{\Omega} \times (0, T_{\max,\varepsilon}); \mathbb{R}^3), \\ P_\varepsilon \in C^{1,0}(\bar{\Omega} \times (0, T_{\max,\varepsilon})) \end{cases} \tag{2.7}$$

with some $q > 3$ and $\alpha \in (\frac{3}{4}, 1)$, such that the functions make up the unique solution of (2.1). In addition, if $T_{\max,\varepsilon} < \infty$, then

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|\rho_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \to \infty \tag{2.8}$$

as $t \to T_{\max,\varepsilon}$.

The presence of the reaction term $-\rho_\varepsilon F_\varepsilon(n_\varepsilon)$ along with the nonnegativity of n_ε and ρ_ε implies some basic but crucial estimates which follow from integration by parts and the maximal principle.

Lemma 2.2. *For each $\varepsilon \in (0,1)$, we have*

$$\frac{d}{dt}\int_\Omega n_\varepsilon(\cdot,t) \le 0, \quad \frac{d}{dt}\int_\Omega \rho_\varepsilon(\cdot,t) \le 0 \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{2.9}$$

and

$$\int_\Omega n_\varepsilon(\cdot,t) \le \int_\Omega n_0, \quad \int_\Omega \rho_\varepsilon(\cdot,t) \le \int_\Omega \rho_0 \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{2.10}$$

as well as

$$\int_\Omega n_\varepsilon(\cdot,t) - \int_\Omega \rho_\varepsilon(\cdot,t) = \int_\Omega n_0 - \int_\Omega \rho_0 \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{2.11}$$

$$\|\rho_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|\rho_0\|_{L^\infty(\Omega)} \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{2.12}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|c_0\|_{L^\infty(\Omega)}, \|\rho_0\|_{L^\infty(\Omega)}\} := M_c \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}). \tag{2.13}$$

Proof. Integrating both n_ε-equation and ρ_ε-equation over Ω, we obtain

$$\frac{d}{dt}\int_\Omega n_\varepsilon(\cdot,t) = \frac{d}{dt}\int_\Omega \rho_\varepsilon(\cdot,t) = -\int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon) \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{2.14}$$

which implies (2.9) thanks to the nonnegativity of $\rho_\varepsilon F_\varepsilon(n_\varepsilon)$, and thus (2.10) is valid upon integrations of (2.9) over $(0,t)$ for any $t \in (0, T_{\max,\varepsilon})$. Moreover, (2.11) is a straightforward result from integrating (2.14) over $(0,t)$ for each $t \in (0, T_{\max,\varepsilon})$. As for (2.12), it follows from an application of the maximum principle to ρ_ε-equation. With (2.12) at hand, we can finally infer (2.13) from the comparison principle. $\square$

Now, we provide an inequality constructed in [15, Lemma 2.7 iv] which plays a crucial role in detecting the evolution of $\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}$ in the sequel.

Lemma 2.3. *For each positive $\omega \in C^2(\bar{\Omega})$ satisfying $\frac{\partial \omega}{\partial \nu} = 0$ on $\partial\Omega$, there exist positive constants k_0 and M_0 such that*

$$-2\int_\Omega \frac{|\Delta\omega|^2}{\omega} + \int_\Omega \frac{|\nabla\omega|^2 \Delta\omega}{\omega^2} \le -k_0 \int_\Omega \omega |D^2 \ln\omega|^2 - k_0 \int_\Omega \frac{|\nabla\omega|^4}{\omega^3} + M_0 \int_\Omega \omega. \tag{2.15}$$

3. A quasi-energy structure of (2.1)

For deriving higher regularity properties, which is used as a foundation of the first iterative bootstrap procedure in the next section, we resort to constructing a quasi-energy functional.

Lemma 3.1. *For each $\varepsilon \in (0,1)$ and any $t \in (0, T_{\max,\varepsilon})$, we can find $M_1 > 0$ and $M_2 > 0$ such that*

$$\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon + \frac{2C_D}{m^2}\int_\Omega |\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}|^2 \le \zeta \int_\Omega |\nabla c_\varepsilon|^4 + M_1 \ \text{ if } \ \frac{10}{9} < m \le 2, \tag{3.1}$$

and

$$\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} + \frac{C_D(m-2)}{2(m-1)}\int_\Omega |\nabla (n_\varepsilon+\varepsilon)^{m-1}|^2 \le \zeta \int_\Omega |\nabla c_\varepsilon|^4 + M_2 \ \text{ if } \ m > 2 \tag{3.2}$$

with arbitrary $\zeta > 0$.

Proof. In view of n_ε-equation in (2.1), $\nabla \cdot u_\varepsilon = 0$, (1.9) and the nonnegativity of $\rho_\varepsilon F_\varepsilon(n_\varepsilon)$, we integrate by parts to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega n_\varepsilon \ln n_\varepsilon =& \int_\Omega \ln n_\varepsilon \big(\nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon) - \nabla\cdot(n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon) - \rho_\varepsilon F_\varepsilon(n_\varepsilon) \\ & - u_\varepsilon \cdot \nabla n_\varepsilon\big) - \int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon) \\ \le & - C_D \int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \int_\Omega F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon - \int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon)\ln n_\varepsilon\end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max,\varepsilon})$. From (2.12) and the fact that

$$s \ln s \geq -\frac{1}{e} \qquad \text{for all } \; s > 0, \tag{3.4}$$

we can infer that

$$-\int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon) \ln n_\varepsilon \leq -\int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon) \ln F_\varepsilon(n_\varepsilon) \leq \frac{|\Omega|}{e} \|\rho_0\|_{L^\infty(\Omega)} \tag{3.5}$$

for all $t \in (0, T_{\max,\varepsilon})$. Invoking Young's inequality, we also have

$$\begin{aligned}
&\int_\Omega F_\varepsilon'(n_\varepsilon) \nabla n_\varepsilon \cdot \nabla c_\varepsilon \\
\leq &\frac{C_D}{4} \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 + \frac{1}{C_D} \int_\Omega (n_\varepsilon + \varepsilon)^{2-m} |\nabla c_\varepsilon|^2 \\
\leq &\frac{C_D}{4} \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 + \zeta \int_\Omega |\nabla c_\varepsilon|^4 + \frac{1}{4C_D^2 \zeta} \int_\Omega (n_\varepsilon + \varepsilon)^{4-2m}
\end{aligned} \tag{3.6}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $\zeta > 0$ is arbitrary. In the case when $\frac{10}{9} < m < \frac{3}{2}$, we actually have $\frac{2}{m} < \frac{4(2-m)}{m} < 6$, and thereby an application of the Gagliardo–Nirenberg inequality along with (2.10) provides $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}
\int_\Omega (n_\varepsilon + \varepsilon)^{4-2m} = &\left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{4(2-m)}{m}}(\Omega)}^{\frac{4(2-m)}{m}} \\
\leq &C_1 \left(\left\| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^2(\Omega)}^{a} \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)}^{1-a} + \left\| (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right\|_{L^{\frac{2}{m}}(\Omega)} \right)^{\frac{4(2-m)}{m}} \\
\leq &C_2 \left(\int_\Omega \left| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right|^2 + 1 \right)^{\frac{9-6m}{3m-1}}
\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $a = \frac{3m(3-2m)}{2(2-m)(3m-1)} \in (0, 1)$. Since $m > \frac{10}{9}$ implies $\frac{9-6m}{3m-1} < 1$, we employ Young's inequality once more to derive

$$\begin{aligned}
\frac{1}{4C_D^2 \zeta} \int_\Omega (n_\varepsilon + \varepsilon)^{4-2m} \leq &\frac{C_2}{4C_D^2 \zeta} \left(\int_\Omega \left| \nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}} \right|^2 + 1 \right)^{\frac{9-6m}{3m-1}} \\
\leq &\frac{C_D}{m^2} \int_\Omega |\nabla (n_\varepsilon + \varepsilon)^{\frac{m}{2}}|^2 + C_3 \\
= &\frac{C_D}{4} \int_\Omega (n_\varepsilon + \varepsilon)^{m-2} |\nabla n_\varepsilon|^2 + C_3
\end{aligned} \tag{3.8}$$

with some $C_3 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Whereas for $\frac{3}{2} \le m \le 2$, it is clear that $0 \le 4 - 2m \le 1$, whence the Hölder inequality combined with (2.10) entails

$$\frac{1}{4C_D^2\zeta}\int_\Omega (n_\varepsilon+\varepsilon)^{4-2m} \le C_4 \tag{3.9}$$

with certain $C_4 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Letting $C_5 := \max\{C_3, C_4\}$, we deduce from (3.8) and (3.9) that

$$\frac{1}{4C_D^2\zeta}\int_\Omega (n_\varepsilon+\varepsilon)^{4-2m} \le \frac{C_D}{4}\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + C_5 \tag{3.10}$$

for all $t \in (0, T_{\max,\varepsilon})$. Thereupon, a combination of (3.6) and (3.10) implies

$$\int_\Omega F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \le \frac{C_D}{2}\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|^2 + \zeta\int_\Omega |\nabla c_\varepsilon|^4 + C_5 \tag{3.11}$$

for all $t \in (0, T_{\max,\varepsilon})$. Substituting (3.3) and (3.5) into (3.11), we achieve (3.1) with the choice of $M_1 := \frac{|\Omega|}{e}\|\rho_0\|_{L^\infty(\Omega)} + C_5$.

Next, for $m > 2$, we test n_ε-equation against $(m-1)(n_\varepsilon+\varepsilon)^{m-2}$ and obtain from (1.9) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} =& -(m-1)(m-2)\int_\Omega (n_\varepsilon+\varepsilon)^{m-3}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\\
&+(m-1)(m-2)\int_\Omega n_\varepsilon(n_\varepsilon+\varepsilon)^{m-3}F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
&-(m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}\rho_\varepsilon F_\varepsilon(n_\varepsilon)\\
\le& -C_D(m-1)(m-2)\int_\Omega (n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2\\
&+(m-1)(m-2)\int_\Omega n_\varepsilon(n_\varepsilon+\varepsilon)^{m-3}F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
&-(m-1)\int_\Omega (n_\varepsilon+\varepsilon)^{m-2}\rho_\varepsilon F_\varepsilon(n_\varepsilon)
\end{aligned} \tag{3.12}$$

for all $t \in (0, T_{\max,\varepsilon})$, where in conjunction with (2.5) two applications of Young's inequality yield

$$
\begin{aligned}
&(m-1)(m-2)\int_\Omega n_\varepsilon(n_\varepsilon+\varepsilon)^{m-3}F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon\\
\le&(m-1)(m-2)\int_\Omega(n_\varepsilon+\varepsilon)^{m-2}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon|\\
\le&\frac{C_D(m-1)(m-2)}{2}\int_\Omega(n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2+\frac{(m-1)(m-2)}{2C_D}\int_\Omega|\nabla c_\varepsilon|^2\\
\le&\frac{C_D(m-1)(m-2)}{2}\int_\Omega(n_\varepsilon+\varepsilon)^{2m-4}|\nabla n_\varepsilon|^2+\zeta\int_\Omega|\nabla c_\varepsilon|^4+\frac{(m-1)^2(m-2)^2}{16\zeta C_D^2}
\end{aligned}
\tag{3.13}
$$

for all $t\in(0,T_{\max,\varepsilon})$. Thanks to the nonnegativity of $(m-1)\int_\Omega(n_\varepsilon+\varepsilon)^{m-2}\rho_\varepsilon F_\varepsilon(n_\varepsilon)$, (3.2) thus follows from (3.12) and (3.13) with $M_2:=\frac{(m-1)^2(m-2)^2}{16\zeta C_D^2}$.

Aided by the uniform L^∞-bounds in (2.13), the integral $\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}$ generated in the following detection does not only make it possible to offset the integral appearing on the right-hand side of both (3.1) and (3.2) but also provides a chance to obtain the spatio-temporal estimates of $\int_t^{t+1}\int_\Omega|\nabla c_\varepsilon|^4$ in the sequel. □

Lemma 3.2. *For all $\varepsilon\in(0,1)$ and each $t\in(0,T_{\max,\varepsilon})$, it is valid that*

$$
\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{3k_0}{4}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\le\frac{4M_c}{k_0}\int_\Omega|\nabla u_\varepsilon|^2+\int_\Omega\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon}+M_0M_c|\Omega|
\tag{3.14}
$$

with M_c defined as in Lemma 2.2 and k_0 as well as M_0 provided in Lemma 2.3.

Proof. Due to $c_\varepsilon>0$ guaranteed by Lemma 2.1, it follows from c_ε-equation in (2.1) and integration by parts that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}=&2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla c_{\varepsilon t}}{c_\varepsilon}-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}c_{\varepsilon t}\\
=&2\int_\Omega\frac{\nabla c_\varepsilon}{c_\varepsilon}\cdot\nabla\left(\Delta c_\varepsilon-c_\varepsilon+\rho_\varepsilon-u_\varepsilon\cdot\nabla c_\varepsilon\right)\\
&-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\left(\Delta c_\varepsilon-c_\varepsilon+\rho_\varepsilon-u_\varepsilon\cdot\nabla c_\varepsilon\right)\\
=&-2\int_\Omega\frac{|\Delta c_\varepsilon|^2}{c_\varepsilon}+\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+2\int_\Omega\frac{\nabla c_\varepsilon\cdot\nabla\rho_\varepsilon}{c_\varepsilon}\\
&-\int_\Omega\frac{\rho_\varepsilon|\nabla c_\varepsilon|^2}{c_\varepsilon^2}+2\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)
\end{aligned}
\tag{3.15}
$$

for all $t \in (0, T_{\max,\varepsilon})$. By Young's inequality, we obtain

$$2\int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla \rho_\varepsilon}{c_\varepsilon} \le \int_\Omega \frac{\rho_\varepsilon |\nabla c_\varepsilon|^2}{c_\varepsilon^2} + \int_\Omega \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} \tag{3.16}$$

for all $t \in (0, T_{\max,\varepsilon})$. In view of $\nabla \cdot u_\varepsilon = 0$, we integrate by parts again to derive

$$\begin{aligned}
2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon \cdot \nabla c_\varepsilon) =& 2\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) - 2\int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla(u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon} \\
=& 2\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) - 2\int_\Omega \frac{\nabla c_\varepsilon \cdot \nabla u_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon} - 2\int_\Omega \frac{u_\varepsilon \cdot D^2 c_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon}
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, where

$$\begin{aligned}
\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) =& -\int_\Omega |\nabla c_\varepsilon|^2 u_\varepsilon \cdot \nabla \cdot \left(\frac{1}{c_\varepsilon}\right) = \int_\Omega \frac{u_\varepsilon \cdot \nabla |\nabla c_\varepsilon|^2}{c_\varepsilon} \\
=& 2\int_\Omega \frac{u_\varepsilon \cdot D^2 c_\varepsilon \cdot \nabla c_\varepsilon}{c_\varepsilon}
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, whence again in light of Young's inequality, we have

$$\begin{aligned}
2\int_\Omega \frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon \cdot \nabla c_\varepsilon) - \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon \cdot \nabla c_\varepsilon) =& -2\int_\Omega \frac{\nabla c_\varepsilon \cdot (\nabla u_\varepsilon \cdot \nabla c_\varepsilon)}{c_\varepsilon} \\
\le& \frac{k_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{4}{k_0}\int_\Omega c_\varepsilon |\nabla u_\varepsilon|^2 \\
\le& \frac{k_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{4M_c}{k_0}\int_\Omega |\nabla u_\varepsilon|^2
\end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max,\varepsilon})$. Apart from that, Lemma 2.3 together with (2.13) implies

$$\begin{aligned}
-2\int_\Omega \frac{|\Delta c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon \le& -k_0\int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 - k_0\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + M_0\int_\Omega c_\varepsilon \\
\le& -k_0\int_\Omega c_\varepsilon |D^2 \ln c_\varepsilon|^2 - k_0\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + M_0 M_c |\Omega|
\end{aligned} \tag{3.18}$$

for all $t \in (0, T_{\max,\varepsilon})$. Inserting (3.16)-(3.18) into (3.15), we obtain

$$\frac{d}{dt}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \le -k_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2 - \frac{3k_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{4M_c}{k_0}\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} + M_0M_c|\Omega| \tag{3.19}$$

for all $t \in (0, T_{\max,\varepsilon})$, and thus (3.14) can be achieved immediately by dropping the integral $k_0\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2$ thanks to its nonnegativity.

Now, let us detect the evolution of $\int_\Omega \rho_\varepsilon\ln\rho_\varepsilon$ and $\int_\Omega |u_\varepsilon|^2$, respectively. □

Lemma 3.3. *For any $\varepsilon \in (0, 1)$ and all $t \in (0, T_{\max,\varepsilon})$, we have*

$$\frac{d}{dt}\int_\Omega \rho_\varepsilon\ln\rho_\varepsilon + \int_\Omega \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} \le \frac{1}{e}\int_\Omega n_0, \tag{3.20}$$

and moreover, there exists $M_3 > 0$ such that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le \begin{cases} \dfrac{k_0}{6M_c}\cdot\dfrac{D_1}{m^2}\displaystyle\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + M_3 & \text{if } \dfrac{10}{9} < m \le 2,\\ \dfrac{k_0}{6M_c}\cdot\dfrac{D_1(m-2)}{4(m-1)}\displaystyle\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + M_3 & \text{if } m > 2,\end{cases} \tag{3.21}$$

where $M_c > 0$ is provided by Lemma 2.2.

Proof. Upon integration by parts, we derive from ρ_ε-equation in (2.1), $\nabla\cdot u_\varepsilon = 0$ and the nonnegativity of $\int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon$ that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho_\varepsilon\ln\rho_\varepsilon &= \int_\Omega \ln\rho_\varepsilon\cdot\Big(\Delta\rho_\varepsilon - \rho_\varepsilon F_\varepsilon(n_\varepsilon) - u_\varepsilon\cdot\nabla\rho_\varepsilon\Big) - \int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\\ &= -\int_\Omega \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} - \int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\ln\rho_\varepsilon - \int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\\ &\le -\int_\Omega \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} - \int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\ln\rho_\varepsilon\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, where from (3.4), (2.5) and (2.10), we can infer that

$$-\int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\ln\rho_\varepsilon \le \frac{1}{e}\int_\Omega F_\varepsilon(n_\varepsilon) \le \frac{1}{e}\int_\Omega n_\varepsilon \le \frac{1}{e}\int_\Omega n_0$$

for all $t \in (0, T_{\max,\varepsilon})$, and thus (3.20) holds.

Next, according to the reasoning of [32, Lemma 3.5], a standard testing procedure with u_ε as a testing function applied to u_ε-equation in (2.1) entails

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 = \int_\Omega (n_\varepsilon + \rho_\varepsilon)u_\varepsilon \cdot \nabla\phi \tag{3.22}$$

for all $t \in (0, T_{\max,\varepsilon})$. Due to $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ in the three-dimensional context, we can find $C_1 > 0$ satisfying

$$\|u_\varepsilon\|_{L^6(\Omega)} \le C_1\|\nabla u_\varepsilon\|_{L^2(\Omega)} \tag{3.23}$$

for all $t \in (0, T_{\max,\varepsilon})$. It follows from the Hölder inequality, (3.23), Minkwoski's inequality, Young's inequality, (2.12) as well as $(\xi+\eta)^2 \le 2(\xi^2+\eta^2)$ for all $\xi \ge 0$ and $\eta \ge 0$ that

$$\begin{aligned}
&\int_\Omega (n_\varepsilon + \rho_\varepsilon)u_\varepsilon \cdot \nabla\phi \\
\le&\|u_\varepsilon\|_{L^6(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}\|n_\varepsilon+\rho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \\
\le&C_1\|\nabla u_\varepsilon\|_{L^2(\Omega)}\|\nabla\phi\|_{L^\infty(\Omega)}(\|\rho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} + \|n_\varepsilon+\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}) \\
\le&\frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}(\|\rho_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} + \|n_\varepsilon+\varepsilon\|_{L^{\frac{6}{5}}(\Omega)})^2 \\
\le&\frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}(\|\rho_\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} + \|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)}) \\
\le&\frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|^{\frac{5}{3}} + C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)}
\end{aligned} \tag{3.24}$$

for all $t \in (0, T_{\max,\varepsilon})$. For $\frac{10}{9} < m \le 2$, we have $\frac{2}{m} < \frac{12}{5m} < 6$, which allows for an application of the Gagliardo–Nirenberg inequality along with (2.10) to provide $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} =&\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\|^{\frac{4}{m}}_{L^{\frac{12}{5m}}(\Omega)} \\
\le&C_2\left(\|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\|^{\frac{m}{2(3m-1)}}_{L^2(\Omega)}\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\|^{\frac{5m-2}{2(3m-1)}}_{L^{\frac{2}{m}}(\Omega)} + \|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}\right)^{\frac{4}{m}} \\
\le&C_3\left(\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + 1\right)^{\frac{1}{3m-1}}
\end{aligned} \tag{3.25}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $\frac{1}{3m-1} < 1$ due to $m > \frac{10}{9}$. Therefore, by Young's inequality, we have

$$\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \le \zeta_1\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + C_4 \tag{3.26}$$

for all $t \in (0, T_{\max,\varepsilon})$, where $\zeta_1 > 0$ is arbitrary and $C_4 := C_4(\zeta_1) > 0$. With $\zeta_1 := \frac{1}{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}} \cdot \frac{k_0}{12M_c} \cdot \frac{C_D}{m^2}$, substituting (3.26) into (3.24) yields

$$\int_\Omega (n_\varepsilon+\rho_\varepsilon)u_\varepsilon\cdot\nabla\phi \le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{k_0}{12M_c}\cdot\frac{C_D}{m^2}\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + C_5 \tag{3.27}$$

with $C_5 := C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}(C_4+\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|^{\frac{5}{3}})$ for all $t\in(0,T_{\max,\varepsilon})$, and thereby it follows from (3.22) and (3.27) that

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le \frac{k_0}{6M_c}\cdot\frac{C_D}{m^2}\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + 2C_5 \tag{3.28}$$

for all $t\in(0,T_{\max,\varepsilon})$. Whereas for $m>2$, it is clear that $\frac{1}{m-1}<\frac{6}{5(m-1)}<6$, whence again by the Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned}
\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} &= \left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|^{\frac{2}{m-1}}_{L^{\frac{6}{5(m-1)}}(\Omega)}\\
&\le C_6\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right\|^{\frac{6m-11}{6m-7}}_{L^2(\Omega)}\left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|^{\frac{4}{6m-7}}_{L^{\frac{1}{m-1}}(\Omega)} + \left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}\right)^{\frac{2}{m-1}}\\
&\le C_7\left(\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2+1\right)^{\frac{1}{m-1}\cdot\frac{6m-11}{6m-7}}
\end{aligned} \tag{3.29}$$

for all $t\in(0,T_{\max,\varepsilon})$. Since $m>2$ implies $0<\frac{1}{m-1}\cdot\frac{6m-11}{6m-7}<1$, we make use of Young's inequality to derive

$$\|n_\varepsilon+\varepsilon\|^2_{L^{\frac{6}{5}}(\Omega)} \le \zeta_2\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + C_8 \tag{3.30}$$

with $\zeta_2 := \frac{1}{C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}}\cdot\frac{k_0}{12M_c}\cdot\frac{C_D(m-2)}{4(m-1)}$ and $C_8 := C_8(\zeta_2)>0$ for all $t\in(0,T_{\max,\varepsilon})$, which along with (3.26) implies

$$\int_\Omega (n_\varepsilon+\rho_\varepsilon)u_\varepsilon\cdot\nabla\phi \le \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \frac{k_0}{12M_c}\cdot\frac{C_D(m-2)}{4(m-1)}\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + C_9 \tag{3.31}$$

for all $t\in(0,T_{\max,\varepsilon})$, where $C_9 := C_1^2\|\nabla\phi\|^2_{L^\infty(\Omega)}(C_8+\|\rho_0\|^2_{L^\infty(\Omega)}|\Omega|^{\frac{5}{3}})$. Combining (3.22) with (3.31), we attain

$$\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2 \le \frac{k_0}{6M_c}\cdot\frac{C_D(m-2)}{4(m-1)}\int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + 2C_9 \tag{3.32}$$

for all $t\in(0,T_{\max,\varepsilon})$. As a consequence of (3.28) and (3.32), (3.21) is valid by a choice of $M_3 := \max\{2C_5, 2C_9\}$.

Now, the quasi-energy structure mentioned above can be constructed by collecting Lemmas 3.1–3.3, which yields the following estimates. $\square$

Lemma 3.4. *Let $m > \frac{10}{9}$. Then one can find $C > 0$ independent of $\varepsilon \in (0,1)$ such that*

$$\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |u_\varepsilon|^2 \le C \quad \textit{for all} \quad t \in (0, T_{\max,\varepsilon}) \tag{3.33}$$

and

$$\begin{aligned}&\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^4 + \int_t^{t+1}\int_\Omega |\nabla \rho_\varepsilon|^2 + \int_t^{t+1}\int_\Omega |\nabla u_\varepsilon|^2 + \begin{cases} \displaystyle\int_t^{t+1}\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 & \textit{for } \ \frac{10}{9} < m \le 2 \\ \displaystyle\int_t^{t+1}\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{m-1}\right|^2 & \textit{for } \ m > 2 \end{cases} \\ &\le C \quad \textit{for each} \quad t \in (0, T_{\max,\varepsilon}-1).\end{aligned} \tag{3.34}$$

Proof. In the case when $\frac{10}{9} < m \le 2$, we pick $\zeta = \frac{k_0}{2\|c_0\|^3_{L^\infty(\Omega)}}$ in Lemma 3.1 and deduce from Lemmas 3.1–3.3 that

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\int_\Omega \rho_\varepsilon \ln \rho_\varepsilon + \frac{6M_c}{k_0}\int_\Omega |u_\varepsilon|^2\right\} \\ &+ \frac{C_D}{m^2}\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{k_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} + \frac{M_c}{k_0}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le M_1 + M_0M_c|\Omega| + \frac{2}{e}\int_\Omega n_0 + \frac{6M_cM_3}{k_0}\end{aligned} \tag{3.35}$$

for all $t \in (0, T_{\max,\varepsilon})$ with $M_3 > 0$ as given in Lemma 3.3. Similarly, for $m > 2$, Lemma 3.1 with $\zeta = \frac{k_0}{2\|c_0\|^3_{L^\infty(\Omega)}}$ together with Lemmas 3.2 and 3.3 entails

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\int_\Omega \rho_\varepsilon \ln \rho_\varepsilon + \frac{6M_c}{k_0}\int_\Omega |u_\varepsilon|^2\right\} \\ &+ \frac{C_D(m-2)}{4(m-1)}\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{m-1}\right|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{k_0}{4}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} + \frac{M_c}{k_0}\int_\Omega |\nabla u_\varepsilon|^2 \\ &\le M_2 + M_0M_c|\Omega| + \frac{2}{e}\int_\Omega n_0 + \frac{6M_cM_3}{k_0}\end{aligned} \tag{3.36}$$

for all $t \in (0, T_{\max,\varepsilon})$. Choosing $M := \max\left\{1, \frac{m^2}{C_D}, \frac{4(m-1)}{C_D(m-2)}, \frac{4}{k_0}, \frac{k_0}{M_c}, M_1 + M_2 + M_0M_c|\Omega| + \frac{2}{e}\int_\Omega n_0 + \frac{6M_cM_3}{k_0}\right\}$, we infer from (3.35) and (3.36) that

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega n_\varepsilon \ln n_\varepsilon + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\int_\Omega \rho_\varepsilon \ln \rho_\varepsilon + \frac{6M_c}{k_0}\int_\Omega |u_\varepsilon|^2\right\}\\&+\frac{1}{M}\left\{\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} + \int_\Omega |\nabla u_\varepsilon|^2\right\}\\&\le M\end{aligned} \tag{3.37}$$

for all $t \in (0, T_{\max,\varepsilon})$, and that

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\int_\Omega \rho_\varepsilon \ln \rho_\varepsilon + \frac{6M_c}{k_0}\int_\Omega |u_\varepsilon|^2\right\}\\&+\frac{1}{M}\left\{\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{m-1}\right|^2 + \int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} + \int_\Omega |\nabla u_\varepsilon|^2\right\}\\&\le M\end{aligned} \tag{3.38}$$

for all $t \in (0, T_{\max,\varepsilon})$, respectively. In light of (2.10), the facts that $\xi \ln \xi \le \frac{3}{2}\xi^{\frac{5}{3}}$ for all $\xi \ge 0$, and that $\frac{2}{m} < \frac{10}{3m} < 6$ as well as $0 < \frac{2}{3m-1} < 1$ due to $\frac{10}{9} < m \le 2$, we invoke the Gagliardo–Nirenberg inequality and Young's inequality to attain

$$\begin{aligned}\int_\Omega n_\varepsilon \ln n_\varepsilon &\le \frac{3}{2}\int_\Omega n_\varepsilon^{\frac{5}{3}}\\&\le \frac{3}{2}\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{10}{3m}}(\Omega)}^{\frac{10}{3m}}\\&\le C_1\left(\left\|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^2(\Omega)}^{\frac{6m}{5(3m-1)}}\left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{9m-5}{5(3m-1)}} + \left\|(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right\|_{L^{\frac{2}{m}}(\Omega)}\right)^{\frac{10}{3m}}\\&\le C_2\left(\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + 1\right)^{\frac{2}{3m-1}}\\&\le C_2\int_\Omega \left|\nabla (n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + C_2 + 1\end{aligned} \tag{3.39}$$

with some $C_1 > 0$ and $C_2 > 0$ for all $t \in (0, T_{\max,\varepsilon})$. Likewise, we can find positive constants C_3 and C_4 fulfilling

$$\int_\Omega \rho_\varepsilon \ln \rho_\varepsilon \le \frac{3}{2}\int_\Omega \rho_\varepsilon^{\frac{5}{3}}$$

$$
\begin{aligned}
&= \frac{3}{2}\|\rho_\varepsilon^{\frac{1}{2}}\|^{\frac{10}{3}}_{L^{\frac{10}{3}}(\Omega)} \\
&\le C_3\left(\|\nabla\rho_\varepsilon^{\frac{1}{2}}\|^2_{L^2(\Omega)}\|\rho_\varepsilon^{\frac{1}{2}}\|^{\frac{4}{3}}_{L^2(\Omega)} + \|\rho_\varepsilon^{\frac{1}{2}}\|^{\frac{10}{3}}_{L^2(\Omega)}\right) \qquad (3.40)\\
&\le C_4 \int_\Omega \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} + C_4
\end{aligned}
$$

for all $t \in (0, T_{\max,\varepsilon})$. Whereas for $m > 2$, another application of the Gagliardo–Nirenberg inequality along with Young's inequality provides $C_5 > 0$ and $C_6 > 0$ such that

$$
\begin{aligned}
\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} =& \left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|_{L^1(\Omega)} \\
\le& C_5\left(\left\|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right\|^{\frac{6m-12}{6m-7}}_{L^2(\Omega)}\left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|^{\frac{5}{6m-7}}_{L^{\frac{1}{m-1}}(\Omega)} + \left\|(n_\varepsilon+\varepsilon)^{m-1}\right\|_{L^{\frac{1}{m-1}}(\Omega)}\right) \\
\le& C_6 \int_\Omega \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + C_6 \qquad (3.41)
\end{aligned}
$$

for all $t \in (0, T_{\max,\varepsilon})$. In addition, thanks to $u_\varepsilon = 0$ on $\partial\Omega$, we have the Poincaré inequality

$$
\int_\Omega |u_\varepsilon|^2 \le C_P \int_\Omega |\nabla u_\varepsilon|^2 \qquad (3.42)
$$

with certain $C_P > 0$ for all $t \in (0, T_{\max,\varepsilon})$.

Now, we define

$$
g_\varepsilon(t) := \int_\Omega \left\{ n_\varepsilon \ln n_\varepsilon + \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\rho_\varepsilon \ln\rho_\varepsilon + \frac{6M_c}{k_0}|u_\varepsilon|^2 \right\}(\cdot, t)
$$

for all $t \in (0, T_{\max,\varepsilon})$ and

$$
h_\varepsilon(t) := \int_\Omega \left\{ \left|\nabla(n_\varepsilon+\varepsilon)^{\frac{m}{2}}\right|^2 + \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon} + |\nabla u_\varepsilon|^2 \right\}(\cdot, t)
$$

for all $t \in (0, T_{\max,\varepsilon})$, as well as

$$
y_\varepsilon(t) := \int_\Omega \left\{ (n_\varepsilon+\varepsilon)^{m-1} + \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + 2\rho_\varepsilon \ln\rho_\varepsilon + \frac{6M_c}{k_0}|u_\varepsilon|^2 \right\}(\cdot, t)
$$

for all $t \in (0, T_{\max,\varepsilon})$, and

$$z_\varepsilon(t) := \int\limits_\Omega \left\{ \left|\nabla(n_\varepsilon+\varepsilon)^{m-1}\right|^2 + \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{|\nabla \rho_\varepsilon|^2}{\rho_\varepsilon} + |\nabla u_\varepsilon|^2 \right\}(\cdot,t)$$

for all $t \in (0, T_{\max,\varepsilon})$. Collecting (3.39)–(3.42), we can obtain $C_8 > 0$ and $C_9 > 0$ such that

$$g_\varepsilon(t) \le C_8 h_\varepsilon(t) + C_8$$

for all $t \in (0, T_{\max,\varepsilon})$, as well as

$$y_\varepsilon(t) \le C_9 z_\varepsilon(t) + C_9$$

for all $t \in (0, T_{\max,\varepsilon})$, which in conjunction with (3.37) and (3.38) entails that

$$g'_\varepsilon(t) + \frac{1}{2MC_8} g_\varepsilon(t) + \frac{1}{2M} h_\varepsilon(t) \le \frac{1}{2M} + M := C_{10} \tag{3.43}$$

for all $t \in (0, T_{\max,\varepsilon})$, and that

$$y'_\varepsilon(t) + \frac{1}{2MC_8} y_\varepsilon(t) + \frac{1}{2M} z_\varepsilon(t) \le C_{10} \tag{3.44}$$

for all $t \in (0, T_{\max,\varepsilon})$. Therefore, by an ODE comparison argument, we achieve

$$g_\varepsilon(t) \le C_{11} := \max\{ \sup_{\varepsilon\in(0,1)} g_\varepsilon(0), 2MC_8C_{10}\}$$

for all $t \in (0, T_{\max,\varepsilon})$, and

$$y_\varepsilon(t) \le C_{12} := \max\{ \sup_{\varepsilon\in(0,1)} y_\varepsilon(0), 2MC_9C_{10}\}$$

for all $t \in (0, T_{\max,\varepsilon})$, which combined with (2.13) implies (3.33). In the final, (3.34) follows directly from (2.13) by integrating (3.43) and (3.44) over $(t, t+1)$ for any $t \in (0, T_{\max,\varepsilon} - 1)$, respectively. □

4. A priori estimates for n_ε

4.1. Preparation for iterations

In this portion, we aim to clarify the relationship between the regularity index of n_ε and that of ∇c_ε by means of a similar procedure as performed in [30, Lemma 4.1], which is regarded as a recursion formula for iterations in the sequel.

Lemma 4.1. *Let $m > 1$, $p_* \ge 1$ and $q \ge 2$ fulfill*

$$p \le 2q(m - 1 + \frac{p_*}{3}) - (m - 1 + \frac{2p_*}{3}). \tag{4.1}$$

Then for each $L > 0$ there exists $C = C(p, p_*, q, L) > 0$ such that if for certain $\varepsilon \in (0, 1)$, both

$$\int_\Omega n_\varepsilon^{p_*}(\cdot, t) \le L \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{4.2}$$

and

$$\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon(\cdot, t)|^{2q} \le L \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{4.3}$$

are valid, then we have

$$\int_\Omega n_\varepsilon^{p}(\cdot, t) \le C \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{4.4}$$

Proof. It follows from a direct computation that $2q(m - 1 + \frac{p_*}{3}) - (m - 1 + \frac{2p_*}{3}) = (2q - 1)(m - 1) + \frac{2p_*}{3}(q - 1) \ge 3(m - 1)$, which allows for a hypothesis that $p \ge \max\{p_*, m - 1\}$ without loss of generality. In light of (1.9), (2.2), (2.5) as well as the nonnegativity of n_ε and ρ_ε, we test n_ε-equation by pn_ε^{p-1} and make use of Young's inequality to obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n_\varepsilon^p =& - p(p-1)\int_\Omega D_\varepsilon(n_\varepsilon)n_\varepsilon^{p-2}|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega n_\varepsilon^{p-1}F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&- p\int_\Omega \rho_\varepsilon n_\varepsilon^{p-1}F_\varepsilon(n_\varepsilon) \\
\le& - C_D p(p-1)\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2 + p(p-1)\int_\Omega n_\varepsilon^{p-1}F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
\le& - \frac{C_D p(p-1)}{2}\int_\Omega n_\varepsilon^{m+p-3}|\nabla n_\varepsilon|^2 + \frac{p(p-1)}{2C_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 \\
=& - \frac{2C_D p(p-1)}{(m+p-1)^2}\int_\Omega \left|\nabla n_\varepsilon^{\frac{m+p-1}{2}}\right|^2 + \frac{p(p-1)}{2C_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2
\end{aligned} \tag{4.5}$$

for all $t \in (0, T_{\max,\varepsilon})$, where by the Hölder inequality,

$$\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 \le \left\{\int_\Omega n_\varepsilon^{(p-m+1)\cdot\frac{q}{q-1}}\right\}^{\frac{q-1}{q}} \cdot \left\{\int_\Omega |\nabla c_\varepsilon|^{2q}\right\}^{\frac{1}{q}} \tag{4.6}$$

holds for all $t \in (0, T_{\max,\varepsilon})$. In the case when $(p - m + 1)\cdot\frac{q}{q-1} \le p_*$, the Hölder inequality along with (4.2) provides $C_1 > 0$ such that

$$\left\{\int_\Omega n_\varepsilon^{(p-m+1)\cdot\frac{q}{q-1}}\right\}^{\frac{q-1}{q}} \leq C_1 \tag{4.7}$$

for all $t \in (0, T_{\max,\varepsilon})$. If, inversely, $(p-m+1)\cdot\frac{q}{q-1} > p_*$, then from the assumption $p \geq \max\{p_*, m-1\}$ and the fact that $\frac{q}{q-1} \leq 2$ due to $q \geq 2$, we deduce that

$$\frac{2(p-m+1)}{p+m-1}\cdot\frac{q}{q-1} \leq \frac{2q}{q-1} \leq 4 < 6.$$

This implies $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q(p-m+1)}{(q-1)(p+m-1)}}(\Omega) \hookrightarrow L^{\frac{2p_*}{p+m-1}}(\Omega)$, whereupon combining with (4.2), we employ the Gagliardo–Nirenberg inequality to obtain $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
\left\{\int_\Omega n_\varepsilon^{(p-m+1)\cdot\frac{q}{q-1}}\right\}^{\frac{q-1}{q}} =& \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2q(p-m+1)}{(q-1)(p+m-1)}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}} \\
\leq& C_2 \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{q[3(p+m-1)-p_*]}} \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p_*}{p+m-1}}(\Omega)}^{(1-a)\cdot\frac{2(p-m+1)}{p+m-1}} \\
&+ C_2 \left\| n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^{\frac{2p_*}{p+m-1}}(\Omega)}^{\frac{2(p-m+1)}{p+m-1}} \\
\leq& C_3 \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{q[3(p+m-1)-p_*]}} + C_3
\end{aligned} \tag{4.8}$$

for all $t \in (0, T_{\max,\varepsilon})$ with $a := \frac{3(p+m-1)[q(p-m+1-p_*)+p_*]}{q(p-m+1)[3(p+m-1)-p_*]} \in [0,1]$. In conjunction with (4.6) and (4.8), an application of Young's inequality entails

$$\begin{aligned}
\frac{p(p-1)}{2C_D}\int_\Omega n_\varepsilon^{p-m+1}|\nabla c_\varepsilon|^2 \leq& C_4 \left\{ \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{q[3(p+m-1)-p_*]}} + 1 \right\} \cdot \left\{ \int_\Omega |\nabla c_\varepsilon|^{2q} \right\}^{\frac{1}{q}} \\
\leq& 2^{-\frac{q}{q-1}} \cdot \frac{2C_D p(p-1)}{(m+p-1)^2} \left\{ \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{q[3(p+m-1)-p_*]}} + 1 \right\}^{\frac{q}{q-1}} \\
&+ C_5 \int_\Omega |\nabla c_\varepsilon|^{2q} \\
\leq& \frac{C_D p(p-1)}{(m+p-1)^2} \left\{ \left\| \nabla n_\varepsilon^{\frac{p+m-1}{2}} \right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{(q-1)[3(p+m-1)-p_*]}} + 1 \right\} \\
&+ C_5 \int_\Omega |\nabla c_\varepsilon|^{2q}
\end{aligned} \tag{4.9}$$

with $C_4 > 0$ and $C_5 > 0$ for all $t \in (0, T_{\max,\varepsilon})$, where the third inequality follows from the fact that $(\xi + \eta)^{\frac{q}{q-1}} \le 2^{\frac{1}{q-1}}(\xi^{\frac{q}{q-1}} + \eta^{\frac{q}{q-1}})$ for all $\xi \ge 0$ and $\eta \ge 0$. Since

$$\begin{aligned}
&\frac{6[q(p-m+1-p_*)+p_*]}{(q-1)[3(p+m-1)-p_*]} - 2\\
=&\frac{6}{(q-1)[3(p+m-1)-p_*]}\left\{p - [2q(m-1+\frac{p_*}{3}) - (m-1+\frac{2p_*}{3})]\right\}\\
\le&0
\end{aligned}$$

by (4.1), we again make use of Young's inequality to obtain

$$\begin{aligned}
\frac{C_D p(p-1)}{(m+p-1)^2}\left\{\left\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}^{\frac{6[q(p-m+1-p_*)+p_*]}{(q-1)[3(p+m-1)-p_*]}} + 1\right\} \le{}& \frac{C_D p(p-1)}{(m+p-1)^2}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2\\
&+\frac{2C_D p(p-1)}{(m+p-1)^2}
\end{aligned} \tag{4.10}$$

for all $t \in (0, T_{\max,\varepsilon})$. Collecting (4.5), (4.9) and (4.10), we derive

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{C_D p(p-1)}{(m+p-1)^2}\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 \le C_5\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{2C_D p(p-1)}{(m+p-1)^2} \tag{4.11}$$

for all $t \in (0, T_{\max,\varepsilon})$. Recalling $m > 1$ and $p \ge \max\{p_*, m-1\}$, we can see that $\frac{2p_*}{p+m-1} \le \frac{2p}{p+m-1} < 2 < 6$, which warrants the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{p+m-1}}(\Omega) \hookrightarrow L^{\frac{2p_*}{p+m-1}}(\Omega)$, whence another application of the Gagliardo–Nirenberg inequality together with (4.2) provides positive constants C_6 and C_7 such that

$$\begin{aligned}
\int_\Omega n_\varepsilon^p =&\left\|n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}}\\
\le&C_6\left\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}^{b\cdot\frac{2p}{p+m-1}}\left\|n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p_*}{p+m-1}}(\Omega)}^{(1-b)\cdot\frac{2p}{p+m-1}} + C_6\left\|n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2p_*}{p+m-1}}(\Omega)}^{\frac{2p}{p+m-1}}\\
\le&C_7\left\|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right\|_{L^2(\Omega)}^{b\cdot\frac{2p}{p+m-1}} + C_7
\end{aligned} \tag{4.12}$$

with $b := \frac{3(p+m-1)(1-\frac{p_*}{p})}{(q-1)[3(p+m-1)-p_*]} \in (0,1)$ for all $t \in (0, T_{\max,\varepsilon})$. Due to $b \cdot \frac{2p}{p+m-1} = \frac{6(p-p_*)}{3(p-p_*)+2p_*+3(m-1)} < 2$, we apply Young's inequality to (4.12) and have

$$\int_\Omega n_\varepsilon^p \le C_7\int_\Omega \left|\nabla n_\varepsilon^{\frac{p+m-1}{2}}\right|^2 + 2C_7$$

for all $t \in (0, T_{\max,\varepsilon})$. This along with (4.11) implies

$$\frac{d}{dt}\int_\Omega n_\varepsilon^p + \frac{C_D p(p-1)}{C_7(m+p-1)^2}\int_\Omega n_\varepsilon^p \le C_5\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{4C_D p(p-1)}{(m+p-1)^2} \tag{4.13}$$

for all $t \in (0, T_{\max,\varepsilon})$. From (4.3), we deduce that

$$\begin{aligned}\int_t^{t+1}\left\{C_5\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{4C_D p(p-1)}{(m+p-1)^2}\right\} &\le C_5\int_t^{t+1}\int_\Omega |\nabla c_\varepsilon|^{2q} + \frac{4C_D p(p-1)}{(m+p-1)^2}\\ &\le C_5 L + \frac{4C_D p(p-1)}{(m+p-1)^2}\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon} - 1)$, and thus letting $z(t) := \int_\Omega n_\varepsilon^p(\cdot, t)$ and $g(t) := C_5\int_\Omega |\nabla c_\varepsilon(\cdot, t)|^{2q} + \frac{4C_D p(p-1)}{(m+p-1)^2}$, we infer from (4.13) and [24, Lemma 3.4] that there exists $C_8 > 0$ such that $z(t) \le C_8$ for all $t \in (0, T_{\max,\varepsilon})$, which entails (4.4). $\square$

4.2. The first iteration for $L^\infty((0, T_{\max,\varepsilon}); L^p(\Omega))$-estimates of n_ε for $p < 9(m-1)$

Based on the regularity of ∇c_ε provided in Lemma 3.4, we can achieve the uniform L^p-bounds on n_ε in $(0, T_{\max,\varepsilon})$ by repeated applications of the iterative criterion (4.1) with fixed $q = 2$ provided that $m > \frac{10}{9}$. As enlightenments for the reasoning herein, the original idea can be found in precedents [26,30].

Lemma 4.2. *Let $m > \frac{10}{9}$. Then for each $p \in [1, 9(m-1))$, one can find $C(p) > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le C(p) \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{4.14}$$

Proof. Define a sequence $(p_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ fulfilling

$$p_{j+1} := \frac{2}{3}p_j + 3(m-1) \qquad \text{for all } j \in \mathbb{N} \tag{4.15}$$

with the initial term $p_0 := 1$, then the hypothesis $m > \frac{10}{9}$ ensures that $p_j \nearrow 9(m-1)$ as $j \to \infty$. Thus, for deriving (4.14), it is sufficient to verify the validity of (4.14) with $p = p_j$ for each $j \in \mathbb{N}$ thanks to an interpolation reasoning. From (2.10), we know that (4.14) holds with $p = p_0$, whereupon by an inductive argument we only need to prove that (4.14) is valid with $p = p_{j+1}$ provided that $\int_\Omega n_\varepsilon^{p_j}(\cdot, t) \le C_1(j)$ holds with some $C_1(j) > 0$ for $j \in \mathbb{N}$. In light of (3.34), this can be achieved by an application of Lemma 4.1 with $p_* := p_j$ and $q := 2$, and thus completes the proof. $\square$

4.3. An improvement on regularity for ∇c_ε

Observing from Lemmas 3.4, 4.1 and 4.2, one can find that merely relying on the condition (4.3) with $q = 2$ might be inadequate for deriving higher regularity on n_ε, so that in view of the relationship between p and q as shown in (4.1), it is essential to improve the estimates of ∇c_ε at first. To achieve this, we need the following two well-established lemmas, where the first one provided in [28] shows the improved estimates for u_ε based on known regularity of n_ε in the context of 3d-Stokes system, and the second one asserted in [18] reveals the spatio-temporal regularity that ∇c_ε can reach whenever n_ε has the same regularity as that in the first one.

Lemma 4.3. *Let $N = 3,\ \ p \in [1, \infty)$ and $q \in [1, \infty]$ be such that*

$$\begin{cases} q < \dfrac{3p}{2(3-2p)} & \text{if } p \le \dfrac{3}{2}, \\ q \le \infty & \text{if } p > \dfrac{3}{2}. \end{cases} \tag{4.16}$$

Then for all $L > 0$ there exists $C = C(p, q, L) > 0$ such that if

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \le L \ \text{ for all } \ t \in (0, T_{\max,\varepsilon}), \tag{4.17}$$

then

$$\|u_\varepsilon(\cdot, t)\|_{L^{2q}(\Omega)} \le C \ \text{ for all } \ t \in (0, T_{\max,\varepsilon}). \tag{4.18}$$

Lemma 4.4. *Let $N = 3,\ \ p \in [1, \infty)$ and $q \in [1, \infty]$ satisfy*

$$\begin{cases} q < \dfrac{3p}{2(3-2p)} & \text{if } p \le \dfrac{3}{2}, \\ q \le \infty & \text{if } p > \dfrac{3}{2}. \end{cases} \tag{4.19}$$

Then for each $L > 0$ one can find $C = C(p, q, L) > 0$ such that if

$$\int_\Omega n_\varepsilon^p(\cdot, t) \le L \ \text{ for all } \ t \in (0, T_{\max,\varepsilon}), \tag{4.20}$$

then

$$\int_t^{t+1} \int_\Omega |\nabla c_\varepsilon|^{2q} \le C \ \text{ for all } \ t \in (0, T_{\max,\varepsilon} - 1). \tag{4.21}$$

Due to the same structure appearing in c-equation of (1.1), the proof of Lemma 4.4 is almost same as that in [18], so here we omit it.

4.4. The second iteration for $L^\infty((0,T_{\max,\varepsilon});L^p(\Omega))$-estimates of n_ε for any $p>1$

Noting that

$$9(m-1)\begin{cases}\le \frac{3}{2} & \text{for } m\le \frac{7}{6},\\ > \frac{3}{2} & \text{for } m>\frac{7}{6},\end{cases} \tag{4.22}$$

we intend to execute the second iteration for $m\le\frac{7}{6}$ as well as $m>\frac{7}{6}$, respectively.

Lemma 4.5. *Let $m>\frac{7}{6}$. Then for each $p>1$ there exists $C(p)>0$ such that*

$$\int_\Omega n_\varepsilon^p(\cdot,t)\le C(p) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.23}$$

Proof. Since $m>\frac{7}{6}$ implies $9(m-1)>\frac{3}{2}$, Lemma 4.2 warrants the existence of some p^* close to $9(m-1)$ fulfilling $p^*>\frac{3}{2}$ and certain $C(p^*)>0$ such that

$$\int_\Omega n_\varepsilon^{p^*}(\cdot,t)\le C(p^*) \qquad \text{for all } t\in(0,T_{\max,\varepsilon}).$$

Thus, a combination of Lemmas 4.3–4.4 allows for a choice of arbitrarily large $q>2$ in (4.21), which in conjunction with Lemma 4.1 shows (4.23) holds for any $p>1$.

Now, let us consider the more complex case that $m\le\frac{7}{6}$. Firstly, we try to identify the starting point for a second iterative argument by developing the regularity achieved in Lemma 4.2. □

Lemma 4.6. *Let $\frac{10}{9}<m\le\frac{7}{6}$, and let $p_*\in[1,9(m-1))$. Then for any $p>1$ fulfilling*

$$p<\frac{3p_*}{3-2p_*}\Big(m-1+\frac{p_*}{3}\Big)-\Big(m-1+\frac{2p_*}{3}\Big) \tag{4.24}$$

and each chosen $L>0$ there exists $C(p,L)>0$ such that if for certain $\varepsilon\in(0,1)$

$$\int_\Omega n_\varepsilon^{p_*}(\cdot,t)\le L \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}) \tag{4.25}$$

holds, then

$$\int_\Omega n_\varepsilon^{p}(\cdot,t)\le C(p,L) \qquad \textit{for all } t\in(0,T_{\max,\varepsilon}). \tag{4.26}$$

Proof. Abbreviating $f(q):=2q(m-1+\frac{p_*}{3})-(m-1+\frac{2p_*}{3})$, we can see that f is monotone increasing with respect to q by computing $f'(q)=2(m-1+\frac{p_*}{3})>0$ due to $m>\frac{10}{9}$. Moreover, since (4.25) implies the validity of (4.21) for any $q<\frac{3p_*}{2(3-2p_*)}$ by Lemma 4.4, we infer from Lemma 4.1 and the monotonicity of f that (4.26) holds for all p complying with (4.24).

As shown in Lemma 4.4 and also in the reasoning of Lemma 4.5, the derivation of (4.4) for arbitrary $p > 1$ merely relies on the existence of $p_* > \frac{3}{2}$ fulfilling (4.2). In fact, in the case when $m = \frac{7}{6}$, this can be achieved by an application of Lemma 4.6. □

Lemma 4.7. *Let $m = \frac{7}{6}$. Then one can find $p^* > \frac{3}{2}$ and $C(p^*) > 0$ such that*

$$\int_\Omega n_\varepsilon^{p^*}(\cdot,t) \le C(p^*) \qquad \textit{for all } \ t \in (0, T_{\max,\varepsilon}). \tag{4.27}$$

Proof. Since $9(m-1) = \frac{3}{2}$ due to $m = \frac{7}{6}$, it can be inferred from Lemma 4.2 that for any $p_* \in [1, \frac{3}{2})$ there exists $C(p_*) > 0$ such that

$$\int_\Omega n_\varepsilon^{p_*}(\cdot,t) \le C(p_*) \qquad \text{for all } \ t \in (0, T_{\max,\varepsilon}). \tag{4.28}$$

In particular, if we choose $p_* := \frac{8}{7} < \frac{3}{2}$, an elementary calculation entails

$$\left\{ \frac{3p_*}{3-2p_*}(m-1+\frac{p_*}{3}) - (m-1+\frac{2p_*}{3}) \right\}\bigg|_{p_*=\frac{8}{7}} = \frac{84}{55} > \frac{3}{2}.$$

Therefore, Lemma 4.6 ensures that (4.27) is indeed valid for some $p^* > \frac{3}{2}$.

Whereas for $\frac{10}{9} < m < \frac{7}{6}$, the availability of the condition (4.24) in verifying the existence of $p^* > \frac{3}{2}$ requires $m > \frac{37}{33}$ as stated in Theorem 1.1. □

Lemma 4.8. *For $\frac{37}{33} < m < \frac{7}{6}$, let*

$$\varrho(p) := \frac{3p}{3-2p}(m-1+\frac{p}{3}) - (m-1+\frac{2p}{3}), \qquad p \in \mathbb{R}\backslash \left\{\frac{3}{2}\right\}. \tag{4.29}$$

Then

$$\varrho\Big(9(m-1)\Big) > 9(m-1) \quad \textit{is equivalent to} \quad \frac{37}{33} < m < \frac{7}{6}, \tag{4.30}$$

and there exist $\eta_1(p) > 0$ and $\Lambda > 1$ such that

$$\varrho(p) \ge \Lambda p \quad \textit{for all } \ p \in \Big(9(m-1) - \eta_1(p), \frac{3}{2}\Big). \tag{4.31}$$

Proof. Upon an elementary computation, we have

$$\begin{aligned}&\varrho\Big(9(m-1)\Big)-9(m-1)\\=&\frac{27(m-1)}{3-18(m-1)}\cdot\left(m-1+\frac{9(m-1)}{3}\right)-\left(m-1+\frac{18(m-1)}{3}\right)-9(m-1)\\=&\frac{4(m-1)(33m-37)}{7-6m},\end{aligned}$$

which shows the validity of (4.30). Now, we abbreviate

$$\tilde{\varrho}(p):=\frac{\varrho(p)}{p}\quad\text{for }\ p>0.$$

From (4.30), it is clear that $C_1:=\tilde{\varrho}\Big(9(m-1)\Big)-1>0$, whence a continuity argument warrants the existence of $\eta_1:=\eta_1(p)>0$ such that $9(m-1)-\eta_1(p)>0$ and that

$$\tilde{\varrho}(p)\geq\Lambda:=1+\frac{C_1}{2}\quad\text{for all }\ p\in\Big(9(m-1)-\eta_1(p),9(m-1)\Big].\tag{4.32}$$

Moreover, due to $m>1$, another computation shows that

$$\tilde{\varrho}'(p)=\left(\frac{6}{(3-2p)^2}+\frac{1}{p^2}\right)(m-1)+\frac{3}{(3-2p)^2}>0\quad\text{for all }\ p\in\mathbb{R}\backslash\left\{\frac{3}{2}\right\},$$

from which and (4.32) we infer that $\tilde{\varrho}\geq\Lambda$ on $\Big(9(m-1)-\eta_1(p),\frac{3}{2}\Big)$, and thus (4.31) holds.

Next, we try to show that in the case when $\frac{37}{33}<m<\frac{7}{6}$ the index p^* in (4.27) can still achieve over $\frac{3}{2}$ by a second iteration. $\square$

Lemma 4.9. *Let $\frac{37}{33}<m<\frac{7}{6}$. Then there exist $p^*>\frac{3}{2}$ and $C(p^*)>0$ such that*

$$\int_\Omega n_\varepsilon^{p^*}(\cdot,t)\leq C(p^*)\qquad\textit{for all }\ t\in(0,T_{\max,\varepsilon}).\tag{4.33}$$

Proof. Due to $m>\frac{37}{33}$, with $\varrho:\mathbb{R}\setminus\{\frac{3}{2}\}\to\mathbb{R}$ defined as in Lemma 4.8, we derive from an elementary computation that $9(m-1)>\frac{12}{11}$ and that

$$\begin{aligned}\varrho\left(\frac{12}{11}\right)=&\frac{3\cdot\frac{12}{11}}{3-2\cdot\frac{12}{11}}(m-1+\frac{\frac{12}{11}}{3})-(m-1+\frac{2\cdot\frac{12}{11}}{3})\\=&3(m-1)+\frac{8}{11}\\>&\frac{12}{11}.\end{aligned}\tag{4.34}$$

By choosing

$$p_0 := \frac{12}{11} \tag{4.35}$$

without loss of generality, we thus obtain a recursive sequence defined as

$$p_j := \varrho(p_{j-1}), \qquad j \in \mathbb{N}_+ = \{1, 2, 3, ...\}. \tag{4.36}$$

Therefore, for some given $0 < C_1 < \min\{\frac{5}{11}, \tilde{\varrho}\left(\frac{12}{11}\right) - 1\}$, from the monotonicity of $\tilde{\varrho}$ in $\mathbb{R} \setminus \{\frac{3}{2}\}$ and an inductive reasoning, we infer that

$$p_j \geq \left(1 + \frac{C_1}{2}\right)^j \cdot p_0 \qquad \text{for all} \quad j \in \mathbb{N}_+. \tag{4.37}$$

According to the reasoning of Lemma 4.7, for achieving (4.33) for some $p^* > \frac{3}{2}$, it is sufficient for us to find some $j_0 \in \mathbb{N}$ such that

$$\frac{8}{7} < p_{j_0} < \frac{3}{2}. \tag{4.38}$$

For the interval $I := \left(\frac{1}{\log_{\frac{22}{21}}\left(1+\frac{C_1}{2}\right)}, \frac{1}{\log_{\frac{4}{3}}\left(1+\frac{C_1}{2}\right)}\right)$, since

$$\frac{1}{\log_{\frac{4}{3}}\left(1 + \frac{C_1}{2}\right)} - \frac{1}{\log_{\frac{22}{21}}\left(1 + \frac{C_1}{2}\right)} = \frac{\ln\frac{4}{3} - \ln\frac{22}{21}}{\ln\left(1 + \frac{C_1}{2}\right)} = \frac{\ln\left(1 + \frac{1}{2}\cdot\frac{6}{11}\right)}{\ln\left(1 + \frac{C_1}{2}\right)} > 1$$

due to $C_1 < \frac{5}{11}$, we make sure that there exists at least one integer lying in I, which we denote by j_0. Then recalling (4.35), we have

$$\frac{8}{7} < \left(1 + \frac{C_1}{2}\right)^{j_0} \cdot p_0 < \frac{16}{11} < \frac{3}{2}.$$

This along with (4.37) shows that the p_{j_0} we obtain by iteration fulfills (4.38). Thus, in light of (4.36), Lemma 4.6 warrants that

$$\int_\Omega n_\varepsilon^{\frac{8}{7}}(\cdot, t) \leq C_2 \qquad \text{for all} \;\; t \in (0, T_{\max,\varepsilon})$$

with some $C_2 > 0$, from which (4.33) follows in accordance with the reasoning of Lemma 4.7.

With Lemmas 4.7 and 4.9 at hand, we can achieve the same estimates as Lemma 4.5 for $\frac{37}{33} < m \leq \frac{7}{6}$. $\square$

Lemma 4.10. *Let* $\frac{37}{33} < m \le \frac{7}{6}$. *Then for each* $p > 1$ *one can find* $C(p) > 0$ *such that*

$$\int_\Omega n_\varepsilon^p(\cdot,t) \le C(p) \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{4.39}$$

Proof. From the arguments of Lemma 4.5, we can see that (4.39) is actually implied by the exploration of some $p^* > \frac{3}{2}$ such that

$$\int_\Omega n_\varepsilon^{p^*}(\cdot,t) \le C_1 \qquad \text{for all } t \in (0, T_{\max,\varepsilon}), \tag{4.40}$$

which has been verified in Lemmas 4.7 and 4.9 for $m = \frac{7}{6}$ and $\frac{37}{33} < m < \frac{7}{6}$, respectively. □

5. Global solvability of (2.1)

In light of the arguments established in the precedents [29,30], the uniform $L^p(\Omega)$-estimates of n_ε in $(0, T_{\max,\varepsilon})$ allow for further improving on the regularity properties with respect to c_ε, ρ_ε and u_ε.

Lemma 5.1. *Let* $m > \frac{37}{33}$. *Then for each* $q > 1$ *there exists some* $C(q) > 0$ *independent of* $\varepsilon \in (0, 1)$ *such that*

$$\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^q \le C(q) \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{5.1}$$

and

$$\int_\Omega |\nabla \rho_\varepsilon(\cdot,t)|^q \le C(q) \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{5.2}$$

as well as

$$\int_\Omega |u_\varepsilon(\cdot,t)|^q \le C(q) \qquad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{5.3}$$

Moreover, for all $\varepsilon \in (0, 1)$ *there exists* $C > 0$ *such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{5.4}$$

and

$$\|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \text{for all } t \in (0, T_{\max,\varepsilon}) \tag{5.5}$$

with some fixed $\alpha \in (0, \frac{3}{4})$.

Proof. As a direct consequence, (5.3) follows from Lemmas 4.5, 4.10 and 4.3. Based on (2.12), (2.13), Lemmas 4.5, 4.10 and (5.3), the derivation of (5.1) and (5.2) can proceed along the reasoning of [18, Lemma 5.1]. With the aid of (5.1)–(5.3), (5.4) is valid by an application of a Moser-type iterative technique as established in [1,25]. Therefore, we can finally achieve (5.5) according to the arguments of [19, Lemma 3.4].

Combining (2.8) with Lemma 5.1, we can obtain the following results on global solvability. □

Proposition 5.2. *Let $m > \frac{37}{33}$. Then (2.1) is globally solvable in the sense that there exists some $C(q) > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|\rho_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(q) \tag{5.6}$$

for all $t \in (0, \infty)$.

Proof. Proposition 5.2 results from Lemma 2.1 and Lemma 5.1 immediately. □

6. Further regularity properties

Aided by Lemma 5.1, it is possible for us to explore further regularity properties for $(n_\varepsilon, c_\varepsilon, \rho_\varepsilon, u_\varepsilon)$, which might provide the compactness that is essential whether for the construction of the global solutions of (1.1), (1.6) and (1.7) by extraction procedures or for the detection of the convergence of the solutions to some spatial homogeneous equilibrium by an Ehrling-type argument.

Without reliance on the regularity of the respective initial data, the derivation of the following Hölder continuity is mainly based on maximal Sobolev regularity properties and suitable embedding conclusions. To avoid repetition, readers can refer to [18, Lemmas 5.4-5.6] for detailed reasoning.

Lemma 6.1. *One can find certain $\mu \in (0, 1)$ which allows for existence of some $C > 0$ independent of $\varepsilon \in (0, 1)$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{C^\mu(\bar\Omega)} \le C \qquad \textit{for all} \quad t \ge 0, \tag{6.1}$$

and that

$$\|c_\varepsilon\|_{C^\mu(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all} \quad t \ge 0 \tag{6.2}$$

as well as

$$\|\rho_\varepsilon\|_{C^\mu(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all} \quad t \ge 0, \tag{6.3}$$

and that for each $\tau > 0$, we can choose certain $C(\tau) > 0$ independent of $\varepsilon \in (0, 1)$ satisfying

$$\|\nabla c_\varepsilon\|_{C^\mu(\bar\Omega\times[t,t+1])} \le C(\tau) \qquad \textit{for all} \quad t \ge \tau \tag{6.4}$$

as well as

$$\|\nabla\rho_\varepsilon\|_{C^\mu(\bar\Omega\times[t,t+1])}\le C(\tau)\qquad \textit{for all}\quad t\ge\tau. \tag{6.5}$$

Now, let us provide two time regularity results on n_ε by means of a standard testing procedure. Throughout the sequel, we will use the abbreviation $M_n := \sup_{\varepsilon\in(0,1)}\|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$.

Lemma 6.2. *Let $T>0$. Then one can find $C(T)>0$ independent of $\varepsilon\in(0,1)$ such that*

$$\int_0^T \|\partial_t n_\varepsilon^m(\cdot,t)\|_{(W_0^{1,\infty}(\Omega))^*}dt\le C(T). \tag{6.6}$$

Moreover, for all $\varepsilon\in(0,1)$ there exists $C>0$ fulfilling

$$\|n_\varepsilon(\cdot,t)-n_\varepsilon(\cdot,s)\|_{(W_0^{2,2}(\Omega))^*}\le C|t-s|\quad \textit{for all}\ \ t\ge 0\ \ \textit{and}\ \ s\ge 0. \tag{6.7}$$

Proof. Recalling (1.9) and denoting $C_1 := \|D\|_{L^\infty(0,M_n)}+2$, for some fixed $t\in(0,T)$ and certain $\varphi\in C_0^\infty(\Omega)$, we deduce from n_ε-equation in (2.1) upon integration by parts and Young's inequality that in the case when $\frac{37}{33}<m\le 2$

$$\begin{aligned}
\left|\frac{1}{m}\int_\Omega \partial_t n_\varepsilon^m(\cdot,t)\cdot\varphi\right| &= \left|\int_\Omega n_\varepsilon^{m-1}\left\{\nabla\cdot\big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon - n_\varepsilon u_\varepsilon\big) - F_\varepsilon(n_\varepsilon)\rho_\varepsilon\right\}\cdot\varphi\right| \\
&= \Bigg|-(m-1)\int_\Omega n_\varepsilon^{m-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\varphi - \int_\Omega n_\varepsilon^{m-1}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi \\
&\quad + (m-1)\int_\Omega n_\varepsilon^{m-1}F_\varepsilon'(n_\varepsilon)\left(\nabla n_\varepsilon\cdot\nabla c_\varepsilon\right)\varphi + \int_\Omega n_\varepsilon^m F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi \\
&\quad + \frac{1}{m}\int_\Omega n_\varepsilon^m u_\varepsilon\cdot\nabla\varphi - \int_\Omega n_\varepsilon^{m-1}F_\varepsilon(n_\varepsilon)\rho_\varepsilon\varphi\Bigg| \\
&\le \Bigg\{C_1(m-1)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + C_1\int_\Omega n_\varepsilon^{m-1}|\nabla n_\varepsilon| \\
&\quad + (m-1)\int_\Omega n_\varepsilon^{m-1}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon| + \int_\Omega n_\varepsilon^m|\nabla c_\varepsilon| \\
&\quad + \frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| + \int_\Omega n_\varepsilon^m\rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)} \\
&\le \Bigg\{C_1(m-1)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + C_1(m-1)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C_1}{4(m-1)}\int_\Omega n_\varepsilon^m + C_1(m-1)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{m-1}{4C_1}\int_\Omega n_\varepsilon^m|\nabla c_\varepsilon|^2\\
&+\int_\Omega n_\varepsilon^m|\nabla c_\varepsilon| + \frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| + \int_\Omega n_\varepsilon^m\rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)}\\
&\le\Bigg\{3C_1(m-1)\int_\Omega n_\varepsilon^{m-2}|\nabla n_\varepsilon|^2 + \frac{C_1M_n^m|\Omega|}{4(m-1)} + \frac{M_n^m(m-1)}{4C_1}\int_\Omega|\nabla c_\varepsilon|^2\\
&+M_n^m\int_\Omega|\nabla c_\varepsilon| + \frac{M_n^m}{m}\int_\Omega|u_\varepsilon| + M_n^m\int_\Omega\rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)}
\end{aligned}
$$

for all $\varepsilon\in(0,1)$, whereas for $m>2$,

$$
\begin{aligned}
\left|\frac{1}{m}\int_\Omega\partial_t n_\varepsilon^m(\cdot,t)\cdot\varphi\right| &= \Bigg|-(m-1)\int_\Omega n_\varepsilon^{m-2}D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2\varphi - \int_\Omega n_\varepsilon^{m-1}D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon\cdot\nabla\varphi\\
&+(m-1)\int_\Omega n_\varepsilon^{m-1}F_\varepsilon'(n_\varepsilon)\,(\nabla n_\varepsilon\cdot\nabla c_\varepsilon)\,\varphi + \int_\Omega n_\varepsilon^m F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi\\
&+\frac{1}{m}\int_\Omega n_\varepsilon^m u_\varepsilon\cdot\nabla\varphi - \int_\Omega n_\varepsilon^{m-1}F_\varepsilon(n_\varepsilon)\rho_\varepsilon\varphi\Bigg|\\
&\le\Bigg\{C_DC_1(m-1)\int_\Omega n_\varepsilon^{2m-4}|\nabla n_\varepsilon|^2 + C_D\int_\Omega n_\varepsilon^{2m-2}|\nabla n_\varepsilon|\\
&+(m-1)\int_\Omega n_\varepsilon^{m-1}|\nabla n_\varepsilon|\cdot|\nabla c_\varepsilon| + \int_\Omega n_\varepsilon^m|\nabla c_\varepsilon|\\
&+\frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| + \int_\Omega n_\varepsilon^m\rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)}\\
&\le\Bigg\{C_DC_1(m-1)\int_\Omega n_\varepsilon^{2m-4}|\nabla n_\varepsilon|^2 + C_DC_1(m-1)\int_\Omega n_\varepsilon^{2m-4}|\nabla n_\varepsilon|^2\\
&+\frac{C_D}{4C_1(m-1)}\int_\Omega n_\varepsilon^{2m} + C_DC_1(m-1)\int_\Omega n_\varepsilon^{2m-4}|\nabla n_\varepsilon|^2\\
&+\frac{m-1}{4C_DC_1}\int_\Omega n_\varepsilon^2|\nabla c_\varepsilon|^2
\end{aligned}
$$

$$+\int_\Omega n_\varepsilon^m|\nabla c_\varepsilon| + \frac{1}{m}\int_\Omega n_\varepsilon^m|u_\varepsilon| + \int_\Omega n_\varepsilon^m \rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)}$$

$$\le\Bigg\{3C_DC_1(m-1)\int_\Omega n_\varepsilon^{2m-4}|\nabla n_\varepsilon|^2 + \frac{C_DM_n^{2m}|\Omega|}{4C_1(m-1)} + \frac{M_n^2(m-1)}{4C_DC_1}\int_\Omega|\nabla c_\varepsilon|^2$$

$$+M_n^m\int_\Omega|\nabla c_\varepsilon| + \frac{M_n^m}{m}\int_\Omega|u_\varepsilon| + M_n^m\int_\Omega\rho_\varepsilon\Bigg\}\cdot\|\varphi\|_{W^{1,\infty}(\Omega)}$$

for all $\varepsilon\in(0,1)$, which in conjunction with (3.34) and (5.6) entails (6.6). Next, let

$$H_\varepsilon(s) := \int_0^s D_\varepsilon(\sigma)d\sigma, \tag{6.8}$$

then

$$H_\varepsilon(n_\varepsilon)\le C_2 := M_n\cdot(\|D\|_{L^\infty(0,M_n)}+2) = M_n\cdot C_1 \tag{6.9}$$

in $\Omega\times(0,\infty)$ for all $\varepsilon\in(0,1)$. Therefore, we integrate by parts and invoke the Hölder inequality to obtain

$$\begin{aligned}\left|\int_\Omega \partial_t n_\varepsilon(\cdot,t)\cdot\varphi\right| &= \left|\int_\Omega \left\{\nabla\cdot\big(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon - n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon - n_\varepsilon u_\varepsilon\big) - F_\varepsilon(n_\varepsilon)\rho_\varepsilon\right\}\cdot\varphi\right| \\ &= \left|\int_\Omega H_\varepsilon(n_\varepsilon)\Delta\varphi + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla c_\varepsilon\cdot\nabla\varphi + \int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\varphi - \int_\Omega F_\varepsilon(n_\varepsilon)\rho_\varepsilon\varphi\right| \\ &\le C_2\int_\Omega|\Delta\varphi| + M_n\int_\Omega|\nabla c_\varepsilon|\cdot|\nabla\varphi| + M_n\int_\Omega|u_\varepsilon|\cdot|\nabla\varphi| + M_n\int_\Omega\rho_\varepsilon|\varphi| \\ &\le C_3\left\{|\Omega|^{\frac{1}{2}} + \|\nabla c_\varepsilon\|_{L^2(\Omega)} + \|u_\varepsilon\|_{L^2(\Omega)} + \|\rho_\varepsilon\|_{L^2(\Omega)}\right\}\cdot\|\varphi\|_{W^{2,2}(\Omega)}\end{aligned}$$

with $C_3 := C_2 + M_n$ for all $\varepsilon\in(0,1)$ and each $t>0$, whence in view of (5.6), we can find $C_4>0$ such that

$$\|\partial_t n_\varepsilon(\cdot,t)\|_{(W_0^{2,2}(\Omega))^*}\le C_4 \quad \text{for all } \varepsilon\in(0,1) \text{ and any } t>0,$$

which yields (6.7). □

7. Global boundedness for (1.1), (1.6) and (1.7)

Based on Proposition 5.2 and the regularity properties provided in Lemmas 6.1–6.2, we can construct the global bounded solutions as asserted in Theorem 1.1 in terms of the following natural notion.

Definition 7.1. A quadruple of functions (n, c, ρ, u) is called a global weak solution of (1.1), (1.6) and (1.7), if it complies with

$$\begin{cases} n \in L^1_{loc}(\bar{\Omega} \times [0,\infty)), \\ c \in L^\infty_{loc}(\bar{\Omega} \times [0,\infty)) \bigcap L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ \rho \in L^\infty_{loc}(\bar{\Omega} \times [0,\infty)) \bigcap L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \\ u \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega; \mathbb{R}^3)), \end{cases} \tag{7.1}$$

and $n \geq 0$, $c \geq 0$, $\rho \geq 0$ in $\Omega \times (0,\infty)$, as well as

$$H(n),\ n|\nabla c|,\ n|u|,\ c|u| \text{ and } \rho|u| \text{ belong to } L^1_{loc}(\bar{\Omega} \times [0,\infty)), \tag{7.2}$$

where $H(s) := \int_0^s D(\sigma)d\sigma$, if $\nabla \cdot u = 0$ in the distributional sense, if

$$\begin{aligned} -\int_0^\infty \int_\Omega n\psi_t - \int_\Omega n_0\psi(\cdot,0) = &-\int_0^\infty \int_\Omega H(n)\Delta\psi + \int_0^\infty \int_\Omega n\nabla c \cdot \nabla\psi \\ &+ \int_0^\infty \int_\Omega nu \cdot \nabla\psi - \int_0^\infty \int_\Omega \rho n\psi \end{aligned} \tag{7.3}$$

for each $\psi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$ such that $\frac{\partial\psi}{\partial\nu} = 0$, if

$$\begin{aligned} -\int_0^\infty \int_\Omega c\psi_t - \int_\Omega c_0\psi(\cdot,0) = &-\int_0^\infty \int_\Omega \nabla c \cdot \nabla\psi - \int_0^\infty \int_\Omega c\psi + \int_0^\infty \int_\Omega \rho\psi \\ &+ \int_0^\infty \int_\Omega cu \cdot \nabla\psi \end{aligned} \tag{7.4}$$

and

$$-\int_0^\infty \int_\Omega \rho\psi_t - \int_\Omega \rho_0\psi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla\rho \cdot \nabla\psi - \int_0^\infty \int_\Omega \rho n\psi + \int_0^\infty \int_\Omega \rho u \cdot \nabla\psi \tag{7.5}$$

for each $\psi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$, as well as if in addition

$$-\int_0^\infty\int_\Omega u\cdot\psi_t-\int_\Omega u_0\psi(\cdot,0)=-\int_0^\infty\int_\Omega \nabla u\cdot\nabla\psi+\int_0^\infty\int_\Omega(\rho+n)\nabla\phi\cdot\psi \tag{7.6}$$

for each $\psi\in C_0^\infty\left(\bar\Omega\times[0,\infty);\mathbb{R}^3\right)$ satisfying $\nabla\cdot\psi\equiv 0$.

Within this context, the verification of the global solvability of (1.1), (1.6) and (1.7) relies on standard extraction procedures.

Lemma 7.2. *Let $m>\frac{37}{33}$. Then there exists $(\varepsilon_l)_{l\in\mathbb{N}}\subset(0,1)$ fulfilling $\varepsilon_l\searrow 0$ as $l\to\infty$, a null set $\aleph\subset(0,\infty)$ and a quadruple of functions (n,c,ρ,u) such that as $\varepsilon_l\searrow 0$*

$$n_\varepsilon\to n\qquad \text{a.e. in } \Omega \text{ for each } t\in(0,\infty)\backslash\aleph, \tag{7.7}$$

$$n_\varepsilon\overset{*}{\rightharpoonup} n\qquad \text{in } L^\infty(\Omega\times(0,\infty)), \tag{7.8}$$

$$n_\varepsilon\to n\qquad \text{in } C^0_{loc}\Big([0,\infty);(W_0^{2,2}(\Omega))^*\Big), \tag{7.9}$$

$$c_\varepsilon\to c\ \text{ in }\ C^0_{loc}\left(\bar\Omega\times[0,\infty)\right), \tag{7.10}$$

$$c_\varepsilon\overset{*}{\rightharpoonup} c\ \text{ in }\ L^\infty((0,\infty);W^{1,q}(\Omega))\ \text{ for each }\ q\in(1,\infty), \tag{7.11}$$

$$\nabla c_\varepsilon\to\nabla c\ \text{ in }\ C^0_{loc}\left(\bar\Omega\times[0,\infty)\right), \tag{7.12}$$

$$\rho_\varepsilon\to\rho\ \text{ in }\ C^0_{loc}\left(\bar\Omega\times[0,\infty)\right), \tag{7.13}$$

$$\rho_\varepsilon\overset{*}{\rightharpoonup}\rho\ \text{ in }\ L^\infty((0,\infty);W^{1,q}(\Omega))\ \text{ for each }\ q\in(1,\infty), \tag{7.14}$$

$$\nabla\rho_\varepsilon\to\nabla\rho\ \text{ in }\ C^0_{loc}\left(\bar\Omega\times[0,\infty)\right), \tag{7.15}$$

$$u_\varepsilon\to u\ \text{ in }\ C^0_{loc}\left(\bar\Omega\times[0,\infty)\right), \tag{7.16}$$

$$u_\varepsilon\overset{*}{\rightharpoonup} u\ \text{ in }\ L^\infty\left(\Omega\times(0,\infty)\right), \tag{7.17}$$

and

$$\nabla u_\varepsilon\rightharpoonup\nabla u\ \text{ in }\ L^2_{loc}\left(\bar\Omega\times[0,\infty)\right). \tag{7.18}$$

In addition, n,c,ρ and u make up a global weak solution of (1.1), (1.6) and (1.7) in the sense of Definition 7.1.

Proof. Thanks to the embedding $W^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$, Lemma 6.2 implies that $(\partial_t n_\varepsilon^m)_{\varepsilon\in(0,1)}$ is bounded in $L^2_{loc}\left([0,\infty);(W^{3,2}(\Omega))^*\right)$. Moreover, (3.34) along with (5.4) shows the boundedness of $(n_\varepsilon^m)_{\varepsilon\in(0,1)}$ in $L^2_{loc}\left([0,\infty);(W^{1,2}(\Omega))\right)$. Therefore, according to Aubin–Lions lemma [27], we can extract $(\varepsilon_l)_{l\in\mathbb{N}}\subset(0,1)$ fulfilling $\varepsilon_l\searrow 0$ as $l\to\infty$ such that $n_\varepsilon^m\to n^m$ is valid a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_l\searrow 0$ with certain nonnegative function n defined on $\Omega\times(0,\infty)$, and thus (7.7) results from Fubini–Tonelli theorem immediately. As direct consequences of (5.4) and (6.7), respectively, both (7.8) and (7.9) follow from further extractions. By means of Arzelà–Ascoli theorem, we can infer (7.10)–(7.18) from the regularity properties achieved in Lemmas 3.4, 5.1 and 6.1.

Recalling (1.9), we can see clearly that (7.1), (7.2) and the divergence-free property of u are valid from (7.7)–(7.18), and whereafter the derivation of (7.3)–(7.6) proceeds along standard testing procedures. $\square$

8. Stabilization. Proof of Theorem 1.1

8.1. Basic decay properties

By an elementary observation, one can see that the presence of the reaction term ρn in (1.1) implies natural decay properties with respect to the quantities nc and ∇c, which is viewed as the cornerstone in detecting the convergence of each component as time goes infinity. Throughout the remaining parts, we will suppose that $m > \frac{37}{33}$ and that (n, c, ρ, u) represents the global solutions established in Lemma 7.2 without special instructions.

Lemma 8.1. *There exist some $\varepsilon_* \in (0, 1)$ and $C > 0$ fulfilling*

$$\int_0^\infty \int_\Omega n_\varepsilon c_\varepsilon \leq C \quad \textit{for all} \;\; \varepsilon \in (0, \varepsilon_*) \tag{8.1}$$

and

$$\int_0^\infty \int_\Omega |\nabla \rho_\varepsilon|^2 \leq C \quad \textit{for all} \;\; \varepsilon \in (0, \varepsilon_*) \tag{8.2}$$

as well as

$$\int_0^\infty \int_\Omega |\nabla c_\varepsilon|^2 \leq C \quad \textit{for all} \;\; \varepsilon \in (0, \varepsilon_*). \tag{8.3}$$

Proof. For $M_n > 0$ defined as in Section 6, we can find certain $\varepsilon_* \in (0, 1)$ sufficiently small such that $M_n \leq \frac{1}{\varepsilon_*}$, whence it can be inferred from (2.4) that $F_\varepsilon(n_\varepsilon) \equiv n_\varepsilon$ over $\Omega \times (0, \infty)$ provided that $\varepsilon \in (0, \varepsilon_*)$. Upon an integration of ρ_ε-equation in (2.1) over Ω, we have

$$\int_\Omega \rho_\varepsilon(\cdot, t) + \int_0^t \int_\Omega \rho_\varepsilon n_\varepsilon = \int_\Omega \rho_0 \quad \text{for all} \;\; \varepsilon \in (0, \varepsilon_*) \;\; \text{and any} \;\; t > 0,$$

and thereby (8.1) holds. With ρ_ε as a testing function, an application of a standard testing procedure to ρ_ε-equation entails

$$\frac{1}{2}\int_\Omega \rho_\varepsilon^2(\cdot, t) + \int_0^t \int_\Omega |\nabla \rho_\varepsilon|^2 = \frac{1}{2}\int_\Omega \rho_0^2 - \int_0^t \int_\Omega \rho_\varepsilon F_\varepsilon(n_\varepsilon) \leq \frac{1}{2}\int_\Omega \rho_0^2 \quad \text{for all} \;\; \varepsilon \in (0, \varepsilon_*)$$
$$\text{and any} \;\; t > 0,$$

which implies (8.2). In view of c_ε-equation in (2.1), (2.13) and (2.14), we deduce from integration by parts and Young's inequality that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2 &= \int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)\cdot(c_{\varepsilon t}-\bar{\rho}_{\varepsilon t})\\
&= \int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)\cdot\Big(\Delta c_\varepsilon - c_\varepsilon+\rho_\varepsilon-u_\varepsilon\cdot\nabla c_\varepsilon+\frac{1}{|\Omega|}\int_\Omega \rho_\varepsilon n_\varepsilon\Big)\\
&= -\int_\Omega |\nabla c_\varepsilon|^2-\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2-\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)(\bar{\rho}_\varepsilon-\rho_\varepsilon)\\
&\quad+\frac{1}{|\Omega|}\int_\Omega c_\varepsilon\int_\Omega \rho_\varepsilon n_\varepsilon-\bar{\rho}_\varepsilon\int_\Omega \rho_\varepsilon n_\varepsilon\\
&\le -\int_\Omega |\nabla c_\varepsilon|^2-\frac{1}{2}\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2+\frac{1}{2}\int_\Omega (\bar{\rho}_\varepsilon-\rho_\varepsilon)^2+\frac{1}{|\Omega|}\int_\Omega c_\varepsilon\int_\Omega \rho_\varepsilon n_\varepsilon\\
&\le -\int_\Omega |\nabla c_\varepsilon|^2-\frac{1}{2}\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2+\frac{1}{2}\int_\Omega (\bar{\rho}_\varepsilon-\rho_\varepsilon)^2+M_c\int_\Omega \rho_\varepsilon n_\varepsilon
\end{aligned}$$

for all $\varepsilon\in(0,\varepsilon_*)$ and any $t>0$, whence along with the nonnegativity of $\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2$ an application of the Poincaré inequality further yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (c_\varepsilon-\bar{\rho}_\varepsilon)^2+\int_\Omega |\nabla c_\varepsilon|^2 &\le \frac{1}{2}\int_\Omega (\rho_\varepsilon-\bar{\rho}_\varepsilon)^2+M_c\int_\Omega \rho_\varepsilon n_\varepsilon\\
&\le \frac{C_P}{2}\int_\Omega |\nabla \rho_\varepsilon|^2+M_c\int_\Omega \rho_\varepsilon n_\varepsilon
\end{aligned}\tag{8.4}$$

for all $\varepsilon\in(0,\varepsilon_*)$ and any $t>0$. Therefore, upon integrating (8.4) on $(0,t)$ for each $t>0$, (8.3) follows from (8.1) and (8.2). □

8.2. Convergence of spatial averages

As consequences of the uniform boundedness of n and the basic decay information derived above, we can show the following large time asymptotic properties with respect to the spatial averages of n and ρ by a similar reasoning as that of [8, Lemma 4.2], nevertheless the convergence obtained here is in the sense of $(0,\infty)\backslash\aleph\ni t\to\infty$ with the null set $\aleph$ provided in Lemma 7.2.

Lemma 8.2. *Let $\aleph\subset(0,\infty)$ be the null set as given in Lemma 7.2. Then*

$$\int_\Omega n(\cdot,t)\to\Big\{\int_\Omega n_0-\int_\Omega \rho_0\Big\}_+ \quad as\ \ (0,\infty)\backslash\aleph\ni t\to\infty \tag{8.5}$$

and

$$\int_\Omega \rho(\cdot,t) \to \left\{\int_\Omega \rho_0 - \int_\Omega n_0\right\}_+ \quad \textit{as} \;\; (0,\infty)\backslash\aleph \ni t \to \infty. \tag{8.6}$$

Proof. From (7.7) and (7.13), we can infer that there exists $(\varepsilon_l)_{l\in\mathbb{N}} \subset (0, \varepsilon_*)$ satisfying $\varepsilon_l \searrow 0$ as $l \to \infty$, such that

$$\rho_\varepsilon n_\varepsilon \to \rho n \quad \text{as} \;\; \varepsilon = \varepsilon_l \searrow 0 \;\; \text{a.e. in} \;\; \Omega \;\; \text{for each} \;\; t \in (0,\infty)\backslash\aleph \tag{8.7}$$

with $\aleph$ as provided in Lemma 7.2, whereupon by the dominated convergence theorem

$$\int_\Omega \rho_\varepsilon n_\varepsilon \to \int_\Omega \rho n \quad \text{as} \;\; \varepsilon = \varepsilon_l \searrow 0 \;\; \text{for each} \;\; t \in (0,\infty)\backslash\aleph.$$

Thanks to (8.1), this together with Fatou's lemma implies

$$\int_0^\infty \int_\Omega \rho n \le C_1 \tag{8.8}$$

with some $C_1 > 0$. Similarly, in light of (7.15) and (8.2), another application of Fatou's lemma provides $C_2 > 0$ fulfilling

$$\int_0^\infty \int_\Omega |\nabla \rho|^2 \le C_2. \tag{8.9}$$

Therefore, according to [33, Lemma 4.2], we can choose $(t_j)_{j\in\mathbb{N}_+} \subset (0,\infty)$ satisfying $t_j \to \infty$ as $j \to \infty$ such that

$$\int_{t_j}^{t_j+1} \int_\Omega \rho n \to 0 \tag{8.10}$$

and

$$\int_{t_j}^{t_j+1} \int_\Omega |\nabla \rho|^2 \to 0 \tag{8.11}$$

as $j \to \infty$. Note that

$$\int_{t_j}^{t_j+1}\int_\Omega \rho n = \int_{t_j}^{t_j+1}\int_\Omega (\rho-\bar\rho)n + \int_{t_j}^{t_j+1}\bar\rho\int_\Omega n$$
$$= \int_{t_j}^{t_j+1}\int_\Omega (\rho-\bar\rho)n + \frac{1}{|\Omega|}\int_{t_j}^{t_j+1}\left(\int_\Omega \rho\right)\left(\int_\Omega n\right)$$

for all $j \in \mathbb{N}$, where invoking the Hölder inequality and the Poincaré inequality, we deduce from the uniform boundedness of n and (8.11) that

$$\left|\int_{t_j}^{t_j+1}\int_\Omega (\rho-\bar\rho)n\right| \le \int_{t_j}^{t_j+1} \|\rho(\cdot,s)-\bar\rho(\cdot,s)\|_{L^2(\Omega)}\|n(\cdot,s)\|_{L^2(\Omega)}ds$$
$$\le M_n|\Omega|^{\frac12}C_P\int_{t_j}^{t_j+1}\|\nabla\rho(\cdot,s)\|_{L^2(\Omega)}ds$$
$$\le M_n|\Omega|^{\frac12}C_P\left(\int_{t_j}^{t_j+1}\int_\Omega|\nabla\rho|^2\right)^{\frac12} \to 0 \quad \text{as } j\to\infty,$$

and thus combining with (8.10) shows that

$$\int_{t_j}^{t_j+1}\left(\int_\Omega \rho\right)\left(\int_\Omega n\right) \to 0 \quad \text{as } j\to\infty. \tag{8.12}$$

For $\int_\Omega \rho_0 \ge \int_\Omega n_0$, it is clear from (2.11) that

$$\int_\Omega \rho_\varepsilon(\cdot,t) \ge \int_\Omega n_\varepsilon(\cdot,t) \quad \text{for all } t>0,$$

whence by the dominated convergence theorem, we infer from (7.7) and (7.13) that

$$\int_\Omega \rho(\cdot,t) \ge \int_\Omega n(\cdot,t) \quad \text{for each } t\in(0,\infty)\backslash\aleph,$$

which together with (8.12) implies

$$\int_{t_j}^{t_j+1}\left(\int_\Omega n\right)^2 \to 0 \quad \text{as } j\to\infty. \tag{8.13}$$

In view of (2.9), we see that for all $t, s \in (0, \infty)$ satisfying $t > s$

$$\int_\Omega n_\varepsilon(\cdot, t) \le \int_\Omega n_\varepsilon(\cdot, s)$$

holds, whereupon again by (7.7) along with the dominated convergence theorem we have

$$\int_\Omega n(\cdot, t) \le \int_\Omega n(\cdot, s) \quad \text{for all } \ t, s \in (0, \infty)\backslash\aleph \ \text{ fulfilling } \ t > s.$$

In light of (8.13), this entails

$$\int_\Omega n(\cdot, t) \le \left\{ \int_{t_j}^{t_j+1} \left(\int_\Omega n \right)^2 \right\}^{\frac{1}{2}} \to 0 \quad \text{for each } \ t \in (t_j + 1, \infty)\backslash\aleph \ \text{ as } \ j \to \infty.$$

Consequently, (8.5) is achieved for $\int_\Omega \rho_0 \ge \int_\Omega n_0$. Since the dominated convergence theorem in conjunction with (2.11) implies

$$\int_\Omega \rho(\cdot, t) - \int_\Omega n(\cdot, t) = \int_\Omega \rho_0 - \int_\Omega n_0 \quad \text{for each } \ t \in (0, \infty)\backslash\aleph,$$

which along with (8.5) yields (8.6). As for the case $\int_\Omega \rho_0 < \int_\Omega n_0$, both (8.5) and (8.6) can be derived from a similar reasoning. □

8.3. Convergence of ρ

Based on the Hölder regularity properties provided by Lemma 6.1, the convergence of the component ρ asserted in Theorem 1.1 can be achieved by applications of Lemmas 8.1–8.2.

Lemma 8.3. *We have*

$$\rho \to \rho_\infty \quad in \ \ W^{1,\infty}(\Omega) \ \ as \ \ t \to \infty. \tag{8.14}$$

Proof. From (8.9) and the Poincaré inequality, we can find some $C_1 > 0$ such that

$$\int_0^\infty \|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|^2_{L^2(\Omega)} dt \le C_1. \tag{8.15}$$

Since the Hölder continuity property presented in (6.3) implies the uniform continuity of $0 \le t \to \|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^2(\Omega)}$, it can be inferred from (8.15) and a standard reasoning as [2, Theorem 1.1] that

$$\|\rho(\cdot, t) - \bar{\rho}(\cdot, t)\|_{L^2(\Omega)} \to 0 \quad \text{as} \ \ t \to \infty,$$

which allows for a choice of $t' > 0$ such that for arbitrary $\delta > 0$

$$\|\rho(\cdot,t) - \bar{\rho}(\cdot,t)\|_{L^2(\Omega)} \le \frac{\delta}{2} \quad \text{for all } \ t > t'.$$

Moreover, in light of (8.6), we can find some $t'' > 0$ fulfilling

$$|\bar{\rho}(\cdot,t) - \rho_\infty|^2 = \frac{1}{|\Omega|^2}\left|\int_\Omega \rho - \left\{\int_\Omega \rho_0 - \int_\Omega n_0\right\}_+\right|^2 \le \frac{\delta^2}{4|\Omega|} \quad \text{for each } \ t \in (t'',\infty)\backslash\aleph.$$

Thereupon, let $t_* := \max\{t', t''\}$, then

$$\begin{aligned} \int_\Omega |\rho(\cdot,t) - \rho_\infty|^2 \le& 2\int_\Omega |\rho(\cdot,t) - \bar{\rho}(\cdot,t)|^2 + 2\int_\Omega |\bar{\rho}(\cdot,t) - \rho_\infty|^2 \\ \le& \frac{\delta^2}{2} + \frac{\delta^2}{2} = \delta^2 \quad \text{for each } \ t \in (t_*,\infty)\backslash\aleph. \end{aligned} \tag{8.16}$$

Now, we try to ensure that (8.16) remains to hold for any $t > t_*$. Thanks to the density of $(t_*,\infty)\backslash\aleph$ in (t_*,∞), there exists $(t_j)_{j\in\mathbb{N}} \subset (t_*,\infty)\backslash\aleph$ such that $t_j \to t$ as $j \to \infty$. Since (8.16) makes it possible to extract a subsequence $(t_{j_k})_{k\in\mathbb{N}}$ of $(t_j)_{j\in\mathbb{N}}$ such that $\rho(\cdot,t_{j_k}) - \rho_\infty \rightharpoonup z$ in $L^2(\Omega)$ as $k \to \infty$, which also indicates that $\rho(\cdot,t_{j_k}) - \rho_\infty \to z$ in $L^2(\Omega)$ as $k \to \infty$ because $L^2(\Omega)$ is a Hilbert space, we conclude from the uniform continuity of $t \mapsto \|\rho(\cdot,t) - \bar{\rho}(\cdot,t)\|_{L^2(\Omega)}$ that actually $z = \rho(\cdot,t) - \rho_\infty$, whence

$$\|\rho(\cdot,t) - \rho_\infty\|_{L^2(\Omega)} \le \liminf_{k\to\infty} \|\rho(\cdot,t_{j_k}) - \rho_\infty\|_{L^2(\Omega)} \le \delta,$$

which shows

$$\rho(\cdot,t) - \rho_\infty \to 0 \quad \text{in } \ L^2(\Omega) \ \text{ as } \ t \to \infty. \tag{8.17}$$

In view of (6.3) and (6.5), there exist some $C_2 > 0$ and $\mu \in (0,1)$ such that

$$\|\rho(\cdot,t) - \rho_\infty\|_{C^{1+\mu}(\bar{\Omega})} \le C_2 \quad \text{for all } \ t > t_*. \tag{8.18}$$

Thanks to the embedding $C^{1+\mu}(\bar{\Omega}) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow L^2(\Omega)$, where the first one is compact, an application of an Ehrling-type lemma provides some $C_3 > 0$ such that

$$\|\rho(\cdot,t) - \rho_\infty\|_{W^{1,\infty}(\Omega)} \le \frac{\delta}{2C_2}\|\rho(\cdot,t) - \rho_\infty\|_{C^{1+\mu}(\bar{\Omega})} + C_3\|\rho(\cdot,t) - \rho_\infty\|_{L^2(\Omega)} \quad \text{for all } \ t > t_*,$$

where from (8.17) we can pick $t^* > t_*$ such that

$$\|\rho(\cdot,t) - \rho_\infty\|_{L^2(\Omega)} \le \frac{\delta}{2C_3} \quad \text{for all } \ t > t^*,$$

and thereby

$$\|\rho(\cdot,t)-\rho_\infty\|_{W^{1,\infty}(\Omega)} \le \frac{\delta}{2}+\frac{\delta}{2}=\delta \quad \text{for all } \ t>t^*,$$

as claimed. □

8.4. Convergence of c

With the aid of the convergence of ρ, a standard testing procedure along with an interpolation type argument as Lemma 8.3 can yield the following stabilization of c as declared in Theorem 1.1.

Lemma 8.4. *The solution component c fulfills*

$$c \to \rho_\infty \quad in \ \ W^{1,\infty}(\Omega) \ \ as \ \ t\to\infty. \tag{8.19}$$

Proof. In light of (6.2) and (6.4), it is sufficient for us to verify the convergence in $L^2(\Omega)$ according to the reasoning of Lemma 8.3. Testing c_ε-equation in (2.1) by $c_\varepsilon-\rho_\infty$, we make use of Young's inequality to have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (c_\varepsilon-\rho_\infty)^2+\int_\Omega |\nabla c_\varepsilon|^2 &= \int_\Omega (c_\varepsilon-\rho_\infty)(-c_\varepsilon+\rho_\varepsilon)\\
&= -\int_\Omega (c_\varepsilon-\rho_\infty)^2+\int_\Omega (c_\varepsilon-\rho_\infty)(\rho_\varepsilon-\rho_\infty)\\
&\le -\frac{1}{2}\int_\Omega (c_\varepsilon-\rho_\infty)^2+\frac{1}{2}\int_\Omega (\rho_\varepsilon-\rho_\infty)^2\\
&\qquad \text{for all } \ t>0 \ \text{ and } \ \varepsilon\in(0,1),
\end{aligned}$$

whence an ODE comparison argument yields

$$\int_\Omega (c_\varepsilon-\rho_\infty)^2 \le e^{-t}\int_\Omega (c_0-\rho_\infty)^2+\int_0^t e^{-(t-s)}\int_\Omega (\rho_\varepsilon-\rho_\infty)^2 \quad \text{for all } \ t>0 \ \text{ and } \ \varepsilon\in(0,1), \tag{8.20}$$

where by (2.12)

$$\begin{aligned}
\int_0^t e^{-(t-s)}\int_\Omega (\rho_\varepsilon-\rho_\infty)^2 &\le (\|\rho_0\|_{L^\infty(\Omega)}+\rho_\infty)^2\cdot|\Omega|\cdot\int_0^t e^{-(t-s)}ds\\
&= (\|\rho_0\|_{L^\infty(\Omega)}+\rho_\infty)^2\cdot|\Omega|\cdot(1-e^{-t})\\
&\le (\|\rho_0\|_{L^\infty(\Omega)}+\rho_\infty)^2\cdot|\Omega| \quad \text{for all } \ t>0 \ \text{ and } \ \varepsilon\in(0,1).
\end{aligned}$$

Thereupon, thanks to the boundedness of $\int_\Omega (c_\varepsilon - \rho_\infty)^2$ guaranteed by (2.13) as well as the pointwise continuity properties implied by (7.10) and (7.13), an application of the dominated convergence theorem to (8.20) entails

$$\int_\Omega (c-\rho_\infty)^2 \le e^{-t}\int_\Omega (c_0-\rho_\infty)^2 + \int_0^t e^{-(t-s)}\int_\Omega (\rho-\rho_\infty)^2 \quad \text{for all } t>0. \tag{8.21}$$

Now, we let $K := (M_c+\rho_\infty)^2 \cdot |\Omega|$, which bounds both $\int_\Omega (c-\rho_\infty)^2$ and $\int_\Omega (\rho-\rho_\infty)^2$ from above for all $t \ge 0$ by (2.12), (2.13), (7.10) and (7.13), then for given $\delta > 0$ there exists some $j_* \in \mathbb{N}$ fulfilling

$$e^{-j_*}K \le \frac{\delta}{3}.$$

In addition, Lemma 8.3 allows for a choice of $t_* > j_*$ sufficiently large such that

$$\int_t^{t+1}\int_\Omega (\rho-\rho_\infty)^2 \le \frac{\delta}{3j_*} \quad \text{for all } t \ge t_* - j_*.$$

Therefore, (8.21) can be rewritten as

$$\begin{aligned}
\int_\Omega (c-\rho_\infty)^2 \le& e^{-t}\int_\Omega (c_0-\rho_\infty)^2 + \int_0^{t-j_*} e^{-(t-s)}\int_\Omega (\rho-\rho_\infty)^2 \\
&+ \sum_{k=1}^{j_*}\int_{t-j_*+k-1}^{t-j_*+k} e^{-(t-s)}\int_\Omega (\rho-\rho_\infty)^2 \\
\le& Ke^{-t} + K\int_0^{t-j_*} e^{-(t-s)}ds + \sum_{k=1}^{j_*}\int_{t-j_*+k-1}^{t-j_*+k}\int_\Omega (\rho-\rho_\infty)^2 \\
\le& Ke^{-j_*} + K(e^{-j_*}-e^{-t}) + j_*\cdot\frac{\delta}{3j_*} \\
\le& 2Ke^{-j_*} + \frac{\delta}{3} \\
\le& \delta \quad \text{for all } t > t_*,
\end{aligned}$$

which verifies

$$c-\rho_\infty \to 0 \quad \text{in } L^2(\Omega) \text{ as } t\to\infty,$$

and thus (8.19) follows from a similar argument as that of Lemma 8.3. □

8.5. Convergence of n

Since the presence of degenerate diffusion in n_ε-equation of (2.1) makes it impossible to derive the decay information with respect to n_ε as exhibited in (8.2) and (8.3) for ρ_ε and c_ε, respectively, we thus turn to resort to a quasi-energy structure being similar as that constructed in [30] to show that the quantity $\int_\Omega (n_\varepsilon - n_\infty)^2$ would keep small during a certain short-time.

Lemma 8.5. *There exist some $C > 0$ and $\varepsilon_* \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_*)$ and each chosen $t_* \geq 0$*

$$\int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\infty\Big)^2 \leq C \cdot \left\{ \int_\Omega \Big(n_\varepsilon(\cdot,t_*) - n_\infty\Big)^2 + \int_\Omega |\nabla c_\varepsilon(\cdot,t_*)|^2 + \int_{t_*}^t \int_\Omega \rho_\varepsilon n_\varepsilon + \sup_{s\in(t_*,t_*+1)} \int_\Omega |\nabla \rho_\varepsilon(\cdot,s)|^2 \right\} \quad \textit{for all } \ t \in (t_*, t_*+1). \tag{8.22}$$

Proof. Firstly, let us choose certain $\varepsilon_* \in (0, 1)$ sufficiently small such that $M_n := \sup_{\varepsilon\in(0,1)} \|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))} \leq \frac{1}{\varepsilon_*}$, then for all $\varepsilon \in (0, \varepsilon_*)$, (2.3) together with (2.4) shows $F_\varepsilon(n_\varepsilon) \equiv n_\varepsilon$. Next, testing n_ε-equation in (2.1) by $n_\varepsilon - n_\infty$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (n_\varepsilon - n_\infty)^2 &= -\int_\Omega D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega \rho_\varepsilon n_\varepsilon(n_\varepsilon - n_\infty) \\ &\leq \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon + n_\infty \int_\Omega \rho_\varepsilon n_\varepsilon \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*). \end{aligned} \tag{8.23}$$

Let

$$I_\varepsilon(s) := \int_0^s \sigma F_\varepsilon'(\sigma) d\sigma, \qquad s > 0,$$

then upon integration by parts and $\int_\Omega \Delta c_\varepsilon = 0$ for all $t > 0$,

$$\begin{aligned} \int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon &= \int_\Omega \nabla I_\varepsilon(n_\varepsilon) \cdot \nabla c_\varepsilon \\ &= -\int_\Omega I_\varepsilon(n_\varepsilon)\Delta c_\varepsilon \\ &= -\int_\Omega \Big(I_\varepsilon(n_\varepsilon) - I_\varepsilon(n_\infty)\Big) \cdot \Delta c_\varepsilon \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*). \end{aligned} \tag{8.24}$$

Since (2.5) implies $0 \le I_\varepsilon'(s) \le s$, we recall the definition of $M_n > 0$ introduced in Section 6 and infer from the mean value theorem that

$$\begin{aligned}\big|I_\varepsilon(n_\varepsilon(x,t)) - I_\varepsilon(n_\infty)\big| \le& \|I_\varepsilon'\|_{L^\infty(0,M_n)}|n_\varepsilon(x,t) - n_\infty|\\ \le& M_n|n_\varepsilon(x,t) - n_\infty| \quad \text{for all } \ x \in \Omega, t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*),\end{aligned}$$

which along with an application of Young's inequality to the right-hand side of (8.24) shows that

$$\begin{aligned}\int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon \le& M_n \int_\Omega |n_\varepsilon - n_\infty| \cdot |\Delta c_\varepsilon|\\ \le& \frac{1}{2}\int_\Omega (n_\varepsilon - n_\infty)^2 + \frac{M_n^2}{2}\int_\Omega |\Delta c_\varepsilon|^2 \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*).\end{aligned}$$

Thus, (8.23) can be rewritten as

$$\begin{aligned}\frac{d}{dt}\int_\Omega (n_\varepsilon - n_\infty)^2 \le& \int_\Omega (n_\varepsilon - n_\infty)^2 + M_n^2 \int_\Omega |\Delta c_\varepsilon|^2\\ &+ 2n_\infty \int_\Omega \rho_\varepsilon n_\varepsilon \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*).\end{aligned} \tag{8.25}$$

For absorbing the second integral on the right-hand side, we test c_ε-equation by $-\Delta c_\varepsilon$ and employ Young's inequality to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 =& \int_\Omega (u_\varepsilon \cdot \nabla c_\varepsilon)\Delta c_\varepsilon - \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega \nabla \rho_\varepsilon \cdot \nabla c_\varepsilon\\ \le& \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 + \frac{1}{2}\int_\Omega |u_\varepsilon|^2 \cdot |\nabla c_\varepsilon|^2 - \frac{1}{2}\int_\Omega |\nabla c_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla \rho_\varepsilon|^2\\ \le& \frac{1}{2}\int_\Omega |\Delta c_\varepsilon|^2 + \frac{M_u^2}{2}\int_\Omega |\nabla c_\varepsilon|^2\\ &+ \frac{1}{2}\int_\Omega |\nabla \rho_\varepsilon|^2 \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*),\end{aligned}$$

that is

$$\frac{d}{dt}\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\Delta c_\varepsilon|^2 \le M_u^2 \int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega |\nabla \rho_\varepsilon|^2 \quad \text{for all } \ t > 0 \ \text{ and } \ \varepsilon \in (0, \varepsilon_*), \tag{8.26}$$

where we set $M_u := \sup_{\varepsilon \in (0,1)} \|u_\varepsilon\|_{C(\bar\Omega \times (0,\infty))}$ according to Lemma 6.1. It follows from a combination of (8.25) and (8.26) that

$$\frac{d}{dt}\left\{\int_\Omega (n_\varepsilon - n_\infty)^2 + M_n^2 \int_\Omega |\nabla c_\varepsilon|^2\right\}$$
$$\le \int_\Omega (n_\varepsilon - n_\infty)^2 + M_n^2 M_u^2 \int_\Omega |\nabla c_\varepsilon|^2 + 2n_\infty \int_\Omega \rho_\varepsilon n_\varepsilon \tag{8.27}$$
$$+ M_n^2 \int_\Omega |\nabla \rho_\varepsilon|^2 \quad \text{for all } \ t>0 \ \text{ and } \ \varepsilon \in (0,\varepsilon_*).$$

Denote $y(t) := \int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\infty\Big)^2 + M_n^2 \int_\Omega |\nabla c_\varepsilon(\cdot,t)|^2, t \ge 0$, then (8.27) yields

$$y'(t) \le C_1 y(t) + C_2 \left\{\int_\Omega \rho_\varepsilon n_\varepsilon + \int_\Omega |\nabla \rho_\varepsilon|^2\right\} \quad \text{for all } \ t>0 \ \text{ and } \ \varepsilon \in (0,\varepsilon_*)$$

with $C_1 := \max\{1, M_u^2\}$ and $C_2 := \max\{2n_\infty, M_n^2\}$. We thus make use of an ODE comparison to have

$$\begin{aligned} y(t) \le & e^{C_1(t-t_*)} y(t_*) + C_2 \int_{t_*}^t e^{C_1(t-s)} \left\{\int_\Omega \rho_\varepsilon(\cdot,s) n_\varepsilon(\cdot,s) + \int_\Omega |\nabla \rho_\varepsilon(\cdot,s)|^2\right\} \\ \le & e^{C_1} y(t_*) + C_2 e^{C_1} \left\{\int_{t_*}^t \int_\Omega \rho_\varepsilon n_\varepsilon + \sup_{s\in(t_*,t_*+1)} \int_\Omega |\nabla \rho_\varepsilon(\cdot,s)|^2\right\} \end{aligned}$$

for all $t \in (t_*, t_*+1)$ and $\varepsilon \in (0,\varepsilon_*)$, which implies (8.22).

Now, let us consider the asymptotic property of n_ε in two cases. For $\int_\Omega n_0 \le \int_\Omega \rho_0$ i.e. $n_\infty = 0$, Lemma 8.2 shows the convergence of n toward n_∞ in $L^1(\Omega)$ except for a null set of time, whereas for $\int_\Omega n_0 > \int_\Omega \rho_0$ i.e. $n_\infty > 0$, the verification of the convergence to n_∞ relies on an application of L^p testing procedure as well as the estimate (8.3) in Lemma 8.1. □

Lemma 8.6. *For $\aleph \subset (0,\infty)$ as provided in Lemma 7.2, if $\int_\Omega n_0 \le \int_\Omega \rho_0$, then*

$$n(\cdot,t) \to n_\infty \quad \text{in} \ \ L^1(\Omega) \ \ \text{as} \ \ (0,\infty)\backslash\aleph \ni t \to \infty, \tag{8.28}$$

whereas if $\int_\Omega n_0 > \int_\Omega \rho_0$, then

$$n(\cdot,t) \to n_\infty \quad \text{in} \ \ L^2(\Omega) \ \ \text{as} \ \ (0,\infty)\backslash\aleph \ni t \to \infty. \tag{8.29}$$

Proof. In the case when $\int_\Omega n_0 \le \int_\Omega \rho_0$, it is clear that $n_\infty = \frac{1}{|\Omega|}\{\int_\Omega n_0 - \int_\Omega \rho_0\}_+ = 0$, whence for $\aleph \subset (0,\infty)$ as given in Lemma 7.2, (8.5) implies

$$\|n(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega n(\cdot,t) \to 0 \quad \text{as} \ \ (0,\infty)\backslash\aleph \ni t \to \infty,$$

as claimed. $\square$

For $\int_\Omega n_0 > \int_\Omega \rho_0$, Lemma 8.5 guarantees the existence of $C_1 > 0$ such that for each $t_* > 0$ and all $\varepsilon \in (0, \varepsilon_*)$

$$\int_\Omega \Big(n_\varepsilon(\cdot,t) - n_\infty\Big)^2 \le C_1 \cdot \left\{ \int_\Omega \Big(n_\varepsilon(\cdot,t_*) - n_\infty\Big)^2 + \int_\Omega |\nabla c_\varepsilon(\cdot,t_*)|^2 \right.$$
$$\left. + \int_{t_*}^t \int_\Omega \rho_\varepsilon n_\varepsilon + \sup_{s\in(t_*,t_*+1)} \int_\Omega |\nabla \rho_\varepsilon(\cdot,s)|^2 \right\}$$

holds for any $t \in (t_*, t_*+1)$. In view of the convergences $c_\varepsilon \to c$, $\rho_\varepsilon \to \rho$ in $C^0_{loc}\big(\bar\Omega \times [0,\infty)\big)$ and $\nabla c_\varepsilon \to \nabla c$, $\nabla\rho_\varepsilon \to \nabla\rho$ in $C^0_{loc}\big(\bar\Omega \times [1,\infty)\big)$ as $\varepsilon = \varepsilon_l \searrow 0$ by Lemma 7.2 as well as the boundedness of $(n_\varepsilon(\cdot,t_*) - n_\infty)_{\varepsilon\in(0,1)}$ by Lemma 5.1, we take $\varepsilon = \varepsilon_l \searrow 0$ and infer from (7.7) and the dominated convergence theorem that

$$\int_\Omega \Big(n(\cdot,t) - n_\infty\Big)^2 \le C_1 \cdot \left\{ \int_\Omega \Big(n(\cdot,t_*) - n_\infty\Big)^2 + \int_\Omega |\nabla c(\cdot,t_*)|^2 \right.$$
$$\left. + \int_{t_*}^t \int_\Omega \rho n + \sup_{s\in(t_*,t_*+1)} \int_\Omega |\nabla \rho(\cdot,s)|^2 \right\} \tag{8.30}$$

for any $t_* \in (1,\infty) \setminus \aleph$ and each $t \in (t_*, t_*+1) \setminus \aleph$. For the sake of estimating the right-hand side of (8.30) properly, we pick some $\theta \ge \max\{1, m-1\}$ and test n_ε-equation in (2.1) by $n_\varepsilon^{2\theta-m}$ to obtain

$$\begin{aligned}
\frac{1}{2\theta-m+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\theta-m+1} =& -(2\theta-m)\int_\Omega n_\varepsilon^{2\theta-m-1} D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 \\
&+ (2\theta-m)\int_\Omega n_\varepsilon^{2\theta-m} F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon - \int_\Omega \rho_\varepsilon n_\varepsilon^{2\theta-m} F_\varepsilon(n_\varepsilon) \\
\le& -(2\theta-m)\int_\Omega n_\varepsilon^{2\theta-m-1} D_\varepsilon(n_\varepsilon)|\nabla n_\varepsilon|^2 \\
&+ (2\theta-m)\int_\Omega n_\varepsilon^{2\theta-m} F'_\varepsilon(n_\varepsilon)\nabla n_\varepsilon \cdot \nabla c_\varepsilon
\end{aligned}$$

for each $t > 0$ and all $\varepsilon \in (0,1)$ thanks to the nonnegativity of $\int_\Omega \rho_\varepsilon n_\varepsilon^{2\theta-m} F_\varepsilon(n_\varepsilon)$. Combining with (1.9), (2.2) and (2.5), we employ Young's inequality to have

$$\frac{1}{2\theta-m+1}\frac{d}{dt}\int_\Omega n_\varepsilon^{2\theta-m+1} \le -C_D(2\theta-m)\int_\Omega n_\varepsilon^{2\theta-2}|\nabla n_\varepsilon|^2 + (2\theta-m)\int_\Omega n_\varepsilon^{2\theta-m}|\nabla n_\varepsilon| \cdot |\nabla c_\varepsilon|$$

$$\le -C_D(2\theta-m)\int_\Omega n_\varepsilon^{2\theta-2}|\nabla n_\varepsilon|^2+\frac{C_D(2\theta-m)}{2}\int_\Omega n_\varepsilon^{2\theta-2}|\nabla n_\varepsilon|^2$$
$$+\frac{2\theta-m}{2C_D}\int_\Omega n_\varepsilon^{2\theta-2m+2}|\nabla c_\varepsilon|^2$$
$$\le -\frac{C_D(2\theta-m)}{2}\int_\Omega n_\varepsilon^{2\theta-2}|\nabla n_\varepsilon|^2+\frac{M_n^{2\theta-2m+2}(2\theta-m)}{2C_D}\int_\Omega |\nabla c_\varepsilon|^2$$

for any $t>0$ and all $\varepsilon\in(0,1)$ with M_n introduced as in Section 6, whereupon an integration over $(0,t)$ yields

$$\frac{1}{2\theta-m+1}\int_\Omega n_\varepsilon^{2\theta-m+1}(\cdot,t)+\frac{C_D(2\theta-m)}{2}\int_0^t\int_\Omega n_\varepsilon^{2\theta-2}|\nabla n_\varepsilon|^2$$
$$\le\frac{1}{2\theta-m+1}\int_\Omega n_0^{2\theta-m+1}+\frac{M_n^{2\theta-2m+2}(2\theta-m)}{2C_D}\int_0^t\int_\Omega|\nabla c_\varepsilon|^2$$

for all $t>0$ and $\varepsilon\in(0,1)$. In view of (8.3), we can find certain $\varepsilon_*\in(0,1)$ and $C_2>0$ such that

$$\int_0^\infty\int_\Omega|\nabla n_\varepsilon^\theta|^2\le C_2\quad\text{for all }\ \varepsilon\in(0,\varepsilon_*),$$

which in conjunction with the Poincaré inequality provides some $C_3>0$ satisfying

$$\int_0^\infty\left\|n_\varepsilon^\theta(\cdot,t)-\gamma_\varepsilon^\theta(t)\right\|_{L^2(\Omega)}^2dt\le C_3\quad\text{for all }\ \varepsilon\in(0,\varepsilon_*),$$

where $\gamma_\varepsilon(t):=\left\{\frac{1}{|\Omega|}\int_\Omega n_\varepsilon^\theta(\cdot,t)\right\}^{\frac{1}{\theta}}$ for $t>0$ and $\varepsilon\in(0,1)$. Since (7.7) together with the dominated convergence theorem implies

$$\left\|n_\varepsilon^\theta(\cdot,t)-\gamma_\varepsilon^\theta(t)\right\|_{L^2(\Omega)}^2\to\left\|n^\theta(\cdot,t)-\gamma^\theta(t)\right\|_{L^2(\Omega)}^2\quad\text{as }\ \varepsilon=\varepsilon_l\searrow 0$$

with $\gamma(t):=\left\{\frac{1}{|\Omega|}\int_\Omega n^\theta(\cdot,t)\right\}^{\frac{1}{\theta}}$ for $t\in(0,\infty)\setminus\aleph$, we thus make use of Fatou's lemma with respect to time on $(0,\infty)$ to infer that

$$\int_0^\infty\left\|n^\theta(\cdot,t)-\gamma^\theta(t)\right\|_{L^2(\Omega)}^2dt\le C_3.\tag{8.31}$$

In light of (7.7) and (7.13), another application of the dominated convergence theorem to (2.11) entails that as $\varepsilon = \varepsilon_l \searrow 0$

$$\int_\Omega n(\cdot,t) - \int_\Omega \rho(\cdot,t) = \int_\Omega n_0 - \int_\Omega \rho_0 \quad \text{for each } \ t \in (0,\infty) \setminus \aleph,$$

which implies

$$\int_\Omega n(\cdot,t) \geq \int_\Omega n_0 - \int_\Omega \rho_0 \quad \text{for each } \ t \in (0,\infty) \setminus \aleph$$

due to the nonnegativity of $\int_\Omega \rho(\cdot,t)$ for all $t > 0$. Thus, the Hölder inequality warrants that in the case when $\int_\Omega n_0 > \int_\Omega \rho_0$

$$|\Omega| n_\infty = \int_\Omega n_0 - \int_\Omega \rho_0 \leq \int_\Omega n(\cdot,t) \leq |\Omega|^{1-\frac{1}{\theta}} \cdot \left\{ \int_\Omega n^\theta(\cdot,t) \right\}^{\frac{1}{\theta}} \quad \text{for each } \ t \in (0,\infty) \setminus \aleph.$$

This shows $n_\infty \leq \gamma(t)$ for each $t \in (0,\infty) \setminus \aleph$. From the fact that $|\xi^\theta - \eta^\theta| \geq \eta^{\theta-1} \cdot |\xi - \eta|$ for any $\xi \geq 0$ and $\eta \geq 0$, it follows that

$$\begin{aligned} \left| n^\theta(\cdot,t) - \gamma^\theta(t) \right|^2 \geq & \gamma^{2\theta-2}(t) \cdot |n(\cdot,t) - \gamma(t)|^2 \\ \geq & n_\infty^{2\theta-2} \cdot |n(\cdot,t) - \gamma(t)|^2 \end{aligned}$$

a.e. in Ω for each $t \in (0,\infty) \setminus \aleph$, whence (8.31) implies

$$\int_0^\infty \left\| n(\cdot,t) - \gamma(t) \right\|_{L^2(\Omega)}^2 dt \leq C_4 \tag{8.32}$$

with $C_4 := n_\infty^{2-2\theta} \cdot C_3$. Recalling (8.5), (8.8) and Lemmas 8.3–8.4, we combine with (8.32) to infer that for given $\delta > 0$ there exists some $t' > 1$ large enough such that

$$\int_{t'-1}^\infty \left\| n(\cdot,t) - \gamma(t) \right\|_{L^2(\Omega)}^2 dt \leq \frac{\delta}{24 C_1} \tag{8.33}$$

and

$$\left| \int_\Omega n(\cdot,t) - |\Omega| n_\infty \right| \leq \sqrt{\frac{\delta |\Omega|}{24 C_1}} \quad \text{for each } \ t \in (t'-1,\infty) \setminus \aleph \tag{8.34}$$

as well as

$$\int_{t'-1}^{\infty}\int_{\Omega} \rho n \le \frac{\delta}{4C_1}, \tag{8.35}$$

and that

$$\int_{\Omega} |\nabla c(\cdot,t)|^2 \le \frac{\delta}{4C_1} \quad \text{for any} \;\; t > t'-1 \tag{8.36}$$

as well as

$$\int_{\Omega} |\nabla \rho(\cdot,t)|^2 \le \frac{\delta}{4C_1} \quad \text{for any} \;\; t > t'-1. \tag{8.37}$$

Since for each $t > t'$ (8.33) allows for a choice of certain $t_* = t_*(t) \in (t-1,t) \setminus \aleph$ fulfilling

$$\int_{\Omega} \left| n(\cdot,t_*) - \gamma(t_*)\right|^2 \le \frac{\delta}{24C_1}, \tag{8.38}$$

we invoke the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |\Omega| \cdot |\overline{n(\cdot,t_*)} - \gamma(t_*)| =& \left| \int_{\Omega} \Big(n(\cdot,t_*) - \gamma(t_*) \Big) \right| \\ \le & |\Omega|^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \Big(n(\cdot,t_*) - \gamma(t_*) \Big)^2 \right\}^{\frac{1}{2}} \\ \le & \sqrt{\frac{\delta|\Omega|}{24C_1}}, \end{aligned}$$

and thereby

$$\int_{\Omega} \Big(\overline{n(\cdot,t_*)} - \gamma(t_*) \Big)^2 \le \frac{\delta}{24C_1}. \tag{8.39}$$

Moreover, due to $t_* \in (t-1,t) \setminus \aleph \subset (t'-1,\infty) \setminus \aleph$, (8.34) entails

$$\int_{\Omega} \Big(\overline{n(\cdot,t_*)} - n_\infty \Big)^2 \le \frac{\delta}{24C_1}. \tag{8.40}$$

Collecting (8.38)–(8.40), we have

$$\begin{aligned}\int_\Omega \Big(n(\cdot,t_*)-n_\infty\Big)^2 \le & 2\int_\Omega \Big(n(\cdot,t_*)-\gamma(t_*)\Big)^2+2\int_\Omega \Big(\gamma(t_*)-\overline{n(\cdot,t_*)}\Big)^2 \\ & +2\int_\Omega \Big(\overline{n(\cdot,t_*)}-n_\infty\Big)^2 \le \frac{\delta}{4C_1}.\end{aligned} \tag{8.41}$$

Since $t\in(t_*,t_*+1)\setminus\aleph$, it follows from (8.30), (8.35)–(8.37) and (8.41) that

$$\int_\Omega \Big(n(\cdot,t)-n_\infty\Big)^2 \le C_1\cdot\left\{\frac{\delta}{4C_1}+\frac{\delta}{4C_1}+\frac{\delta}{4C_1}+\frac{\delta}{4C_1}\right\}=\delta,$$

which verifies (8.29) thanks to the arbitrariness of δ.

Based on Lemma 8.6, an interpolation type argument in conjunction with the continuity property of n attained in Lemma 6.2 enable us to achieve the stabilization of n in the sense asserted in Theorem 1.1.

Corollary 8.7. *For any $p\ge 1$, we have*

$$n(\cdot,t)\to n_\infty \quad in\ \ L^p(\Omega)\ \ as\ \ t\to\infty. \tag{8.42}$$

Proof. Here we only prove (8.42) for $\int_\Omega n_0\le\int_\Omega\rho_0$, while for $\int_\Omega n_0>\int_\Omega\rho_0$ the reasoning can proceed along a similar procedure. In light of the boundedness of Ω, it is sufficient to verify the validity of (8.42) for $p>1$. By the Hölder inequality and (7.8), we have

$$\|n(\cdot,t)\|_{L^p(\Omega)}\le\|n(\cdot,t)\|_{L^\infty(\Omega)}^{\frac{p-1}{p}}\|n(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{p}}\le M_n^{\frac{p-1}{p}}\|n(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{p}}\quad \text{for any}\ \ t>0$$

with M_n defined as in Section 6, whence for arbitrary $\delta>0$ (8.28) in Lemma 8.6 warrants the existence of certain $t'>0$ fulfilling

$$\|n(\cdot,t)\|_{L^p(\Omega)}\le\delta\quad \text{for each}\ \ t\in(t',\infty)\setminus\aleph. \tag{8.43}$$

Next, we try to show that (8.43) actually holds for any $t>t'$. Due to the density of $(t',\infty)\setminus\aleph$ in (t',∞), we can take $(t_j)_{j\in\mathbb{N}}\subset(t',\infty)\setminus\aleph$ such that $t_j\to t$ as $j\to\infty$. Since the boundedness of $\|n(\cdot,t_j)\|_{L^p(\Omega)}$ guaranteed by (8.43) allows for an extraction of a subsequence $(t_{j_k})_{k\in\mathbb{N}}$ from $(t_j)_{j\in\mathbb{N}}$ fulfilling $n(\cdot,t_{j_k})\rightharpoonup z$ in $L^p(\Omega)$ as $k\to\infty$, this implies $n(\cdot,t_{j_k})\to z$ in $(W_0^{2,2}(\Omega))^*$ as $k\to\infty$, which in conjunction with the generalized continuity property of n with respect to t exhibited in Lemma 6.2 shows the identity between z and $n(\cdot,t)$. As a result,

$$\|n(\cdot,t)\|_{L^p(\Omega)}\le\liminf_{k\to\infty}\|n(\cdot,t_{j_k})\|_{L^p(\Omega)}\le\delta,$$

as desired. □

8.6. Decay of u

In the final, by means of standard regularity theory and the convergence of ρ and n, we are able to detect the asymptotic behavior of u as time goes to infinity.

Lemma 8.8. *We have*

$$u(\cdot,t)\to 0 \quad in\ \ L^\infty(\Omega)\ \ as\ \ t\to\infty. \tag{8.44}$$

Proof. Throughout the proof, we use $\mathcal{P}$ to denote the Helmholtz projection from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$. Since the set $W^{2,\infty}(\Omega)$ lies in the kernel of $\mathcal{P}$, and since $n_\infty, \rho_\infty\in\mathbb{R}$, we can rewrite u_ε-equation in (2.1) as

$$u_{\varepsilon t}+Au_\varepsilon=\mathcal{P}\Big[(n_\varepsilon(\cdot,t)-n_\infty)\nabla\phi\Big]+\mathcal{P}\Big[(\rho_\varepsilon(\cdot,t)-\rho_\infty)\nabla\phi\Big],\quad x\in\Omega,\ t>0,$$

whence by variation-of-constants formula, we further have

$$\begin{aligned}u_\varepsilon(\cdot,t)=&e^{-tA}u_0+\int_0^t e^{-(t-s)A}\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-n_\infty)\nabla\phi\Big]ds\\&+\int_0^t e^{-(t-s)A}\mathcal{P}\Big[(\rho_\varepsilon(\cdot,s)-\rho_\infty)\nabla\phi\Big]ds\quad\text{for any }\ t>0.\end{aligned}$$

Due to $D(A^\alpha)\hookrightarrow L^\infty(\Omega)$ with some $\alpha\in(\frac{3}{4},1)$ by a known embedding result ([11,12]), an application of the regularized properties of the analytic semigroup $(e^{-tA})_{t\geq 0}$ ([10,23]) yields that

$$\begin{aligned}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\leq&C_1\Big\|A^\alpha e^{-tA}u_0+\int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-n_\infty)\nabla\phi\Big]ds\\&+\int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(\rho_\varepsilon(\cdot,s)-\rho_\infty)\nabla\phi\Big]ds\Big\|_{L^2(\Omega)}\\\leq&C_2\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)}\\&+C_2\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}\Big\|\mathcal{P}\Big[(n_\varepsilon(\cdot,s)-n_\infty)\nabla\phi\Big]\Big\|_{L^2(\Omega)}ds\\&+C_2\int_0^t (t-s)^{-\alpha}e^{-\lambda(t-s)}\Big\|\mathcal{P}\Big[(\rho_\varepsilon(\cdot,s)-\rho_\infty)\nabla\phi\Big]\Big\|_{L^2(\Omega)}ds\end{aligned}\tag{8.45}$$

for any $t>0$ with $C_1>0$, $C_2>0$ and certain $\lambda>0$. From (1.10) and (5.4), we can infer that

$$\begin{aligned} &C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\left[(n_\varepsilon(\cdot,s)-n_\infty)\nabla\phi\right] \right\|_{L^2(\Omega)} ds \\ \le& C_2(M_n+n_\infty)\|\nabla\phi\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2}} \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} ds \\ \le& C_2(M_n+n_\infty)\|\nabla\phi\|_{L^\infty(\Omega)}|\Omega|^{\frac{1}{2}} \cdot C_3 \quad \text{for any } t>0, \end{aligned} \tag{8.46}$$

where $M_n = \sup_{\varepsilon\in(0,1)} \|n_\varepsilon\|_{L^\infty(\Omega\times(0,\infty))}$ has been defined in Section 6 and $C_3 := \int_0^\infty \sigma^{-\alpha} e^{-\lambda\sigma} d\sigma$. Recalling (2.12), the integrability of $\int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \|\mathcal{P}[(\rho_\varepsilon(\cdot,t) - \rho_\infty)\nabla\phi]\|_{L^2(\Omega)} ds$ can be also verified through a similar reasoning. Moreover, again based on (2.12) and (5.4), we make use of the dominated convergence theorem along with (7.7) and (7.13) to deduce that as $\varepsilon=\varepsilon_l \searrow 0$ $\|n_\varepsilon(\cdot,t)-n_\infty\|_{L^2(\Omega)} \to \|n(\cdot,t)-n_\infty\|_{L^2(\Omega)}$ for each $t\in(0,\infty)\setminus\aleph$ and $\|\rho_\varepsilon(\cdot,t)-\rho_\infty\|_{L^2(\Omega)} \to \|\rho(\cdot,t)-\rho_\infty\|_{L^2(\Omega)}$ for all $t>0$. Thus, in view of (7.16) and the integrability warranted by the reasoning of (8.46), we once more apply the dominated convergence theorem to the right-hand side of (8.45) and infer that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le& C_2\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} \\ &+ C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\left[(n(\cdot,s)-n_\infty)\nabla\phi\right] \right\|_{L^2(\Omega)} ds \\ &+ C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\left[(\rho(\cdot,s)-\rho_\infty)\nabla\phi\right] \right\|_{L^2(\Omega)} ds \quad \text{for any } t>0. \end{aligned} \tag{8.47}$$

In accordance with (14.9) in [10, p.160], the first term on the right-hand side of (8.47) can be controlled as

$$\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} \le \frac{C_4}{t^\alpha} e^{-\epsilon t} \|u_0\|_{L^2(\Omega)} \quad \text{for any } t>0$$

with some $C_4>0$ and $\epsilon>0$, whereupon for given $\delta>0$ we can find some $t_*>1$ sufficiently large such that

$$\|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} \le \frac{\delta}{5C_2} \quad \text{for any } t>t_*. \tag{8.48}$$

Apart from that, Lemma 8.3 together with Corollary 8.7 also guarantees the existence of $t^*>t_*$ such that for arbitrary $t>t^*$

$$\|n(\cdot,t)-n_\infty\|_{L^2(\Omega)} \le \frac{\delta}{5C_2C_3\|\nabla\phi\|_{L^\infty(\Omega)}} \quad \text{and} \quad \|\rho(\cdot,t)-\rho_\infty)\|_{L^2(\Omega)} \le \frac{\delta}{5C_2C_3\|\nabla\phi\|_{L^\infty(\Omega)}}. \tag{8.49}$$

Now, let

$$t_1 := t^* + \left(\frac{5C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (M_n + n_\infty)|\Omega|^{\frac{1}{2}}}{\delta\lambda} \right)^{\frac{1}{\alpha}}$$

and

$$t_2 := t^* + \left(\frac{5C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (\|\rho_0\|_{L^\infty(\Omega)} + \rho_\infty)|\Omega|^{\frac{1}{2}}}{\delta\lambda} \right)^{\frac{1}{\alpha}},$$

then thanks to (8.49), for any $t > t_1$, we have

$$\begin{aligned}
& C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\Big[(n(\cdot,s) - n_\infty)\nabla\phi\Big] \right\|_{L^2(\Omega)} ds \\
\leq & C_2 \left\{ \int_0^{t^*} (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\Big[(n(\cdot,s) - n_\infty)\nabla\phi\Big] \right\|_{L^2(\Omega)} ds \right. \\
& \left. + \int_{t^*}^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\Big[(n(\cdot,s) - n_\infty)\nabla\phi\Big] \right\|_{L^2(\Omega)} ds \right\} \\
\leq & C_2 \left\{ \int_0^{t^*} (t-s)^{-\alpha} e^{-\lambda(t-s)} \|n(\cdot,s) - n_\infty\|_{L^2(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} ds \right. \\
& \left. + \int_{t^*}^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \|n(\cdot,s) - n_\infty\|_{L^2(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} ds \right\} \\
\leq & C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (M_n + n_\infty)|\Omega|^{\frac{1}{2}} (t - t^*)^{-\alpha} \int_0^t e^{-\lambda(t-s)} ds \\
& + C_2 \|\nabla\phi\|_{L^\infty(\Omega)} \cdot \frac{\delta}{5C_2C_3 \|\nabla\phi\|_{L^\infty(\Omega)}} \int_{t^*}^t (t-s)^{-\alpha} e^{-\lambda(t-s)} ds \\
\leq & C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (M_n + n_\infty)|\Omega|^{\frac{1}{2}} (t - t^*)^{-\alpha} \cdot \frac{e^{-\lambda(t-t_1)}}{\lambda} \cdot (1 - e^{-\lambda t_1}) \\
& + C_2C_3 \|\nabla\phi\|_{L^\infty(\Omega)} \cdot \frac{\delta}{5C_2C_3 \|\nabla\phi\|_{L^\infty(\Omega)}} \\
< & C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (M_n + n_\infty)|\Omega|^{\frac{1}{2}} \cdot \frac{\delta\lambda}{5C_2 \|\nabla\phi\|_{L^\infty(\Omega)} (M_n + n_\infty)|\Omega|^{\frac{1}{2}}} \cdot \frac{1}{\lambda} + \frac{\delta}{5} = \frac{2\delta}{5}.
\end{aligned}$$

Similarly, for each $t > t_2$,

$$C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \left\| \mathcal{P}\left[(\rho(\cdot,s)-\rho_\infty)\nabla\phi\right]\right\|_{L^2(\Omega)} ds < \frac{2\delta}{5}$$

also holds. Consequently, setting $t_0 := \max\{t_1, t_2\}$, we collect (8.47)–(8.49) to derive

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} < \delta \quad \text{for any } \; t > t_0,$$

which entails (8.44) thanks to the arbitrariness of $\delta > 0$. □

8.7. *Proof of Theorem 1.1*

The derivation of our main results relies on a combination of the precedent detections.

Proof of Theorem 1.1. The conclusion on global solvability in the weak sense along with the regularity properties that the solution fulfills in (1.12) has been claimed by Proposition 5.2. The stabilization exhibited in (1.13) precisely follows from Lemma 8.3, Lemma 8.4, Corollary 8.7 and Lemma 8.8. □

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