

On Caccioppoli's inequalities of Stokes equations and Navier-Stokes equations near boundary

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Abstract

We study Caccioppoli's inequalities of the non-stationary Stokes equations and Navier-Stokes equations. Our analysis is local near boundary and we prove that, in contrast to the interior case, the Caccioppoli's inequalities of the Stokes equations and the Navier-Stokes equations, in general, fail near boundary.

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1. Introduction

We consider first non-stationary Stokes equations near flat boundary

$$w_t - \Delta w + \nabla \pi = 0 \quad \operatorname{div} w = 0 \quad \text{in } Q_{(0,1),1}^+ := B_{0,1}^+ \times (0, 1), \quad (1.1)$$

where $B_{0,r}^+ := \{x = (x', x_n) \in \mathbb{R}^n : |x'| < r, x_n > 0\}$ for $r > 0$. Here no-slip boundary condition is given on the flat boundary, i.e.

$$w = 0 \quad \text{on } \Sigma := (B_{0,1} \cap \{x_n = 0\}) \times (0, 1), \quad (1.2)$$

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where $B_{0,r} := \{x \in \mathbb{R}^n : |x| < r\}$ for $r > 0$. We emphasize that boundary conditions are prescribed only on flat boundary of $Q_{(0,1),1}^+$, not on the rounded boundary, $\{x \in \mathbb{R}^n : |x| = 1, x_n > 0\} \times (0, 1)$. From now on, we denote $Q_{(0,1),1}^+$ and $B_{0,1}^+$ by Q_1^+ and B_1^+ , respectively, unless any confusion is to be expected.

We can compare similar situation to the heat equation, i.e.

$$v_t - \Delta v = 0 \quad \text{in } Q_1^+$$

with homogeneous boundary condition

$$v = 0 \quad \text{on } \Sigma.$$

It is then well-known that the following priori estimate, so called Caccioppoli's inequality, is available:

$$\|\nabla v\|_{L^2(Q_{\frac{1}{2}}^+)} \leq c \|v\|_{L^2(Q_1^+)}, \quad (1.3)$$

where $Q_{\frac{1}{2}}^+ := B_{\frac{1}{2}}^+ \times (\frac{3}{4}, 1)$ and c is independent of v . For the Stokes equations (1.1)–(1.2), an energy estimate shows

$$\|\nabla w\|_{L^2(Q_{\frac{1}{2}}^+)} \leq c \left(\|w\|_{L^2(Q_1^+)} + \|\pi\|_{L^2(Q_1^+)} \right), \quad (1.4)$$

and, to the authors' knowledge, the Caccioppoli's inequality as in (1.3) is not known for the Stokes equations (1.1)–(1.2), namely it is unknown whether or not the following inequality is available;

$$\|\nabla w\|_{L^2(Q_{\frac{1}{2}}^+)} \leq c \|w\|_{L^2(Q_1^+)}, \quad (1.5)$$

where c is independent of w .

We remark that it was shown in [11] that the maximum of normal derivatives of tangential components for Stokes equations are not controlled by the righthand side of (1.5). More precisely, an example of the Stokes equations (1.1)–(1.2) was constructed such that $\sup_{Q_{1/2}^+} |D_{x_3} w|$ is not

bounded, although $\int_{Q_1^+} |w|^2$ is finite. Furthermore, solutions of the Stokes equations (1.1)–(1.2) may not be even Hölder continuous, unless corresponding pressures are integrable (see [11, Remark 6] and compare to [14]). A simplified example was constructed in [15] as the form of shear flow with bounded velocity field in a half-space to the Stokes, and Navier-Stokes equations with no-slip boundary conditions and the gradient of the solution is, however, is unbounded near boundary.

We can also consider the Navier-Stokes equations near flat boundary, i.e.

$$u_t - \Delta u + \nabla p = -\operatorname{div}(u \otimes u), \quad \operatorname{div} u = 0 \quad \text{in } Q_1^+$$

with no-slip boundary condition

$$u = 0 \quad \text{on } \Sigma.$$

Again, we emphasize that homogeneous boundary conditions are assigned only on flat boundary of Q_1^+ . We can ask whether or not the following Caccioppoli type's inequality of the Navier-Stokes equations is satisfied:

$$\|\nabla u\|_{L^2(Q_1^+)}^2 \leq c \left(\|u\|_{L^2(Q_1^+)}^2 + \|u\|_{L^3(Q_1^+)}^3 \right), \quad (1.6)$$

where c is independent of u . The main point of the above inequality is that pressure does not appear in the righthand side.

Our main goal is to show that it is not, in general, true to obtain near boundary the Caccioppoli's inequality (1.5) of the Stokes equations and Caccioppoli type's inequality (1.6) of the Navier-Stokes equations. More precisely, we construct sequences of smooth solutions (for example $W_2^{2,1}(Q_1^+)$ -solutions) of the Stokes equations and the Navier-Stokes equations such that righthand-sides of (1.5) and (1.6) are uniformly bounded but left-hand sides are not uniformly bounded, i.e. L^2 -norms of gradient of velocities, say $\|\nabla u_n\|_{L^2(Q_1^+)}^2$, tend to infinity by passing

them to the limit (see Theorem 1.1).

For the interior case, it was independently shown in [4], [9] and [19] by different method of proofs that the Caccioppoli's inequality (1.5) of the Stokes equations is valid in the interior.

The Caccioppoli type's inequality (1.6) was also extended in the interior to *suitable* weak solutions of some nonlinear fluid equations, e.g. the Navier-Stokes equations [19], Magnetohydrodynamics equations [5] and non-Newtonian fluid flow [10].

We remark that the Caccioppoli type's inequality of the stationary case was obtained in the interior or near boundary (refer to e.g. [7], [12] and [18]).

Our main result reads as follows:

Theorem 1.1. *The Caccioppoli's inequality (1.5) of the Stokes equations and Caccioppoli type's inequality (1.6) of the Navier-Stokes equations, in general, fail near boundary.*

We remark that our analysis is only local near boundary. Our construction of sequences of solutions failing (1.5) or (1.6) are caused by non-local behavior of solutions and singular boundary conditions away from flat boundary. Therefore, such construction would not be applicable to the Stokes and the Navier-Stokes equations in domains with no-slip conditions on boundaries everywhere.

We briefly give the mainstream of how we show Theorem 1.1.

Firstly, we construct, in a half-space, a very weak solution of the Stokes equations whose normal derivatives of tangential components is not locally square integrable near boundary. More precisely, we use the explicit formula (3.7) of the Stokes equations with a boundary condition given in (3.9) and (3.10). In particular, among all split terms of the normal derivative of tangential component, it turns out that the following integral (see (3.15)) is not square integrable in a local neighborhood near boundary (see Proposition 3.2).

$$D_{x_n} \int_0^t \int_{\mathbb{R}^{n-1}} B_{in}(x' - y', x_n, t - s) g_n(y', s) dy' ds, \quad i = 1, 2, \dots, n-1,$$

where B_{in} is given in (3.5). Indeed, one crucial estimate is

$$\int_{\mathbb{R}^{n-1}} e^{-\frac{|X'-z'|^2}{t-s}} \frac{z_1}{|z'|^n} dz' \gtrsim (t-s)^{\frac{n-1}{2}} \left(1 - e^{-\frac{c}{t-s}}\right), \quad |X'| \geq \frac{1}{2},$$

which causes the unbounded L^2 -norm of normal derivative in $B_r(0) \times (t_0, t_0 + r^2)$ (see (3.26), (3.28) and (3.29)). On the other hand, we can see that the solution is integrable in $L^4(0, \infty; L^p(\mathbb{R}_+^n))$ for $n/(n-1) < p < \infty$ (see Proposition 3.2).

Next step is to regularize the boundary data that is originally singular so that corresponding solutions of the Stokes equations are regular. If the Caccioppoli's inequality near boundary is valid, then the regular solutions should satisfy the inequality. If this is the case, by passing to the limit, the gradient of the limit solution must be square integrable near boundary, which leads to a contradiction (see for details Subsection 4.1 in Section 4).

For the Navier-Stokes equations, we consider the similar situation as in the Stokes equations. If we denote by w the solution of the Stokes equations mentioned above, we look for a solution u of the form $u = w + v$ such that v solving the following perturbed Navier-Stokes equations:

$$v_t - \Delta v + \nabla \pi = -\operatorname{div}((v+w) \otimes (v+w)), \quad \operatorname{div} v = 0$$

with zero boundary and zero initial data. In fact, we construct a weak solution v whose gradient is square integrable, in case that the size of w is assumed to be a sufficiently small. Therefore, similar argument as in Stokes equations yields that the Caccioppoli's inequality near boundary fails in general for the Navier-Stokes equations as well (see for details Subsection 4.2 in Section 4).

This paper is organized as follows. In Section 2, we recall some known results and introduce new estimates that are useful for our purpose. Section 3 is devoted to constructing a very weak solution in a half space such that L^2 -norm of its gradient is not bounded. In Section 4, we present the proof of Theorem 1.1.

2. Preliminaries

For notations, we denote $x = (x', x_n)$, where the symbol $'$ means the coordinate up to $n-1$, that is, $x' = (x_1, x_2, \dots, x_{n-1})$. We write $D_{x_i}u$ as the partial derivative of u with respect to x_i , $1 \leq i \leq n$, i.e., $D_{x_i}u(x) = \frac{\partial}{\partial x_i}u(x)$. Throughout this paper we denote by c various generic positive constant and by $c(*, \dots, *)$ depending on the quantities appearing in the parenthesis.

Next, we introduce notions of *very weak solutions* for the Stokes equations and the Navier-Stokes equations with non-zero boundary values in a half-space \mathbb{R}_+^n . To be more precise, we consider first the following Stokes equations in \mathbb{R}_+^n with non-zero boundary values:

$$w_t - \Delta w + \nabla \pi = \operatorname{div} F, \quad \operatorname{div} w = 0, \quad \text{in } \mathbb{R}_+^n \times (0, \infty) \quad (2.1)$$

with

$$w|_{t=0} = 0, \quad w|_{x_n=0} = g. \quad (2.2)$$

We now define very weak solutions of (2.1)–(2.2).

Definition 2.1. Let $g \in L_{\text{loc}}^1(\partial\mathbb{R}_+^n \times (0, T))$ and $F \in L_{\text{loc}}^1(\mathbb{R}_+^n \times (0, T))$. A vector field $u \in L_{\text{loc}}^1(\mathbb{R}_+^n \times (0, T))$ is called a very weak solution of the Stokes equations (2.1)–(2.2), if the following equality is satisfied:

$$-\int_0^T \int_{\mathbb{R}_+^n} u \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}_+^n} (u \cdot \Phi_t - F : \nabla \Phi) dx dt - \int_0^T \int_{\partial \mathbb{R}_+^n} g \cdot D_{x_n} \Phi dx' dt$$

for each $\Phi \in C_c^2(\overline{\mathbb{R}_+^n} \times [0, T])$ with

$$\operatorname{div} \Phi = 0, \quad \Phi|_{\partial \mathbb{R}_+^n \times (0, T)} = 0, \quad \Phi(\cdot, T) = 0. \quad (2.3)$$

In addition, for each $\Psi \in C_c^1(\overline{\mathbb{R}_+^n})$ with $\Psi|_{\partial \mathbb{R}_+^n} = 0$

$$\int_{\mathbb{R}_+^n} u(x, t) \cdot \nabla \Psi(x) dx = 0 \quad \text{for all } 0 < t < T. \quad (2.4)$$

We recall existence results of the Stokes equations (2.1)–(2.2).

Proposition 2.2. ([2, Theorem 1.2]) Let $1 < p, q < \infty$. Assume that $g \in L^q(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))$ and $F \in L^{q_1}(0, \infty; L^{p_1}(\mathbb{R}_+^n))$, where p_1, q_1 satisfy $n/p_1 + 2/q_1 = n/p + 2/q + 1$ with $q_1 \leq q$ and $p_1 \leq p$. Then, there is the unique very weak solution $w \in L^q(0, \infty; L^p(\mathbb{R}_+^n))$ of the Stokes equations (2.1)–(2.2) such that w satisfies

$$\|w\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \leq c \left(\|F\|_{L^{q_1}(0, \infty; L^{p_1}(\mathbb{R}_+^n))} + \|g\|_{L^q(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))} \right).$$

Here, $\dot{B}_{pp}^\alpha(\mathbb{R}^{n-1})$ is homogeneous Besov space in \mathbb{R}^{n-1} (see [1] for definition and properties of homogeneous Besov spaces).

The estimate of the following proposition may be known to the experts, but we couldn't find it in the literature. Since it will be used to prove Theorem 1.1, we give its details in Appendix A.

Proposition 2.3. Let $1 < p, q < \infty$, $F \in L^q(0, \infty; L^p(\mathbb{R}_+^n))$, $F|_{x_n=0} \in L^q(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))$ and $g = 0$. Then, there is the unique very weak solution w of the Stokes equations (2.1)–(2.2) such that w satisfies

$$\|\nabla w\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \leq c \left(\|F\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|F|_{x_n=0}\|_{L^q(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))} \right). \quad (2.5)$$

In [8], the authors showed (2.5) with additional condition $\operatorname{div} F = 0$ and $F_{in}|_{x_n=0} = 0$, $i = 1, \dots, n$. In [13], the authors showed (2.5) with condition $p = q$, then the second term of the right-hand side is dropped.

We also consider the boundary value problem of the Navier-Stokes equations in a half-space, namely

$$u_t - \Delta u + \nabla p = -\operatorname{div}(u \otimes u), \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}_+^n \times (0, \infty) \quad (2.6)$$

with

$$u|_{t=0} = 0, \quad u|_{x_n=0} = g. \quad (2.7)$$

We mean by very weak solutions of the Navier-Stokes equations (2.6)–(2.7) as follows:

Definition 2.4. Let $g \in L^1_{\text{loc}}(\partial\mathbb{R}^n_+ \times (0, T))$. A vector field $u \in L^2_{\text{loc}}(\mathbb{R}^n_+ \times (0, T))$ is called a very weak solution of the non-stationary Navier-Stokes equations (2.6)–(2.7), if the following equality is satisfied:

$$-\int_0^T \int_{\mathbb{R}^n_+} u \cdot \Delta \Phi dx dt = \int_0^T \int_{\mathbb{R}^n_+} (u \cdot \Phi_t + (u \otimes u) : \nabla \Phi) dx dt - \int_0^T \int_{\partial\mathbb{R}^n_+} g \cdot D_{x_n} \Phi dx' dt$$

for each $\Phi \in C^2_c(\overline{\mathbb{R}^n_+} \times [0, T])$ satisfying (2.3). In addition, for each $\Psi \in C^1_c(\overline{\mathbb{R}^n_+})$ with $\Psi|_{\partial\mathbb{R}^n_+} = 0$, u satisfies (2.4).

3. Stokes equations with boundary data in a half-space

We let N and Γ be the fundamental solutions to the Laplace equation and the heat equation, respectively, i.e.

$$N(x) = \begin{cases} -\frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3 \\ \frac{1}{2\pi} \ln |x|, & n = 2 \end{cases} \quad \Gamma(x, t) = \begin{cases} \frac{1}{\sqrt{4\pi t^n}} e^{-\frac{|x|^2}{4t}}, & t > 0, \\ 0, & t < 0, \end{cases}$$

where ω_n is the measure of the unit sphere in \mathbb{R}^n . For convenience, we introduce a tensor L_{ij} and a scalar function A defined by

$$L_{ij}(x, t) = D_{x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} D_{z_n} \Gamma(z, t) D_{x_i} N(x - z) dz, \quad i, j = 1, 2, \dots, n, \quad (3.1)$$

$$A(x, t) = \int_{\mathbb{R}^{n-1}} \Gamma(z', 0, t) N(x' - z', x_n) dz'.$$

The Poisson kernel (K, π) of the Stokes equations is given as

$$K_{ij}(x' - y', x_n, t) = -2\delta_{ij} D_{x_n} \Gamma(x' - y', x_n, t) - L_{ij}(x' - y', x_n, t) + \delta_{jn} \delta(t) D_{x_i} N(x' - y', x_n), \quad (3.2)$$

$$\pi_j(x' - y', x_n, t) = -2\delta(t) D_{x_j} D_{x_n} N(x' - y', x_n) + 4D_{x_n} D_{x_n} A(x' - y', x_n, t) + 4D_t D_{x_j} A(x' - y', x_n, t), \quad (3.3)$$

where $\delta(t)$ is the Dirac delta function and δ_{ij} is the Kronecker delta function.

We recall the following relations on L (see [16]):

$$\sum_{i=1}^n L_{ii} = -2D_{x_n}\Gamma, \quad L_{in} = L_{ni} + B_{in} \text{ if } i \neq n, \quad (3.4)$$

where

$$B_{in}(x, t) = - \int_{\mathbb{R}^{n-1}} D_{x_n}\Gamma(x' - z', x_n, t) D_{z_i}N(z', 0) dz'. \quad (3.5)$$

Furthermore, we remind estimates of L_{ij} defined in (3.1) (see [16]).

$$|D_{x_n}^{l_0} D_{x'}^{k_0} D_t^{m_0} L_{ij}(x, t)| \leq \frac{c}{t^{m_0+\frac{1}{2}}(|x|^2 + t)^{\frac{1}{2}n+\frac{1}{2}k_0}(x_n^2 + t)^{\frac{1}{2}l_0}}, \quad (3.6)$$

where $1 \leq i \leq n$ and $1 \leq j \leq n-1$.

It is shown in [16] that the solution (w, π) of the Stokes equations (2.1)–(2.2) with $F = 0$ is expressed by

$$w_i(x, t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t-s) g_j(y', s) dy' ds, \quad (3.7)$$

$$\pi(x, t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^{n-1}} \pi_j(x' - y', x_n, t-s) g_j(y', s) dy' ds, \quad (3.8)$$

where (K_{ij}, π_j) is Poisson kernel of the Stokes equations given in (3.2) and (3.3).

Next, we will construct a solution w of Stokes equations via (3.7) and (3.8) for a certain g such that L^2 -norm of ∇w is not bounded. For convenience, we denote

$$A = \{y' = (y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^{n-1} \mid 1 < |y'| < 2, -2 < y_i < -1, 1 \leq i \leq n-1\}.$$

We note that if $y' \in A$, then $|y'| \leq 2 < -2y_i$ for all $1 \leq i \leq n-1$. Let

$$g_n^1(y') = \chi_A(y'), \quad g_n^2(s) = \left(s - \frac{1}{16}\right)^{-\frac{1}{4}} \left| \ln \left(s - \frac{1}{16}\right) \right|^{-\frac{1}{4}-\epsilon} \chi_{\frac{1}{16} < s < \frac{1}{4}}(s), \quad (3.9)$$

where ϵ is a fixed constant with $0 < \epsilon \leq \frac{1}{4}$ and χ is characteristic function. We introduce a non-zero boundary data $g : \mathbb{R}^{n-1} \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ with only n -th component defined by

$$g(y', s) = (0, \dots, 0, g_n(y', s)) = (0, \dots, 0, \alpha g_n^1(y') g_n^2(s)), \quad (3.10)$$

where $\alpha > 0$ is determined late on. Fig. 1 is two dimensional cartoon for A that is the support of g_n^1 .

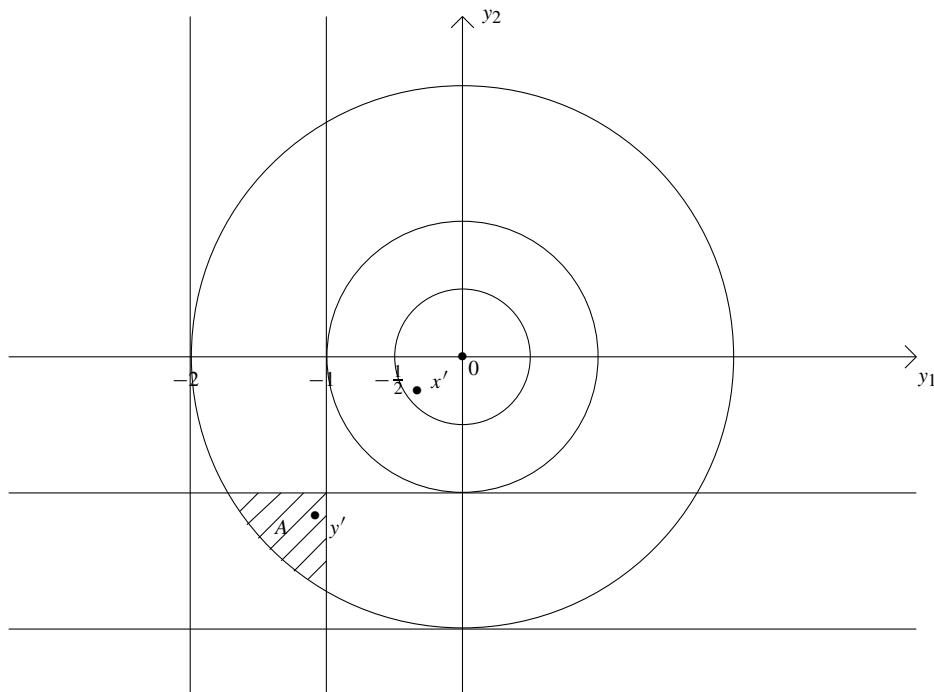


Fig. 1. A is the support of g_n^1 in \mathbb{R}^2 .

Remark 3.1. We note that $g \in L^4(0, T; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))$ for $\frac{n}{n-1} < p < \infty$. In fact, for $1 < r < \infty$ satisfying $-\frac{n-1}{r} = -\frac{1}{p} - \frac{n-1}{p}$, we can see that (refer to e.g. [1, Theorem 6.5.1])

$$\|g(t)\|_{\dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1})} \leq c \|g(t)\|_{L^r(\mathbb{R}^{n-1})} < \infty.$$

Proposition 3.2. Let w be a solution of the Stokes equations (2.1)–(2.2) with $F = 0$ defined by (3.7), where the boundary data g is given in (3.9) and (3.10). Let $\frac{n}{n-1} < p < \infty$, $t_0 = \frac{1}{16}$ and $r \leq \frac{1}{2}$. Then, w satisfies

$$\|w\|_{L^4(0, \infty; L^p(\mathbb{R}_+^n))} < \infty, \quad (3.11)$$

$$\int_{t_0}^{t_0+r^2} \int_{B(0,r)} |D_{x_n} w_i(x, t)|^2 dx dt = \infty \quad i = 1, \dots, n-1, \quad (3.12)$$

$$\int_{t_0}^{t_0+r^2} \int_{B(0,r)} |D_x w_n(x, t)|^2 dx dt < \infty. \quad (3.13)$$

Proof. We recall Proposition 2.2 and Remark 3.1, which implies that for $\frac{n}{n-1} < p < \infty$

$$\|w\|_{L^4(0,\infty;L^p(\mathbb{R}_+^n))} \leq c_p \|g\|_{L^4(0,\infty;\dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))} \leq c \|g\|_{L^4(0,\infty;L^r(\mathbb{R}^{n-1}))} < \infty.$$

Hence, the first inequality (3.11) is immediate.

Next, we show the estimate (3.12). Since the arguments are similar, we only prove the case of $i = 1$. It follows from (3.2) and (3.7) that

$$\begin{aligned} w_1(x, t) &= \int_0^t \int_A L_{1n}(x' - y', x_n, t - s) g_n(y', s) dy' ds - \int_A D_{x_1} N(x' - y', x_n) g_n(y', t) dy' \\ &:= w_1^1(x, t) + w_1^2(x, t). \end{aligned}$$

We note that $|x' - y'| \geq \frac{1}{2}$ for $|x'| \leq \frac{1}{2}$ and $y' \in A$, and so, for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$ we have

$$|\nabla w_1^2(x, t)| \leq c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4} - \epsilon}. \quad (3.14)$$

Since $L_{1n} = L_{n1} + B_{1n}$ by the second equality of (3.4), we divide $w_1^1 = w_1^{11} + w_1^{12}$ by

$$\begin{aligned} w_1^1(x, t) &= \int_0^t \int_{\mathbb{R}^{n-1}} L_{n1}(x' - y', x_n, t - s) g_n(y', s) dy' ds \\ &\quad + \int_0^t \int_{\mathbb{R}^{n-1}} B_{1n}(x' - y', x_n, t - s) g_n(y', s) dy' ds \\ &:= w_1^{11}(x, t) + w_1^{12}(x, t). \end{aligned} \quad (3.15)$$

Now, we estimate of w_1^{11} . We assume that $t - t_0 < x_n^2$. Note that $|x' - y'| \geq \frac{1}{2}$ for $x \in B(0, \frac{1}{2})$ and $y' \in A$. Due to (3.6), for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$, we have

$$\begin{aligned} |D_x w_1^{11}(x, t)| &\leq c \int_{t_0}^t \int_{1 \leq |y'| \leq 2} \frac{1}{(t - s)^{\frac{1}{2}} (x_n^2 + t - s)^{\frac{1}{2}} (|x' - y'|^2 + x_n^2 + t - s)^{\frac{n}{2}}} \\ &\quad \times (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4} - \epsilon} dy' ds \\ &\leq c x_n^{-1} \int_{t_0}^t \frac{1}{(t - s)^{\frac{1}{2}}} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4} - \epsilon} ds \\ &= c x_n^{-1} \int_{t_0}^{\frac{t_0+t}{2}} \frac{1}{(t - s)^{\frac{1}{2}}} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4} - \epsilon} ds \end{aligned}$$

$$\begin{aligned}
& + cx_n^{-1} \int_{\frac{t_0+t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds \\
& = I_1 + I_2.
\end{aligned} \tag{3.16}$$

Since $t - t_0 < x_n^2$, we have

$$\begin{aligned}
I_2 & \leq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} \int_{\frac{t_0+t}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \\
& = c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}.
\end{aligned} \tag{3.17}$$

On the other hand, noting that for $0 < a < \frac{1}{2}$, we observe

$$c_1 a^{\frac{3}{4}} |\ln a|^{-\frac{1}{4}-\epsilon} \leq \int_0^a s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds \leq c_2 a^{\frac{3}{4}} |\ln a|^{-\frac{1}{4}-\epsilon}. \tag{3.18}$$

Indeed, via Hospital's Theorem, we have

$$\lim_{a \rightarrow 0} \frac{\int_0^a s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds}{a^{\frac{3}{4}} |\ln a|^{-\frac{1}{4}-\epsilon}} = \lim_{a \rightarrow 0} \frac{a^{-\frac{1}{4}} |\ln a|^{-\frac{1}{4}-\epsilon}}{\frac{3}{4} a^{-\frac{1}{4}} |\ln a|^{-\frac{1}{4}-\epsilon} - (\frac{1}{4} + \epsilon) a^{-\frac{1}{4}} |\ln a|^{-\frac{5}{4}-\epsilon}} = \frac{4}{3}.$$

Since $a^{\frac{3}{4}} |\ln a|^{-\frac{1}{4}-\epsilon}, \int_0^a s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds > 0$ for $0 < a < \frac{1}{2}$, which implies (3.18).

Due to (3.18), since $t - t_0 < x_n^2$, we obtain

$$\begin{aligned}
I_1 & \leq c(t-t_0)^{-1} \int_{t_0}^{\frac{t_0+t}{2}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds \\
& \leq c(t-t_0)^{-1} \int_0^{\frac{t-t_0}{2}} s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds \leq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}.
\end{aligned} \tag{3.19}$$

Summing up (3.16), (3.17) and (3.19), for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$ and $t - t_0 < x_n^2$, we have

$$|D_x w_1^{11}(x, t)| \leq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}. \tag{3.20}$$

Next, we estimate w_1^{12} . Reminding (3.5), we note that

$$D_{x_n} w_1^{12}(x, t) = c_n \int_{t_0}^t \int_{\mathbb{R}^{n-1}} g_n(y', s) (t-s)^{-\frac{n+2}{2}} \left(-2 + \frac{4x_n^2}{t-s} \right)$$

$$\times e^{-\frac{z_1^2}{t-s}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'-z'|^2}{t-s}} \frac{z_1}{|z'|^n} dz' dy' ds. \quad (3.21)$$

For fixed $X' = x' - y'$, we divide \mathbb{R}^{n-1} by three disjoint sets D_1 , D_2 and D_3 defined by

$$D_1 = \left\{ z' \in \mathbb{R}^{n-1} : |X' - z'| \leq \frac{1}{10} |X'| \right\},$$

$$D_2 = \left\{ z' \in \mathbb{R}^{n-1} : |z'| \leq \frac{1}{10} |X'| \right\}, \quad D_3 = \mathbb{R}^{n-1} \setminus (D_1 \cup D_2).$$

We then split the following integral into three terms as follows:

$$\int_{\mathbb{R}^{n-1}} e^{-\frac{|X'-z'|^2}{t-s}} \frac{z_1}{|z'|^n} dz' = \int_{D_1} \cdots + \int_{D_2} \cdots + \int_{D_3} \cdots := J_1 + J_2 + J_3.$$

Since $\int_{D_2} \frac{z_1}{|z'|^n} dz' = 0$, we have $\int_{D_2} \frac{z_1}{|z'|^n} e^{-\frac{|X'-z'|^2}{t-s}} dz' = \int_{D_2} \frac{z_1}{|z'|^n} (e^{-\frac{|X'-z'|^2}{t-s}} - e^{-\frac{|X'|^2}{t-s}}) dz'$. Using the Mean-value Theorem, we have

$$|J_2| = \left| \int_{D_2} \frac{z_1}{|z'|^n} (e^{-\frac{|X'-z'|^2}{t-s}} - e^{-\frac{|X'|^2}{t-s}}) dz' \right| \leq c(t-s)^{-1} |X'| e^{-c \frac{|X'|^2}{t-s}} \int_{D_2} \frac{1}{|z'|^{n-2}} dz'$$

$$\leq c(t-s)^{-1} |X'|^2 e^{-c \frac{|X'|^2}{t-s}} \leq c e^{-c \frac{|X'|^2}{t-s}}. \quad (3.22)$$

Since $\int_{|z'|>a} e^{-|z'|^2} dz' \leq c_1 e^{-c_2 a^2}$, $a > 0$, we have

$$|J_3| \leq \frac{c}{|X'|^{n-1}} \int_{D_3} e^{-\frac{|z'-X'|^2}{t-s}} dz' \leq \frac{c}{|X'|^{n-1}} \int_{\{|z'-X'| \geq \frac{1}{10} |X'|\}} e^{-\frac{|z'-X'|^2}{t-s}} dz'$$

$$= \frac{c(t-s)^{\frac{n-1}{2}}}{|X'|^{n-1}} \int_{\{|z'| \geq \frac{1}{10} \frac{|X'|}{\sqrt{t-s}}\}} e^{-|z'|^2} dz'$$

$$\leq \frac{c(t-s)^{\frac{n-1}{2}}}{|X'|^{n-1}} e^{-c \frac{|X'|^2}{t-s}} \leq c e^{-c \frac{|X'|^2}{t-s}}. \quad (3.23)$$

Since $\frac{1}{2} \leq |X'| \leq 2$, from (3.22) and (3.23), we have

$$|J_2|, |J_3| \leq c e^{-c \frac{1}{(t-s)}} \quad (3.24)$$

Now, we estimate J_1 . We note that for $|x'| < \frac{1}{2}$ and $y' \in A$ we see that $X_1 = x_1 - y_1 \geq -\frac{1}{2} + 1 = \frac{1}{2}$ and $\frac{1}{5} |X'| \leq \frac{1}{5} (|x'| + |y'|) \leq \frac{1}{5} \cdot \frac{5}{2} \leq X_1$. Then, for $|X' - z'| \leq \frac{1}{10} |X'|$, we have

$$\begin{aligned} z_1 &= z_1 - X_1 + X_1 \geq X_1 - |z_1 - X_1| \geq X_1 - |X' - z'| \\ &\geq X_1 - \frac{1}{10}|X'| \geq \frac{1}{5}|X'| - \frac{1}{10}|X'| = \frac{1}{10}|X'|. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} J_1 &= \int_{D_1} e^{-\frac{|X'-z'|^2}{t-s}} \frac{z_1}{|z'|^n} dz' \geq \frac{c}{|X'|^{n-1}} \int_{D_1} e^{-\frac{|X'-z'|^2}{t-s}} dz' \\ &= \frac{c(t-s)^{\frac{n-1}{2}}}{|X'|^{n-1}} \int_{\{|z'| \leq \frac{1}{10} \frac{|X'|}{\sqrt{t-s}}\}} e^{-|z'|^2} dz' \\ &\geq \frac{c(t-s)^{\frac{n-1}{2}}}{|X'|^{n-1}} \geq c(t-s)^{\frac{n-1}{2}}. \end{aligned} \quad (3.25)$$

Combining (3.24) and (3.25), we obtain

$$\int_{\mathbb{R}^{n-1}} e^{-\frac{|x'-y'-z'|^2}{(t-s)}} \frac{z_1}{|z'|^n} dz' \geq c \left((t-s)^{\frac{n-1}{2}} - e^{-c \frac{1}{(t-s)}} \right) \geq c(t-s)^{\frac{n-1}{2}} \left(1 - e^{-c \frac{1}{(t-s)}} \right). \quad (3.26)$$

Noting that $-2 + \frac{4x_n^2}{t-s} > 2$ for $t - t_0 < x_n^2$ and $t_0 < s < t$, it follows from (3.21) and (3.26) that

$$\begin{aligned} |D_{x_n} w_1^{12}(x, t)| &\geq c \int_{t_0}^t \int_A (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} (t - s)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-s}} \left(1 - e^{-c \frac{1}{(t-s)}} \right) dy' ds \\ &\geq c \int_{t_0}^t (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} (t - s)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-s}} ds - c. \end{aligned}$$

Splitting the above integral over intervals $(t_0, \frac{1}{2}(t + t_0))$ and $(\frac{1}{2}(t + t_0), t)$, we first compute

$$\begin{aligned} &\int_{t_0}^{\frac{1}{2}(t+t_0)} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} (t - s)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-s}} ds \\ &\geq c(t - t_0)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-t_0}} \int_{t_0}^{\frac{1}{2}(t+t_0)} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} ds \\ &\geq c(t - t_0)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-t_0}} \int_0^{\frac{1}{2}(t-t_0)} s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds \end{aligned}$$

$$\geq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} e^{-\frac{x_n^2}{t-t_0}}.$$

For the third inequality, we use $\int_0^a s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds \geq ca^{\frac{3}{4}} |\ln a|^{-\frac{1}{4}-\epsilon}$ for $a \leq \frac{1}{2}$ (see (3.18)).

On the other hand, since $\int_a^\infty s^{\frac{1}{2}} e^{-s} ds \geq ca^{\frac{1}{2}} e^{-a}$ for $a > 1$, we have

$$\begin{aligned} & \int_{\frac{1}{2}(t+t_0)}^t (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} (t-s)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-s}} ds \\ & \geq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} \int_{\frac{1}{2}(t+t_0)}^t (t-s)^{-\frac{3}{2}} e^{-\frac{x_n^2}{t-s}} ds \\ & = c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} \int_0^{\frac{1}{2}(t-t_0)} s^{-\frac{3}{2}} e^{-\frac{x_n^2}{s}} ds \\ & \geq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} x_n^{-1} \int_{\frac{2x_n^2}{t-t_0}}^\infty s^{\frac{1}{2}} e^{-s} ds \\ & \geq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} e^{-c\frac{x_n^2}{t-t_0}}. \end{aligned}$$

Adding the estimates above, for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{2})$ and $t-t_0 \leq x_n^2$, we have

$$|D_{x_n} w_1^{12}(x, t)| \geq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} e^{-\frac{x_n^2}{t-t_0}} - c. \quad (3.27)$$

From (3.14), (3.20) and (3.27), for $t-t_0 \leq x_n^2$ and $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{2})$, we get

$$|D_{x_n} w_1(x, t)| \geq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} e^{-\frac{x_n^2}{t-t_0}} - c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}. \quad (3.28)$$

Since

$$\int_{t_0}^{\frac{1}{4}} \int_{\{(t-t_0)^{\frac{1}{2}} \leq x_n \leq 1\}} (t-t_0)^{-\frac{1}{2}} |\ln(t-t_0)|^{-\frac{1}{2}-2\epsilon} dx_n dt < \infty,$$

from (3.28), for any ϵ with $0 < \epsilon \leq \frac{1}{4}$, we obtain

$$\int_{t_0}^{t_0+r^2} \int_{B_r(0)} |D_{x_n} w_1(x, t)|^2 dx dt$$

$$\begin{aligned}
&\geq c \int_{t_0}^{t_0+r^2} \int_{\sqrt{t-t_0}}^r \int_{|x'|<r} (t-t_0)^{-\frac{3}{2}} |\ln(t-t_0)|^{-\frac{1}{2}-2\epsilon} e^{-\frac{x_n^2}{t-t_0}} dx' dx_n dt - c \\
&\geq cr^{n-1} \int_0^{r^2} \int_1^{\frac{r}{\sqrt{t}}} t^{-1} |\ln t|^{-\frac{1}{2}-2\epsilon} e^{-x_n^2} dx_n dt - c \\
&\geq cr^{n-1} \int_0^{\frac{1}{4}r^2} t^{-1} |\ln t|^{-\frac{1}{2}-2\epsilon} dt - c = \infty.
\end{aligned} \tag{3.29}$$

Therefore, we complete the proof of (3.12).

It remains to prove (3.13). It follows from (3.2) and (3.7) that

$$\begin{aligned}
w_n(x, t) &= \int_0^t \int_A D_{x_n} \Gamma(x' - y', x_n, t - s) g_n(y', s) dy' ds \\
&\quad + \int_0^t \int_A L_{nn}(x' - y', x_n, t - s) g_n(y', s) dy' ds - \int_A D_{x_n} N(x' - y', x_n) g_n(y', t) dy' \\
&:= w_n^1(x, t) + w_n^2(x, t) + w_n^3(x, t).
\end{aligned} \tag{3.30}$$

As the same reason with (3.14), we obtain

$$\int_{t_0}^{t_0+r^2} |D_x w_n^3(x, t)|^2 dx dt < \infty. \tag{3.31}$$

We note that by the first equality (3.4), the kernel of w_n^2 also satisfies (3.6). With the same estimate (3.20), for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$ and $t - t_0 < x_n^2$, we have

$$|D_x w_n^2(x, t)| \leq c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon}. \tag{3.32}$$

Now, we assume that $x_n^2 < t - t_0$. From (3.6), for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$, we have

$$\begin{aligned}
|D_x w_n^2(x, t)| &\leq c \int_{t_0}^t \frac{1}{(t-s)^{\frac{1}{2}} (x_n^2 + t-s)^{\frac{1}{2}}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds \\
&\leq c \int_{t_0}^{t-x_n^2} \frac{1}{(t-s)^{\frac{1}{2}} (x_n^2 + t-s)^{\frac{1}{2}}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds
\end{aligned}$$

$$\begin{aligned}
& + c \int_{t-x_n^2}^t \frac{1}{(t-s)^{\frac{1}{2}}(x_n^2+t-s)^{\frac{1}{2}}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds \\
& = II_1 + II_2.
\end{aligned} \tag{3.33}$$

Let $2x_n^2 < t - t_0$ ($\frac{t+t_0}{2} < t - x_n^2$). Note that for $t - x_n^2 < s < t$, we have $(s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} \leq c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon}$. Hence, we have

$$\begin{aligned}
II_2 & \leq cx_n^{-1} (t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} \int_{t-x_n^2}^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \\
& = c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon}.
\end{aligned} \tag{3.34}$$

Since $\frac{t+t_0}{2} < t - x_n^2$, from (3.18), we have

$$\begin{aligned}
II_1 & \leq c(t - t_0)^{-1} \int_{t_0}^{\frac{t+t_0}{2}} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} ds \\
& \quad + c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} \int_{\frac{t+t_0}{2}}^{t-x_n^2} (t-s)^{-1} ds \\
& = c(t - t_0)^{-1} \int_0^{\frac{t-t_0}{2}} s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds + c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} \ln\left(\frac{2x_n^2}{t - t_0}\right) \\
& \leq c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} + c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} \ln\left(\frac{2x_n^2}{t - t_0}\right).
\end{aligned} \tag{3.35}$$

Hence, summing up (3.33), (3.34) and (3.35), for $(x, t) \in B(0, \frac{1}{2}) \times (t_0, \frac{1}{4})$ and $2x_n^2 < t - t_0$, we have

$$|D_x w_n^2(x, t)| \leq c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4}-\epsilon} \left(1 + \ln\left(\frac{2x_n^2}{t - t_0}\right)\right). \tag{3.36}$$

Next, we assume that $x_n^2 < t - t_0 < 2x_n^2$ ($\frac{t+t_0}{2} > t - x_n^2 > t_0$). From (3.18), we have

$$II_1 \leq c(t - t_0)^{-1} \int_{t_0}^{t-x_n^2} (s - t_0)^{-\frac{1}{4}} |\ln(s - t_0)|^{-\frac{1}{4}-\epsilon} ds$$

$$\begin{aligned}
&\leq c(t-t_0)^{-1} \int_0^{t-t_0} s^{-\frac{1}{4}} |\ln s|^{-\frac{1}{4}-\epsilon} ds \\
&\leq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon},
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
II_2 &\leq c(t-t_0)^{-1} \int_{t-x_n^2}^{\frac{t+t_0}{2}} (s-t_0)^{-\frac{1}{4}} |\ln(s-t_0)|^{-\frac{1}{4}-\epsilon} ds \\
&\quad + c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} \int_{\frac{t+t_0}{2}}^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \\
&\leq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}.
\end{aligned} \tag{3.38}$$

From the estimates (3.33), (3.37) and (3.38), for $(x, t) \in B_{\frac{1}{2}} \times (t_0, \frac{1}{4})$ and $x_n^2 < t - t_0 < 2x_n^2$

$$|D_x w_n^2(x, t)| \leq c(t-t_0)^{-\frac{1}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon}. \tag{3.39}$$

From (3.32), (3.36) and (3.39), we have

$$\begin{aligned}
\int_{t_0}^{t_0+r^2} \int_{B_r} |\nabla w_n^2(x, t)|^2 dx dt &\leq c \int_{t_0}^{t_0+r^2} \int_0^{\sqrt{\frac{t-t_0}{2}}} (t-t_0)^{-\frac{1}{2}} |\ln(t-t_0)|^{-\frac{1}{2}-2\epsilon} \ln^2\left(\frac{2x_n^2}{t-t_0}\right) dx_n dt \\
&\quad + \int_{t_0}^{t_0+r^2} \int_0^r (t-t_0)^{-\frac{1}{2}} |\ln(t-t_0)|^{-\frac{1}{2}-2\epsilon} dx_n dt < \infty.
\end{aligned} \tag{3.40}$$

Similarly, we have

$$\int_{t_0}^{t_0+r^2} \int_{B_r} |\nabla w_n^1(x, t)|^2 dx dt < \infty. \tag{3.41}$$

Hence, from (3.30), (3.31), (3.40) and (3.41), we complete the proof of (3.13), and thus we deduce the proposition. \square

Remark 3.3. From the estimates of the proof of Proposition 3.2 (in particular w_1^{12}), for $(x, t) \in B_{\frac{1}{2}} \times (t_0, \frac{1}{4})$, we can obtain

$$|D_{x_n} w_i(x, t)| \leq c(t-t_0)^{-\frac{3}{4}} |\ln(t-t_0)|^{-\frac{1}{4}-\epsilon} e^{-\frac{x_n^2}{t-t_0}}$$

$$+ c(t - t_0)^{-\frac{1}{4}} |\ln(t - t_0)|^{-\frac{1}{4} - \epsilon} \ln\left(\frac{2x_n^2}{t - t_0}\right) \chi_{x_n^2 \leq t - t_0} \quad i = 1, \dots, n - 1$$

and which implies $D_{x_n} w_i \in L^4(t_0, \frac{1}{4}; L^p(B_{\frac{1}{2}}))$ for any $1 \leq p < 2$.

4. Proof of Theorem 1.1

4.1. Stokes equations

We take a sequence $\{g_{n,k} | k = 1, 2, \dots\}$ with $g_{n,k} \in C_c^\infty(A \times (t_0, \frac{1}{4}))$ such that $g_{n,k}$ goes to g_n in $L^4(0, \infty; L^p(\mathbb{R}^{n-1}))$ as $k \rightarrow \infty$, where g_n is defined in (3.10). Let $\tilde{g}_k = (0, g_{nk})$ and w_k be solution of the Stokes equations (2.1)–(2.2) defined by (3.7). Since the boundary data \tilde{g}_k are functions in $C_c^\infty(\mathbb{R}^{n-1} \times (0, \infty))$, we can obtain that w_k is a classical function, e.g. $w_k \in \dot{C}^{l, \frac{l}{2}}(\overline{\mathbb{R}^n_+} \times [0, \infty)) \cap \dot{W}_p^{l, \frac{l}{2}}(\mathbb{R}^n_+ \times [0, \infty))$ for any $l \in \mathbb{N}$. Note that $w_k = 0$ on $B_{\frac{1}{2}} \times (t_0, \frac{1}{4})$.

Suppose that the Caccioppoli's inequality holds for smooth solutions of the Stokes equations, that is, w_k satisfies the following inequality for $k \in \mathbb{N}$;

$$\int_{t_0}^{t_0 + \frac{1}{4}r^2} \int_{B_{\frac{1}{2}r}^+(0)} |\nabla w_k(x, t)|^2 dx dt \leq cr^{-2} \int_{t_0}^{t_0 + r^2} \int_{B_r^+(0)} |w_k(x, t)|^2 dx dt, \quad (4.1)$$

where c is independent of solutions. Due to Proposition 2.2, we note that w_k converges to u in $L^4(0, \infty; L^2(\mathbb{R}^n_+))$ and on the other hand, by the above inequality (4.1), ∇w_k converges to ∇w in $L^2(B_{\frac{1}{2}r}^+(0) \times (t_0, t_0 + r^2))$. Hence, we also get

$$\int_{t_0}^{t_0 + \frac{1}{4}r^2} \int_{B_{\frac{1}{2}r}^+(0)} |\nabla w(x, t)|^2 dx dt \leq cr^{-2} \int_{t_0}^{t_0 + r^2} \int_{B_r^+(0)} |w(x, t)|^2 dx dt \leq C.$$

This is, however, contrary to Proposition 3.2. Therefore, the Caccioppoli's inequality is not true for the Stokes equations near boundary. We complete the proof of the Theorem 1.1 for the case of Stokes equations.

4.2. Navier-Stokes equations

Let g be a boundary data defined in (3.10) and w be a solution of the Stokes equations (2.1)–(2.2) defined by (3.7). By the result of Proposition 3.2, for $\frac{n}{n-1} < p < \infty$, we have

$$\|w\|_{L^4(0, \infty; L^p(\mathbb{R}^n_+))} \leq c \|g\|_{L^4(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))} \leq c\alpha, \quad \|\nabla w\|_{L^2(B_r \times (t_0, t_0 + r^2))} = \infty \quad (4.2)$$

for all $r > 0$, where $\alpha > 0$ is defined in (3.10).

Next, we consider the following perturbed Navier-Stokes equations in $\mathbb{R}_+^n \times (0, \infty)$:

$$v_t - \Delta v + \nabla q + \operatorname{div} (v \otimes v + v \otimes w + w \otimes v) = -\operatorname{div} (w \otimes w), \quad \operatorname{div} v = 0 \quad (4.3)$$

with homogeneous initial and boundary data, i.e.

$$v(x, 0) = 0, \quad v(x, t) = 0 \text{ on } \{x_n = 0\}. \quad (4.4)$$

Our aim is to establish the existence of solution v for (4.3) satisfying $v \in L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))$ and $\nabla v \in L^2(0, \infty; L^n(\mathbb{R}_+^n))$. In order to do that, we consider the iterative scheme for (4.3), which is given as follows: For a positive integer $m \geq 1$

$$\begin{aligned} v_t^{m+1} - \Delta v^{m+1} + \nabla q^{m+1} &= -\operatorname{div} (v^m \otimes v^m + v^m \otimes w + w \otimes v^m + w \otimes w), \\ \operatorname{div} v^{m+1} &= 0 \end{aligned}$$

with homogeneous initial and boundary data, i.e. $v^{m+1}(x, 0) = 0$ and $v^{m+1}(x, t) = 0$ on $\{x_n = 0\}$. We set $v^1 = 0$. We then have, due to Proposition 2.3, we have

$$\begin{aligned} \|\nabla v^2\|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} &\leq c \left(\| |w|^2 \|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} + \| w \otimes w |_{x_n=0} \|_{L^2(0, \infty; \dot{B}_{nn}^{-\frac{1}{n}}(\mathbb{R}^{n-1}))} \right) \\ &\leq c \left(\| w \|^2_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} + \| w \otimes w |_{x_n=0} \|_{L^2(0, \infty; L^{n-1}(\mathbb{R}^{n-1}))} \right) \\ &\leq c \left(\| w \|^2_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} + \| g \|^2_{L^4(0, \infty; L^{2(n-1)}(\mathbb{R}^{n-1}))} \right). \end{aligned} \quad (4.5)$$

On the other hand, from Proposition 2.2, we have

$$\|v^2\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} \leq c \|w \otimes w\|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} \leq c \|w\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))}^2. \quad (4.6)$$

By (4.2), we have $A := \|w\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} + \|g\|_{L^4(0, \infty; L^{2(n-1)}(\mathbb{R}^{n-1}))} \leq c\alpha$, where $\alpha > 0$ is defined in (3.10). Taking $\alpha > 0$ small such that $A < \frac{1}{4c}$, where c is the constant in (4.5)–(4.6) such that

$$\|\nabla v^2\|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} + \|v^2\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} < A.$$

Then, iterative arguments show that

$$\begin{aligned} &\|\nabla v^{m+1}\|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} \\ &\leq c \left(\| |v^m|^2 + |v^m w| + |w|^2 \|_{L^2(0, \infty; L^n(\mathbb{R}_+^n))} + \| w \otimes w |_{x_n=0} \|_{L^2(0, \infty; \dot{B}_{nn}^{-\frac{1}{n}}(\mathbb{R}^{n-1}))} \right) \\ &\leq 2c \left(\|v^m\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))}^2 + \|w\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))}^2 + \|g\|_{L^4(0, \infty; L^{2(n-1)}(\mathbb{R}^{n-1}))}^2 \right) \\ &\leq 4cA^2 < A. \end{aligned} \quad (4.7)$$

Similarly, we note that

$$\begin{aligned} \|v^{m+1}\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))} &\leq c\| |v^m|^2 + |v^m w| + |w|^2 \|_{L^2(0,\infty;L^n(\mathbb{R}_+^n))} \\ &\leq 2c \left(\|v^m\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))}^2 + \|w\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))}^2 \right) \leq 4cA^2 < A. \end{aligned} \quad (4.8)$$

We denote $V^{m+1} := v^{m+1} - v^m$ and $Q^{m+1} := q^{m+1} - q^m$ for $m \geq 1$. We then see that (V^{m+1}, Q^{m+1}) solves

$$\begin{aligned} V_t^{m+1} - \Delta V^{m+1} + \nabla Q^{m+1} &= -\operatorname{div} \left(V^m \otimes v^m + v^{m-1} \otimes V^m + V^m \otimes w + w \otimes V^m \right), \\ \operatorname{div} V^{m+1} &= 0, \end{aligned}$$

with homogeneous initial and boundary data, i.e. $V^{m+1}(x, 0) = 0$ and $V^{m+1}(x, t) = 0$ on $\{x_n = 0\}$. Taking sufficiently small $\alpha > 0$ such that $A < \frac{1}{6c}$, from (4.7) and (4.8), we obtain

$$\begin{aligned} &\|\nabla V^{m+1}\|_{L^2(0,\infty;L^n(\mathbb{R}_+^n))} + \|V^{m+1}\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))} \\ &\leq c \left\| |V^m v^m| + |V^m v^{m-1}| + |V^m w| \right\|_{L^2(0,\infty;L^n(\mathbb{R}_+^n))} \\ &\leq 3cA \|V^m\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))} < \frac{1}{2} \|V^m\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))}. \end{aligned}$$

Therefore, $(v^m, \nabla v^m)$ converges to $(v, \nabla v)$ in $L^4(0, \infty; L^{2n}(\mathbb{R}_+^n)) \times L^2(0, \infty; L^n(\mathbb{R}_+^n))$ such that v solves in the sense of distributions

$$\begin{aligned} v_t - \Delta v + \nabla \Pi &= -\operatorname{div} (v \otimes v + v \otimes w + v \otimes w + w \otimes w), \\ \operatorname{div} v &= 0, \end{aligned}$$

with homogeneous initial and boundary data, i.e. $v(x, 0) = 0$ and $v(x, t) = 0$ on $\{x_n = 0\}$.

We then set $u := v + w$ and $p = \pi + q$, which becomes a very weak solution of the Navier-Stokes equations in $\mathbb{R}_+^n \times (0, \infty)$, namely

$$u_t - \Delta u + \nabla p = -\operatorname{div} (u \otimes u), \quad \operatorname{div} u = 0,$$

with boundary data $u(x, t) = g(x)$ on $\{x_n = 0\}$ and homogeneous initial data $u(x, 0) = 0$ such that

$$\|u\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))} \leq c, \quad \|\nabla u\|_{L^2(B_r \times (t_0, t_0+r^2))} = \infty \quad \forall r > 0. \quad (4.9)$$

Similarly, as in case of Stokes equations, we take $g_{n,k} \in C_c^\infty(A \times (t_0, \frac{1}{4}))$ such that $g_{n,k}$ goes to g_n in $L^4(0, \infty; L^{2n}(\mathbb{R}^{n-1}))$ as $k \rightarrow \infty$. We denote $\tilde{g}_k = (0, g_{n,k})$. Let w_k be a solution of Stokes equations with boundary data \tilde{g}_k and v_k be a solution of (4.3)–(4.4) with replacement of w by w_k . We recall that w_k is smooth vector field such that w_k converges to w in $L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))$.

We also observe that v_k is smooth vector field such that

$$\|\nabla v_k\|_{L^2(0,\infty;L^n(\mathbb{R}_+^n))} + \|v_k\|_{L^4(0,\infty;L^{2n}(\mathbb{R}_+^n))} \leq c.$$

By weak compactness, there is a subsequence of $\{v_k\}$, redefined as v_k , and $v \in L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))$ such that v_k weakly converges in $L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))$ to v . This implies that $u_k = w_k + v_k$ weakly converges to $u = w + v$ in $L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))$.

Suppose that the Caccioppoli's inequality holds for smooth solutions for the Navier-Stokes equations, that is, for $k \in \mathbb{N}$, u_k is assumed to satisfy the following inequality;

$$\begin{aligned} \int_{t_0}^{t_0 + \frac{1}{4}r^2} \int_{B_{\frac{1}{2}r}^+(0)} |\nabla u_k(x, t)|^2 dx dt &\leq cr^{-2} \int_{t_0}^{t_0 + r^2} \int_{B_r^+(0)} |u_k(x, t)|^2 dx dt \\ &\quad + cr^{-1} \int_{t_0}^{t_0 + r^2} \int_{B_r^+(0)} |u_k(x, t)|^3 dx dt, \end{aligned}$$

where $c > 0$ is independent of k . Since $\|u_k\|_{L^4(0, \infty; L^{2n}(\mathbb{R}_+^n))} \leq c$ for all k , this leads to

$$\int_{t_0}^{t_0 + \frac{1}{4}r^2} \int_{B_{\frac{1}{2}r}^+(0)} |\nabla u(x, t)|^2 dx dt \leq c < \infty.$$

This is, however, contrary to (4.9), and therefore, the Caccioppoli's inequality is not true for the Navier-Stokes equations near boundary. This completes the proof of Theorem 1.1. \square

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Appendix A. Proof of Proposition 2.3

A.1. Helmholtz projection in half space

It is well known that the Helmholtz projection \mathbb{P} in half space \mathbb{R}_+^n is given by

$$\mathbb{P}f = f - \nabla \mathbb{Q}_1 f - \nabla \mathbb{Q}_2 f = f - \nabla \mathbb{Q}f, \quad (\text{A.1})$$

where $\mathbb{Q}_1 f$ and $\mathbb{Q}_2 f$ satisfy the following equations;

$$\Delta \mathbb{Q}_1 f = \operatorname{div} f, \quad \mathbb{Q}_1 f|_{x_n=0} = 0$$

and

$$\Delta \mathbb{Q}_2 f = 0, \quad D_{x_n} \mathbb{Q}_2 f|_{x_n=0} = (f_n - D_{x_n} \mathbb{Q}_1 f)|_{x_n=0}.$$

Note that $\mathbb{Q}_1 f$ and $\mathbb{Q}_2 f$ are represented by

$$\mathbb{Q}_1 f(x) = - \int_{\mathbb{R}_+^n} \nabla_y (N(x-y) - N(x-y^*)) \cdot f(y) dy, \quad (\text{A.2})$$

$$\mathbb{Q}_2 f(x) = \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) (f_n(y', 0) - D_{y_n} \mathbb{Q}_1 f(y', 0)) dy', \quad (\text{A.3})$$

where $y^* = (y', -y_n)$. Note that $(\mathbb{P} f)_n|_{x_n=0} = 0$.

Lemma A.1. *Let $f_i = \operatorname{div} F_i$ and $f = (f_1, \dots, f_n)$. Then,*

$$\begin{aligned} \mathbb{Q}_1 f(x) &= \sum_{k \neq n} D_{x_k} \mathbb{Q}_1 F_k(x) + D_{x_n} A(x), \\ \mathbb{Q}_2 f(x) &= \sum_{k \neq n} D_{x_k} \mathbb{Q}_2 F_k(x) - \sum_{k \neq n} D_{x_k} B_k^1(x) - \sum_{k \neq n} D_{x_k}^2 B^2(x), \end{aligned}$$

where

$$\begin{aligned} A(x) &= - \int_{\mathbb{R}_+^n} \nabla_y (N(x-y) + N(x-y^*)) \cdot F_n(y) dy + 2B_n^1(x), \\ B_k^1(x) &= - \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) F_{kn}(y', 0) dy', \quad 1 \leq k \leq n, \\ B^2(x) &= \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) A(y', 0) dy'. \end{aligned}$$

Proof. From (A.2), we have

$$\begin{aligned} \mathbb{Q}_1 f(x) &= - \sum_{k \neq n} D_{x_k} \int_{\mathbb{R}_+^n} \nabla_y (N(x-y) - N(x-y^*)) \cdot F_k(y) dy \\ &\quad - D_{x_n} \int_{\mathbb{R}_+^n} \nabla_y (N(x-y) + N(x-y^*)) \cdot F_n(y) dy \\ &\quad + 2D_{x_n} \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) F_{nn}(y', 0) dy' \\ &:= \sum_{k \neq n} D_{x_k} \mathbb{Q}_1 F_k(x) + D_{x_n} A(x). \end{aligned} \quad (\text{A.4})$$

Since $\Delta A = \operatorname{div} F_n = f_n$, from (A.4), we have $D_{y_n} \mathbb{Q}_1 f(y) = D_{y_n} \sum_{k \neq n} D_{y_k} \mathbb{Q}_1 F_k(y) + f_n(y) - \Delta' A(y)$. Hence, we have

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} N(x' - y', x_n) D_{y_n} \mathbb{Q}_1 f(y', 0) dy' &= \sum_{k \neq n} D_{x_k} \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) D_{y_n} \mathbb{Q}_1 F_k(y', 0) dy' \\
&+ \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) f_n(y', 0) dy' \\
&+ \sum_{k \neq n} D_{x_k} \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) D_{y_k} A(y', 0) dy'.
\end{aligned} \tag{A.5}$$

Hence, from (A.3) and (A.5), we have

$$\begin{aligned}
\mathbb{Q}_2 f(x) &= - \sum_{k \neq n} D_{x_k} \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) D_{y_n} \mathbb{Q}_1 F_k(y', 0) dy' \\
&- \sum_{k \neq n} D_{x_k} \int_{\mathbb{R}^{n-1}} N(x' - y', x_n) D_{y_k} A(y', 0) dy'.
\end{aligned}$$

We complete the proof. \square

A.2. Proof of Proposition 2.3

To prove Proposition 2.3, we use the following Proposition.

Proposition A.2. ([2, Proposition 3.2]) Let $1 < p, q < \infty$. Let $\Gamma *' g(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', x_n, t - \tau) g(y', \tau) dy' d\tau$. Then

$$\|D_x \Gamma *' g\|_{L^q(\mathbb{R}; L^p(\mathbb{R}_+^n))} \leq c \|g\|_{L^q(\mathbb{R}; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))}.$$

We consider the Stokes equations

$$\begin{aligned}
v_t - \Delta v + \nabla \Pi &= f, & \operatorname{div} v &= 0, & \text{in } \mathbb{R}_+^n \times (0, \infty), \\
v|_{t=0} &= 0, & v|_{x_n=0} &= 0,
\end{aligned} \tag{A.6}$$

where $f = \operatorname{div} F$.

Let $f = \mathbb{P}f + \nabla \mathbb{Q}f$ be a decomposition of f defined (A.1). Note that $(\mathbb{P}f)_n|_{x_n=0} = 0$. We define (v, Π_0) by

$$\begin{aligned}
v_i(x, t) &= \int_0^t \int_{\mathbb{R}_+^n} G_{ij}(x, y, t - \tau) (\mathbb{P}f)_j(y, \tau) dy d\tau, \\
\Pi_0(x, t) &= \int_0^t \int_{\mathbb{R}_+^n} P(x, y, t - \tau) \cdot (\mathbb{P}f)(y, \tau) dy d\tau,
\end{aligned} \tag{A.7}$$

where G and P are defined by

$$G_{ij} = \delta_{ij}(\Gamma(x - y, t) - \Gamma(x - y^*, t)) + 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} \frac{\partial N(x - z)}{\partial x_i} \Gamma(z - y^*, t) dz, \quad (\text{A.8})$$

$$P_j(x, y, t) = 4(1 - \delta_{jn}) \frac{\partial}{\partial x_j} \left[\int_{\mathbb{R}^{n-1}} \frac{\partial N(x' - z', x_n)}{\partial x_n} \Gamma(z' - y', y_n, t) dz' \right. \\ \left. + \int_{\mathbb{R}^{n-1}} N(x' - z', x_n) \frac{\partial \Gamma(z' - y', y_n, t)}{\partial y_n} dz' \right].$$

From [16], (v, Π_0) satisfies

$$v_t - \Delta v + \nabla \Pi_0 = \mathbb{P} f, \quad \operatorname{div} v = 0, \text{ in } \mathbb{R}_+^n \times (0, \infty), \\ v|_{t=0} = 0, \quad v|_{x_n=0} = 0.$$

Let $\Pi = \Pi_0 + \mathbb{Q} f$. Then, (v, Π) is solution of (A.6).

Let $1 < p, q < \infty$. In Section 3 in [3], the authors showed that v defined by (A.7) has the following estimate;

$$\|\nabla v\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \leq c(\|\nabla \Gamma * \mathbb{P} f\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|\nabla \Gamma^* * \mathbb{P} f\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))}), \quad (\text{A.9})$$

where $\Gamma^* * f(x, t) = \int_0^t \int_{\mathbb{R}_+^n} \Gamma(x - y^*, t - \tau) f(y, \tau) dy d\tau$.

Lemma A.3. Let $1 < p, q < \infty$. Let $F \in L^q(0, \infty, L^p(\mathbb{R}_+^n))$. Then,

$$\|\nabla \Gamma * \mathbb{P}(\operatorname{div} F)\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|\nabla \Gamma^* * \mathbb{P}(\operatorname{div} F)\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \\ \leq c(\|F\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|F|_{x_n=0}\|_{L^q(0, \infty; B_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))}).$$

Proof. Since the proofs are exactly same, we only prove the case of $\Gamma^* * \mathbb{P}(\operatorname{div} F)$.

Since $(\mathbb{P} \operatorname{div} F)_j(t) = \operatorname{div} F_j(t) - D_{x_j} \mathbb{Q} \operatorname{div} F(t)$, for $1 \leq j \leq n-1$, we have

$$(\Gamma^* * (\mathbb{P} \operatorname{div} F))_j(x, t) = \int_0^t \int_{\mathbb{R}_+^n} \nabla_y \Gamma(x - y^*, t - \tau) \cdot F_j(y, \tau) dy d\tau \\ + \int_0^t \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', x_n, t - \tau) F_{jn}(y', 0, \tau) dy' d\tau \\ + \int_0^t \int_{\mathbb{R}_+^n} D_{y_j} \Gamma(x - y^*, t - \tau) \mathbb{Q}(\operatorname{div} F)(y, \tau) dy d\tau.$$

Note that $D_{x_n}^2 A(t) = -\Delta' A(t) + \operatorname{div} F_n(t)$. From Lemma A.1, we have

$$\begin{aligned} \Gamma^* * (\mathbb{P} \operatorname{div} F)_n(x, t) &= \int_0^t \int_{\mathbb{R}_+^n} \nabla_y \Gamma(x - y^*, t - \tau) \cdot F_n(y, \tau) dy d\tau dz' \\ &\quad + \int_0^t \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', x_n, t - \tau) F_{nn}(y', 0, \tau) dy' d\tau \\ &\quad + \sum_{1 \leq k \leq n-1} D_{x_k} \int_0^t \int_{\mathbb{R}_+^n} \Gamma(x - y^*, t - \tau) D_{y_n} \mathbb{Q} F_k(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}_+^n} \nabla' \Gamma(x - y^*, t - \tau) \cdot \nabla' A(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}_+^n} \nabla' \Gamma(x - y^*, t - \tau) \cdot \nabla' D_{y_n} B_k^1(y, \tau) dy d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}_+^n} \nabla' \Gamma(x - y^*, t - \tau) \cdot \nabla' D_{y_n} B^2(y, \tau) dy d\tau. \end{aligned}$$

Due to parabolic type's Calderon-Zygmund Theorem and Proposition A.2, for $1 < p, q < \infty$, we have

$$\begin{aligned} \|\nabla \Gamma^* * (\mathbb{P} \operatorname{div} F)\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} &\leq \left(\|D_x \mathbb{Q} F\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|D_x A\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \right. \\ &\quad + \|D_x B_k^1 F_k\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|D_x^2 B^2\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} \\ &\quad \left. + \|F\|_{L^q(0, \infty; L^p(\mathbb{R}_+^n))} + \|F|_{x_n=0}\|_{L^q(0, \infty; \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1}))} \right). \end{aligned}$$

It is well known (see e.g. [17]) that $DB_k^1(t)$ are bounded from $\dot{B}_{pp}^{l-\frac{1}{p}}(\mathbb{R}^{n-1})$ to $\dot{W}_p^l(\mathbb{R}_+^n)$, $l \in \mathbb{N} \cup \{0\}$ so that

$$\|D_x B_k^1(t)\|_{\dot{W}_p^l(\mathbb{R}_+^n)} \leq c \|F(t)|_{x_n=0}\|_{\dot{B}_{pp}^{l-\frac{1}{p}}(\mathbb{R}^{n-1})}. \quad (\text{A.10})$$

By Calderon-Gygmund Theorem and (A.10), we have

$$\begin{aligned} \|D_x \mathbb{Q}_1 F(t)\|_{L^p(\mathbb{R}_+^n)} &\leq c \|F(t)\|_{L^p(\mathbb{R}_+^n)}, \\ \|D_x A(t)\|_{L^p(\mathbb{R}_+^n)} &\leq c \|F(t)\|_{L^p(\mathbb{R}_+^n)} + c \|F(t)|_{x_n=0}\|_{\dot{B}_{pp}^{k-\frac{1}{p}}(\mathbb{R}^{n-1})}, \end{aligned}$$

$$\begin{aligned} \|D_x^2 B^2(t)\|_{L^p(\mathbb{R}_+^n)} &\leq c \|A(t)|_{x_n=0}\|_{\dot{B}_{pp}^{1-\frac{1}{p}}(\mathbb{R}^{n-1})} \leq c \|D_x A(t)\|_{L^p(\mathbb{R}_+^n)} \\ &\leq c \|F(t)\|_{L^p(\mathbb{R}_+^n)} + c \|F(t)|_{x_n=0}\|_{\dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1})}. \end{aligned}$$

Finally, from (A.10), we have

$$\|D_x \mathbb{Q}_2 F_k(t)\|_{L^p(\mathbb{R}_+^n)} \leq c \|(F_k(t) - \nabla \mathbb{Q}_1 F_k(t))_n|_{x_n=0}\|_{\dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1})}.$$

Since $F_k(t) - \nabla \mathbb{Q}_1 F_k(t)$ is in $L^p(\mathbb{R}_+^n)$ and divergence free in \mathbb{R}_+^n , its normal component has trace $(F_k(t) - \nabla \mathbb{Q}_1 F_k(t))_n|_{x_n=0} \in \dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1})$ (see [6]). Hence, we have

$$\|(F_k(t) - \nabla \mathbb{Q}_1 F_k(t))_n|_{x_n=0}\|_{\dot{B}_{pp}^{-\frac{1}{p}}(\mathbb{R}^{n-1})} \leq \|F_k(t) - \nabla \mathbb{Q}_1 F_k(t)\|_{L^p(\mathbb{R}_+^n)} \leq c \|F_k(t)\|_{L^p(\mathbb{R}_+^n)}.$$

Therefore, we complete the proof of Lemma A.3. \square

With the aid of the estimate (A.9) and Lemma A.3, Proposition 2.3 is immediate.

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