

Parabolic equations in Musielak - Orlicz spaces with discontinuous in time N -function

Miroslav Bulíček^{a,1}, Piotr Gwiazda^{b,2}, Jakub Skrzeczkowski^{c,*,3}

^a *Mathematical Institute, Faculty of Mathematics and Physics, Charles University, Sokolovska 83, 186 75, Prague, Czech Republic*

^b *Institute of Mathematics of Polish Academy of Sciences, Jana i Jędrzeja Śniadeckich 8, 00-656 Warsaw, Poland*

^c *Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Stefana Banacha 2, 02-097 Warsaw, Poland*

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Abstract

We consider a parabolic PDE with Dirichlet boundary condition and monotone operator A with non-standard growth controlled by an N -function depending on time and spatial variable. We do not assume continuity in time for the N -function. Using an additional regularization effect coming from the equation, we establish the existence of weak solutions and in the particular case of isotropic N -function, we also prove their uniqueness. This general result applies to equations studied in the literature like $p(t, x)$ -Laplacian and double-phase problems.

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* Corresponding author.

E-mail addresses: mbul8060@karlin.mff.cuni.cz (M. Bulíček), pgwiazda@mimuw.edu.pl (P. Gwiazda), jakub.skrzeczkowski@student.uw.edu.pl (J. Skrzeczkowski).

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1. Introduction

1.1. PDEs in Musielak - Orlicz spaces

This paper focuses on study of parabolic equations having the form

$$u_t(t, x) = \operatorname{div} A(t, x, \nabla u(t, x)) + f(t, x) \text{ in } (0, T) \times \Omega, \quad (1.1)$$

completed by the homogeneous Dirichlet boundary condition and the initial value $u_0(x)$. Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain, T denotes the length of time interval, $f : (0, T) \times \Omega \rightarrow \mathbb{R}$ is a measurable bounded function and A is a monotone operator with coercivity and growth controlled by a so - called N -function $M : (0, T) \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ (see Definition 1.2), i.e. for almost all $(t, x) \in (0, T) \times \Omega$ and all $\xi \in \mathbb{R}^d$, we have:

$$M(t, x, \xi) + M^*(t, x, A(t, x, \xi)) \leq c A(t, x, \xi) \cdot \xi + h(t, x) \quad (1.2)$$

where M^* denotes the convex conjugate to M (see Definition 1.3) and $h \in L^1((0, T) \times \Omega)$. Originally, problem (1.1) was solved with $M(t, x, \xi) = |\xi|^p$ where $1 < p < \infty$. In this classical setting, (1.2) implies that A , understood as a map

$$L^p(0, T; W_0^{1,p}(\Omega)) \ni u \mapsto A(t, x, \nabla u) \in \left(L^p\left(0, T; W_0^{1,p}(\Omega)\right) \right)^*,$$

is a bounded continuous operator and standard approaches (Galerkin method and compactness in Sobolev-Bochner spaces) applies (see [10,32] and references therein) showing that the Sobolev space is an appropriate functional setting for problem (1.1). However, if the N -function M appearing in (1.2) has not a polynomial growth with respect to ξ and is (t, x) -dependent, one has to look for a solution u such that its gradient ∇u belongs to the Musielak - Orlicz space $L_M((0, T) \times \Omega)$, i.e. the space of measurable functions $\xi : (0, T) \times \Omega \rightarrow \mathbb{R}^d$ which satisfy

$$\int_{(0,T) \times \Omega} M\left(t, x, \frac{\xi(t, x)}{\lambda}\right) dt dx < \infty$$

for some $\lambda > 0$, see Definition 1.6. First results in this direction were focused on function M being independent of (t, x) and direction of ξ , i.e. $M(t, x, \xi) = N(|\xi|)$. Under the additional coercivity estimate $t^2 \ll N(t)$ and the so-called Δ_2 condition for convex conjugate, i.e.

$$N^*(2t) \leq k N^*(t) \quad (1.3)$$

for some constant k , this case was treated in [21,37]. Another approach, introduced in [22], assumed growth bound $N(t) \ll t^{d/(d-1)}$ and condition $N(Cts) \leq N(t)N(s)$ to be satisfied by N . Briefly speaking, condition (1.3) provides a characterization of appropriate dual spaces (see [1, Theorem 8.20]) and allows to extract weakly-* converging subsequences from bounded sequences. Similar methods have been used to study existence of solutions to (1.1) with data “below the duality”, i.e. $f \in L^1((0, T) \times \Omega)$, see [30].

Another approach is based on looking for hypothesis on M implying that $C_0^\infty((0, T) \times \Omega)$ is a dense subset of $L_M((0, T) \times \Omega)$ (at least in the sense of modular convergence, see Definition 1.8)

so that one can test (1.1) with the solution itself. It is a classical fact that for variable Lebesgue spaces (i.e. $M(t, x, \xi) = |\xi|^{p(t,x)}$) some continuity of p in (t, x) is in general necessary (see [18, Example 6.12]). Density argument was first exploited to establish well-posedness of (1.1) for $M(t, x, \xi) = N(|\xi|)$ in [23] and it was extended later to cover more and more general functions M without assumption of the form (1.3) but with some sort of continuity hypothesis with respect to (t, x) [15, 16, 29, 40, 41] with the most general condition given in [16]. We remark that similar progress has been made for elliptic equations and we refer the reader to the excellent review [12] discussing PDEs in Musielak - Orlicz spaces in detail.

We want to emphasize here that all papers mentioned above have a disadvantage on the continuity assumption of N -function $M(t, x, \xi)$ with respect to t . However, this cannot be optimal. One can consider the PDE of the form:

$$u_t = \begin{cases} \operatorname{div} \nabla u & \text{in } (0, 1] \times \Omega, \\ \operatorname{div} (|\nabla u|^2 \nabla u) & \text{in } (1, 2] \times \Omega, \end{cases}$$

which can be solved piecewisely (first on time interval $(0, 1]$ and then on $(1, 2]$) so one can develop well-posedness theory. We remark that in the recent monograph [4, Section 2.2] there is an example of degenerated parabolic equation

$$u_t - \operatorname{div}(|u|^{\gamma(t,x)} \nabla u) = f, \quad (1.4)$$

where the exponent $\gamma(t, x)$ satisfies bounds $-1 < \gamma_- \leq \gamma(t, x) \leq \gamma_+ < \infty$ and $\nabla \gamma \in L^2((0, T) \times \Omega)$. Then, (1.4) has at least one bounded weak solution. Moreover, if

$$\gamma_- > 0 \quad \text{and} \quad \operatorname{ess\,sup}_{x \in \overline{\Omega}} |\nabla \gamma(t, x)| \in L^2(0, T),$$

the solution becomes unique. However, these results are strongly based on the particular form of the operator in (1.4). Finally, let us remark that many problems that are of current interests can be studied in the framework

In this paper we establish the existence of solutions to (1.1) in the Musielak - Orlicz space $L_M((0, T) \times \Omega)$ without any assumption on continuity of $M(t, x, \xi)$ with respect to t (see Theorem 1.23). Moreover, for isotropic N -functions of the form $M(t, x, |\xi|)$ we obtain the uniqueness in a given class.⁴ The main features of our work are:

- In contrast to works described above, we do not try to approximate *every* function in modular topology but only the distributional solution to (1.1). Using the equation satisfied by the solution, we can retrieve the missing regularity in time and proceed without continuity with respect to time assumption for $M(t, x, \xi)$. Similar approaches have been used for renormalized solutions to the transport equation, see [19, Section 2.1].
- Existence result is deduced by using only the local versions of standard methods: the energy equality (2.20) and the monotonicity method in Section 2.4.
- Uniqueness result is based on the global energy equality (4.5) that can be deduced from the local one.

⁴ Note that in case of spatially boundary conditions, we have the uniqueness of a weak solution even without any structural assumption on M .

We remark that (1.1) generalizes a great variety of parabolic problems and we refer to [16, Corollary 1.1-1.2, Example 1.1-1.2] for a long list of examples with associated N -functions. This includes double phase problems where the N -function M is trapped between two power-type functions. Such equations have been studied by Marcellini [33,34] and they are still subject of active research [6,7,17].

Finally, we would like to emphasize that the studied problem has not only a theoretical background but can find an application in physically well-motivated problems whenever one considers rapid changes of the underlying equations with respect to time variable. As a prototypic example may serve the flow of incompressible electrorheological fluids (see [20] or [38] for more details). These fluids are described by the system of equations:

$$\begin{aligned}\operatorname{div} \mathbf{v} &= 0, \\ \partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p + \mathbf{g} + \nabla \mathbf{E} \cdot \mathbf{P},\end{aligned}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ denotes the velocity of the fluid, \mathbf{S} is the viscous stress tensor, \mathbf{E} is the electrical intensity and \mathbf{P} is the polarization. Note that in the case of no electric field present we have

$$\mathbf{S} \sim D(\mathbf{v}) \text{ where } D(\mathbf{v}) = \frac{1}{2} \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T \right).$$

But, when we apply an electric field, the viscous stress changes dramatically and behaves like $\mathbf{S} \sim |D(\mathbf{v})|^{r(t,x)} D(\mathbf{v})$ with some function $r(t, x)$. Hence, it is evident that we are now in the case corresponding to the choice of N -function $M(t, x, \xi) = |\xi|^{r(t,x)}$, where $r(t, x)$ is discontinuous with respect to time variable.

1.2. Musielak - Orlicz spaces

In this subsection we briefly recall theory of Musielak - Orlicz spaces. For detailed discussion, we refer the reader to the classical book [36] as well as to a modern presentation [14] aimed at applications in PDEs.

In what follows, $\Omega \subset \mathbb{R}^d$ denotes a bounded domain and $T > 0$ is arbitrary. We set $\Omega_T := (0, T) \times \Omega$.

Definition 1.1 (*Young function*). We say that $m : [0, \infty) \rightarrow [0, \infty)$ is a Young function if the following holds true:

- (Y1) $m(s) = 0 \iff s = 0$,
- (Y2) m is convex,
- (Y3) m is superlinear, i.e. $\lim_{s \rightarrow 0} \frac{m(s)}{s} = 0$ and $\lim_{s \rightarrow \infty} \frac{m(s)}{s} = \infty$.

Definition 1.2 (N -function). We say that $M : \Omega_T \times \mathbb{R}^d \rightarrow \mathbb{R}$ is N -function if the following holds true:

- (M1) $M(t, x, \xi) = M(t, x, -\xi)$ for a.e. $(t, x) \in \Omega_T$ and all $\xi \in \mathbb{R}^d$,
- (M2) $M(t, x, \xi)$ is a Carathéodory function, i.e. for a.e. $(t, x) \in \Omega_T$, the mapping $\mathbb{R}^d \ni \xi \mapsto M(t, x, \xi)$ is continuous and for all $\xi \in \mathbb{R}^d$, the mapping $\Omega_T \ni (t, x) \mapsto M(t, x, \xi)$ is measurable,

- (M3) for a.e. $(t, x) \in \Omega_T$, the map $\mathbb{R}^d \ni \xi \mapsto M(t, x, \xi)$ is convex,
 (M4) there exist two Young functions m_1, m_2 such that for almost all $(t, x) \in \Omega_T$ and all $\xi \in \mathbb{R}^d$ we have

$$m_1(|\xi|) \leq M(t, x, \xi) \leq m_2(|\xi|).$$

Definition 1.3 (Convex conjugate). Let m be a Young function. Then, we define its convex conjugate m^* as

$$m^*(s) = \sup_{t \in [0, \infty)} (st - m(t)).$$

Similarly, if M is an N -function, we define its convex conjugate M^* as

$$M^*(t, x, \eta) = \sup_{\xi \in \mathbb{R}^d} (\xi \cdot \eta - M(t, x, \xi)).$$

Lemma 1.4 (Properties of N -functions). Let m be a Young function and M be an N -function. Then:

- (N1) function $\frac{m(t)}{t}$ is nondecreasing,
 (N2) m^* is a Young function,
 (N3) M^* is an N -function,
 (N4) $\lim_{|\xi| \rightarrow 0} \operatorname{ess\,sup}_{(t,x) \in \Omega_T} \frac{M(t,x,\xi)}{|\xi|} = 0$ and $\lim_{|\xi| \rightarrow \infty} \operatorname{ess\,inf}_{(t,x) \in \Omega_T} \frac{M(t,x,\xi)}{|\xi|} = \infty$,
 (N5) if $f_n : \Omega_T \rightarrow \mathbb{R}^d$ is a sequence of functions and $\int_{\Omega_T} M(t, x, f_n(t, x)) \, dt \, dx \leq C$ independently of n , then $\{f_n\}_{n \in \mathbb{N}}$ is equi-integrable,
 (N6) if $f_n : \Omega_T \rightarrow \mathbb{R}^d$ is a sequence of functions and $\int_{\Omega_T} M(t, x, f_n(t, x)) \, dt \, dx \leq C$ for some $C > 1$ then $\|f_n\|_{L_M} \leq C$,
 (N7) if $f_n : \Omega_T \rightarrow \mathbb{R}^d$ is a sequence of functions such that $f_n \rightarrow f$ a.e. in Ω_T and $\|f_n\|_\infty \leq C$ independently of n , then $\int_{\Omega_T} M(t, x, f_n(t, x)) \, dt \, dx \rightarrow \int_{\Omega_T} M(t, x, f(t, x)) \, dt \, dx$.

Proof. Let $t \leq s$. By convexity of m , we have

$$\frac{m(t)}{t} = \frac{1}{t} m\left(\frac{t}{s}s + \left(1 - \frac{t}{s}\right)0\right) \leq \frac{1}{t} \frac{t}{s} m(s) = \frac{m(s)}{s},$$

which proves (N1).

To see property (N2), we observe directly from Definition 1.3 that $m^*(0) = 0$ as $m \geq 0$ and $m(0) = 0$. The convexity of m^* follows as it is a supremum of affine maps. Hence, it remains to check (Y3) in Definition 1.1. For any $\lambda > 0$

$$\liminf_{s \rightarrow \infty} \frac{m^*(s)}{s} \geq \frac{\lambda s - m(\lambda)}{s} \geq \lambda$$

which proves $\lim_{s \rightarrow \infty} \frac{m^*(s)}{s} = \infty$. Now, let $\delta > 0$ and $s \in (0, \delta)$ be arbitrary. Then,

$$\frac{m^*(s)}{s} = \sup_{t \in [0, \infty)} \left(t - \frac{m(t)}{s}\right) = \sup_{t \in [0, \infty)} t \left(1 - \frac{m(t)}{t} \frac{1}{s}\right) \leq \sup_{t \in [0, \infty)} t \left(1 - \frac{m(t)}{t} \frac{1}{\delta}\right)$$

However, for t such that $\frac{m(t)}{t} \geq \delta$, the maximized expression is negative. By property (N1) and (Y3) in Definition 1.1, we find t_δ , such that $\frac{m(t_\delta)}{t_\delta} = \delta$ and we get that

$$\frac{m^*(s)}{s} \leq \sup_{t \in [0, t_\delta]} t \left(1 - \frac{m(t)}{t} \frac{1}{\delta} \right) \leq t_\delta.$$

We claim that $t_\delta \rightarrow 0$ as $\delta \rightarrow 0$. For if not, $C_2 \geq t_\delta \geq C_1 > 0$ for some constants C_1 and C_2 . But then

$$\delta = \frac{m(t_\delta)}{t_\delta} \geq \frac{m(C_1)}{C_2} > \frac{m(0)}{C_2} = 0,$$

since m is strictly increasing and $m(0) = 0$. This proves (N2). To see (N3), we observe that

$$m_1(|\xi|) \leq M(t, x, \xi) \leq m_2(\xi) \implies m_2^*(|\xi|) \leq M^*(t, x, \xi) \leq m_1^*(\xi).$$

Since m_1^* and m_2^* are Young functions, the conclusion follows. Property (N4) is a consequence of (M4) in Definition 1.2 and superlinearity of Young functions (Y3). To deduce (N5), we note that

$$\int_{\Omega_T} m_1(|f_n(t, x)|) \, dt \, dx \leq C$$

and it is well-known that such bound for superlinear function m_1 is equivalent to uniform integrability on bounded domains, see [3, Proposition 1.27]. Property (N6) follows by convexity:

$$\int_{\Omega_T} M\left(t, x, \frac{f_n(t, x)}{C}\right) \, dt \, dx \leq \frac{1}{C} \int_{\Omega_T} M(t, x, f_n(t, x)) \, dt \, dx \leq 1.$$

Finally, as Young function is increasing, property (N7) follows by Dominated Convergence Theorem. \square

Remark 1.5. In previous works on PDEs in Musielak - Orlicz spaces, N -functions were defined slightly differently using combination of conditions in Definition 1.1, Definition 1.2 and Lemma 1.4 (see, for instance, [11, 15, 16]). We believe that Definition 1.2 makes our work more accessible for readers not familiar with this setting.

Definition 1.6 (Musielak - Orlicz space $L_M(\Omega_T)$). Let M be an N - function. Then, the Musielak - Orlicz space $L_M(\Omega_T)$ is defined as

$$L_M(\Omega_T) = \left\{ \xi : \Omega_T \rightarrow \mathbb{R}^d : \text{there is } \lambda > 0 \text{ such that } \int_{\Omega_T} M\left(t, x, \frac{\xi(t, x)}{\lambda}\right) \, dt \, dx < \infty \right\}.$$

This is a Banach space equipped with the norm

$$\|\xi\|_{L_M} = \inf \left\{ \lambda > 0 : \int_{\Omega_T} M \left(t, x, \frac{\xi(t, x)}{\lambda} \right) dt dx \leq 1 \right\}. \quad (1.5)$$

If m is a Young function, we can similarly define the Musielak - Orlicz space $L_m(\Omega_T)$.

The following form of the Young and the Hölder inequalities are true in Musielak-Orlicz spaces (see [42, Lemma 2.4]):

Lemma 1.7. *Let M be an N -function and M^* be its convex conjugate. Then, for all $\xi \in L_M(\Omega_T)$ and $\eta \in L_{M^*}(\Omega_T)$:*

$$(I1) \quad \int_{\Omega_T} \xi(t, x) \eta(t, x) dt dx \leq \int_{\Omega_T} M(t, x, \xi(t, x)) dt dx + \int_{\Omega_T} M^*(t, x, \eta(t, x)) dt dx,$$

$$(I2) \quad \int_{\Omega_T} \xi(t, x) \eta(t, x) dt dx \leq 2 \|\xi\|_{L_M} \|\eta\|_{L_{M^*}}.$$

As convergence in norm in space $L_M(\Omega_T)$ seems to be too strong for applications in PDEs, we introduce the concept of modular convergence.

Definition 1.8 (*Modular convergence in $L_M(\Omega_T)$*). We say that sequence of functions $\{\xi_n\}_{n \in \mathbb{N}} \subset L_M(\Omega_T)$ converges to ξ modularly if there exists $\lambda > 0$ such that

$$\int_{\Omega_T} M \left(t, x, \frac{\xi_n(t, x) - \xi(t, x)}{\lambda} \right) dt dx \rightarrow 0.$$

We write $\xi_n \xrightarrow{M} \xi$. By convexity, it follows that if $\{\xi_n\}_{n \in \mathbb{N}} \subset L_M(\Omega_T)$ and $\xi_n \xrightarrow{M} \xi$ then $\xi \in L_M(\Omega_T)$.

Note that modularly converging sequences converge in $L^1(\Omega_T)$ and so, they have a subsequence converging a.e. As in the case of classical Lebesgue spaces, simple functions are dense in $L_M(\Omega_T)$ with respect to the modular convergence:

Lemma 1.9 (*Density of simple functions*). *Let $\xi \in L_M(\Omega_T)$. Then, there is a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ of simple functions such that $\xi_n \xrightarrow{M} \xi$.*

Due to Vitali Convergence Theorem (cf. [26, Exercise 15, Section 6.1]), we have the following characterization of modular convergence and its corollary.

Theorem 1.10. *Let $\{\xi_n\}_{n \in \mathbb{N}} \subset L_M(\Omega_T)$ and $\xi \in L_M(\Omega_T)$. Then, $\xi_n \xrightarrow{M} \xi$ if and only if the following hold:*

(V1) $\{\xi_n\}_{n \in \mathbb{N}}$ converges to ξ in measure,

(V2) $\left\{ M \left(t, x, \frac{\xi_n}{\lambda} \right) \right\}_{n \in \mathbb{N}}$ is uniformly equi-integrable for some $\lambda > 0$.

Corollary 1.11. Let $\{\varphi_j\}_{j \in \mathbb{N}} \subset L_M(\Omega_T)$ and $\{\phi_j\}_{j \in \mathbb{N}} \subset L_{M^*}(\Omega_T)$. Suppose that $\varphi_j \xrightarrow{M} \varphi$ and $\phi_j \xrightarrow{M^*} \phi$. Then, $\varphi_j \phi_j \rightarrow \varphi \phi$ in $L^1(\Omega_T)$.

Proof. By Theorem 1.10, $\varphi_j \rightarrow \varphi$ and $\phi_j \rightarrow \phi$ in measure, and so $\varphi_j \cdot \phi_j \rightarrow \varphi \cdot \phi$ also in measure. To conclude, we have to prove uniform integrability of $\{\varphi_j \cdot \phi_j\}$. However, by Young's inequality, for any $Q \subset \Omega_T$:

$$\int_Q \frac{\varphi_j(t, x) \cdot \phi_j(t, x)}{\lambda} dt dx \leq \int_Q M\left(t, x, \frac{\varphi_j(t, x)}{\lambda}\right) dt dx + \int_Q M^*\left(t, x, \frac{\phi_j(t, x)}{\lambda}\right) dt dx. \quad (1.6)$$

Again, Theorem 1.10 implies existence of $\lambda_1, \lambda_2 > 0$ such that sequences $\left\{M\left(t, x, \frac{\varphi_j(x)}{\lambda_1}\right)\right\}$ and $\left\{M^*\left(t, x, \frac{\phi_j(x)}{\lambda_2}\right)\right\}$ are uniformly integrable. Taking $\lambda = \max(\lambda_1, \lambda_2)$ in (1.6), we conclude the proof. \square

Finally, we discuss some compactness results allowing to extract converging subsequences.

Definition 1.12 (Subspace $E_M(\Omega_T)$). $E_M(\Omega_T)$ is a closure of bounded functions in the norm (1.5).

It is easy to see by approximation with simple functions that $E_M(\Omega_T)$ is separable. Therefore, [42, Theorem 2.6] and the Banach-Alaoglu-Bourbaki Theorem (cf. [9, Theorem 3.16 and Corollary 3.30]) yields:

Lemma 1.13. We have the following duality characterization $(E_M(\Omega_T))^* = L_{M^*}(\Omega_T)$. In particular, if $\{\xi_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L_{M^*}(\Omega_T)$, it has a weakly-* converging subsequence.

For Young functions, we also define Orlicz–Sobolev spaces and we recall their basic properties (cf. [1, Chapter 8]).

Definition 1.14 (Orlicz–Sobolev space). Let $m : \mathbb{R} \rightarrow \mathbb{R}$ be a Young function. We define Orlicz–Sobolev spaces $W_0^1 L_m(\Omega_T)$ as

$$W_0^1 L_m(\Omega_T) = \left\{ \xi \in L^1(0, T; W_0^{1,1}(\Omega)) : \|\xi\|_{L_m}, \|\nabla \xi\|_{L_m} < \infty \right\}$$

and equip it with the norm

$$\|\xi\|_{W^1 L_m} = \|\xi\|_{L_m} + \|\nabla \xi\|_{L_m}.$$

We also consider its subset $W_0^1 E_m(\Omega_T)$:

$$W_0^1 E_m(\Omega_T) = \left\{ \xi \in W_0^1 L_m : \xi \in E_m(\Omega_T) \text{ and } \nabla \xi \in E_m(\Omega_T) \right\}$$

Lemma 1.15 (Properties of $W_0^1 E_m(\Omega_T)$ and $W_0^1 L_m(\Omega_T)$). Spaces $W_0^1 E_m(\Omega_T)$ and $W_0^1 L_m(\Omega_T)$ have the following properties:

- (P1) $W_0^1 E_m(\Omega_T)$ is separable,
- (P2) space $C_0^\infty((0, T) \times \Omega)$ is dense in $W_0^1 E_m(\Omega_T)$ with respect to $\|\cdot\|_{L_m}$ norm,
- (P3) (Poincaré inequality, cf. [13, Corollary 4.1]) there are constants c_1 and c_2 such that for all $u \in W_0^1 L_m(\Omega_T)$,

$$\int_{\Omega_T} m(c_1|u|) \, dt \, dx \leq c_2 \int_{\Omega_T} m(|\nabla u|) \, dt \, dx.$$

In particular, $\|\nabla u\|_{L_m}$ is an equivalent norm on $W_0^1 L_m(\Omega_T)$.

1.3. Main result

We start with assumptions on \mathcal{N} -function M and operator A .

Assumption 1.16 (Assumptions on M). We assume that $M : \Omega_T \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an N -function. Moreover, we assume that there is a function $\Theta : (0, T) \times [0, 1] \times [0, \infty) \rightarrow [0, \infty)$, which is nondecreasing with respect to the second and the third variable, such that

$$\forall C > 1 \, \forall \delta_0 > 0 \, \exists R > 0 \text{ such that for a.e. } t \in (0, T) \text{ and all } \delta \leq \delta_0 \\ \text{there holds } \Theta(t, \delta, C\delta^{-1}) \leq R.$$

This function describes relation between $M(t, x, \xi)$ and $M_Q(t, \xi) = \operatorname{ess\,inf}_{x \in \Omega \cap 5Q} M(t, x, \xi)$, where $Q \subset \mathbb{R}^d$ is an arbitrary cube and $5Q$ is a cube with the same center as Q with five times longer edge. More precisely, we assume that there exists $\xi_0 \in \mathbb{R}$ and $\delta_0 > 0$ such that for every cube $Q \subset \mathbb{R}^d$ with edge $\delta \in (0, \delta_0)$ and all $\xi \in \mathbb{R}^d$ with $|\xi| > \xi_0$ we have

$$\frac{M(t, x, \xi)}{M_Q^{**}(t, \xi)} \leq \Theta(t, \delta, |\xi|), \quad (1.7)$$

where M_Q^{**} is the second convex conjugate to M_Q , see Definition 1.3.

We remark that Assumption 1.16 mimics the one made in [16], namely

$$\frac{M(t, x, \xi)}{M_{Q,I}^{**}(\xi)} \leq \Theta(\delta, |\xi|),$$

where $M_{Q,I}(\xi) = \operatorname{ess\,inf}_{x \in \Omega \cap 3Q, t \in I \cap (0, T)} M(t, x, \xi)$, Q is a cube with edge of length δ , I is a subinterval of \mathbb{R} with $|I| \leq \delta$ and function Θ satisfies:

$$\forall C > 0 \, \forall \delta_0 > 0 \, \exists R > 0 \text{ such that for a.e. } t \in (0, T) \text{ and all } \delta \leq \delta_0 \text{ there holds } \Theta(\delta, C\delta^{-d}) \leq R.$$

We also note that our assumption is equivalent to condition (A1') in [31, Definition 4.1.1] while the latter to the condition (A1-n) in [31, Chapter 7.3].

On the one hand, the relaxed regularity in time allows for N -functions which are merely measurable in time. On the other hand, we need to control the quantity $\Theta(t, \delta, C\delta^{-1})$ rather than $\Theta(\delta, C\delta^{-d})$ which results in better exponents regimes for some well-known examples of N -functions, see Example 1.18. This improvement is based on the observation that in the approximation result one needs to approximate in the modular topology functions of the form

$$\nabla(T_k(u) + \varphi) \text{ where } \nabla u \in L_M(\Omega_T), \varphi \in C_c^\infty(\Omega_T)$$

The observation described above can be easily implemented in the previous works on this topic cf. [15,16].

Remark 1.17. In the particular case of an isotropic N -function $M(t, x, |\xi|)$, Assumption 1.16 boils down to existence of the function $\Theta : (0, T) \times [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ which is nondecreasing with respect to second and third variable such that

$$\limsup_{\delta \rightarrow 0^+} \Theta(t, \delta, C\delta^{-1}) \text{ is bounded uniformly in time } t \in (0, T) \quad (1.8)$$

and

$$\frac{M(t, x, r)}{M(t, y, r)} \leq \Theta(t, |x - y|, r).$$

See [16, Lemma A.4] for the proof.

Example 1.18. We list here N -functions satisfying Assumptions 1.16. For the proof, we refer to Appendix A.1.

(E1) $M(t, x, \xi) = |\xi|^{p(t,x)}$ with $1 < p_- \leq p(t, x) \leq p_+ < \infty$ and $p(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$. Here, $C_{\log}(\Omega)$ is the space of log-Hölder continuous functions on Ω , i.e. functions $v : \Omega \rightarrow \mathbb{R}$ such that

$$|v(x) - v(y)| \leq \frac{C}{\log|x - y|}$$

for all $x, y \in \Omega$ and some constant C . Note that only very low regularity of $p(t, x)$ in time is required.

(E2) $M(t, x, \xi) = |\xi|^{p(t,x)} + a(t, x) |\xi|^{q(t,x)}$ where

- $1 < p_- \leq p(t, x) < p_+ < \infty$, $1 < q_- \leq q(t, x) < q_+ < \infty$,
- $p(t, x), q(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$,
- $a(t, x) \in L^\infty(0, T; C^\alpha(\Omega))$ for some $\alpha \in (0, 1)$ and $a \geq 0$,
- $q(t, x) - p(t, x) \leq \alpha$.

Here, $C^\alpha(\Omega)$ is the space of α -Hölder continuous functions on Ω . We stress that only very low regularity of $p(t, x)$ and $q(t, x)$ in time is required. We also observe that for $p_- < d$, our admissible regime of exponents is better than $q(t, x) - p(t, x) \leq \frac{\alpha p_-}{d}$ known from [16].

Assumption 1.19 (Assumptions on A). We assume that $A : \Omega_T \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies:

- (A1) A is a Carathéodory's function, i.e. for a.e. $(t, x) \in \Omega_T$, map $\mathbb{R}^d \ni \xi \mapsto A(t, x, \xi)$ is continuous and for all $\xi \in \mathbb{R}^d$, map $\Omega_T \ni (t, x) \mapsto A(t, x, \xi)$ is measurable,
- (A2) (coercivity and growth bound) there is a constant c and function $h \in L^\infty(\Omega_T)$ such that for all $\xi \in \mathbb{R}^d$ and a.e. $(t, x) \in \Omega_T$:

$$M(t, x, \xi) + M^*(t, x, A(t, x, \xi)) \leq c A(t, x, \xi) \cdot \xi + h(t, x),$$

- (A3) (monotonicity) for all $\eta, \xi \in \mathbb{R}^d$ and a.e. $(t, x) \in \Omega_T$:

$$(A(t, x, \xi) - A(t, x, \eta)) \cdot (\xi - \eta) \geq 0,$$

- (A4) for a.e. $(t, x) \in \Omega_T$ we have $A(t, x, 0) = 0$.

Remark 1.20. In classical papers, condition (A4) could be deduced from coercivity and growth bounds. Here, (A2) implies only that

$$0 \leq M^*(t, x, A(t, x, 0)) \leq h(t, x).$$

We believe that (A4) can be waived. Nevertheless, we make this assumption as it is natural and it simplifies many technical computations.

Example 1.21. We list here functions \mathcal{A} corresponding to N -functions in Example 1.18 which satisfy Assumptions 1.19. For the proof, we refer to Appendix A.2.

- (F1) $A(t, x, \xi) = |\xi|^{p(t,x)-2} \xi$ leads to the equation with $p(t, x)$ -Laplacian

$$u_t(t, x) = \operatorname{div} \left[|\nabla u(t, x)|^{p(t,x)-2} \nabla u(t, x) \right] + f(t, x)$$

and the governing N -function $M(t, x, \xi)$ is given by (E1) in Example 1.18. Such problems have been considered recently for instance in [2,5] under assumption that $p(t, x)$ is log-Hölder continuous jointly in t and x . In our setting, we only need $p(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$.

- (F2) $A(t, x, \xi) = |\xi|^{p(t,x)-2} \xi + a(t, x) |\xi|^{q(t,x)-2} \xi$ leads to the double phase problem

$$u_t(t, x) = \operatorname{div} \left[|\nabla u(t, x)|^{p(t,x)-2} \nabla u(t, x) + a(t, x) |\nabla u(t, x)|^{q(t,x)-2} \nabla u(t, x) \right] + f(t, x).$$

Such problems were studied with variational methods [8,35] but mostly with constant or only x -dependent exponents. The case of $p(t, x)$ and $q(t, x)$ which are log-Hölder continuous jointly in t and x was studied in [16]. Our theory requires only $p(t, x), q(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$.

Lemma 1.22. Let A satisfy Assumption 1.19. Then, for every $K > 0$, there exists a constant $C(K)$ depending on K such that $|A(t, x, \xi)| \leq C(K)$ for a.e. $(t, x) \in \Omega_T$ and all $\xi \in \mathbb{R}^d$ fulfilling $|\xi| \leq K$.

Proof. Let $|\xi| \leq K$. Assumption (A2) implies that

$$M^*(t, x, A(t, x, \xi)) \leq c A(t, x, \xi) \cdot \xi + h(t, x). \quad (1.9)$$

Let m be a Young function such that $m(|\xi|) \leq M^*(t, x, \xi)$ for a.e. $(t, x) \in \Omega_T$ as in point (M4) in Definition 1.2. If $|A(t, x, \xi)| \leq 1$, the assertion follows by choosing $C(K) \geq 1$. Otherwise, (1.9) implies

$$\frac{m(|A(t, x, \xi)|)}{|A(t, x, \xi)|} \leq c |\xi| + \|h\|_\infty \leq c K + \|h\|_\infty.$$

Since map $s \mapsto \frac{m(s)}{s}$ is nondecreasing (property (N1) in Lemma 1.4) and m is superlinear (property (Y3) in Definition 1.1), the assertion follows. \square

Next, we define a function space relevant for the problem (1.1) as follows:

$$V_T^M = \left\{ u : \Omega_T \rightarrow \mathbb{R} \text{ such that } u \in L^1(0, T; W_0^{1,1}(\Omega)), \right. \\ \left. \nabla u \in L_M(\Omega_T) \text{ and } u \in L^\infty(0, T; L^2(\Omega)) \right\}.$$

The main results of this paper read:

Theorem 1.23 (Existence of solutions). *Suppose that Assumptions 1.16 and 1.19 are satisfied. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$. Then, there exists $u \in V_T^M(\Omega)$ which is a weak solution to (1.1). More precisely, there exists $u \in V_T^M(\Omega)$ such that $A(t, x, \nabla u) \in L_{M^*}(\Omega_T)$ and for all $\varphi \in C_0^\infty([0, T] \times \Omega)$, there holds:*

$$-\int_{\Omega_T} u(t, x) \partial_t \varphi(t, x) \, dt \, dx - \int_{\Omega} u_0(x) \varphi(0, x) \, dx + \\ + \int_{\Omega_T} A(t, x, \nabla u) \cdot \nabla \varphi(t, x) \, dt \, dx = \int_{\Omega_T} f(t, x) \varphi(t, x) \, dt \, dx.$$

In addition, u satisfies the global energy inequality, i.e. for all $t \in [0, T]$ there holds

$$\frac{1}{2} \int_{\Omega} \left[u^2(t, x) - u_0^2(x) \right] \, dx \leq - \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x)) \cdot \nabla u(s, x) \, dx \, ds \\ + \int_0^t \int_{\Omega} f(s, x) u(s, x) \, dx \, ds. \quad (1.10)$$

Theorem 1.24 (Uniqueness of solutions). *Let all assumptions of Theorem 1.23 be satisfied. Moreover, suppose that the N -function M is isotropic, i.e. it is of the form $M(t, x, |\xi|)$. Then, weak solution to (1.1) is unique and it satisfies the energy equality, i.e. for all $t \in [0, T]$ there holds*

$$\begin{aligned}
\frac{1}{2} \int_{\Omega} \left[u^2(t, x) - u_0^2(x) \right] dx &= - \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x)) \cdot \nabla u(s, x) dx ds \\
&\quad + \int_0^t \int_{\Omega} f(s, x) u(s, x) dx ds.
\end{aligned} \tag{1.11}$$

2. Auxiliary theory and results

2.1. Smooth approximation

In this section we prove that if $u \in V_T^M(\Omega)$, then u can be approximated in the modular topology of the gradients. We formulate this result locally in Ω but we remark that the similar approach has already been used in [16, Theorem 3.1], where approximation was performed globally for Lipschitz domains Ω by using a decomposition on star-shaped sets, see [27, Lemma II.1.3].

First, we recall the definition of truncation and mollification operators:

Definition 2.1 (*Truncation*). Function

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{otherwise,} \end{cases}$$

is called truncation at level k . We also denote by G_k its primitive function, i.e. we set

$$G_k(s) = \int_0^s T_k(\sigma) d\sigma.$$

Definition 2.2 (*Mollification with respect to the spatial variable*). Let $\eta : \mathbb{R}^d \rightarrow \mathbb{R}$ be a standard regularizing kernel, i.e. η is a smooth nonnegative function compactly supported in a ball of radius one and fulfills $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Then, we set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^d} \eta\left(\frac{x}{\varepsilon}\right)$ and for arbitrary $u : \Omega \rightarrow \mathbb{R}$ and $\Omega' \Subset \Omega$, we define $u^\varepsilon : \Omega' \rightarrow \mathbb{R}$ as

$$u^\varepsilon(x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x - y) u(y) dy.$$

Furthermore, if $u : \Omega_T \rightarrow \mathbb{R}$, then u^ε denotes mollification in space, i.e.

$$u^\varepsilon(t, x) = \int_{\mathbb{R}^d} \eta_\varepsilon(x - y) u(t, y) dy.$$

Definition 2.3 (*Mollification with respect to time*). Let $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ be a standard regularizing kernel, i.e. ζ is a smooth nonnegative function compactly supported in a ball of radius one and fulfills $\int_{\mathbb{R}} \zeta(x) dx = 1$. Then, we set $\zeta_\varepsilon(x) = \frac{1}{\varepsilon} \zeta\left(\frac{x}{\varepsilon}\right)$ and for arbitrary $u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, we define $S^\varepsilon u : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ as

$$S^\varepsilon u(t, x) = \int_{\mathbb{R}} \zeta_\varepsilon(t-s) u(s, x) \, ds.$$

For properties of mollified functions, the reader may consult [24, Appendix C.4]. Finally, we formulate the approximative properties of the mollifications defined above, which is the most essential tool used in the paper.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^d$, $\psi : \Omega \rightarrow \mathbb{R}$ be compactly supported satisfying $0 \leq \psi \leq 1$ and $u \in V_T^M(\Omega)$. Suppose that Assumption 1.16 is satisfied. Then, there exists $\varepsilon_0 > 0$:*

- (S1) $(T_k(u^\varepsilon)\psi)^\varepsilon \in L^1(0, T; C_0^\infty(\Omega))$ for all $\varepsilon \in (0, \varepsilon_0)$,
- (S2) $T_k(u^\varepsilon)\psi \rightarrow T_k(u)\psi$ a.e. in Ω_T and in $L^1(0, T; L^1(\Omega))$ as $\varepsilon \rightarrow 0^+$,
- (S3) $\nabla(T_k(u^\varepsilon)\psi)^\varepsilon \xrightarrow{M} \nabla(T_k(u)\psi)$ as $\varepsilon \rightarrow 0^+$, where the modular convergence \xrightarrow{M} is defined in Definition 1.8.

The key estimate needed for the proof of Theorem 2.4 is formulated in the following lemma.

Lemma 2.5. *Suppose that Assumption 1.16 is satisfied, $v : \Omega_T \rightarrow \mathbb{R}^d$ and $v \in L_M(\Omega_T)$ with $\int_{\Omega_T} M(t, x, v(t, x)) \, dt \, dx < \infty$. Assume that $v = \nabla u + \varphi$ for some $u \in V_T^M(\Omega)$ and $\varphi \in L^\infty(\Omega_T)$. Then, there is a constant C such that for any compactly supported $\psi : \Omega \rightarrow \mathbb{R}$ with $0 \leq \psi \leq 1$ and for all $k \in \mathbb{N}$,*

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_T} M(t, x, (\mathbb{1}_{|u^\varepsilon| \leq k} v^\varepsilon(t, x) \psi(x))^\varepsilon) \, dt \, dx &\leq \\ &\leq \int_{\Omega_T} m_2(|v(t, x)| \psi(x)) \mathbb{1}_{|v(t, x)| \psi(x) \leq \xi_0} \, dt \, dx + C \int_{\Omega_T} M(t, x, v(t, x)) \, dt \, dx, \end{aligned}$$

where ξ_0 is a constant from Assumption 1.16 and m_2 is a Young function as in (M4) in Definition 1.2.

Remark 2.6. Since $v \in L_M(\Omega_T)$, the condition $\int_{\Omega_T} M(t, x, v(t, x)) \, dt \, dx < \infty$ can be always satisfied by considering appropriate scaling if necessary.

Proof of Lemma 2.5. To shorten all formulas, we denote $z_\varepsilon(t, x) = (\mathbb{1}_{|u^\varepsilon| \leq k} v^\varepsilon(t, x) \psi(x))^\varepsilon$ and write:

$$\begin{aligned} \int_{\Omega_T} M(t, x, (\mathbb{1}_{|u^\varepsilon| \leq k} v^\varepsilon(t, x) \psi(x))^\varepsilon) \, dt \, dx &\leq \\ &\leq \int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| \leq \xi_0} \, dt \, dx + \int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| > \xi_0} \, dt \, dx. \end{aligned} \tag{2.1}$$

For the first term, we use (M4) in Definition 1.2 to observe:

$$\int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| \leq \xi_0} dt dx \leq \int_{\Omega_T} m_2(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| \leq \xi_0} dt dx$$

and so, by (N7) in Lemma 1.4 we get

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| \leq \xi_0} dt dx \leq \int_{\Omega_T} m_2(|v(t, x)|\psi(x)) \mathbb{1}_{|v(t, x)|\psi(x) \leq \xi_0} dt dx. \quad (2.2)$$

Hence, it is sufficient to focus on the second term in (2.1). Let $\{Q_j\}_{j=1}^{N_\varepsilon}$ be a family of closed cubes with edge ε such that $\text{int} Q_j \cap \text{int} Q_i = \emptyset$ for $i \neq j$ and $\Omega \subset \bigcup_{i=1}^{N_\varepsilon} Q_i$. Moreover, let $3Q_i$ and $5Q_i$ be the cubes with the same center as Q_i and edges 3ε and 5ε , respectively. Then,

$$\begin{aligned} \int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| > \xi_0} dt dx &= \\ &= \sum_{i=1}^{N_\varepsilon} \int_0^T \int_{Q_i \cap \Omega} \frac{M(t, x, z_\varepsilon(t, x))}{M_{Q_i}^{**}(t, z_\varepsilon(t, x))} M_{Q_i}^{**}(t, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| > \xi_0} dx dt, \end{aligned}$$

where $M_{Q_i}^{**}$ is defined in Assumption 1.16. Note that we assume that $v = \nabla u + \varphi$ for some $u \in V_T^M(\Omega)$ and $\varphi \in L^\infty(\Omega_T)$. We note that

$$z_\varepsilon(t, x) = (\nabla T_k(u^\varepsilon(t, x))\psi(x))^\varepsilon + (\mathbb{1}_{|u^\varepsilon| \leq k} \varphi^\varepsilon(t, x)\psi(x))^\varepsilon := z_\varepsilon^1(t, x) + z_\varepsilon^2(t, x).$$

Clearly, using Young's convolutional inequality, we have $|z_\varepsilon^2(t, x)| \leq \|\varphi\|_\infty \|\psi\|_\infty$. Moreover,

$$z_\varepsilon^1(t, x) = -(T_k(u^\varepsilon) \operatorname{div} \psi) * \eta_\varepsilon(t, x) + (T_k(u^\varepsilon)\psi) * \nabla \eta_\varepsilon(t, x)$$

so applying Young's convolutional inequality we have:

$$|z_\varepsilon^1(t, x)| \leq k \|\operatorname{div} \psi\|_\infty + \frac{k \|\psi\|_\infty \|\nabla \eta_\varepsilon\|_1}{\varepsilon}.$$

We conclude that $|z_\varepsilon(t, x)| \leq \frac{C(k, \varphi, \eta)}{\varepsilon}$ for $\varepsilon < 1$ and therefore, using (1.7), we get that for $x \in Q_i \cap \Omega$ the following inequality

$$\frac{M(t, x, z_\varepsilon(t, x))}{M_{Q_i}^{**}(t, z_\varepsilon(t, x))} \leq \Theta\left(t, \delta, \frac{C(k, \varphi, \eta)}{\varepsilon}\right) \leq C$$

holds true for sufficiently small ε . Consequently,

$$\int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| > \xi_0} dt dx \leq C \sum_{i=1}^{N_\varepsilon} \int_0^T \int_{Q_i \cap \Omega} M_{Q_i}^{**}(t, z_\varepsilon(t, x)) dx dt. \quad (2.3)$$

To estimate the right hand side in the above inequality, we focus on each summand separately. Using Jensen's and Young's convolutional inequalities we deduce:

$$\begin{aligned}
 & \int_0^T \int_{Q_i \cap \Omega} M_{Q_i}^{**}(t, z_\varepsilon(t, x)) \, dx \, dt \\
 &= \int_0^T \int_{Q_i \cap \Omega} M_{Q_i}^{**} \left(t, \int_{B(0, \varepsilon)} \eta_\varepsilon(y) \left(v^\varepsilon(t, x-y) \psi(x-y) \mathbb{1}_{|u^\varepsilon| \leq k}(x-y) \right) dy \right) dx \, dt \\
 &\leq \int_0^T \int_{Q_i \cap \Omega} \int_{B(0, \varepsilon)} \eta_\varepsilon(y) M_{Q_i}^{**}(t, v^\varepsilon(t, x-y) \psi(x-y) \mathbb{1}_{|u^\varepsilon| \leq k}(x-y)) \, dy \, dx \, dt \\
 &\leq \int_0^T \int_{\mathbb{R}^d} \int_{B(0, \varepsilon)} \eta_\varepsilon(y) M_{Q_i}^{**}(t, v^\varepsilon(t, x-y) \psi(x-y) \mathbb{1}_{3Q_i \cap \Omega}(x-y)) \, dy \, dx \, dt \\
 &\leq \int_0^T \int_{\mathbb{R}^d} M_{Q_i}^{**}(t, v^\varepsilon(t, x) \psi(x) \mathbb{1}_{3Q_i \cap \Omega}(x)) \, dx \, dt = \int_0^T \int_{3Q_i \cap \Omega} M_{Q_i}^{**}(t, v^\varepsilon(t, x) \psi(x)) \, dx \, dt,
 \end{aligned} \tag{2.4}$$

where we used the fact that $\|\eta_\varepsilon\|_{L^1} = 1$ and the fact that $M_{Q_i}^{**}(t, \xi) = 0 \iff \xi = 0$. Next, by convexity of $\xi \mapsto M_{Q_i}^{**}(t, \xi)$ and thanks to $0 \leq \psi(x) \leq 1$, we can simply estimate the last term as

$$\int_0^T \int_{3Q_i \cap \Omega} M_{Q_i}^{**}(t, v^\varepsilon(t, x) \psi(x)) \, dx \, dt \leq \int_0^T \int_{3Q_i \cap \Omega \cap \text{supp}(\psi)} M_{Q_i}^{**}(t, v^\varepsilon(t, x)) \, dx \, dt.$$

Then, repeating the procedure from (2.4), we deduce

$$\int_0^T \int_{3Q_i \cap \Omega \cap \text{supp}(\psi)} M_{Q_i}^{**}(t, v^\varepsilon(t, x)) \, dx \, dt \leq \int_0^T \int_{5Q_i \cap \Omega \cap \text{supp}(\psi)} M_{Q_i}^{**}(t, v(t, x)) \, dx \, dt.$$

Finally, as $M_{Q_i}(t, \xi) = \text{ess inf}_{x \in \Omega \cap 5Q_i} M(t, x, \xi)$ and since $M_{Q_i}^{**}(t, \xi) \leq M_{Q_i}(t, \xi)$, we can estimate each summand by the above inequality to get:

$$\int_0^T \int_{5Q_i \cap \Omega \cap \text{supp}(\psi)} M_{Q_i}^{**}(t, v(t, x)) \, dx \, dt \leq \int_0^T \int_{5Q_i \cap \Omega} M(t, x, v(t, x)) \, dx \, dt.$$

Coming back to (2.3), we obtain

$$\int_{\Omega_T} M(t, x, z_\varepsilon(t, x)) \mathbb{1}_{|z_\varepsilon(t, x)| > \xi_0} dt dx \leq \int_{\Omega_T} M(t, x, v(t, x)) dt dx \quad (2.5)$$

for some possibly different constant C which can be increased due to integration over repeating parts of overlapping cubes $\{5Q_i\}_{i=1}^{N_\varepsilon}$. Combining (2.2) with (2.5), we finish the proof. \square

Proof of Theorem 2.4. First two properties follow from properties of mollification and continuity of the truncation. To show also the third property, we first compute:

$$\nabla (T_k(u^\varepsilon)\psi)^\varepsilon = (\mathbb{1}_{|u^\varepsilon| \leq k} (\nabla u)^\varepsilon \psi)^\varepsilon + (T_k(u^\varepsilon) \nabla \psi)^\varepsilon.$$

Then, due to (N7) in Lemma 1.4, $(T_k(u^\varepsilon) \nabla \psi)^\varepsilon \xrightarrow{M} T_k(u) \nabla \psi$ and so, it is sufficient to focus only on the first term. Using Lemma 1.9, we find a sequence of simple functions $\{\varphi_n\}_{n \in \mathbb{N}}$ such that $\varphi_n \rightarrow \nabla u$ a.e. and $\varphi_n \xrightarrow{M} \nabla u$ as $n \rightarrow \infty$, i.e. there is $\tilde{\lambda} > 0$ such that

$$\int_{\Omega_T} M\left(t, x, \frac{\nabla u(t, x) - \varphi_n(t, x)}{\tilde{\lambda}}\right) dt dx \rightarrow 0.$$

Then, for some $\lambda_1, \lambda_2, \lambda_3$ to be chosen later, $\lambda = \lambda_1 + \lambda_2 + \lambda_3$ and some $n \in \mathbb{N}$ we write:

$$\begin{aligned} \int_{\Omega_T} M\left(t, x, \frac{(\mathbb{1}_{|u^\varepsilon| \leq k} (\nabla u)^\varepsilon \psi)^\varepsilon - \mathbb{1}_{|u| \leq k} \nabla u \psi}{\lambda}\right) dt dx &\leq \\ &\leq \frac{\lambda_1}{\lambda} \int_{\Omega_T} M\left(t, x, \frac{(\mathbb{1}_{|u^\varepsilon| \leq k} (\nabla u)^\varepsilon \psi)^\varepsilon - (\mathbb{1}_{|u^\varepsilon| \leq k} (\varphi_n)^\varepsilon \psi)^\varepsilon}{\lambda_1}\right) dt dx + \\ &+ \frac{\lambda_2}{\lambda} \int_{\Omega_T} M\left(t, x, \frac{(\mathbb{1}_{|u^\varepsilon| \leq k} (\varphi_n)^\varepsilon \psi)^\varepsilon - \mathbb{1}_{|u| \leq k} \varphi_n \psi}{\lambda_2}\right) dt dx \\ &+ \frac{\lambda_3}{\lambda} \int_{\Omega_T} M\left(t, x, \mathbb{1}_{|u| \leq k} \psi \frac{\varphi_n - \nabla u}{\lambda_3}\right) dt dx =: A^{n, \varepsilon} + B^{n, \varepsilon} + C^{n, \varepsilon}. \end{aligned}$$

Using (N7) in Lemma 1.4, for any $n \in \mathbb{N}$ and $\lambda_2 > 0$, $\limsup_{\varepsilon \rightarrow 0} B^{n, \varepsilon} = 0$. Also, we note that

$$\begin{aligned} \frac{\lambda_1}{\lambda} \int_{\Omega_T} M\left(t, x, \frac{(\mathbb{1}_{|u^\varepsilon| \leq k} (\nabla u)^\varepsilon \psi)^\varepsilon - (\mathbb{1}_{|u^\varepsilon| \leq k} (\varphi_n)^\varepsilon \psi)^\varepsilon}{\lambda_1}\right) dt dx &\leq \\ &\leq \frac{\lambda_1}{\lambda} \int_{\Omega_T} M\left(t, x, \frac{(\mathbb{1}_{|u^\varepsilon| \leq k} (\nabla u - \varphi_n)^\varepsilon \psi)^\varepsilon}{\lambda_1}\right) dt dx. \end{aligned}$$

Therefore, if we choose $\lambda_1 = \lambda_3 = \tilde{\lambda}$ and use Lemma 2.5, we obtain

$$\limsup_{\varepsilon \rightarrow 0} (A^{n,\varepsilon} + C^{n,\varepsilon}) \leq \int_{\Omega_T} M \left(t, x, \mathbb{1}_{|u| \leq k} \psi \frac{\varphi_n - \nabla u}{\tilde{\lambda}} \right) dt dx + \int_{\Omega_T} m_2 \left(\left| \frac{\varphi_n - \nabla u}{\tilde{\lambda}} \right| \psi(x) \right) \mathbb{1}_{\left| \frac{\varphi_n - \nabla u}{\tilde{\lambda}} \right| \psi(x) \leq \xi_0} dt dx.$$

Since $\varphi_n \rightarrow \nabla u$ a.e. in Ω_T and $\varphi_n \xrightarrow{M} \nabla u$, we conclude the proof. \square

2.2. Regularization of the operator

In this section, we formulate well-posedness theory for parabolic equations in Musielak-Orlicz spaces with Young functions. This allows us to construct solution to our problem by a limiting procedure. The following result was proven by Elmahi and Meskine [23, Theorem 2] using Galerkin's approximation and mollification as in Section 2.1 (however here N -function is homogeneous and isotropic so the result can be established significantly easier).

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with segment property. Let $m : \mathbb{R} \rightarrow \mathbb{R}$ be a Young function. Suppose that $a : \Omega_T \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies:*

- (R1) *a is a Carathéodory's function, i.e. for a.e. $(t, x) \in \Omega_T$, map $\mathbb{R}^d \ni \xi \mapsto a(t, x, \xi)$ is continuous and for all $\xi \in \mathbb{R}^d$, map $\Omega_T \ni (t, x) \mapsto a(t, x, \xi)$ is measurable,*
- (R2) *there are $c \in E_{m^*}(\Omega_T)$ with $c \geq 0$ and nonnegative constant β and γ such that*

$$|a(t, x, \xi)| \leq \beta \left(c(t, x) + (m^*)^{-1}(m(\gamma|\xi|)) \right),$$

- (R3) *there are $d \in L^1(\Omega_T)$ and nonnegative constants α and λ such that*

$$a(t, x, \xi) \cdot \xi + d(t, x) \geq \alpha m \left(\frac{|\xi|}{\lambda} \right),$$

- (R4) *a is strongly monotone, i.e. for all $\eta, \xi \in \mathbb{R}^d$ and a.e. $(t, x) \in \Omega_T$:*

$$(a(t, x, \xi) - a(t, x, \eta)) \cdot (\xi - \eta) > 0.$$

Then, the problem

$$u_t = \operatorname{div} a(t, x, \nabla u) + g$$

with $u(0, x) = u_0(x) \in L^\infty(\Omega_T)$, $u(t, x) = 0$ for $x \in \partial\Omega$ and $g \in L^\infty(\Omega_T)$ has the unique weak solution $u \in C((0, T); L^2(\Omega)) \cap W^1 L_m(\Omega_T)$ (see Definition 1.14).

Using Theorem 2.7, one can define a sequence approximating solutions to (1.1) as follows:

Lemma 2.8. *Suppose A satisfies Assumption 1.19, M is an N -function and m is a Young function such that $M(t, x, \xi) \leq m(|\xi|)$. For $\theta \in (0, 1]$, consider regularized operator*

$$A_\theta(t, x, \xi) = A(t, x, \xi) + \theta \nabla_\xi m(|\xi|). \quad (2.6)$$

Then, there exists a weak solution to the problem

$$u_t^\theta = \operatorname{div} A_\theta(t, x, \nabla u^\theta) + g \quad (2.7)$$

with $u(0, x) = u_0(x) \in L^\infty(\Omega_T)$, $u(t, x) = 0$ for $x \in \partial\Omega$ and $g \in L^\infty(\Omega_T)$. More precisely,

$$u^\theta \in C((0, T); L^2(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega)).$$

Moreover, u^θ satisfies the global energy equality:

$$\begin{aligned} \int_{\Omega} [(u^\theta(t, x))^2 - (u_0(x))^2] dx = \\ - \int_0^t \int_{\Omega} A^\theta(s, x, \nabla u^\theta(s, x)) \cdot \nabla u^\theta(s, x) dx ds + \int_0^t \int_{\Omega} g(s, x) u^\theta(s, x) dx ds. \end{aligned} \quad (2.8)$$

We also have bounds which are uniform in θ :

- (C1) sequence $\{u^\theta\}_\theta$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$,
- (C2) sequence $\{\nabla u^\theta\}_\theta$ is uniformly bounded in $L_M(\Omega_T)$,
- (C3) sequence $\{A(t, x, \nabla u^\theta)\}_\theta$ is uniformly bounded in $L_{M^*}(\Omega_T)$,
- (C4) sequence $\{\theta m^*(\nabla_\xi m(|\nabla u^\theta|))\}_\theta$ is uniformly bounded in $L^1(\Omega_T)$.

Proof. First, we observe from the definition of the convex conjugate that

$$\nabla_\xi m(|\xi|) \cdot \xi = m(|\xi|) + m^*(|\nabla_\xi m(|\xi|)|). \quad (2.9)$$

We also note that $\nabla_\xi m(|\xi|) = m'(|\xi|) \frac{\xi}{|\xi|}$ so that $\nabla_\xi m(|\xi|) \xi \geq 0$. Let us check that assumptions of Theorem 2.7 are satisfied with operator (2.6) controlled by N -function m . Assumption (R1) is fulfilled trivially. To verify (R2), we use (2.9), (A2) in Assumption 1.19 and the convexity, to obtain:

$$\begin{aligned} c A_\theta(t, x, \xi) \cdot \xi &\geq M(t, x, \xi) + M^*(t, x, A(t, x, \xi)) - h(t, x) + c \theta \nabla_\xi m(|\xi|) \cdot \xi \\ &\geq 0 + m^*(|A(t, x, \xi)|) - h(t, x) + c \theta m^*(|\nabla_\xi m(|\xi|)|) \\ &\geq 2 \min(1, c) \left(\frac{1}{2} m^*(|A(t, x, \xi)|) + \frac{1}{2} m^*(\theta |\nabla_\xi m(|\xi|)|) \right) - |h(t, x)| \\ &\geq 2 \min(1, c) m^* \left(\frac{1}{2} |A_\theta(t, x, \xi)| \right) - |h(t, x)|. \end{aligned} \quad (2.10)$$

On the other hand, by Young's inequality

$$c A_\theta(t, x, \xi) \cdot \xi \leq \min(1, c) m \left(\frac{c}{\min^2(1, c)} |\xi| \right) + \min(1, c) m^* \left(\frac{1}{2} |A_\theta(t, x, \xi)| \right). \quad (2.11)$$

Hence, we combine (2.10) and (2.11) to deduce

$$\min(1, c) m^* \left(\frac{1}{2} |A_\theta(t, x, \xi)| \right) \leq \min(1, c) m \left(\frac{c}{\min^2(1, c)} |\xi| \right) + |h(t, x)|.$$

Next, we abbreviate $c_1 = 1/\min(1, c)$ and $c_2 = \frac{c}{\min^2(1, c)}$. Furthermore, since m^* is increasing and convex, then $(m^*)^{-1}$ is increasing and concave. Moreover $(m^*)^{-1}(0) = 0$ so $(m^*)^{-1}$ is sub-additive and therefore

$$\frac{1}{2} |A_\theta(t, x, \xi)| \leq (m^*)^{-1} \left(m(|\xi|) + c_1 |h(t, x)| \right) \leq (m^*)^{-1} (m(|\xi|)) + (m^*)^{-1} (c_1 |h(t, x)|),$$

which proves (R2) since $h \in L^\infty(\Omega_T)$. Then, repeating computation in (2.10) and applying (2.9) we deduce:

$$\begin{aligned} c A_\theta(t, x, \xi) \cdot \xi &\geq M(t, x, \xi) + M^*(t, x, A(t, x, \xi)) - h(t, x) + c \theta \nabla_\xi m(|\xi|) \cdot \xi \\ &\geq c \theta m(|\xi|) - h(t, x), \end{aligned} \quad (2.12)$$

which proves (R3). Finally, (R4) follows easily as the function m can be always assumed to be strictly convex (otherwise, one can add a strictly convex function to m). Therefore, Theorem 2.7 applies so we conclude that for each $\theta \in (0, 1]$ there is a unique solution u^θ as desired. Moreover, energy equality (2.8) is valid.

Now, we intend to establish uniform estimates (C1)–(C4). Let m_1 be a Young function such that $m_1(|\xi|) \leq M(t, x, \xi)$ as in point (M4) in Definition 1.2. We estimate by using the Hölder inequality:

$$\int_{\Omega_t} f(s, x) u^\theta(s, x) \, ds \, dx \leq \|f\|_\infty \int_{\Omega_t} |u^\theta(s, x)| \, ds \, dx \leq \|f\|_\infty \int_{\Omega_t} (|u^\theta(s, x)|^2 + 1) \, ds \, dx. \quad (2.13)$$

Using energy equality (2.8) and noting that $A^\theta(s, x, \nabla u^\theta(s, x)) \cdot \nabla u^\theta(s, x) \geq 0$ we deduce that for a.e. $t \in (0, T)$

$$\int_{\Omega} (u^\theta(t, x))^2 \, dx \leq \int_{\Omega} (u_0(x))^2 \, dx + \|f\|_\infty \int_0^t \int_{\Omega} (|u^\theta(s, x)|^2 + 1) \, dx \, ds.$$

Therefore, Grönwall's lemma implies that u^θ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$. Moreover, (A2) in Assumption 1.19 leads to the estimate:

$$\begin{aligned} \int_{\Omega_t} M^*(s, x, A(s, x, \nabla u^\theta(s, x))) \, ds \, dx + \int_{\Omega_t} M(s, x, \nabla u^\theta(s, x)) \, ds \, dx - \int_{\Omega_t} h(s, x) \, ds \, dx \leq \\ \leq c \int_{\Omega_t} A(s, x, \nabla u^\theta(s, x)) \cdot \nabla u^\theta(s, x) \, ds \, dx. \end{aligned}$$

As $\int_{\Omega} (u^\theta(t, x))^2 \, dx$ and $\int_{\Omega_t} f(s, x) u^\theta(s, x) \, ds \, dx$ are uniformly bounded, we deduce from energy equality (2.8) that for a.e. $(t, x) \in \Omega_T$, the quantity

$$\begin{aligned} \int_{\Omega_t} M^*(s, x, A(s, x, \nabla u^\theta(s, x))) \, ds \, dx + \int_{\Omega_t} M(s, x, \nabla u^\theta(s, x)) \, ds \, dx + \\ + \int_{\Omega_t} \theta \nabla_\xi m(|\nabla u^\theta(s, x)|) \cdot \nabla u^\theta(s, x) \, ds \, dx \leq C(f, h, u_0), \end{aligned}$$

the constant $C(f, h, u_0)$ is independent of θ . Due to (N6) in Lemma 1.4, we have that $\{\nabla u^\theta\}_{\theta \in (0,1]}$ is uniformly bounded in $L_M(\Omega_T)$ and $\{A(t, x, \nabla u^\theta)\}_\theta$ is uniformly bounded in $L_{M^*}(\Omega_T)$. Finally, using (2.9) we deduce that sequence $\{\theta m^*(\nabla_\xi m(|\nabla u^\theta|))\}_{\theta \in (0,1]}$ is uniformly bounded in $L^1(\Omega_T)$. \square

Thanks to the uniform bounds established in Lemma 2.8, we can now let $\theta \rightarrow 0$ in (2.7). The starting point for this limiting procedure is the observation that the approximative term vanishes in the limit, which is formulated in the next lemma.

Lemma 2.9. *Under notation and assumptions of Lemma 2.8, for any $\varphi : \Omega_T \mapsto \mathbb{R}^d$ such that $\varphi \in L^\infty(\Omega_T; \mathbb{R}^d)$, we have*

$$\lim_{\theta \rightarrow 0} \int_{\Omega_T} \theta \nabla_\xi m(|\nabla u^\theta|) \cdot \varphi \, dt \, dx = 0.$$

Proof. This was also proved in [16] but it was not formulated as a separate result so we provide the proof here. Consider $\Omega_T^R = \{(t, x) \in \Omega_T : |\nabla u^\theta| \leq R\}$ and write

$$\int_{\Omega_T} |\theta \nabla_\xi m(|\nabla u^\theta|)| \, dx = \int_{\Omega_T^R} |\theta \nabla_\xi m(|\nabla u^\theta|)| \, dx + \int_{\Omega_T \setminus \Omega_T^R} |\theta \nabla_\xi m(|\nabla u^\theta|)| \, dx. \quad (2.14)$$

For any $R > 0$, the first term converges to 0 as $\theta \rightarrow 0$. Note that by convexity,

$$m^*(\theta \nabla_\xi m(|\nabla u^\theta|)) \leq m^*(\nabla_\xi m(|\nabla u^\theta|))$$

so that due to (N5) in Lemma 1.4, sequence $\{\theta \nabla_\xi m(|\nabla u^\theta|)\}_\theta$ is uniformly integrable. Therefore, as $R \rightarrow \infty$, the second term in (2.14) tends to 0 and the conclusion follows. \square

The next result deals with the time derivatives of u^θ and will be used to deduce the pointwise convergence.

Lemma 2.10. *Under notation and assumptions of Lemma 2.8, for every $\theta > 0$, we have $\partial_t u^\theta \in (W^1 E_m(\Omega_T))^*$ where m is defined in Lemma 2.8. Moreover, for all $\varphi \in W^1 E_m(\Omega_T)$ we have the following inequality:*

$$(\partial_t u^\theta, \varphi) \leq C \|\varphi\|_{W^1 L_m}, \quad (2.15)$$

where the constant C is independent of θ .

Proof. First, let $\varphi \in C_0^\infty((0, T) \times \Omega)$. By the weak formulation of (2.7) we have

$$\begin{aligned} - \int_{\Omega_T} u^\theta(t, x) \partial_t \varphi(t, x) \, dt \, dx + \int_{\Omega_T} A(t, x, \nabla u^\theta) \cdot \nabla \varphi(t, x) \, dt \, dx + \\ + \int_{\Omega_T} \theta_n \nabla_\xi m(|\nabla u^\theta|) \cdot \nabla \varphi \, dt \, dx = \int_{\Omega_T} f(t, x) \varphi(t, x) \, dt \, dx. \end{aligned}$$

Thus, we can estimate the left hand side using Lemma 1.7 as follows:

$$\begin{aligned} \left| \int_{\Omega_T} u^\theta(t, x) \partial_t \varphi(t, x) \, dt \, dx \right| \leq \|A(t, x, \nabla u^\theta)\|_{L_m^*} \|\nabla \varphi\|_{L_m} + \\ + \theta_n \|\nabla_\xi m(|\nabla u^\theta|)\|_{L_m^*} \|\nabla \varphi\|_{L_m} + |\Omega_T| m^*(\|f\|_\infty) \|\varphi\|_{L_m}. \end{aligned}$$

Note that $M(t, x, \xi) \leq m(|\xi|)$ implies $m^*(|\xi|) \leq M^*(t, x, \xi)$ and so,

$$\|A(t, x, \nabla u^\theta)\|_{L_m^*} \leq \|A(t, x, \nabla u^\theta)\|_{L_{M^*}}.$$

Therefore, we can use uniform bounds provided by Lemma 2.8 and this (after application of the Poincaré inequality from Lemma 1.15) concludes the proof of (2.15) for $\varphi \in C_0^\infty((0, T) \times \Omega)$. The general case follows by the density (in norm!) of $C_0^\infty((0, T) \times \Omega)$ in $W_0^1 E_m(\Omega_T)$ (cf. (P2) in Lemma 1.15). \square

Finally, note that uniform bounds in Lemma 2.8 guarantee the existence of subsequences (that we do not relabel) converging weakly-* in appropriate spaces (cf. Lemma 1.13). We will also need stronger compactness provided by the following result.

Lemma 2.11. *Under notation and assumptions of Lemma 2.8, the sequence $\{u^\theta\}_{\theta \in (0, 1]}$ is relatively compact in $L^1(0, T; L^1(\Omega))$. In particular, it has a subsequence converging a.e. in Ω_T .*

Proof. We recall a version of Aubin-Lions Lemma (cf. [39]):

Aubin-Lions Lemma. Let X_0 , X and X_1 be Banach spaces such that X_0 is compactly embedded in X and X is continuously embedded in X_1 . Suppose that sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is bounded in $L^q(0, T; X)$ and $L^1(0, T; X_0)$. Moreover, assume that sequence of distributional

time derivatives $\{\partial_t f_n\}$ is bounded in $L^1(0, T; X_1)$. Then, $\{f_n\}_{n \in \mathbb{N}}$ is relatively compact in $L^p(0, T; X)$ for any $1 \leq p < q$.

We want to apply this result with $X_0 = W_0^{1,1}(\Omega)$, $X = L^1(\Omega)$ and $X_1 = W^{-2,r}(\Omega)$ for r such that $W_0^{2,r}(\Omega)$ is continuously embedded in $C^1(\Omega)$ ($r > d$ is sufficient, cf. [28, Corollary 7.11]).

- By Rellich-Kondrachov Theorem (or Arzela-Ascoli Theorem if $d = 1$), X_0 is compactly embedded in X .
- Let $f \in L^1(\Omega)$. Then, for $\varphi \in W_0^{2,r}(\Omega)$,

$$\left| \int_{\Omega} f \varphi \right| \leq \|f\|_{L^1} \|\varphi\|_{L^\infty} \leq C \|f\|_{L^1} \|\varphi\|_{W^{2,r}},$$

for some constant C so that X is continuously embedded in X_1 .

- Sequence $\{u^\theta\}_{\theta \in (0,1]}$ is uniformly bounded in $L^\infty(0, T; L^2(\Omega))$ and $\{\nabla u^\theta\}_{\theta \in (0,1]}$ is uniformly bounded in $L_{M^*}(\Omega_T)$. In particular, $\{u^\theta\}_{\theta \in (0,1]}$ is uniformly bounded in $L^1(0, T; W_0^{1,1}(\Omega))$ and $L^2(0, T; L^1(\Omega))$.
- Let $\varphi \in L^\infty(0, T; W_0^{2,r}(\Omega))$ with $\|\varphi\|_{L^\infty(0,T;W_0^{2,r}(\Omega))} \leq 1$ and the plan is to prove that $(\partial_t u^\theta, \varphi)$ is uniformly bounded in φ and $\theta \in (0, 1]$. By the choice of r , there is a constant C such that $|\varphi| \leq C$ and $|\nabla \varphi| \leq C$. In particular, $\varphi \in W_0^1 E_m(\Omega_T)$ and $\|\varphi\|_{W^1 L_m} \leq C$ for some possibly different constant C . Using Lemma 2.10, we establish assertion. By duality, this shows that $\partial_t u^\theta$ is uniformly bounded in $L^1(0, T; W^{-2,r}(\Omega))$.

Aubin-Lions Lemma implies that $\{u^\theta\}_{\theta \in [0,1]}$ is relatively compact in $L^1(0, T; L^1(\Omega))$. \square

2.3. Equation $u_t = \operatorname{div} \alpha + f$ for $\alpha \in L_{M^*}(\Omega_T)$ and $f \in L^\infty(\Omega_T)$

In this section we study the equation

$$u_t = \operatorname{div} \alpha + f$$

or more precisely, the following identity required to be satisfied for all $\varphi \in C_0^\infty([0, T) \times \Omega)$:

$$\begin{aligned} - \int_{\Omega_T} u(t, x) \partial_t \varphi(t, x) \, dt \, dx - \int_{\Omega} u_0(x) \varphi(0, x) \, dx + \\ + \int_{\Omega_T} \alpha(t, x) \cdot \nabla \varphi(t, x) \, dt \, dx = \int_{\Omega_T} f(t, x) \varphi(t, x) \, dt \, dx, \end{aligned} \quad (2.16)$$

which is obtained in Section 3 as the limit of (2.7). For $u : \Omega_T \rightarrow \mathbb{R}$ solving (2.16), we write \tilde{u} to denote its extension:

$$\tilde{u}(t, x) = \begin{cases} 0 & \text{for } t > T, \\ u(t, x) & \text{for } t \in (0, T], \\ u_0(x) & \text{for } t \leq 0. \end{cases} \quad (2.17)$$

We also extend α and f to be zero for $t \in \mathbb{R} \setminus (0, T)$:

$$\bar{\alpha}(t, x) = \begin{cases} \alpha(t, x) & \text{for } t \in (0, T), \\ 0 & \text{for } t \in \mathbb{R} \setminus (0, T), \end{cases} \quad \bar{f}(t, x) = \begin{cases} f(t, x) & \text{for } t \in (0, T), \\ 0 & \text{for } t \in \mathbb{R} \setminus (0, T). \end{cases} \quad (2.18)$$

Our goal is to obtain some form of energy equality which will be crucial in developing the existence theory for (1.1). Classical approach (cf. [16]) was based on appropriate mollification in space and time which required some continuity assumptions on $M(t, x, \xi)$ both in t and x . Below, we show that mollification of the solution u only in space has already Sobolev regularity in space and time.

Lemma 2.12. *Suppose that $u \in V_T^M(\Omega)$, $\alpha \in L_{M^*}(\Omega_T)$ and $f \in L^\infty(\Omega_T)$. Consider extensions \tilde{u} , $\bar{\alpha}$ and \bar{f} defined in (2.17) and (2.18). Then,*

$$-\int_{\Omega} \int_{-T}^T \tilde{u}(t, x) \partial_t \varphi(t, x) \, dt \, dx = -\int_{\Omega} \int_{-T}^T \bar{\alpha}(t, x) \cdot \nabla \varphi(t, x) \, dt \, dx = \int_{\Omega} \int_{-T}^T \bar{f}(t, x) \varphi(t, x) \, dt \, dx, \quad (2.19)$$

for arbitrary $\varphi \in C_0^\infty((-T, T) \times \Omega)$. Moreover, $\tilde{u}^\varepsilon \in W^{1,1}((-T, T) \times \Omega')$ where $\Omega' \Subset \Omega$.

Proof. To verify (2.19), let $\varphi \in C_0^\infty((-T, T) \times \Omega)$. We compute using (2.16):

$$\begin{aligned} -\int_{\Omega} \int_{-T}^T \tilde{u}(t, x) \partial_t \varphi(t, x) \, dt \, dx &= \\ &= -\int_{\Omega} \int_{-T}^0 \tilde{u}(t, x) \partial_t \varphi(t, x) \, dt \, dx - \int_{\Omega} \int_0^T \tilde{u}(t, x) \partial_t \varphi(t, x) \, dt \, dx = \\ &= -\int_{\Omega} u_0(x) \varphi(0, x) \, dx - \int_{\Omega} \int_0^T u(t, x) \partial_t \varphi(t, x) \, dt \, dx = \\ &= -\int_{\Omega} \int_{-T}^T \bar{\alpha}(t, x) \cdot \nabla \varphi(t, x) \, dt \, dx + \int_{\Omega} \int_{-T}^T \bar{f}(t, x) \varphi(t, x) \, dt \, dx. \end{aligned}$$

Mollifying (2.19) in space (by testing with mollified test function), we deduce $\partial_t u^\varepsilon \in L^1((-T, T) \times \Omega')$ proving the Sobolev regularity in time. Asserted regularity in space is obvious. \square

Remark 2.13. Extension procedure above can be applied to obtain that $u^\varepsilon \in W^{1,1}((-M, T) \times \Omega')$ for any $0 < M < T$. However, we only need Sobolev regularity on $(-\delta, T) \times \Omega'$ for some $\delta > 0$ which can be arbitrarily small.

Lemma 2.14 (Local energy equality). Suppose that $u \in V_T^M(\Omega)$ is a solution to (2.16) with $\alpha \in L_{M^*}(\Omega_T)$, $f \in L^\infty(\Omega_T)$ and Assumption 1.16 is satisfied. Then, for arbitrary $k \in \mathbb{N}$, for arbitrary $\psi \in C_0^\infty(\Omega)$ fulfilling $0 \leq \psi \leq 1$ and for a.e. $t \in (0, T)$, the following energy equality is satisfied:

$$\begin{aligned} \int_{\Omega} \psi(x) [G_k(u(t, x)) - G_k(u_0(x))] dx &= \\ &= - \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla [T_k(u(s, x)) \psi(x)] dx ds + \int_0^t \int_{\Omega} f(s, x) T_k(u(s, x)) \psi(x) dx ds, \end{aligned} \quad (2.20)$$

where the function G_k and the function T_k are defined in Definition 2.1.

Proof. For $s_1, s_2 \in \mathbb{R}$ and $\tau > 0$ we define the approximation of $\mathbb{1}_{[s_1, s_2]}$:

$$\gamma_{s_1, s_2}^\tau(s) = \begin{cases} 0 & \text{for } s \leq s_1 - \tau \text{ or } s \geq s_2 + \tau, \\ 1 & \text{for } s \in [s_1, s_2], \\ \text{affine} & \text{for } s \in [s_1 - \tau, s_1] \cup [s_2, s_2 + \tau]. \end{cases}$$

Let $\psi \in C_0^\infty(\Omega)$, $k \in \mathbb{N}$, $\varepsilon, \delta, \tau$ be small positive parameters and $\eta, \beta \in (0, T)$. Consider test function in (2.19):

$$\varphi_{\eta, \beta}^{\delta, \tau, \varepsilon}(t, x) = \left(S^\delta \left(T_k(\tilde{u}^\varepsilon(t, x)) \psi(x) \gamma_{-\eta, \beta}^\tau(t) \right) \right)^\varepsilon \in C_0^\infty((-T, T) \times \Omega),$$

see Definitions 2.2 and 2.3 for mollification operators and Definition 2.1 for truncation T_k . Note that since $\psi \in C_0^\infty(\Omega)$, mollification in space is well-defined for sufficiently small $\varepsilon > 0$.

Now, we want to take limits in (2.19): first $\delta \rightarrow 0$, then $\tau \rightarrow 0$ and finally $\varepsilon \rightarrow 0$. We denote:

$$\begin{aligned} A_{\eta, \beta}^{\delta, \tau, \varepsilon} &= - \int_{\Omega} \int_{-T}^T \tilde{u}(t, x) \partial_t \varphi_{\eta, \beta}^{\delta, \tau, \varepsilon}(t, x) dt dx, \\ B_{\eta, \beta}^{\delta, \tau, \varepsilon} &= - \int_{\Omega} \int_{-T}^T \bar{\alpha}(t, x) \cdot \nabla \varphi_{\eta, \beta}^{\delta, \tau, \varepsilon}(t, x) dt dx, \\ C_{\eta, \beta}^{\delta, \tau, \varepsilon} &= \int_{\Omega} \int_{-T}^T \bar{f}(t, x) \varphi_{\eta, \beta}^{\delta, \tau, \varepsilon} dt dx, \end{aligned}$$

and we study each term separately.

Term $A_{\eta, \beta}^{\delta, \tau, \varepsilon}$. Note that Sobolev derivatives and mollification commute so using Sobolev regularity in time from Lemma 2.12:

$$A_{\eta,\beta}^{\delta,\tau,\varepsilon} = \int_{\Omega} \int_{-T}^T \partial_t \tilde{u}^\varepsilon(t, x) \left(S^\delta \left(T_k(\tilde{u}^\varepsilon(t, x)) \psi(x) \gamma_{-\eta,\beta}^\tau(t) \right) \right) dt dx.$$

Using Dominated Convergence (we still have $\varepsilon > 0$),

$$\lim_{\tau \rightarrow 0} \lim_{\delta \rightarrow 0} A_{\eta,\beta}^{\delta,\tau,\varepsilon} = \int_{\Omega} \int_{-\eta}^{\beta} \partial_t \tilde{u}^\varepsilon(t, x) T_k(\tilde{u}^\varepsilon(t, x)) \psi(x) dt dx =: A_{\eta,\beta}^\varepsilon.$$

As function $G(s) = \int_0^s T_k(\sigma) d\sigma$ is C^1 with uniformly bounded derivative so standard chain rule for Sobolev maps [28, Theorem 7.8] together with Sobolev regularity in time from Lemma 2.12 shows that $G(\tilde{u}^\varepsilon(t, x))\psi(x)$ is in $W^{1,1}((-T, T) \times \Omega)$, in particular it has Sobolev derivative in time. Moreover,

$$\partial_t G(\tilde{u}^\varepsilon(t, x)) = T_k(\tilde{u}^\varepsilon(t, x)) \partial_t \tilde{u}^\varepsilon(t, x)$$

Therefore, we can write:

$$A_{\eta,\beta}^\varepsilon = \int_{\Omega} \int_{-\eta}^{\beta} \partial_t G(\tilde{u}^\varepsilon(t, x)) dt \psi(x) dx.$$

Now, using absolute continuity on lines for Sobolev maps [25, Theorem 4.21], fundamental theorem of calculus applies for a.e. $x \in \Omega$ and $\eta, \beta \in (0, T)$ so we obtain

$$A_{\eta,\beta}^\varepsilon = \int_{\Omega} [G_k(\tilde{u}^\varepsilon(\beta, x)) - G_k(\tilde{u}^\varepsilon(-\eta, x))] \psi(x) dx.$$

However, using definition of extension (2.17), this can be rewritten as

$$A_{\eta,\beta}^\varepsilon = \int_{\Omega} [G_k(\tilde{u}^\varepsilon(\beta, x)) - G_k(u_0^\varepsilon(x))] \psi(x) dx.$$

Note that this step would not be achieved without extension for negative times as then, absolute continuity of Sobolev functions could be only applied for almost all times in $(0, T)$. Finally, using a.e. convergence of mollification and Dominated Convergence Theorem,

$$\lim_{\varepsilon \rightarrow 0} A_{\eta,\beta}^\varepsilon = \int_{\Omega} [G_k(\tilde{u}(\beta, x)) - G_k(u_0(x))] \psi(x) dx$$

for almost all $\beta > 0$.

Term $B_{\eta,\beta}^{\delta,\tau,\varepsilon}$. First, we use commuting properties of mollification to write:

$$\begin{aligned} B_{\eta,\beta}^{\delta,\tau,\varepsilon} &= - \int_{\Omega} \int_{-T}^T \bar{\alpha}^\varepsilon(t, x) \cdot \nabla S^\delta \left(T_k(\tilde{u}^\varepsilon(t, x)) \psi(x) \gamma_{-\eta,\beta}^\tau(t) \right) dt dx \\ &= \int_{\Omega} \int_{-T}^T \operatorname{div} \bar{\alpha}^\varepsilon(t, x) \psi(x) S^\delta \left(T_k(\tilde{u}^\varepsilon(t, x)) \gamma_{-\eta,\beta}^\tau(t) \right) dt dx. \end{aligned}$$

Note that as $\delta \rightarrow 0$ and $\tau \rightarrow 0$, $S^\delta \left(T_k(\tilde{u}^\varepsilon(t, x)) \gamma_{-\eta,\beta}^\tau(t) \right) \rightarrow T_k(\tilde{u}^\varepsilon(t, x)) \mathbb{1}_{[-\eta,\beta]}(t)$ a.e. in $(-T, T) \times \Omega'$ for $\Omega' \Subset \Omega$. As $\operatorname{div} \bar{\alpha}^\varepsilon(t, x) \psi(x) \in L^1(0, T; C_0^\infty(\Omega))$, we use Dominated Convergence Theorem to obtain

$$\lim_{\tau \rightarrow 0} \lim_{\delta \rightarrow 0} B_{\eta,\beta}^{\delta,\tau,\varepsilon} = \int_{\Omega} \int_{-\eta}^{\beta} \operatorname{div} \bar{\alpha}^\varepsilon(t, x) \psi(x) T_k(\tilde{u}^\varepsilon(t, x)) dt dx := B_{\eta,\beta}^\varepsilon.$$

Then, we write:

$$B_{\eta,\beta}^\varepsilon = - \int_{\Omega} \int_{-\eta}^{\beta} \bar{\alpha}(t, x) \cdot \nabla (\psi(x) T_k(\tilde{u}^\varepsilon(t, x)))^\varepsilon dt dx.$$

Due to Theorem 2.4, $\nabla (\psi(x) T_k(\tilde{u}^\varepsilon(t, x)))^\varepsilon \xrightarrow{M} \nabla (\psi(x) T_k(\tilde{u}(t, x)))$ so using Corollary 1.11 we finally conclude

$$\begin{aligned} B_{\eta,\beta}^{\eta,\tau,\varepsilon} &\rightarrow - \int_{\Omega} \int_{-\eta}^{\beta} \bar{\alpha}(t, x) \cdot \nabla (\psi(x) T_k(\tilde{u}(t, x))) dt dx \\ &= - \int_{\Omega} \int_0^{\beta} \alpha(t, x) \cdot \nabla (\psi(x) T_k(u(t, x))) dt dx. \end{aligned}$$

Term $C_{\eta,\beta}^{\delta,\tau,\varepsilon}$. This is the easiest part. Note that $\varphi_{\eta,\beta}^{\delta,\tau,\varepsilon} \rightarrow T_k(\tilde{u}(t, x)) \psi(x) \mathbb{1}_{[-\eta,\beta]}(t)$ a.e. in $(-T, T) \times \Omega$ as $\delta \rightarrow 0$, $\tau \rightarrow 0$ and $\varepsilon \rightarrow 0$. Moreover, since $f \in L^\infty(\Omega_T)$ and $|\varphi_{\eta,\beta}^{\delta,\tau,\varepsilon}| \leq k$, we use Dominated Convergence Theorem to deduce

$$C_{\eta,\beta}^{\delta,\tau,\varepsilon} \rightarrow \int_{\Omega} \int_{-\eta}^{\beta} \bar{f}(t, x) T_k(\tilde{u}(t, x)) \psi(x) dt dx = \int_{\Omega} \int_0^{\beta} f(t, x) T_k(u(t, x)) \psi(x) dt dx.$$

Finally, we obtain (2.20) for $t = \beta$ concluding the proof. \square

Remark 2.15. The same energy equality as (2.20) is satisfied by the solution to (2.7). Indeed, as the operator (2.6) is controlled by a Young function, Assumption 1.16 is satisfied. Therefore, for $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi(x) \leq 1$ and a.e. $t \in (0, T)$:

$$\begin{aligned} & \int_{\Omega} \psi(x) [G_k(u^\theta(t, x)) - G_k(u_0(x))] \, dx = \\ &= - \int_0^t \int_{\Omega} A_\theta(s, x, \nabla u^\theta) \cdot \nabla [T_k(u^\theta(s, x)) \psi(x)] \, dx \, ds + \int_0^t \int_{\Omega} f(s, x) T_k(u^\theta(s, x)) \psi(x) \, dx \, ds. \end{aligned} \quad (2.21)$$

Note that u^θ also satisfies the global energy equality (2.8), see Lemma 2.8.

2.4. Local version of monotonicity method

The following procedure allows us to identify weak-* limit of $A(t, x, \nabla u_n)$. We formulate here its local version and provide the proof that is almost identical to the global case presented in [16, Lemma A.5].

Lemma 2.16. *Let A satisfy Assumption 1.19 and M be an N -function. Assume that there are $\alpha \in L_{M^*}(\Omega_T; \mathbb{R}^d)$ and $\xi \in L_M(\Omega_T; \mathbb{R}^d)$ such that*

$$\int_{\Omega_T} (\alpha - A(t, x, \eta)) \cdot (\xi - \eta) \psi(x) \, dt \, dx \geq 0 \quad (2.22)$$

for all $\eta \in L^\infty(\Omega_T; \mathbb{R}^d)$ and $\psi \in C_0^\infty(\Omega)$ with $0 \leq \psi \leq 1$. Then,

$$A(t, x, \xi) = \alpha(t, x) \text{ a.e. in } \Omega_T.$$

Proof. Consider subsets $\Omega_T^k = \{(t, x) \in \Omega_T : |\xi(t, x)| \leq k\}$ and note that if $j < i$ then $\Omega_T^j \subset \Omega_T^i$. We use the assumption (2.22) with $\eta = \xi \mathbb{1}_{\Omega_T^i} + h z \mathbb{1}_{\Omega_T^j}$ where $h > 0$ and $z \in L^\infty(\Omega_T; \mathbb{R}^d)$ and we obtain

$$\int_{\Omega_T} \left(\alpha - A(t, x, \xi \mathbb{1}_{\Omega_T^i} + h z \mathbb{1}_{\Omega_T^j}) \right) \cdot (\xi - \xi \mathbb{1}_{\Omega_T^i} - h z \mathbb{1}_{\Omega_T^j}) \psi(x) \, dt \, dx \geq 0.$$

Considering integral on Ω_T^i and $\Omega_T \setminus \Omega_T^i$ we deduce

$$\int_{\Omega_T \setminus \Omega_T^i} (\alpha - A(t, x, 0)) \cdot \xi \psi(x) \, dt \, dx + h \int_{\Omega_T^i} (A(t, x, \xi + h z) - \alpha) \cdot z \psi(x) \, dt \, dx \geq 0.$$

Note that $A(s, x, 0) = 0$ due to (A4) in Assumption 1.19. Therefore, by integrability, the first term tends to 0 as $i \rightarrow \infty$. Therefore,

$$\int_{\Omega_T^j} (A(t, x, \xi + h z) - \alpha) \cdot z \psi(x) \, dt \, dx \geq 0.$$

Now, we want to let $h \rightarrow 0$. We have convergence $A(t, x, \xi + h z) \rightarrow A(t, x, \xi)$ due to (A1) in Assumption 1.19. Moreover, $\xi + h z$ is uniformly bounded on Ω_T^j . Therefore, (N7) in Lemma 1.4 implies:

$$\int_{\Omega_T^j} (A(t, x, \xi) - \alpha) \cdot z \psi(x) \, dt \, dx \geq 0.$$

Finally, choosing $z(t, x) = -\frac{A(t, x, \xi) - \alpha(t, x)}{|A(t, x, \xi) - \alpha(t, x)|} \mathbb{1}_{A(t, x, \xi) - \alpha(t, x) \neq 0}$, we deduce

$$A(t, x, \xi) = \alpha(t, x) \quad \text{for a.e. } (t, x) \in \Omega_T^j \cap \text{supp} \psi.$$

Since j and ψ are arbitrary, the assertion follows. \square

3. Proof of existence result (Theorem 1.23)

Consider sequence of solutions $\{u^\theta\}_{\theta \in (0, 1]}$ to the regularized problem (2.7). Using Lemma 2.11 as well as uniform bounds from Lemmata 2.8 and 1.13, we can extract a subsequence denoted with $u_n := u^{\theta_n}$ and $\theta_n \rightarrow 0$ such that:

- $u_n \rightarrow u$ in $L^1(0, T; L^1(\Omega))$ and a.e. in Ω_T ,
- $u_n \xrightarrow{*} u$ weakly-* in $L^\infty(0, T; L^2(\Omega))$,
- $\nabla u_n \xrightarrow{*} \nabla u$ weakly-* in $L_M(\Omega_T)$,
- $u_n \rightharpoonup u$ weakly in $L^1(0, T; W^{1,1}(\Omega))$,
- $A(\cdot, \cdot, \nabla u_n) \xrightarrow{*} \alpha$ weakly-* in $L_{M^*}(\Omega_T)$,

for some $u \in V_T^M(\Omega)$ and $\alpha \in L_{M^*}(\Omega_T)$.

For solutions to the regularized problem (2.7) we have the weak formulation. Namely, for all $\varphi \in C_0^\infty([0, T) \times \Omega)$:

$$\begin{aligned} - \int_{\Omega_T} u_n(t, x) \partial_t \varphi(t, x) \, dt \, dx - \int_{\Omega} u_0(x) \varphi(0, x) \, dx + \int_{\Omega_T} A(t, x, \nabla u_n) \cdot \nabla \varphi(t, x) \, dt \, dx + \\ + \int_{\Omega_T} \theta_n \nabla_\xi m(|\nabla u_n|) \cdot \nabla \varphi \, dt \, dx = \int_{\Omega_T} f(t, x) \varphi(t, x) \, dt \, dx. \end{aligned} \quad (3.1)$$

Using Lemma 2.9, we can pass to the limit with $n \rightarrow \infty$ (or $\theta_n \rightarrow 0$) in (3.1) to obtain:

$$\begin{aligned}
& - \int_{\Omega_T} u(t, x) \partial_t \varphi(t, x) \, dt \, dx - \int_{\Omega} u_0(x) \varphi(0, x) \, dx = \\
& = - \int_{\Omega_T} \alpha \cdot \nabla \varphi(t, x) \, dt \, dx + \int_{\Omega_T} f(t, x) \varphi(t, x) \, dt \, dx.
\end{aligned} \tag{3.2}$$

Thanks to (3.2), the theory from Section 2.3 can be applied and by using Lemma 2.14 we obtain that for $\psi \in C_0^\infty(\Omega)$ with $0 \leq \psi \leq 1$ and a.e. $t \in (0, T)$:

$$\begin{aligned}
& \int_{\Omega} \psi(x) [G_k(u(t, x)) - G_k(u_0(x))] \, dx = - \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla (T_k(u(s, x))) \psi(x) \, dx \, ds \\
& - \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla \psi(x) T_k(u(s, x)) \, dx \, ds + \int_0^t \int_{\Omega} f(s, x) T_k(u(s, x)) \psi(x) \, dx \, ds.
\end{aligned} \tag{3.3}$$

Due to Remark 2.15, a similar energy equality holds for sequence $\{u_n\}_{n \in \mathbb{N}}$:

$$\begin{aligned}
& \int_{\Omega} \psi(x) [G_k(u_n(t, x)) - G_k(u_0(x))] \, dx = \\
& = - \int_0^t \int_{\Omega} A_{\theta_n}(s, x, \nabla u_n) \cdot \nabla [T_k(u_n) \psi(x)] \, dx \, ds + \int_0^t \int_{\Omega} f(s, x) T_k(u_n) \psi(x) \, dx \, ds.
\end{aligned} \tag{3.4}$$

We note that the term with operator $A_{\theta_n}(s, x, \nabla u_n)$ can be decomposed into four parts:

- $\int_0^t \int_{\Omega} A(s, x, \nabla u_n) \cdot \nabla (T_k(u_n)) \psi(x) \, dx \, ds$
- $\int_0^t \int_{\Omega} A(s, x, \nabla u_n) \cdot \nabla \psi(x) T_k(u_n) \, dx \, ds$ which, due to $A(s, x, \nabla u_n) \xrightarrow{*} \alpha$, $u_n \rightarrow u$ a.e. and Dominated Convergence Theorem, converges to $\int_0^t \int_{\Omega} \alpha \cdot \nabla \psi(x) T_k(u) \, dx \, ds$,
- $\int_0^t \int_{\Omega} \theta_n \nabla_{\xi} m(|\nabla u_n|) \cdot \nabla (T_k(u_n)) \psi(x) \, dx \, ds$, which is nonnegative,
- $\int_0^t \int_{\Omega} \theta_n \nabla_{\xi} m(|\nabla u_n|) \cdot \nabla \psi(x) T_k(u_n) \, dx \, ds$, converging to 0 due to Lemma 2.9.

Therefore, (3.4) implies:

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \int_0^t \int_{\Omega} A(s, x, \nabla u_n) \cdot \nabla (T_k(u_n(s, x))) \psi(x) \, dx \, ds \leq \\
& \leq - \int_0^t \int_{\Omega} \alpha \cdot \nabla \psi(x) T_k(u(s, x)) \, dx \, ds - \int_{\Omega} \psi(x) [G_k(u(t, x)) - G_k(u_0(x))] \, dx \\
& + \int_0^t \int_{\Omega} f(s, x) T_k(u(s, x)) \psi(x) \, dx \, ds,
\end{aligned}$$

which combined with (3.3) yields:

$$\limsup_{n \rightarrow \infty} \int_0^t \int_{\Omega} A(s, x, \nabla u_n) \cdot \nabla (T_k(u_n)) \psi(x) \, dx \, ds \leq \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla (T_k(u)) \psi(x) \, dx \, ds. \quad (3.5)$$

Now, let $\alpha_k = \alpha \mathbb{1}_{|u(t,x)| < k}$. We claim that for any $k \in \mathbb{N}$, $\eta \in L^\infty(\Omega_T; \mathbb{R}^d)$, $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1$ and a.e. $t \in (0, T)$:

$$\int_{\Omega} \int_0^t (\alpha_k - A(s, x, \eta)) \cdot (\nabla T_k(u) - \eta) \psi(x) \, ds \, dx \geq 0. \quad (3.6)$$

Indeed, by monotonicity ((A3) in Assumption 1.19) we have that

$$\int_{\Omega} \int_0^t (A(s, x, \nabla T_k(u_n)) - A(s, x, \eta)) \cdot (\nabla T_k(u_n) - \eta) \psi(x) \, ds \, dx \geq 0.$$

By denoting $\Omega_t = (0, t) \times \Omega$, we see that:

- $\int_{\Omega_t} A(s, x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n) \psi \, ds \, dx = \int_{\Omega_T} A(s, x, \nabla u_n) \cdot \nabla T_k(u_n) \psi \, ds \, dx$ since we have $\nabla [T_k(u_n)] = \nabla u_n \mathbb{1}_{|u_n| < k}$,
- $\int_{\Omega_t} A(s, x, \nabla T_k(u_n)) \cdot \eta \psi \, ds \, dx \rightarrow \int_{\Omega_t} \alpha \mathbb{1}_{|u(s,x)| < k} \cdot \eta \psi \, ds \, dx = \int_{\Omega_t} \alpha_k \cdot \eta \psi \, ds \, dx$. Indeed, we can write $A(s, x, \nabla T_k(u_n)) = A(s, x, \nabla u_n) \mathbb{1}_{|u_n(s,x)| < k}$ and pass to the limit with n using $A(s, x, \nabla u_n) \xrightarrow{*} \alpha(s, x)$ and $u_n \rightarrow u$ a.e.,
- $\int_{\Omega_t} A(s, x, \eta) \cdot \nabla T_k(u_n) \psi \, ds \, dx \rightarrow \int_{\Omega_t} A(s, x, \eta) \cdot \nabla T_k(u) \psi \, ds \, dx$ due to $\nabla u_n \xrightarrow{*} \nabla u$ and $u_n \rightarrow u$ a.e.

Therefore, (3.6) follows. By monotonicity trick (Lemma 2.16), $\alpha_k(t, x) = A(t, x, \nabla T_k(u))$ for any $k \in \mathbb{N}$ and this finally implies $\alpha = A(t, x, u)$ concluding the proof of existence.

Finally, to establish global energy inequality (1.10), we note that for a.e. $t \in (0, T)$

$$\int_{\Omega} u^2(t, x) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} (u^n(t, x))^2 \, dx \quad (3.7)$$

as L^2 norm is weakly lower semicontinuous (since it is strongly continuous and convex). We claim that

$$\liminf_{n \rightarrow \infty} \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u_n(s, x) \, dx \, ds \geq \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x)) \cdot \nabla u(s, x) \, dx \, ds. \quad (3.8)$$

Indeed, let $k \in \mathbb{N}$. We can write

$$\begin{aligned}
& \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u_n(s, x) \, dx \, ds = \\
& = \int_0^t \int_{\Omega} \left[A(s, x, \nabla u_n(s, x)) - A(s, x, \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k}) \right] \\
& \quad \cdot \left[\nabla u_n(s, x) - \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k} \right] \, dx \, ds \\
& + \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k}) \cdot \left[\nabla u_n(s, x) - \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k} \right] \, dx \, ds \\
& + \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k},
\end{aligned}$$

where the first term is nonnegative due to (A3) in Assumption 1.19. Recall that we already know that $\nabla u_n \xrightarrow{*} \nabla u$ weakly-* in $L_M(\Omega_T)$ and $A(\cdot, \cdot, \nabla u_n) \xrightarrow{*} A(\cdot, \cdot, \nabla u)$ weakly-* in $L_{M^*}(\Omega_T)$. Lemma 1.22 implies that the map $(s, x) \mapsto A(s, x, \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k})$ is bounded. Therefore,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u_n(s, x) \, dx \, ds & \geq \\
& \geq \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k}) \cdot \nabla u(s, x) \mathbb{1}_{|\nabla u| \geq k} \, dx \, ds \\
& + \int_0^t \int_{\Omega} A(s, x, \nabla u(s, x)) \cdot \nabla u(s, x) \mathbb{1}_{|\nabla u| \leq k},
\end{aligned}$$

where the first term vanished due to presence of two characteristic functions $\mathbb{1}_{|\nabla u| \geq k}$ and $\mathbb{1}_{|\nabla u| \leq k}$. Finally, we let $k \rightarrow \infty$ and deduce (3.8).

By energy equality for the regularized problem (2.8), we have:

$$\begin{aligned}
\int_{\Omega} [(u_n(t, x))^2 - (u_0(x))^2] \, dx & = - \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u_n(s, x) \, dx \, ds \\
& - \int_0^t \int_{\Omega} \theta_n \nabla_{\xi} m(|\nabla u_n|) \cdot \nabla u_n(s, x) \, dx \, ds + \int_0^t \int_{\Omega} f(s, x) u_n(s, x) \, dx \, ds.
\end{aligned}$$

We note that $\int_0^t \int_{\Omega} \theta_n \nabla_{\xi} m(|\nabla u_n|) \cdot \nabla u_n(s, x) \, dx \, ds \geq 0$ so that

$$\begin{aligned} \int_{\Omega} [(u_n(t, x))^2 - (u_0(x))^2] dx &\leq - \int_0^t \int_{\Omega} A(s, x, \nabla u_n(s, x)) \cdot \nabla u_n(s, x) dx ds \\ &\quad + \int_0^t \int_{\Omega} f(s, x) u_n(s, x) dx ds. \end{aligned}$$

Using (3.7) and (3.8), we let $n \rightarrow \infty$ and conclude the proof of the energy inequality (1.10) for a.e. $t \in (0, T)$. Finally, as the map $[0, T) \ni t \mapsto u(t, \cdot) \in L^2(\Omega)$ is weakly continuous, energy inequality holds for all $t \in [0, T)$.

4. Proof of uniqueness result (Theorem 1.24)

To obtain the uniqueness of a weak solution, it is standard in the theory of parabolic equations to test the equation for the difference of solutions with the difference of solutions itself. In the Musielak-Orlicz framework, it is unfortunately not so straightforward. In fact, we want to improve the result of Lemma 2.14, where we showed the local energy equality, to the global energy equality, i.e. we want to remove the presence of the cut-off function. Next lemma shows that under the additional structural hypothesis on M (the radial symmetry), such procedure can be made rigorous.

Lemma 4.1. *Let Ω be a Lipschitz domain. Suppose that the N -function M is isotropic (as in assumptions of Theorem 1.24) and Assumption 1.16 is satisfied. Then, there is a family of functions $\{\psi_j\}_{j \in \mathbb{N}}$ compactly supported in Ω and fulfilling $\psi_j \rightarrow 1$ as $j \rightarrow \infty$, such that if $u \in L^\infty(0, T; L^\infty(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega))$ with $\nabla u \in L_M(\Omega_T)$, we have*

$$\int_0^t \int_{\Omega} M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds \rightarrow 0 \text{ as } j \rightarrow \infty,$$

where the constant C_u can be chosen as $C_u = C \|\nabla u\|_{L_M}$ where C depends only on Ω .

Proof. Since Ω is a Lipschitz domain, we can flatten the boundary locally with bi-Lipschitz homeomorphisms so that using appropriate partition of unity, $\partial\Omega$ can be assumed to be flat. This argument relies heavily on the isotropy of M as otherwise it is not clear if $\nabla u \in L_M(\Omega_T)$ implies $\nabla(u \circ \Psi) \in L_M(\Omega_T)$ for some bi-Lipschitz homeomorphism Ψ .

Let $\Omega_j = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \frac{1}{j}\}$ so that $\Omega_j \nearrow \Omega$ as $j \rightarrow \infty$. Moreover, let $\psi_j \in C_0^\infty(\Omega)$ such that $\psi_j = 1$ on Ω_j . Note that $\nabla \psi_j = 0$ on Ω_j and $|\nabla \psi_j| \leq Cj$ for some constant C . We cover $\Omega \setminus \Omega_j$ with the family of disjoint cubes $\{Q_m^j\}_{m=1}^{N_j}$ with edge of length $\frac{1}{j}$. Then, we write for some constant C_u to be chosen later:

$$\begin{aligned}
& \int_0^t \int_{\Omega} M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds = \\
&= \int_0^t \int_{\Omega \setminus \Omega_j} M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds \\
&\leq \sum_{m=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds \\
&\leq \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} \frac{M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right)}{M_{Q_m^j}^{**}\left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right)} M_{Q_m^j}^{**}\left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds.
\end{aligned} \tag{4.1}$$

Note that $\left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \leq \frac{j \|u\|_{\infty}}{C_u}$ so that we can apply Assumption 1.16 and deduce:

$$\limsup_{j \rightarrow \infty} \frac{M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right)}{M_{Q_m^j}^{**}\left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right)} \leq C.$$

Therefore, (4.1) reads:

$$\begin{aligned}
& \int_0^t \int_{\Omega} M\left(s, x, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds \\
&\leq C \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**}\left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds.
\end{aligned} \tag{4.2}$$

Now, for any $x = (x_1, x_2, \dots, x_d) \in Q_m^j$, we write $x^* = (x_1, x_2, \dots, 0)$ for its projection on the face of the cube Q_m^j sticking to the boundary $\partial\Omega$ (see Fig. 1). Note that we assumed that the face of the cube is perpendicular to the axis of the last variable x_d which is not restrictive and can be obtained by choosing appropriate straightening bi-Lipschitz homeomorphisms.

For a.e. $x \in Q_m^j$, using absolute continuity on lines for Sobolev maps (cf. Theorem 4.21 in [25]), we can write:

$$u(s, x) = \int_0^{x_d} u_{x_d}(s, x_1, x_2, \dots, r) dr$$

where u_{x_d} denotes derivative with respect to the last variable. Note that since $|x_d| \leq \frac{1}{j}$, $|u(s, x)|$ can be bounded as

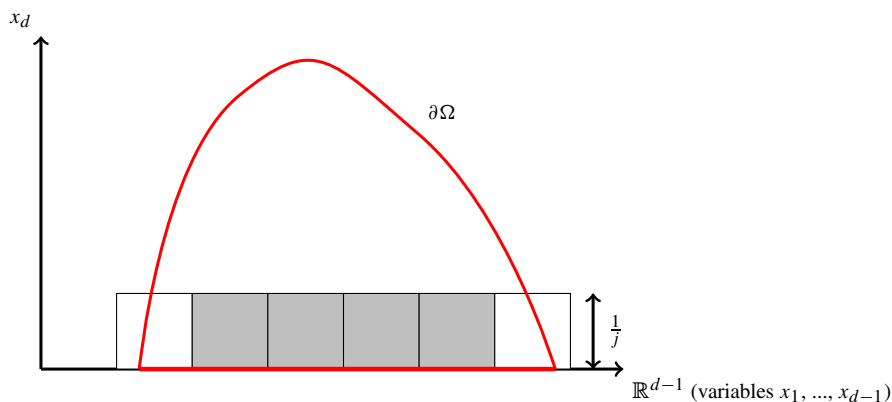


Fig. 1. The boundary $\partial\Omega$ with some part of it flattened after change of coordinates. Gray cubes from the family $\{Q_m^j\}_{m=1}^{N_j}$ correspond to the area that is relevant for further computations after application of partition of unity.

$$|u(s, x)| \leq \int_0^{x_d} |u_{x_d}(s, x_1, x_2, \dots, x_{d-1}, r)| \, dr \leq \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, r)| \, dr.$$

Using this inequality in (4.2), we can continue as follows:

$$\begin{aligned} \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**} \left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx \, ds &\leq \\ &\leq \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**} \left(s, \frac{|\nabla \psi_j(x)|}{C_u} \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, r)| \, dr \right) dx \, ds \\ &= \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**} \left(s, \frac{C}{C_u} j \int_0^{\frac{1}{j}} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, r)| \, dr \right) dx \, ds \\ &\leq \sum_{i=1}^{N_j} \int_0^{\frac{1}{j}} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} j M_{Q_m^j}^{**} \left(s, \frac{C}{C_u} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, r)| \right) dx \, ds \, dr \end{aligned}$$

where we used the bound $|\nabla \psi_j(x)| \leq Cj$ and Jensen's inequality. Note that the integrand does not depend on x_d and so, the integral over this variable cancels with the factor j . Finally, as cube has edge of length $\frac{1}{j}$, Fubini's theorem implies

$$\begin{aligned}
& \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**} \left(s, \left| \frac{\nabla \psi_j(x) u(s, x)}{C_u} \right| \right) dx ds \leq \\
& = \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M_{Q_m^j}^{**} \left(s, \frac{C}{C_u} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, x_d)| \right) dx ds \\
& \leq \sum_{i=1}^{N_j} \int_0^t \int_{Q_m^j \cap \Omega \setminus \Omega_j} M \left(s, x, \frac{C}{C_u} |\nabla u(s, x_1, x_2, \dots, x_{d-1}, x_d)| \right) dx ds \\
& = \int_0^t \int_{\Omega \setminus \Omega_j} M \left(s, x, \frac{C}{C_u} |\nabla u(s, x)| \right) dx ds.
\end{aligned}$$

Now, as $\nabla u \in L_M(\Omega_T)$, we can choose $C_u = C \|\nabla u\|_{L_M}$ so that the integral

$$\int_0^t \int_{\Omega} M \left(s, x, \frac{C}{C_u} |\nabla u(s, x)| \right) dx ds$$

is finite and the conclusion follows by $\Omega_j \nearrow \Omega$ as $j \rightarrow \infty$. \square

Lemma 4.2 (Global energy equality). *Under assumptions of Lemma 2.14 and Theorem 1.24, the following energy equality is satisfied for a.e. $t \in (0, T)$:*

$$\begin{aligned}
& \int_{\Omega} [G_k(u(t, x)) - G_k(u_0(x))] dx = \\
& = - \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla [T_k(u(s, x))] dx ds + \int_0^t \int_{\Omega} f(s, x) T_k(u(s, x)) dx ds.
\end{aligned} \tag{4.3}$$

Proof. The main idea is to consider local energy equality (2.20) with a sequence of cut-off functions $\{\psi_j\}_{j \in \mathbb{N}}$ from Lemma 4.1.

Note that, as $j \rightarrow \infty$, $\psi_j \rightarrow 1$ in Ω . Therefore, to conclude (4.3) from (2.20) we only have to establish:

$$\int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla \psi_j(x) T_k(u(s, x)) dx ds \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Since $\nabla \psi_j(x) = 0$ for $x \in \Omega_j$, we write for some constant C_α and C_u to be chosen later:

$$\begin{aligned}
\int_0^t \int_{\Omega \setminus \Omega_j} \alpha(s, x) \cdot \nabla \psi_j(x) T_k(u(s, x)) \, dx \, ds &\leq C_\alpha C_u \int_0^t \int_{\Omega \setminus \Omega_j} M^* \left(s, x, \left| \frac{\alpha(s, x)}{C_\alpha} \right| \right) \, dx \, ds + \\
&+ C_\alpha C_u \int_0^t \int_{\Omega \setminus \Omega_j} M \left(s, x, \left| \frac{\nabla \psi_j(x) T_k(u(s, x))}{C_u} \right| \right) \, dx \, ds,
\end{aligned} \tag{4.4}$$

where we have applied Young's inequality (Lemma 1.7). Since $\alpha \in L_{M^*}(\Omega_T)$, there is C_α so that $M^* \left(s, x, \left| \frac{\alpha(s, x)}{C_\alpha} \right| \right) \, dx \, ds < \infty$. Choosing such C_α , the first integral on the (RHS) of (4.4) tends to 0 as $j \rightarrow \infty$ due to integrability of the integrand. Moreover, the second integral on the (RHS) of (4.4) converges to 0 due to Lemma 4.1. \square

Remark 4.3. Using Dominated Convergence Theorem, (4.3) implies that for a.e. $t \in (0, T)$:

$$\frac{1}{2} \int_{\Omega} \left[u^2(t, x) - u_0^2(x) \right] \, dx = - \int_0^t \int_{\Omega} \alpha(s, x) \cdot \nabla u(s, x) \, dx \, ds + \int_0^t \int_{\Omega} f(s, x) u(s, x) \, dx \, ds. \tag{4.5}$$

Proof of Theorem 1.24. Energy equality (1.11) follows from Remark 4.3. Now, suppose there are two solutions u and v to (1.1). Then, their difference satisfies weak formulation for

$$(u - v)_t = \operatorname{div} [A(t, x, \nabla u) - A(t, x, \nabla v)]$$

with zero initial condition. Using (4.5) with $\alpha(t, x) = A(t, x, \nabla u) - A(t, x, \nabla v)$, we obtain for a.e. $t \in (0, T)$:

$$\frac{1}{2} \int_{\Omega} (u(t, x) - v(t, x))^2 \, dx = - \int_0^t \int_{\Omega} \left[A(s, x, \nabla u) - A(s, x, \nabla v) \right] \cdot \left[\nabla u(s, x) - \nabla v(s, x) \right] \, dx \, ds$$

which due to weak monotonicity (A3) in Assumption 1.19 implies $u = v$ a.e. in Ω_T . \square

Appendix A. Proofs concerning Examples 1.18 and 1.21

A.1. Example 1.18

Let $M(t, x, \xi) = |\xi|^{p(t, x)}$. We want to establish condition (1.8) in Remark 1.17. Fix $t \in (0, T)$ and $x, y \in \Omega$. When $|\xi| > 1$

$$\frac{M(t, x, \xi)}{M(t, y, \xi)} = |\xi|^{p(t, x) - p(t, y)} \leq |\xi|^{|p(t, x) - p(t, y)|} \leq |\xi|^{-\frac{C}{\log|x-y|}}$$

since $p(t, x) \in L^\infty(0, T; C_{\log}(\Omega))$. We let $\Theta(t, \delta, \xi) := |\xi|^{-\frac{C}{\log \delta}}$ so that for all $\tilde{C} > 1$ we have

$$\Theta(t, \delta, \tilde{C}\delta^{-1}) \leq (\tilde{C}\delta^{-1})^{-\frac{C}{\log \delta}} = (\delta/\tilde{C})^{\frac{C}{\log \delta}} = e^{\frac{C}{\log \delta} \log(\delta/\tilde{C})} = e^C e^{-\frac{C \log \tilde{C}}{\log \delta}}$$

so that $\limsup_{\delta \rightarrow 0} \Theta(t, \delta, \tilde{C}\delta^{-1})$ is bounded.

Similarly, let $M(t, x, \xi) = |\xi|^{p(t,x)} + a(t, x) |\xi|^{q(t,x)}$ and suppose that $q(t, y) - p(t, y) \leq \alpha$. Then, for $|\xi| > 1$ we have

$$\begin{aligned} \frac{M(t, x, \xi)}{M(t, y, \xi)} &= \frac{|\xi|^{p(t,x)} + a(t, x) |\xi|^{q(t,x)}}{|\xi|^{p(t,y)} + a(t, y) |\xi|^{q(t,y)}} = \frac{|\xi|^{q(t,x)} |\xi|^{p(t,x)-q(t,x)} + a(t, x)}{|\xi|^{q(t,y)} |\xi|^{p(t,y)-q(t,y)} + a(t, y)} \leq \\ &\leq |\xi|^{q(t,x)-q(t,y)} \left[\frac{|\xi|^{p(t,x)-q(t,x)}}{|\xi|^{p(t,y)-q(t,y)}} + \frac{a(t, x) - a(t, y)}{|\xi|^{p(t,y)-q(t,y)}} + 1 \right] \\ &\leq |\xi|^{q(t,x)-q(t,y)} \left[|\xi|^{p(t,x)-p(t,y)} |\xi|^{q(t,y)-q(t,x)} + \frac{a(t, x) - a(t, y)}{|\xi|^{p(t,y)-q(t,y)}} + 1 \right] \\ &\leq |\xi|^{-\frac{C}{\log|x-y|}} \left[|\xi|^{-\frac{C}{\log|x-y|}} |\xi|^{-\frac{C}{\log|x-y|}} + |a|_\alpha |x-y|^\alpha |\xi|^{q(t,y)-p(t,y)} + 1 \right] \\ &\leq |\xi|^{-\frac{C}{\log|x-y|}} \left[|\xi|^{-\frac{C}{\log|x-y|}} |\xi|^{-\frac{C}{\log|x-y|}} + |a|_\alpha |x-y|^\alpha |\xi|^\alpha + 1 \right] \end{aligned}$$

where $|a|_\alpha$ is a constant such that $|a(t, x) - a(t, y)| \leq |a|_\alpha |x - y|^\alpha$. Hence, we define

$$\Theta(t, \delta, \xi) = |\xi|^{-\frac{C}{\log \delta}} \left[|\xi|^{-\frac{C}{\log \delta}} |\xi|^{-\frac{C}{\log \delta}} + |a|_\alpha \delta^\alpha |\xi|^\alpha + 1 \right].$$

We have already seen that $(\tilde{C}\delta^{-1})^{-\frac{C}{\log \delta}}$ is bounded when $\delta \rightarrow 0$. It follows that $\Theta(t, \delta, \tilde{C}\delta^{-1})$ is bounded for such δ .

A.2. Example 1.21

In both examples, the only nontrivial condition in Assumption 1.19 is (A2) (growth and coercivity). For (F1), we study $A(t, x, \xi) = |\xi|^{p(t,x)-2} \xi$ with $M(t, x, \xi) = |\xi|^{p(t,x)} = A(t, x, \xi) \cdot \xi$ so that we only need to verify

$$M^*(t, x, A(t, x, \xi)) \leq C A(t, x, \xi) \cdot \xi$$

for some numerical constant C . As we know that $M^*(t, x, \xi) \leq C |\xi|^{p'(t,x)}$ where $p'(t, x)$ is Hölder conjugate of $p(t, x)$ (i.e. $\frac{1}{p(t,x)} + \frac{1}{p'(t,x)} = 1$) we have

$$\begin{aligned} M^*(t, x, A(t, x, \xi)) &\leq C \left| |\xi|^{p(t,x)-2} \xi \right|^{p'(t,x)} = C |\xi|^{(p(t,x)-1)p'(t,x)} \\ &= C |\xi|^{p(t,x)} = C A(t, x, \xi) \cdot \xi. \end{aligned}$$

For (F2), we have $A(t, x, \xi) = |\xi|^{p(t,x)-2} \xi + a(t, x) |\xi|^{q(t,x)-2} \xi$ and $M(t, x, \xi) = |\xi|^{p(t,x)} + a(t, x) |\xi|^{q(t,x)}$. Again, since $A(t, x, \xi) \cdot \xi = M(t, x, \xi)$ we only have to prove

$$M^*(t, x, A(t, x, \xi)) \leq C A(t, x, \xi) \cdot \xi$$

for some numerical constant C . Using Definition 1.3,

$$\begin{aligned} M^*(t, x, A(t, x, \xi)) &= \sup_{\eta \in \mathbb{R}^d} \{ \eta A(t, x, \xi) - M(t, x, \xi) \} \\ &\leq \sup_{\eta \in \mathbb{R}^d} \left\{ \eta \cdot \left(|\xi|^{p(t,x)-2} \xi + a(t, x) |\xi|^{q(t,x)-2} \xi \right) \right. \\ &\quad \left. - \left(|\xi|^{p(t,x)} + a(t, x) |\xi|^{q(t,x)} \right) \right\} \\ &\leq \sup_{\eta \in \mathbb{R}^d} \left\{ \eta \cdot \xi |\xi|^{p(t,x)-2} - |\xi|^{p(t,x)} \right\} \\ &\quad + a(t, x) \sup_{\eta \in \mathbb{R}^d} \left\{ \eta \cdot \xi |\xi|^{q(t,x)-2} - |\xi|^{q(t,x)} \right\} \end{aligned}$$

We introduce auxiliary notation $M_1(t, x, \xi) = |\xi|^{p(t,x)}$, $A_1(t, x, \xi) = |\xi|^{p(t,x)-2} \xi$ as well as $M_2(t, x, \xi) = |\xi|^{q(t,x)}$, $A_2(t, x, \xi) = |\xi|^{q(t,x)-2} \xi$ and we recognize that

$$\begin{aligned} M^*(t, x, A(t, x, \xi)) &\leq M_1^*(t, x, A_1(t, x, \xi)) + a(t, x) M_2^*(t, x, A_2(t, x, \xi)) \\ &\leq A_1(t, x, \xi) \cdot \xi + a(t, x) A_2(t, x, \xi) \cdot \xi = \mathcal{A}(t, x, \xi) \cdot \xi \end{aligned}$$

which is justified by computations for the variable exponent case.

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