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Spatial Heterogeneity of Resources versus Lotka–Volterra Dynamics

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The problem is motivated by a consideration of two phenotypes of a species in a strongly heterogenous spatial environment. The phenotypes vary in their diffusion rates and interspecific interactions. The aim is to investigate the relative strengths of the diffusion and reaction effects. The problem is thus of competing species type, but there are many questions which arise which are not covered by standard theory. We investigate, in particular, the stability of the equilibria and the existence of coexistence solutions with emphasis on cases where the spatial variation of the environment becomes large. We discuss briefly the implications of our results for the principle of competitive exclusions and for the question of the evolution of diffusion discussed in Dockery *et al.* (*J. Math. Biol.* **37** (1998), 61–83). © 2002 Elsevier Science (USA)

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1. INTRODUCTION

That ecology and evolution are fundamentally influenced by the spatial characteristics of the environment is well accepted. As an example of this one may consider the paradox of diversity. Simple models such as the Lotka–Volterra system

$$\begin{aligned} \dot{u} &= u(1 - u - bv), \\ \dot{v} &= v(1 - cu - v) \end{aligned} \tag{1.1}$$

which do not include any spatial component give rise to the principle of competitive exclusion; "when two species compete for the same limited resource one of the species usually becomes extinct" [11]. On the other hand, the common observation is that in a wide variety of habitats a multitudes of species coexist. This leads to the "paradox of enrichment" [7], which can be explained away, at least in part, by expanding the model to include spatial effects. Of course, once spatial components are introduced, dispersal rates become a central feature [2].

Unfortunately, our understanding of cause and effect in this more general situation is poor. The reason for this appears to be fourfold. First, the number of variables in realistic ecological and environmental models are enormous. Second, spatial heterogeneities occur at all scales of the environment [9]. Third, obtaining precise data for these variables from field studies is extremely difficult [14]. Finally, the current mathematical techniques for handling models which incorporate both spatial and dynamical properties seem to be inadequate. Given this state of affairs, the strategy of this paper is to consider an extremely simple model in the hopes of elucidating some basic biological principles and identifying some fundamental mathematical issues. With this in mind we model space as a continuous variable rather than as a number of discrete patches. In line with this, we use the simplest dispersal model consistent with a continuous spatial variable, namely diffusion.

The range of questions which could be asked even in this greatly simplified setting is too wide to be accommodated by any single model, therefore we consider a more specific motivation. How does spatial heterogeneity of resources affect the balance between competitive strength and rates of dispersal? To study this we will consider the following system of reaction–diffusion equations:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mu \Delta u + u[1 + \gamma\beta(x) - u - bv], \\ \frac{\partial v}{\partial t} &= v \Delta v + v[1 + \gamma\beta(x) - cu - v] \end{aligned} \tag{1.2}$$

defined on the domain $\Omega \times (0, \infty)$ with zero Neumann boundary conditions:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \tag{1.3}$$

where $\partial/\partial n$ denotes differentiation in the direction of the outward normal. Since the variables u and v are meant to represent the densities of two phenotypes of a species it must be assumed that $u(x, 0) \geqslant 0$ and $v(x, 0) \geqslant 0$ for all $x \in \Omega$.

Observe that $\gamma \geqslant 0$ is the parameter that measures the degree of spatial heterogeneity: $\gamma = 0$ leads to a perfectly homogeneous level of resources in the environment; while large γ indicates that these levels vary dramatically. In order to make sure that we are measuring the effects of the heterogeneity rather than the total carrying capacity of the environment, we typically assume that $\int_\Omega \beta = 0$.

The dispersal rates of the species are given by μ and ν. Because of the symmetry of the model, we can, without loss of generality, assume that

$$\nu \geqslant \mu. \tag{1.4}$$

The final point to be made is that the parameters b and c which come from the original Lotka–Volterra model indicate the relative strength of competition. In particular, $b > c$ implies that v is the superior competitor.

As will become clear in a moment, the following functions play a crucial role in our analysis. Let $\tilde{u}$ and $\tilde{v}$ be the unique positive solutions in Ω satisfying

$$\mu \Delta \tilde{u} + \tilde{u}[1 + \gamma\beta(x) - \tilde{u}] = 0 \tag{1.5}$$

and

$$\nu \Delta \tilde{v} + \tilde{v}[1 + \gamma\beta(x) - \tilde{v}] = 0, \tag{1.6}$$

respectively, together with zero Neumann boundary conditions.

The analysis of this paper begins with the observation that if the local interaction of the species with the environment is identical, i.e., $b = c = 1$, then in the context of spatial heterogeneity in the level of resources slow dispersal rates are advantageous. More precisely, the following theorem is true.

THEOREM 1.1 (Dockery *et al.* [3]). *Assume that $b = c = 1$, $\nu > \mu$, $\gamma > 0$, and β is not constant. Then,*

1. *The only equilibria solutions to* (1.2) *are the semi-trivial solutions* $(\tilde{u}, 0)$ *and* $(0, \tilde{v})$ *and the trivial solution* $(0, 0)$.

2. $(\tilde{u}, 0)$ *is a hyperbolic attractor and furthermore is the global attractor for the set of positive initial conditions.*

3. $(0, \tilde{v})$ *is unstable.*

Clearly, by a simple comparison argument this result remains true for $b<1$ and $c>1$. Intuitively, this means that the faster diffuser is a weaker competitor, from which it follows immediately that $(\tilde{u}, 0)$ is stable. The main interest thus falls on cases where (b, c) does not lie in the semi-infinite strip $(0,1] \times [1, \infty)$. The use of arguments similar to those presented in [3] allows us to conclude that the results remain true for small perturbations in b and c, that is there exists $\delta > 0$ (dependent on μ, v and γ) such that if $(b, c) \in (0, 1+\delta) \times (1-\delta, \infty)$, then the conclusions of Theorem 1.1 still hold.

At this point, it is important to contrast this result with that of the spatially homogeneous model obtained by setting $\gamma = 0$. As can be seen by studying the Lotka–Volterra equations (1.1), if $b, c > 1$ then both $(\tilde{u}, 0)$ and $(0, \tilde{v})$ are stable solutions to (1.2), while $b, c < 1$ implies they are both unstable. Furthermore, for $b > 1$ and $c < 1$, $(0, \tilde{v})$ is the global attractor while $(\tilde{u}, 0)$ is unstable.

The goal of this paper is to try to understand this dichotomy between the spatially homogeneous and spatially heterogeneous models. In particular, we are interested in describing the set of parameter values, μ, v, b, c, and γ, for which:

1. $(\tilde{u}, 0)$ is locally stable or unstable;
2. $(0, \tilde{v})$ is locally stable or unstable;
3. existence/nonexistence of coexisting positive steady-state solutions to (1.2) holds;
4. $(\tilde{u}, 0)$ or $(0, \tilde{v})$ is the global attractor.

Before considering the precise statements of the mathematical results, observe that a concrete problem leading to the study of spatial properties of resources in ecology is that of habitat destruction and the ensuing loss of species [16]. Obviously, local destruction of a habitat can be viewed as the introduction of a major heterogeneity in the resources. Therefore, the results of this paper may give an indication of the relationship between the possible competitive strengths and dispersal rates of species which persist. A crucial question in this context is which species will survive and which are driven to extinction. This has, of course, been studied by many authors (see [15] and references therein). At the heart of the analysis of [15] is the assumption that the greater the competitive superiority the lower the dispersal rate of the species. Given a complicated ecosystem this may be a valid assumption. However, as is indicated in the analysis of this paper, the spatial heterogeneity of resources in and of itself has the effect of forcing rapid dispersers to be competitively superiority. To put this another way, the slower diffuser can sustain a penalty, i.e., can be competitively inferior, and still dominate. This is exactly the opposite relationship to that postulated in [15].

Now consider a scenario in which the habitat destruction is taking place on a similar time scale as the evolution of the competitive traits. (Given the time scales on which competitive adaptations have been observed in finches on the Galápagos Islands this may be a reasonable assumption in some circumstances [5].) In this case, it is conceivable that the environmental heterogeneities induced by the destruction itself will have an impact on the relationship between strength of competition and dispersal rates. Such changes will in turn influence any analysis of the form described in [15].

We now turn to a more detailed description of the mathematics of this paper and begin by stating our assumptions:

(H1) Ω is a bounded open subset of R^m with C^2 boundary $\partial\Omega$. v, μ, γ, b and c are positive constants and $v > \mu$.

(H2) β is not constant, $\int_\Omega \beta = 0$ (unless explicitly specified to the contrary), $\Gamma = \{x \in \bar{\Omega}\colon \beta(x) = 0\}$ does not intersect with $\partial\Omega$, and $\beta \in C^2(\bar{\Omega})$.

(H3) For $n = 1$, Γ is a union of finite number of points, denoted by $x_1, \dots, x_k$, and $\beta'(x_i) \neq 0$ for $1 \leqslant i \leqslant k$. For $n \geqslant 2$, Γ is a union of finite number of disjoint C^1 closed hypersurfaces in R^m, and $\nabla\beta$ does not vanish on Γ.

In the next section, we will consider the question of the stability of $(\tilde{u}, 0)$. This is essentially a question concerning a principle eigenvalue which leads to the following proposition. Set

$$c_* = -\inf_{\substack{\varphi \in H^1(\Omega) \\ \varphi \neq 0}} \frac{\int_\Omega [v|\nabla\varphi|^2 - (1 + \gamma\beta)\varphi^2]}{\int_\Omega \tilde{u}\varphi^2}. \tag{1.7}$$

PROPOSITION 1.2. *$(\tilde{u}, 0)$ is stable if $c > c_*$, and unstable if $c < c_*$.*

Hence, it is interesting to know the qualitative properties of c_* in terms of the parameters μ, v and γ. For simplicity, we shall fix μ and think of c_* as a function of v and γ only.

THEOREM 1.3. *The constant c_* satisfies the relation $0 < c_* < 1$, $\lim_{\gamma \to 0} c_* = 1$, and $\lim_{\gamma/v \to +\infty} c_* = 1$ (Fig. 1).*

The analysis of the stability of $(0, \tilde{v})$ is considerably more delicate and will be dealt with in Section 3. Of course, this too is a problem involving a principle eigenvalue and hence

$$b_* = -\inf_{\substack{\varphi \in H^1 \\ \varphi \neq 0}} \frac{\int_\Omega [\mu|\nabla\varphi|^2 - (1 + \gamma\beta)\varphi^2]}{\int_\Omega \tilde{v}\varphi^2} \tag{1.8}$$

plays an important role.

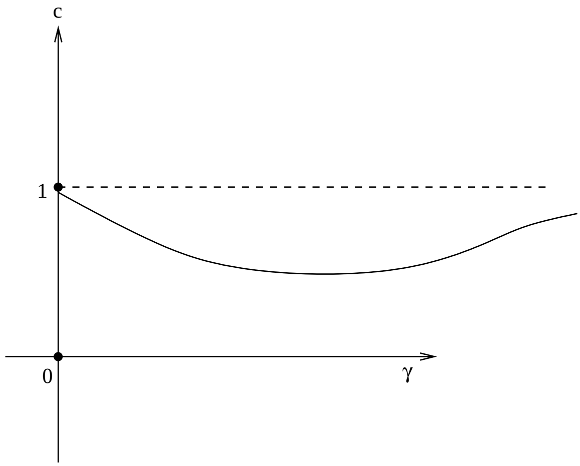

FIG. 1. A typical graph of c_*. For $c > c_*$, $(\tilde{u}, 0)$ is stable.

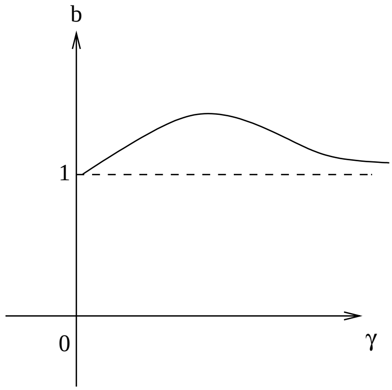

FIG. 2. The graph of b_*. For $b > b_*$, $(0, \tilde{v})$ is stable.

PROPOSITION 1.4. $(0, \tilde{v})$ *is stable if* $b > b_*$ *and unstable if* $b < b_*$ (Fig. 2).

Again, we are interested in the qualitative properties of b_*. It follows from Theorem 1.1 that $b_* > 1$, and clearly $\lim_{\gamma \to 0} b_* = 1$. Furthermore, we shall show that the following holds.

THEOREM 1.5. $\lim_{\gamma/\nu \to +\infty} b_* = 1$.

The implications of Theorems 1.3 and 1.5 are noteworthy. Consider for example a fixed $b > 1$ but not too large and treat γ as a free parameter. Increasing γ, which is equivalent to increasing the spatial heterogeneity, leads to the destabilization of $(0, \tilde{v})$. This is not surprising in view of Theorem 1.1. However, as γ increases further, $(0, \tilde{v})$ *regains* its stability.

Since $b_* \approx 1$ for large and small γ, there is a maximal value for b_*. It seems to be a difficult problem to obtain further detailed information, for example the number of local maxima and the location and values of the maxima. Some asymptotic results concerning this are presented in Section 3.

In Section 4, we turn our attention to the problem of coexistence. In particular, we are interested in understanding whether spatial heterogeneity of resources can lead to coexistence. Propositions 1.2 and 1.4 along with the fact that (1.2) is a monotone system [13] immediately gives rise to the following result.

THEOREM 1.6. *If $b < b_*$ and $c < c_*$, then there exists a stable coexistence equilibrium to* (1.2).

This theorem is yet another example of how spatial effects can overcome the paradox of enrichment. In particular, in the Lotka–Volterra model, $b > 1$ and $c < 1$ always leads to the extinction of u. However, in the case of (1.2) we have coexistence at these parameter values as long as $b < b_*$ and $c < c_*$.

Of course, it is also interesting to understand coexistence in terms of the diffusion parameters. In Section 4, an analysis is presented of the bifurcation from $(0, \tilde{v})$ as b passes through b_*, the results being summarized in Fig. 3. There exists $\bar{d} \in (0, \nu)$ such that for $\mu > \bar{d}$, an unstable branch of solutions bifurcates off, but for $\mu < \bar{d}$, at least a pair of solutions is produced by the bifurcation.

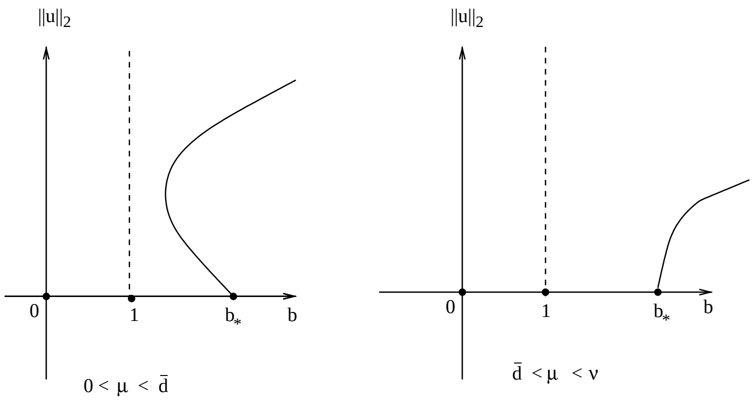

FIG. 3. Typical bifurcation diagrams for different ranges of μ. Note that for the range $0 < \mu < \bar{d}$, the bifurcating branch is initially stable.

The final section of the paper deals with the global dynamics of the system. Our primary interest is in determining the domains of attraction of the semi-trivial solutions $(\tilde{u}, 0)$ and $(0, \tilde{v})$ and in discovering when each of these solutions is a global attractor. Since (1.2) is a monotone system, the answers to these questions follow from understanding the existence of interior equilibria. Using this we shall prove the following two theorems.

Let

$$\Sigma = \{(b, c) \in R_+^2 \colon (\tilde{u}, 0) \text{ is the global attractor of } (1.2)\}. \tag{1.9}$$

In the more interesting case $b > c$, one observes that $(b, c) \in \Sigma$ implies that the superior competitor, in the sense of the reaction system, is incapable of surviving because of the spatial heterogeneity of the resource.

THEOREM 1.7. *If* $\gamma/v \to +\infty$, *then* $\Sigma \to (0, 1] \times [1, +\infty)$.

On the other hand, if $\gamma/v \to 0$ and $\gamma \to +\infty$, the set Σ can be arbitrarily large. More precisely, we have

THEOREM 1.8. *For all* $\varepsilon > 0$, *there exists* $C(\varepsilon) > 0$ *large, independent of* γ, v, b, c, *such that if* $\min\{\gamma, v/\gamma\} \geqslant C(\varepsilon)$, *then*

$$(0, \varepsilon^{-1}] \times [\varepsilon, +\infty) \subset \Sigma. \tag{1.10}$$

2. STABILITY OF $(\tilde{u}, 0)$ AND RELATED MATTERS

In view of Proposition 1.2, the stability of the semi-trivial equilibrium $(\tilde{u}, 0)$ is completely determined by c_*. After proving this proposition, we establish some basic estimates which yield the shape of the graph of c_* as a function of γ—see Fig. 1.

Proof of Proposition 1.2. Recall that c_* is given by (1.7). By the definition of c_*, $\exists \varphi_* > 0$ such that

$$v\Delta\varphi_* + (1 + \gamma\beta - c_*\tilde{u})\varphi_* = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi_*}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.1}$$

For the stability of $(\tilde{u}, 0)$, consider the linear eigenvalue problem

$$v\Delta\varphi + (1 + \gamma\beta - c\tilde{u})\varphi = -\lambda_1\varphi, \qquad \varphi > 0 \quad \text{in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\lambda_1 = (c - c_*)\frac{\int_\Omega \tilde{u}\varphi\varphi_*}{\int_\Omega \varphi\varphi_*}, \tag{2.3}$$

and this proves Proposition 1.2. ∎

We begin our analysis of the shape of c_* by describing its behavior in the neighborhood of $\gamma = 0$. With this in mind let $\{\xi_i(x)\}_{i=0}^{\infty}$ be an orthonormal collection of eigenfunctions for $-\Delta$ on Ω, with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$, i.e.,

$$\Delta\xi_i + \lambda_i\xi_i = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial \xi_i}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.4}$$

Expanding β in terms of ξ_i, one obtains

$$\beta(x) = \sum_{i=1}^{\infty} a_i\xi_i(x) \tag{2.5}$$

in the L^2 sense, where $a_0 = 0$ since $\int_\Omega \beta = 0$.

Proposition 2.1. *$c_*(\gamma) < 1$ for any $\gamma > 0$, and $c_*(\gamma) = 1 + c_2\gamma^2 + o(\gamma^2)$ for $0 < \gamma \ll 1$ where*

$$c_2 = \frac{\mu}{|\Omega|}\left(\frac{\mu}{\nu} - 1\right)\sum_{i=1}^{\infty}\frac{a_i^2\lambda_i}{(1+\lambda_i\mu)^2} < 0.$$

Proof. It follows from the comparison principle for eigenvalues that $c_*(\gamma) < 1$ for any $\gamma \neq 0$. For $0 < \gamma \ll 1$, since $\lim_{\gamma\to 0} c_*(\gamma) = 1$,

$$c_*(\gamma) = 1 + c_1\gamma + c_2\gamma^2 + o(\gamma^2), \tag{2.6}$$

and it is clear from the previous remark that $c_1 = 0$. By definition (1.7) of c_*, we need to consider the following eigenvalue problem with $\varphi > 0$ on Ω:

$$\nu\Delta\varphi + [1 + \gamma\beta(x) - c_*\tilde{u}]\varphi = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.7}$$

Again, for $0 < \gamma \ll 1$, we have

$$\varphi = 1 + \gamma\varphi_1 + \gamma^2\varphi_2 + o(\gamma^2), \tag{2.8}$$

$$\tilde{u} = 1 + \gamma\theta_1 + \gamma^2\theta_2 + o(\gamma^2). \tag{2.9}$$

By comparison of the powers of γ we see that

$$\mu\Delta\theta_1 - \theta_1 = -\beta \qquad \text{in } \Omega, \quad \left.\frac{\partial\theta_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.10}$$

$$\mu\Delta\theta_2 - \theta_2 = \theta_1(\theta_1 - \beta) \qquad \text{in } \Omega, \quad \left.\frac{\partial\theta_2}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.11}$$

$$v\Delta\varphi_1 = \theta_1 - \beta \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{2.12}$$

$$v\Delta\varphi_2 = \theta_2 + c_2 + \varphi_1(\theta_1 - \beta) \qquad \text{in } \Omega, \qquad \left.\frac{\partial\varphi_2}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.13}$$

By (2.10) we have

$$\theta_1(x) = \sum_{i=1}^{\infty} \frac{a_i}{1 + \lambda_i\mu}\, \xi_i. \tag{2.14}$$

From (2.10) and (2.12),

$$\varphi_1(x) = \frac{\mu}{v}\, \theta_1, \tag{2.15}$$

and from (2.13) and (2.11), respectively,

$$\int_\Omega c_2 = \int_\Omega (\beta - \theta_1)\varphi_1 - \int_\Omega \theta_2 \tag{2.16}$$

and

$$\int_\Omega \theta_2 = \int_\Omega \theta_1(\beta - \theta_1). \tag{2.17}$$

It follows from (2.15) to (2.17) that

$$c_2 = \frac{1}{|\Omega|}\left(\frac{\mu}{v} - 1\right) \int_\Omega (\beta - \theta_1)\theta_1. \tag{2.18}$$

Using (2.5) and (2.14) we get

$$\beta - \theta_1 = \sum_{n=1}^{\infty} \frac{a_i\lambda_i\mu}{1 + \lambda_i\mu}\, \xi_i. \tag{2.19}$$

Thus from (2.18) and (2.19),

$$
\begin{aligned}
c_2 &= \frac{\mu}{|\Omega|}\Big(\frac{\mu}{\nu} - 1\Big)\int_\Omega \left(\sum_{n=1}^{\infty} \frac{a_i\lambda_i}{1+\lambda_i\mu}\,\xi_i\right)\left(\sum_{j=1}^{\infty} \frac{a_j}{1+\lambda_j\mu}\,\xi_j\right) \\
&= \frac{\mu}{|\Omega|}\Big(\frac{\mu}{\nu} - 1\Big)\int_\Omega \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{a_ia_j\lambda_i}{(1+\lambda_i\mu)(1+\lambda_j\mu)}\,\xi_i\xi_j \\
&= \frac{\mu}{|\Omega|}\Big(\frac{\mu}{\nu} - 1\Big)\sum_{i=1}^{\infty} \frac{a_i^2\lambda_i}{(1+\lambda_i\mu)^2} < 0. \quad \blacksquare
\end{aligned}
$$

Understanding c_* for large values of γ is more difficult, therefore for simplicity, we fix μ and think of c_* as functions of ν and γ only.

PROPOSITION 2.2. (i) $0 < c_* < 1$;

(ii) $\lim_{\substack{\nu/\gamma\to+\infty \\ \gamma\to+\infty}} c_* = 0$;

(iii) $\lim_{\gamma/\nu\to+\infty} c_* = 1$.

Proof. (i) The inequality $c_* < 1$ is proved in Proposition 2.1. Letting $\varphi = 1$ in (1.7), we see that

$$
-c_* \leqslant \frac{-\int_\Omega (1+\gamma\beta)}{\int_\Omega \tilde{u}} = \frac{-|\Omega|}{\int_\Omega \tilde{u}}, \tag{2.20}
$$

which implies that $c_* \geqslant |\Omega|/\int_\Omega \tilde{u} > 0$.

(ii) Let φ_* be the (unique) solution of (2.1) such that $\max_{\bar{\Omega}} \varphi_* = 1$ and $\varphi_* > 0$ in $\bar{\Omega}$. We show that $\varphi_* \to 1$ as $\nu/\gamma \to +\infty$ and $\gamma \to +\infty$: rewrite (2.1) as

$$
\Delta\varphi_* + \varphi_*\left(\frac{1}{\nu} + \frac{\gamma}{\nu}\beta - c_*\frac{\tilde{u}}{\gamma}\frac{\gamma}{\nu}\right) = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi_*}{\partial n}\right|_{\partial\Omega} = 0. \tag{2.21}
$$

By (1.5) and the Maximum Principle [12], $\|\tilde{u}\|_\infty \leqslant 1 + \gamma \max_{\bar{\Omega}} \beta$. Therefore, by (2.21) and standard elliptic regularity [4], $\varphi_* \to \varphi$ in $C^1(\bar{\Omega})$, where $\Delta\varphi = 0$ in Ω, $\max_{\bar{\Omega}} \varphi = 1$, $\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0$. Hence $\varphi_* \to 1$ in $C^1(\bar{\Omega})$.

Now dividing (2.1) by γ and integrating over Ω, we have

$$
c_* \int_\Omega \varphi_* \frac{\tilde{u}}{\gamma} = \int_\Omega \varphi_*\left(\frac{1}{\gamma} + \beta\right). \tag{2.22}
$$

By Proposition A.1 in Appendix A, $\tilde{u}/\gamma \to \beta_+(x)$. Passing to the limit in (2.22), since $\int_\Omega \beta = 0$ and $\varphi_* \to 1$, we see that $c_* \to 0$ as $\gamma \to \infty$ and $\nu/\gamma \to \infty$.

(iii) We first establish the following assertion.

CLAIM. *For any* $\tilde{c} \in (0,1)$, *consider the linear eigenvalue problem*

$$\nu\Delta\varphi + \varphi(1+\gamma\beta - \tilde{c}\tilde{u}) = -\lambda_1\varphi \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0, \quad \varphi > 0 \text{ in } \bar{\Omega}. \tag{2.23}$$

Then there exist c_1, c_2, *both positive and independent of* ν *and* γ, *such that if* $\nu/\gamma \leqslant c_1$, *we have* $\lambda_1 \leqslant -(1-\tilde{c})c_2\gamma < 0$.

Recall that λ_1 can be characterized as

$$\lambda_1 = \inf_{\substack{\varphi\in H^1\\ \varphi\neq 0}} \frac{\int_\Omega [\nu|\nabla\varphi|^2 + (-1-\gamma\beta+\tilde{c}\tilde{u})\varphi^2]}{\int_\Omega \varphi^2}. \tag{2.24}$$

By Proposition A.1 in Appendix A,

$$\frac{\tilde{u}}{\gamma} \leqslant \beta_+(x) + c_3\left(\frac{\mu}{\gamma}\right)^{1/3} \tag{2.25}$$

provided that γ is sufficiently large. Hence by (2.24),

$$\frac{\lambda_1}{\gamma} \leqslant \frac{\int_\Omega(\tilde{c}\beta_+ - \beta)\varphi^2 + \int_\Omega[\frac{\nu}{\gamma}|\nabla\varphi|^2 + (c_3(\frac{\mu}{\gamma})^{1/3} - \frac{1}{\gamma})\varphi^2]}{\int_\Omega \varphi^2} \tag{2.26}$$

for any $\varphi \in H^1(\Omega)$, $\varphi \neq 0$. Let φ be chosen in the following way: $\varphi \geqslant 0$, $\varphi \not\equiv 0$ and $\operatorname{supp}\varphi \subset \{x \in \Omega : \beta(x) > 0\}$. Then it is easy to see that there exist c_1, $c_2 > 0$ such that if $\nu/\gamma \leqslant c_1$, $\lambda_1/\gamma \leqslant -c_2(1-\tilde{c})$. This proves the assertion.

We now show that (iii) follows from our assertion. For any $\varepsilon > 0$, let $\tilde{c} = 1-\varepsilon$. From (2.1) and (2.23),

$$-\lambda_1 \int_\Omega \varphi\varphi_* = (c_* - (1-\varepsilon)) \int_\Omega \tilde{u}\varphi\varphi_*. \tag{2.27}$$

By our assertion, there exists $c_1(\varepsilon) > 0$ such that if $\gamma/\nu \geqslant c_1(\varepsilon)$, then $\lambda_1 < 0$. This implies that $c_* \geqslant 1-\varepsilon$ if $\gamma/\nu \geqslant c_1(\varepsilon)$. But $c_* < 1$ from (i). Hence $c_* \to 1$ if $\gamma/\nu \to +\infty$. ∎

Remark 2.3. Parts (i) and (iii) hold for $\int_\Omega \beta \geqslant 0$, but (ii) only holds for $\int_\Omega \beta = 0$. In fact, if $\int_\Omega \beta > 0$, part (ii) fails. More precisely, if $\int_\Omega \beta > 0$, then there exists some positive constant c_3 such that $c_* \geqslant c_3$ for any γ and ν.

3. THE STABILITY OF $(0, \tilde{v})$

In this case, the stability is determined by b_* (defined by (1.8)). Theorem 1.5 states that $\lim_{\gamma/\nu\to\infty} b_* = 1$, but this result is much harder to prove than the analogous result for c_*. The reason is that we have to show that $b_* \leqslant 1+\varepsilon$ for sufficiently large γ/ν. This is equivalent to establishing a lower bound for a linear eigenvalue problem and a lower bound is usually harder to find than an upper bound (used in the case of the estimate of c_*). Since the proof of Proposition 1.4 is almost identical to that of Proposition 1.2, we omit it. Similarly, the proof of the following proposition which characterizes b_* for $\gamma \approx 0$ is essentially the same as that of Proposition 2.1.

PROPOSITION 3.1. *$b_*(\gamma) > 1$ for any $\gamma > 0$, and $b_*(\gamma) = 1 + b_2\gamma^2 + o(\gamma^2)$ for $0 < \gamma \ll 1$ where*

$$b_2 = \frac{\nu}{|\Omega|}\left(\frac{\nu}{\mu} - 1\right)\sum_{i=1}^{\infty}\frac{a_i^2\lambda_i}{(1+\lambda_i\nu)^2} > 0.$$

Next we turn to the proof of Theorem 1.5, i.e., $b_* \to 1$ as $\gamma/\nu \to +\infty$. As in the proof of Proposition 2.2(iii), Theorem 1.5 follows from an estimate for a certain principal eigenvalue λ_1, in this case defined by Eq. (3.1). In fact it suffices to show that $\lambda_1 > 0$, but we present a stronger result, which is itself of interest and scarcely more difficult to prove. In the rest of this section, $c_1, c_2, \ldots$ will denote positive constants independent of ν and γ.

THEOREM 3.2. *Let λ_1 be the principal eigenvalue for the problem*

$$-\mu\Delta\varphi - [1 + \gamma\beta - b\tilde{v}]\varphi = \lambda\varphi \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.1}$$

Then given $b > 1$, there exist positive constants c_1, c_2 such that $\lambda_1 \geqslant c_2\gamma^{2/3}\nu^{1/3}$ for $\gamma/\nu \geqslant c_1$.

Proof.

Step 1. Let x_γ satisfy $\varphi(x_\gamma) = \max_{\bar{\Omega}}\varphi > 0$, where φ is given by (3.1). Then there exist c_3 and c_4 such that if $\gamma/\nu \geqslant c_3$, we have

$$\operatorname{dist}(x_\gamma, \Gamma) \leqslant c_4\left(\frac{\nu}{\gamma}\right)^{1/3}, \tag{3.2}$$

where $\Gamma = \{x \in \Omega : \beta(x) = 0\}$.

To prove this assertion, we first observe that

$$1 + \gamma\beta(x_\gamma) - b\tilde{v}(x_\gamma) + \lambda_1 \geqslant 0. \tag{3.3}$$

There are two cases in the proof of (3.3): $x_\gamma \in \Omega$ or $x_\gamma \in \partial\Omega$. If $x_\gamma \in \Omega$, from the Maximum Principle, $\Delta\varphi(x_\gamma) \leqslant 0$, and (3.3) follows from (3.1). If $x_\gamma \in \partial\Omega$ and (3.3) fails, by the continuity of β and $\tilde{v}$, there exists a small open ball denoted by B such that $B \subset \Omega$, $\bar{B} \cap \partial\Omega = \{x_\gamma\}$ and $1 + \gamma\beta(x) - b\tilde{v}(x) + \lambda_1 < 0$ in $\bar{B}$. By (3.1), $\Delta\varphi > 0$ in B. The Hopf Boundary Lemma implies that $\frac{\partial\varphi}{\partial n}(x_\gamma) > 0$, which contradicts $\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0$. Hence (3.3) holds.

By Proposition A.1,

$$\frac{\tilde{v}(x_\gamma)}{\gamma} \geqslant \beta_+(x_\gamma) - c_5\left(\frac{v}{\gamma}\right)^{1/3}. \tag{3.4}$$

By (3.3) and (3.4), we have

$$b\beta_+(x_\gamma) - \beta(x_\gamma) \leqslant c_6\left(\frac{v}{\gamma}\right)^{1/3} + \frac{\lambda_1}{\gamma}. \tag{3.5}$$

We claim that there exist c_7 and c_8 such that if $\gamma/v \geqslant c_7$, then $\lambda_1 \leqslant c_8\gamma^{2/3}v^{1/3}$. To prove this assertion, observe that λ_1 can be characterized by

$$\lambda_1 = \inf_{\substack{\varphi\in H^1 \\ \varphi\neq 0}} \frac{\int_\Omega [\mu|\nabla\varphi|^2 - (1 + \gamma\beta - b\tilde{v})\varphi^2]}{\int_\Omega \varphi^2}. \tag{3.6}$$

Choose the following test function φ:

$$\varphi(x) = \begin{cases} 0 & \text{if } \beta(x) \leqslant 0 \text{ or } \operatorname{dist}(x,\Gamma) \geqslant 2(\frac{v}{\gamma})^{1/3}, \\ (\frac{\gamma}{v})^{1/3}\operatorname{dist}(x,\Gamma) & \text{if } \beta(x) \geqslant 0 \text{ and } 0 \leqslant \operatorname{dist}(x,\Gamma) \leqslant (\frac{v}{\gamma})^{1/3}, \\ 2 - (\frac{\gamma}{v})^{1/3}\operatorname{dist}(x,\Gamma) & \text{if } \beta(x) \geqslant 0 \text{ and} \\ & \quad (\frac{v}{\gamma})^{1/3} \leqslant \operatorname{dist}(x,\Gamma) \leqslant 2(\frac{v}{\gamma})^{1/3}. \end{cases} \tag{3.7}$$

By Proposition A.1,

$$\frac{\tilde{v}}{\gamma} \leqslant \beta_+ + c_9\left(\frac{v}{\gamma}\right)^{1/3}. \tag{3.8}$$

Using (3.6)–(3.8) we can check that $\lambda_1 \leqslant c_8\gamma^{2/3}v^{1/3}$ provided that $\gamma/v \gg 1$.

By (3.5) and the previous assertion, we have

$$b\beta_+(x_\gamma) - \beta(x_\gamma) \leqslant c_{10}\left(\frac{v}{\gamma}\right)^{1/3}. \tag{3.9}$$

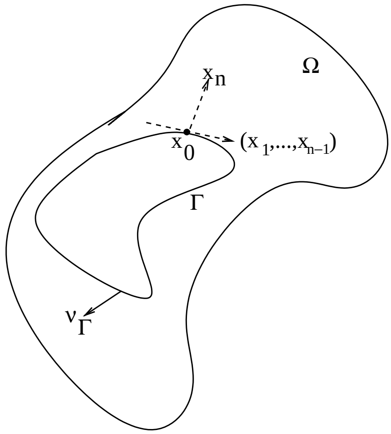

FIG. 4. Illustration of the choice of coordinates in proof of Step 2.

If $\beta(x_\gamma) \geqslant 0$, then

$$0 \leqslant \beta(x_\gamma) \leqslant c_{11}\left(\frac{v}{\gamma}\right)^{1/3}; \tag{3.10}$$

if $\beta(x_\gamma) \leqslant 0$, we have

$$-c_{10}\left(\frac{v}{\gamma}\right)^{1/3} \leqslant \beta(x_\gamma) \leqslant 0. \tag{3.11}$$

Since $\nabla\beta$ does not vanish on Γ, we see that (3.2) holds provided that $\gamma/v \gg 1$.

Step 2. By (3.2), passing to a subsequence if necessary, we may assume that $x_\gamma \to x_0 \in \Gamma$ as $\gamma/v \to +\infty$. After translation and rotation, we may assume that x_0 is the origin, the normal to Γ at x_0 is $(0, \ldots, 0, 1)$ (Fig. 4), and near the origin, $\beta(x) = \frac{\partial\beta}{\partial x_m}(0)x_m + O(|x|^2)$. Without loss of generality, we may take $\frac{\partial\beta}{\partial x_m}(0) = 1$.

Choose $\tilde{x}_\gamma \in \Gamma$ such that $|x_\gamma - \tilde{x}_\gamma| = \mathrm{dist}(x_\gamma, \Gamma)$. Set $x_\gamma = \tilde{x}_\gamma + (v/\gamma)^{1/3} y_\gamma$. By (3.2) we see that $|y_\gamma| \leqslant c_4$. Hence we may assume that $y_\gamma \to y^* \in R^m$. Dividing (3.3) by $\gamma^{2/3}v^{1/3}$, we have

$$\frac{1}{\gamma^{2/3}v^{1/3}} + \left(\frac{\gamma}{v}\right)^{1/3}\beta\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) - b\,\frac{\tilde{v}(\tilde{x}_\gamma + (\frac{v}{\gamma})^{1/3} y_\gamma)}{\gamma^{2/3}v^{1/3}} \geqslant \frac{-\lambda_1}{\gamma^{2/3}v^{1/3}}. \tag{3.12}$$

We first show that

$$\left(\frac{\gamma}{v}\right)^{1/3}\beta\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) \to y_m^* \tag{3.13}$$

as $\gamma/v \to \infty$, where $y^* = (y_1^*, \ldots, y_m^*)$. Recalling that $\beta(\tilde{x}_\gamma) = 0$ we have

$$\beta\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) = \int_0^1 \frac{\partial\beta}{\partial t}\left(\tilde{x}_\gamma + t\left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) dt$$
$$= \left(\frac{v}{\gamma}\right)^{1/3} \int_0^1 \nabla\beta\left(\tilde{x}_\gamma + t\left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) \cdot y_\gamma \, dt.$$

Hence there exists $t_\gamma \in (0, 1)$ such that

$$\left(\frac{\gamma}{v}\right)^{1/3} \beta\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) = \nabla\beta\left(\tilde{x}_\gamma + t_\gamma\left(\frac{v}{\gamma}\right)^{1/3} y_\gamma\right) \cdot y_\gamma$$
$$\to \nabla\beta(0) \cdot y^* = y_m^*,$$

since $\tilde{x}_\gamma \to 0$, $|y_\gamma| \leqslant c_4$, and we have assumed that $\frac{\partial\beta}{\partial x_m}(0) = 1$.

Set

$$\tilde{w}(y) = \frac{\tilde{v}\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y\right)}{\gamma^{2/3} v^{1/3}}. \tag{3.14}$$

We claim that $\tilde{w} \to w$ uniformly in any compact subset of R^m, where $w > 0$ satisfies the following in R^m:

$$\begin{cases} \Delta w + w(y_m - w) = 0, \\ (y_m)_+ - c_{12} \leqslant w(y) \leqslant (y_m)_+ + c_{12}. \end{cases} \tag{3.15}$$

To prove this, observe that $\tilde{w}$ satisfies

$$\Delta\tilde{w} + \tilde{w}\left[\frac{1}{\gamma^{2/3} v^{1/3}} + \left(\frac{\gamma}{v}\right)^{1/3} \beta\left(\tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y\right) - \tilde{w}\right] = 0 \qquad \text{in } \Omega_\gamma, \tag{3.16}$$

where

$$\Omega_\gamma = \left\{ y \in R^m \colon \tilde{x}_\gamma + \left(\frac{v}{\gamma}\right)^{1/3} y \in \Omega \right\}. \tag{3.17}$$

By Proposition A.1, we have

$$\left(\frac{\gamma}{v}\right)^{1/3}\beta_+\left(\tilde{x}_\gamma+\left(\frac{v}{\gamma}\right)^{1/3}y\right)-c_{12}\leqslant\tilde{w}(y)$$
$$\leqslant\left(\frac{\gamma}{v}\right)^{1/3}\beta_+\left(\tilde{x}_\gamma+\left(\frac{v}{\gamma}\right)^{1/3}y\right)+c_{12}. \quad (3.18)$$

By applying elliptic regularity on $\tilde{w}$ in any compact subset of R^m and a diagonal process, we see that, passing to a subsequence if necessary, $\tilde{w}\to w$ uniformly in any compact subset of R^m. Passing to the limit in (3.16) and (3.18), since

$$\left(\frac{\gamma}{v}\right)^{1/3}\beta\left(\tilde{x}_\gamma+\left(\frac{v}{\gamma}\right)^{1/3}y\right)\to y_m \quad (3.19)$$

(the proof is exactly the same as that of (3.13)), we see that w satisfies (3.15). Since $w\geqslant 0$ and $w\not\equiv 0$, the Maximum Principle ensures that $w>0$ in R^m. This proves our claim.

Claim. *If $\gamma/v\gg 1$, then $\lambda_1\geqslant -c_{13}\gamma^{2/3}v^{1/3}$, where $c_{13}>0$ is independent of γ and v.*

This follows easily from (3.2) and (3.3) for we have

$$\frac{-\lambda_1}{\gamma}\leqslant\frac{1}{\gamma}+\beta(x_\gamma)\leqslant\frac{1}{\gamma}+c_{14}\left(\frac{v}{\gamma}\right)^{1/3}. \quad (3.20)$$

Step 3. We can now establish Theorem 3.2. To this end, we argue by contradiction: assume that our assertion $\lambda_1\geqslant c_2\gamma^{2/3}v^{1/3}$ fails. By passing to a subsequence if necessary, we may assume that $\lambda_1/(\gamma^{2/3}v^{1/3})\to -c_{15}$ for some $c_{15}\geqslant 0$. Now passing to the limit in (3.12), we have

$$y_m^*-bw(y^*)\geqslant c_{15}\geqslant 0. \quad (3.21)$$

However, (3.21) contradicts the following assertion.

Claim. *For any $b>1$, $y_m-bw(y)<0$ for any $y\in R^m$.*

We argue by contradiction: suppose not, by (3.15) we see that given $b>1$, $y_m-bw(y)$ is bounded from above in R^m. Hence we may assume that

$$\sup_{R^m}(y_m-bw(y))=a\geqslant 0. \quad (3.22)$$

Therefore there exists a sequence $\{y^{(k)}\}_{k=1}^\infty$ such that $y_m^{(k)}-bw(y(k))\to a$, where $y^{(k)}=(y_1^{(k)},\ldots,y_m^{(k)})$. It is easy to see that $\{y_m^{(k)}\}_{k=1}^\infty$ is bounded from below; on the other hand, since $w(y) \geqslant (y_m)_+ - c_{12}$, $\{y_m^{(k)}\}_{k=1}^{\infty}$ is bounded from above. Hence, we may assume that $y_m^{(k)} \to \hat{y} \in R^1$. Set

$$w^{(k)}(y) = w(y_1 + y_1^{(k)}, \ldots, y_{m-1} + y_{m-1}^{(k)}, y_m). \tag{3.23}$$

Then $w^{(k)}$ satisfies (3.15) as well. By elliptic regularity and a diagonal process, $w^{(k)}(y) \to \hat{w}(y)$ (in C^2) in any compact subset of R^m, and $\hat{w}$ still satisfies (3.15). By the Maximum Principle, $\hat{w} > 0$ in R^m. Since $y_m^{(k)} - bw(y^{(k)}) \to a$, $w(y^{(k)}) = w^{(k)}(0, \ldots, 0, y_m^{(k)})$, $y_m^{(k)} \to \hat{y}$, we see that

$$\hat{y} - b\hat{w}(0, \ldots, 0, \hat{y}) = a \geqslant 0. \tag{3.24}$$

Also, since $y_m - bw^{(k)}(y) \leqslant a$, passing to the limit we see that

$$\sup_{R^m}(y_m - b\hat{w}(y)) \leqslant a. \tag{3.25}$$

In other words, the function $y_m - b\hat{w}(y)$ attains its maximum at $(0, \ldots, 0, \hat{y})$. Hence $\Delta(y_m - b\hat{w}(y))|_{y=(0,\ldots,0,\hat{y})} \leqslant 0$. That is, $\Delta w(0, \ldots, 0, \hat{y}) \geqslant 0$. Note that $\hat{w}$ satisfies (3.15), i.e., $\Delta\hat{w} + \hat{w}(y_m - \hat{w}) = 0$. As $\hat{w} > 0$ in R^m, we see that $\hat{y} \leqslant \hat{w}(0, \ldots, 0, \hat{y})$. By (3.24) we have

$$b\hat{w}(0, \ldots, 0, \hat{y}) \leqslant b\hat{w}(0, \ldots, 0, \hat{y}) + a = \hat{y} \leqslant \hat{w}(0, \ldots, 0, \hat{y}). \tag{3.26}$$

Since $\hat{w}(0, \ldots, 0, \hat{y}) > 0$, we have $b \leqslant 1$, which contradicts the assumption $b > 1$. This proves our claim, which in turn yields Theorem 3.2. ∎

Remark 3.3. (i) The assumption $\int_\Omega \beta(x)\, dx = 0$ is unnecessary in Theorem 3.2. Note that Proposition A.1 in the appendix does not require $\int_\Omega \beta = 0$ either.

(ii) The proof of Theorem 1.5 depends crucially on condition (H3), which is used in the construction of a sub-solution. However, there are some grounds for supposing that the result may be true even if a weaker condition replaces the restriction that $\nabla\beta$ does not vanish on Γ. For in the special case when $\Omega \subset \mathbb{R}$, it is possible to relax this condition and assume a condition on β'' at a zero of β. We leave open the question of generalizing this result to higher dimensions.

To understand the following results, it is helpful to discuss the "shadow" problem obtained by letting $v \to \infty$. For fixed γ, since $\int_\Omega \beta = 0$, $\tilde{v} \to 1$ uniformly as $v \to \infty$. Therefore, it is plausible that, at least for any compact set of γ, $b_*(\gamma, v)$ is close to $b_*(\gamma, \infty)$, where $b_*(\gamma, \infty)$ is defined by

$$b_*(\gamma, \infty) = -\inf_{\substack{\varphi \in H^1 \\ \varphi \neq 0}} \frac{\int_\Omega [\mu |\nabla \varphi|^2 - (1 + \gamma\beta)\varphi^2]}{\int_\Omega \varphi^2}. \tag{3.27}$$

We are interested in the connection/difference between $b_*(\gamma, v)$ and $b_*(\gamma, \infty)$ for $v \gg 1$. It turns out that, as shown in the next few results, $b_*(\gamma, v) \approx b_*(\gamma, \infty)$ for $\gamma = o(\sqrt{v})$, but $b_*(\gamma, v)$ behaves differently from $b_*(\gamma, \infty)$ if $\gamma \geqslant O(\sqrt{v})$. A detailed description of $b_*(\gamma, v)$ is given by Theorem 3.4 and Proposition 3.5.

In the following, we write b_* as $b_*(\gamma, v)$ to denote its dependence on γ and v. We also need to assume $\int_\Omega \beta(x)\,dx = 0$ from now on. The case $\int_\Omega \beta(x)\,dx > 0$ is quite different.

THEOREM 3.4. *Let φ^* be the unique solution of $-\Delta\varphi^* = \beta$ in Ω, $\frac{\partial \varphi^*}{\partial n}|_{\partial\Omega} = 0$ and $\int_\Omega \varphi^*(x)\,dx = 0$. Then*

(i) $$\lim_{v \to +\infty} \frac{\max_{0<\gamma<\infty} b_*(\gamma, v)}{\sqrt{v}} = \frac{\max_{\bar{\Omega}} \beta |\Omega|^{1/2}}{2\|\nabla \varphi^*\|_2}.$$

(ii) *If $\gamma^* = \gamma^*(v)$ satisfies $b_*(\gamma^*, v) = \max_{0<\gamma<\infty} b_*(\gamma, v)$, then*

$$\lim_{v \to +\infty} \frac{\gamma^*}{\sqrt{v}} = \frac{|\Omega|^{1/2}}{\|\nabla \varphi^*\|_2}.$$

To establish Theorem 3.4, we need some preliminary results about qualitative properties of $b_*(\gamma, v)$.

PROPOSITION 3.5. *The following hold*:

(i) $\lim_{\gamma^2/v \to 0^+} b_*(\gamma, v)/b_*(\gamma, \infty) = 1$;

(ii) $\forall \eta > 0$, $\exists k(\eta) > 0$ *large, independent of γ and v, such that if $\eta\sqrt{v} \leqslant \gamma \leqslant v/\eta$, then*

$$\frac{1}{k(\eta)} \leqslant \frac{\gamma^2}{v} \frac{b_*(\gamma, v)}{b_*(\gamma, \infty)} \leqslant k(\eta). \tag{3.28}$$

To prove Proposition 3.5, we need the following two lemmas which are also useful in Section 5.

LEMMA 3.6. $\exists c_{17}$ *and* $c_{18} > 0$, *independent of γ and v, such that if* $\min\{v/\gamma, \gamma\} \geqslant c_{17}$, *then*

$$1 - c_{18}\frac{\gamma}{\nu} \leqslant \tilde{v}(x) \leqslant 1 + c_{18}\frac{\gamma^2}{\nu} \qquad \forall x \in \bar{\Omega}. \tag{3.29}$$

Proof. Set

$$\bar{v} = \left(1 + c_{19}\frac{\gamma^2}{\nu}\right)\left(1 + \frac{\gamma}{\nu}\varphi^*\right). \tag{3.30}$$

We claim that for large c_{19} which is independent of γ and ν, $\bar{v}$ is a super-solution of (1.6) provided that ν/γ, γ are sufficiently large. By direct calculation,

$$\frac{\nu}{\gamma}\Delta\bar{v} + \bar{v}\left(\frac{1}{\gamma} + \beta - \frac{\bar{v}}{\gamma}\right) = \left(1 + c_{19}\frac{\gamma^2}{\nu}\right)\frac{\gamma}{\nu}\left[-c_{19} + \beta\varphi^* - \frac{1}{\gamma}\varphi^*\left(1 + 2c_{19}\frac{\gamma^2}{\nu}\right)\right.$$
$$\left. - \frac{1}{\nu}(\varphi^*)^2\left(1 + c_{19}\frac{\gamma^2}{\nu}\right)\right]. \tag{3.31}$$

Set $c_{19} = \|\beta\varphi^*\|_\infty + 1$. Then we have

$$\frac{\nu}{\gamma}\Delta\bar{v} + \bar{v}\left(\frac{1}{\gamma} + \beta - \frac{\bar{v}}{\gamma}\right) \leqslant \left(1 + c_{19}\frac{\gamma^2}{\nu}\right)\frac{\gamma}{\nu}\left[-1 + c_{20}\left(\frac{\gamma}{\nu} + \frac{1}{\nu} + \frac{\gamma^2}{\nu^2} + \frac{1}{\gamma}\right)\right] \leqslant 0$$

provided that $\min\{\nu/\gamma, \gamma\} \gg 1$. This shows that $\bar{v}$ is a super-solution of (1.6) and

$$\tilde{v} \leqslant \bar{v} = \left(1 + c_{19}\frac{\gamma^2}{\nu}\right)\left(1 + \frac{\gamma}{\nu}\varphi^*\right) \leqslant 1 + c_{18}\frac{\gamma^2}{\nu}. \tag{3.32}$$

We now find a sub-solution $\underline{v}$ of (1.6). Let φ_* be the unique solution of

$$\Delta\varphi_* = \int_\Omega \beta\varphi^* - \beta\varphi^*, \qquad \int_\Omega \varphi_* = 0, \qquad \left.\frac{\partial\varphi_*}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.33}$$

Set

$$\underline{v} = 1 + \frac{\gamma}{\nu}\varphi^* + \frac{\gamma^2}{\nu^2}\varphi_*. \tag{3.34}$$

Then by direct calculation,

$$
\begin{aligned}
&\frac{v}{\gamma}\,\Delta\underline{v} + \underline{v}\left(\frac{1}{\gamma} + \beta - \frac{\underline{v}}{\gamma}\right)\\
&\quad \geqslant \frac{\gamma}{v}\,(\Delta\varphi_* + \beta\varphi^*) - c_{21}\left(\frac{1}{v} + \frac{\gamma}{v^2} + \frac{\gamma^2}{v^3} + \frac{\gamma^2}{v^2}\right)\\
&\quad = \frac{\gamma}{v}\left[\int_\Omega \beta\varphi^* - c_{21}\left(\frac{1}{\gamma} + \frac{1}{v} + \frac{\gamma}{v^2} + \frac{\gamma}{v}\right)\right]\\
&\quad = \frac{\gamma}{v}\left[\int_\Omega |\nabla\varphi^*|^2 - c_{21}\left(\frac{1}{\gamma} + \frac{1}{v} + \frac{\gamma}{v^2} + \frac{\gamma}{v}\right)\right] \geqslant 0 \qquad (3.35)
\end{aligned}
$$

provided that $\min\{\gamma, v/\gamma\} \geqslant 1$, where the last equality follows from $\int_\Omega \beta\varphi^* = \int_\Omega |\nabla\varphi^*|^2$. This implies that $\underline{v}$ is a sub-solution and $\tilde{v} \geqslant \underline{v} \geqslant 1 - c_{18}\gamma/v$.

Lemma 3.7. *$\forall\eta > 0$, $\exists c_{22}(\eta) > 0$ large, independent of γ and v, such that if $\eta\sqrt{v} \leqslant \gamma \leqslant v/\eta$, then*

$$
\frac{1}{c_{22}(\eta)}\frac{\gamma^2}{v} \leqslant \tilde{v}(x) \leqslant c_{22}(\eta)\,\frac{\gamma^2}{v} \qquad \forall x \in \bar{\Omega}. \qquad (3.36)
$$

Proof. We first show that $\|\tilde{v}\|_{L^\infty(\Omega)} \leqslant c_{24}(\eta)\gamma^2/v$ for some $c_{24}(\eta)$ which is independent of γ and v. Set $\hat{v} = |\Omega|^{-1}\int_\Omega \tilde{v}$.

Claim 1. $\|\tilde{v} - \hat{v}\|_\infty \leqslant c_{25}(\eta)(\gamma/v)\|\tilde{v}\|_\infty$.

To prove this assertion, rewrite the equation for $\tilde{v}$ as

$$
\begin{cases} -\Delta(\tilde{v} - \hat{v}) = f := \frac{\gamma}{v}\,\tilde{v}(\frac{1}{\gamma} + \beta - \frac{\tilde{v}}{\gamma}) & \text{in } \Omega,\\ \frac{\partial}{\partial n}(\tilde{v} - \hat{v})|_{\partial\Omega} = 0. \end{cases} \qquad (3.37)
$$

Multiplying (3.37) by $\tilde{v} - \hat{v}$ and integrating, since $\int_\Omega(\tilde{v} - \hat{v}) = 0$, by the Hölder inequality and the Poincaré inequality we see that $\|\tilde{v} - \hat{v}\|_{W^{1,2}(\Omega)} \leqslant c^*_{25}\|f\|_2$. By the Sobolev Embedding Theorem, the L^p estimates [4] and standard bootstrap arguments, we have $\|\tilde{v} - \hat{v}\|_\infty \leqslant c_{26}\|f\|_\infty$. By the Maximum Principle, $\|\tilde{v}/\gamma\|_\infty \leqslant c_{27}$. Hence $\|f\|_\infty \leqslant c_{28}(\eta)(\gamma/v)\|\tilde{v}\|_\infty$, from which it follows that $\|\tilde{v} - \hat{v}\|_\infty \leqslant c_{25}(\eta)(\gamma/v)\|\tilde{v}\|_\infty$. This proves Claim 1.

Claim 2. $\|\tilde{v}\|_\infty \int_\Omega(\tilde{v}/\|\tilde{v}\|_\infty)^2 \leqslant c_{29}(\eta)\gamma^2/v$.

To prove this assertion, integrate (1.6) and divide it by $\|\tilde{v}\|_\infty$. We have

$$\begin{aligned}
\|\tilde{v}\|_\infty \int_\Omega \left(\frac{\tilde{v}}{\|\tilde{v}\|_\infty}\right)^2 &= \int_\Omega \frac{\tilde{v}}{\|\tilde{v}\|_\infty} + \gamma \int_\Omega \frac{\tilde{v}}{\|\tilde{v}\|_\infty}\,\beta \\
&= \int_\Omega \frac{\tilde{v}}{\|\tilde{v}\|_\infty} + \frac{\gamma}{\|\tilde{v}\|_\infty} \int_\Omega (\tilde{v} - \hat{v})\beta \\
&\leqslant |\Omega| + \frac{\gamma}{\|\tilde{v}\|_\infty} \|\tilde{v} - \hat{v}\|_{\infty(\Omega)} \int_\Omega |\beta| \\
&\leqslant |\Omega| + c_{25} \int_\Omega |\beta| \frac{\gamma^2}{v} \\
&\leqslant c_{29}(\eta) \frac{\gamma^2}{v}. \qquad (3.38)
\end{aligned}$$

This proves Claim 2, where the condition $\int_\Omega \beta = 0$ and Claim 1 have been used.

Note that $\tilde{v}/\|\tilde{v}\|_\infty$ satisfies

$$\begin{aligned}
&\Delta\left(\frac{\tilde{v}}{\|\tilde{v}\|_\infty}\right) + \frac{\tilde{v}}{\|\tilde{v}\|_\infty}\left(\frac{1}{v} + \frac{\gamma}{v}\beta - \frac{\tilde{v}}{\gamma}\frac{\gamma}{v}\right) = 0 \qquad \text{in } \Omega, \\
&\frac{\partial}{\partial n}\left(\frac{\tilde{v}}{\|\tilde{v}\|_\infty}\right)\bigg|_{\partial\Omega} = 0. \qquad (3.39)
\end{aligned}$$

Since

$$\left\|\frac{1}{v} + \frac{\gamma}{v}\beta - \frac{\tilde{v}}{\gamma}\frac{\gamma}{v}\right\|_\infty \leqslant c_{30}(\eta), \qquad (3.40)$$

by the global Harnack inequality (see [10]), we have

$$\min_{\bar{\Omega}} \frac{\tilde{v}}{\|\tilde{v}\|_\infty} \geqslant c_{31}(\eta) \max_{\bar{\Omega}} \frac{\tilde{v}}{\|\tilde{v}\|_\infty} = c_{31}(\eta). \qquad (3.41)$$

It follows from (3.41) and Claim 2 that $\|\tilde{v}\|_\infty \leqslant (c_{29}/c_{31}^2)(\gamma^2/v)$; this is the required upper bound on $\tilde{v}$.

Next, we show that $\min_{\bar{\Omega}} \tilde{v} \geqslant c_{32}(\eta)\gamma^2/v$ if $\eta\sqrt{v} \leqslant \gamma \leqslant v/\eta$. Since (3.41) implies that $\min_{\bar{\Omega}} \tilde{v} \geqslant c_{31} \max_{\bar{\Omega}} \tilde{v}$, it suffices to show that $\max_{\bar{\Omega}} \tilde{v} \geqslant c_{33}\gamma^2/v$ for some $c_{33} > 0$. We shall argue by contradiction: suppose that there exist $\eta_0 > 0$, $\{(\gamma_i, v_i)\}_{i=1}^\infty$ satisfying $\eta_0\sqrt{v_i} \leqslant \gamma_i \leqslant v_i/\eta_0$, $v_i\|\tilde{v}_i\|_\infty/\gamma_i^2 \to 0$, where $\tilde{v}_i$ satisfies

$$v_i \Delta\tilde{v}_i + \tilde{v}_i(1 + \gamma_i\beta - \tilde{v}_i) = 0 \qquad \text{in } \Omega, \quad \frac{\partial\tilde{v}_i}{\partial n}\bigg|_{\partial\Omega} = 0. \qquad (3.42)$$

Set $\hat{v}_i = |\Omega|^{-1} \int_\Omega \tilde{v}_i$. Integrating (3.42) and dividing it by $\gamma_i^2 \|\tilde{v}_i\|_\infty / v_i$, after some rearrangement we have

$$\frac{v_i}{\gamma_i^2} \int_\Omega \frac{\tilde{v}_i}{\|\tilde{v}_i\|_\infty} + \frac{v_i}{\gamma_i \|\tilde{v}_i\|_\infty} \int_\Omega (\tilde{v}_i - \hat{v}_i)\beta = \frac{v_i}{\gamma_i^2} \|\tilde{v}_i\|_\infty \int_\Omega \left(\frac{\tilde{v}_i}{\|\tilde{v}_i\|_\infty}\right)^2. \quad (3.43)$$

Dividing (3.42) by $\tilde{v}_i$ and integrating, we have

$$\int_\Omega \tilde{v}_i = |\Omega| + \gamma_i \int_\Omega \beta + v_i \int_\Omega \frac{\Delta \tilde{v}_i}{\tilde{v}_i} = |\Omega| + v_i \int_\Omega \frac{|\nabla \tilde{v}_i|^2}{\tilde{v}_i^2}, \quad (3.44)$$

i.e., $\int_\Omega \tilde{v}_i \geqslant |\Omega|$. Therefore $\|\tilde{v}_i\|_\infty \geqslant 1$, which together with $v_i \|\tilde{v}_i\|_\infty / \gamma_i^2 \to 0$ implies that $v_i/\gamma_i^2 \to 0$. Passing to the limit in (3.43), we have, as $i \to +\infty$,

$$\frac{v_i}{\gamma_i \|\tilde{v}_i\|_\infty} \int_\Omega \beta(\tilde{v}_i - \hat{v}_i) \to 0. \quad (3.45)$$

Set $w_i = (\tilde{v}_i - \hat{v}_i)/\|\tilde{v}_i - \hat{v}_i\|_\infty$. Then w_i satisfies

$$\begin{cases} (\frac{v_i}{\gamma_i \|\tilde{v}_i\|_\infty} \|\tilde{v}_i - \hat{v}_i\|_\infty)\Delta w_i + \frac{\tilde{v}_i}{\|\tilde{v}_i\|_\infty}(\frac{1}{\gamma_i} + \beta - \frac{\tilde{v}_i}{\gamma_i}) = 0 & \text{in } \Omega, \\ \frac{\partial w_i}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad \|w_i\|_\infty = 1. & \end{cases} \quad (3.46)$$

We consider two different cases:

Case 1: $v_i/\gamma_i \to +\infty$. For this case, it is easy to see that $\tilde{v}_i/\|\tilde{v}_i\|_\infty \to 1$ uniformly. Moreover, $\tilde{v}_i/\gamma_i \to 0$ uniformly. To see the last assertion, observe that $\tilde{v}_i/\gamma_i$ is uniformly bounded by the Maximum Principle. Since γ_i/v_i is bounded, by (3.42) and elliptic regularity, we see that $\tilde{v}_i/\gamma_i$ is uniformly bounded in the $C^{2,\alpha}$ norm for some $\alpha > 0$. Hence, passing to some subsequence if necessary, we may assume that $\tilde{v}_i/\gamma_i \to v_0$ in C^2, where $\Delta v_0 = 0$ in Ω and $\frac{\partial v_0}{\partial n}|_{\partial\Omega} = 0$. Hence v_0 is a nonnegative constant. If $v_0 > 0$, dividing (3.42) by γ_i^2 and integrating in Ω, we have $\int_\Omega v_0(\beta - v_0) = 0$, which implies that $v_0 = |\Omega|^{-1} \int_\Omega \beta = 0$. Contradiction! Hence $\tilde{v}_i/\gamma_i \to 0$ uniformly.

If $v_i \|\tilde{v}_i - \hat{v}_i\|_\infty / (\gamma_i \|\tilde{v}_i\|_\infty) \to +\infty$, then by (3.46), $w_i \to w$ and w satisfies $\Delta w = 0$, $\frac{\partial w}{\partial n}|_{\partial\Omega} = 0$, $\|w\|_\infty = 1$ and $\int_\Omega w = 0$ since $\int_\Omega w_i = 0$ and $\|w_i\|_\infty = 1$. However, such w clearly does not exist. If $v_i \|\tilde{v}_i - \hat{v}_i\|_\infty / (\gamma_i \|\tilde{v}_i\|_\infty) \to 0$, multiplying (3.46) by any C^2 function φ with $\frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0$, integrating in Ω, we have

$$\frac{v_i}{\gamma_i \|\tilde{v}_i\|_\infty} \|\tilde{v}_i - \hat{v}_i\|_\infty \int_\Omega w_i \Delta\varphi + \int_\Omega \frac{v_i}{\|\tilde{v}_i\|_\infty} \varphi \left(\frac{1}{\gamma_i} + \beta - \frac{\tilde{v}_i}{\gamma_i}\right) = 0. \quad (3.47)$$

Passing to the limit in (3.47), since $v_i/\|\tilde{v}_i\|_\infty \to 1$, $\tilde{v}_i/\gamma_i \to 0$, $\gamma_i \to +\infty$ (because $v_i/\gamma_i^2 \to 0$ as proved previously and $v_i > \mu > 0$), we have $\int_\Omega \beta\varphi = 0, z$ which is impossible. Therefore we may assume that $v_i \|\tilde{v}_i - \hat{v}_i\|_\infty / (\gamma_i \|\tilde{v}_i\|_\infty) \to a \in (0, +\infty)$.

By standard elliptic regularity we see that $w_i \to w$, where

$$a\Delta w + \beta = 0, \qquad \int_\Omega w = 0, \qquad \|w\|_\infty = 1, \qquad \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.48}$$

Passing to the limit in (3.45) we have $\int_\Omega \beta w = 0$, which is impossible since by (3.48) we have

$$\int_\Omega \beta w = -a \int_\Omega w \cdot \Delta w = a \int_\Omega |\nabla w|^2 > 0. \tag{3.49}$$

This completes the discussion of Case 1.

Case 2: $v_i/\gamma_i \to d \in (0, +\infty)$. (Note that $d > 0$ because $v_i/\gamma_i \geqslant \eta_0 > 0$.) Since $v_i/\gamma_i^2 \to 0$, we see that $\gamma_i \to +\infty$. For this case, $\tilde{v}_i/\gamma_i \to v_0$, $\tilde{v}_i/\|\tilde{v}_i\|_\infty \to v_0/\|v_0\|_\infty$, where v_0 is the unique solution of

$$d\Delta v_0 + v_0(\beta - v_0) = 0 \qquad \text{in } \Omega, \qquad \left.\frac{\partial v_0}{\partial n}\right|_{\partial\Omega} = 0. \tag{3.50}$$

As in Case 1, we can show that, passing to a subsequence if necessary,

$$\frac{v_i}{\gamma_i \|\tilde{v}_i\|_\infty} \|\tilde{v}_i - \hat{v}_i\|_\infty \to a \in (0, \infty), \tag{3.51}$$

and $w_i \to w$, where w satisfies

$$a\Delta w + \frac{v_0}{\|v_0\|_\infty}(\beta - v_0) = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0, \quad \int_\Omega w = 0. \tag{3.52}$$

By (3.50) and (3.52) we see that $w = \tau v_0$ for some $\tau > 0$. However, this is impossible since $\int_\Omega w = 0$. This contradiction implies that $\tilde{v}(x) \geqslant c_{34}\gamma^2/v$ for any $x \in \bar{\Omega}$ and some $c_{34} > 0$. ∎

Proof of Proposition 3.5. Part (i) follows from (1.8) and (3.27) since by Lemma 3.6, $\tilde{v} \to 1$ uniformly as $\gamma^2/v \to 0$. Part (ii) follows from (1.8), (3.27) and Lemma 3.7. ∎

Proof of Theorem 3.4. *Step* 1: $\lim_{\substack{\gamma/\sqrt{v} \to 0 \\ v \to +\infty}} b_*(\gamma, v)/\sqrt{v} = 0$.

By part (i) of Proposition 3.5,

$$\lim_{\gamma/\sqrt{v} \to 0} \frac{b_*(\gamma, v)}{b_*(\gamma, \infty)} = 1. \tag{3.53}$$

In particular, $b_*(\gamma, v)/b_*(\gamma, \infty)$ is uniformly bounded if $\gamma/\sqrt{v} \ll 1$. By the definition of $b_*(\gamma, \infty)$, it is easy to show that $b_*(\gamma, v)/(1 + \gamma)$ is uniformly bounded for any $\gamma \geqslant 0$. Therefore, if $\gamma/\sqrt{v} \to 0$ and $v \to +\infty$,

$$\frac{b_*(\gamma, v)}{\sqrt{v}} = \frac{b_*(\gamma, v)}{b_*(\gamma, \infty)} \frac{b_*(\gamma, \infty)}{1+\gamma} \frac{1+\gamma}{\sqrt{v}} \to 0. \tag{3.54}$$

Step 2: $\lim_{\substack{\gamma/\sqrt{v} \to \infty \\ v \to +\infty}} b_*(\gamma, v)/\sqrt{v} = 0$.

If $\gamma/v \to +\infty$, by Theorem 1.5 we have $b_*(\gamma, v) \to 1$, which implies that $b_*(\gamma, v)/\sqrt{v} \to 0$; hence we may assume that $\gamma \leqslant v/\eta$ for some $\eta > 0$. Since $\gamma/\sqrt{v} \to +\infty$, we apply part (ii) of Proposition 3.5 to see that $(\gamma^2/v)(b_*(\gamma, v)/b_*(\gamma, \infty))$ is uniformly bounded. Hence, if $\gamma/\sqrt{v} \to +\infty$ and $v \to +\infty$,

$$\frac{b_*(\gamma, v)}{\sqrt{v}} = \left(\frac{\gamma^2}{v} \frac{b_*(\gamma, v)}{b_*(\gamma, \infty)}\right) \frac{b_*(\gamma, \infty)}{\gamma} \frac{\sqrt{v}}{\gamma} \to 0. \tag{3.55}$$

Step 3. In view of Steps 1 and 2, we may assume that $v \to +\infty$ and $\gamma/\sqrt{v} \to s \in (0, +\infty)$.

CLAIM. *$\tilde{v} \to 1 + s^2|\Omega|^{-1} \int_\Omega |\nabla \varphi^*|^2$ uniformly, where φ^* is defined as in Theorem 3.4 and $\tilde{v}$ is defined by (1.6).*

By Lemma 3.7 we see that there exists $c > 0$, independent of v and γ, such that $c \leqslant \tilde{v}(x) \leqslant 1/c$ for any $x \in \bar{\Omega}$. Since $v \to +\infty$ and $\gamma/v \to 0$, by standard elliptic regularity, $\tilde{v} \to \bar{v}$ uniformly, where $\bar{v}$ is some positive constant. To establish the claim it is thus enough to show that

$$\bar{v} = 1 + \frac{s^2}{|\Omega|} \int_\Omega |\nabla \varphi^*|^2. \tag{3.56}$$

Set $\hat{v} = |\Omega|^{-1} \int_\Omega \tilde{v}$. We first prove the following estimate: $\exists c > 0$, independent of v and γ, such that

$$c \leqslant \sqrt{v} \|\tilde{v} - \hat{v}\|_{L^\infty(\Omega)} \leqslant \frac{1}{c}. \tag{3.57}$$

To show (3.57), we rewrite (1.6) as

$$(\sqrt{v}\|\tilde{v} - \hat{v}\|_\infty)\Delta\left(\frac{\tilde{v} - \hat{v}}{\|\tilde{v} - \hat{v}\|_\infty}\right) + \frac{\tilde{v}(1-\tilde{v})}{\sqrt{v}} + \frac{\gamma}{\sqrt{v}}\beta\tilde{v} = 0 \qquad \text{in } \Omega. \tag{3.58}$$

If $\sqrt{v}\|\tilde{v} - \hat{v}\|_\infty \to 0$, multiplying (3.58) by any $\varphi \in C^2(\bar{\Omega})$ such that $\frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0$, and integrating over Ω, we have

$$(\sqrt{v}\|\tilde{v} - \hat{v}\|_\infty) \int_\Omega \frac{\tilde{v} - \hat{v}}{\|\tilde{v} - \hat{v}\|_\infty} \Delta\varphi + \frac{1}{\sqrt{v}} \int_\Omega \tilde{v}(1-\tilde{v})\varphi + \frac{\gamma}{\sqrt{v}} \int_\Omega \beta\tilde{v}\varphi = 0. \tag{3.59}$$

Passing to the limit in (3.59), we get $\int_\Omega \beta\varphi = 0$ for any $\varphi \in C^2(\bar{\Omega})$ and $\frac{\partial \varphi}{\partial n}|_{\partial\Omega} = 0$, which is clearly impossible. Hence $\sqrt{v}||\tilde{v} - \hat{v}||_\infty \not\to 0$.

If $\sqrt{v}||\tilde{v} - \hat{v}||_\infty \to +\infty$, by (3.58) we see that $(\tilde{v} - \hat{v})/||\tilde{v} - \hat{v}||_\infty \to \psi$, where ψ satisfies $\Delta\psi = 0$ in Ω, $\frac{\partial \psi}{\partial n}|_{\partial\Omega} = 0$, $\int_\Omega \psi = 0$, $||\psi||_\infty = 1$. However, such a ψ does not exist. This proves (3.57).

By (3.57), we may assume that, passing to some subsequence if necessary, $\sqrt{v}||\tilde{v} - \hat{v}||_\infty \to \tau \in (0, \infty)$. Again by (3.58), we may assume that $(\tilde{v} - \hat{v})/||\tilde{v} - \hat{v}||_\infty \to \psi$, where ψ satisfies

$$\tau\Delta\psi + s\bar{v}\beta = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0, \quad \int_\Omega \psi = 0. \tag{3.60}$$

By the definition of φ^*, we have $\psi = (s\bar{v}/\tau)\varphi^*$. Now integrating (1.6), after some rearrangement we find that

$$\int_\Omega \tilde{v}(1 - \tilde{v}) + \frac{\gamma}{\sqrt{v}}(\sqrt{v}||\tilde{v} - \hat{v}||_\infty) \int_\Omega \beta \frac{\tilde{v} - \hat{v}}{||\tilde{v} - \hat{v}||_\infty} = 0. \tag{3.61}$$

Passing to the limit in (3.61), we have

$$\bar{v}(1 - \bar{v})|\Omega| + s\tau \int_\Omega \beta\psi = 0. \tag{3.62}$$

Since $\psi = (s\bar{v}/\tau)\varphi^*$, from (3.62)

$$\bar{v} = 1 + \frac{s^2}{|\Omega|}\int_\Omega \beta\varphi^* = 1 + \frac{s^2}{|\Omega|}\int_\Omega |\nabla\varphi^*|^2, \tag{3.63}$$

which proves (3.56) and thus the claim.

Step 4. We show that $\lim_{\substack{\gamma/\sqrt{v}\to s \\ v\to+\infty}} b_*(\gamma, v)/\sqrt{v} = f(s)$, where

$$f(s) := \frac{(\max_{\bar{\Omega}} \beta)s}{1 + (|\Omega|^{-1}\int_\Omega |\nabla\varphi^*|^2)s^2}. \tag{3.64}$$

Recall that

$$\frac{b_*(\gamma, v)}{\sqrt{v}} = -\inf_{\substack{\varphi\neq 0 \\ \varphi\in H^1}} \frac{\int_\Omega[\frac{\mu}{\sqrt{v}}|\nabla\varphi|^2 - (\frac{1}{\sqrt{v}} + \frac{\gamma}{\sqrt{v}}\beta)\varphi^2]}{\int_\Omega \tilde{v}\varphi^2}. \tag{3.65}$$

Since $\tilde{v} \to 1 + s^2|\Omega|^{-1}\int_\Omega |\nabla\varphi^*|^2$, $v \to +\infty$ and $\gamma/\sqrt{v} \to s$, we see that $b_*(\gamma, v)/\sqrt{v} \to f(s)$ uniformly in v and $\gamma/\sqrt{v}$. For $0 \leqslant s < +\infty$, it is easy to check that f attains the maximum $\max_{\bar{\Omega}} \beta|\Omega|^{1/2}/(2||\nabla\varphi^*||_2)$ at $s = |\Omega|^{1/2}/||\nabla\varphi^*||_2$. This proves both parts (i) and (ii) of the theorem. ∎

Remark 3.8. If $\int_\Omega \beta(x)\,dx > 0$, then Theorem 3.4 fails. More precisely, if $\int_\Omega \beta > 0$, then there exists some positive constant c such that $b_* \leqslant c$ for any γ and v.

Remark 3.9. It would be interesting to know whether for any $v > \mu$, $b_*(\gamma, v)$ has a unique local maximum (which would then be the global maximum) for $0 \leqslant \gamma < \infty$. It would also be interesting to know the rate of decay of $b_*(\gamma, v)$ as $\gamma \to \infty$.

4. COEXISTENCE OF POSITIVE STEADY STATES

In this section, we shall discuss the coexistence of positive steady states to (1.2). Theorem 1.6 is concerned with the case where both semi-trivial steady states $(\tilde{u}, 0)$ and $(0, \tilde{v})$ are unstable, i.e., $b < b_*$ and $c < c_*$, respectively. Since Theorem 1.6 follows from Propositions 1.2, 1.4 and the fact that (1.2) is a monotone system (see, e.g., [13]), we omit its proof. Of course in addition, if $b > b_*$ and $c > c_*$, a (not necessarily stable) coexistence state also exists. If $1 < b < b_*$ and $c > 1$, from Propositions 1.2 and 1.4, $(\tilde{u}, 0)$ is stable and $(0, \tilde{v})$ is unstable. However, it is interesting to note that nonetheless $(\tilde{u}, 0)$ may not be the global attractor for the interior; as can be seen from Fig. 3 there will be both stable and unstable coexistence states if μ and γ are small enough. This follows from Theorem 4.1, and the main purpose of this section is to prove Theorem 4.1 and give some applications to the coexistence of steady states. Throughout this section, we shall assume that $b = c > 1$ and use b as the bifurcation parameter.

By the local bifurcation theorem [1], positive steady states of (1.2) bifurcate from $(u, v) = (0, \tilde{v})$ at $b = b_*$. Moreover, all positive steady states of (1.2) near $(u, v, b) = (0, \tilde{v}, b_*)$ can be represented as

$$(u(s), v(s), b(s)) = (s\varphi + O(s^2), \tilde{v} + s\psi + O(s^2), b_* + s\lambda(0) + O(s^2)) \tag{4.1}$$

for $0 < s \ll 1$, where φ, ψ satisfy

$$\mu\Delta\varphi + [1 + \gamma\beta(x) - b_*\tilde{v}]\varphi = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.2}$$

$$v\Delta\psi + \psi(1 + \gamma\beta - 2\tilde{v}) = b_*\tilde{v}\varphi \qquad \text{in } \Omega, \quad \left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.3}$$

It is crucial to determine the sign of $\lambda(0)$, which will in turn yield the bifurcation direction and stability of solution branch near $(u, v, b) = (0, \tilde{v}, b_*)$. The following result gives a complete understanding of $\lambda(0)$ when $0 < \gamma \ll 1$. In general, determining the sign of $\lambda(0)$ is a difficult problem.

THEOREM 4.1. *There exists* $\bar{d} \in (0, v)$ *depending only on* v, β *and* Ω, *and* $\bar{\gamma} > 0$ *depending on* μ, v, β *and* Ω *such that if* $0 < \gamma \leqslant \bar{\gamma}$, $\mathrm{sign}(\lambda(0)) = \mathrm{sign}(\mu - \bar{d})$.

Proof. As $\gamma \to 0$, we know that $\tilde{v} \to 1$, $b_* \to 1$ and $\varphi \to 1$. Therefore, it is easy to check that $\psi \to -1$ as $\gamma \to 0$. Consider henceforth the range $0 < \gamma \ll 1$, and recall that $b_* = 1 + b_2\gamma^2 + O(\gamma^3)$. Also

$$\varphi = 1 + \gamma\varphi_1 + \gamma^2\varphi_2 + o(\gamma^2), \tag{4.4}$$

$$\psi = -1 + \gamma\psi_1 + \gamma^2\psi_2 + o(\gamma^2), \tag{4.5}$$

$$\tilde{v} = 1 + \gamma\theta_1 + \gamma^2\theta_2 + o(\gamma^2), \tag{4.6}$$

where θ_1, θ_2, φ_1, φ_2, ψ_1 and ψ_2 are given by

$$v\Delta\theta_1 - \theta_1 = -\beta \qquad \text{in } \Omega, \quad \left.\frac{\partial\theta_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.7}$$

$$v\Delta\theta_2 - \theta_2 = \theta_1(\theta_1 - \beta) \qquad \text{in } \Omega, \quad \left.\frac{\partial\theta_2}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.8}$$

$$\mu\Delta\varphi_1 = \theta_1 - \beta \qquad \text{in } \Omega, \qquad \int_\Omega \varphi_1 = 0, \quad \left.\frac{\partial\varphi_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.9}$$

$$\mu\Delta\varphi_2 = \theta_2 + b_2 + \varphi_1(\theta_1 - \beta) \qquad \text{in } \Omega, \quad \left.\frac{\partial\varphi_2}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.10}$$

$$v\Delta\psi_1 - \psi_1 = \varphi_1 + \beta - \theta_1 \qquad \text{in } \Omega, \quad \left.\frac{\partial\psi_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.11}$$

$$\begin{aligned} v\Delta\psi_2 - \psi_2 = {}& \theta_1\varphi_1 - \theta_2 + b_2 - \beta\psi_1 \\ & + 2\theta_1\psi_1 + \varphi_2 \qquad \text{in } \Omega, \quad \left.\frac{\partial\psi_2}{\partial n}\right|_{\partial\Omega} = 0. \end{aligned} \tag{4.12}$$

By using (4.1) and the equation for u, we find that

$$\lambda(0)\int_\Omega \tilde{v}\varphi^2 = -\int_\Omega \varphi^2(\varphi + b_*\psi). \tag{4.13}$$

For $\gamma \ll 1$, direct calculation gives

$$\int_\Omega \varphi^2(\varphi + b_*\psi) = \gamma \int_\Omega (\varphi_1 + \psi_1) + \gamma^2 \int_\Omega (\varphi_2 + \psi_2 - b_2 + 2\varphi_1^2 + 2\varphi_1\psi_1) + o(\gamma^2). \quad (4.14)$$

Integrating (4.11) we have

$$\int_\Omega (\psi_1 + \varphi_1 + \beta - \theta_1) = 0. \quad (4.15)$$

Since $\int_\Omega(\beta - \theta_1) = 0$, it follows that $\int_\Omega(\varphi_1 + \psi_1) = 0$. That is, the first term on the right-hand side of (4.14) vanishes. It remains to calculate the second term: integrating (4.12) we get

$$\int_\Omega (\varphi_2 + \psi_2) = \int_\Omega (-\theta_1\varphi_1 + \theta_2 - b_2 + \beta\psi_1 - 2\theta_1\psi_1). \quad (4.16)$$

From (4.7) and (4.9),

$$\varphi_1(x) = \frac{v}{\mu}\theta_1, \quad (4.17)$$

and from (4.10) and (4.8), respectively,

$$\int_\Omega b_2 = \int_\Omega (\beta - \theta_1)\varphi_1 - \int_\Omega \theta_2 \quad (4.18)$$

and

$$\int_\Omega \theta_2 = \int_\Omega \theta_1(\beta - \theta_1). \quad (4.19)$$

It follows from (4.14), (4.16), (4.18) and(4.19) that

$$-\mathrm{sign}[\lambda(0)] = \mathrm{sign}\Big[\int_\Omega (\theta_1\varphi_1 + 3\beta\theta_1 - 3\theta_1^2 - 2\beta\varphi_1 + \beta\psi_1 - 2\theta_1\psi_1 + 2\varphi_1^2 + 2\varphi_1\psi_1)\Big]. \quad (4.20)$$

Multiplying (4.7) by ψ_1, multiplying (4.11) by θ_1, subtracting them and integrating, we have

$$\int_\Omega \beta\psi_1 = \int_\Omega (-\theta_1\varphi_1 + \theta_1^2 - \beta\theta_1). \quad (4.21)$$

It follows from (4.20) and (4.21) that

$$-\mathrm{sign}[\lambda(0)] = \mathrm{sign}\left[\int_\Omega (\theta_1 - \varphi_1)(\beta - \theta_1 - \varphi_1 - \psi_1)\right]. \tag{4.22}$$

By (4.11), $\varphi_1 + \psi_1 = v\Delta\psi_1 - (\beta - \theta_1)$. Therefore

$$\begin{aligned}
&\int_\Omega (\theta_1 - \varphi_1)(\beta - \theta_1 - \varphi_1 - \psi_1) \\
&\quad = \int_\Omega (\theta_1 - \varphi_1)[2(\beta - \theta_1) - v\Delta\psi_1] \\
&\quad = \left(1 - \frac{v}{\mu}\right)\int_\Omega \theta_1[2(\beta - \theta_1) - v\Delta\psi_1] \qquad \text{(by (4.17))} \\
&\quad = \left(1 - \frac{v}{\mu}\right)\int_\Omega (\beta - \theta_1)(2\theta_1 + \psi_1) \qquad \text{(by (4.11))}.
\end{aligned} \tag{4.23}$$

By (4.11) and (4.17) we have

$$v\Delta\psi_1 - \psi_1 = \beta + \left(\frac{v}{\mu} - 1\right)\theta_1 \qquad \text{in } \Omega, \quad \left.\frac{\partial\psi_1}{\partial n}\right|_{\partial\Omega} = 0, \tag{4.24}$$

which becomes on using (4.7),

$$\psi_1 = -\theta_1 + \left(\frac{v}{\mu} - 1\right)(v\Delta - 1)^{-1}\theta_1. \tag{4.25}$$

In (4.23) substitute for $(\beta - \theta_1)$ from (4.7). Then from (4.22), (4.23), and (4.25)

$$\mathrm{sign}[\lambda(0)] = \mathrm{sign}\left[\int_\Omega |\nabla\theta_1|^2 + \left(\frac{v}{\mu} - 1\right)\int_\Omega \nabla\theta_1 \cdot \nabla w\right], \tag{4.26}$$

where w satisfies

$$v\Delta w - w = \theta_1 \qquad \text{in } \Omega, \quad \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0. \tag{4.27}$$

Claim. $\int_\Omega \nabla\theta_1 \cdot \nabla w < 0$.

To prove our assertion, by (2.5) and (4.7) we have

$$\theta_1 = \sum_{i=1}^{\infty} \frac{a_i}{1 + \lambda_i v}\,\xi_i \tag{4.28}$$

in the L^2 sense. By (4.27) it is easy to check that

$$w = -\sum_{i=1}^{\infty} \frac{a_i}{(1+\lambda_i v)^2}\,\xi_i. \tag{4.29}$$

Therefore

$$\int_\Omega \nabla\theta_1 \cdot \nabla w = -\sum_{i=1}^{\infty} \frac{a_i^2 \lambda_i}{(1+\lambda_i v)^3} < 0. \tag{4.30}$$

Set

$$\bar{d} = v \Big/ \left[1 - \int_\Omega |\nabla\theta_1|^2 \Big/ \int_\Omega \nabla\theta_1 \cdot \nabla w\right] \in (0, v). \tag{4.31}$$

We see that $\lambda(0) < 0$ where $0 < \mu < \bar{d}$, and $\lambda(0) > 0$ when $\bar{d} < \mu < v$. This proves Theorem 4.1. ∎

Remark 4.2. Similar bifurcation analysis can be carried out for the general case $c = (1-\eta) + \eta b$, where η is any fixed positive number and b is still the bifurcation parameter. Note that to avoid technicality we only discuss the case $\eta = 1$ in Theorem 4.1.

Finally, we give some applications of Theorem 4.1 to the coexistence of steady states of (1.2).

THEOREM 4.3. *Set*

$$\Lambda = \{b = c > 1\text{: (1.2) } \textit{has a coexistence positive steady state}\}. \tag{4.32}$$

Then $\Lambda \supset (b_*, +\infty)$. *Moreover, if* $\gamma \ll 1$ *and* $0 < \mu \leqslant \bar{d}$, $\exists \underline{b} \in (1, b_*)$ *such that* $\Lambda \supset [\underline{b}, +\infty)$, *and* (1.2) *has at least one stable positive steady-state solution for any* $b \in [\underline{b}, b_*)$.

Remark 4.4. It is interesting to note that when $\underline{b} < b < b_*$, $(0, \tilde{v})$ is unstable and $(\tilde{u}, 0)$ is stable. However, for $\gamma \ll 1$ and $\mu \leqslant \bar{d} < v$, $(\tilde{u}, 0)$ is *not* the global attractor and surprisingly, there could be stable steady states for this range of b.

Proof of Theorem 4.3. We know that solutions bifurcate from $(u, v, b) = (0, \tilde{v}, b_*)$. By standard global bifurcation techniques we can show that there is a global branch of steady states connecting $(0, \tilde{v}, b_*)$ and $(u_\infty, v_\infty, \infty)$ for some u_∞, v_∞. This in particular implies that $\Lambda \supset (b_*, +\infty)$. When $\gamma \ll 1$ and $0 < \mu \leqslant \bar{d}$ we know that $\lambda(0) < 0$ (Theorem 4.1). By the standard local bifurcation theorem

[1] and exchange of stability of $(0, \tilde{v})$ at $b = b_*$, we see that $\exists \underline{b} \in (1, b_*)$ such that (1.2) has at least one stable solution for $\underline{b} \leqslant b < b_*$. This together with the global bifurcation argument implies that $\Lambda \supset [\underline{b}, +\infty)$. Since these bifurcation techniques are rather standard, we do not give the details here. ∎

5. THE GLOBAL ATTRACTIVITY OF $(\tilde{u}, 0)$

Consider the set

$$\Sigma = \{(b, c) \in R_+^2\colon\ (\tilde{u}, 0) \text{ is the global attractor of } (1.2)\}. \tag{5.1}$$

By Theorem 1.1, [3], we know that $\Sigma \supset (0, 1] \times [1, +\infty)$. Theorem 1.7 is a direct consequence of (5.1), Proposition 1.2, Theorem 1.3, Proposition 1.4 and Theorem 1.5.

To prove Theorem 1.8, the following preliminary results are needed.

LEMMA 5.1. *$\exists c_1 > 0$ and $c_2 > 0$, independent of γ and v, such that if $\min\{\gamma, v/\gamma\} \geqslant c_1$, then $b_*(\gamma, v) \geqslant c_2 \min\{\gamma, v/\gamma\}$.*

Proof. We start by noting that

$$\lim_{\gamma \to \infty} \frac{b_*(\gamma, \infty)}{\gamma} = \max_{\bar{\Omega}} \beta. \tag{5.2}$$

The proof of (5.2) is essentially the same as in [8].

To establish Lemma 5.1, we argue by contradiction. Since $b_*(\gamma, v) \geqslant 1$, we may suppose that there are sequences $\{\gamma_i\}_{i=1}^{\infty}$, $\{v_i\}_{i=1}^{\infty}$ such that $\gamma_i \to +\infty$, $v_i/\gamma_i \to +\infty$, and $b_*(\gamma_i, v_i)/\min\{\gamma_i, v_i/\gamma_i\} \to 0$.

If $\gamma_i^2/v_i \to 0$, part (i) of Proposition 3.5 implies that $b_*(\gamma_i, v_i)/b_*(\gamma_i, \infty) \to 1$. Noting that

$$\frac{b_*(\gamma_i, v_i)}{\min\{\gamma_i, v_i/\gamma_i\}} = \frac{b_*(\gamma_i, v_i)}{\gamma_i} \to 0, \tag{5.3}$$

we see that $b_*(\gamma_i, \infty)/\gamma_i \to 0$ as $\gamma_i \to +\infty$, which contradicts (5.2). Therefore, we may assume that $\gamma_i^2/v_i \geqslant c_3$ for some $c_3 > 0$. Since $\gamma_i/v_i \to 0$, (ii) of Proposition 3.5 implies that

$$\frac{\gamma_i^2}{v_i} \frac{b_*(\gamma_i, v_i)}{b_*(\gamma_i, \infty)} \geqslant c_4 > 0. \tag{5.4}$$

By (5.2) and (5.4), we see that $\gamma_i b_*(\gamma_i, v_i)/v_i \geqslant c > 0$, which contradicts the following assertion.

CLAIM. $b_*(\gamma_i, v_i)\gamma_i/v_i \to 0$.

To prove this assertion, observe that if $\gamma_i^2 \leqslant v_i$, then

$$\frac{b_*(\gamma_i, v_i)\gamma_i}{v_i} \leqslant \frac{b_*(\gamma_i, v_i)}{\gamma_i} = \frac{b_*(\gamma_i, v_i)}{\min\{\gamma_i, v_i/\gamma_i\}} \to 0; \tag{5.5}$$

if $\gamma_i^2 \geqslant v_i$, then

$$\frac{b_*(\gamma_i, v_i)\gamma_i}{v_i} = \frac{b_*(\gamma_i, v_i)}{\min\{\gamma_i, v_i/\gamma_i\}} \to 0. \tag{5.6}$$

This finishes the proof of Lemma 5.1. ∎

COROLLARY 5.2. *There exist* $c_5 > 0$ *and* $c_6 > 0$, *independent of* γ *and* v, *such that if* $\min\{\gamma, v/\gamma\} \geqslant c_5$ *and* $b \leqslant c_6 \min\{\gamma, v/\gamma\}$, *then* $(0, \tilde{v})$ *is unstable.*

Proof. This follows from Lemma 5.1 and Proposition 1.4. ∎

LEMMA 5.3. *Suppose that* $(0, \tilde{v})$ *is unstable, and* (1.2) *has a positive steady-state solution with parameters* $(\hat{b}, \hat{c})$. *Then for any* $0 < c < \hat{c}$, (1.2) *has at least one positive steady-state solution with parameters* $(\hat{b}, c)$.

Proof. The system is competitive, and we may therefore use the sub-super-solution method, see [6]. Denote by $(\hat{u}, \hat{v})$ the positive steady-state solution of (1.2) with $(b, c) = (\hat{b}, \hat{c})$. Since $(0, \tilde{v})$ is unstable, the following linear eigenvalue problem:

$$\mu\Delta\varphi + \varphi(1 + \gamma\beta - \hat{b}\tilde{v}) = -\lambda_1\varphi, \qquad \varphi > 0, \quad \left.\frac{\partial\varphi}{\partial n}\right|_{\partial\Omega} = 0 \tag{5.7}$$

has a solution with $\lambda_1 < 0$. Set

$$(\bar{u}, \underline{u}) = (\hat{u}, \delta\varphi), \qquad (\bar{v}, \underline{v}) = (\tilde{v}, \hat{v}). \tag{5.8}$$

It is easy to check that $\bar{u} > \underline{u}$ provided that $\delta > 0$ is sufficiently small, and $\bar{v} \geqslant \underline{v}$ since we have $\hat{v} \leqslant \tilde{v}$. Moreover, if δ is small, one can check that $(\bar{u}, \underline{u})$, $(\bar{v}, \underline{v})$ are super-sub-solutions of (1.2) with $(b, c) = (\hat{b}, c)$ for any $0 < c \leqslant \hat{c}$. This implies that for any $c < \hat{c}$, (1.2) with $b = \hat{b}$ has a positive steady-state solution. ∎

COROLLARY 5.4. $\exists c_5 > 0$ *and* $c_6 > 0$, *independent of* γ *and* v, *such that if* $\min\{\gamma, v/\gamma\} \geqslant c_5$, $\hat{b} < c_6 \min\{\gamma, v/\gamma\}$, *and* (1.2) *has a positive steady-state solution with* $(b, c) = (\hat{b}, \hat{c})$, *then* (1.2) *has at least one positive steady-state solution for* $b = \hat{b}$ *and any* c *such that* $0 < c < \hat{c}$.

Proof. This follows from Corollary 5.2 and Lemma 5.3. ∎

LEMMA 5.5. *$\forall \eta > 0$, $\exists c_7(\eta) > 0$ and $c_8(\eta) > 0$, independent of γ, v, b and c, such that if $\min\{\gamma, v/\gamma\} \geqslant c_7(\eta)$, $c \geqslant \eta$, and $b \leqslant c_8(\eta)\min\{\gamma, v/\gamma\}$, then (1.2) has no positive steady-state solution.*

Proof. We argue by contradiction: suppose that $\exists \eta_0 > 0$ and sequences $\{b_i\}$, $\{c_i\}$, $\{\gamma_i\}$, $\{v_i\}$, $\{u_i\}$, $\{v_i\}$ such that $c_i \geqslant \eta_0$, $\gamma_i \to +\infty$, $v_i/\gamma_i \to +\infty$, $b_i/\min\{\gamma_i, v_i/\gamma_i\} \to 0$, and (u_i, v_i) are positive solutions of

$$\begin{cases} \mu\Delta u_i + u_i(1 + \gamma_i\beta - u_i - b_i v_i) = 0 & \text{in } \Omega, \\ v_i\Delta v_i + v_i(1 + \gamma_i\beta - c_i u_i - v_i) = 0 & \text{in } \Omega, \\ \frac{\partial u_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

We may assume that $c_i = \eta_0$. For otherwise, by Corollary 5.4, we may use the parameters $(b, c, v, \gamma) = (b_i, \eta_0, v_i, \gamma_i)$ and work with the corresponding solutions (u_i, v_i) of (5.9).

Step 1: *$v_i/\|v_i\|_\infty \to 1$ uniformly*. Set $\varphi_i = v_i/\|v_i\|_\infty$. Then φ_i satisfies

$$\Delta\varphi_i + \varphi_i\left(\frac{1}{v_i} + \frac{\gamma_i}{v_i}\beta - \eta_0\frac{\gamma_i}{v_i}\frac{u_i}{\gamma_i} - \frac{\gamma_i}{v_i}\frac{v_i}{\gamma_i}\right) = 0, \qquad \left.\frac{\partial\varphi_i}{\partial n}\right|_{\partial\Omega} = 0. \tag{5.10}$$

By the Maximum Principle, $\|u_i\|_\infty \leqslant k\gamma_i$ and $\|v_i\|_\infty \leqslant k\gamma_i$ for some k independent of i. By elliptic regularity, $\varphi_i \to \varphi$ in $C^1(\bar{\Omega})$, where φ satisfies $\Delta\varphi = 0$, $\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = 0$, $\|\varphi\|_\infty = 1$. Hence $\varphi = 1$, i.e., $\varphi_i \to 1$ uniformly.

Step 2: $b_i\|v_i\|_\infty/\gamma_i \to 0$. Observe that v_i is a sub-solution of

$$v_i\Delta w_i + w_i(1 + \gamma_i\beta - w_i) = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial w_i}{\partial n}\right|_{\partial\Omega} = 0. \tag{5.11}$$

Hence $v_i \leqslant w_i$. There are two possibilities to consider:

(i) $\gamma_i^2/v_i \to 0$. For this case,

$$\frac{b_i}{\gamma_i} = \frac{b_i}{\min\{\gamma_i, v_i/\gamma_i\}} \to 0. \tag{5.12}$$

Since $\gamma_i^2/v_i \to 0$, by Lemma 3.6 we see that

$$\|w_i\|_\infty \leqslant 1 + c_{18}\frac{\gamma_i^2}{v_i} \leqslant k \tag{5.13}$$

for some constant k. Hence

$$\frac{b_i\|v_i\|_\infty}{\gamma_i} \leqslant \frac{b_i\|w_i\|_\infty}{\gamma_i} \leqslant k\frac{b_i}{\gamma_i} \to 0. \tag{5.14}$$

(ii) $\gamma_i^2/v_i \geqslant \xi^2 > 0$ for some $\xi > 0$. Since $\xi\sqrt{v_i} \leqslant \gamma_i \leqslant v_i/\xi$, by Lemma 3.7 we see that $\|w_i\|_\infty \leqslant k\gamma_i^2/v_i$. Hence

$$\frac{b_i\|v_i\|_\infty}{\gamma_i} \leqslant \frac{b_i\|w_i\|_\infty}{\gamma_i} \leqslant k\frac{b_i\gamma_i}{v_i}. \tag{5.15}$$

We claim that $b_i\gamma_i/v_i \to 0$: if $\gamma_i^2 \leqslant v_i$, then

$$\frac{b_i\gamma_i}{v_i} = \frac{b_i}{\gamma_i}\frac{\gamma_i^2}{v_i} \leqslant \frac{b_i}{\gamma_i} = \frac{b_i}{\min\{v_i, v_i/\gamma_i\}} \to 0; \tag{5.16}$$

if $\gamma_i^2 \geqslant v_i$, then

$$\frac{b_i\gamma_i}{v_i} = \frac{b_i}{\min\{v_i, v_i/\gamma_i\}} \to 0. \tag{5.17}$$

Hence $b_i\|v_i\|_\infty/\gamma_i \to 0$.

Step 3. $\forall \varepsilon > 0$, $u_i/\gamma_i \geqslant (\beta - \varepsilon)_+$ *for sufficiently large* i. Note that u_i satisfies

$$-\mu\Delta u_i = u_i\left[1 + \gamma_i\left(\beta - \frac{b_i v_i}{\gamma_i}\right) - u_i\right] \geqslant u_i\left[1 + \gamma_i\left(\beta - \frac{\varepsilon}{2}\right) - u_i\right], \tag{5.18}$$

where the last inequality follows from Step 2, provided that $i \gg 1$. Hence $u_i \geqslant w_i$, where w_i satisfies

$$\mu\Delta w_i + w_i\left[1 + \gamma_i\left(\beta - \frac{\varepsilon}{2}\right) - w_i\right] = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial w_i}{\partial n}\right|_{\partial\Omega} = 0. \tag{5.19}$$

By Proposition A.1 in the appendix,

$$\frac{w_i}{\gamma_i} \geqslant \left(\beta - \frac{\varepsilon}{2}\right)_+ - c\left(\frac{\mu}{\gamma_i}\right)^{1/3}. \tag{5.20}$$

Hence, for sufficiently large i,

$$\frac{u_i}{\gamma_i} \geqslant \frac{w_i}{\gamma_i} \geqslant (\beta - \varepsilon)_+. \tag{5.21}$$

Step 4. Integrating the equation for v_i, and dividing it by $\gamma_i\|v_i\|_\infty$, we have

$$\frac{1}{\gamma_i}\int_\Omega \frac{v_i}{\|v_i\|_\infty} + \int_\Omega \frac{v_i}{\|v_i\|_\infty}\beta \geqslant \eta_0 \int_\Omega \frac{v_i}{\|v_i\|_\infty}\frac{u_i}{\gamma_i} \geqslant \eta_0 \int_\Omega \frac{v_i}{\|v_i\|_\infty}(\beta - \varepsilon)_+. \tag{5.22}$$

Passing to the limit in (5.22), as $v_i/\|v_i\|_\infty \to 1$ and $\int \beta = 0$, we have $\int_\Omega (\beta - \varepsilon)_+ \leqslant 0$ for any $\varepsilon > 0$, which is obviously a contradiction. This completes the proof of Lemma 5.5. ∎

Proof of Theorem 1.8. It suffices to check the following for $c \geqslant \varepsilon$ and $b \leqslant \varepsilon^{-1}$.

(a) $(\tilde{u}, 0)$ *is stable*. By (ii) of Proposition 2.2, $\forall \varepsilon > 0$, $\exists c_1(\varepsilon) > 0$ such that if $\min\{\gamma, v/\gamma\} \geqslant c_1(\varepsilon)$, then $c_* < \varepsilon$. Then by Proposition 1.2, if $c \geqslant \varepsilon$, $(\tilde{u}, 0)$ is stable.

(b) $(0, \tilde{v})$ *is unstable*. $\forall \varepsilon > 0$, by Lemma 5.1, if $\min\{\gamma, v/\gamma\} \geqslant \max\{c_1, 1/(c_2\varepsilon)\}$, we have $b \leqslant \varepsilon^{-1} \leqslant \min\{\gamma, v/\gamma\} c_2 < b_*(\gamma, v)$, which implies that $(0, \tilde{v})$ is unstable (Proposition 1.4).

(c) *Equation* (1.2) *has no positive steady-state solution*. This follows easily from Lemma 5.5.

Since (1.2) is a monotone system, we know that (a)–(c) imply that $(\tilde{u}, 0)$ is the global attractor for (1.2). ∎

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APPENDIX A

Let $v(x)$ be the unique positive solution of

$$d\Delta v + v(1 + \gamma\beta(x) - v) = 0 \qquad \text{in } \Omega, \quad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.1}$$

PROPOSITION A.1. *Suppose that* (H1)–(H3) *hold except that the condition* $\int_\Omega \beta = 0$ *is relaxed. Then* $\exists c_1 > 0$ *and* $c_2 > 0$ *large such that if* $\gamma/d \geqslant c_1$ *and* $\gamma \geqslant c_1$,

$$\left\| \frac{v}{\gamma} - \beta_+ \right\|_{L^\infty(\Omega)} \leqslant c_2 \left(\frac{d}{\gamma}\right)^{1/3}. \tag{A.2}$$

Proof. The proof is based on a super-sub-solution method [4], and we will construct explicit weak super-sub-solutions to (A.1).

Set

$$\frac{\bar{v}}{\gamma} = \frac{\sqrt{\beta^2 + \tau_1(\frac{d}{\gamma})^{2/3}} + \beta}{2} + z\left(x; \frac{d}{\gamma}, \tau_2\right), \tag{A.3}$$

where $\tau_1 > 0$, $\tau_2 > 0$ are to be determined, and z is defined by

$$z\left(x; \frac{d}{\gamma}, \tau_2\right) = \begin{cases} (\frac{d}{\gamma})^{1/3} \quad \exp[-\tau_2 \operatorname{dist}(x, \partial\Omega)(\frac{\gamma}{d})^{1/3}] \times \\ \qquad \times k_{\delta_0}(\operatorname{dist}(x, \partial\Omega)) & \text{if } \operatorname{dist}(x, \partial\Omega) \leqslant \delta_0, \\ 0 & \text{if } \operatorname{dist}(x, \partial\Omega) \geqslant \delta_0, \end{cases} \tag{A.4}$$

where $\delta_0 > 0$ small is chosen such that $\mathrm{dist}(x, \partial\Omega)$ is C^2 as long as $\mathrm{dist}(x, \partial\Omega) \leqslant \delta_0$. Here $k_\delta(\cdot) \in C^2(\mathbf{R}, [0, 1])$ and

$$k_\delta(\ell) = \begin{cases} 1, & |\ell| \leqslant \frac{\delta}{2}, \\ 0, & |\ell| > \delta. \end{cases} \tag{A.5}$$

The role of z is to ensure $\frac{\partial \bar{v}}{\partial n} \geqslant 0$ on $\partial\Omega$. Define the operator L by setting

$$Lv = \frac{d}{\gamma}\Delta v + v\left(\frac{1}{\gamma} + \beta - v\right). \tag{A.6}$$

We need to check that $L(\bar{v}/\gamma) \leqslant 0$ in Ω:

$$L\left(\frac{\bar{v}}{\gamma}\right) \leqslant \frac{d}{\gamma}\Delta\left(\frac{\sqrt{\beta^2 + \tau_1(\frac{d}{\gamma})^{2/3}} + \beta}{2}\right) + \frac{d}{\gamma}\Delta z + \frac{1}{\gamma}\frac{\bar{v}}{\gamma} - \frac{\tau_1}{4}\left(\frac{d}{\gamma}\right)^{2/3}. \tag{A.7}$$

It is straightforward to check that

$$\Delta\left(\frac{\sqrt{\beta^2 + \tau_1(\frac{d}{\gamma})^{2/3}} + \beta}{2}\right) \leqslant c_3\left(1 + \frac{1}{\tau_1^{1/2}(\frac{d}{\gamma})^{1/3}}\right), \tag{A.8}$$

$$\left|\frac{d}{\gamma}\Delta z\right| \leqslant c_4\tau_2^2\left(\frac{d}{\gamma}\right)^{2/3}, \tag{A.9}$$

$$\frac{1}{\gamma}\frac{\bar{v}}{\gamma} \leqslant c_5\frac{d}{\gamma}. \tag{A.10}$$

By (A.7)–(A.10) we see that

$$L\left(\frac{\bar{v}}{\gamma}\right) \leqslant \left(\frac{d}{\gamma}\right)^{2/3}\left[-\frac{\tau_1}{2} + c_4\tau_2^2 + \frac{c_3}{\tau_1^{1/2}} + (c_3 + c_5)\left(\frac{d}{\gamma}\right)^{1/3}\right] \leqslant 0 \tag{A.11}$$

provided that d/γ is sufficiently small and τ_1 is suitably large. Note that the choice of τ_1 depends on τ_2 at this stage. However, note that on $\partial\Omega$, $\frac{\partial}{\partial n}(\bar{v}/\gamma) \geqslant \tau_2 - c_6 \max_{\partial\Omega}|\nabla\beta|$. By choosing τ_2 and then τ_1, we see that $\bar{v}/\gamma$ satisfies $L(\bar{v}/\gamma) \leqslant 0$ in Ω and $\frac{\partial}{\partial n}(\bar{v}/\gamma) \geqslant 0$ on $\partial\Omega$ provided that d/γ is sufficiently small. That is, $\bar{v}$ is a super-solution of (A.1) where $d/\gamma \leqslant c_7$ for some $c_7 > 0$ small, where c_7 is independent of d and γ.

We now construct the sub-solution $\underline{v}$ of (A.1). Recall that $\Gamma = \{x \in \Omega : \beta(x) = 0\}$, and let $\Gamma_\delta = \{x \in \Omega : \beta(x) = \delta\}$ for $\delta > 0$ small,

$D_\delta = \{x \in \Omega : 0 < \beta(x) < \delta\}$. Let h_δ be the unique solution of

$$\begin{cases} \Delta h_\delta = 0 & \text{in } D_\delta, \\ h_\delta = 0 & \text{on } \Gamma, \\ h_\delta = 1 & \text{on } \Gamma_\delta. \end{cases} \tag{A.12}$$

Since $\nabla\beta$ does not vanish on Γ, $\exists c_8$ and $c_9 > 0$ such that $c_8 \operatorname{dist}(x, \Gamma) \leqslant \beta(x) \leqslant c_9 \operatorname{dist}(x, \Gamma)$. Then $\|\nabla h_\delta\|_{L^\infty(D_\delta)} \leqslant c_{10}/\delta$ for some positive constant c_{10} provided that δ is sufficiently small. This implies that

$$h_\delta(x) \leqslant \frac{c_{11}}{\delta} \operatorname{dist}(x, \Gamma) \leqslant \frac{c_{12}}{\delta} \beta(x)$$

for every $x \in \bar{D}_\delta$ and some constants c_{11} and c_{12}. Now set

$$\frac{\underline{v}}{\gamma} = \begin{cases} 0 & \text{if } \beta(x) \leqslant 0, \\ (\frac{d}{\gamma})^{1/3} h_{\tau_3(\frac{d}{\gamma})^{1/3}} & \text{if } 0 \leqslant \beta(x) \leqslant \tau_3(\frac{d}{\gamma})^{1/3}, \\ \beta - (\tau_3 - 1)(\frac{d}{\gamma})^{1/3} - \frac{z(x, (\frac{d}{\gamma})^{1/3}, \tau_4)}{2} & \text{if } \beta \geqslant \tau_3(\frac{d}{\gamma})^{1/3}. \end{cases} \tag{A.13}$$

For $\beta(x) \leqslant 0$, $L\underline{v} = 0$. On Γ, since $\frac{\partial h_\delta}{\partial n_1}|_\Gamma \leqslant 0$, where n_1 is the outward normal vector on Γ as part of the boundary of the domain D_δ, we know that $\frac{\partial \underline{v}}{\partial n_1}|_\Gamma \leqslant 0$, which is required to ensure $\underline{v}$ is a weak sub-solution. For $x \in D_\delta$, since h_δ is harmonic, we see that

$$\begin{aligned} L\left(\frac{\underline{v}}{\gamma}\right) &= \frac{\underline{v}}{\gamma}\left(\frac{1}{\gamma} + \beta - \left(\frac{d}{\gamma}\right)^{1/3} h_{\tau_3(\frac{d}{\gamma})^{1/3}}\right) \\ &\geqslant \frac{\underline{v}}{\gamma}\left(\beta - \left(\frac{d}{\gamma}\right)^{1/3} \frac{c_{12}}{\tau_3(\frac{d}{\gamma})^{1/3}} \beta\right) \\ &= \frac{\underline{v}}{\gamma} \beta\left(1 - \frac{c_{12}}{\tau_3}\right) \geqslant 0 \end{aligned} \tag{A.14}$$

provided that $\tau_3 \geqslant c_{12}$. On $\beta(x) = \tau_3(d/\gamma)^{1/3}$,

$$\left|\frac{\partial}{\partial n_2}\left(\frac{d}{\gamma}\right)^{1/3} h_{\tau_3(\frac{d}{\gamma})^{1/3}}\right| \leqslant \left(\frac{d}{\gamma}\right)^{1/3} |\nabla h_{\tau_3(\frac{d}{\gamma})^{1/3}}| \leqslant \frac{c_{10}}{\tau_3} \leqslant \frac{\partial}{\partial n_2} \beta = \frac{\partial}{\partial n_2}\left(\frac{\underline{v}}{\gamma}\right),$$

where n_2 is the outward normal vector on $\partial D_\delta/\Gamma := \{x \colon \beta(x) = \tau_3(d/\gamma)^{1/3}\}$. Note that for $d/\gamma \ll 1$, $\frac{\partial\beta(x)}{\partial n_2} \geqslant c_{13} > 0$ for some small constant c_{13} and any x such that $\beta(x) = \tau_3(d/\gamma)^{1/3}$. This is due to our assumption $\frac{\partial\beta(x)}{\partial n_1} < 0$ for any

$x \in \Gamma$. This ensures that $\underline{v}/\gamma$ is a weak solution in the region $0 \leqslant \beta(x) \leqslant \tau_3(d/\gamma)^{1/3}$. Finally, consider the region $\beta(x) \geqslant \tau_3(d/\gamma)^{1/3}$. By definition (A.6),

$$L\left(\frac{\underline{v}}{\gamma}\right) = \frac{d}{\gamma}\Delta\beta - \frac{d}{2\gamma}\Delta z + \left[\beta - (\tau_3 - 1)\left(\frac{d}{\gamma}\right)^{1/3} - \frac{z}{2}\right] \times \left(\frac{1}{\gamma} + (\tau_3 - 1)\left(\frac{d}{\gamma}\right)^{1/3} + \frac{z}{2}\right). \tag{A.15}$$

It is easy to see that, for some constant c_{15},

$$\frac{d}{\gamma}\Delta\beta \geqslant -\frac{d}{\gamma}\|\Delta\beta\|_{L^\infty(\Omega)}, \qquad -\frac{d}{2\gamma}\Delta z \geqslant -c_{15}\tau_4^2\left(\frac{d}{\gamma}\right)^{2/3}, \tag{A.16}$$

$$\beta - (\tau_3 - 1)\left(\frac{d}{\gamma}\right)^{1/3} - \frac{z}{2} \geqslant \frac{1}{2}\left(\frac{d}{\gamma}\right)^{1/3}, \tag{A.17}$$

since $\beta \geqslant \tau_3(d/\gamma)^{1/3}$ and $z \leqslant (d/\gamma)^{1/3}$. By (A.16)–(A.18), we have

$$L\left(\frac{\underline{v}}{\gamma}\right) \geqslant \left(\frac{d}{\gamma}\right)^{2/3}\left(\frac{\tau_3 - 1}{2} - c_{15}\tau_4^2 - \left(\frac{d}{\gamma}\right)^{1/3}\|\Delta\beta\|_{L^\infty(\Omega)}\right) > 0 \tag{A.18}$$

provided that τ_3 is suitably large (depending on τ_4 only); on the other hand, by choosing τ_4 suitably large, we have $\frac{\partial \underline{v}}{\partial n}|_{\partial\Omega} \leqslant 0$. This ensures that $\underline{v}$ is a weak sub-solution of (A.1) provided that $d/\gamma \ll 1$. It is easy to check that if d/γ is sufficiently small,

$$\beta_+(x) - c_2\left(\frac{d}{\gamma}\right)^{1/3} \leqslant \frac{\underline{v}}{\gamma} \leqslant \frac{\bar{v}}{\gamma} \leqslant \beta_+(x) + c_2\left(\frac{d}{\gamma}\right)^{1/3} \tag{A.19}$$

for some $c_2 > 0$ large, independent of d and γ. Since (A.1) has a unique solution v, (A.2) follows from (A.19).

Remark A.2. The upper bound $\beta_+(x) + c_2(d/\gamma)^{1/3}$ seems to be optimal, while the lower bound $\beta_+(x) - c_2(d/\gamma)^{1/3}$ may be improved to $\beta_+(x) - c_2(d/\gamma)^{1/2}$ or even $\beta_+(x) - c_2(d/\gamma)^{2/3}$. However, it is unknown whether the lower bound $\beta_+ - c_2(d/\gamma)^{2/3}$ holds.

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