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Asymptotic energy concentration in the phase space of the weak solutions to the Navier–Stokes equations

Takahiro Okabe

Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan

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ABSTRACT

We study the asymptotic behavior of the energy of weak solutions of Navier–Stokes equations as $t \to \infty$. We characterize the space of the initial data which causes a concentration of the kinetic energy in the phase space. Moreover, an explicit convergence rate is obtained.

1. Introduction

In this paper we study the asymptotic behavior of the energy of weak solutions to the Navier–Stokes equations in $\mathbb{R}^n$, $n \geqslant 2$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla p = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = a, & \text{in } \mathbb{R}^n, \end{cases} \tag{N-S}$$

where $u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x,t) \in \mathbb{R}^n \times (0,\infty)$, while $a = a(x) = (a_1(x), \ldots, a_n(x))$ is a given initial velocity vector field.

E-mail address: sa6m08@math.tohoku.ac.jp.

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For the existence of weak solutions of (N-S), Leray [4] constructed a *turbulent solution* on $\mathbb{R}^3$ which satisfies the strong energy inequality and he proposed the problem whether or not $\lim_{t\to\infty}\|u(t)\|_{L^2(\mathbb{R}^3)}=0$ (for the definition of the turbulent solution, see Definition 2.1). Masuda [5] first gave a partial answer to Leray's problem and clarified that the strong energy inequality plays an important role for L^2-decay of weak solutions. Kato [3] proved the L^r-decay properties for strong solutions on $\mathbb{R}^n$ with small initial data (for the definition of the strong solution, see Definition 2.2). He also constructed a turbulent solution in $\mathbb{R}^4$. Using the uniqueness criterion for weak solutions given by Serrin [8], we may identify the turbulent solution with the strong solution after some definite time.

For L^2-decay property of solutions of (N-S), there are many results. Kajikiya and Miyakawa [2] proved that there exists a weak solution of (N-S) which behaves like the Stokes flow $e^{-tA}a$ asymptotically, where A denotes the Stokes operator. Later, Wiegner [11] proved that turbulent solutions have the L^2-decay property in Kajikiya and Miyakawa [2]. On the other hand, Schonbek [7] proved that there exists a weak solution which has the lower bound of L^2-decay.

Recently, another aspect of asymptotic behavior of the energy of solutions has been investigated. Skalák [9,10] proved the asymptotic energy concentration in the following sense:

$$\lim_{t\to\infty}\frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2}=1 \tag{1.1}$$

under the assumption that $\limsup_{t\to\infty}\|A^{1/2}u(t)\|_2/\|u(t)\|_2<\infty$ for the strong solution of (N-S), where $\{E_\lambda\}_{\lambda\geqslant 0}$ is the spectral decomposition of the Stokes operator A. With the Fourier transformation, we have $E_\lambda u(\xi,t)=\chi_{\{|\xi|\leqslant\sqrt{\lambda}\}}\hat{u}(\xi,t)$, where $\chi_{\{|\xi|\leqslant\sqrt{\lambda}\}}$ denotes the characteristic function on the set $\{|\xi|\leqslant\sqrt{\lambda}\}$. Hence the neighborhood near $\xi=0$ in the phase space plays an important role for the behavior of the energy of solutions asymptotically.

The purpose of the present paper is to characterize the set of initial values that causes (1.1). For this aim, we consider the set of initial data that causes a lower bound of the energy of solutions. More precisely, we introduce the set

$$K_{m,\alpha}^{\delta}=\left\{\phi\in L^2;\ \left|\hat{\phi}(\xi)\right|\geqslant\alpha|\xi|^m\ \text{for}\ |\xi|\leqslant\delta\right\} \tag{1.2}$$

for $\alpha,\delta>0$ and $m\geqslant 0$. The set $K_{m,\alpha}^{\delta}$ is a generalization of the set given by Schonbek [7]. We prove that if the initial data a belongs to $K_{m,\alpha}^{\delta}$, then the turbulent solution satisfies the energy concentration such as (1.1). Furthermore, the explicit convergence rate of $u(t)$ in (1.1) is shown.

In Section 2, we shall give our main results. Section 3 is devoted to preparing some lemmas, some of which were shown in the previous papers, Schonbek [7], Borchers and Miyakawa [1], Kajikiya and Miyakawa [2], Wiegner [11] and Kato [3]. However, we give an independent proof. In particular, the difference of decay rates between the solutions of (N-S) and Stokes flows is clarified, which yields the lower bound of the L^2-decay of the solution of (N-S). In Section 4, we prove main results. In Appendix A, we introduce the initial values which satisfy the assumption of Theorem 1 and of Theorem 2.

2. Results

Before stating our results we introduce some function spaces and give our definition of *turbulent solutions* of (N-S). $C_{0,\sigma}^{\infty}$ denotes the set of all C^∞-real vector functions ϕ with compact support in $\mathbb{R}^n$ such that $\operatorname{div}\phi=0$. L_σ^r is the closure of $C_{0,\sigma}^{\infty}$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ is the inner product in L^2. L^r stands for the usual (vector-valued) L^r-space over $\mathbb{R}^n$, $1\leqslant r\leqslant\infty$. $H_{0,\sigma}^1$ is the closure of $C_{0,\sigma}^{\infty}$ with respect to the norm $\|\phi\|_{H^1}=\|\phi\|_2+\|\nabla\phi\|_2$, where $\nabla\phi=(\partial\phi_i/\partial x_j)_{i,j=1,\dots,n}$. When X is a Banach space, we denote by $\|\cdot\|_X$ the norm on X. $C^m([t_1,t_2];X)$ and $L^r(t_1,t_2;X)$ are the usual Banach spaces, where $m=0,1,\dots,$ and t_1 and t_2 are real numbers such that $t_1<t_2$. In this paper we denote by C various constants.

Definition 2.1. Let $a \in L^2_\sigma$. A measurable function u defined on $\mathbb{R}^n \times (0, \infty)$ is called a turbulent solution of (N-S) if

(i) $u \in L^\infty(0, \infty; L^2_\sigma) \cap L^2(0, T; H^1_{0,\sigma})$ for all $0 < T < \infty$;
(ii) the relation

$$\int_0^T \bigl[-(u, \partial\phi/\partial t) + (\nabla u, \nabla\phi) + (u \cdot \nabla u, \phi)\bigr]\, dt = \bigl(a, \phi(0)\bigr)$$

holds for almost all T and all $\phi \in C^1([0, T); H^1_{0,\sigma} \cap L^n)$ such that $\phi(\cdot, T) = 0$;
(iii) the strong energy inequality

$$\bigl\|u(t)\bigr\|_2^2 + 2\int_s^t \bigl\|\nabla u(\tau)\bigr\|_2^2\, d\tau \leqslant \bigl\|u(s)\bigr\|_2^2 \tag{2.1}$$

holds for almost all $s \geqslant 0$, including $s = 0$, and all $t > s$.

We call a function u satisfying the above conditions (i) and (ii) a weak solution of (N-S). We can redefine any weak solution $u(t)$ of (N-S) on a set of measure zero of the time interval $(0, \infty)$ so that $u(t)$ is weakly continuous in t with values in L^2_σ. Moreover, such a redefined weak solution u satisfies for each $0 \leqslant s < t$,

$$\int_s^t \bigl[-(u, \partial\phi/\partial t) + (\nabla u, \nabla\phi) + (u \cdot \nabla u, \phi)\bigr]\, d\tau = -\bigl(u(t), \phi(t)\bigr) + \bigl(u(s), \phi(s)\bigr) \tag{2.2}$$

for all $\phi \in C^1([s, t]; H^1_{0,\sigma} \cap L^n)$, see Prodi [6]. The existence of turbulent solutions for $n = 3$ and $n = 4$ was given by Leray [4] and Kato [3], respectively.

Let us define the Stokes operator A_r in L^r_σ. We have the following Helmholtz decomposition:

$$L^r = L^r_\sigma \oplus G^r, \quad 1 < r < \infty,$$

where $G^r = \{\nabla p \in L^r;\ p \in L^r_{\mathrm{loc}}\}$. P_σ denotes the projection operator from L^r onto L^r_σ. The Stokes operator A_r is defined by $A_r = -P_\sigma \Delta$ with domain $D(A_r) = H^{2,r} \cap L^r_\sigma$. A_2 is nonnegative and self-adjoint operator on L^2_σ. For simplicity, A denotes the Stokes operator A_r if we have no possibility of confusion. $\{E_\lambda\}_{\lambda \geqslant 0}$ denotes the spectral decomposition of the nonnegative self-adjoint operator A.

Let us introduce the definition of strong solution of (N-S).

Definition 2.2. Let $n < r < \infty$, $a \in L^n_\sigma$. A measurable function u defined on $\mathbb{R}^n \times (0, \infty)$ is called a global strong solution of (N-S) if

$$u \in C\bigl([0, \infty); L^n_\sigma\bigr) \cap C\bigl((0, \infty); L^r\bigr), \tag{2.3}$$

$$\frac{\partial u}{\partial t}, Au \in C\bigl((0, \infty); L^n_\sigma\bigr), \tag{2.4}$$

and u satisfies

$$\frac{\partial u}{\partial t} + Au + P_\sigma(u \cdot \nabla u) = 0, \quad t > 0.$$

Now our results read:

Theorem 1. *Let* $2 \leqslant n \leqslant 4$, *and let* $r > 1$ *and* $m \geqslant 0$ *be*

(i) *for* $n = 2$,

$$1 < r < \frac{4}{3}, \qquad 0 \leqslant m < \frac{4}{r} - 3,$$

(ii) *for* $n = 3, 4$,

$$1 < r < \frac{n}{n-1}, \qquad 0 \leqslant m < \frac{n}{r} - (n-1).$$

Suppose that $K^{\delta}_{m,\alpha}$ *is the same as* (1.2). *If* $a \in L^r_\sigma \cap L^2_\sigma \cap K^{\delta}_{m,\alpha}$ *for some* $\alpha, \delta > 0$, *then for every turbulent solution* $u(t)$ *there exist* $T > 0$ *and* $C(n, r, m, \delta, \alpha, a) > 0$ *such that*

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \leqslant \frac{C}{\lambda} t^{-(n/r-n+1-m)} \tag{2.5}$$

holds for all λ *and for all* $t > T$.

Theorem 2. *Let* $n \geqslant 5$, *and let* $r > 1$ *and* $m \geqslant 0$ *be*

$$1 < r < \frac{n}{n-1}, \qquad 0 \leqslant m < \frac{n}{r} - (n-1).$$

Then there exists $\gamma > 0$ *such that if* $a \in L^r_\sigma \cap L^n_\sigma \cap K^{\delta}_{m,\alpha}$ *for some* $\alpha, \delta > 0$ *and if a satisfies* $\|a\|_n \leqslant \gamma$, *then there exists a unique global strong solution* $u(t)$ *with the following property. There exist* $T > 0$ *and* $C(n, r, m, \delta, \alpha, a) > 0$ *such that*

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \leqslant \frac{C}{\lambda} t^{-(n/r-n+1-m)} \tag{2.6}$$

holds for all λ *and for all* $t > T$.

Remark 3. Skalák [10] proved energy concentration (1.1) under the assumption $\limsup_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 < \infty$. From the assumption of Theorem 1 and of Theorem 2, we can show that $\lim_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 = 0$. On the other hand, our advantage seems to characterize the set of initial data which causes an energy concentration. Moreover, we get the explicit convergence rate of (1.1). We introduce the set $K^{\delta}_{m,\alpha}$ of initial data that causes (1.1), especially, causes the lower bound of the L^2-decay of the solutions of (N-S). (See also Schonbek [7].)

Remark 4. It seems to be an interesting question whether the similar energy concentration occurs in exterior domains where the Poincaré inequality does not hold. For that purpose, we need to obtain the lower bound of $\|u(t)\|_2$ as $t \to \infty$, which will be discussed in the forthcoming paper.

3. Preliminaries

Lemma 3.1. *Let* $n \geqslant 2$, *and let r be*

(i) *for* $n = 2$, $1 < r < 2$,
(ii) *for* $n \geqslant 3$, $1 < r \leqslant n/(n-1)$.

If $a \in L^r_\sigma \cap L^2_\sigma$ *then every turbulent solution* $u(t)$ *of* (N-S) *lies in* L^r_σ *for all* $t > 0$.

Proof. For each $\varphi \in C_{0,\sigma}^{\infty}$ we put $\phi(\tau) = e^{-(t-\tau)A}\varphi$. We substitute ϕ for the test function in (2.2) and obtain

$$\big(u(t), \varphi\big) = \big(u(s), e^{-(t-s)A}\varphi\big) - \int_s^t \big(u \cdot \nabla u, e^{-(t-\tau)A}\varphi\big)\, d\tau \tag{3.1}$$

for all $t > s \geqslant 0$. Since $u(t)$ is weakly continuous and uniformly bounded with respect to t, by the Hölder inequality we have

$$\begin{aligned} \big|\big(u(0), e^{-tA}\varphi\big)\big| &= \big|\big(a, e^{-tA}\varphi\big)\big| \\ &\leqslant \|a\|_r \big\|e^{-tA}\varphi\big\|_{r'} \\ &\leqslant \|a\|_r \|\varphi\|_{r'}. \end{aligned} \tag{3.2}$$

First, we consider the case $n = 2$. Let $1 \leqslant q \leqslant r$. By the Hölder and the Gagliardo–Nirenberg inequalities we have

$$\begin{aligned} \big|\big(u \cdot \nabla u, e^{-(t-\tau)A}\varphi\big)\big| &\leqslant \big|\big(u \cdot \nabla e^{-(t-\tau)A}\varphi, u\big)\big| \\ &\leqslant \|u\|_{2q}^2 \big\|\nabla e^{-(t-\tau)A}\varphi\big\|_{q'} \\ &\leqslant \|u\|_2^{2/q} \|\nabla u\|_2^{2-2/q} (t-\tau)^{-(1/r'-1/q')-1/2} \|\varphi\|_{r'} \\ &\leqslant \|a\|_2^{2/q} (t-\tau)^{-(1/q-1/r)-1/2} \|\nabla u\|_2^{2-2/q} \|\varphi\|_{r'}. \end{aligned} \tag{3.3}$$

Noting that $1 - q/r + q/2 < 1$, we obtain

$$\begin{aligned} \int_0^t \|\nabla u\|_2^{2-\frac{2}{q}} (t-\tau)^{-(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\, d\tau &\leqslant \left(\int_0^t \|\nabla u\|_2^2\, d\tau\right)^{\frac{1}{q'}} \left(\int_0^t (t-\tau)^{-1+\frac{q}{r}-\frac{q}{2}}\, d\tau\right)^{\frac{1}{q}} \\ &< C_t \|a\|_2^{2/q'}, \end{aligned}$$

where constant C_t depends on t, which implies

$$\left|\int_s^t \big(u \cdot \nabla u,\ e^{-(t-\tau)A}\varphi\big)\, d\tau\right| \leqslant C_t \|a\|_2^{2/q} \|a\|_2^{2/q'} \|\varphi\|_{r'}. \tag{3.4}$$

So (3.2), (3.4) and (3.1) with $s = 0$ yield by duality,

$$\begin{aligned} \big\|u(t)\big\|_r &= \sup_{\varphi \in L_\sigma^{r'}} \frac{|(u(t), \varphi)|}{\|\varphi\|_{r'}} \\ &\leqslant \|a\|_r + C_t \|a\|_2^2. \end{aligned}$$

Next we consider the case $n \geqslant 3$. Since $2 < 2r'/(r'-2) < 2n/(n-2)$, by the Hölder and the Sobolev inequalities we have

$$\begin{aligned}\left|\left(u\cdot\nabla u, e^{-(t-\tau)A}\varphi\right)\right| &\leqslant C\|\varphi\|_{r'}\|u\|_{2r'/(r'-2)}\|\nabla u\|_2\\ &\leqslant C\|\varphi\|_{r'}\|u\|_2^{1-n/r'}\|u\|_{2n/(n-2)}^{n/r'}\|\nabla u\|_2\\ &\leqslant C\|\varphi\|_{r'}\|u\|_2^{1-n/r'}\|\nabla u\|_2^{1+n/r'}\\ &\leqslant C\|\varphi\|_{r'}\|a\|_2^{1-n/r'}\|\nabla u\|_2^{1+n/r'}.\end{aligned}\tag{3.5}$$

Since

$$\int_0^t \|\nabla u\|_2^{1+n/r'}\,d\tau \leqslant t^{1/2-n/2r'}\left(\int_0^t \|\nabla u\|_2^2\,d\tau\right)^{1/2+n/2r'},$$

we have with some constant C_t

$$\left|\int_s^t \left(u\cdot\nabla u, e^{-(t-\tau)A}\varphi\right)d\tau\right| \leqslant C_t\|a\|_2^2\|\varphi\|_{r'}.$$

So the same argument as the case $n=2$ can be applied. The proof of Lemma 3.1 is complete. □

Lemma 3.2. *Let $n\geqslant 2$ and put $v(t)=e^{-tA}a$. If $a\in L^2_\sigma\cap K^\delta_{m,\alpha}$, then $v(t)$ satisfies*

$$\left\|v(t)\right\|_2 \geqslant Ct^{-\frac{1}{2}(m+\frac{n}{2})}\tag{3.6}$$

for all $t\geqslant 1$, where C depends on m,n,δ,α.

Proof. By Plancherel's theorem and changing variables we have

$$\begin{aligned}\left\|v(t)\right\|_2^2 = \left\|\hat v(t)\right\|_2^2 &\geqslant \int_{|\xi|\leqslant\delta} e^{-2t|\xi|^2}|\hat a|^2\,d\xi\\ &\geqslant \alpha^2\int_{|\xi|\leqslant\delta} e^{-2t|\xi|^2}|\xi|^{2m}\,d\xi\\ &= \alpha^2\left|\mathbb{S}^{n-1}\right|\int_0^\delta e^{-2t\rho^2}\rho^{2m+n-1}\,d\rho\\ &= \frac{\alpha^2|\mathbb{S}^{n-1}|}{(2t)^{-(m+n/2)}}\int_0^{\sqrt{2t}\,\delta} e^{-s^2}s^{2m+n-1}\,ds\\ &\geqslant Ct^{-(m+n/2)},\end{aligned}$$

where we put

$$C=\frac{\alpha^2|\mathbb{S}^{n-1}|}{2^{m+n/2}}\int_0^{\sqrt{2}\,\delta} e^{-s^2}s^{2m+n-1}\,ds.$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *There is a constant C depending only on n such that*

$$\left\| E_\lambda P_\sigma[u\cdot\nabla v]\right\| \leqslant C\lambda^{(n+2)/4}\|u\|_2\|v\|_2 \tag{3.7}$$

holds for all $\lambda>0$ and for all $u, v\in H^1_{0,\sigma}$ with $u\cdot\nabla v\in L^2$.

The proof is given by Kajikiya and Miyakawa [2].

The following lemma is originally proved by Wiegner [11]. However, we here give an independent proof which is based on the spectral decomposition of A.

Lemma 3.4. *Let $1<r<2$ and $a\in L^r_\sigma\cap L^2_\sigma$. We put $v(t)=e^{-tA}a$. Then every turbulent solution $u(t)$ of* (N-S) *has the following property*:

$$\left\|u(t)-v(t)\right\|_2=\begin{cases} O(t^{-(n/r-n/4-1/2)}), & n(\frac{1}{r}-\frac{1}{2})<1,\\ O(t^{-(n/4+1/2)}\log t), & n(\frac{1}{r}-\frac{1}{2})=1,\\ O(t^{-(n/4+1/2)}), & n(\frac{1}{r}-\frac{1}{2})>1,\end{cases}$$

as $t\to\infty$.

Proof. Let $w(t)=u(t)-v(t)$. Since $u(t)$ and $v(t)$ satisfy strong the energy inequality (2.1), we obtain

$$\begin{aligned}
&\left\|w(t)\right\|_2^2+2\int_s^t\left\|\nabla w(\tau)\right\|_2^2d\tau\\
&\quad=\left\|u(t)\right\|_2^2+\left\|v(t)\right\|_2^2-2\big(u(t),v(t)\big)+2\int_s^t\left[\left\|\nabla u(\tau)\right\|_2^2+\left\|\nabla v(\tau)\right\|_2^2-2\big(\nabla u(\tau),\nabla v(\tau)\big)\right]d\tau\\
&\quad\leqslant\left\|u(s)\right\|_2^2+\left\|v(s)\right\|_2^2-2\big(u(t),v(t)\big)-4\int_s^t\big(\nabla u(\tau),\nabla v(\tau)\big)\,d\tau
\end{aligned}\tag{3.8}$$

for almost all $s\geqslant 0$, including $s=0$, and all $t>s$. We substitute $\phi(\tau)=v(\tau)$ for the test function in (2.2) and obtain

$$\big(u(t),v(t)\big)+2\int_s^t\big(\nabla u(\tau),\nabla v(\tau)\big)\,d\tau+\int_s^t\big(u(\tau)\cdot\nabla u(\tau),v(\tau)\big)\,d\tau=\big(u(s),v(s)\big) \tag{3.9}$$

for almost all $s\geqslant 0$, including $s=0$, and all $t>s$, since $dv/dt=-Av$. Hence (3.8) and (3.9) yield

$$\left\|w(t)\right\|_2^2+2\int_s^t\left\|\nabla w(\tau)\right\|_2^2d\tau\leqslant\left\|w(s)\right\|_2^2+2\int_s^t\big(u(\tau)\cdot\nabla u(\tau),v(\tau)\big)\,d\tau \tag{3.10}$$

for almost all $s\geqslant 0$, including $s=0$, and all $t>s$. We estimate the last term in (3.10). Since $(u\cdot\nabla v,v)=0$, by the Hölder and the Young inequalities we have

$$\begin{aligned}\big|\big(u(\tau)\cdot\nabla u(\tau), v(\tau)\big)\big| &= \big|\big(u(\tau)\cdot\nabla w(\tau), v(\tau)\big)\big|\\ &\leqslant \|u(\tau)\|_2\|\nabla w(\tau)\|_2\|v(\tau)\|_\infty\\ &\leqslant C\tau^{-n/2r}\|u(\tau)\|_2\|\nabla w(\tau)\|_2\|a\|_r\\ &\leqslant \frac{1}{2}\|\nabla w(\tau)\|_2^2 + \frac{C}{2}\|u(\tau)\|_2^2\|a\|_r^2\tau^{-n/r}. \end{aligned}\tag{3.11}$$

Hence (3.10) and (3.11) yield

$$\|w(t)\|_2^2 + \int_s^t \|\nabla w(\tau)\|_2^2\,d\tau \leqslant \|w(s)\|_2^2 + C\int_s^t \tau^{-n/r}\|u(\tau)\|_2^2\|a\|_r^2\,d\tau \tag{3.12}$$

for almost all $s \geqslant 0$, including $s = 0$, and all $t > s$. Let λ be any smooth positive function on $(0,\infty)$. From (3.12) and the estimate

$$\begin{aligned}\|\nabla w(\tau)\|_2^2 = \|A^{1/2}w(\tau)\|_2^2 &= \int_0^\infty \rho\,d\|E_\rho w\|_2^2\\ &\geqslant \int_{\lambda(\tau)}^\infty \rho\,d\|E_\rho w\|_2^2\\ &\geqslant \lambda(\tau)\big[\|w(\tau)\|_2^2 - \|E_{\lambda(\tau)}w(\tau)\|_2^2\big],\end{aligned}$$

we obtain

$$\begin{aligned}&\|w(t)\|_2^2 + \int_s^t \lambda(\tau)\|w(\tau)\|_2^2\,d\tau\\ &\quad\leqslant \|w(s)\|_2^2 + \int_s^t \lambda(\tau)\|E_{\lambda(\tau)}w(\tau)\|_2^2\,d\tau + C\int_s^t \tau^{-n/r}\|u(\tau)\|_2^2\|a\|_r^2\,d\tau.\end{aligned}\tag{3.13}$$

To estimate the term $\|E_{\lambda(t)}w(t)\|_2$ we go back to (2.2). Since $E_\lambda\varphi \in L^n$ for all $\varphi \in C_{0,\sigma}^\infty$ and for all $\lambda > 0$, we may choose $\phi(\tau) = e^{-(t-\tau)A}E_{\lambda(t)}\varphi$ as the test function of (2.2). It follows that

$$\begin{aligned}\big(E_{\lambda(t)}w(t),\varphi\big) &= \big(u(s),\phi(s)\big) - \big(v(s),\phi(s)\big) - \int_s^t \big(u(\tau)\cdot\nabla u(\tau),\phi(\tau)\big)\,d\tau\\ &= \big(w(s), e^{-(t-s)A}E_{\lambda(t)}\varphi\big) - \int_s^t \big(u(\tau)\cdot\nabla u(\tau),\phi(\tau)\big)\,d\tau\end{aligned}\tag{3.14}$$

for all $t > s \geqslant 0$. By Lemma 3.3 we have

$$\begin{aligned}\big|\big(u(\tau)\cdot\nabla u(\tau),\phi(\tau)\big)\big| &= \big|\big(E_{\lambda(t)}P_\sigma\big[u(\tau)\cdot\nabla u(\tau)\big], e^{-(t-\tau)A}\varphi\big)\big|\\ &= \big\|E_{\lambda(t)}P_\sigma\big[u(\tau)\cdot\nabla u(\tau)\big]\big\|_2\big\|e^{-(t-\tau)A}\varphi\big\|_2\\ &\leqslant C\lambda(t)^{(n+2)/4}\|u(\tau)\|_2^2\|\varphi\|_2.\end{aligned}\tag{3.15}$$

Since $w(t)$ is weakly continuous in L^2_σ and $\|w(t)\|_2$ is bounded with respect to t, we have by (3.14) and (3.15) with $s=0$ that

$$\left\|E_{\lambda(t)}w(t)\right\|_2 \leqslant C\lambda(t)^{(n+2)/4}\int_0^t \left\|u(\tau)\right\|_2^2\,d\tau. \tag{3.16}$$

Hence (3.13) and (3.16) yield

$$\begin{aligned}
&\left\|w(t)\right\|_2^2+\int_s^t \lambda(\tau)\left\|w(\tau)\right\|_2^2\,d\tau \\
&\quad\leqslant \left\|w(s)\right\|_2^2+C\int_s^t \lambda(\tau)^{(n+4)/2}\left(\int_0^\tau \left\|u(\sigma)\right\|_2^2\,d\sigma\right)^2 d\tau + C\int_s^t \tau^{-n/r}\left\|u(\tau)\right\|_2^2\|a\|_r^2\,d\tau.
\end{aligned} \tag{3.17}$$

In (3.17) we put

$$y(t)=\left\|w(t)\right\|_2^2,$$
$$g(t,s)=C\left[\int_s^t \lambda(\tau)^{(n+4)/2}\left(\int_0^\tau \left\|u(\sigma)\right\|_2^2\,d\sigma\right)^2 d\tau+\int_s^t \tau^{-n/r}\left\|u(\tau)\right\|_2^2\,d\tau\right],$$

and obtain

$$y(t)-g(t,s)+\int_s^t \lambda(\tau)y(\tau)\,d\tau \leqslant y(s) \tag{3.18}$$

for a.e. $s\in(0,t)$. We now want to apply Gronwall's lemma to (3.18) with respect to s. Consider the function $h(s)=\int_s^t \lambda(\tau)y(\tau)\,d\tau$, which is almost everywhere differentiable in $(0,t)$ with $h'\in L^\infty((\delta,t))$ for small $\delta>0$. From (3.18) we have

$$h'(\tau)=-\lambda(\tau)y(\tau)\leqslant -\lambda(\tau)\big[y(t)+h(\tau)-g(t,\tau)\big]. \tag{3.19}$$

Let $H\geqslant 0$ be a solution of the equation $H'(\tau)=\lambda(\tau)H(\tau)$. Multiplying (3.19) by H and then integrating over $[s,t]$, we have

$$\big(H(t)-H(s)\big)y(t)\leqslant H(s)h(s)+\int_s^t H'(\tau)g(t,\tau)\,d\tau, \tag{3.20}$$

since $h(t)=0$. Applying (3.18) to the right-hand side in (3.20) and integrating by parts, we obtain

$$H(t)y(t)\leqslant H(s)y(s)-\int_s^t H(\tau)\frac{\partial g}{\partial\tau}(t,\tau)\,d\tau, \tag{3.21}$$

since $g(t,t)=0$. Now choose $\lambda(\tau)=m\tau^{-1}$, $m>0$, so that $H(\tau)=\tau^m$. Since (3.21) holds for almost every $s\geqslant 0$ and since $y(s)$ is bounded, by taking m sufficiently large we can pass the limit $s\to 0$ in (3.21) to obtain

$$t^m\|w(t)\|_2^2\leqslant C\left[t^{m-n/2-2}\int_0^t\left(\int_0^\tau\|u(\sigma)\|_2^2\,d\sigma\right)^2 d\tau+t^{m-n/r}\int_0^t\|u(\tau)\|_2^2\,d\tau\right]. \tag{3.22}$$

Now we note that the turbulent solution of (N-S) becomes the strong solution of (N-S) after some definite time. So by the energy inequality and by the decay estimate proved by Kato [3], for each turbulent solution $u(t)$, the estimate

$$\|u(t)\|_2\leqslant Ct^{-n(1/r-1/2)/2}$$

holds, if the initial data $a\in L^r_\sigma\cap L^2_\sigma$, $1<r<2$.

First we consider the case $n(1/r-1/2)<1$. Since $\|u(\tau)\|_2^2\leqslant C\tau^{-n(1/r-1/2)}$, (3.22) yields

$$\|w(t)\|_2^2\leqslant Ct^{1+n/2-2n/r}.$$

We next consider the case $n(1/r-1/2)=1$. Since $\|u(\tau)\|_2^2\leqslant C(1+\tau)^{-1}$ we have

$$\begin{aligned}\|w(t)\|_2^2&\leqslant C\left[t^{-n/2-1}\bigl(\log(1+t)\bigr)^2+t^{-n/r}\log(1+t)\right]\\&\leqslant Ct^{-n/2-1}\bigl(\log(1+t)\bigr)^2\end{aligned}$$

for large t.

Finally we consider the case $n(1/r-1/2)>1$. Since $\|u(\tau)\|_2^2\leqslant(1+\tau)^{-1-\beta}$ for some $\beta>0$, and so $\int_0^\infty\|u(\tau)\|_2^2\,d\tau<\infty$, (3.22) gives

$$\|w(t)\|_2^2\leqslant C\left[t^{-1-n/2}+t^{-n/r}\right]. \tag{3.23}$$

Since $1+n/2<n/r$, we obtain the desired result. The proof of Lemma 3.4 is complete. □

Lemma 3.5. *Let $n\geqslant 2$, and let r and m be as*

(i) $1<r\leqslant 2n/(n+2)$, $0\leqslant m<1$,

or

(ii) $2n/(n+2)<r<2n/(n+1)$, $0\leqslant m<2n/r-n-1$.

If $a\in L^r_\sigma\cap L^2_\sigma\cap K^\delta_{m,\alpha}$ for some $\alpha,\delta>0$, then for every turbulent solution $u(t)$ of (N-S) *there exist $T>1$ and constant C such that*

$$\|u(t)\|_2\geqslant Ct^{-\frac{1}{2}(m+\frac{n}{2})} \tag{3.24}$$

for all $t\geqslant T$.

Proof. Let $v(t)=e^{-tA}a$. It suffices to show that

$$\lim_{t\to\infty}\frac{\|u-v\|_2}{\|v\|_2}=0, \tag{3.25}$$

due to Lemma 3.2. Indeed, under the condition (3.25), there exists $T > 1$ such that

$$\frac{\|u(t) - v(t)\|_2}{\|v\|_2} \leqslant \frac{1}{2}$$

for all $t > T$. Hence by the triangle inequality and Lemma 3.2 we have

$$\begin{aligned}
\|u(t)\|_2 &\geqslant \|v(t)\|_2 - \|u(t) - v(t)\|_2 \\
&= \|v(t)\|_2 \left(1 - \frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2}\right) \\
&\geqslant \frac{1}{2}\|v(t)\|_2 \\
&\geqslant Ct^{-\frac{1}{2}(m+\frac{n}{2})}
\end{aligned}$$

for all $t \geqslant T$.

Now it remains to prove (3.25). First we consider the case $1 < r < 2n/(n+2)$. The assumption (i) implies $n(1/r - 1/2) > 1$ and $(m + n/2)/2 < (n/4 + 1/2)$. Hence from Lemmas 3.2 and 3.4 it follows that

$$\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} \leqslant C\frac{t^{-(n/4+1/2)}}{t^{-(m+n/2)/2}} \to 0,$$

as $t \to \infty$.

Next we consider the case $r = 2n/(n+2)$. Since $m < 1$ and $n(1/r - 1/2) = 1$ by the assumption (i), we have

$$\begin{aligned}
\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} &\leqslant C\frac{t^{-(n/4+1/2)}\log t}{t^{-(m+n/2)/2}} \\
&= C\frac{t^{(-n/4+1/2)}}{t^{-(m+n/2)/2-\varepsilon/2}}\left(\frac{\log t}{t^{\varepsilon/2}}\right) \\
&\to 0,
\end{aligned}$$

as $t \to \infty$.

Finally we consider the case $2n/(n+2) < r < 2n/(n+1)$. The assumption (ii) implies $n(1/r - 1/2) < 1$ and $(m + n/2)/2 < n/r - n/4 - 1/2$. Hence we obtain

$$\frac{\|u(t) - v(t)\|_2}{\|v(t)\|_2} \leqslant C\frac{t^{-(n/r-n/4-1/2)}}{t^{-(m+n/2)/2}} \to 0,$$

as $t \to \infty$. The proof of Lemma 3.5 is complete. □

Lemma 3.6. *Let* $2 \leqslant n \leqslant 4$. *Let* r *and* m *be as*

(i) *for* $n = 2$,

$$1 < r < \frac{4}{3}, \qquad 0 \leqslant m < \frac{4}{r} - 3,$$

(ii) *for* $n \geqslant 3$,

$$1 < r < \frac{n}{n-1}, \qquad 0 \leqslant m < \frac{n}{r} - (n-1).$$

If $a \in L^r_\sigma \cap L^2_\sigma \cap K^\delta_{0,\sigma}$, *then every turbulent solution* $u(t)$ *of* (N-S) *satisfies*

$$\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2} \leqslant O\big(t^{-(n/r-n+1-m)}\big), \tag{3.26}$$

as $t \to \infty$.

Proof. By the well-known Leray's structure theorem, every turbulent solution of (N-S) becomes the strong solution after some definite time. Furthermore, it is shown by Kato [3] that the strong solution $u(t)$ decays in the same way as the Stokes flow $e^{-tA}a$ as $t \to \infty$. Since $a \in L^r_\sigma \cap L^2_\sigma$ for $1 < r < 2$, we have $\|\nabla u(t)\|_2 \leqslant Ct^{-n(1/r-1/2)/2-1/2}$ for sufficiently large t. Hence by Lemma 3.5, we obtain (3.26). $\square$

4. Proof of main results

4.1. Proof of Theorem 1

As we mentioned above, turbulent solutions of (N-S) become strong solutions after some definite time. So for the turbulent solution $u(t)$ of (N-S) there exists $T_* > 0$ such that $u(t)$ is strong solution of (N-S) on $[T_*, \infty)$. Hence we have the energy identity

$$\frac{d}{dt}\big\|u(t)\big\|_2^2 + 2\big\|A^{1/2}u(t)\big\|_2^2 = 0 \tag{4.1}$$

for $t \geqslant T_*$. For any fixed $\lambda > 0$, the second term in (4.1) is estimated from below as

$$\begin{aligned}\big\|A^{1/2}u(t)\big\|_2^2 &= \int_0^\infty \rho\, d\|E_\rho u\|_2^2 \geqslant \int_\lambda^\infty \rho\, d\|E_\rho u\|_2^2 \\ &\geqslant \lambda \int_\lambda^\infty d\|E_\rho u\|_2^2 \geqslant \frac{\lambda}{2}\big(\big\|u(t)\big\|_2^2 - \big\|E_\lambda u(t)\big\|_2^2\big).\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we have

$$\frac{d}{dt}\big\|u(t)\big\|_2^2 + \lambda\big\|u(t)\big\|_2^2 \leqslant \lambda\big\|E_\lambda u(t)\big\|_2^2. \tag{4.3}$$

Dividing both sides of (4.3) by $\lambda\|u(t)\|_2^2$, we obtain

$$\frac{\frac{d}{dt}\|u(t)\|_2^2}{\lambda\|u(t)\|_2^2} + 1 \leqslant \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2}. \tag{4.4}$$

On the other hand, by (4.1), we have $(d/dt)\|u(t)\|_2^2 = -2\|A^{1/2}u(t)\|_2^2 = -2\|\nabla u(t)\|_2^2$, from which and (4.4) it follows that

$$1 - \frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2} \leqslant \frac{2}{\lambda}\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2}.$$

Hence by Lemma 3.6, there exists T such that

$$\left|\frac{\|E_\lambda u(t)\|_2^2}{\|u(t)\|_2^2}-1\right| \leqslant \frac{C}{\lambda} t^{-(n/r-n+1-m)}$$

for all $t \geqslant T$. This completes the proof of Theorem 1.

4.2. *Proof of Theorem 2*

Take γ sufficiently small so that if $\|a\|_n \leqslant \gamma$ there exists a unique global strong solution $u(t)$ of (N-S) satisfying the following decay properties

$$\big\|u(t)\big\|_2 = O\big(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})}\big),$$
$$\big\|\nabla u(t)\big\|_2 = O\big(t^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{2})-\frac{1}{2}}\big),$$

as $t \to \infty$. Since the strong solution becomes necessarily the turbulent solution and since Lemma 3.6 does work for $n \geqslant 5$ under the assumption of Theorem 2, we see that the same arguments in the proof of Theorem 1 can be applied. The proof of Theorem 2 is complete.

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Appendix A. Example of initial values

We can construct the initial values satisfying the assumption of Theorem 1 and of Theorem 2 when the dimension n is even.

Let $n = 2k$ for some $k \in \mathbb{Z}$. For each $1 \leqslant j \leqslant k$, we put

$$M_j = \begin{pmatrix} 0 & R_{2j} \\ -R_{2j-1} & 0 \end{pmatrix},$$

where $R_j = (\partial/\partial x_j)(-\Delta)^{-1/2}$ denotes the Riesz operator. Then we set

$$a = \begin{pmatrix} M_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M_k \end{pmatrix} \begin{pmatrix} e^{-|x|^2} \\ \vdots \\ e^{-|x|^2} \end{pmatrix}.$$

With the Fourier transformation, we have

$$\hat{a}(\xi) = Ce^{-|\xi|^2/4}\left(-i\frac{\xi_2}{|\xi|}, i\frac{\xi_1}{|\xi|}, \dots, -i\frac{\xi_{2k}}{|\xi|}, i\frac{\xi_{2k-1}}{|\xi|}\right). \tag{A.1}$$

Since $i\xi \cdot \hat{a}(\xi) = 0$, we obtain $\operatorname{div} a = 0$.

On the other hand, for any fixed $\delta > 0$, (A.1) yields

$$\big|\hat{a}(\xi)\big| \geqslant Ce^{-\delta^2/4}$$

for all $|\xi| \leqslant \delta$. So it follows that $a \in K^{\delta}_{0,\alpha}$ for some $\alpha, \delta > 0$.

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